Bidding Against an Unknown Number of Competitors Sharing Affiliated Information*

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Abstract

In the general symmetric model of Milgrom and Weber, equilibrium bidding is analyzed with a stochastic number of bidders. The equilibrium strategies generalize the known expressions in a coherent way. For the equilibrium bid function of the first price auction, an interpretation involving 'marginal winning probabilities' is proposed. With a generalized version of the linkage principle, the well-known revenue ranking theorems extend to a stochastic number of bidders. As an application, we show that the seller's generically optimal information policy regarding the number of competitors is concealing the information.

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1. Introduction

Most of the results in auction theory have been obtained under the assumption that the exact number of bidders is fixed and mutual (or common) knowledge among the participants. Yet in most auctions, potential bidders do not know how many competing bids will be submitted. In particular in sealed bid auctions, the exact number of competing bids is hardly ever known.[1]

A weaker assumption is that the distribution only that governs the number of competitors is common knowledge. Those potential buyers interested in tendering a bid share implicit common knowledge regarding the competitive structure they are part of, such as the number of firms present in the relevant market, the scope of each firm's activities, cooperative arrangements and legal relationships between the players, etc. Whenever not all potential bidders can profitably submit a costly bid, participation in symmetric equilibrium must be randomized (Milgrom, 1981). The equilibrium distribution over the number of submitted bids then provides a reduced-form description of the underlying competitive structure and of the associated priors. The structure of economic relations, for example, forms 'a Poisson environment' if participation decisions are taken independently and if the equilibrium participation probability of a single firm is small while there is a large number of potential buyers.[2]

McAfee and McMillan (1987a) and Matthews (1987) pioneered the analysis of auctions with a stochastic number of bidders. The emphasis was put on risk-averse bidders and their indirect expected utility in the equilibrium of a revelation mechanism. Harstad, Kagel, and Levin (1990) and Piccione and Tan (1996) consider explicit bid functions for risk neutral bidders facing an uncertain number of (informed) competitors. While McAfee and McMillan, and Harstad et al. exclusively studied models where the bidders have independent private information on the object in sale, Matthews also explored the case of affiliated private information, and Piccione and Tan considered bidders with partially asymmetric information on an unknown common value. This paper analyzes equilibrium bidding for symmetrically informed, risk neutral bidders in the general symmetric model of Milgrom and Weber (1982).

When the number of actual competitors is uncertain to a bidder, his win-

[1] Even when the number of participating traders is observable there may be substantial uncertainty on the number of 'actual bidders' because some of the participants can be intermediaries; e.g., see the Dutch auctions for French eggplants analyzed by Laffont et al. (1995, pp.967-70).

[2] McAfee and Vincent (1992, p.515) note that 'commonly, the negative binomial distribution is used to model stochastic numbers of participants.'

ning probability is a mixture distribution of the respective winning probabilities against different numbers of rivals. Certainly, this alone does not imply that the equilibrium bidding functions must be simple *weighted averages* of the respective strategies against a fixed number of competitors. Still, such a relation is known to hold true with independent private information of the bidders, and it has been suggested to hold generally (Harstad et al., 1990).[3] However, the 'surprising' description as weighted averages does not extend to correlated private information. A general explanation is significantly simpler: in a symmetric equilibrium with monotone strategies, a bidder must win against the average highest competing type, where the average is taken with respect to the random size of the pool of competitors. As the bidders care about winning on average, the associated equilibrium bids then condition on the average highest competing type.

This intuition indeed captures the essence of all the effects implied by a competitency of random size. In the same way as the equilibrium strategies generalize the well-known bidding functions in a straightforward way, the linkage principle extends to a more general version.[4] As a consequence, the central revenue rankings from Milgrom and Weber (1982) persist with a stochastic number of bidders. For the equilibrium strategy of the first price auction, whose form was originally obtained by Wilson (1977), we offer an intuitive interpretation in terms of marginal equilibrium winning probabilities.

An important problem of practical relevance is the choice of the information policy by the auctioneer. It is known from Milgrom and Weber that revealing any information on the object sold further increases expected revenue to the auctioneer. However, this result is reverted with respect to information on the number of competing biders. If the bid-taker knows the number of bids that will be submitted, and if the bidders share affiliated private information, Matthews (1987) found that it is better for the seller always to conceal this number. Essentially, what is revealed becomes negatively affiliated to the bidders' private information.[5]

[3] *"Clearly, an equilibrium bid facing this 'numbers uncertainty' will be [a] weighted average of the bids that would have been chosen for each number of rivals. What are the weights?"* (Harstad et al., 1990, p.35). For a review of the results, see Wilson (1992, p.236).

[4] For an exposition, see Milgrom and Weber (1982, p.1110), or Milgrom (1987).

[5] Intuitively, if a random vector is affiliated it is more likely that larger values in some coordinates realize jointly together with smaller values in other coordinates. The concept of affiliation was introduced into economics and formalized generally by Milgrom and Weber (1982). A continuously distributed random vector is called *affiliated* if its density f is such that $\log f$ is supermodular: $f(\mathbf{v} \vee \mathbf{v}')\, f(\mathbf{v} \wedge \mathbf{v}') \geq f(\mathbf{v})\, f(\mathbf{v}')\ \forall \mathbf{v}' \neq \mathbf{v}$, $\vee$ ($\wedge$) the coordinatewise maximum (minimum) operation. Negative affiliation is defined by the reverse inequality.

Allowing for reporting policies that exploit the fact that 'inverse' reports are in turn positively affiliated with the bidders' private information, we show that there is no information policy that generically improves upon the concealment policy in terms of expected revenue.

The rest of this section reviews the related literature. Section 2 provides an illustrating example. Sections 3 and 4 describe the model and some technical results used subsequently. Sections 5 and 6, respectively, derive and interpret the symmetric equilibrium strategies of the second and first price auctions. Section 7 considers the revenue rankings for a stochastic participancy and generalizes the argument. Section 8 asks for the seller's optimal information policy regarding the number of competitors.

Related Literature

Bidding games with a stochastic participancy[6] have been studied only in specializations of Milgrom and Weber's general symmetric model; for brief reference, we use the following abbreviations. If the bidders' privately received signals about the resource's value are independently and identically distributed variates that coincide with their valuations, this is the case of *symmetric independent private valuations* (henceforth: *SIPV case).* The simplest model that assumes the object's idiosyncratic value to a bidder is a non-decreasing symmetric functional of all the received signals is *the SIAS case (symmetric, independent, aggregated signals);* already in such a model, equilibrium bidding avoids a winner's curse. *The SAPV case* assumes identically distributed, but affiliated private valuations. Finally, in *the CISCV case*, the bidders obtain conditionally independent, symmetrically distributed signals on an unknown common value.[7]

The original studies have concerned *risk-averse bidders* and an auctioneer that is informed on the exact number of bidders. In the SIPV case and for bidders that are constantly absolute risk averse, McAfee and McMillan (1987a) have shown that by always concealing the number of bidders the seller will raise expected revenue from the first price auction, compared to a revealing policy. Matthews (1987) condenses the arguments and generalizes the results to different degrees of absolute risk aversion. In the SIPV case, the concealment (revealing) policy

[6] Auctions with a stochastic participancy due to symmetric, endogenous bidder entry have been analyzed for the revenue impact of screening policies; for some reference see Section 8.1. Here, only the continuation game following some unmodelled entry stage game is considered.

[7] In this 'mineral rights model', signals are informative under the assumption that their conditional density exhibits the monotone likelihood ratio property; see Milgrom and Weber (1982, p.1099) for details.

is preferred by bidders with increasing (decreasing) absolute risk aversion.[8] In the SAPV case, the revealing policy is preferred by bidders with non-increasing absolute risk aversion. Dyer, Kagel and Levin (1989) experimentally investigated the revenue-enhancing potential of a concealment policy for the first price auction in the SIPV case. With actual participancies of three and six bidders, respectively, average winning bids were higher indeed under the concealment policy (though by approximately 2% only).

For *risk neutral bidders*, McAfee and McMillan generalized the work of Myerson (1981) to a bidding participancy whose size and composition is uncertain. If only the size of the participancy is uncertain, any auction awarding the object to the bidder with the highest bid from each potential set of participants implements the optimal revelation mechanism. Therefore in the SIPV case, revenue equivalence across the usual auction forms persists as long as the bidders' (posterior) beliefs on the composition of the participancy (derived from common priors) are symmetric. Unless, the first price auction almost surely does not implement the optimal auction any more. For the SIAS case, explicit equilibrium bid functions have been obtained by Harstad et al. (1990) for the first and second price auction. The bid functions are explained as certain weighted averages, and they are concluded to be revenue equivalent. From the present analysis, these results occur as corollaries. Finally, in the CISCV framework, Piccione and Tan (1996) have analyzed a semi-symmetric, mixed equilibrium of the first price auction with both uninformed and informed bidders, where the number of informed bidders is uncertain but the total number of bidders is fixed and known.

2. A Simple Example

Consider the SIPV case, valuations being distributed with density $f(\cdot)$ on $[0, \bar{v}]$, and the total number N of bidders[9] being distributed with probability weights p. Assume each submitted bid is paid to the seller. Then if all his competitors bid according to the (strictly increasing, continuous) function β, type v of bidder 1

[8]Building upon the bidders' corresponding ranking derived by Matthews for the 'fixed-n case', Smith and Levin (1996) show that with risk averse bidders, decreasing absolute risk aversion can imply higher expected revenue from the *second price* than from the first price auction with symmetric equilibrium entry of bidders.

[9]Throughout, random variables are denoted by capital, and their realizations by small letters.

obtains

$$\pi(a|v) = v \cdot G(\beta^{-1}(a)) - a, \text{ where } G(\cdot) \equiv \sum_{n=1}^{\infty} p(n) F(\cdot)^{n-1}$$

upon bidding $a \geq 0$. Thus, the symmetric equilibrium bids are $\beta(v) = \int_0^v \tau dG(\tau)$.[10] Let $\hat{V}(n) = max_{j=2,\ldots,n}\{V_j\}$. Then with $n-1$ competitors, bidder 1, in equilibrium, wins with probability $F_{\hat{V}(n)}(v)$. Consider the distribution function G, and let $\hat{V}$ be the random variable characterized by this distribution (up to the null sets of the measure induced by the cdf G). As $E_N[F_{\hat{V}(N)}] = G$, where $F_{\hat{V}(n)}$ is the distribution function of $\hat{V}(n)$, write $\hat{V} = E_N[\hat{V}(N)]$. Using this short-hand notation, a compact form of β is:

$$\beta(v) = E_{\hat{V}}[\hat{V} \cdot \mathbf{1}_{\{\hat{V} \leq v\}}].$$

If only the winning bid is paid to the auctioneer, similar arguments show that the symmetric equilibrium strategy b is:

$$b(v) = E_{\hat{V}}[\hat{V} \mid \hat{V} \leq v].$$

Thus, expected revenue from the first price and the all pay auction is the same also with a stochastic number of bidders. Obviously, the strategies are completely parallel to their 'fixed-n' analogues, because von-Neumann-Morgenstern utility is a linear functional of the distribution of the prospect evaluated. The equilibrium bid functions for a stochastic number of bidders must therefore be the same functionals of the (expected) winning probability as their fixed-n counterparts. By contrast, Harstad et al. (1990) have characterized b as:

$$b(v) = E_N\left[E_{\hat{V}(N)}\left[\hat{V}(N) \mid \hat{V}(N) \leq v\right] \mid \hat{V} \leq v\right].$$

3. The Model

An asset is sold to a random-sized set of risk neutral bidders, each of whom submits one (scalar) bid at the stage where the number of competitors is unknown. The

[10] Integrating the first order condition $\beta'(v) = vG'(v)$ subject to $\beta(0) = 0$ yields the expression. No bidder can improve by unilateral deviation because $\pi_a(\beta(v)|v) = 0$ and $\pi_a(\beta(\tilde{v})|v) = vG'(\tilde{v})/b'(\tilde{v}) - 1 = v/\tilde{v} - 1$ has the sign of $v - \tilde{v}$.

actual number of (total) participants is commonly perceived as a random variable N with support $I = \{1\} \cup J$, $\emptyset \neq J \subseteq \{2, 3, ...\}$,[11] and probability weights $p_N(\cdot)$ such that $\sum_{n \in I} p_N(n) = 1$ and $\sum_{n \in I} n\, p_N(n) < \infty$.

For each realization $n \in I$ of N, n potential participants each privately receive a signal $v_i \in [0, \bar{v}]$, $\bar{v} \in (0, \infty)$, $1 \leq i \leq n$, on the value of the resource in sale. The hidden characteristics of the resource are represented by a random vector $\mathbf{S}$ mapping into $\Re^m$, and a bidder's idiosyncratic valuation is determined by a joint realization of $\mathbf{S}$ and the signals $\mathbf{V}^{(n)}$.[12] Specifically, if i is of type v and his $n-1$ competitors are of type $\mathbf{v}_{-i}^{(n-1)}$, i's valuation is given by $U_n(v_i; \mathbf{v}_{-i}^{(n-1)}; \mathbf{s})$, where the common functional U_n: $\Re_+ \times \Re_+^{n-1} \times \Re^m \rightarrow \Re_+$ has the following properties.

Assumption 1. *$\forall n \in J$, U_n is symmetric in its second block of $n-1$ arguments, non-decreasing, and strictly increasing in its first argument.*

Assumption 2. *(i) $\forall n \in J$, the joint distribution of $\mathbf{\Sigma}^{(n)} \equiv (\mathbf{V}^{(n)}, \mathbf{S})$ is symmetric and has finite marginal expectations with respect to the coordinate variables of $\mathbf{V}^{(n)}$. (ii) $\forall n \in J$, $F_{\mathbf{\Sigma}^{(n)}}$ admits a density $f_{\mathbf{\Sigma}^{(n)}}$ that is affiliated, strictly affiliated at least for one $n \in J$.*

Assumption 3. *$(\mathbf{\Sigma}^{(n)}, N)$ has a product distribution $f_{(\mathbf{\Sigma}^{(n)},N)}(\boldsymbol{\sigma}^{(n+m)}, n) = f_{\mathbf{\Sigma}^{(n)}}(\boldsymbol{\sigma}^{(n+m)})\, p_N(n)$ on $[0, \bar{v}]^n \times \Re^m \times I \quad (n \in I)$.*

Assumption 2(ii) is not needed technically; it only serves to emphasize a difference between the fixed-n and the stochastic-n case. Apart from Assumptions 2 and 3, we do not restrict the distributions $\{F_{\mathbf{\Sigma}^{(n)}}\}_{n \in J}$ in any way.

Let $\hat{V}(n) = max\{V_2, .., V_n\}$. Setting $\hat{V}(1) \equiv 0$, so that $F_{\hat{V}(1)}(\cdot) \equiv 1$ a.e., $\hat{V}(n)$ is well-defined on $I \ni n$. By symmetry, $\hat{V}(n)$ is the highest type of any bidder's competitors when the number of these is $n-1$, and $\hat{V}(N)$ is then a two stage random variable that represents the (random) highest type from among a *random* number $N-1$ of competitors.

The following suggestive notation is used. $\vec{\mathbf{V}}_{-1}^{(n-1)}$ denotes the vector of the order statistics of $\mathbf{V}_{-1}^{(n-1)}$, the stochastically largest component of which is $\hat{V}(n)$. Let $\vec{\mathbf{V}}_{-1}^{(n-2)} \equiv \vec{\mathbf{V}}_{-1}^{(n-1)} \backslash \hat{V}(n)$ be the $(n-2)$-vector $\vec{\mathbf{V}}_{-1}^{(n-1)}$ without its first coordinate

[11] Since for each participant being an active bidder is a self-evident event (he knows the event whenever it obtains), it is mutually self-evident, hence common knowledge among all bidders that the total number of them is *at least* one.

[12] The dimension of the vector of signals is indicated in small brackets. $\mathbf{V}_{-i}^{(n-1)}$ is understood as the random vector consisting of the $n-1$ components $(V_j)_{j \neq i}$.

variable, let $\vec{\mathbf{\Sigma}}_{-1}^{(n-1)} \equiv (\vec{\mathbf{V}}_{-1}^{(n-1)}, \mathbf{S})$ and $\vec{\mathbf{\Sigma}}_{-1}^{(n-2)} \equiv (\vec{\mathbf{V}}_{-1}^{(n-2)}, \mathbf{S})$. As $\mathbf{\Sigma}^{(n)}$ is affiliated, also $(V_1, \vec{\mathbf{\Sigma}}_{-1}^{(n-1)}) = (V_1, \hat{V}(n), \vec{\mathbf{\Sigma}}_{-1}^{(n-2)})$ is (Theorem 2 of Milgrom and Weber, 1982).

Define the real-valued random variable $W_n = U_n(V_1; \mathbf{V}_{-1}^{(n-1)}; \mathbf{S})$ and the function w_n: $[0, \bar{v}]^2 \to \Re_+$ by

$$w_n(v, \hat{v}) = E_{\mathbf{\Sigma}^{(n)}}[W_n \mid \hat{V} = \hat{v}, V_1 = v].$$

The function $w_n(\cdot, \cdot)$ is non-decreasing because an affiliated random vector has the 'conditional monotone regression endowment': its conditional expectation is positively dependent on the realizations of each conditioning coordinate variable. Milgrom and Weber (1982, Theorems 23(iii), 24, and 5) show that affiliation of a random vector is in fact characterized by its conditional monotone regression endowment with respect to all weak inequality conditions imposed on any selection of its coordinate variables. As an illustration consider the SIAS case, where

$$\begin{aligned} w_n(v, \hat{v}) &= E_{\mathbf{V}^{(n)}}[U_n(V_1; \mathbf{V}_{-1}^{(n-1)}) \mid V_1 = v, \hat{V}(n) = \hat{v}] \\ &= E_{\vec{\mathbf{V}}_{-1}^{(n-1)}}[U_n(v; \vec{\mathbf{V}}_{-1}^{(n-1)}) \mid \hat{V}(n) = \hat{v}]. \end{aligned}$$

As $\vec{\mathbf{V}}_{-1}^{(n-1)} = (\hat{V}(n), \vec{\mathbf{V}}_{-1}^{(n-2)})$ is (non-trivially) affiliated already if $\mathbf{V}^{(n)}$ is a vector of independent variates,[13] it follows with Theorem 5 of Milgrom and Weber that $D_2 w_n(\cdot, \cdot) \geq 0$. Hence, the fact that winning is informative on the value of the object obtained must be incorporated in the equilibrium bids already in the SIAS case, in order to avoid a winner's curse $(E[w_n(v, \hat{V}(n)) \mid \hat{V}(n) \leq v] \leq E[w_n(v, \hat{V}(n))])$.

4. Uncertain Number of Bidders

The highest type from among a bidder's competitors when there are n actual bidders is $\hat{V}(n)$. Consider now a random variable $\hat{V}$ such that $\hat{V}(n) = \hat{V} | \{N = n\}$; i.e., $\hat{V}$ is defined such that the joint density of $(N, \hat{V})$ takes the form

$$f_{(N,\hat{V})|V_1}(n, \hat{v}|v) = f_{\hat{V}(n)|V_1}(\hat{v}|v)\, p_N(n). \tag{4.1}$$

[13] The density of $\vec{\mathbf{V}}_{-1}^{(n-1)}$ is $f_{\vec{\mathbf{V}}_{-1}^{(n-1)}}(\mathbf{v}) = (n-1)! \prod_{i=1}^{n-1} f_1(v_i) \cdot \mathbf{1}_{\{v_1 \geq \dots \geq v_{n-1}\}}(\mathbf{v})$, where the indicator function obeys the affiliation inequality (Proposition 3.11 of Karlin and Rinott, 1980).

Then for the marginal distribution function of $\hat{V}|\{V_1 = v\}$:[14]

$$
\begin{aligned}
F_{\hat{V}|V_1}(\hat{v}|v) &= \int_0^{\hat{v}} \sum_{n \in I} f_{(N,\hat{V})|V_1}(n, t|v)\, dt \\
&= \sum_{n \in I} F_{\hat{V}(n)|V_1}(\hat{v}|v)\, p_N(n) \\
&= E_N[F_{\hat{V}(N)|V_1}(\hat{v}|v)]. \qquad (4.2)
\end{aligned}
$$

In symmetric equilibrium of a standard auction, the distribution function of the variate $\hat{V}|\{V_1 = v\}$, evaluated at v, is then the *expected* winning probability of a type-v bidder against a competitency of random size. Therefore, the random variable $\hat{V}$ characterized by the distribution (4.2) is the *expected highest type of 1's competitors.* Whenever we will indicate this by the short-hand notation $\hat{V} = E_N[\hat{V}(N)]$, this is done with the understanding that the indicated relation holds for the distribution functions of these random variables.

Example 1. Consider the SIPV case, and suppose the total number N_0 of potentially participating bidders is known to be Poisson distributed on $\{0, 1, 2, ...\}$ with parameter $\lambda > 0$. Each participating bidder considers the random variable $N = N_0|\{N_0 \geq 1\}$, described by the probability weights

$$p_N(n) = p_{N_0|\{N_0 \geq 1\}}(n) = \frac{e^{-\lambda}}{1 - e^{-\lambda}} \frac{\lambda^n}{n!} \cdot \mathbf{1}_{\{n \geq 1\}}(n).$$

A straightforward calculation shows that $\hat{V}$ in this case is given by:

$$F_{\hat{V}}(v) = \frac{e^{-\lambda}}{1 - e^{-\lambda}} \frac{1}{F_1(v)} \left[e^{\lambda F_1(v)} - 1 \right].$$

As $\hat{V}$ is distributed according to a convex mixture of the distributions of $\hat{V}(n)$, it seems reasonable that $(\hat{V}, V_1, \vec{\boldsymbol{\Sigma}}_{-1}^{(n-1)})$ is affiliated $\forall n \in J$. Intuitively, the affiliation property of a random vector should be preserved under probability mixtures of its coordinate variables. In fact it is even intensified.

Theorem 1. $(\hat{V}, V_1, \vec{\boldsymbol{\Sigma}}_{-1}^{(n-1)})$ *is strictly affiliated* $(n \in J)$.

[14]Summation and integration can be interchanged because the sum is absolutely converging; since by assumption $\sum_{n \in I} n\, p_N(n) < \infty$, necessarily $\sum_{n \in I} f_{\hat{V}(n)|V_1}(\cdot|\cdot)\, p_N(n) < \infty$.

(Proofs not stated immediately are delegated to the Appendix.) Two useful relations follow immediately from the definition of $\hat{V}$. By (4.1), the conditional probability weights $p_{N|\hat{V},V_1}(\cdot|\hat{v},v)$ of $N|\{\hat{V}=\hat{v},V_1=v\}$ are:

$$p_{N|\hat{V},V_1}(n|\hat{v},v) = \frac{f_{\hat{V}(n)|V_1}(\hat{v}|v)\, p_N(n)}{f_{\hat{V}|V_1}(\hat{v}|v)} \cdot \mathbf{1}_{\{\hat{v},v>0\}}(\hat{v},v). \tag{4.3}$$

Similary, the conditional probability mass function $\check{p}_{N|\hat{V},V_1}(\cdot|\hat{v},v)$ of $N|\{\hat{V} \leq \hat{v} \,|\, V_1 = v\}$ (with distribution function $\check{P}_{N|\hat{V},V_1}(\cdot|\hat{v},v)$) is:

$$\check{p}_{N|\hat{V},V_1}(n|\hat{v},v) = \frac{F_{\hat{V}(n)|V_1}(\hat{v}|v)\, p_N(n)}{F_{\hat{V}|V_1}(\hat{v}|v)} \cdot \mathbf{1}_{\{\hat{v},v>0\}}(\hat{v},v). \tag{4.4}$$

Finally, define the function $\hat{w}(\cdot,\cdot)$: $[0,\bar{v}]^2 \to \Re_+$ as:

$$\hat{w}(v,\hat{v}) = E_N\left[E_{\mathbf{\Sigma}^{(N)}}\left[W_N \mid \hat{V}=\hat{v}, V_1=v\right]\right].$$

In contrast to $w_n(\cdot,\cdot)$, the conditional expectation in $\hat{w}(\cdot,\cdot)$ refers to $\hat{V}$ (instead of to $\hat{V}(n)$); in addition, $\hat{w}(\cdot,\cdot)$ averages $\{\mathbf{\Sigma}^{(n)}, U_n\}_{n\in I}$ over the different $n \in I$.

Theorem 2. *For the function* $\hat{w}(\cdot,\cdot)$:
(i) $E_N[w_N(v,\hat{v}) \,|\, \hat{V}=\hat{v}, V_1=v] = \hat{w}(v,\hat{v})$.
(ii) $D_2\hat{w}(\cdot,\cdot) \geq 0$, $D_1\hat{w}(\cdot,\cdot) > 0$.
(iii) For arbitrary $v,\hat{v}>0$, *if* $\{F_{\mathbf{\Sigma}^{(n)}}, U_n\}_{n\in J}$ *is such that* $w_n(v,\hat{v}) \geq w_{n-1}(v,\hat{v})$, *then* $\hat{w}(v,\hat{v})$ *is uniformly non-decreasing as* N *increases stochastically.*

5. The Second Price Auction

Consider bidder 1 being of type $v>0$. Assume that all his rivals bid according to the continuous and strictly increasing function $b(\cdot)$: $[0,\bar{v}] \to \Re_+$, which is differentiable a.e. Given 1 wins upon bidding $a \in \Re_+$ and there are n participants, he pays the second highest bid $b(\hat{V}(n))|\{\hat{V}(n) \leq b^{-1}(a), V_1=v\}$ and obtains an object of value $W_{1,n}|\{\hat{V}(n) \leq b^{-1}(a), V_1=v\}$. Thus his expected payoff when he only knows that $n \in I$ is:

$$\pi(a|v) = E_N\left[E_{(\mathbf{\Sigma}^{(N)},\hat{V}(N))}\left[\left\{W_N - b(\hat{V}(N))\right\} \cdot \mathbf{1}_{\left\{\hat{V}(N)\leq b^{-1}(a)\right\}} \,|\, V_1=v\right]\right].$$

Consider the candidate strategy $b(v) = \hat{w}(v,v)$ for a symmetric equilibrium $[b_i(v) = b(v)]_{i \in I}$. Theorem 2(ii) implies that $b'(\cdot) > 0$.

Proposition 1. *Suppose that all actual competitors of 1 bid according to the strategy b. If 1 wins upon bidding $a \in \Re_+$, his expected profit is:*

$$\pi(a|v) = E_{\hat{V}} \left[\left(\hat{w}(V_1, \hat{V}) - \hat{w}(\hat{V}, \hat{V}) \right) \cdot \mathbf{1}_{\left\{\hat{V} \leq b^{-1}(a)\right\}} \mid V_1 = v \right]. \tag{5.1}$$

It follows by a straightforward modification of Theorem 6 of Milgrom and Weber (1982) that symmetric bidding according to $b(\cdot)$ forms an equilibrium. Theorems 1 and 2 show that all the properties used in the derivation apply to $\hat{V}|\{V_1 = v\}$.

Theorem 3. *The symmetric equilibrium $[b_i(v) = b(v)]_{i \in I}$ of the second price auction is given by $b(v) = \hat{w}(v,v)$.*

(For a self-contained proof without the use of Proposition 1, see the Appendix.) As indicated in the SIPV example above, the basic modification of the equilibrium strategies with a stochastic participancy is the substitution of the highest competing type $\hat{V}(n)$ by the expected highest competing type $\hat{V}$. With the bidders' valuations arising from an isotonic statistics U_n of all the received signals, this results in a replacement of $w_n(\cdot,\cdot)$ by $\hat{w}(\cdot,\cdot)$. Proposition 1 and Theorem 3 then show that in the symmetric equilibrium of the second price auction each type bids his expected valuation conditional on *the average highest competing type $\hat{V}$ being just as high.* Note that for $b^{-1}(a) = v$, (5.1) is strictly positive by Theorems 1 and 2(ii), and Theorem 5 of Milgrom and Weber (1982).

A bidder in equilibrium thus avoids an *average* winner's curse, averaged over the potential number of competitors. It is not clear intuitively whether an increase in expected competition then intensifies or a releases this average winner's curse, i.e. whether a stochastic increase in N demands higher or lower discounts in the equilibrium bids. Intuition only points to $\hat{V}$ increasing stochastically as N does so. This intuition is true also with affiliated types.

Proposition 2. $\forall v \in (0, \bar{v})$: *(i) $\hat{V}(n)|\{V_1 = v\}$ is strictly stochastically increasing in $n \in I$; (ii) $\hat{V}|\{V_1 = v\}$ is (strictly) stochastically increasing as N increases (strictly) stochastically.*

Now Theorem 2(iii) shows that also the function $\hat{w}(\cdot,\cdot)$ actually shifts upwards as N increases stochastically, whenever the fixed-n bids $w_n(\cdot,\cdot)$ are non-decreasing in n. In conjunction with Proposition 2(ii), it follows that

$$E_{\hat{V}}[\hat{w}(\hat{V},\hat{V})\cdot \mathbf{1}_{\{\hat{V}\leq V_1\}}\,|\,V_1=v]=\int_0^v [1-F_{\hat{V}|V_1}(t|v)]\,d\hat{w}(t,t)$$

is non-decreasing as N increases stochastically ($v>0$). This proves:

Corollary 1. *With fixed-n bids that are non-decreasing in n, expected equilibrium payments from the second price auction do not decrease as the number of bidders increases stochastically.*

Reconsider the SIAS case with two bidders ($n=2$). The symmetric equilibrium bids $w_2(v,v)=\int_{\Re^m} U_2(v;v;\mathbf{s})\,f_{\mathbf{S}}(\mathbf{s})d\mathbf{s}$ are independent of the distribution of $\vec{\mathbf{V}}_{-1}^{(2)}=\hat{V}(2)$, so the symmetric equilibrium is one in dominant strategies even. When the linkages among the types are weakened such that the maximum of the observed signals is a sufficient statistics for the expectation $w_n(v,\hat{V}(n))$ generated by the distribution of $\mathbf{\Sigma}^{(n)}$, an analogous property also obtains in the affiliated case. This was observed for a fixed and known number of bidders by Harstad and Levin (1985), who called the distribution of $\mathbf{\Sigma}^{(n)}$ *maximal attentive* iff $max(\hat{v},\check{v})\leq v$ implies that $w_n(v,\hat{v})=w_n(v,\check{v})$.[15]

Theorem 4. *If the bidders' beliefs are such that $F_{\mathbf{\Sigma}^{(n)}}$ is maximal attentive ($n\in J$), the symmetric equilibrium of the second price auction is an equilibrium in dominant strategies also with a stochastic number of bidders.*

6. The First Price Auction

Let a candidate strategy for a symmetric equilibrium in the first price sealed bid auction be the strictly increasing function $b(\cdot)$: $[0,\bar{v}]\rightarrow\Re_+$. If all his competitors use the strategy $b(\cdot)$, expected profit of a type-v bidder ($v>0$) who bids $a>0$ is:

$$\begin{aligned}\pi(a|v) &= E_N\left[E_{(\mathbf{\Sigma}^{(N)},\hat{V}(N))}\left[\{W_N-a\}\cdot\mathbf{1}_{\{b(\hat{V}(N))\leq a\}}\,|\,V_1=v\right]\right]\\ &= E_N\left[E_{\hat{V}(N)}\left[\left\{w_N(V_1,\hat{V}(N))-a\right\}\cdot\mathbf{1}_{\{\hat{V}(N)\leq b^{-1}(a)\}}\,|\,V_1=v\right]\right]\end{aligned}$$

[15] For a simple example, let $m=1$ and let $\mathbf{V}|\{S=s\}$ be a vector of independent variates. For each $N=n$, assume the marginal distribution of S is uniform on $[0,\bar{v}]$, while $\mathbf{V}^{(n)}|\{S=s\}$ is distributed uniformly on $[0,s]^n$. Then $T_n(\mathbf{v})=\max_{i=1,..n}(v_i)$ is a sufficient statistics for the parameter s of the distribution of $\mathbf{V}^{(n)}|\{S=s\}$. Thus, $F_{\mathbf{\Sigma}^{(n)}}$ is maximal attentive.

$$= \quad E_N \left[\int_0^{b^{-1}(a)} [w_N(v,\tau) - a] \; f_{\hat{V}(N)|V_1}(\tau|v) d\tau \right].$$

In a symmetric Harsanyi-Bayes-Nash equilibrium, $\pi_a(b(v)|v) = 0$, whence:

$$E_N \left[(w_N(v,v) - b(v)) \, f_{\hat{V}(N)|V_1}(v|v) \right] = b'(v) \cdot E_N[F_{\hat{V}(N)|V_1}(v|v)]. \tag{6.1}$$

In addition, in equilibrium the lowest type must bid such that his expected profit, given he wins, is zero:

$$E_N[w_N(V_1, \hat{V}) \,|\, V_1 = 0, \hat{V} = 0] - b(0) = 0,$$

which by Theorem 2(i) yields $b(0) = \hat{w}(0,0)$. The two necessary conditions define the candidate strategies for a symmetric equilibrium solutions to the initial value problem $[b'(v) = \Phi(b(v), v);\; b(0) = \hat{w}(0,0)]$, where the functional $\Phi(b,v)$ is implicitly given by the differential equation (6.1). Before looking for solution functions, let us stay for a moment with condition (6.1).

Recall that $F_{\hat{V}|V_1}(v|v)$ is the equilibrium winning probability of bidder 1 when of type v. If 1 wins, the fictitous bidder having type $\hat{V}$ is then the 'second-highest' bidder only. For $t \leq v$, $F_{\hat{V}|V_1}(t|v)$ is then the probability that the second-highest bidder is of a lower type than t, given that 1 is of type v (seen from 1's perspective). For $t \leq v$ consider the following hazard function $\hat{\lambda}(\cdot|v)$ of $\hat{V}|\{V_1 = v\}$:

$$\hat{\lambda}(t|v) := \lim_{h \searrow 0} \frac{1}{h} \Pr[t - h < \hat{V} \leq t \,|\, \hat{V} \leq t, V_1 = v] = \frac{f_{\hat{V}|V_1}(t|v)}{F_{\hat{V}|V_1}(t|v)}.$$

For $t \leq v$, $\hat{\lambda}(t|v)$ measures the 'instantaneous' likelihood that the second-highest bidder just reaches type t, given it falls short of a marginally higher type, and given the covariate V_1 takes the value v. Thus, $\hat{\lambda}(t|v)$ is the *marginal* probability that the second-highest bidder is of type t, given 1 is of (higher) type v. Hence, $\hat{\lambda}(v|v)$ is just the marginal *winning* probability of type v in equilibrium.

Definition 1. *The marginal (expected) equilibrium winning probability of a type-v bidder is the hazard function $\hat{\lambda}(\cdot|v)$: $(0,v] \to \Re_+$ evaluated at v.* [16,17]

[16] In a procurement setting, where the bidder's types index their costs, $\hat{\lambda}(\cdot|v)$ becomes the 'usual' hazard function of $\hat{V}|\{V_1 = v\}$, with the cdf $F_{\hat{V}|V_1}$ being replaced by the complementary cdf $1 - F_{\hat{V}|V_1}$, and the marginal winning probability is defined (to be zero) for the lowest type. By contrast, here $\lim_{t \searrow 0} \hat{\lambda}(t|v) = \lim_{t \searrow 0} D_1 log F_{\hat{V}|V_1}(t|v) = D_1 log \lim_{t \searrow 0} F_{\hat{V}|V_1}(t|v) = +\infty$.

[17] As remarked by Sergiu Hart, in particular in the (strategically equivalent) Dutch auction

Similarly, let $\lambda_n(v|v) := f_{\hat{V}(n)|V_1}(v|v)/F_{\hat{V}(n)|V_1}(v|v)$ be the marginal winning probability of type v against a fixed number of $n-1$ competitors. Now using (4.2), equation (6.1) determines $\Phi(b(\cdot),\cdot)$ as:

$$\Phi(b(v),v) = E_N[w_N(v,v)\,\lambda_N(v|v)] - b(v)\hat{\lambda}(v|v). \tag{6.2}$$

In view of Definition 1, condition (6.2) then fixes the slope of type v's equilibrium bid at his expected valuation given he obtains the object $[w_N(v,v)]$ times his marginal equilibrium winning probability $[\lambda_N(v|v)]$ against a random number of competitors, expected with respect to the number of these $(E_N[w_N(v,v)\lambda_N(v|v)])$, net of his marginal expected payment given he wins $[b(v)\hat{\lambda}(v|v)]$. Roughly, *the slope of type v's equilibrium bidding schedule equals his marginal expected equilibrium profit given he wins.*

The optimality condition for the equilibrium bidding scheme of the first price auction is then that each type's marginal increase in conditional expected payment given he wins equal his marginal expected profit given he wins. Solving for the functions $b(\cdot)$ reveals that the equilibrium strategy of the first price auction, too, exactly parallels the expression known from the fixed-n case.

Theorem 5. *The symmetric equilibrium $[b_i(v) = b(v)]_{i\in I}$ of the first price auction is given by:*

$$b(v) = \int_0^v \hat{w}(t,t)\,d\hat{L}(t|v), \quad with \quad \hat{L}(t|v) = exp\left\{-\int_t^v \hat{\lambda}(s|s)ds\right\}.$$

With independent types, the integrating probability measure simplifies to $\hat{L}(t|v) = exp\left\{-\int_t^v \frac{d}{ds}[lnF_{\hat{V}}(s)]ds\right\} = F_{\hat{V}}(t)/F_{\hat{V}}(v)$. For the SIAS case, b is hence a simple conditional expectation, which was contrasted above with a result from Harstad et al. (1990). In fact, the expressions obtained are identical.

Corollary 2. *In the SIAS case, the symmetric equilibrium strategy of the first price auction is:*

$$b(v) = E_{\hat{V}}\left[\hat{w}(\hat{V},\hat{V})\,|\,\hat{V}\le v\right]. \tag{6.3}$$

Proposition 3. *The equilibrium strategy (6.3) is identical to:*

$$b(v) = E_N\left[E_{\hat{V}(N)}\left[w_N(\hat{V}(N),\hat{V}(N))\,|\,\hat{V}(N)\le v\right]\,|\,\hat{V}\le v\right].$$

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mechanism with its perpetually decreasing price pointer, the hazard function is relevant literally. In particular, a bidder's marginal winning probability then coincides with his instantaneous winning probability.

Building upon Definition 1, the rest of this section further interprets the equilibrium strategy. Recall that if $S_T(t) = 1 - F_T(t)$ is the survivor function of a (lifetime) random variable $T > 0$ that has a density f_T with associated hazard function $\lambda_T = f_T/S_T$, then because of $-[lnS_T]' = \lambda_T$ the survivor function rewrites as $S_T(t) = exp\{-\int_0^t \lambda_T(s)ds\}$. In our bidding context, the role of the survivor function S_T was played by the equilibrium winning probability $F_{\hat{V}|V_1}(\cdot|v)$ of a type-v bidder, and the original meaning of the 'hazard' function was reversed (cf. Definition 1). Thus, for $t \leq v$, consider the function

$$\hat{S}(t) := exp\left\{\int_0^t \hat{\lambda}(s|s)ds\right\}.$$

As $[ln\hat{S}(t)]' = \hat{\lambda}(t|t)$, $\hat{S}(t)$ is the particular survivor function associated to the hazard function $\hat{\lambda}(t|t)$. By Definition 1, $\hat{S}(t)$ is the cumulated marginal (expected) equilibrium winning probability of type t. Now, for varying t, $t \leq v$,

$$\hat{S}(t)/\hat{S}(v) = \hat{L}(t|v)$$

defines a distribution function on $[0, v]$ (cf. Theorem 5). Intuitively, this is the case because the cumulated marginal equilibrium winning probability of type v exceeds the one of type $t \leq v$ since $\hat{S}'(\cdot) > 0$. To see explicitly what the probability measure induced by the distribution $\hat{L}(\cdot|v)$ describes, consider independent types. Then,

$$\forall t \leq v: \; 1 - \hat{L}(t|v) = 1 - \frac{F_{\hat{V}}(t)}{F_{\hat{V}}(v)} = \frac{F_{\hat{V}}(v) - F_{\hat{V}}(t)}{F_{\hat{V}}(v)} = \Pr[t \leq \hat{V} \mid \hat{V} \leq v]$$

is the conditional equilibrium losing probability of type t given that type $v \geq t$ wins the auction. Hence, $F_{\hat{V}}(t)/F_{\hat{V}}(v)$ is the conditional winning probability of type t given that v wins the auction (and in particular wins against type t). With affiliated types, this probability is given by the ratio $\hat{S}(t)/\hat{S}(v) \cdot \mathbf{1}_{\{t \leq v\}}(t, v)$.

Definition 2. *The conditional (expected) equilibrium winning probability of type t given that type $v \geq t$ wins the auction* is measured by the distribution function $\hat{L}(\cdot|v)$: $[0, \bar{v}] \to [0, 1]$, evaluated at t.

Thus, the equilibrium strategy of the first price auction prescribes that bidder 1 submit the expected valuation $\hat{w}(t, t)$ of a type-t competitor, weighted by the conditional equilibrium winning probability of t given that actually 1 wins the auction, and summed over all these types t that 1 wins against.

More precisely, the conditional expectation from Theorem 5 weights 1's competitors' expected valuations $\hat{w}(t,t)$ with their cumulated marginal winning probabilities (given 1 wins). Intuitively, this is because in symmetric equilibrium each bidder equates the slope of his expected payment scheme (given he wins) with his marginal expected profit (given he wins). By contrast, the expected equilibrium payment of a winning bidder in the second price auction weights the competitors' expected valuations $\hat{w}(t,t)$ with their simple equilibrium winning probability (given 1 wins). To see the difference, note that since $F_{\hat{V}|V_1}(t|v) = exp\{\int_0^v D_1 lnF_{\hat{V}|V_1}(s|v)\, ds\} = exp\{\int_0^t \hat{\lambda}(s|v)ds\}$,

$$\frac{F_{\hat{V}|V_1}(t|v)}{F_{\hat{V}|V_1}(v|v)} = exp\left\{-\int_t^v \hat{\lambda}(s|v)ds\right\} < exp\left\{-\int_t^v \hat{\lambda}(s|s)ds\right\} = \hat{L}(t|v) \qquad (6.4)$$

because $D_2\hat{\lambda}(\cdot|\cdot) > 0$ (cf. Theorem 5). The following is immediate:

Corollary 3. *The winnig bidder's expected equilibrium payment in the second price auction strictly exceeds the one in the first price auction.*

Proof. As the left-hand distribution function in (6.4) is stochastically dominating the right-hand one on their common support $[0,v]$,

$$\begin{aligned} & E_{\hat{V}}\left[w(\hat{V},\hat{V})\,|\,\hat{V} \,\le\, V_1, V_1 = v\right] \\ = & \int_0^v \hat{w}(t,t)\, d\, exp\left\{-\int_t^v \hat{\lambda}(s|v)ds\right\} \\ > & \int_0^v \hat{w}(t,t)\, d\, exp\left\{-\int_t^v \hat{\lambda}(s|s)ds\right\} = b(v). \ \square \end{aligned}$$

7. Expected Revenue

As indicated by Corollary 3, the first and the second price auction auction yield different expected equilibrium revenues. Building upon the ideas of Milgrom and Weber (1982), this section extends results obtained along this line to auctions with a stochastic number of bidders. First, it remains to consider when a bidder's equilibrium expected payment from the first price auction is increasing as competition increases stochastically. In addition to the condition on the fixed-n bids, an additional assumption on the distribution of types is required.

Proposition 4. *Assume that type v's marginal expected equilibrium winning probability $\hat{\lambda}(v|v)$ increases as N increases stochastically ($v > 0$). If $w_n(\cdot,\cdot)$ is non-decreasing in $n \in J$, expected payments in the first price auction are uniformly non-decreasing as N shifts up stochastically.*

The assumption from Proposition 4 is the weakest sufficient condition for the 'reasonable' comparative statics in N. Note that marginal equilibrium winning probabilities $\hat{\lambda}_n(v|v)$ that uniformly increase in n do not imply that the expected marginal winning probability uniformly increases as N increases stochastically.

For independent types, if $w_n(\cdot,\cdot)$ is non-decreasing in $n \in J$, it is guaranteed that the fixed-n bids of the first price auction are non-decreasing in n because $\lambda_{n+1}(v|v) = \lambda_{n+1}(v) = n\lambda_1(v) > (n-1)\lambda_1(v)$. Actually, Corollaries 1 and 2 imply that for independent types, each types's expected equilibrium payment increases as N increases stochastically even without the additional assumption from Proposition 4.

In fact, Corollaries 1 and 2 show that with independent types, the first and the second price auction yield the same expected equilibrium payments from any type $v > 0$. With affiliated types, Milgrom and Weber (1982) showed that the revenue equivalence principle does not apply any more, but is substituted by the *linkage principle*: any additional information that is available in equilibrium and that is linked to (read: affiliated with) a bidder's private information devaluates the latter, reducing the bidder's information rents. As a consequence, expected equilibrium payments become steeper, increasing expected equilibrium revenue to the seller. This linkage principle actually is two-faced. On the one hand, ex ante expected revenue is augmented by additional exogenous information on $\mathbf{\Sigma}$ (see below). On the other hand, interim expected payments increase in auctions that in equilibrium more tightly link the bidders' private information to one another. For extending the results to a stochastic number of bidders, we interpret the interim aspect of the principle as suggested by Riley (1989).

Proposition 5. (Linkage Principle I). *For 1's conditional beliefs on $\hat{V}$,*

$$\textit{(i)}\ D_2 F_{\hat{V}|V_1}(\cdot|v) < 0; \quad \textit{(ii)}\ D_2 \frac{F_{\hat{V}|V_1}(\check{v}|v)}{F_{\hat{V}|V_1}(\hat{v}|v)} < 0 \quad \forall \hat{v} > \check{v} > 0,\ v > 0.$$

According to Proposition 5, the probability distribution of the (expected) highest competing type (strictly) shifts up stochastically as a bidder's type increases. As the bidders' beliefs change with their types in this way, higher types must then bid more aggressively to reach a fixed expected equilibrium profit. Proposition 5(ii) shows that the conditional stochastic dominance relation in the bidders' beliefs about the highest competing type is strong, in that even the distribution $\Pr[\hat{V} \leq \check{v} \,|\, \hat{V} \leq \hat{v}, V_1 = v] \cdot \mathbf{1}_{\{\hat{v} > \check{v}\}}$ increases stochastically in a bidder's type.

For a paradigmatic application, consider the all pay (first price) and the first price (winner-only pays) auction. For a fixed number of bidders, it is known that the former yields higher expected proceeds from sale than the latter (Weber, 1985, p.163; Amann and Leininger, 1995; Krishna and Morgan, 1997). Substituting $\hat{V}(n)$ by $\hat{V}$ and w_n by $\hat{w}$, and appealing to Proposition 1, the symmetric equilibrium strategy β for the all pay auction becomes: [18]

$$\beta(v) = \int_0^v \hat{w}(t,t)\, f_{\hat{V}|V_1}(t|v)\, dt, \tag{7.1}$$

provided that $\hat{w}(v,\cdot) f_{\hat{V}|V_1}(\cdot|v)$ is non-decreasing in v (cf. Theorem 2 of Krishna and Morgan for fixed $n \in J$). Using a third, shorter, direct proof, we obtain:

Theorem 6. *For a stochastic number of bidders, equilibrium expected revenue from the all pay auction strictly exceeds the one from the first price auction.*

Proof. Since from (7.1) and Theorem 5:

$$\beta(v) - b(v) F_{\hat{V}|V_1}(v|v) = \int_0^v \hat{w}(t,t) \left[f_{\hat{V}|V_1}(t|t) - F_{\hat{V}|V_1}(v|v)\, \hat{\lambda}(t|t)\, \hat{L}(t|v) \right] dt,$$

differentiation yields:

$$\begin{aligned} & \beta'(v) - b'(v) F_{\hat{V}|V_1}(v|v) - b(v) \frac{d}{dv} F_{\hat{V}|V_1}(v|v) = \\ = \;& + \int_0^v \hat{w}(t,t)\, \hat{\lambda}(t|t)\, f_{\hat{V}|V_1}(v|v)\, \hat{L}(t|v)\, dt - b(v) \frac{d}{dv} F_{\hat{V}|V_1}(v|v) \\ = \;& \int_0^v \hat{w}(t,t)\, \hat{\lambda}(t|t) \left[f_{\hat{V}|V_1}(v|v) - \frac{d}{dv} F_{\hat{V}|V_1}(v|v) \right] \hat{L}(t|v) dt, \end{aligned}$$

where the bracketed term equals $-D_2 F_{\hat{V}|V_1}(v|v)$ and is strictly positive by Proposition 5(i). □

[18] By Proposition 1, $\pi(a|v) = E_N[E_{\hat{V}(N)}[w_N(V_1, \hat{V}(N)) \cdot \mathbf{1}_{\{\hat{V}(N) \leq \beta^{-1}(a)\}} \,|\, V_1 = v]] - a = E_{\hat{V}}[\hat{w}(V_1, \hat{V}) \cdot \mathbf{1}_{\{\hat{V} \leq \beta^{-1}(a)\}} \,|\, V_1 = v] - a$.

Theorem 6 renders explicit why the all pay version of the first price auction yields larger interim expected payments: since higher types believe that the (average) highest competing type is stochastically larger, in equilibrium they have to bid more only to avoid paying their bids without winning the object.[19]

Although obtained already in Corollary 3, the linkage principle provides an alternative, probably more intuitive way to obtain a similar result regarding the second and the first price auction. The argument is then broadly similar to the original one in Milgrom and Weber (1982, Theorem 15), although simpler. Intuitively, the second price auction yields higher expected equilibrium revenue than the first price auction because in the equilibrium strategy of the latter any direct influence of the bidder's own on the expected highest competing type is 'averaged out'. Using Proposition 5(ii), the result is then obtained by completely different arguments. We thus state for a second time:

Theorem 7. *With a stochastic number of bidders, expected equilibrium revenue from the second price strictly exceeds the one from the first price auction.*

Summarizing, the revenue rankings do not only extend, but even sharpen with a stochastic number of bidders because the linkage among $(V_1, \hat{V})$ is much tighter than among $(V_1, \hat{V}(n))$ for each n such that $n \in J$.[20]

Although the problem is predominant in sealed bid auctions, in open bid auctions too the exact number of active bidders is difficult to assess. Often, for example, additional bids are submitted late in the auctioning process by telephone. Consider the Japanese form of the English auction, as described by Milgrom and Weber (1982, Section 5). Suppose that N has realized to be $n \in J$, where n is unknown to the bidders. The auction proceeds then in n stages $k = 0, 1, ..., n-1$ that represent the number of bidders observed to have quit. Assume the price clock automatically stops when stage $n-1$ has been reached, and the remaining bidder wins the auction at the associated standing price. A strategy in the kth stage game maps the private information hold and the standing prices observed hitherto into a bid b_k: $[0, \bar{v}] \times \Re_+^k \to \Re_+$. The symmetric equilibrium of stage 0

[19] Krishna and Morgan (1997) argue similarly, but distinguish between a "second price" and a "losing bid effect" to explain the various revenue rankings derived. Using Theorems 1 and 2 and Proposition 1, the rankings involving the second-price all-pay auction extend similarly to a stochastic number of bidders, given all the associated assumptions are met when they refer to the (distribution of the) expected highest competing type.

[20] We used Assumption 2(ii) to make this explicit, in replacing weak by strict inequalities in Theorem 1. Without the Assumption, the tightening of the linkage principle is not visible.

becomes:

$$b_0(v) = E_N \left[E_{\vec{\mathbf{\Sigma}}^{(N)}} \left[W_N \mid (V_1, \vec{\mathbf{V}}_{-1}^{(N-1)}) = (v,..,v) \right] \right].$$

If one bidder each has been observed to quit at price p_k in stage $1 \leq k \leq n-1$, given the sequence $\{b_l\}_{l<k}$, then the following, recursively defined stage game strategies forms a symmetric equilibrium:

$$\begin{aligned} & b_k(v; p_1, ..., p_k) = \\ = \;& E_{(\vec{\mathbf{\Sigma}}^{(N)},N)} \left[W_N \mid (V_1, \vec{\mathbf{V}}_{-1}^{(N-1)}) = (v,..,v), \Xi_{k-1}^{(N)}(p_1,..,p_k), .., \Xi_0^{(N)}(p_1), N > k \right] \\ = \;& E_N \left[E_{\vec{\mathbf{\Sigma}}^{(N)}} \left[W_N \mid (V_1, \vec{\mathbf{V}}_{-1}^{(N-1)}) = (v,..,v), \Xi_{k-1}^{(N)}(p_1,..,p_k), .., \Xi_0^{(N)}(p_1), N \right] \mid N > k \right], \end{aligned}$$

where $\Xi_{k-1}^{(N)}(p_1,..,p_k)$ denotes the event $\left\{ V_{(N-k:N-1)} \mid b_{k-1}(V_{(N-k:N-1)}; p_1,..,p_{k-1}) = p_k \right\}$, with $V_{(j:N-1)}$ the jth coordinate variable of $\vec{\mathbf{V}}_{-1}^{(N-1)}$; see Theorem 10 of Milgrom and Weber for n fixed and known.

Two effects contribute to the performance of this auction with respect to ex ante expected revenue. First, the additional information available about the competitors' signals $\mathbf{V}_{-1}^{(n-1)}$ increases ex ante expected equilibrium payments relative to the second price auction, for each $n \in J$. Second, the likely number of competitors is increasing from one stage bid to the next. As the order statistics $\vec{\mathbf{V}}_{-1}^{(N-1)}$ are stochastically increasing in the realizations of N (Proposition 2(i) for the first order statistic), expected equilibrium payments become even larger. Building upon Theorem 11 (Theorem 8, respectively) of Milgrom and Weber, we obtain:

Theorem 8. *With an uncertain number of bidders, ex ante expected equilibrium revenue from the English auction* (R_E) *exceeds the one from the second price auction* (R_2) *by more than when the number of bidders is fixed and known.*

8. Information Policy

In many sealed-bid auctions, participating bidders have to provide some security deposit before being permitted to submit bids. While this practice may be advantageous for various reasons, one consequence is that it allows the bid-taker to be perfectly informed on the number of competing bidders. The seller could then commit, in advance, to (always) reveal the number of competitors.

Whenever the bidders have independent private information, such a policy is equivalent to concealing the information in terms of ex ante expected revenue.

Consider the first price auction, and let $b(\cdot;n)$ be the fixed-n bidding strategy in a symmetric equilibrium. If the bidders in all states of the world know the actual size of the participancy, type v's expected equilibrium payment is:

$$E_N\left[b(v;N)\,F_{\hat{V}(N)}(v)\right] = E_N\left[b(v;N)\,|\,\hat{V}\leq v\right]F_{\hat{V}}(v) = b(v)F_{\hat{V}}(v), \qquad (8.1)$$

by relation (4.4), Proposition 3, and Corollary 2. A similar reasoning obtains for the second price auction.[21]

Proposition 6. *Consider the second price auction in the SIAS case. The revealing and the concealment policy attain the same expected revenue.*

Proof. Always revealing the realizations of N yields an expected payment from type v of:

$$\begin{aligned}
& E_N\left[E_{\hat{V}(N)}\left[w_N(\hat{V}(N),\hat{V}(N))\cdot\mathbf{1}_{\{\hat{V}(N)\leq v\}}\right]\right] \\
= & \sum_{n\in I}p_N(n)\int_0^v w_n(\tau,\tau)\,f_{\hat{V}(n)}(\tau)d\tau \\
= & \int_0^v\sum_{n\in I}p_{N|\hat{V}}(n|\tau)\,w_n(\tau,\tau)\,f_{\hat{V}}(\tau)d\tau \\
= & \int_0^v\hat{w}(\tau,\tau)\,f_{\hat{V}}(\tau)d\tau \\
= & E_{\hat{V}}[\hat{w}(\hat{V},\hat{V})\cdot\mathbf{1}_{\{\hat{V}\leq v\}}]
\end{aligned}$$

using (4.3) and Theorem 2(i). □

To see that with correlated private information the revealing policy does make a difference, observe that Proposition 6 fails, since

$$\int_0^v\sum_{n\in I}p_{N|\hat{V},V_1}(n|\tau,v)\,w_n(\tau,\tau)\,f_{\hat{V}|V_1}(\tau|v)d\tau \neq \int_0^v\hat{w}(\tau,\tau)\,f_{\hat{V}|V_1}(\tau|v)d\tau.$$

Now the linkage principle also applies to the revelation of exogenous information. Milgrom and Weber (1982) show that a policy of publicly revealing any information hold by the auctioneer that is affiliated with $\mathbf{\Sigma}$ increases ex ante expected

[21] Such a result is indicated by Harstad et al. (1990, p.39), who assume that the seller is uninformed of the exact number of participants, but asks for a vector of 'contingent' bids $\{b(;n)\}_{n\in I}$ from each bidder. With expected revenue being the sum of the 'contingent' bids, weighted by the priors $\{p_N(n)\}_{n\in I}$, a revealing policy yields the same revenue.

revenue. By all of the above, these results equally apply to an uncertain number of bidders. We state informally:

Proposition 7 (Linkage Principle II)**.** *A policy of always revealing any information hold by the auctioneer that is positively affiliated with* $\mathbf{\Sigma}$ *increases ex ante expected revenue.*

Proof. Follows from Theorems 8, 12 and 16 of Milgrom and Weber (1982) in conjunction with Theorems 1,2,3,5 and Propositions 1 and 5. For the all pay auction, the reasoning is as in Theorem 16 of Milgrom and Weber. □

According to this aspect of the linkage principle, revealing information that is negatively affiliated to the bidders' private information should then *de*crease ex ante expected revenue to the seller. In fact Matthews (1987) has shown for the SAPV case that always publishing the number of competing bidders prior to the bidding stage lowers expected revenue compared to concealing the information. Interestingly, the result again uses the interim version of the linkage principle. It relies on the observation that with affiliated types, $(N, V_1)|\{\hat{V} \leq \hat{v}\}$ is negatively affiliated ($\hat{v} > 0$). The intuition runs as follows.

The winner owns information on the likely number of participants that makes this number stochastically smaller. Given the event $\{\hat{V} \leq v, V_1 = v\}$, N decreases (strictly) stochastically because $(\hat{V}, \hat{V}(N))$ is (strictly) affiliated and $\hat{V}(N)|\{V_1 = v\}$ is (strictly) stochastically isotonic in N (Proposition 2(i)). With $\check{p}_{N|\hat{V}}(n|v)$ the probability of there being n bidders given type v wins, it follows that $D_2\check{P}_{N|\hat{V}}(\cdot|\cdot) < 0$. Thus, with independent types, a winning bidder infers that competition has been weaker as expected ex ante ($E_N[N \mid \hat{V} \leq v] < E[N]$). But with affiliated types, the inference is altered by the additional fact that a winning bidder of high type $V_1 = v$ believes that also $\hat{V}$ is high as $(V_1, \hat{V})$ is (strictly) affiliated, too. Hence, conditional on $\{\hat{V} \leq v\}$, N and V_1 are *negatively* dependent variates. Lemma 1 of Matthews (1987) shows that $\check{p}_{N|\hat{V},V_1}(\cdot|\hat{v}, \cdot)$ in fact obeys the (weak) negative affiliation inequality. An argument as in Proposition 5(i) shows that then $D_3\check{P}_{N|\hat{V},V_1}(\cdot|\cdot, \cdot) \geq 0$.

Assumption 4. *For the auction considered, the fixed-n equilbrium bids* $b(\cdot; n)$ *are non-decreasing in* $n \in J$: $\Delta b(\cdot; n) = b(\cdot; n) - b(\cdot; n-1) \geq 0$ $[b(\cdot; 0) \equiv 0]$.

With this assumption, assume the seller commits to the revealing policy, and consider a type-v bidder that bids as if he was of type u. As the conditional expected payment function (given he wins) takes the form of (8.1), then

$$\frac{d}{dv}E_N[b(u;N)\,|\,\hat{V}\leq u, V_1=v] = \sum_{n\in I}\Delta b(u;n)\,D_3\left[1-\check{P}_{N|\hat{V},V_1}(n|u,v)\right]\leq 0. \tag{8.2}$$

Since the partial derivative is the only relevant one in symmetric equilibrium, expected equilibrium payments decrease compared to the concealment policy.

But now $(\bar{n}-N,V_1)|\{\hat{V}\leq\hat{v}\}$, where $\bar{n}=supI+1$, is in turn positively affiliated.[22] By (8.2), a policy of always reporting the 'inverse realization' $\bar{n}-n$ of N should thus *in*crease expected equilibrium payments, relative to a concealment policy. The optimal information policy then seems to involve a revelation of information, but of false one.[23]

To investigate this, suppose the seller commits publically to a report function r: $I\to I\cup\{0\}$ to convey information on the realizations of N. Theorem 9 of Milgrom and Weber (1982) implies that we can restrict attention to strictly monotone report functions, since any other function does not fully exploit the impact of revealing affiliated information on expected revenue. As there are only two strictly monotone functions mapping I onto itself, it is enough to consider three reporting policies: the concealment policy ρ_0 identified by $r(n)=0\ \forall n\in I$, the fully revealing policy ρ where $r(\cdot)=id(\cdot)$, and the inverse reporting policy $\bar{\rho}$ given by $r(n)=\bar{n}-n$. To retain generality, suppose the auctioneer can randomize among these choices: each of the three policies $\rho,\bar{\rho}$, and ρ_0 will be used (ex ante) with probabilities $\pi,\bar{\pi}$, and $1-\pi-\bar{\pi}$, respectively ($\pi+\bar{\pi}\leq 1$). The seller then commits to report according to the outcome of a random device that selects each of the policies $p\in\{\rho,\bar{\rho},\rho_0\}$ according to the probabilities $(\pi,\bar{\pi})$ that she has fixed ex ante.

Theorem 9. *Assume the information policy $(\pi,\bar{\pi})\in[0,1]^2$ is chosen independently of the constellation of exogenous parameters $[I,\{F_{\mathbf{\Sigma}(n)},P_N(n),U_n\}_{n\in I}]$. For a measure one of distributions of N, the seller's optimal policy is to have the concealment policy ρ_0 be selected with probability* 1.

[22]Thanks to Preston McAfee for the suggestion. Since $\bar{n}-N$ is distributed with distribution $1-P_N(\bar{n}-n)$ and probability weigths $p_N(\bar{n}-n)$, the affiliation inequality for $\check{p}_{N|\hat{V},V_1}(\cdot|\hat{v},\cdot)$ in (n,v) and (m,u) reverses as $\binom{n}{m}\wedge\binom{v}{u}$ and $\binom{n}{m}\vee\binom{v}{u}$ change places if $\bar{n}-n$ replaces n.

[23]For the sake of the argument, temporarily abstract from the issue of potential ex post verifiability of the information revealed; sealed bids are kept secret also ex post, including their number.

Proof. Consider three revelation mechanisms M_p: $[0,\bar{v}]^{supI} \to [0,1]^{supI} \times \Re^{supI}$, $p \in \{\rho_0, \rho, \bar{\rho}\}$, where M_0 implements the symmetric equilibrium of the given auction under policy ρ_0, and where M_r implement the symmetric equilibrium under policies $r \in \{\rho, \bar{\rho}\}$, respectively. Given the seller has fixed $(\pi, \bar{\pi})$, and given the bidders obtained their private information, the games induced by M_p, $p = \rho, \bar{\rho}, \rho_0$, are played with probabilities $\pi, \bar{\pi}, 1-\pi-\bar{\pi}$, respectively. Suppose bidder 1 is of type v and claims to be of type u $(u, v > 0)$ while all his actual opponents transmit their types truthfully. If under a policy $r \in \{\rho, \bar{\rho}\}$ the bids condition on the reports $r(n)$, 1's expected utility in mechanism M_r is:

$$U_r(u|v) = E_N\left[E_{(\mathbf{\Sigma}^{(N)},\hat{V}(N))}\left[(W_{1,N} - b(u; r(N))) \cdot \mathbf{1}_{\{\hat{V}(N) \le u\}} \,|\, V_1 = v\right]\right],$$

while in M_0, $b(u)$ replaces $b(u; r(N))$ to yield $U_0(u|v)$. Hence, using (4.4),

$$\begin{aligned} U_0(u|v) - U_r(u|v) &= E_N\left[E_{\hat{V}(N)}\left[[b(u; r(N)) - b(u)] \cdot \mathbf{1}_{\{\hat{V}(N) \le u\}} \,|\, V_1 = v\right]\right] \\ &= \sum_{n \in I} p_N(n)[b(u; r(n)) - b(u)] \int_0^u f_{\hat{V}(n)|V_1}(s|v) ds \\ &= \sum_{n \in I} \check{p}_{N|\hat{V},V_1}(n|u,v)[b(u; r(n)) - b(u)] F_{\hat{V}|V_1}(u|v) \\ &= H_r(u|v)\, F_{\hat{V}|V_1}(u|v), \\ \text{where } H_r(u|v) &\equiv E_N[b(u; r(N)) - b(u) \,|\, \hat{V} \le u, V_1 = v]. \end{aligned} \tag{8.3}$$

Thus,

$$D_2\{U_0(u|v) - U_r(u|v)\} = H_r(u|v)\, D_2 F_{\hat{V}|V_1}(u|v) + F_{\hat{V}|V_1}(u|v)\, D_2 H_r(u|v), \tag{8.4}$$

where $D_2 F_{\hat{V}|V_1}(\cdot|\cdot) \le 0$ and where (8.2) shows that:

$$r(n+1) \underset{<}{>} r(n)\ \forall n \in J \Rightarrow D_2 H_r(u|v) \underset{\geq}{\leq} 0.$$

Consider first the choice of $\frac{\pi}{\bar{\pi}}$ given that $\frac{\bar{\pi}=0}{\pi=0}$. As for policy $\frac{\rho}{\bar{\rho}}$, $D_2 H_r(u|v) \underset{\geq}{\leq} 0$, (8.4) is $\substack{\text{negative} \\ \text{positive}}$ whenever $H_r(u|v) \underset{\leq}{\geq} 0$. Comparing the truth-telling equilibria of M_0 and M_r $(\substack{r=\rho \\ r=\bar{\rho}})$, let $U_p(v|v) = U_p(v)$ $(p = 0, r)$. Since $H_r(u|v) \underset{\leq}{\geq} 0$ is equivalent to $U_0(u|v) \underset{\leq}{\geq} U_r(u|v)$ for $u, v > 0$, from the envelope theorem:

$$U_0(v) \underset{\leq}{\geq} U_r(v) \Rightarrow U_0'(v) \underset{\geq}{\leq} U_r'(v) \quad \forall v > 0.$$

Together with $U_0(0) = 0 = U_r(0)$, the implication excludes $U_0(\cdot) \substack{> \\ <} U_r(\cdot)$, so necessarily $U_r(\cdot) \substack{\geq \\ \leq} U_0(\cdot)$. Hence, choosing $\substack{\pi>0 \\ \bar{\pi}<1}$ is suboptimal for the seller if $\substack{\bar{\pi}=0 \\ \pi=0}$.

With the seller's preferences being $\rho \preceq \rho_0 \preceq \bar{\rho}$, $\bar{\pi} = 1$ is then optimal if the policy is effective. But given $\bar{\rho}$ is chosen with probability 1, the realizations of N are updated perfectly from the reports and choosing $\bar{\pi} = 1$ yields the same expected revenue as $\pi = 1$. As the same obtains for any $\bar{\pi} < 1$ such that $\pi = 0$, consider $\pi > 0$ and $\pi + \bar{\pi} < 1$. Once a report $n \in I$ is delivered to the bidders, n is then taken to be the true realization of N with probability

$$\tau(n) \equiv \frac{\pi p_N(n)}{\pi p_N(n) + \bar{\pi}[1 - p_N(n)]} \cdot \mathbf{1}_{\{n \neq \frac{\bar{n}}{2}\}}(n) + \mathbf{1}_{\{n = \frac{\bar{n}}{2}\}}(n).$$

In the bidder's posterior world $[\tilde{I} = \{n, \bar{n}-n\}, \tilde{p}_N(n) = \tau(n), \tilde{p}_N(\bar{n}-n) = 1-\tau(n)]$ following a report of 'n', let $\tilde{b}_{\tau(n)}(\cdot)$ be the associated equilibrium bid function, where the index indicates the dependence of the functional form of $\tilde{b}$ on the beliefs $\tau(n)$. From the argument following (8.3), it is enough to consider 1's conditional expected payment given he wins. Before receiving any particular report, this is then given as:

$$(1 - \pi - \bar{\pi})\, b(u) + (\pi + \bar{\pi}) \left[1 - \mathbf{1}_{\{\frac{\bar{n}}{2} \in I\}} \cdot p_N\left(\frac{\bar{n}}{2}\right)\right] E_N \left[\tilde{b}_{\tau(N)}(u) \,|\, \hat{V} \leq u, V_1 = v\right]$$
$$+(\pi + \bar{\pi}) \cdot \mathbf{1}_{\{\frac{\bar{n}}{2} \in I\}} \cdot p_N\left(\frac{\bar{n}}{2}\right) \cdot E_N \left[b(u; N) \,|\, \hat{V} \leq u, V_1 = v\right]. \tag{8.5}$$

Assume that $\bar{n}/2 \notin I$, so that the third term in (8.5) vanishes. Then for $\pi + \bar{\pi} = 1$ to be optimal for all parameters, necessarily

$$\frac{d}{dv} E_N \left[\tilde{b}_{\tau(N)}(u) \,|\, \hat{V} \leq u, V_1 = v\right] \geq 0. \tag{8.6}$$

Suppose the functions $\tilde{b}_{\tau(\cdot)}(\cdot)$ are uniformly increasing as the distribution of N shifts up stochastically (cf. Corollary 1 and Proposition 4). An analogous argument to (8.2) then implies that for (8.6) to hold, the function $\tau(\cdot)$: $I \to [0, 1]$ be decreasing on I, strictly at least for one $n \in J$, for the selected $(\pi, \bar{\pi}) \in (0, 1)^2$. $\forall(\pi, \bar{\pi})$, this necessary condition is equivalent to:

$$p_N(n) \leq p_N(n-1) \;\; \forall n \in J; \; \exists n \in J\colon p_N(n) < p_N(n-1). \tag{8.7}$$

Since (8.7) only holds for a measure zero of distributions of N, choosing $\pi + \bar{\pi} > 0$ is typically suboptimal, even if $\bar{n}/2 \notin I$. □

Of course, the restriction to policies that are chosen independently of the exogenous parameters $[I, \{F_{\mathbf{\Sigma}^{(n)}}, P_N(n), U_n\}_{n\in I}]$ is binding here. But since committing to a reporting policy implies that the policy choice be robust against imperfect knowledge of the precise economic environment, we believe the restriction is reasonable. In particular, although the distributions $\{F_{\mathbf{\Sigma}^{(n)}}, P_N(n)\}_{n\in I}$ can be estimated to some degree, such estimates are hardly obtainable at the stage where the seller's information policy is chosen.

An intuition for why there is then no 'smart' misinformation policy that generically improves upon the concealment policy is the following. Any candidate policy exploiting the impact of the information revealed must leave some uncertainty on the true state of the world in order to be effective. But then subsequent equilibrium bids will not condition on the exact reports any more, but on the posterior distribution of N induced by the reports. In particular, a bidder does *not* "decide to trust a report 'n' if $\tau(n) \geq 1/2$ and thus bid $b(\cdot; n)$ with probability $\pi Pr[N \,|\, \tau(N) \geq 1/2] + \bar{\pi} Pr[N \,|\, \tau(N) < 1/2]$ and $b(\cdot; \bar{n} - n)$ with complementary probability", but he forms one bid for the remaining distribution on the number of his competitors. Essentially, the bids thus 'close' against the particular information revealed, and only an indirect linkage of the bidder's type to the information revealed remains, which works over the distribution that the equilibrium bid functions condition upon. Thus, at best, any systematic (distribution-free) disinformation policy can increase expected revenue only for some particular priors $\{p_N(n)\}_{n\in I}$.

8.1. Endogenous Entry

If potential bidders incur information acquisition costs in obtaining private information, free entry drives their ex ante expected profits near to zero, up to some residual rents left by the integer constraint on the number of entrants. Since the seller's ex ante expected revenue then coincides with expected social surplus (net of aggregate entrance costs), screening instruments are useful only to extract these residual rents.[24]

If fixed-n bids are non-decreasing in n and not all $supI$ bidders can enter profitably, symmetric equilibria involve mixed entrance strategies. Analyses of

[24]Results along these lines have been obtained by French and McCormick (1984), Engelbrecht-Wiggans (1987), McAfee and McMillan (1987b), Harstad (1990), McAfee and Vincent (1992), Engelbrecht-Wiggans (1993), and Levin and Smith (1994). Burguet and Sákovič (1996) show that 'large' reserve prices (that may even reduce socially optimal entry) increase expected revenue when an object that is left unsold can be re-auctioned subsequently.

the optimal use of screening instruments then require that expected equilibrium payment not decrease if the number of entering bidders shifts upwards stochastically. For the first price auction, this does not follow from the fixed-n bids being non-decreasing in n, but an additional assumption is required (cf. Proposition 4).

Another assumption, however, is not needed to analyze symmetric endogenous bidder entry: to dispose with the problem of a stochastic (equilibrium) participancy in the bidding stage game, it has been assumed that after the entry decisions are taken, the number of (equilibrium) entrants becomes common knowledge.[25]

9. Conclusion

The symmetric equilibrium bid functions for a stochastic number of competitors generalize the known bidding strategies for the fixed-n case without complicating surprises. For the equilibrium strategy of the first price auction, we offered an interpretation that might simplify an intuitive handling of competitive bidding situations where the winner's payment depends on his bid. Since the linkage principle tightens with a stochastic competitency, the fixed-n rankings of different auctions extend to sharper versions. Generally, auction mechanisms that in equilibrium more tightly link the bidders' private information decrease expected information rents relative to mechanisms that leave the competitors' private information more loosely linked and hence more 'valuable'. McAfee and McMillan (1987a) have generalized the findings of Myerson (1981) on the optimal auction for independent types to a competitency of stochastic composition; again the functional characteristics of the optimal revelation scheme carry over to a random participancy. Taken together, the results suggest that the 'fixed-n paradigm' does not entail much loss of generality in the conclusions obtained for simple auctions with risk neutral agents.

The analysis was fundamentally eased by the assumption that the number of participating bidders is independent of their private information and of the underlying common values. Despite of having been invoked throughout in the

[25]See Assumption 4 and notes 5 and 15 in Levin and Smith (1994, p.586f. and p.592), Milgrom (1981, p.935), or Assumption 4 of Smith and Levin (1996). The assumption is extreme and contrary to the idea of mixed strategy Nash equilibrium. To obtain his reduced-form profit function, each potential participant had to know (a summary statistics of) all realizations of the mixing devices used by all his potential competitors. Under such a hypothesis, asymmetric but correlated entry is more plausible. Campbell (1997) argues in favor of asymmetric entry equilibria and shows how coordinating their entry decisions improves bidders.

literature, such an assumption may not be overly realistic. For example, in the 1,264 auctions for wildcat oil leases re·analyzed by McAfee and Vincent (1992, Table I, p.516), mean participation rates are by overall tendency non-decreasing in the tract value. The observation suggests that as N shifts up stochastically, the marginal distribution of $\mathbf{S}$ rather increases stochastically.

However, potentially increasing participation rates with different tract values could also be explained by a change in the bidders' beliefs about the information owned by their competitors; bidders may believe that the probability of more incompletely informed and less 'serious' competitors increases disproportionately in $n \in J$. Since a violation of the independence assumption of N and $\mathbf{S}$ is hence observationally equivalent to the bidders having beliefs $F_{\mathbf{\Sigma}^{(n)}}$ of different functional forms for varying n, there is an identification problem that is hard to overcome empirically. The problem becomes even worse since the valuation functions $\{U_n\}_{n \in J}$, too, can vary in any way over J.

To assess the effect of the assumption, it then seems necessary to study bidding games where the number of bidders is simultaneously determined with their bids in equilibrium. Now if $(N, \mathbf{\Sigma}^{(n)})$ is positively affiliated on $I \ni n$, the linkage principle is sharpened again since larger types' beliefs about the expected highest competing type increase stochastically only because N does so. This should imply revenue rankings of the kind encountered above already with independent types. For affiliated types, the rankings must sharpen even further, because auctions that more tightly link the bidders' types to $\hat{V}$ induce the expected equilibrium winning probability to shift down even more and thus force higher types to bid even higher to reach a fixed level of expected information rent. It is not clear, however, how the equilibrium bid functions will look like.

10. Appendix: Proofs

Proof of Theorem 1.

We first show that $(\hat{V}, V_1)$ is affiliated. As $(V_1, \vec{\mathbf{V}}_{-1}^{(n-1)})$ is affiliated, also the subselection $(\hat{V}(n), V_1)$ is, so the joint density of $(\hat{V}(n), V_1)$ obeys the affiliation inequality $\forall n \in I$. Weighting by $p_n \equiv p_N(n)$ and adding the corresponding inequalities for varying n, it follows that for $u' > u,\ v' > v$:

$$\sum_{n\in I} p_n^2\, f_{\hat{V}(n),V_1}(u',v')\, f_{\hat{V}(n),V_1}(u,v) > \sum_{n\in I} p_n^2\, f_{\hat{V}(n),V_1}(u',v)\, f_{\hat{V}(n),V_1}(u,v'), \quad (10.1)$$

the strict equality because at least one of the original inequalities is strict. As each $f_{\hat{V}(n),V_1}(u_1,v_1)$ and $f_{\hat{V}(m),V_1}(u_2,v_2)$ obeys the affiliation equality $(n \neq m;\ n,m \in I)$, also their product does; in particular, for $v' > v > 0$, $u_i' > u_i$ $(i = 1,2)$:

$$\begin{aligned} & f_{\hat{V}(n),V_1}(u_1',v')\, f_{\hat{V}(m),V_1}(u_2',v')\, f_{\hat{V}(n),V_1}(u_1,v)\, f_{\hat{V}(m),V_1}(u_2,v) \\ \geq\ & f_{\hat{V}(n),V_1}(u_1,v')\, f_{\hat{V}(m),V_1}(u_2',v)\, f_{\hat{V}(n),V_1}(u_1',v)\, f_{\hat{V}(m),V_1}(u_2,v'). \end{aligned}$$

Integration with respect to u_2 and u_1' implies:

$$f_{\hat{V}(m),V_1}(u_2',v')\, f_{\hat{V}(n),V_1}(u_1,v) \geq f_{\hat{V}(n),V_1}(u_1,v')\, f_{\hat{V}(m),V_1}(u_2',v).$$

Set $u_1 = u$, $u_2 = u'$ and assume wlog that $u' > u$. Weighting by $p_n p_m$ and adding the associated inequalities:

$$\sum_{\substack{n,m\in I \\ n\neq m}} p_n p_m f_{\hat{V}(n),V_1}(u,v)\, f_{\hat{V}(m),V_1}(u',v') > \sum_{\substack{n,m\in I \\ n\neq m}} p_n p_m f_{\hat{V}(n),V_1}(u,v')\, f_{\hat{V}(m),V_1}(u',v). \quad (10.2)$$

Adding (10.1) and (10.2), and using

$$\begin{aligned} & \sum_n p_n^2\, f_{\hat{V}(n),V_1}(u',v')\, f_{\hat{V}(n),V_1}(u,v) + \sum_{n\neq m} p_n p_m f_{\hat{V}(n),V_1}(u,v)\, f_{\hat{V}(m),V_1}(u',v') \\ =\ & \left[\sum_n p_n f_{\hat{V}(n),V_1}(u',v')\right]\left[\sum_n p_n f_{\hat{V}(n),V_1}(u,v)\right] \end{aligned}$$

(because each of the sums is absolutely converging), we obtain:

$$f_{\hat{V},V_1}(u',v')\, f_{\hat{V},V_1}(u,v) > f_{\hat{V},V_1}(u',v)\, f_{\hat{V},V_1}(u,v'),$$

as $\sum_{n\in I} p_n f_{(\hat{V}(n),V_1)}(\cdot,\cdot) = f_{(\hat{V},V_1)}(\cdot,\cdot)$ by (4.2). Hence, $(\hat{V}, V_1)$ is strictly affiliated. It follows analogously that $(\hat{V}, \mathbf{\Sigma}^{(n)})$ is strictly affiliated ($n \in I$). Therefore, $(\hat{V}, V_1, \vec{\mathbf{\Sigma}}_{-1}^{(n-1)})$ is strictly affiliated as well. □

Proof of Theorem 2.

(i) Observe that

$$
\begin{aligned}
& E_N\left[E_{\mathbf{\Sigma}^{(N)}}\left[W_N \mid \hat{V}=\hat{v}, V_1=v\right]\right] \\
= \; & \sum_{n\in I} p_N(n) \int_{\Re_+^{n-1}} \int_{\Re^m} U_n(v;\boldsymbol{\tau};\boldsymbol{\sigma})\, f_{\vec{\mathbf{\Sigma}}_{-1}^{(n-1)}|\hat{V},V_1}(\boldsymbol{\tau},\boldsymbol{\sigma}|\hat{v},v)d\boldsymbol{\sigma}d\boldsymbol{\tau} \qquad (10.3)\\
= \; & \sum_{n\in I} \int_{\Re_+^{n-1}} \int_{\Re^m} U_n(v;\boldsymbol{\tau};\boldsymbol{\sigma})\, f_{(N,\vec{\mathbf{\Sigma}}_{-1}^{(N-1)})|\hat{V},V_1}(n;\boldsymbol{\tau},\boldsymbol{\sigma}|\hat{v},v)d\boldsymbol{\sigma}d\boldsymbol{\tau} \\
= \; & \sum_{n\in I} \int_{\Re_+^{n-1}} \int_{\Re^m} U_n(v;\boldsymbol{\tau};\boldsymbol{\sigma}) \frac{f_{(N,\vec{\mathbf{\Sigma}}_{-1}^{(N-1)},\hat{V}(N))|V_1}(n;\boldsymbol{\tau},\boldsymbol{\sigma}|v)\cdot \mathbf{1}_{\{\tau_1=\hat{v}\}}(\boldsymbol{\tau})}{f_{\hat{V}|V_1}(\hat{v}|v)} d\boldsymbol{\sigma}d\boldsymbol{\tau} \\
= \; & \sum_{n\in I} p_N(n) \int_{\Re_+^{n-2}} \int_{\Re^m} U_n(v;\hat{v},\boldsymbol{\tau}';\boldsymbol{\sigma}) \frac{f_{\vec{\mathbf{\Sigma}}_{-1}^{(n-1)}|V_1}(\hat{v},\boldsymbol{\tau}',\boldsymbol{\sigma}|v)}{f_{\hat{V}|V_1}(\hat{v}|v)} d\boldsymbol{\sigma}d\boldsymbol{\tau}' \\
= \; & \sum_{n\in I} p_{N|\hat{V},V_1}(n|\hat{v},v) \int_{\Re_+^{n-2}} \int_{\Re^m} U_n(v;\hat{v},\boldsymbol{\tau}';\boldsymbol{\sigma}) \frac{f_{\vec{\mathbf{\Sigma}}_{-1}^{(n-1)}|V_1}(\hat{v},\boldsymbol{\tau},\boldsymbol{\sigma}|v)}{f_{\hat{V}(n)|V_1}(\hat{v}|v)} d\boldsymbol{\sigma}d\boldsymbol{\tau}' \\
= \; & E_N\left[w_N(v,\hat{v}) \mid \hat{V}=\hat{v}, V_1=v\right],
\end{aligned}
$$

the second equality using Assumption 3, the third due to (4.1), the fourth due to Assumption 3, and the fifth equality using (4.3).

(ii) For fixed $n \in I$, consider the integral (10.3) inside the expectation with respect to N, denoted by $\hat{w}_n(\hat{v},v) = E_{\mathbf{\Sigma}^{(n)}}\left[W_n \mid \hat{V}=\hat{v}, V_1=v\right]$. Since $(V_1, \vec{\mathbf{\Sigma}}_{-1}^{(n-1)}, \hat{V})$ is affiliated (Theorem 1), it follows with Theorem 5 of Milgrom and Weber (1982) that

$$D_1\hat{w}_n(\cdot,\cdot) > 0,\; D_2\hat{w}_n(\cdot,\cdot) \geq 0 \;\; \forall n \in I.$$

It follows that for $i = 2$:

$$0 \leq \sum_{n\in I} D_i\, \hat{w}_n(\cdot,\cdot) = D_i\, E_N[\hat{w}_N(\cdot,\cdot)] = D_i\, \hat{w}(\cdot,\cdot),$$

with strict inequality for $i = 1$.

(iii) Let N_1, N_2 such that $1 - P_{N_1}(\cdot) \geq 1 - P_{N_2}(\cdot)$. Then $\forall n \in I$:

$$\begin{aligned} & \left[\sum_{m>n} p_{N_1}(m) f_{\hat{V}(m)|V_1}(\hat{v}|v)\right] \left[\sum_{m\leq n} p_{N_2}(m) f_{\hat{V}(m)|V_1}(\hat{v}|v)\right] \\ \geq \quad & \left[\sum_{m>n} p_{N_2}(m) f_{\hat{V}(m)|V_1}(\hat{v}|v)\right] \left[\sum_{m\leq n} p_{N_1}(m) f_{\hat{V}(m)|V_1}(\hat{v}|v)\right]. \end{aligned}$$

Multiplying with $\left[\sum_{m\leq n} p_{N_1}(m) f_{\hat{V}(m)|V_1}(\hat{v}|v)\right] \cdot \left[\sum_{m\leq n} p_{N_2}(m) f_{\hat{V}(m)|V_1}(\hat{v}|v)\right]$,

$$\begin{aligned} & \left[\sum_{m\leq n} p_{N_2}(m) f_{\hat{V}(m)|V_1}(\hat{v}|v)\right] \left[\sum_{m\in I} p_{N_1}(m) f_{\hat{V}(m)|V_1}(\hat{v}|v)\right] \\ \geq \quad & \left[\sum_{m\in I} p_{N_2}(m) f_{\hat{V}(m)|V_1}(\hat{v}|v)\right] \left[\sum_{m\leq n} p_{N_1}(m) f_{\hat{V}(m)|V_1}(\hat{v}|v)\right], \end{aligned}$$

so that, using (4.3),

$$\sum_{m\leq n} p_{N_2|\hat{V},V_1}(m|\hat{v},v) \geq \sum_{m\leq n} p_{N_1|\hat{V},V_1}(m|\hat{v},v).$$

Thus, if $w_n(v,\hat{v}) - w_{n-1}(v,\hat{v}) \geq 0$ ($n \in I$; setting $w_0(\cdot|\cdot) = 0$), it follows that:

$$\sum_{n\in I} [w_n(v,\hat{v}) - w_{n-1}(v,\hat{v})] \left[P_{N_2|\hat{V},V_1}(n|\hat{v},v) - P_{N_1|\hat{V},V_1}(n|\hat{v},v)\right] \geq 0,$$

which is equivalent to:

$$\sum_{n\in I} w_n(v,\hat{v}) p_{N_1|\hat{V},V_1}(n|\hat{v},v) \geq \sum_{n\in I} w_n(v,\hat{v}) p_{N_2|\hat{V},V_1}(n|\hat{v},v).$$

In view of part (i), this proves the third claim. □

Proof of Proposition 1.

Since for the bivariate expectation

$$E_{(\vec{\boldsymbol{\Sigma}}^{(n)},\hat{V}(n))} \left[W_n \cdot \hat{V}(n)\right] = E_{\hat{V}(n)} \left[\hat{V}(n) \cdot E_{\vec{\boldsymbol{\Sigma}}^{(n)}}[W_n \mid \hat{V}(n)]\right], \qquad (10.4)$$

abbreviating $b^{-1}(a) = u$, we have:

$$\begin{aligned}
& E_N\left[E_{\vec{\mathbf{\Sigma}}^{(N)}}\left[W_N \cdot \mathbf{1}_{\left\{\hat{V}(N)\leq u\right\}} \mid V_1 = v\right]\right] \\
= \;& E_N\left[E_{\hat{V}(N)}\left[\mathbf{1}_{\left\{\hat{V}(N)\leq u\right\}} \cdot E_{\vec{\mathbf{\Sigma}}_{-1}^{(N-1)}}\left[W_N \mid \hat{V}(N), V_1\right] \mid V_1 = v\right]\right] \\
= \;& \sum_{n\in I} p_N(n) \int_0^u w_n(v,t)\, f_{\hat{V}(n)|V_1}(t|v)dt \\
= \;& \int_0^u \left[\sum_{n\in I} p_{N|\hat{V},V_1}(n|t,v)\, w_n(v,t)\right] f_{\hat{V}|V_1}(t|v)dt \\
= \;& \int_0^u \hat{w}(v,t)\, f_{\hat{V}|V_1}(t|v)dt \\
= \;& E_{\hat{V}}\left[\hat{w}(V_1,\hat{V}) \cdot \mathbf{1}_{\left\{\hat{V}\leq u\right\}} \mid V_1 = v\right],
\end{aligned}$$

where the third and fourth equality make use of (4.3) and Theorem 2(i), respectively. This shows the part of the claim regarding the expected value of the object obtained by the winning bidder. Substituting Z_N for W_N, where $Z_n \equiv U_n(\hat{V}(n); \mathbf{V}_{-1}^{(n-1)}; \mathbf{S})$, it follows analogously that the winner's expected payment is:

$$E_{\hat{V}}\left[\hat{w}(\hat{V},\hat{V}) \cdot \mathbf{1}_{\left\{\hat{V}\leq b^{-1}(a)\right\}} |V_1 = v\right].$$

Thus, the winning bidder's profit is composed as claimed. □

Proof of Theorem 3.

For bidder 1's expected profit, if he bids $a \in \Re_+$ while his competitors conform to an arbitrary strictly increasing (continuous) candidate strategy $b(\cdot)$:

$$\begin{aligned}
& \pi(a|v) = \\
= \;& E_N\left[E_{(\mathbf{\Sigma}^{(N)},\hat{V}(N))}\left[\left\{W_N - b(\hat{V}(N))\right\} \cdot \mathbf{1}_{\left\{\hat{V}(N)\leq b^{-1}(a)\right\}} \mid V_1 = v\right]\right] \\
= \;& E_N\left[E_{\hat{V}(N)}\left[E_{\vec{\mathbf{\Sigma}}^{(N)}}\left[\left\{W_N - b(\hat{V}(N))\right\} \cdot \mathbf{1}_{\left\{\hat{V}(N)\leq b^{-1}(a)\right\}} \mid \hat{V}(N), V_1\right] \mid V_1 = v\right]\right] \\
= \;& E_N\left[E_{\hat{V}(N)}\left[\left\{w_N(V_1,\hat{V}(N)) - b(\hat{V}(N))\right\} \cdot \mathbf{1}_{\left\{\hat{V}(N)\leq b^{-1}(a)\right\}} \mid V_1 = v\right]\right] \\
= \;& E_N\left[\int_0^{b^{-1}(a)} \left[w_N(v,t) - b(t)\right]\; f_{\hat{V}(N)|V_1}(t|v)dt\right], \qquad (10.5)
\end{aligned}$$

the second equality because of (10.4). The necessary condition $\pi_a(b(v)|v) = 0$ for a symmetric Harsanyi-Bayes-Nash equilibrium is:

$$\sum_{n \in I} p_N(n)\,[w_n(v,v) - b(v)]\; f_{\hat{V}(n)|V_1}(v|v) = 0.$$

Rearranging, using (4.2) and (4.3), and appealing to Theorem 2(i):

$$\begin{aligned} b(v) &= \frac{\sum_{n \in I} p_N(n)\; f_{\hat{V}(n)|V_1}(v|v)\, w_n(v,v)}{\sum_{n \in I} p_N(n)\; f_{\hat{V}(n)|V_1}(v|v)} \\ &= \sum_{n \in I} p_{N|\hat{V},V_1}(n|v,v)\, w_n(v,v) \\ &= \hat{w}(v,v). \end{aligned}$$

To see that using $b(\cdot)$ is a best response of bidder 1 against $[b_j(\cdot) = b(\cdot)]_{1 \neq j \in I}$, rewrite (10.5) as:

$$\begin{aligned} \pi(a|v) &= \int_0^{b^{-1}(a)} \sum_{n \in I} p_N(n)\,[w_n(v,t) - b(t)]\; f_{\hat{V}(n)|V_1}(t|v)dt \\ &= \int_0^{b^{-1}(a)} \left[\sum_{n \in I} f_{(N,\hat{V})|V_1}(n,t|v)\, w_n(v,t) - f_{\hat{V}|V_1}(t|v)\, \hat{w}(t,t) \right] dt \qquad (10.6) \end{aligned}$$

using (4.1) in the first and (4.2) in the second term of the sum. Then for the integrand $\xi(v,t)$ from (10.6), using (4.3) and Theorem 2(i):

$$\begin{aligned} sgn\{\xi(v,\tau)\} &= sgn \left\{ \sum_{n \in I} f_{(N,\hat{V})|V_1}(n,t|v)\, w_n(v,t) - f_{\hat{V}|V_1}(t|v)\, \hat{w}(t,t) \right\} \\ &= sgn \left\{ \sum_{n \in I} p_{N|\hat{V},V_1}(n|t,v)\, w_n(v,t) - \hat{w}(t,t) \right\} \\ &= sgn\,\{\hat{w}(v,t) - \hat{w}(t,t)\} \\ &= sgn\{v - t\}, \end{aligned}$$

the last equation because of Theorem 2(ii). Hence (10.6) is maximized by a bid a^* such that $v = b^{-1}(a^*)$, so type v's best response against his competitors using the strategy b is uniquely determined as $b(v) = \hat{w}(v,v)$. □

Proof of Proposition 2.

(i) We have to show that $F_{\hat{V}(n)|V_1}(\cdot|v)$ is strictly decreasing in $n \in I$ $(v > 0)$. Since the claim holds trivially when $\mathbf{V}^{(n)}$ is a vector of independent variates $(F_{\hat{V}(n)|V_1}(\cdot|\cdot) = F_1(\cdot)^{n-1})$, it remains to consider the strictly affiliated case. According to Theorem 23(ii) of Milgrom and Weber (1982), a random vector $\mathbf{V}^{(n+1)}$ is strictly affiliated if for any pair of concordantly non-decreasing functions g, h: $\Re^{n+1} \to \Re$, sublattice $S \subset \Re^{n+1}$:

$$E[g(\mathbf{V}^{(n+1)})h(\mathbf{V}^{(n+1)}) \,|\, S] > E[g(\mathbf{V}^{(n+1)}) \,|\, S] \cdot E[h(\mathbf{V}^{(n+1)}) \,|\, S]. \tag{10.7}$$

For each $n \in I$, let $\hat{\mathbf{v}}^{(n)}$ be the n-vector consisting of the value $\hat{v} \in (0, \bar{v})$ in each coordinate. Choose $S = \{(V_2, .., V_{n-1}) \leq \hat{\mathbf{v}}^{(n-2)} \,|\, V_1 = v\}$, $g(\mathbf{V}^{(n+1)}) = \mathbf{1}_{\{V_n \leq \hat{v}\}}$, $h(\mathbf{V}^{(n+1)}) = \mathbf{1}_{\{V_{n+1} \leq \hat{v}\}}$, and note that $\mathbf{1}_S(\hat{\mathbf{v}})$ is affiliated. Using that $\mathbf{V}^{(n+1)}$ is a vector of exchangeable variates, (10.7) yields:

$$\frac{Pr[(V_2, .., V_{n+1}) \leq \hat{\mathbf{v}}^{(n)} \,|\, V_1 = v]}{Pr[(V_2, .., V_{n-1}) \leq \hat{\mathbf{v}}^{(n-2)} \,|\, V_1 = v]} > \left(\frac{Pr[(V_2, .., V_n) \leq \hat{\mathbf{v}}^{(n-1)} \,|\, V_1 = v]}{Pr[(V_2, .., V_{n-1}) \leq \hat{\mathbf{v}}^{(n-2)} \,|\, V_1 = v]}\right)^2,$$

which is equivalent to:

$$\frac{F_{(V_2,..,V_{n+1})|V_1}(\mathbf{v}^{(n)}|v)}{F_{(V_2,..,V_n)|V_1}(\mathbf{v}^{(n-1)}|v)} > \frac{F_{(V_2,..,V_n)|V_1}(\mathbf{v}^{(n-1)}|v)}{F_{(V_2,..,V_{n-1})|V_1}(\mathbf{v}^{(n-2)}|v)}.$$

Since $F_{\hat{V}(n)|V_1}(\hat{v}|v) = F_{(V_2,..,V_n)|V_1}(\hat{\mathbf{v}}^{(n)}|v)$, it follows that the function

$$\varphi(n) = \frac{F_{\hat{V}(n)|V_1}(\hat{v}|v)}{F_{\hat{V}(n-1)|V_1}(\hat{v}|v)}$$

is strictly monotonically increasing in $n \in I$ (where $F_{\hat{V}(0)|V_1}(\cdot|v) \equiv 1$). Since $\varphi(n)$ approaches unity as n approaches infinity, $\varphi(n) < 1$ $\forall n < \infty$. This is the claim.

(ii) Let N_1, N_2 be such that $P_{N_1}(\cdot) < P_{N_2}(\cdot)$, let

$$\Delta_{\hat{V}(n)|V_1}(\hat{v}|v) = F_{\hat{V}(n-1)|V_1}(\hat{v}|v) - F_{\hat{V}(n)|V_1}(\hat{v}|v),$$

where $\underset{n\to\infty}{lim}\, \Delta_{\hat{V}(n)|V_1}(\cdot|v) = 0$, and let $\hat{V}_i = E_{N_i}[\hat{V}(N_i)]$, $i = 1, 2$. Then,

$$F_{\hat{V}_1|V_1}(\hat{v}|v) \;\; = \;\; \sum_{n\in I} p_{N_1}(n)\, F_{\hat{V}(n)|V_1}(\hat{v}|v)$$

$$
\begin{aligned}
&= \sum_{n\in I}\sum_{m\geq n}\Delta_{\hat V(m)|V_1}(\hat v|v)p_{N_1}(n)\\
&= \sum_{m\in I}\Delta_{\hat V(m)|V_1}(\hat v|v)\sum_{n<m}p_{N_1}(n)\\
&< \sum_{m\in I}\Delta_{\hat V(m)|V_1}(\hat v|v)\sum_{n<m}p_{N_1}(n)\\
&= F_{\hat V_2|V_1}(\hat v|v),
\end{aligned}
$$

because $\Delta_{\hat V(m)|V_1}(\hat v|v) > 0\ \forall m < \infty$ by part (i). □

Proof of Theorem 4.

It follows from Theorem 2 that $\hat w(v,\hat v) = \hat w(v,\check v)$ if $max(\hat v,\check v) \leq v$. Consider type v of bidder 1 ($v > 0$) who can choose to bid $b(v) = \hat w(v,v)$ or $\Re_+ \ni b \neq b(v)$. Suppose that 1's most serious competitor employs a strictly increasing strategy $\beta(\cdot)$ such that $\beta(0) = \hat w(0,0)$, so that the highest competing bid is $\beta(\hat V(N))$. (All bids $\beta(0) \neq \hat w(0,0)$ are weakly dominated.) Note that for a constant $c \in \Re_+$, $\beta(\hat V(N)) = c$ implies that $\hat V(n) \overset{a.s.}{=} \beta^{-1}(c)\ \forall n \in J$, so that necessarily $\hat V \overset{a.s.}{=} \beta^{-1}(c)$. In particular, using Theorem 2(ii), $\beta(\hat V(N)) = \hat w(0,0) \Rightarrow \hat V \overset{a.s.}{=} 0$.

First consider the case $\hat w(0,0) < b < b(v)$, where the decisive event for 1's choice is $\Xi := \{\hat V(N) \,|\, b < \beta(\hat V(N)) < b(v)\}$ (ignoring null events, as ties). Given Ξ, bidding b earns zero expected profit while $b(v)$ yields:

$$
\begin{aligned}
\pi(b(v)|v,\Xi) &= E_N\left[w_N(V_1,\hat V(N)) \,|\, \Xi, V_1 = v\right] - E_N\left[E_{\hat V(N)}\left[\beta(\hat V(N)) \,|\, \Xi, V_1 = v\right]\right]\\
{}_{a.s.} &> E_N\left[w_N(V_1,\hat V(N)) \,|\, \Xi, V_1 = v\right] - E_N\left[E_{\hat V(N)}\left[b(v) \,|\, V_1 = v\right]\right]\\
&\geq E_N\left[w_N(V_1,\hat V(N)) \,|\, \beta(\hat V(N)) = \hat w(0,0), V_1 = v\right] - b(v)\\
{}_{a.s.} &= \hat w(v,0) - b(v) = 0,
\end{aligned}
$$

the first inequality using the definition of Ξ; the second inequality due to Proposition 1; the second equality invoking Theorem 2(i) for $\hat V \overset{a.s.}{=} 0$; and the last equality by maximal attentiveness (as $b(v) > \hat w(0,0)$, necessarily $v > 0$).

Now consider the choice of $b > b(v)$, where $b \leq v$ and the decisive event for 1 is $\Xi' := \{\hat V(N) \,|\, b > \beta(\hat V(N)) > b(v)\}$. Suppose that $\beta(\cdot)$ is such that $\beta(v) > v$. Clearly for any of 1's competitors playing such a strategy is dominated by bids such that $\beta'(\cdot) \leq 1$. However if bidding $b(v)$ is best for bidder 1 given Ξ' and $\beta(v) > v$, it follows that a fortiori $b(v)$ is best against any strategy β such that

$\beta'(\cdot) \leq 1$ (because then the winning probability for bidder 1 increases while the expected price he pays decreases). Conditional on Ξ', bidding b then yields an expected profit of:

$$\begin{aligned}
\pi(b|v,\Xi') &= E_N\left[w_N(V_1,\hat{V}(N))\,|\,\Xi',V_1=v\right] - E_N\left[E_{\hat{V}(N)}\left[\beta(\hat{V}(N))\,|\,\Xi',V_1=v\right]\right] \\
&\leq \hat{w}(v,\beta^{-1}(b)) - E_N\left[E_{\hat{V}(N)}\left[\beta(\hat{V}(N))\,|\,\Xi',V_1=v\right]\right] \\
{}_{a.s.} &< \hat{w}(v,v) - b(v) = 0,
\end{aligned}$$

the first inequality from the definition of Ξ' and Proposition 1 in conjunction with Theorem 2(i) for $\hat{V} \overset{a.s.}{=} \beta^{-1}(b)$, and the second inequality by maximal attentiveness for $v \geq \beta^{-1}(v) \geq \beta^{-1}(b)$ and the definition of Ξ'.

As $\Xi \cup \Xi'$ exhausts the events where bidder 1 may have strictly positive profit from bidding either b or $b(v)$, bidding $b(v)$ is optimal for any competing strategy $\beta(\cdot)$. To see that $[b(\cdot),...,b(\cdot)]$ is an equilibrium it remains to note that $b(\cdot)$ has the properties of $\beta(\cdot)$ used. □

Proof of Theorem 5.

Reconsider the necessary condition (6.2), where $\lambda_n(v|v), \hat{\lambda}(v|v), w_n(v,v)$ are continuous in v and $|\Phi_b(b,v)| = \hat{\lambda}(v|v) < \infty\ \forall v > 0$. Although $|\Phi_b(b,v)|$ is unbounded at $v = 0$, the initial value problem $[b'(v) = \Phi(b(v),v);\ b(0) = \hat{w}(0,0)]$ has a unique solution because $\Phi(b,v)$ is linear in b (e.g. Walter, 1972, p.23). Rearranging, multiplying by $exp\{\int_0^v \hat{\lambda}(s|s)ds\}$, and integrating subject to $b(0) = \hat{w}(0,0)$, (6.2) yields:

$$\begin{aligned}
b(v) &= \hat{w}(0,0)\,\hat{L}(0|v) + \int_0^v E_N\left[w_N(t,t)\,f_{\hat{V}(N)|V_1}(t|t)\right]\frac{\hat{L}(t|v)}{F_{\hat{V}|V_1}(t|t)}dt, \\
\text{where}\ \ \hat{L}(t|v) &\equiv exp\{-\int_t^v \hat{\lambda}(s|s)ds\}. \qquad (10.8)
\end{aligned}$$

$\hat{L}(\cdot|v)$ is a proper distribution function on $[0,v]$, because $\hat{L}(v|v) = 1$, $D_1\hat{L}(\cdot|v) > 0$, and $\hat{L}(0|v) = 0$ since $\lim_{s\to 0} \hat{\lambda}(s|s) = +\infty$. Thus, the integration constant in (10.8) vanishes. Using that $d\hat{L}(t|v) = \hat{\lambda}(t|t)\hat{L}(t|v)dt$,

$$b(v) = \int_0^v E_N\left[w_N(t,t)\frac{f_{\hat{V}(N)|V_1}(t|t)}{F_{\hat{V}|V_1}(t|t)}\right]\hat{L}(t|v)dt$$

$$
\begin{aligned}
&= \int_0^v \sum_{n\in I} \frac{p_N(n)\, f_{\hat{V}(n)|V_1}(t|t)}{f_{\hat{V}|V_1}(t|t)} w_n(t,t)\, d\hat{L}(t|v) \\
&= \int_0^v \sum_{n\in I} p_{N|\hat{V},V_1}(n|t,t)\, w_n(t,t)\, d\hat{L}(t|v) \\
&= \int_0^v \hat{w}(t,t)\, d\hat{L}(t|v),
\end{aligned}
$$

the last equality from Theorem 2(i). By Theorem 2(ii) and since the integrating probability measure $\hat{L}(\cdot|v)$ is strictly decreasing in v, $b(\cdot)$ is strictly increasing indeed. It remains to show that $sgn\{\pi_a(b(v_0)|v)\} = sgn\{v - v_0\}$. The argument is parallel to Milgrom and Weber (1982, p.1108): fix v_0, choose $v \neq v_0$ $(v_0, v > 0)$, and rewrite the necessary condition (6.1) for $b(v_0)$ as:

$$
\begin{aligned}
\pi_a(b(v_0)|v) &= -F_{\hat{V}|V_1}(v_0|v) + \frac{1}{b'(v_0)} \sum_{n\in I} p_N(n)\, f_{\hat{V}(n)|V_1}(v_0|v)\, [w_n(v,v_0) - b(v_0)] \\
&= \frac{F_{\hat{V}|V_1}(v_0|v)}{b'(v_0)} \left[\sum_{n\in I} p_{N|\hat{V},V_1}(n|v_0,v)\, \hat{\lambda}(v_0|v)\, [w_n(v,v_0) - b(v_0)] - b'(v_0) \right],
\end{aligned}
$$

using (4.3) and the definition of $\hat{\lambda}(\cdot,\cdot)$. As the factor in front of the bracket is strictly positive,

$$
sgn\{\pi_a(b(v_0)|v)\} = sgn\left\{ \hat{\lambda}(v_0|v)\, [\hat{w}(v,v_0) - b(v_0)] - b'(v_0) \right\} = sgn\{v - v_0\},
$$

the first equality by Theorem 2(i) and the second equality (for varying v) because of Theorem 2(ii) and $D_2\hat{\lambda}(\cdot|\cdot) > 0$. To see the last fact, integrate the affiliation inequality $f_{\hat{V}|V_1}(t|v)\, f_{\hat{V}|V_1}(s|u) > f_{\hat{V}|V_1}(s|v)\, f_{\hat{V}|V_1}(t|u)$ for $v > u > 0$ and $t > s > 0$ over $\{s \,|\, s < t\}$ and rearrange terms. □

Proof of Proposition 3.

For the equilibrium strategy from Corollary 2:

$$
\begin{aligned}
b(v) &= \int_0^v \hat{w}(t,t) \frac{f_{\hat{V}}(t)}{F_{\hat{V}}(v)} dt \\
\text{(by Theorem 2(i))} &= \int_0^v E_N \left[w_N(t,t) \,|\, \hat{V} = t \right] \frac{f_{\hat{V}}(t)}{F_{\hat{V}}(v)} dt
\end{aligned}
$$

$$
\begin{aligned}
\text{(by (4.3))} \quad &= \quad \sum_{n\in I} \frac{p_N(n)}{F_{\hat{V}}(v)} \int_0^v w_n(t,t)\, f_{\hat{V}(n)}(t)dt \\
\text{(by (4.4))} \quad &= \quad \sum_{n\in I} \check{p}_{N|\hat{V}}(n|v)\, E_{\hat{V}(n)} \left[w_n(\hat{V}(n),\hat{V}(n)) \mid \hat{V}(n) \le v \right]. \quad \square
\end{aligned}
$$

Proof of Proposition 4.

Recall from Corollary 3 that since $D_2\hat{\lambda}(\cdot|\cdot) > 0$,

$$
F_{\hat{V}|V_1}(v|v) = exp\left\{ \int_0^v \hat{\lambda}(s|v)ds \right\} > exp\left\{ \int_0^v \hat{\lambda}(s|s)ds \right\}
$$

for any distribution of N and for any $v > 0$. Thus,

$$
\begin{aligned}
b(v)F_{\hat{V}|V_1}(v|v) \quad &= \quad \int_0^v \hat{w}(t,t)\, \hat{\lambda}(t|t)\, \hat{L}(t|v)\, dt \cdot F_{\hat{V}|V_1}(v|v) \\
&> \quad \int_0^v \hat{w}(t,t)\, \hat{\lambda}(t|t)\, exp\left\{ -\int_t^v \hat{\lambda}(s|s)ds \right\} exp\left\{ \int_0^v \hat{\lambda}(s|s)ds \right\} dt \\
&= \quad \int_0^v \hat{w}(t,t)\;\, d\, exp\left\{ \int_0^t \hat{\lambda}(s|s)ds \right\}.
\end{aligned}
$$

Consider the last expression. By Theorem 2(iii), if $w_n(\cdot,\cdot) \ge w_{n-1}(\cdot,\cdot)$ $(n \in J)$, the integrand is uniformly increasing as N shifts up stochastically. If the expected marginal winning probability is non-decreasing as N increases stochastically, also the integrating function is. It follows that the first expression, which majorizes the last one $\forall v > 0$, cannot decrease as N shifts up stochastically. $\square$

Proof of Proposition 5.

(i) Strict affiliation of $(\hat{V}, V_1)$ implies that $\forall \hat{v} > \check{v}, v > v' > 0$:

$$
f_{\hat{V}|V_1}(\hat{v}|v)\, f_{\hat{V}|V_1}(\check{v}|v') > f_{\hat{V}|V_1}(\check{v}|v)\, f_{\hat{V}|V_1}(\hat{v}|v')
$$

whenever the conditional density exists around a neighborhood of $v' > 0$. Integrating over $\{\check{v} \mid \check{v} \le \hat{v}\}$ and rearranging, we obtain:

$$
\frac{d}{d\hat{v}} log \left\{ \frac{F_{\hat{V}|V_1}(\hat{v}|v)}{F_{\hat{V}|V_1}(\hat{v}|v')} \right\} > 0 \;\; (v > v'),
$$

which implies that

$$\frac{d}{d\hat{v}}\left\{\frac{F_{\hat{V}|V_1}(\hat{v}|v)}{F_{\hat{V}|V_1}(\hat{v}|v')}\right\} > 0 \ \ (v > v').$$

As $\lim_{\hat{v}\to\infty} \frac{F_{\hat{V}|V_1}(\hat{v}|v)}{F_{\hat{V}|V_1}(\hat{v}|v')} = 1$, it follows that for $\hat{v} < \infty$:

$$F_{\hat{V}|V_1}(\hat{v}|v) < F_{\hat{V}|V_1}(\hat{v}|v') \quad (v > v').$$

Passing to the limit as v' approaches v yields the claim.

(ii) Recall from Theorem 5 that $D_2\hat{\lambda}(\cdot|\cdot) > 0$. Rewriting the relation as

$$D_2 D_1 log F_{\hat{V}|V_1}(\cdot|\cdot) = D_1 D_2 log F_{\hat{V}|V_1}(\cdot|\cdot) > 0$$

implies that for $\hat{v} > \check{v} > 0, v > 0$:

$$D_2 log F_{\hat{V}|V_1}(\hat{v}|v) > D_2 log F_{\hat{V}|V_1}(\check{v}|v).$$

Thus, $\forall \hat{v} > \check{v}$:

$$sgn\left\{D_2 log \frac{F_{\hat{V}|V_1}(\check{v}|v)}{F_{\hat{V}|V_1}(\hat{v}|v)}\right\} = sgn\left\{D_2 \frac{F_{\hat{V}|V_1}(\check{v}|v)}{F_{\hat{V}|V_1}(\hat{v}|v)}\right\} < 0. \ \square$$

Proof of Theorem 7.

Consider two revelation mechanisms implementing the symmetric equilibrium $[b_M(\cdot), .., b_M(\cdot)]$ of the M-price auctions $M = 1, 2$, respectively. Suppose in each mechanism bidder 1 reports $u > 0$ when being of type $v > 0$, while all of his competitors announce truthfully. If 1's conditional expected payment in mechanism M, given he wins, is $P_M(u|v)$, then $P_1(u|v) = b(u)$ with $b(\cdot)$ from Theorem 5, and by Theorem 3 and Proposition 1:

$$\begin{aligned} P_2(u|v) &= E_{\hat{V}}[\hat{w}(\hat{V}, \hat{V}) \,|\, \hat{V} \leq u, V_1 = v] \\ &= \frac{\hat{w}(u,u) - \hat{w}(0,0)}{F_{\hat{V}|V_1}(u|v)} - \int_0^u \frac{d}{dt}\hat{w}(t,t)\frac{F_{\hat{V}|V_1}(t|v)}{F_{\hat{V}|V_1}(u|v)}\, dt. \end{aligned} \tag{10.9}$$

By Proposition 5(i), the first term of (10.9) has a positive partial derivative with respect to v; by Proposition 5(ii), the same applies to the second term of (10.9), where $\frac{d}{dt}\hat{w}(t,t) > 0$ from Theorem 2(ii). Hence, $D_2P_2(u|v) > 0$, and the envelope theorem implies that

$$\frac{d}{dv}\{P_2(v|v) - P_1(v|v)\} = D_2P_2(v|v) > 0.$$

As $P_2(0|0) = \hat{w}(0,0) = P_1(0|0)$, it follows that $P_2(v|v) > P_1(v|v)\ \forall v > 0$. □

Proof of Theorem 8.

Extending the function $w_n(\cdot,\cdot)$ to $n \geq 2$ arguments, let

$$w_n(\mathbf{v}^{(n)}) := E_{\mathbf{\Sigma}^{(n)}}\left[W_n \,|\, V_1 = v_1, \vec{\mathbf{V}}_{-1}^{(n-1)} = \mathbf{v}_{-1}^{(n-1)}\right],$$

and note that then

$$\begin{aligned} w_n(\hat{v},\hat{v}) &= E_{\mathbf{\Sigma}^{(n)}}\left[W_n \,|\, V_1 = \hat{v}, \hat{V}(n) = \hat{v}\right] \\ &= E_{\mathbf{\Sigma}^{(n)}}\left[w_n(V_1, \hat{V}(n), \vec{\mathbf{V}}_{-1}^{(n-2)}) \,|\, V_1 = \hat{v}, \hat{V}(n) = \hat{v}\right] \\ &= E_{\mathbf{\Sigma}^{(n)}}\left[w_n(\hat{V}(n), \hat{V}(n), \vec{\mathbf{V}}_{-1}^{(n-2)}) \,|\, V_1 = \hat{v}, \hat{V}(n) = \hat{v}\right] \\ &\leq E_{\mathbf{\Sigma}^{(n)}}\left[w_n(\hat{V}(n), \hat{V}(n), \vec{\mathbf{V}}_{-1}^{(n-2)}) \,|\, V_1 = v, \hat{V}(n) = \hat{v},\ v \geq \hat{v}\right], \end{aligned} \tag{10.10}$$

the inequality from Theorem 5 of Milgrom and Weber (1982). Now let $\tilde{V}_n = E_N[\hat{V}(N) \,|\, N \geq n]$, and define $\tilde{w}(\cdot,\cdot)$: $[0,\bar{v}]^2 \to \Re$ in analogy to $\hat{w}(\cdot,\cdot)$, but with $\tilde{V}_n$ replacing $\hat{V}$ and $E_N[\cdot]$ substituted by $E_N[\cdot \,|\, N \geq n]$. Similarly,

$$\tilde{w}(\tilde{V}_n, \tilde{V}_n, \tilde{\mathbf{\Xi}}_n) = E_N\left[E_{\mathbf{\Sigma}^{(N)}}\left[w_N(V_1, \hat{V}(N), \vec{\mathbf{V}}_{-1}^{(N-2)}) \,|\, V_1, \tilde{V}_n\right] \,|\, N \geq n\right],$$

where $\tilde{\mathbf{\Xi}}_n$ represents the refined information $\vec{\mathbf{V}}_{-1}^{(N-2)}|\{N \geq n\}$ available in the stage game equilibrium of the last stage. Then from Proposition 1; since $\tilde{V}_n$ is stochasticaly larger than $\hat{V}\ \ \forall n \in I$; because of relation (10.10); and by extending Theorem 2(i) to $\tilde{w}(\cdot,\cdot,\tilde{\mathbf{\Xi}}_n)$:

$$\begin{aligned} R_2 &= E_{(V_1,\hat{V})}\left[\hat{w}(\hat{V},\hat{V}) \,|\, V_1 \geq \hat{V}\right] \\ &\leq E_{(V_1,\tilde{V}_n)}\left[\tilde{w}(\tilde{V}_n,\tilde{V}_n) \,|\, V_1 \geq \tilde{V}_n\right] \\ &\leq E_{(V_1,\tilde{V}_n)}\left[E_{(V_1,\tilde{V}_n)}\left[\tilde{w}(\tilde{V}_n,\tilde{V}_n,\tilde{\mathbf{\Xi}}_n) \,|\, V_1, \tilde{V}_n\right] \,|\, V_1 \geq \tilde{V}_n\right] \\ &= E_{(V_1,\tilde{V}_n)}\left[\tilde{w}(\tilde{V}_n,\tilde{V}_n,\tilde{\mathbf{\Xi}}_n) \,|\, V_1 \geq \tilde{V}_n\right] = R_E, \end{aligned}$$

the last equality by adapting Proposition 1 to $\tilde{w}(\cdot,\cdot,\tilde{\mathbf{\Xi}}_n)$ with the distribution $\{p_{N|\{N\geq n\},\tilde{V}_n,\tilde{\mathbf{\Xi}}_n,V_1}(n|\tilde{\mathbf{v}}^{(n-1)},v_1)\}_{n\in I}$. $\square$

11. References

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