The Mirrlees-Problem Revisited

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December 1997

Abstract

Optimal incentive schemes need not be complicated. In a hidden action model with lognormally distributed output, Mirrlees (1974) illustrates that the first-best outcome can be approached arbitrarily closely by a suitably chosen sequence of step functions. The present paper shows that this result extends to any probability distribution that satisfies two conditions: 1) a convexity condition which ensures that the first-order approach is valid, and 2) a likelihood ratio condition which implies that low output values are a reliable signal that the agent has shirked. Both conditions are met by the normal, lognormal, gamma, beta, chi-squared, Weibull, t-, and F-distribution.

*This is a revised version of an earlier paper entitled "Approximating the First-Best in Agency Models with Step Function Contracts," Working Papers in Economics E-96-4, Hoover Institution, which was written while I was visiting Stanford University. Financial support from Schweizerischer Nationalfonds and Deutsche Forschungsgemeinschaft, Sonderforschungsbereich 504 is gratefully acknowledged. I would like to thank Heinz Müller, Klaus Schmidt, and Dimitri Vayanos for helpful comments and suggestions.

1 Introduction

The optimal sharing rule under moral hazard is typically very complex as it depends in a non-trivial fashion on the relationship between output and the likelihood ratio (Holmström (1979), Grossman and Hart (1983)). And yet, simple incentive schemes can perform extremely well under certain conditions. For instance, Mirrlees (1974) shows that the first-best outcome can be approached arbitrarily closely by a suitably chosen sequence of step functions if output is lognormally distributed (since a solution does not exist, this is sometimes called the 'Mirrlees-problem'). The example by Mirrlees is special in many respects: 1) output is lognormally distributed, 2) it is implicitly assumed that the first-order approach is valid, and 3) the example is phrased in the context of a production economy with a large number of agents. In the meantime, further examples in the same spirit have been constructed and the Mirrlees-problem has become an integral part of most contract theory courses.[1] However, to our knowledge, a general version of the Mirrlees-problem does not exist.

The present paper is an attempt to fill this gap. Based on the standard principal-agent model, we show that two conditions are sufficient for step functions to be near first-best optimal: 1) a convexity condition, which ensures that the first-order approach is valid, and 2) a likelihood ratio condition, which implies that low output values are a reliable signal that the agent has shirked. In our main theorem, we construct a sequence of step functions which has the property that each step function is individually rational and implements the first-best action. As the cutoff approaches the lower bound of the support of the underlying probability distribution, the payment to the right of the cutoff converges to the first-best payment and the payment to the left of the cutoff ('the penalty') goes to $-\infty$. Since the agent is only punished for output values that indicate with probability close to 1 that he has shirked, the risk of erroneous punishment becomes negligible. In the limit, the agent receives the first-best payment with probability 1 and is punished infinitely hard with probability 0. Given the assumption that the principal is risk-neutral, this result is asymptotically first-best efficient.

The remainder of the paper is organized as follows: section 2 presents the model and shows that a convexity condition and a likelihood ratio condition are sufficient for step functions to be near first-best optimal. In section 3, we show that both conditions are satisfied by a wide range of known probability distributions. Section 4 concludes.

[1] To our knowledge, all these examples use normally distributed output.

2 The Model and Main Result

The model is the standard principal-agent model with hidden action. At time 0, principal and agent agree on a sharing rule $s : X \to \mathcal{R}$ which maps outcomes into payments to the agent. Subsequently, the agent chooses the parameter a of a probability distribution with lower bound $\underline{x} \in \mathcal{R}$ and outcomes $x \in X \subseteq \mathcal{R}$. The parameter a is unobservable and takes values in some open set $A \subseteq \mathcal{R}_+$. Additionally, the agent incurs effort cost $g(a)$, where $g'(\cdot) > 0$ and $g''(\cdot) \geq 0$. At time 1, x is observed and the agent receives $s(x)$. The principal is risk-neutral and the agent has additively separable von Neumann-Morgenstern utility $U(w, a) = u(w) - g(a)$, where $u(\cdot)$ is defined on $W = (\underline{w}, \infty)$ with $u'(\cdot) > 0$, $u''(\cdot) < 0$ and $\underline{w} \in [-\infty, \infty)$. The following four assumptions are crucial:

1. X is invariant with respect to a.
2. $\lim_{w \to \underline{w}} u(w) = -\infty$.
3. Condition LR: $\lim_{x \to \underline{x}} \frac{f_a(x|a)}{f(x|a)} = -\infty \; \forall a \in A$.
4. Condition C: $\exists\, K \in X$ such that $F_{aa}(\overline{x}\,|a) > 0 \; \forall \overline{x} < K$ and $a \in A$.

Assumption 1 ensures that the solution to the principal's problem is nontrivial. If the agent could choose the support of x, the first-best outcome could be attained by means of a forcing contract (Harris and Raviv (1979)). Assumption 2 requires that the agent's utility for wealth be unbounded from below. This assumption is met by many utility functions such as log-utility or negative exponential utility. Assumption 3 states that the likelihood ratio should go to $-\infty$ as x approaches the lower bound of the support. The role of this assumption is discussed at the end of this section. Finally, assumption 4 requires that the CDF be strictly convex in a at the left tail of the distribution. As is shown in the following section, conditions LR and C are innocuous and are satisfied by many known probability distributions.

The principal's first-best expected utility is

$$E\left[x\,|a^*_{FB}\right] - u^{-1}\left(W_A + g\left(a^*_{FB}\right)\right), \tag{1}$$

where $u^{-1}(\cdot)$ is the inverse of $u(\cdot)$, W_A is the agent's reservation utility, and a^*_{FB} is the first-best action. We will now show that (1) can be approached arbitrarily closely by a suitably chosen sequence of step functions. That is, we assume that $s(x)$ takes the form

$$s(x) = \begin{cases} \underline{s} \text{ if } x < \overline{x} \\ \overline{s} \text{ if } x \geq \overline{x} \end{cases} \tag{2}$$

and confine ourselves to finding the values of $\underline{s}$, $\overline{s}$, and $\overline{x}$ that minimize the cost of implementing a^*_{FB} subject to the agent's individual rationality and incentive compatibility constraint. Formally,

$$\max_{\underline{s},\overline{s},\overline{x}} E\left[x\left|a^*_{FB}\right.\right] - \underline{s}F\left(\overline{x}\left|a^*_{FB}\right.\right) - \overline{s}\left(1 - F\left(\overline{x}\left|a^*_{FB}\right.\right)\right) \tag{3}$$

s.t.

$$u\left(\underline{s}\right)F\left(\overline{x}\left|a^*_{FB}\right.\right) + u\left(\overline{s}\right)\left(1 - F\left(\overline{x}\left|a^*_{FB}\right.\right)\right) - g\left(a^*_{FB}\right) \geq W_A \tag{4}$$

and

$$a^*_{FB} \in \arg\max_{a\in A} \; u\left(\underline{s}\right)F\left(\overline{x}\left|a\right.\right) + u\left(\overline{s}\right)\left(1 - F\left(\overline{x}\left|a\right.\right)\right) - g\left(a\right). \tag{5}$$

Since A is an infinite set, (5) constitutes a continuum of incentive compatibility constraints and standard solution techniques are not applicable. Following a tradition in the literature, we employ the first-order approach and replace (5) with the agent's first-order condition at $a = a^*_{FB}$

$$\left[u\left(\underline{s}\right) - u\left(\overline{s}\right)\right]F_a\left(\overline{x}\left|a^*_{FB}\right.\right) - g'\left(a^*_{FB}\right) = 0. \tag{6}$$

While (5) requires that a^*_{FB} be a global maximum, (6) merely states that a^*_{FB} be a stationary point of the agent's problem. Hence, working with (6) instead of (5) is typically invalid unless it can be ensured that the agent's objective function is globally concave. The following proposition shows that condition C is sufficient for the validity of the first-order approach.

Proposition 2.1: For all step functions with $\overline{s} > \underline{s}$ and $\overline{x} < K$, the agent's problem is strictly concave in a.

Proof: The agent's problem is

$$\max_a \left[u\left(\underline{s}\right) - u\left(\overline{s}\right)\right]F\left(\overline{x}\left|a\right.\right) + u\left(\overline{s}\right) - g\left(a\right). \tag{7}$$

The result follows then immediately from $g''(\cdot) \geq 0$ and assumption 4. ■

Proposition 2.1 states that any step function that satisfies (6), $\overline{s} > \underline{s}$ and $\overline{x} < K$ is incentive compatible, i.e. it satisfies (5). Note that the reverse may not be true, i.e. there may exist step functions that satisfy (5) but not $\overline{s} > \underline{s}$ or $\overline{x} < K$.[2] In this case, the substitution of (6), $\overline{s} > \underline{s}$ and $\overline{x} < K$ for (5)

[2]Whether the reverse also holds depends on the underlying probability distribution. For instance, if the distribution satisfies both first-order stochastic dominance (FOSD) and the convexity of the distribution function condition (CDFC), any step function that satisfies (4) (with equality) and (5) automatically satisfies $\overline{s} > \underline{s}$ and $\overline{x} < K$ for $K = \sup X$, which implies that there is no loss of generality from using the first-order approach.

implies a reduction of the set of admissible step functions. As is shown later, however, this has no implications for the optimal solution. We can therefore replace the principal's original problem (3)-(5) with the relaxed problem

$$\max_{\underline{s},\overline{s},\overline{x}} E\left[x\left|a_{FB}^{*}\right.\right]-\underline{s}F\left(\overline{x}\left|a_{FB}^{*}\right.\right)-\overline{s}\left(1-F\left(\overline{x}\left|a_{FB}^{*}\right.\right)\right) \tag{8}$$

s.t.

$$u\left(\underline{s}\right)F\left(\overline{x}\left|a_{FB}^{*}\right.\right)+u\left(\overline{s}\right)\left(1-F\left(\overline{x}\left|a_{FB}^{*}\right.\right)\right)-g\left(a_{FB}^{*}\right)\geq W_{A}; \tag{9}$$

$$\left[u\left(\underline{s}\right)-u\left(\overline{s}\right)\right]F_{a}\left(\overline{x}\left|a_{FB}^{*}\right.\right)-g'\left(a_{FB}^{*}\right)=0, \tag{10}$$

$$\overline{s}>\underline{s}, \tag{11}$$

and

$$\overline{x}<K. \tag{12}$$

Standard results show that the individual rationality constraint (9) must hold with equality in equilibrium. This implies that we can solve (9) (with equality) and (10) explicitly for $\underline{s}$ and $\overline{s}$ as a function of $\overline{x}$:

$$\underline{s}\left(\overline{x}\right)=u^{-1}\left(W_{A}+g\left(a_{FB}^{*}\right)+g'\left(a_{FB}^{*}\right)\frac{1-F\left(\overline{x}\left|a_{FB}^{*}\right.\right)}{F_{a}\left(\overline{x}\left|a_{FB}^{*}\right.\right)}\right) \tag{13}$$

and

$$\overline{s}\left(\overline{x}\right)=u^{-1}\left(W_{A}+g\left(a_{FB}^{*}\right)-g'\left(a_{FB}^{*}\right)\frac{F\left(\overline{x}\left|a_{FB}^{*}\right.\right)}{F_{a}\left(\overline{x}\left|a_{FB}^{*}\right.\right)}\right). \tag{14}$$

The following proposition states that we can simplify the principal's relaxed problem (8)-(12) further using (13)-(14).

Proposition 2.2: There exists a value $H\in X$ such that for all $\overline{x}<H$, (13)-(14) implies (9)-(11).

Proof: Reversing the steps that lead to (13)-(14) shows that (13)-(14) implies (10) and (9) with equality and therefore (9)-(10). By assumption 3, $\lim\limits_{x\to\underline{x}}\frac{f_{a}(x|a)}{f(x|a)}=-\infty$ for all $a\in A$, which implies that there exists a value $H\in X$ such that

$$F_{a}\left(\overline{x}\left|a_{FB}^{*}\right.\right)=\int_{\underline{x}}^{\overline{x}}\frac{f_{a}\left(x\left|a_{FB}^{*}\right.\right)}{f\left(x\left|a_{FB}^{*}\right.\right)}f\left(x\left|a_{FB}^{*}\right.\right)dx \tag{15}$$

is negative for all $\overline{x}<H$. From $F_{a}\left(\overline{x}\left|a_{FB}^{*}\right.\right)<0$ and the fact that $u^{-1}(\cdot)$ is increasing it then follows that for all $\overline{x}<H$, (13)-(14) implies (11). ■

By propositions 2.1 and 2.2, any step function that satisfies (13)-(14) and $\overline{x}<\min[H,K]$ is individually rational and incentive compatible, i.e.

it implements the first-best action a_{FB}^{*}. As in the case of proposition 2.1, the reverse of proposition 2.2 is typically not true.[3] It is therefore likely that substituting (13)-(14) and $\overline{x} < H$ for (9)-(11) implies that the set of admissible step functions has shrunk. However, as is shown in theorem 2.1 below, this has once again no effect on the optimal solution. Replacing (9)-(11) with (13)-(14) and $\overline{x} < H$ and inserting (13)-(14) into the objective function (8) yields

$$\begin{aligned} &\max_{\overline{x}} E\left[x\left|a_{FB}^{*}\right.\right] \\ &-u^{-1}\left(W_A + g\left(a_{FB}^{*}\right) + g'\left(a_{FB}^{*}\right)\frac{1-F\left(\overline{x}\left|a_{FB}^{*}\right.\right)}{F_a\left(\overline{x}\left|a_{FB}^{*}\right.\right)}\right)F\left(\overline{x}\left|a_{FB}^{*}\right.\right) \\ &-u^{-1}\left(W_A + g\left(a_{FB}^{*}\right) - g'\left(a_{FB}^{*}\right)\frac{F\left(\overline{x}\left|a_{FB}^{*}\right.\right)}{F_a\left(\overline{x}\left|a_{FB}^{*}\right.\right)}\right)\left(1-F\left(\overline{x}\left|a_{FB}^{*}\right.\right)\right) \end{aligned} \tag{16}$$

s.t.

$$\overline{x} < \min\left[H, K\right]. \tag{17}$$

We are now in the position to state our main result. By letting $\overline{x} \to \underline{x}$, the principal can approach the first-best outcome arbitrarily closely. In (16), the expression in the second row vanishes in the limit, and the expression in the third row converges to $-u^{-1}\left(W_A + g\left(a_{FB}^{*}\right)\right)$. What remains is the first best expected utility (1). Incidentally, this also implies that the constraint (17) is not binding, which is why the additional restrictions on the constraint set in propositions 2.1 and 2.2 are of no consequence.

Theorem 2.1: The principal's problem (16)-(17) has no solution. As $\overline{x} \to \underline{x}$, the first-best outcome can be approached asymptotically.

Proof: The proof consists in showing that (16) converges to (1) as $\overline{x} \to \underline{x}$, in which case (17) is automatically satisfied. Consider first the third row. By l'Hôpital's rule and assumption 3,

$$\begin{aligned} \lim_{\overline{x}\to\underline{x}} \frac{F\left(\overline{x}\left|a_{FB}^{*}\right.\right)}{F_a\left(\overline{x}\left|a_{FB}^{*}\right.\right)} &= \lim_{\overline{x}\to\underline{x}} \frac{f\left(\overline{x}\left|a_{FB}^{*}\right.\right)}{f_a\left(\overline{x}\left|a_{FB}^{*}\right.\right)} \\ &= 0, \end{aligned} \tag{18}$$

from which it follows that the third row converges to $-u^{-1}\left(W_A + g\left(a_{FB}^{*}\right)\right)$.

[3]If the underlying probability distribution satisfies FOSD, then (9) with equality and (10) imply (13)-(14) and $\overline{x} < H$ since we can set $H = \sup X$. However, it is unlikely that all step functions that satisfy (9)-(10) automatically satisfy (9) with equality.

Next, consider the second row. We can distinguish between two cases: 1) $u^{-1}(\cdot)$ is bounded from below (e.g. if $u(w) = \ln w$), and 2) $u^{-1}(\cdot)$ is unbounded from below (e.g. if $u(w) = -\exp\{-rw\}$). Case 1) is trivial and implies that the expression in the second row converges to 0 (as $\overline{x} \to \underline{x}$, $(1 - F(\overline{x}|a^*_{FB}))/F_a(\overline{x}|a^*_{FB})$ tends to $-\infty$, which implies that $u^{-1}(\cdot)$ converges to a finite value). Case 2) is harder since the limit of $u^{-1}(\cdot)F(\overline{x}|a^*_{FB})$ is indeterminate of the form '$-\infty 0$'. We can evaluate this limit by replacing $u^{-1}(\cdot)$ with its tangent. Define

$$v \equiv W_A + g(a^*_{FB}) + g'(a^*_{FB}) \frac{1 - F(\overline{x}|a^*_{FB})}{F_a(\overline{x}|a^*_{FB})} \tag{19}$$

and denote by $\overline{v}$ the value of v at some fixed point $\overline{x} = \overline{\overline{x}}$. Since $u^{-1}(\cdot)$ is increasing and strictly convex in v, the tangent of $u^{-1}(\cdot)$ at $v = \overline{v}$ lies strictly below the graph of $u^{-1}(\cdot)$ at all $v \neq \overline{v}$. Therefore, if the product of $F(\overline{x}|a^*_{FB})$ with the tangent of $u^{-1}(\cdot)$ converges to 0, the product of $F(\overline{x}|a^*_{FB})$ with $u^{-1}(\cdot)$ must also converge to 0. The product of $F(\overline{x}|a^*_{FB})$ with the tangent of $u^{-1}(\cdot)$ at $v = \overline{v}$ is

$$F(\overline{x}|a^*_{FB}) \left(u^{-1}(\overline{v}) + u^{-1\prime}(\overline{v})[v - \overline{v}] \right), \tag{20}$$

where

$$v - \overline{v} = g'(a^*_{FB}) \left[\frac{1 - F(\overline{x}|a^*_{FB})}{F_a(\overline{x}|a^*_{FB})} - \frac{1 - F(\overline{\overline{x}}|a^*_{FB})}{F_a(\overline{\overline{x}}|a^*_{FB})} \right]. \tag{21}$$

Taking limits, we get

$$\begin{aligned} & \lim_{\overline{x} \to \underline{x}} F(\overline{x}|a^*_{FB}) \left(u^{-1}(\overline{v}) + u^{-1\prime}(\overline{v})[v - \overline{v}] \right) \qquad (22) \\ = \; & u^{-1\prime}(\overline{v}) g'(a^*_{FB}) \lim_{\overline{x} \to \underline{x}} \frac{F(\overline{x}|a^*_{FB})}{F_a(\overline{x}|a^*_{FB})}, \end{aligned}$$

which is equal to zero because

$$\begin{aligned} \lim_{\overline{x} \to \underline{x}} \frac{F(\overline{x}|a^*_{FB})}{F_a(\overline{x}|a^*_{FB})} &= \lim_{\overline{x} \to \underline{x}} \frac{f(\overline{x}|a^*_{FB})}{f_a(\overline{x}|a^*_{FB})} \\ &= 0 \end{aligned} \tag{23}$$

by l'Hôpital's rule and assumption 3. Consequently, the second row converges to 0. In conjunction with our earlier result that the third row converges to $-u^{-1}(W_A + g(a^*_{FB}))$, this implies that (16) converges to (1). The nonexistence of a solution follows from the fact that the limit of $u(\underline{s}(\overline{x}))$ as $\overline{x} \to \underline{x}$ is $u(u^{-1}(-\infty)) = u(\underline{w})$, which is not defined. ■

The proof reveals that condition LR is the driving force behind theorem 2.1. Condition LR requires that the likelihood ratio f_a/f tends to $-\infty$ as $\overline{x} \to \underline{x}$, which implies that low values of x are an extremely accurate signal that the agent has shirked (Milgrom (1981)). As $\overline{x}$ approaches $\underline{x}$, step functions exploit this property in an increasingly optimal fashion. Since the agent is only penalized for values of x that indicate with probability close to 1 that he has shirked, the risk of erroneous punishment (and consequently also the risk pemium that must be paid to the agent) is minimized. In the limit, the agent is punished infinitely hard with probability 0 and receives the first-best wage $u^{-1}\left(W_A + g\left(a^*_{FB}\right)\right)$ with probability 1. Incidentally, this limit is conceptionally equivalent to the shifting support environment studied by Harris and Raviv (1979) where the first-best outcome can be obtained by means of a forcing contract.

Technically, the difficulty of the proof consists in showing that the product $\underline{s}(\overline{x}) F(\overline{x}|a^*_{FB})$ converges to 0 as $\overline{x} \to \underline{x}$. If $u^{-1}(\cdot)$ is bounded from below, this is trivially the case. If $u^{-1}(\cdot)$ is unbounded from below, $\underline{s}(\overline{x})$ must decrease "sufficiently slowly" since $\lim_{\overline{x}\to\underline{x}} \underline{s}(\overline{x}) = -\infty$. An upper bound for the speed at which $\underline{s}(\overline{x})$ decreases can be obtained from (13). Since $u^{-1}(\cdot)$ is increasing and strictly convex, a given decrease in $\overline{x}$ reduces $u^{-1}(\cdot)$ by less than it reduces v defined in (19). Thus, if $vF(\overline{x}|a^*_{FB})$ converges to 0, meaning that $F(\overline{x}|a^*_{FB})$ tends to 0 faster than v tends to $-\infty$, $\underline{s}(\overline{x}) F(\overline{x}|a^*_{FB})$ must also converge to 0. Fortunately, the limit of $vF(\overline{x}|a^*_{FB})$ is easy to compute and equal to $\lim_{\overline{x}\to\underline{x}} f(\overline{x}|a^*_{FB})/f_a(\overline{x}|a^*_{FB})$, which is 0 by condition LR.

3 Distributions

The near first-best result rests primarily on three assumptions: 1) assumption 2, which requires that the agent's utility function is unbounded from below, 2) condition LR, which requires that the likelihood ratio f_a/f goes to $-\infty$ as $\overline{x} \to \underline{x}$, and 3) condition C, which ensures that the CDF is strictly convex in a for all $\overline{x}$ below some critical value K. As is shown below, conditions C and LR hold for a wide range of probability distributions. Incidentally, the counterpart of condition C for general sharing rules is known as the convexity of the distribution function condition (CDFC) (Rogerson (1985)) and is met by virtually no known probability distribution.

Proposition 3.1: Conditions LR and C are satisfied by the normal, lognormal, gamma, beta, chi-squared, Weibull, t-, and F-distribution.

Proof: See appendix A.

From the proof of theorem 2.1 it follows that condition LR is the driving force behind the near first-best result. It is therefore of interest to learn for what distributions this condition is not satisfied.

Proposition 3.2: Condition LR is not satisfied for the Poisson, binomial, geometric, exponential, and Pareto distribution.

Proof: See appendix B.

Two important distributions have been left out: the uniform and the Cauchy distribution. The uniform distribution violates assumption 1 since its parameters affect the support of the distribution. As for the Cauchy distribution, this distribution has no parameters and does consequently not fit into the parameterized distribution framework.

4 Conclusion

In this paper, we have shown that inefficiencies arising from moral hazard can be eliminated almost completely if the underlying probability distribution satisfies two conditions: 1) a convexity condition, which ensures that the first-order approach is valid, and 2) a likelihood ratio condition, which implies that low output values are very informative with respect to the agent's action choice. Furthermore, we have shown that both conditions are satisfied by the normal, lognormal, gamma, beta, chi-squared, Weibull, t-, and F-distribution.

The near first-best result leaves room for two interpretations: on the one hand, it can be viewed as an example for the efficacy of simple incentive schemes. On the other hand, it can be viewed as a pathological example for the nonexistence of a solution. Obviously, we sympathize with the first interpretation as it provides an explanation for the frequent use of bonus and penalty clauses in real-world incentive contracts.

5 Appendix A: Proof of Proposition 3.1

Normal distribution. The density function is

$$f(x\,|a,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\frac{(x-a)^2}{\sigma^2}\right\}. \tag{A.1}$$

Condition LR is met since

$$\frac{f_a(x\,|a,\sigma)}{f(x\,|a,\sigma)} = \frac{x-a}{\sigma^2} \to -\infty \text{ as } x \to -\infty. \tag{A.2}$$

We will now verify that condition C also holds. By Leibniz's rule,

$$\begin{aligned} F_{aa}(\overline{x}\,|a,\sigma) &= \int_{-\infty}^{\overline{x}} \frac{(x-a)^2-\sigma^2}{\sigma^4} f(x\,|a,\sigma)\,dx \qquad \text{(A.3)} \\ &\equiv \lim_{b\to-\infty} \int_{b}^{\overline{x}} \frac{(x-a)^2-\sigma^2}{\sigma^4} f(x\,|a,\sigma)\,dx \end{aligned}$$

provided the limit exists and is finite. Define

$$z \equiv \frac{x-a}{\sigma}; \qquad \text{(A.4)}$$

$$\overline{z} \equiv \frac{\overline{x}-a}{\sigma}; \qquad \text{(A.5)}$$

$$\overline{b} \equiv \frac{b-a}{\sigma}; \qquad \text{(A.6)}$$

and

$$A \equiv \frac{1}{\sigma^2\sqrt{2\pi}}. \qquad \text{(A.7)}$$

Using (A.4)-(A.7), we can write (A.3) as

$$\begin{aligned} F_{aa}(\overline{x}\,|a,\sigma) &= \lim_{\overline{b}\to-\infty} A\int_{\overline{b}}^{\overline{z}} \left[z^2-1\right]\exp\left\{-\frac{1}{2}z^2\right\}dz \qquad \text{(A.8)} \\ &= \lim_{\overline{b}\to-\infty} A\int_{\overline{b}}^{\overline{z}} z^2\exp\left\{-\frac{1}{2}z^2\right\}dz - \frac{1}{\sigma^2}F(\overline{x}\,|a,\sigma). \end{aligned}$$

Integration by parts yields

$$\begin{aligned} &\lim_{\overline{b}\to-\infty} A\int_{\overline{b}}^{\overline{z}} z^2\exp\left\{-\frac{1}{2}z^2\right\}dz \qquad \text{(A.9)} \\ &= -A\overline{z}\exp\left\{-\frac{1}{2}\overline{z}^2\right\} + \frac{1}{\sigma^2}F(\overline{x}\,|a,\sigma) \end{aligned}$$

since by l'Hôpital's rule

$$\begin{aligned} \lim_{\overline{b}\to-\infty} \overline{b}\exp\left\{-\frac{1}{2}\overline{b}^2\right\} &= \lim_{\overline{b}\to-\infty} \frac{1}{\overline{b}\exp\left\{\frac{1}{2}\overline{b}^2\right\}} \qquad \text{(A.10)} \\ &= 0. \end{aligned}$$

From (A.8) and (A.9) it follows that

$$\begin{aligned} F_{aa}(\overline{x}\,|a,\sigma) &= -A\overline{z}\exp\left\{-\frac{1}{2}\overline{z}^2\right\} \qquad \text{(A.11)} \\ &= -\frac{\overline{x}-a}{\sigma^2}f(\overline{x}\,|a,\sigma), \end{aligned}$$

which implies that $F_{aa}(\overline{x}\,|a,\sigma) > 0$ for all $\overline{x} < a$.

Lognormal distribution. The density function is

$$f(x\,|a,\sigma) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\frac{(\ln x - a)^2}{\sigma^2}\right\}. \tag{A.12}$$

Condition LR holds because

$$\frac{f_a(x\,|a,\sigma)}{f(x\,|a,\sigma)} = \frac{\ln x - a}{\sigma^2} \to -\infty \text{ as } x \to 0. \tag{A.13}$$

Condition C is obviously satisfied since $\ln x$ is normally distributed. To see this, set $u \equiv \ln x$ and $\overline{u} \equiv \ln \overline{x}$. The CDF can then be written as

$$F(\overline{x}\,|a,\sigma) = \int_{-\infty}^{\overline{u}} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\frac{(u-a)^2}{\sigma^2}\right\} du. \tag{A.14}$$

By (A.11), $F_{aa}(\overline{x}\,|a,\sigma) > 0$ for all $\overline{u} < a$, which implies that $F(\cdot\,|a,\sigma)$ is strictly convex in a for all $\overline{x} < \exp\{a\}$.

Gamma distribution. The density function is

$$f(x\,|a,b) = \frac{1}{\Gamma(a)} b^a x^{a-1} \exp\{-bx\} \tag{A.15}$$

with gamma function

$$\Gamma(a) \equiv \int_0^\infty x^{a-1} \exp\{-x\}\, dx. \tag{A.16}$$

Condition LR is met since

$$\frac{f_a(x\,|a,b)}{f(x\,|a,b)} = \ln b + \ln x - \frac{\Gamma'(a)}{\Gamma(a)} \to -\infty \text{ as } x \to 0. \tag{A.17}$$

Moreover, condition C holds since

$$F_{aa}(\overline{x}\,|a,b) = \int_0^{\overline{x}} \left[\left(\frac{f_a(x\,|a,b)}{f(x\,|a,b)}\right)^2 - \frac{d}{da}\frac{\Gamma'(a)}{\Gamma(a)}\right] f(x\,|a,b)\, dx. \tag{A.18}$$

By (A.17), $\left(\frac{f_a(x|a,b)}{f(x|a,b)}\right)^2 - \frac{d}{da}\frac{\Gamma'(a)}{\Gamma(a)} \to \infty$ as $x \to 0$, which implies that there exists a critical value K such that $F_{aa}(\overline{x}\,|a,b) > 0$ for all $\overline{x} < K$.

Beta distribution. The density function is

$$f(x\,|a,b) = \frac{1}{B(a,b)} x^{a-1} (1-x)^{b-1} \tag{A.19}$$

with beta function

$$B(a,b) \equiv \int_0^1 x^{a-1}(1-x)^{b-1}\,dx. \tag{A.20}$$

Condition LR is satisfied since

$$\frac{f_a(x\,|a,b)}{f(x\,|a,b)} = \ln x - \frac{B_a(a,b)}{B(a,b)} \to -\infty \text{ as } x \to 0. \tag{A.21}$$

Additionally, condition C holds because

$$F_{aa}(\overline{x}\,|a,b) = \int_0^{\overline{x}} \left[\left(\frac{f_a(x\,|a,b)}{f(x\,|a,b)}\right)^2 - \frac{d}{da}\frac{B_a(a,b)}{B(a,b)}\right] f(x\,|a,b)\,dx. \tag{A.22}$$

By (A.21), $\left(\frac{f_a(x|a,b)}{f(x|a,b)}\right)^2 - \frac{d}{da}\frac{B_a(a,b)}{B(a,b)} \to \infty$ as $x \to 0$. Consequently, there exists a critical value K such that $F_{aa}(\overline{x}\,|a,b) > 0$ for all $\overline{x} < K$.

Chi-squared distribution. The density function is

$$f(x\,|a) = \frac{1}{\Gamma\left(\frac{a}{2}\right)} 2^{\frac{-a}{2}} x^{\frac{a}{2}-1} \exp\left\{-\frac{x}{2}\right\}. \tag{A.23}$$

Define $\varphi(a) \equiv \Gamma\left(\frac{a}{2}\right)^{-1} 2^{\frac{-a}{2}}$. Condition LR holds since

$$\frac{f_a(x\,|a)}{f(x\,|a)} = \frac{1}{2}\ln x + \frac{\varphi'(a)}{\varphi(a)} \to -\infty \text{ as } x \to 0. \tag{A.24}$$

Moreover,

$$F_{aa}(\overline{x}\,|a) = \int_0^{\overline{x}} \left[\left(\frac{f_a(x\,|a)}{f(x\,|a)}\right)^2 + \frac{d}{da}\frac{\varphi'(a)}{\varphi(a)}\right] f(x\,|a)\,dx. \tag{A.25}$$

Therefore, condition C is met since $\left(\frac{f_a(x|a)}{f(x|a)}\right)^2 + \frac{d}{da}\frac{\varphi'(a)}{\varphi(a)} \to \infty$ as $x \to 0$ by (A.24), which implies that there exists a critical value K such that $F_{aa}(\overline{x}\,|a) > 0$ for all $\overline{x} < K$.

Weibull distribution. The density function is

$$f(x\,|a,b) = b^{-a} a x^{a-1} \exp\left\{-\left(\frac{x}{b}\right)^a\right\}, \tag{A.26}$$

and the corresponding likelihood ratio is

$$\frac{f_a(x\,|a,b)}{f(x\,|a,b)} = \ln x - \ln b + \frac{1}{a} - \left(\frac{x}{b}\right)^a \ln\left(\frac{x}{b}\right). \tag{A.27}$$

Condition LR is satisfied since

$$\lim_{x\to 0}\frac{\ln\left(\frac{x}{b}\right)}{\left(\frac{x}{b}\right)^{-a}} = \lim_{x\to 0} -\frac{1}{a}\left(\frac{x}{b}\right)^{a} = 0 \tag{A.28}$$

by l'Hôpital's rule. Condition C is also satisfied since

$$F_{aa}(\overline{x}\,|a,b) = \left[\ln\left(\frac{\overline{x}}{b}\right)\right]^2\left(\frac{\overline{x}}{b}\right)^a \exp\left\{-\left(\frac{\overline{x}}{b}\right)^a\right\}\left[1-\left(\frac{\overline{x}}{b}\right)^a\right] \tag{A.29}$$

is positive for all $\overline{x} < b$.

T-distribution. The density function is

$$f(x\,|a) = \frac{1}{\Gamma\left(\frac{a}{2}\right)\sqrt{a\pi}}\left(1+\frac{x^2}{a}\right)^{-\frac{a+1}{2}}\Gamma\left(\frac{a+1}{2}\right). \tag{A.30}$$

Define $\lambda(a) \equiv \frac{1}{\Gamma\left(\frac{a}{2}\right)\sqrt{a\pi}}\Gamma\left(\frac{a+1}{2}\right)$. The likelihood ratio is

$$\frac{f_a(x\,|a)}{f(x\,|a)} = \frac{1}{2}\frac{x^2(a+1)}{a^2+ax^2} - \frac{1}{2}\ln\left(1+\frac{x^2}{a}\right) + \frac{\lambda'(a)}{\lambda(a)}, \tag{A.31}$$

which goes to $-\infty$ as $x \to -\infty$ since

$$\lim_{x\to-\infty}\frac{1}{2}\frac{x^2(a+1)}{a^2+ax^2} = \frac{a+1}{2a} \tag{A.32}$$

by l'Hôpital's rule. Hence, condition LR is met. In addition, we have

$$F_{aa}(\overline{x}\,|a) = \int_{-\infty}^{\overline{x}}\left[\left(\frac{f_a(x\,|a)}{f(x\,|a)}\right)^2 + \frac{d}{da}\frac{f_a(x\,|a)}{f(x\,|a)}\right] f(x\,|a)\,dx, \tag{A.33}$$

where

$$\frac{d}{da}\frac{f_a(x\,|a)}{f(x\,|a)} = \frac{x^2}{a^2+ax^2} - \frac{x^2(a+1)(4a+2x^2)}{(2a^2+2ax^2)^2} + \frac{d}{da}\frac{\lambda'(a)}{\lambda(a)}. \tag{A.34}$$

Using l'Hôpital's rule yields

$$\lim_{x\to-\infty}\frac{x^2}{a^2+ax^2} = \frac{1}{a} \tag{A.35}$$

and

$$\lim_{x\to-\infty}\frac{x^2(a+1)(4a+2x^2)}{(2a^2+2ax^2)^2} = \frac{a+1}{2a^2}, \tag{A.36}$$

which implies that $\left(\frac{f_a(x|a)}{f(x|a)}\right)^2 + \frac{d}{da}\frac{f_a(x|a)}{f(x|a)} \to \infty$ as $x \to -\infty$. Thus, there exists a critical value K such that $F_{aa}(\overline{x}|a) > 0$ for all $\overline{x} < K$, which in turn implies that condition C holds.

F-distribution. The density function is

$$f(x|a,b) = \frac{1}{B\left(\frac{a}{2},\frac{b}{2}\right)}\left(\frac{a}{b}\right)^{\frac{a}{2}} x^{\frac{a}{2}-1}\left(1+\frac{ax}{b}\right)^{-\frac{a+b}{2}}. \tag{A.37}$$

Define $\delta(a,b) \equiv \frac{1}{B\left(\frac{a}{2},\frac{b}{2}\right)}\left(\frac{a}{b}\right)^{\frac{a}{2}}$. Condition LR is satisfied because

$$\frac{f_a(x|a,b)}{f(x|a,b)} = \frac{1}{2}\ln x - \frac{1}{2}\ln\left(1+\frac{ax}{b}\right) - \frac{1}{2}\frac{(a+b)x}{b+ax} + \frac{\delta_a(a,b)}{\delta(a,b)} \tag{A.38}$$

tends to $-\infty$ as $x \to 0$. Likewise, condition C holds since

$$F_{aa}(\overline{x}|a,b) = \int_{-\infty}^{\overline{x}}\left[\left(\frac{f_a(x|a,b)}{f(x|a,b)}\right)^2 + \frac{d}{da}\frac{f_a(x|a,b)}{f(x|a,b)}\right] f(x|a,b)\,dx, \tag{A.39}$$

where

$$\frac{d}{da}\frac{f_a(x|a,b)}{f(x|a,b)} = -\frac{x}{b+ax} + \frac{1}{2}\frac{(a+b)x^2}{(b+ax)^2} + \frac{d}{da}\frac{\delta_a(a,b)}{\delta(a,b)}. \tag{A.40}$$

Consequently, $\left(\frac{f_a(x|a,b)}{f(x|a,b)}\right)^2 + \frac{d}{da}\frac{f_a(x|a,b)}{f(x|a,b)} \to \infty$ as $x \to 0$, which implies that there exists a critical value K such that $F(\overline{x}|a,b)$ is strictly convex in a for all $\overline{x} < K$. ■

6 Appendix B: Proof of Proposition 3.2

Poisson distribution. The mass function is

$$f(x|a) = \frac{a^x}{x!}\exp\{-a\}. \tag{B.1}$$

Condition LR is not satisfied since

$$\frac{f_a(x|a)}{f(x|a)} = \frac{x}{a} - 1 \to -1 \text{ as } x \to 0. \tag{B.2}$$

Binomial distribution. The mass function is

$$f(x|a,n) = \binom{n}{x} a^x (1-a)^{n-x}, \tag{B.3}$$

from which it follows that

$$\frac{f_a(x|a,n)}{f(x|a,n)} = \frac{x}{a} - \frac{n-x}{1-a} \to -\frac{n}{1-a} \text{ as } x \to 0. \tag{B.4}$$

Therefore, condition LR does not hold.

Geometric distribution. The mass function is

$$f(x|a) = a(1-a)^x. \tag{B.5}$$

Condition LR is not satisfied since

$$\frac{f_a(x|a)}{f(x|a)} = \frac{1}{a} - \frac{x}{1-a} \to \frac{1}{a} \text{ as } x \to 0. \tag{B.6}$$

Exponential distribution. The density function is

$$f(x|a) = a\exp\{-ax\}, \tag{B.7}$$

and the corresponding likelihood ratio is

$$\frac{f_a(x|a)}{f(x|a)} = \frac{1}{a} - x \to \frac{1}{a} \text{ as } x \to 0, \tag{B.8}$$

which implies that condition LR does not hold.

Pareto distribution. The density function is

$$f(x|a,\overline{x}) = \frac{a\overline{x}^a}{x^{a+1}}. \tag{B.9}$$

Condition LR is not met since

$$\frac{f_a(x|a,\overline{x})}{f(x|a,\overline{x})} = \frac{1}{a} + \ln\frac{\overline{x}}{x} \to \frac{1}{a} \text{ as } x \to \overline{x}, \tag{B.10}$$

which completes the proof. ■

7 References

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