Randomization in Dynamic Principal-Agent Problems

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Abstract

In a seminal paper, Holmström and Milgrom (1987) examine a principal-agent problem in which an agent controls the drift of a Brownian motion. Given that the agent can revise his control continuously, they show that the optimal sharing rule is linear in aggregated output. In this paper, we examine the case where control revisions take place in arbitrarily small discrete time intervals. We show that the first-best outcome can be approached asymptotically by a random spot check in conjunction with a step function. The central message of this paper is therefore that in agency problems of the sort studied by Holmström and Milgrom, linear sharing rules may not always be optimal. Random spot checks are widely used in practice and play an important role in the area of quality control.

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1 Introduction

Most real-world incentive schemes are simple, and yet they cannot be explained within the framework of the standard principal-agent model. As was shown by Grossman and Hart (1983), even basic properties such as monotonicity are hard to assure unless one is willing to accept assumptions that are extremely restrictive. Based on this weakness of existing models, Holmström and Milgrom (1987) have developed a principal-agent model in which the optimal sharing rule is linear. In this model, an agent continuously revises the drift of a Brownian motion during a fixed time interval in response to observations of the history of the process. The agent's technology is time- and state-invariant. Moreover, principal and agent both have exponential utility in order to abstract from wealth effects. In a stationary environment like this, linear sharing rules provide uniform incentives since the marginal reward for an additional unit of output is the same at any time.

The central message of this paper is that in agency problems of the sort studied by Holmström and Milgrom, linear sharing rules may not always be optimal. Unlike Holmström and Milgrom who assume that the agent can revise his control continuously, we look at the case where control revisions take place in arbitrarily small discrete time intervals. Incidentally, discrete-time control revisions are consistent with the intention of Holmström and Milgrom to "investigate a situation in which the agent takes actions very frequently in time" (p. 316, see also p. 322). Our main idea is simple. Suppose the relevant time interval is partitioned into n subintervals of equal length and the agent can change his control only at the beginning of each subinterval. In this setting, the principal can create uniform incentives by randomly selecting a subinterval ex post and applying a step function to the output produced in this subinterval. Since the agent does not know in advance which of the subintervals will be selected, his n control problems are identical and he will choose a constant control. This in turn implies that we can replace the dynamic problem with a much simpler static problem in which the agent selects the mean of a normally distributed random variable and the principal is a priori restricted to step functions. As was first shown by Mirrlees (1974) in a related context, the principal can then approach the first-best outcome asymptotically by choosing the cutoff arbitrarily small.

As it turns out, randomization is sufficient, but not necessary for near first-best optimality: the first-best solution can also be approximated by applying a step function to each subinterval separately. In practice, however, measuring output is costly, which implies that detailed checks are Pareto-dominated by random spot checks. This result is related to work by Mookherjee and Png (1989) who study a static principal-agent model in the spirit of

Grossman and Hart (1983) with costly state verification. They show that if the agent's output can only be observed at a cost, optimal audits by the principal must be random. As in our model, randomization is efficient as it achieves a reduction in audit costs without simultaneously distorting the agent's incentives.

Recently, the Holmström-Milgrom model has been extended by several authors. Schättler and Sung (1993) derive necessary and sufficient conditions for the validity of the first-order approach in the continuous-time principal-agent problem for a wide class of stochastic processes. As a special case, they rederive Holmström and Milgrom's linearity result. In a companion paper, Schättler and Sung (1997) show that the minutest deviation from the Brownian model such as the introduction of a time- or state-dependent technology leads to solutions that are nonlinear. Based on the work by Schättler and Sung (1993), Sung (1995) shows that the linearity result continues to hold if the agent is allowed to control both the drift and the diffusion rate of the outcome process. Harris and Bolton (1997) generalize the Brownian model in several directions and characterize properties of the first-best solution. Finally, Müller (1997a) explicitly derives the first-best sharing rule and shows that it is also linear in aggregated output.

In their paper, Holmström and Milgrom examine three models: a static model, a discrete-time model, and a continuous-time model ("the Brownian model"). By the way, the discrete-time model analyzed by Holmström and Milgrom differs from the one analyzed here. In the former model, the agent controls a discrete-time multinomial process whereas here, he controls the drift of a continuous-time process in discrete time intervals. Hellwig and Schmidt (1997) link the discrete- and continuous-time models in a unified framework and explicitly derive the Brownian model as the limit of a sequence of discrete-time models. Furthermore, they provide sufficient conditions for a discrete-time analogue of the continuous-time linearity result. In a companion paper, Hellwig (1997) derives a linearity result similar to the one in Holmström and Milgrom (1987) in a mean-variance framework with endogenous drift and diffusion rate.

The rest of the paper is organized as follows: section 2 presents the Brownian model with continuous-time control revisions and the linearity result derived by Holmström and Milgrom. In section 3, we turn to the Brownian model with discrete-time control revisions and show that the first-best outcome can be approached arbitrarily closely with a random spot check in conjunction with a step function. Section 4 derives some comparative statics results. In section 5, we show that a detailed check of the agent's entire output is also near first-best optimal and discuss the role of measurement cost and unbounded penalties. Section 6 concludes.

2 The Continuous-Time Problem

In this section, we briefly review the Brownian model developed by Holmström and Milgrom (1987, section 4) and state their main result. The notation is primarily adopted from Schättler and Sung (1993). For ease of exposition, we confine ourselves to the case of one-dimensional Brownian motion. At time 0, principal and agent agree on a sharing rule which specifies a payment from the principal to the agent at time 1. The sharing rule may depend on a stochastic outcome process X defined on the time interval $[0, 1]$ which satisfies $X_0 = 0$ and is publicly observable. Formally, X is governed by a stochastic differential equation of the form

$$dX_t = f(u_t)\, dt + \sigma dB_t, \tag{1}$$

where $f(u_t)$ is the instantaneous mean, $u_t = u_t(t, X)$ is the agent's control at time t, σ is the diffusion rate, and B is a standard Brownian motion. The principal receives the end-of-period output X_1. Besides, the principal can observe the outcome process X, but not the agent's control u. Let $(\Omega, \mathcal{F}, P)$ denote the underlying probability space. The control u is an $\mathcal{F}_t-$predictable process with values in some open bounded control set $U \subseteq \mathcal{R}_+$. That is, the agent's control can be revised continuously during the time interval $[0, 1]$ and may depend on the history of X in $[0, t]$, but not on the future $(t, 1]$. Denote the class of all such processes by $\mathcal{U}$. The "production function" $f(\cdot)$ is bounded with derivatives $f'(\cdot) > 0$ and $f''(\cdot) < 0$, and the diffusion rate lies in some bounded subset of $\mathcal{R}_{++}$. The agent incurs effort cost $c(u_t)$, where $c(\cdot)$ is bounded with derivatives $c'(\cdot) > 0$ and $c''(\cdot) \geq 0$. Finally, principal and agent both have negative exponential von Neumann-Morgenstern utility with coefficient of risk aversion R and r, respectively.

Denote by W_A the agent's certainty equivalent at time 0. The principal's problem is to choose a sharing rule S and a control u that maximize her expected utility subject to the agent's participation and incentive compatibility constraint. She solves

$$\max_{S,u} E\left[-\exp\left\{-R\left(X_1 - S\right)\right\}\right] \tag{2}$$

s.t.

$$dX_t = f(u_t)\, dt + \sigma dB_t, \tag{3}$$

$$E\left[-\exp\left\{-r\left(S - \int_0^1 c(u_t)\, dt\right)\right\}\right] \geq -\exp\left\{-rW_A\right\}, \tag{4}$$

and

$$u \in \arg\max_{\hat{u}\in\mathcal{U}} E\left[-\exp\left\{-r\left(S - \int_0^1 c(\hat{u}_t)\, dt\right)\right\}\right] \tag{5}$$

The solution derived by Holmström and Milgrom (1987) and Schättler and Sung (1993) consists of two parts: 1) The optimal control is time-invariant, i.e. $u_t^* = u^*$ for all t, and 2) the optimal sharing rule is a linear function of aggregated output, i.e.

$$S^* = K + \frac{c'(u^*)}{f'(u^*)} X_1, \tag{6}$$

where K is a constant. Given the constancy of the optimal control, the solution to the dynamic problem (2)-(5) corresponds to the solution of a static principal-agent problem in which the agent selects the mean of a normally distributed random variable and the principal is constrained to linear sharing rules. This correspondence between the dynamic model and the much simpler static model has been exploited in several papers (e.g. Holmström and Milgrom (1990, 1991), Dixit et. al (1996)).

The linearity result is driven by a delicate balance of Brownian motion, CARA utility, and a time- and state-invariant technology. Given the stationarity of the environment, linear sharing rules apply a uniform incentive pressure: at any time, the marginal reward for an additional unit of output is the same, regardless of the output that has been accumulated in the past. Consequently, the agent will exert a constant level of effort.

3 The Discrete-Time Problem

In this section, we examine a variant of the Brownian model in which actions are taken in arbitrarily small discrete time intervals. In the discrete-time problem, the interval $[0, 1]$ is partitioned into n subintervals of equal length $\Delta t = \frac{1}{n}$. The agent's control can only be revised at $t \in \left\{0, \frac{1}{n}, ..., \frac{n-1}{n}\right\}$, i.e. it can only be revised at the beginning of a subinterval and remains constant throughout the entire subinterval. Observe that the agent still controls a continuous-time stochastic process. However, since the cumulative output produced in a subinterval is normally distributed, the discrete-time model is conceptionally equivalent to a model in which the agent repeatedly chooses the mean of a normally distributed random variable.

Denote the increments of the processes X and B which are produced in the subinterval $[t, t + \Delta t]$ by the forward differences $\Delta X_t \equiv X_{t+\Delta t} - X_t$ and $\Delta B_t \equiv B_{t+\Delta t} - B_t$, respectively. For convenience, let us also assume that the principal is risk neutral. The principal's discrete-time problem is

$$\max_{S,u} E[X_1 - S] \tag{7}$$

s.t.

$$\Delta X_t = f(u_t)\,\Delta t + \sigma \Delta B_t, \tag{8}$$

$$E\left[-\exp\left\{-r\left(S - \sum_{t=0}^{\frac{n-1}{n}} c(u_t)\,\Delta t\right)\right\}\right] \geq -\exp\{-rW_A\}, \tag{9}$$

and

$$u \in \arg\max_{\hat{u}\in\mathcal{U}} E\left[-\exp\left\{-r\left(S - \sum_{t=0}^{\frac{n-1}{n}} c(\hat{u}_t)\,\Delta t\right)\right\}\right]. \tag{10}$$

In the remainder of this section, we will show that the principal can approach the first-best outcome, i.e. the solution to (7)-(9), arbitrarily closely. Since the principal is risk-neutral, the first-best sharing rule is a constant which compensates the agent for his cumulative effort and the opportunity cost of participating. Using results from Müller (1997a), we obtain

Proposition 3.1: The first-best sharing rule is

$$S_{FB}^* = W_A + c(u_{FB}^*), \tag{11}$$

and the first-best control is constant, unique, and implicitly defined by the equality of marginal productivity and marginal cost

$$f'(u_{FB}^*) = c'(u_{FB}^*). \tag{12}$$

Proof: In the absence of incentive constraints, this problem is separable and can be solved as a static risk-sharing problem. Define net compensation as $Z = S - \sum_{t=0}^{\frac{n-1}{n}} c(u_t)\,\Delta t$. Summing up (8) and inserting the result in (7), we can write the principal's first-best problem as

$$\max_{Z,u} E\left[\sigma B_1 + \sum_{t=0}^{\frac{n-1}{n}} [f(u_t) - c(u_t)]\,\Delta t - Z\right] \tag{13}$$

s.t.

$$E[-\exp\{-rZ\}] \geq -\exp\{-rW_A\}. \tag{14}$$

Standard results show that Z must be non-random and that (14) must hold with equality. This implies

$$f'(u_{tFB}^*) = c'(u_{tFB}^*) \tag{15}$$

for all t, and

$$S_{FB}^* = W_A + \sum_{t=0}^{\frac{n-1}{n}} c(u_{tFB}^*)\,\Delta t. \tag{16}$$

Finally, from (15), $f''(\cdot) < 0$ and $c''(\cdot) \geq 0$ it follows that $u_{tFB}^* = u_{FB}^*$ for all t, i.e. the first-best control is constant and unique. ■

If the agent could choose his control only once, aggregated output X_1 would be normally distributed with mean $f(u)$ and variance σ^2. The principal could then approach the first-best outcome asymptotically by using a step function or "Mirrlees scheme" (Mirrlees (1974), Müller (1997b)). In their survey article, Hart and Holmström (1987, p. 93) argue that this is no longer true if the agent can adjust his control frequently in response to observations of the history of the process and conclude that "the optimality of step functions is highly sensitive to the assumption that the agent chooses his labor input only once". We will show later that this is not necessarily correct. First, however, let us briefly sketch their argument.

Suppose the principal uses a step function based on aggregated output X_1. At the beginning of the last subinterval, X_1 is normally distributed with mean $f\left(u_{\frac{n-1}{n}}\right)\Delta t + X_{\frac{n-1}{n}}$ and variance $\sigma^2 \Delta t$. This implies that the agent's choice of $u_{\frac{n-1}{n}}$ depends on the state of the process $X_{\frac{n-1}{n}}$. Working backwards, it can be shown that in each subinterval, the agent's optimal control is state-dependent and therefore stochastic. Thus, step functions based on aggregated output fail to provide constant intertemporal incentives. Instead, they allow agents to act strategically in response to observations of past perfomance. This fact is well known in the sales business where salesmen who have either already reached their bonus target or are far from reaching it tend to expend less effort than those who are close to their bonus target. We will now show that the principal can counteract such strategic behavior on the part of the agent by using a randomizing device. Randomization can occur in two ways: ex ante (at time 0) or ex post (at time 1).

Definition 3.1: Ex ante-randomization. At time 0, the principal secretly determines a subinterval with a symmetric randomizing device. Subsequently, she secretly measures the output ΔX produced in this subinterval and compensates the agent with a step function based on ΔX.

The problem with definition 3.1 is that both the randomization and the output measurement must be secret and verifiable ex post. If the agent knew the measurement interval in advance, he would work hard only in this subinterval and shirk in the remaining time. In practice, one can conceive of ex-ante randomization as a spot check by a supervisor who checks the agent's output through a one-sided mirror or at a later stage in the production process where the agent is not present. The problem of secrecy can be avoided if the principal randomizes ex post.

Definition 3.2: Ex post-randomization. The principal first observes the entire output process and then determines a measurement interval at time 1 with a symmetric randomizing device. Subsequently, the agent is compensated with a step function based on the measured output ΔX.

The assumption regarding the observability of the output process can be relaxed further since all the principal needs to know is the set of n increments $\Delta X_0, ..., \Delta X_{\frac{n-1}{n}}$ but not the order in which they were generated. In practice, ex post-randomization occurs in the form of quality control where a small subset of the output is selected and the payment to the producer (e.g. a worker or supplier) is based on the quality of the selected output. The introduction of a randomizing device eliminates any strategic advantage on the part of the agent from observing the history of the process. Since the agent's technology is time- and state-invariant and wealth effects do not perpetuate in time, the agent faces the same control problem in each subinterval. This in turn implies that his overall control problem can be expressed as a simple multivariate optimization problem. For instance, if $n = 2$ and the principal employs a step function with cutoff $\overline{\Delta X}$ and payments $\underline{s}$ and $\overline{s}$, the agent's overall control problem is

$$\begin{aligned}
\max_{u_0, u_{\frac{1}{2}}} & -\frac{1}{2} F\left(\overline{\Delta X}\middle| u_0\right) \exp\left\{-r\left(\underline{s} - c(u_0)\Delta t - c\left(u_{\frac{1}{2}}\right)\Delta t\right)\right\} \\
& -\frac{1}{2}\left(1 - F\left(\overline{\Delta X}\middle| u_0\right)\right) \exp\left\{-r\left(\overline{s} - c(u_0)\Delta t - c\left(u_{\frac{1}{2}}\right)\Delta t\right)\right\} \\
& -\frac{1}{2} F\left(\overline{\Delta X}\middle| u_{\frac{1}{2}}\right) \exp\left\{-r\left(\underline{s} - c(u_0)\Delta t - c\left(u_{\frac{1}{2}}\right)\Delta t\right)\right\} \\
& -\frac{1}{2}\left(1 - F\left(\overline{\Delta X}\middle| u_{\frac{1}{2}}\right)\right) \exp\left\{-r\left(\overline{s} - c(u_0)\Delta t - c\left(u_{\frac{1}{2}}\right)\Delta t\right)\right\}.
\end{aligned} \tag{17}$$

In (17), the first two rows represent the agent's expected utility if the first subinterval is the measurement interval, weighted with the probability $\frac{1}{2}$ that this subinterval is chosen. Analogously, the last two rows represent the weighted expected utility if the second subinterval is the measurement interval. Note that despite the symmetry of the problem, the agent may choose a different control in each subinterval if at the optimum he is indifferent between two or more values of u_t. It therefore remains to be shown that the solution with respect to each u_t is unique.[1]

Proposition 3.2: Suppose the principal randomizes either ex ante or ex post. Then there exists a real number H such that for all step functions with $\overline{s} > \underline{s}$ and $\overline{\Delta X} \leq H$ the agent's optimal control is constant.

Proof: See appendix.

[1]We thank Martin Hellwig for pointing this out.

From a methodological viewpoint, the role of randomization is similar to that of linear sharing rules. Given the constancy of the optimal control, the dynamic problem with randomization is equivalent to a much simpler static problem in which the agent selects the mean of a normally distributed random variable ΔX and the principal is a priori restricted to step functions. Unlike linear sharing rules, however, step functions optimally exploit the information contained in the tails of the normal distribution, which is why the first-best solution can be approached arbitrarily closely. We will now prove this claim in several steps. The proof is adopted from Müller (1997b) and is therefore presented in a brief fashion.[2]

Consider the static problem in which the agent controls the random variable ΔX. By proposition 3.2, the principal is a priori restricted to step functions with payments $\overline{s} > \underline{s}$ and cutoff $\overline{\Delta X} \leq H$. Thus, the principal's problem is to find the optimal parameters $\underline{s}$, $\overline{s}$, and $\overline{\Delta X}$ that satisfy $\overline{s} > \underline{s}$ and $\overline{\Delta X} \leq H$ as well as the usual individual rationality and incentive compatibility constraints. Additionally, we assume that the principal wants to implement the first-best control u^*_{FB}. The principal's problem is then

$$\max_{\underline{s},\overline{s},\overline{\Delta X}} E\left[X_1 \left| u^*_{FB}\right.\right] - \underline{s} F\left(\overline{\Delta X} \left| u^*_{FB}\right.\right) - \overline{s}\left(1 - F\left(\overline{\Delta X} \left| u^*_{FB}\right.\right)\right) \tag{18}$$

s.t.

$$\begin{aligned} & -\exp\left\{-r\left(\underline{s} - c\left(u^*_{FB}\right)\right)\right\} F\left(\overline{\Delta X} \left| u^*_{FB}\right.\right) \\ & -\exp\left\{-r\left(\overline{s} - c\left(u^*_{FB}\right)\right)\right\}\left(1 - F\left(\overline{\Delta X} \left| u^*_{FB}\right.\right)\right) \\ \geq \; & -\exp\left\{-rW_A\right\}, \end{aligned} \tag{19}$$

$$\begin{aligned} u^*_{FB} \in \; & \arg\max_{\hat{u} \in U} -\exp\left\{-r\left(\underline{s} - c\left(\hat{u}\right)\right)\right\} F\left(\overline{\Delta X} \left| \hat{u}\right.\right) \\ & -\exp\left\{-r\left(\overline{s} - c\left(\hat{u}\right)\right)\right\}\left(1 - F\left(\overline{\Delta X} \left| \hat{u}\right.\right)\right), \end{aligned} \tag{20}$$

$$\overline{s} > \underline{s}, \tag{21}$$

and

$$\overline{\Delta X} \leq H, \tag{22}$$

where $c\left(u^*_{FB}\right) = nc\left(u^*_{FB}\right)\Delta t$ denotes the cumulative cost of exerting the first-best control in every subinterval. Since U is infinite, we must use the

[2] For further comments, see Müller (1997b). There, we develop an analogous proof for the case of additively separable utility.

first-order approach and replace (20) with the agent's first-order condition

$$\begin{aligned} 0 &= rc'(u^*_{FB})\exp\{-r\underline{s}\}F\left(\overline{\Delta X}\,|u^*_{FB}\right) \\ &+rc'(u^*_{FB})\exp\{-r\overline{s}\}\left(1-F\left(\overline{\Delta X}\,|u^*_{FB}\right)\right) \\ &+\left(\exp\{-r\underline{s}\}-\exp\{-r\overline{s}\}\right)F_u\left(\overline{\Delta X}\,|u^*_{FB}\right). \end{aligned} \tag{23}$$

The substitution of (23) for (20) is invalid unless it can be ensured that the agent's problem is concave. As the following proposition shows, sufficient conditions for the validity of the first-order approach are easy to find.

Proposition 3.3: There exists a real number J such that for all step functions with $\overline{s} > \underline{s}$ and $\overline{\Delta X} \leq J$, the agent's problem is strictly concave.

Proof: The agent's problem is

$$\begin{aligned} &\max_u -\exp\{-r(\underline{s}-c(u))\}F\left(\overline{\Delta X}\,|u\right) \\ &-\exp\{-r(\overline{s}-c(u))\}\left(1-F\left(\overline{\Delta X}\,|u\right)\right) \end{aligned} \tag{24}$$

which can be rearranged as

$$\begin{aligned} &\max_u \left(\exp\{-r\overline{s}\}-\exp\{-r\underline{s}\}\right)\exp\{rc(u)\}F\left(\overline{\Delta X}\,|u\right) \\ &-\exp\{-r(\overline{s}-c(u))\}. \end{aligned} \tag{25}$$

Since $c(\cdot)$ is convex and $\exp\{\cdot\}$ is an increasing convex transformation, $-\exp\{-r(\overline{s}-c(u))\}$ is concave. By assumption, $\overline{s} > \underline{s}$, which implies that (25) is strictly concave if $\exp\{rc(u)\}F\left(\overline{\Delta X}\,|u\right)$ is strictly convex. The rest follows from the proof of proposition 3.2 (note that $J \neq H$ unless $\Delta t = 1$). ■

By proposition 3.3, any step function that satisfies (23), $\overline{s} > \underline{s}$ and $\overline{\Delta X} \leq J$ is incentive compatible, i.e. it satisfies (20). Grossman and Hart (1983) have shown that the individual rationality constraint (19) must be binding in equilibrium. Hence, we can solve (19) (with equality) and (23) explicitly for $\underline{s}$ and $\overline{s}$ as a function of the cutoff $\overline{\Delta X}$ and obtain

$$\underline{s} = W_A + c(u^*_{FB}) + \frac{1}{-r}\ln\left\{1 - rc'(u^*_{FB})\frac{\left(1-F\left(\overline{\Delta X}\,|u^*_{FB}\right)\right)}{F_u\left(\overline{\Delta X}\,|u^*_{FB}\right)}\right\} \tag{26}$$

and

$$\overline{s} = W_A + c(u^*_{FB}) + \frac{1}{-r}\ln\left\{1 + rc'(u^*_{FB})\frac{F\left(\overline{\Delta X}\,|u^*_{FB}\right)}{F_u\left(\overline{\Delta X}\,|u^*_{FB}\right)}\right\}. \tag{27}$$

By Leibniz's rule, $F_u\left(\overline{\Delta X}\,|u^*_{FB}\right)$ is equal to

$$\int_{-\infty}^{\overline{\Delta X}} \frac{\left(\Delta X - f\left(u^*_{FB}\right)\Delta t\right) f'\left(u^*_{FB}\right)}{\sigma^3\sqrt{\Delta t}\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\frac{\left(\Delta X - f\left(u^*_{FB}\right)\Delta t\right)^2}{2\sigma^2\Delta t}\right\} d\Delta X, \tag{28}$$

which is negative. Hence, (27) may not be defined for some $\overline{\Delta X}$. The following proposition shows that this problem can be avoided by imposing an additional restriction on the cutoff $\overline{\Delta X}$.

Proposition 3.4: There exists a real number K such that (27) is defined for all $\overline{\Delta X} \leq K$.

Proof: By l'Hôpital's rule,

$$\begin{aligned} & \lim_{\overline{\Delta X}\to-\infty} \frac{F\left(\overline{\Delta X}\,|u^*_{FB}\right)}{F_u\left(\overline{\Delta X}\,|u^*_{FB}\right)} \\ = & \lim_{\overline{\Delta X}\to-\infty} \frac{\sigma^2}{\left(\overline{\Delta X} - f\left(u^*_{FB}\right)\Delta t\right) f'\left(u^*_{FB}\right)} \\ = & \; 0, \end{aligned} \tag{29}$$

which implies that (27) is defined for sufficiently small values of $\overline{\Delta X}$. ■

An important consequence of proposition 3.4 is that the constraint $\overline{s} > \underline{s}$ can be replaced by $\overline{\Delta X} \leq K$.

Corollary 3.1: For all $\overline{\Delta X} \leq K$, (26)-(27) implies (19), (21), and (23).

Proof: By proposition 3.4, (27) is defined for all $\overline{\Delta X} \leq K$. Reversing the steps that lead to (26)-(27) shows that (26)-(27) implies (23) and (19) with equality and therefore (19). By (28), $F_u\left(\overline{\Delta X}\,|u^*_{FB}\right) < 0$. In conjunction with (26)-(27), this implies (21). ■

Together, proposition 3.3 and corollary 3.1 yield that any step function which satisfies (26)-(27) and $\overline{\Delta X} \leq \min[J, K]$ also satisfies (19)-(21). Replacing (19)-(21) by (26)-(27) and $\overline{\Delta X} \leq \min[J, K]$ and inserting (26)-(27) into the objective function (18), we can rewrite (18)-(22) as[3]

[3]Note that the set defined by (26)-(27) and $\Delta\overline{X} \leq \min[J, K]$ is a subset of (19)-(21). Therefore, replacing (19)-(21) by (26)-(27) and $\Delta\overline{X} \leq \min[J, K]$ reduces the set of admissible step functions. As is shown in theorem 3.1, however, this has no implications for the optimal solution.

$$\max_{\overline{\Delta X}} E\left[X_1 \left| u^*_{FB}\right.\right] - W_A - c\left(u^*_{FB}\right)$$
$$-\frac{1}{-r}\ln\left\{1 - rc'\left(u^*_{FB}\right)\frac{\left(1 - F\left(\overline{\Delta X}\left|u^*_{FB}\right.\right)\right)}{F_u\left(\overline{\Delta X}\left|u^*_{FB}\right.\right)}\right\} F\left(\overline{\Delta X}\left|u^*_{FB}\right.\right) \quad (30)$$
$$-\frac{1}{-r}\ln\left\{1 + rc'\left(u^*_{FB}\right)\frac{F\left(\overline{\Delta X}\left|u^*_{FB}\right.\right)}{F_u\left(\overline{\Delta X}\left|u^*_{FB}\right.\right)}\right\}\left(1 - F\left(\overline{\Delta X}\left|u^*_{FB}\right.\right)\right)$$

s.t.

$$\overline{\Delta X} \leq \min\left[H, J, K\right], \quad (31)$$

where H, J and K are implicitly defined by propositions 3.2-3.4.

We can now state our main theorem. As $\overline{\Delta X}$ goes to $-\infty$, the expressions in the second and third row go to 0 and (30) converges to the first-best utility $E\left[X_1 \left| u^*_{FB}\right.\right] - W_A - c\left(u^*_{FB}\right)$. Hence, the constraint (31) is not binding. For additively separable utility, similar results have been derived by Mirrlees (1974) (for lognormally distributed output) and Müller (1997b) (for a wide class of probability distributions including the normal, lognormal, and gamma distribution).

Theorem 3.1: The principal can approach the first-best solution arbitrarily closely by letting $\overline{\Delta X} \to -\infty$.

Proof: Let us begin with the third row. By (29), $\lim_{\overline{\Delta X}\to-\infty} F\left(\overline{\Delta X}\left|u^*_{FB}\right.\right) = 0$ and $\lim_{\overline{\Delta X}\to-\infty} F\left(\overline{\Delta X}\left|u^*_{FB}\right.\right) \Big/ F_u\left(\overline{\Delta X}\left|u^*_{FB}\right.\right) = 0$, from which it follows that the expression in the third row converges to 0.

Next, consider the second row. Since $\lim_{\overline{\Delta X}\to-\infty} F_u\left(\overline{\Delta X}\left|u^*_{FB}\right.\right) = 0$ and $F_u\left(\overline{\Delta X}\left|u^*_{FB}\right.\right) < 0$, the limit of the second row is indeterminate of the form "$\infty 0$". We can evaluate this limit by replacing $\ln(\cdot)$ with its tangent. Define

$$v \equiv 1 - rc'\left(u^*_{FB}\right)\frac{\left(1 - F\left(\overline{\Delta X}\left|u^*_{FB}\right.\right)\right)}{F_u\left(\overline{\Delta X}\left|u^*_{FB}\right.\right)} \quad (32)$$

and denote by $\overline{v}$ the value of v at some fixed point $\overline{\Delta X} = \overline{\overline{\Delta X}}$. Since $\ln(\cdot)$ is strictly concave in v, the tangent of $\ln(\cdot)$ at $v = \overline{v}$ lies strictly above the graph of $\ln(\cdot)$ at all $v \neq \overline{v}$. Thus, if the product of $F\left(\overline{\Delta X}\left|u^*_{FB}\right.\right)$ with the tangent of $\ln(\cdot)$ converges to 0 as $\overline{\Delta X} \to -\infty$, the product of $F\left(\overline{\Delta X}\left|u^*_{FB}\right.\right)$

with $\ln(\cdot)$ must also converge to 0. The product of $F\left(\overline{\Delta X}\,|u^*_{FB}\right)$ with the tangent of $\ln(\cdot)$ at $v = \overline{v}$ is

$$F\left(\overline{\Delta X}\,|u^*_{FB}\right)\left(\ln\overline{v} + \frac{1}{\overline{v}}\left(v - \overline{v}\right)\right), \tag{33}$$

where

$$v - \overline{v} = rc'\left(u^*_{FB}\right)\left[\frac{\left(1 - F\left(\overline{\overline{\Delta X}}\,|u^*_{FB}\right)\right)}{F_u\left(\overline{\overline{\Delta X}}\,|u^*_{FB}\right)} - \frac{\left(1 - F\left(\overline{\Delta X}\,|u^*_{FB}\right)\right)}{F_u\left(\overline{\Delta X}\,|u^*_{FB}\right)}\right]. \tag{34}$$

As $\overline{\Delta X} \to -\infty$, (33) tends to

$$\begin{aligned} &\lim_{\overline{\Delta X}\to-\infty} F\left(\overline{\Delta X}\,|u^*_{FB}\right)\left(\ln\overline{v} + \frac{1}{\overline{v}}\left(v - \overline{v}\right)\right) \\ = &\lim_{\overline{\Delta X}\to-\infty} F\left(\overline{\Delta X}\,|u^*_{FB}\right)\frac{1}{\overline{v}}rc'\left(u^*_{FB}\right)\frac{-\left(1 - F\left(\overline{\Delta X}\,|u^*_{FB}\right)\right)}{F_u\left(\overline{\Delta X}\,|u^*_{FB}\right)}, \end{aligned} \tag{35}$$

which is equal to 0 because

$$\begin{aligned} \lim_{\overline{\Delta X}\to-\infty}\frac{F\left(\overline{\Delta X}\,|u^*_{FB}\right)}{F_u\left(\overline{\Delta X}\,|u^*_{FB}\right)} &= \lim_{\overline{\Delta X}\to-\infty}\frac{h\left(\overline{\Delta X}\,|u^*_{FB}\right)}{h_u\left(\overline{\Delta X}\,|u^*_{FB}\right)} \\ &= 0, \end{aligned} \tag{36}$$

as was shown in (29). This implies that the second row also converges to 0 and that (30) converges to $E\left[X_1\,|u^*_{FB}\right] - W_A - c\left(u^*_{FB}\right)$. ■

The proof reveals that the near first-best result is driven by a likelihood ratio property of the normal distribution: as $\Delta X \to -\infty$, the likelihood ratio $h_u\left(\cdot\,|u^*_{FB}\right)/\,h\left(\cdot\,|u^*_{FB}\right)$ goes to $-\infty$, which implies that low values of ΔX are very informative with respect to the agent's action choice (Milgrom (1981)) (to avoid confusion with the drift rate $f(\cdot)$, we have denoted the density function by $h(\cdot)$). For further comments, see Müller (1997b).

4 Comparative Statics

In this section, we examine the behavior of the penalty $\underline{s}$ as the interval length Δt goes to 0 (or equivalently, as the number of control revisions goes to ∞). As $\Delta t \to 0$, the probability that a particular interval is chosen as the measurement interval goes to 0. One is therefore inclined to believe that the penalty $\underline{s}$ must simultaneously go to $-\infty$ in order to preserve incentive compatibility. While it is true that $\underline{s} \to -\infty$ as $\Delta t \to 0$, proposition 4.1 below shows that the above intuition is false.

Lemma 4.1: For negative values of $\overline{\Delta X}$, $\lim_{\Delta t \to 0} F\left(\overline{\Delta X} \,|u^*_{FB}\right) = 0$ and $\lim_{\Delta t \to 0} F_u\left(\overline{\Delta X} \,|u^*_{FB}\right) = 0$.

Proof: $F(\cdot\,|u^*_{FB})$ is the CDF of a normally distributed random variable ΔX with mean $f(u^*_{FB})\Delta t$ and variance $\sigma^2 \Delta t$. As $\Delta t \to 0$, the density function $h(\cdot\,|u^*_{FB})$ converges pointwise to 0 for all $\Delta X < 0$, which implies that $F\left(\overline{\Delta X}\,|u^*_{FB}\right) = \int_{-\infty}^{\overline{\Delta X}} h(\Delta X\,|u^*_{FB})\, d\Delta X$ converges to 0 for $\overline{\Delta X} < 0$.

Define $z \equiv \Delta X - f(u^*_{FB})\Delta t / \sigma\sqrt{\Delta t}$ and $\overline{z} \equiv \overline{\Delta X} - f(u^*_{FB})\Delta t \big/ \sigma\sqrt{\Delta t}$. Using these definitions we can write $F_u\left(\overline{\Delta X}\,|u^*_{FB}\right)$ as

$$\begin{aligned}
& \int_{-\infty}^{\overline{\Delta X}} \frac{(\Delta X - f(u^*_{FB})\Delta t) f'(u^*_{FB})}{\sigma^3\sqrt{\Delta t}\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\frac{(\Delta X - f(u^*_{FB})\Delta t)^2}{\sigma^2 \Delta t}\right\} d\Delta X \\
= & \int_{-\infty}^{\overline{z}} \frac{-f'(u^*_{FB})}{\sigma^2\sqrt{2\pi}}(-z)\exp\left\{-\frac{1}{2}z^2\right\} dz \qquad (37) \\
\equiv & \lim_{a \to -\infty} \int_{a}^{\overline{z}} \frac{-f'(u^*_{FB})}{\sigma^2\sqrt{2\pi}}(-z)\exp\left\{-\frac{1}{2}z^2\right\} dz.
\end{aligned}$$

Integrating and taking limits, we obtain

$$\begin{aligned}
F_u\left(\overline{\Delta X}\,|u^*_{FB}\right) &= \lim_{a \to -\infty} \frac{-f'(u^*_{FB})}{\sigma^2\sqrt{2\pi}} \exp\left\{-\frac{1}{2}z^2\right\}\Big|_a^{\overline{z}} \qquad (38) \\
&= \frac{-f'(u^*_{FB})}{\sigma^2\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\overline{z}^2\right\}.
\end{aligned}$$

As $\Delta t \to 0$, $\overline{z}$ goes to $-\infty$, which implies that $F_u\left(\overline{\Delta X}\,|u^*_{FB}\right)$ converges to 0 from below. ■

Proposition 4.1: For negative values of $\overline{\Delta X} < 0$, $\underline{s}$ goes to $-\infty$ as $\Delta t \to 0$.

Proof: Observe that proposition 3.2 holds for any value of Δt. The result follows then immediately from (26) and lemma 4.1. ■

The probability that a particular subinterval is selected is $1/n = \Delta t$. As $\Delta t \to 0$, this probability goes to 0. However, since the randomization device is symmetric, the decrease is uniform in all subintervals and the agent continues to face constant incentives. In other words, the agent will not shirk in one or more subintervals even though the probability that this results in a punishment goes to 0. Consequently, proposition 3.2 continues to hold and we can replace the dynamic problem with a static problem in which the agent controls a normally distributed random variable with mean $f(u^*_{FB})\,\Delta t$ and variance $\sigma^2\Delta t$. The intuition for proposition 4.1 is now obvious. As $\Delta t \to 0$, mean and variance of ΔX both go to 0, and the density function converges pointwise to 0 for all $\Delta X < 0$. Therefore, for negative values of $\overline{\Delta X}$, the probability that the agent is penalized goes to 0, which implies that $\underline{s}$ must go to $-\infty$ in order to preserve incentive compatibility.

5 Discussion and Extensions

5.1 Randomization vs. Detailed Checks

The fact that random spot checks are near first-best optimal is a neat result because such mechanisms can be frequently found in practice. Note, however, that randomization is not necessary to obtain near first-best optimality. As long as measuring output is costless, the principal can also approach the first-best outcome by a detailed check of the agent's entire output. Suppose the agent receives the same step function in each subinterval, i.e. he gets $\underline{s}$ for each subinterval where $\Delta X < \overline{\Delta X}$ and $\overline{s}$ for each subinterval where $\Delta X \geq \overline{\Delta X}$. Since the environment is stationary and wealth effects play no role, the agent's n control problems are identical. As in proposition 3.2, it can then be shown that the optimal control is constant if the solution with respect to each u_t is unique.

Let us illustrate this point for the case $n = 2$. Since the control problem in the second subinterval is independent of the state and control variables in the first subinterval, the agent's overall control problem can be expressed as a simple multivariate optimization problem. The agent solves

$$
\begin{aligned}
&\max_{u_0,u_{\frac{1}{2}}} -\prod_{t=0}^{\frac{1}{2}} F\left(\overline{\Delta X}\middle| u_t\right)\exp\left\{-r\left(2\underline{s}-\sum_{t=0}^{\frac{1}{2}} c(u_t)\,\Delta t\right)\right\}\\
&-\prod_{t=0}^{\frac{1}{2}}\left(1-F\left(\overline{\Delta X}\middle| u_t\right)\right)\exp\left\{-r\left(2\overline{s}-\sum_{t=0}^{\frac{1}{2}} c(u_t)\,\Delta t\right)\right\} \qquad (39)\\
&-F\left(\overline{\Delta X}\middle| u_0\right)\left(1-F\left(\overline{\Delta X}\middle| u_{\frac{1}{2}}\right)\right)\exp\left\{-r\left(\underline{s}+\overline{s}-\sum_{t=0}^{\frac{1}{2}} c(u_t)\,\Delta t\right)\right\}\\
&-F\left(\overline{\Delta X}\middle| u_{\frac{1}{2}}\right)\left(1-F\left(\overline{\Delta X}\middle| u_0\right)\right)\exp\left\{-r\left(\underline{s}+\overline{s}-\sum_{t=0}^{\frac{1}{2}} c(u_t)\,\Delta t\right)\right\}.
\end{aligned}
$$

Denote by $T = \left\{0, \frac{1}{2}\right\}$ the set of times at which the agent can revise his control and consider an arbitrary element $s \in T$. The agent's problem with respect to u_s is

$$\max_{u_s} -F\left(\overline{\Delta X}\middle| u_s\right) \exp\{rc(u_s)\Delta t\}[M - N + P - Q]$$
$$-\exp\{rc(u_s)\Delta t\}[N + Q], \tag{40}$$

where

$$M \equiv F\left(\overline{\Delta X}\middle| u_{t\neq s}\right) \exp\{-r(2\underline{s} - c(u_{t\neq s})\Delta t)\}, \tag{41}$$

$$N \equiv \left(1 - F\left(\overline{\Delta X}\middle| u_{t\neq s}\right)\right) \exp\{-r(2\overline{s} - c(u_{t\neq s})\Delta t)\}, \tag{42}$$

$$P \equiv \left(1 - F\left(\overline{\Delta X}\middle| u_{t\neq s}\right)\right) \exp\{-r(\underline{s} + \overline{s} - c(u_{t\neq s})\Delta t)\}, \tag{43}$$

and

$$Q \equiv F\left(\overline{\Delta X}\middle| u_{t\neq s}\right) \exp\{-r(\underline{s} + \overline{s} - c(u_{t\neq s})\Delta t)\}. \tag{44}$$

By symmetry, the optimal control is constant if (40) is strictly concave in u_s. Since $c(\cdot)$ is convex and $\exp\{\cdot\}$ is an increasing convex transformation, $\exp\{rc(u_s)\Delta t\}$ is convex. In conjunction with $N + Q > 0$, this implies that the expression in the second row is concave. In the appendix it is shown that $F\left(\overline{\Delta X}\middle| u_s\right)\exp\{rc(u_s)\Delta t\}$ is strictly convex for all $\overline{\Delta X}$ below some critical level. Moreover, $P - N > 0$ since $2\overline{s} > \underline{s} + \overline{s}$ and $M - Q > 0$ since $\underline{s} + \overline{s} > 2\underline{s}$. Thus, the expression in the first row is also strictly concave in u_s and the agent's optimal control is constant.

Analogous to section 3, constancy of the optimal control implies that we can replace the dynamic problem with a static problem in which the agent simultaneously determines the (identical) mean of n normally distributed random variables $\Delta X_0, ..., \Delta X_{\frac{n-1}{n}}$. It is now a straightforward exercise to derive the analogues of propositions 3.3-3.4 and theorem 3.1 and to show that the first-best solution can be approached arbitrarily closely by applying a step function to each subinterval separately.

5.2 Measurement Cost

The problem with detailed checks is that they require a great deal of information. For instance, bosses need to check the work of their subordinates continuously and buyers need to check the quality of every single delivered item. In practice, such detailed checks are rare since measuring output is typically costly. If we introduce measurement cost into the Brownian model, detailed checks become prohibitively costly and are Pareto-dominated by random spot checks. To see this, suppose the principal incurs fixed cost of δ for each measured increment ΔX. While detailed checks entail measurement cost of $n\delta$, random spot checks entail measurement cost of only δ. Hence, randomization shifts the upper bound on the principal's expected utility upwards by $n\delta - \delta$, which implies that randomization is strictly optimal.

5.3 Unbounded Penalties

Theorem 3.1 relies heavily on the existence of infinitely large penalties and breaks down if the agent's wealth is bounded from below. Incidentally, the same goes for the linearity result derived by Holmström and Milgrom (1987). There, the second-best sharing rule takes the form $S^* = K + \beta X_1$, where aggregated output X_1 is normally distributed with mean $f(u^*)$ and variance σ^2. Consequently, the agent's payment S^* is also normally distributed and thus unbounded from below.

6 Conclusion

In this paper, we have shown that in dynamic agency problems of the sort studied by Holmström and Milgrom (1987) linear sharing rules are not optimal if the agent takes actions in small but discrete time intervals. The principal can then approach the first-best solution arbitrarily closely by using a step function in conjunction with a random spot check. Our result is driven by two fundamental principles:

1. *Randomization creates uniform incentives*: Since the agent does not know in advance which of the subintervals is the relevant one, he faces uniform incentives and selects a constant control. One implication of this is that we can replace the dynamic problem with a mathematically much simpler static problem in which the agent chooses the mean of a normally distributed random variable and the principal is a priori restricted to step functions.

2. *Step functions and normally distributed output yield near first-best optimality*: An important property of the normal distribution is that low outcomes are a reliable signal that the agent has shirked. Step functions make optimal use of this property by punishing the agent only for output values that indicate with probability close to 1 that he has shirked. Hence, the risk of erroneous punishment (and therefore also the risk premium that must be paid to the agent) becomes negligible and the first-best outcome can be approached arbitrarily closely.

Moreover, we have shown that randomization is sufficient, but not necessary for near first-best optimality: the principal can also approach the first-best solution by applying a step function to each subinterval separately. However, such detailed checks are rare in practice since measuring output is typically costly. By contrast, random spot checks are widely used as they limit measurement cost to a single observation. For instance, in firms, bosses check the work of their subordinates only randomly. In buyer-seller relationships, buyers typically check the quality of a delivery by means of a few randomly selected samples. Additionally, random inspections occur in the military, in traffic checks, and ticket checks in public transport.

7 Appendix: Proof of Proposition 3.2

The agent's problem is

$$\max_{u_0,\ldots,u_{\frac{n-1}{n}}} -\frac{1}{n}\sum_{t=0}^{\frac{n-1}{n}} F\left(\overline{\Delta X}\middle| u_t\right)\exp\left\{-r\left(\underline{s}-\sum_{t=0}^{\frac{n-1}{n}} c\left(u_t\right)\Delta t\right)\right\}$$
$$-\frac{1}{n}\sum_{t=0}^{\frac{n-1}{n}}\left(1-F\left(\overline{\Delta X}\middle| u_t\right)\right)\exp\left\{-r\left(\overline{s}-\sum_{t=0}^{\frac{n-1}{n}} c\left(u_t\right)\Delta t\right)\right\}. \quad \text{(A.1)}$$

By symmetry, the optimal control is constant if the solution with respect to each u_t is unique. We will now show that there exists a real number H such that for all t and step functions with $\overline{s}>\underline{s}$ and $\overline{\Delta X}\leq H$, the agent's problem is strictly concave in u_t. Define by $T=\left\{0,\frac{1}{n},\ldots,\frac{n-1}{n}\right\}$ the set of times at which the agent can revise his control and consider an arbitrary element $s\in T$. The agent's problem with respect to u_s is

$$\max_{u_s} -\frac{1}{n}\exp\left\{rc\left(u_s\right)\Delta t\right\}F\left(\overline{\Delta X}\middle| u_s\right)\left(U-V\right) \quad \text{(A.2)}$$
$$-\frac{1}{n}\exp\left\{rc\left(u_s\right)\Delta t\right\}\left[nV+\sum_{T\setminus\{s\}} F\left(\overline{\Delta X}\middle| u_t\right)\left(U-V\right)\right],$$

where U and V are defined as $U\equiv\exp\left\{-r\left(\underline{s}-\sum_{T\setminus\{s\}} c\left(u_t\right)\Delta t\right)\right\}$ and $V\equiv\exp\left\{-r\left(\overline{s}-\sum_{T\setminus\{s\}} c\left(u_t\right)\Delta t\right)\right\}$.

In the second row, $\overline{s}>\underline{s}$ implies that the term in brackets is strictly positive. Since $c(\cdot)$ is convex and $\exp\{\cdot\}$ is an increasing convex transformation, $\exp\left\{rc\left(u_s\right)\Delta t\right\}$ is also convex and the expression in the second row is concave. The difficult part of the proof consists in showing that the expression in the first row is strictly concave in u_s. Since $U>V$, this is the case if and only if $\exp\left\{rc\left(u_s\right)\Delta t\right\}F\left(\overline{\Delta X}\middle| u_s\right)$ is strictly convex. By Leibniz's rule, the second derivative of $\exp\left\{rc\left(u_s\right)\Delta t\right\}F\left(\overline{\Delta X}\middle| u_s\right)$ with respect to u_s is

$$\int_{-\infty}^{\overline{\Delta X}}\exp\left\{rc\left(u_s\right)\Delta t\right\}g\left(\Delta X\right)h\left(\Delta X\right)d\Delta X, \quad \text{(A.3)}$$

where

$$g\left(\Delta X\right) \equiv \left(rc'\left(u_s\right)\Delta t+\frac{\left(\Delta X-f\left(u_s\right)\Delta t\right)f'\left(u_s\right)\Delta t}{\sigma^2\Delta t}\right)^2 \quad \text{(A.4)}$$

$$+rc''(u_s)\Delta t+\frac{(\Delta X-f(u_s)\Delta t)\, f''(u_s)\Delta t}{\sigma^2\Delta t}-\frac{(f'(u_s)\Delta t)^2}{\sigma^2\Delta t};$$

and where

$$h(\Delta X)\equiv\frac{1}{\sigma\sqrt{\Delta t}\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\frac{(\Delta X-f(u_s)\Delta t)^2}{\sigma^2\Delta t}\right\}. \tag{A.5}$$

is the density function of a normally distributed random variable with mean $f(u_s)\Delta t$ and variance $\sigma^2\Delta t$.

It remains to be shown that there exists a real number H such that for all cutoff values $\overline{\Delta X}\le H$, (A.3) is strictly positive. The improper integral (A.3) is defined as

$$\int_{-\infty}^{\overline{\Delta X}}\exp\{rc(u_s)\Delta t\}\, g(\Delta X)\, h(\Delta X)\, d\Delta X \tag{A.6}$$
$$=\lim_{a\to-\infty}\int_a^{\overline{\Delta X}}\exp\{rc(u_s)\Delta t\}\, g(\Delta X)\, h(\Delta X)\, d\Delta X$$

provided that the limit exists and is finite. Define

$$z\equiv\frac{\Delta X-f(u_s)\Delta t}{\sigma\sqrt{\Delta t}}, \tag{A.7}$$

$$\overline{z}\equiv\frac{\overline{\Delta X}-f(u_s)\Delta t}{\sigma\sqrt{\Delta t}}, \tag{A.8}$$

$$\underline{z}\equiv\frac{a-f(u_s)\Delta t}{\sigma\sqrt{\Delta t}} \tag{A.9}$$

$$A\equiv\frac{1}{\sigma\sqrt{\Delta t}\sqrt{2\pi}}, \tag{A.10}$$

$$B\equiv\exp\{rc(u_s)\Delta t\}\,\sigma\sqrt{\Delta t}, \tag{A.11}$$

$$C\equiv(rc'(u_s)\Delta t)^2+rc''(u_s)\Delta t-\frac{(f'(u_s)\Delta t)^2}{\sigma^2\Delta t}, \tag{A.12}$$

$$D\equiv\frac{2rc'(u_s)\Delta t f'(u_s)\Delta t}{\sigma\sqrt{\Delta t}}, \tag{A.13}$$

$$E\equiv\frac{f''(u_s)\Delta t}{\sigma\sqrt{\Delta t}}, \tag{A.14}$$

and

$$G\equiv\frac{(f'(u_s)\Delta t)^2}{\sigma^2\Delta t}. \tag{A.15}$$

Multiplying out and using (A.7)-(A.15) yields

$$\begin{aligned}
&\lim_{a\to-\infty}\int_a^{\overline{\Delta X}} \exp\left\{rc\left(u_s\right)\Delta t\right\} g\left(\Delta X\right) h\left(\Delta X\right) d\Delta X \\
=\; &\lim_{\underline{z}\to-\infty} AB\int_{\underline{z}}^{\overline{z}} \left(C+\left[D+E\right]z+Gz^2\right)\exp\left\{-\frac{1}{2}z^2\right\}dz.
\end{aligned} \tag{A.16}$$

We will now break up (A.16) into three integrals and evaluate the limit of each integral separately. Consider first the limit

$$\lim_{\underline{z}\to-\infty} ABC\int_{\underline{z}}^{\overline{z}} \exp\left\{-\frac{1}{2}z^2\right\}dz. \tag{A.17}$$

While $\exp\left\{-\frac{1}{2}z^2\right\}$ cannot be integrated, the limit (A.17) is well known and defined as

$$\lim_{\underline{z}\to-\infty} ABC\int_{\underline{z}}^{\overline{z}} \exp\left\{-\frac{1}{2}z^2\right\}dz \equiv C\exp\left\{rc\left(u_s\right)\Delta t\right\}F\left(\overline{\Delta X}\left|u_s\right.\right), \tag{A.18}$$

where $F\left(\cdot\left|u_s\right.\right)$ is the CDF of a normally distributed random variable. Next, consider the limit

$$\lim_{\underline{z}\to-\infty} AB\left[D+E\right]\int_{\underline{z}}^{\overline{z}} z\exp\left\{-\frac{1}{2}z^2\right\}dz. \tag{A.19}$$

Integrating (A.19), we obtain

$$\begin{aligned}
&\lim_{\underline{z}\to-\infty} AB\left[D+E\right]\int_{\underline{z}}^{\overline{z}} z\exp\left\{-\frac{1}{2}z^2\right\}dz \\
=\; &\lim_{\underline{z}\to-\infty} -AB\left[D+E\right]\exp\left\{-\frac{1}{2}z^2\right\}\Big|_{\underline{z}}^{\overline{z}} \\
=\; &-AB\left[D+E\right]\exp\left\{-\frac{1}{2}\overline{z}^2\right\}.
\end{aligned} \tag{A.20}$$

Finally, consider the limit

$$\lim_{\underline{z}\to-\infty} ABG\int_{\underline{z}}^{\overline{z}} z^2\exp\left\{-\frac{1}{2}z^2\right\}dz. \tag{A.21}$$

Integration by parts gives

$$\begin{aligned}
&\lim_{\underline{z}\to-\infty} ABG\int_{\underline{z}}^{\overline{z}} z^2\exp\left\{-\frac{1}{2}z^2\right\}dz \\
=\; &\lim_{\underline{z}\to-\infty} ABG\left[-z\exp\left\{-\frac{1}{2}z^2\right\}\Big|_{\underline{z}}^{\overline{z}}+\int_{\underline{z}}^{\overline{z}}\exp\left\{-\frac{1}{2}z^2\right\}dz\right] \\
=\; &-ABG\overline{z}\exp\left\{-\frac{1}{2}\overline{z}^2\right\}+G\exp\left\{rc\left(u_s\right)\Delta t\right\}F\left(\overline{\Delta X}\left|u_s\right.\right),
\end{aligned} \tag{A.22}$$

since by l'Hôpital's rule

$$\lim_{\underline{z} \to -\infty} \underline{z} \exp\left\{-\frac{1}{2}\underline{z}^2\right\} = \lim_{\underline{z} \to -\infty} \frac{1}{\underline{z} \exp\left\{\frac{1}{2}\underline{z}^2\right\}} = 0. \tag{A.23}$$

Adding up (A.18), (A.20) and (A.22), we can write (A.16) as

$$\begin{aligned} &\lim_{\underline{z} \to -\infty} AB \int_{\underline{z}}^{\overline{z}} \left(C + [D + E]\, z + Gz^2\right) \exp\left\{-\frac{1}{2}z^2\right\} dz \\ = \; & -AB\,[D + E + G\overline{z}] \exp\left\{-\frac{1}{2}\overline{z}^2\right\} \\ & + [C + G] \exp\{rc(u_s)\,\Delta t\}\, F\left(\overline{\Delta X}\,|u_s\right). \end{aligned} \tag{A.24}$$

We are now in the position to evaluate the sign of (A.24). Recall that a strictly positive sign implies that $\exp\{rc(u_s)\,\Delta t\}\, F\left(\overline{\Delta X}\,|u_s\right)$ is strictly convex and that the agent's problem is strictly concave. From $c''(\cdot) \geq 0$ it follows that $[C + G] \exp\{rc(u_s)\,\Delta t\}\, F\left(\overline{\Delta X}\,|u_s\right)$ is strictly positive. The sign of $D + E + G\overline{z}$ is ambiguous because $E + G\overline{z} < 0$ (for negative $\overline{z}$) but $D > 0$. However, as $\overline{\Delta X} \to -\infty$, $\overline{z}$ tends to $-\infty$, which implies that $\overline{z}$ can take arbitrarily large negative values. Hence, there exists a negative real number H such that $D + E + G\overline{z} < 0$ for all $\overline{\Delta X} \leq H$. Since $AB > 0$, it follows that the sign of (A.24) is strictly positive for all $\overline{\Delta X} \leq H$. ■

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