ONE AGAINST ALL IN THE FICTITIOUS PLAY PROCESS*

Aner Sela
Economics Department

University of Mannheim

Seminargebaeude A5

Postfach 103462

D-68131 Mannheim, Germany

* I wish to thank Dov Monderer, Dov Samet, and Karl H. Schlag for helpful comments. Financial support from the Deutsche Forschungsgemeinschaft, Sonderforschungsbereich 504 at the University of Mannheim is gratefully acknowledged.

Abstract

There are only few "positive" results concerning multi-person games with the *fictitious play property*, that is, games in which every *fictitious play* process approaches the set of equilibria. In this paper we characterize classes of multi-person games with the *fictitious play property*. We consider an { EINBETTEN Equation.2 } player game { EINBETTEN Equation.2 } based on { EINBETTEN Equation.2 } two-person sub-games. In each of these sub-games player 0 plays against one of the other players. Player 0 is regulated, so that he must choose the same strategy in all { EINBETTEN Equation.2 } sub-games. We show that if all sub-games are either zero-sum games, weighted potential games, or games with identical payoff functions, then the *fictitious play property* holds for the associated game.

1. Introduction

In the *fictitious play* (*FP*) process proposed by Brown (1951) each player believes that each one of his opponents is using a stationary mixed strategy, which is the empirical distribution of this opponent's past actions. We say that a *FP* process approaches equilibrium, if the belief sequence is as close as we wish to equilibria set after sufficiently late time. A game in which every *FP* process, independent of initial actions and beliefs, approaches equilibrium, is called a *game with the FP property*. Robinson (1951) proved that every two person zero-sum game has the *FP* property. Miyasawa (1961) proved that every two person 2x2 game has the *FP* property under certain indifference breaking rules. In 1964 Shapley gave an example of an ordinal class of 3x3 games without the *FP* property. Milgrom and Roberts (1991) showed that every game which is dominance solvable has the *FP* property. Krishna (1992) proved that if the strategy sets are linearly ordered, then every game with strategic complementarities and diminishing returns has the *FP* property. Monderer and Shapley (1996) proved that every game with identical interests has the *FP* property, where a game with identical interests is a game which is best response equivalent in mixed strategies to a game with identical payoff functions.

There are only few "positive" results concerning the *FP* property in multi-person games. In this paper we characterize classes of multi-person games with the *FP* property.

We study an{ EINBETTEN Equation.2 } player game { EINBETTEN Equation.2 } based on { EINBETTEN Equation.2 } two-person games,{ EINBETTEN Equation.2 }. The players in { EINBETTEN Equation.2 }, are player 0 and player { EINBETTEN Equation.2 }. The payoff function of player { EINBETTEN Equation.2 }in { EINBETTEN Equation.2 } for { EINBETTEN Equation.2 } depends only on his strategy and player 0's strategy. Player 0 has the same strategy set in all { EINBETTEN Equation.2 } sub games { EINBETTEN Equation.2 }, such that he must choose the same strategy in all { EINBETTEN Equation.2 } sub games, and his payoff is the sum of his payoffs in all the sub games. { EINBETTEN Equation.2 } will be called the *compound game*. We analyze the *fictitious play* process in such compound games. We do it by associating with each such game, a two player game that will be called the *reduced game*. We show that a *fictitious play* process approaches equilibrium in the compound ({ EINBETTEN Equation.2 }+1) player game, if and only if it approaches equilibrium in the reduced two player game. This result enables us to identify classes of compound games with the *fictitious play property*. In particular, we show that if all sub-games are either zero-sum games, weighted potential games or games with identical payoff functions then the *fictitious play property* holds for the associated compound game.

2. An Illustration

Consider an industry with a corporation producing and selling a single good in { EINBETTEN Equation.2 } distinct markets. In each of these markets there is also a local firm operating solely in this market. The corporation is regulated, so that it must charge the same price in all the markets. We can model this situation by a Bertrand-like game with { EINBETTEN Equation.2 } players. The strategies of the players are prices (charged by the firm and the corporation) in the local market.

We denote the corporation by 0 and the local firm in market { EINBETTEN Equation.2 }by { EINBETTEN Equation.2 }.

The demand functions in market { EINBETTEN Equation.2 }, { EINBETTEN Equation.2 }, are:

{ **EINBETTEN Equation.2** }.

Where { EINBETTEN Equation.2 } are the prices charged by the corporation and the local firm respectively in market { EINBETTEN Equation.2 }.

The corporation's profit function in market { EINBETTEN Equation.2 }is:

{ EINBETTEN Equation.2 }.

And the firm { EINBETTEN Equation.2 } profit function is:

{ EINBETTEN Equation.2 }.

The corporation's profit is the sum of its profits in all the markets. We call such a game a *compound Bertrand game*.

Consider now a repeated compound Bertrand game in which the firms adopt *fictitious play*'s behavior rules. We are interested whether the agents in a compound Bertrand game learn to play Nash equilibrium strategies by this adaptive play, namely, whether the compound Bertrand game has the *fictitious play property*.

It can be shown that each market (Bertrand oligopoly) is a *game with identical interests*, where a game with identical interests is best response equivalent in mixed strategies to a game with identical payoff functions. As was shown by Monderer and Shapley (1996), every game with identical interests has the *fictitious play property*, and therefore in every local market the agents' beliefs approach equilibrium. In this paper we study whether the agents' beliefs approach equilibrium in the compound Bertrand game as well. This is not a simple question since the corporation operates in all the markets simultaneously and therefore the { EINBETTEN Equation.2 } distinct markets are not completely separated. We show below that compound Bertrand games as well as other classes of games based on the same separable structure, are games with the *fictitious play property*.

3. The Fictitious Play process

The *fictitious play* (*FP*) proposed by Brown (1951) has two different versions. In the first version, each player believes that each one of his opponents is using a stationary mixed strategy, which is the empirical distribution of this opponent's past actions. Such a player will be called a *IFP* player ("I" stands for independent). In the second version, each player believes that his opponents are using a joint correlated mixed strategy, which is the empirical distribution of his opponents' past actions. Such a player will be called a *JFP* player ("J" stands for joint). In two-person games, the concepts of *IFP* player and *JFP* player coincide. Since most of the research on the *FP* process has concentrated in two-person games, the difference between *JFP* and *IFP* has been hardly noticed. In this paper we refer to the *IFP* as *FP*.

Let { EINBETTEN Equation.2 }be the set of players. For each { EINBETTEN Equation.2 }, { EINBETTEN Equation.2 } is the finite strategy set of player { EINBETTEN Equation.2 }. For every { EINBETTEN Equation.2 } we denote { EINBETTEN Equation.2 }. In particular we denote { EINBETTEN Equation.2 } and { EINBETTEN Equation.2 }. Let { EINBETTEN Equation.2 } be player { EINBETTEN Equation.2 }s payoff function, where { EINBETTEN Equation.2 }denotes the set of real numbers. For each finite set { EINBETTEN Equation.2 } we denote by { EINBETTEN Equation.2 } the set of probability measures over { EINBETTEN Equation.2 }. For { EINBETTEN Equation.2 } we denote { EINBETTEN Equation.2 } and { EINBETTEN Equation.2 }. The set of player { EINBETTEN Equation.2 }s mixed strategies { EINBETTEN Equation.2 } is denoted by { EINBETTEN Equation.2 }.

We denote { EINBETTEN Equation.2 } and{ EINBETTEN Equation.2 }. We identify { EINBETTEN Equation.2 } and { EINBETTEN Equation.2 } with extreme points in { EINBETTEN Equation.2 } and { EINBETTEN Equation.2 }respectively.

A *path* in S is a sequence { EINBETTEN Equation.2 }, for { EINBETTEN Equation.2 } of elements in S.

A *belief sequence* { EINBETTEN Equation.2 }, for { EINBETTEN Equation.2 }{ EINBETTEN Equation.2 }, consist of elements of { EINBETTEN Equation.2 }, i.e., { EINBETTEN Equation.2 }, is the belief of player { EINBETTEN Equation.2 }about the other players' strategies at stage { EINBETTEN Equation.2 }. { EINBETTEN Equation.2 } is the belief of player { EINBETTEN Equation.2 }about player { EINBETTEN Equation.2 }s strategy at stage { EINBETTEN Equation.2 }.

A *joint belief sequence* { EINBETTEN Equation.2 }, for { EINBETTEN Equation.2 }, consist of elements of { EINBETTEN Equation.2 }, i.e. { EINBETTEN Equation.2 }, is the belief of player { EINBETTEN Equation.2 }about the joint strategy of the other players at stage{ EINBETTEN Equation.2 }. Let { EINBETTEN Equation.2 } and { EINBETTEN Equation.2 }. Denote by { EINBETTEN Equation.2 }the marginal distribution on { EINBETTEN Equation.2 }. That is ,

{ EINBETTEN Equation.2 }.

A *learning process* is a pair { EINBETTEN Equation.2 }, where { EINBETTEN Equation.2 } is a path in { EINBETTEN Equation.2 }, and { EINBETTEN Equation.2 } is either a belief sequence or a joint belief sequence, such that for every { EINBETTEN Equation.2 } and every player{ EINBETTEN Equation.2 }the strategy { EINBETTEN Equation.2 } is a best response to { EINBETTEN Equation.2 }.

A learning process { EINBETTEN Equation.2 }is a *fictitious play (FP)* process, if for every player { EINBETTEN Equation.2 }, and for every { EINBETTEN Equation.2 }, { EINBETTEN Equation.2 }, (here { EINBETTEN Equation.2 }is a point in { EINBETTEN Equation.2 }).

Note that in a *FP* process { EINBETTEN Equation.2 } for all { EINBETTEN Equation.2 }. We denote by { EINBETTEN Equation.2 }the identical belief of all the players about player { EINBETTEN Equation.2 } strategy at stage { EINBETTEN Equation.2 } .

A *FP* process { EINBETTEN Equation.2 }*approaches equilibrium*, if for every { EINBETTEN Equation.2 } there exist { EINBETTEN Equation.2 }, such that for every { EINBETTEN Equation.2 }, there exists a mixed equilibrium profile { EINBETTEN Equation.2 }, such that, { EINBETTEN Equation.2 } .

We say that a game has the *FP property* , if every *FP* process, independent of initial actions and beliefs, approaches equilibrium.

A learning process { EINBETTEN Equation.2 } is a *joint fictitious play* (*JFP*) process, if for every player { EINBETTEN Equation.2 }, { EINBETTEN Equation.2 } (here { EINBETTEN Equation.2 }is a point in { EINBETTEN Equation.2 }).

Note that in a *JFP* process, for every two players { EINBETTEN Equation.2 }, and for all { EINBETTEN Equation.2 }, { EINBETTEN Equation.2 }. We denote by { EINBETTEN Equation.2 }the identical belief of all the players about player { EINBETTEN Equation.2 }’s strategy at stage { EINBETTEN Equation.2 }.

In two person games, the *FP* and *JFP* processes coincide. It can be shown that in general these processes do not coincide.

4. The Model

{ EINBETTEN Equation.2 } is the set of players. { EINBETTEN Equation.2 } is the finite strategy set of player { EINBETTEN Equation.2 }, and { EINBETTEN Equation.2 }. { EINBETTEN Equation.2 }, is a two-person game (the players are player 0 and player { EINBETTEN Equation.2 }). The payoff functions of the players in { EINBETTEN Equation.2 } are : { EINBETTEN Equation.2 }.

We define an { EINBETTEN Equation.2 } player game { EINBETTEN Equation.2 } as follows:

The payoff of player 0 is defined as the sum of his payoffs in the games { EINBETTEN Equation.2 } , that is,

{ EINBETTEN Equation.2 }

Where { EINBETTEN Equation.2 } is player { EINBETTEN Equation.2 }'s strategy in the joint strategy { EINBETTEN Equation.2 }.

The payoff of player { EINBETTEN Equation.2 }, depends only on his strategy and player 0's strategy in { EINBETTEN Equation.2 }, that is,

{ EINBETTEN Equation.2 }.

The game { EINBETTEN Equation.2 }will be called the *compound game.*

At each stage of a learning process, each of the players in { EINBETTEN Equation.2 } may face a tie problem between several best replies. We assume a complete order on each of the strategy sets { EINBETTEN Equation.2 }, and the following tie-breaking rule :

<u>Assumption (tie breaking rule)</u>: If a player plays according to the *FP* or *JFP* process and he is indifferent among some best replies, he chooses the largest one according to the order on his strategy set.

The compound game { EINBETTEN Equation.2 }will be associated with a two player game { EINBETTEN Equation.2 } that will be called the *reduced game* of { EINBETTEN Equation.2 }. The players in the reduced game will be denoted by 0 (the row player) and by A (the column player).

The strategy set of player 0 in { EINBETTEN Equation.2 } is { EINBETTEN Equation.2 }. That is, player 0 in the compound game { EINBETTEN Equation.2 } and player 0 in its reduced game { EINBETTEN Equation.2 } have the same strategy set.

The strategy set of player { EINBETTEN Equation.2 } is { EINBETTEN Equation.2 }, where { EINBETTEN Equation.2 }.

The payoff function of player 0 in { EINBETTEN Equation.2 } is :

{ *EINBETTEN Equation.2* }.

That is, the payoffs of players 0 are the same in the compound game and in its reduced game. In fact we can say that the same player 0 plays in both games.

The payoff function of player A is :

{ EINBETTEN Equation.2 } .

That is, the payoff of player{ EINBETTEN Equation.2 } in the reduced game is equal to the sum of all the players' (except player 0) payoffs in its compound game.

We assume the following tie-breaking rule for the *FP* process in the reduced game :

Assumption (tie breaking rule): If player{ EINBETTEN Equation.2 } plays according to the *FP* process in a reduced game, and he is indifferent among some best replies, he chooses the strategy { EINBETTEN Equation.2 }, such that for every { EINBETTEN Equation.2 }, { EINBETTEN Equation.2 }, { EINBETTEN Equation.2 } is the best response of player { EINBETTEN Equation.2 }(in the compound game), which is the largest one according to the order of player { EINBETTEN Equation.2 }'s strategy set. [1]

5. Identities

Let { EINBETTEN Equation.2 } be a *FP* process and { EINBETTEN Equation.2 } be a *JFP* process as detailed in section 2. The two processes *define the same beliefs at stage { EINBETTEN Equation.2 }*, if for every player { EINBETTEN Equation.2 }, { EINBETTEN Equation.2 }. That is, the beliefs of all the players in { EINBETTEN Equation.2 } about player { EINBETTEN Equation.2 }'s strategy at stage { EINBETTEN Equation.2 } are identical according to both processes.

Lemma 1: Let { EINBETTEN Equation.2 } be a compound game, and let { EINBETTEN Equation.2 } and { EINBETTEN Equation.2 } be *FP* and *JFP* processes in { EINBETTEN Equation.2 } respectively. If the processes define the

[1] It is always possible to choose such largest strategy because of the additive property of player *A*'s utility function.

same beliefs at { EINBETTEN Equation.2 }, then the processes define the same path in { EINBETTEN Equation.2 }, that is, { EINBETTEN Equation.2 }.

Proof: Note that if { EINBETTEN Equation.2 } and { EINBETTEN Equation.2 } are *FP* and *JFP* processes respectively, that define the same beliefs at stage t, then for every player { EINBETTEN Equation.2 }, { EINBETTEN Equation.2 }, since actually every player { EINBETTEN Equation.2 } (except player 0) plays a two player game, and in such games there is no difference between the processes.

Therefore we proceed to show that player 0 also chooses the same strategy at stage { EINBETTEN Equation.2 } in both cases.

Given a learning process, we denote by { EINBETTEN Equation.2 }, the number of stages that the strategy profile { EINBETTEN Equation.2 } occurred up to stage { EINBETTEN Equation.2 }.

The expected payoff of player 0 according to the *FP* process at stage { EINBETTEN Equation.2 } if he chooses { EINBETTEN Equation.2 } is :

{ EINBETTEN Equation.2 }.

On the other hand, the expected payoff of player 0 according to the *JFP* process at stage { EINBETTEN Equation.2 } if he chooses { EINBETTEN Equation.2 } is :

{ EINBETTEN Equation.2 }

Thus, the expected payoff of player 0 for every strategy is the same in both processes. Since player 0 chooses the best action according to his belief, which is also the largest one according to the order on his strategy set, he chooses the same strategies in both cases. That is, { EINBETTEN Equation.2 }, and this implies that { EINBETTEN Equation.2 }. ■

Let { EINBETTEN Equation.2 } be a *JFP* process defined on a compound game { EINBETTEN Equation.2 } , and { EINBETTEN Equation.2 } be a *JFP* process defined on its reduced game { EINBETTEN Equation.2 }. The processes define *identical beliefs at*
stage { EINBETTEN Equation.2 } if :

1. { EINBETTEN Equation.2 }. That is, the belief of each player { EINBETTEN Equation.2 }, in { EINBETTEN Equation.2 } about player 0’s strategy at stage {

EINBETTEN Equation.2 }, is the same as player { EINBETTEN Equation.2 }'s belief in { EINBETTEN Equation.2 } about player 0's strategy at stage { EINBETTEN Equation.2 }.

2. { EINBETTEN Equation.2 }. That is, the belief of player 0 in { EINBETTEN Equation.2 } about the joint strategy { EINBETTEN Equation.2 } of the other players at stage { EINBETTEN Equation.2 }, is the same as player 0's belief in { EINBETTEN Equation.2 } about player { EINBETTEN Equation.2 }'s strategy { EINBETTEN Equation.2 } at stage { EINBETTEN Equation.2 }.

By the construction of the reduced game we obtain :

Lemma 2: Let { EINBETTEN Equation.2 } be a *JFP* process defined on a compound game { EINBETTEN Equation.2 } , and { EINBETTEN Equation.2 } be a *JFP* process defined on its reduced game { EINBETTEN Equation.2 }. If the processes define identical beliefs at { EINBETTEN Equation.2 }, then the processes define identical beliefs for every { EINBETTEN Equation.2 } .

By lemma 1 and lemma 2 we can learn about the *FP* process in a multi-player ({ EINBETTEN Equation.2 }players) game by analyzing a simpler two player game. In particular we have the following result :

Proposition 3: A *FP* process approaches equilibrium in a compound game { EINBETTEN Equation.2 } if and only if it approaches equilibrium in its reduced game { EINBETTEN Equation.2 }.

We now use proposition 3 to demonstrate by examples that the behavior of the *fictitious play* process in the { EINBETTEN Equation.2 } sub games may not help us in analyzing its behavior in the compound game.

Example 4

{ EINBETTEN Equation.2 }

{ EINBETTEN Equation.2 } and { EINBETTEN Equation.2 } have dominated strategies. Thus, it can be verified that every *FP* process in { EINBETTEN Equation.2 } approaches the unique equilibrium of this game, namely, row 2 and column 1. Likewise, every *FP* process in { EINBETTEN Equation.2 } approaches the unique equilibrium of this game, namely, row 1 and column 1. That is, each of the games { EINBETTEN Equation.2 } has the *FP* property.

Nevertheless, by proposition 3, the compound game of { EINBETTEN Equation.2 }and { EINBETTEN Equation.2 } does not have the *FP* property, since its reduced game after elimination of weakly dominated strategies[2]
is the following game :

{ EINBETTEN Equation.2 }

This game belongs to the class of Shapley's games (Shapley (1964)), and therefore it does not have the *FP* property.

Example 5

{ EINBETTEN Equation.2 }

{ EINBETTEN Equation.2 } and { EINBETTEN Equation.2 } have the *FP* property[3]. In this case the compound game of { EINBETTEN Equation.2 } and { EINBETTEN Equation.2 } has also the *FP* property, since its reduced game is the following game:

{ EINBETTEN Equation.2 }

And this game is best response equivalent in mixed strategies to a zero-sum game, and therefore it has the *FP* property by Robinson (1951).

Example 6

{ EINBETTEN Equation.2 }

{ EINBETTEN Equation.2 }and { EINBETTEN Equation.2 } belong to the class of Shapley's games. Hence, both games don't have the *FP* property. By proposition 3 the

[2] Note that if every row occurred at least once up to stage { EINBETTEN Equation.2 }, then non of the eliminated strategies will be played from this stage on.

[3] These games are actually { EINBETTEN Equation.2 }games.

compound game of { EINBETTEN Equation.2 }and { EINBETTEN Equation.2 } does not have the *FP* property either, since its reduced game after elimination of weakly dominated strategies is the following game:

{ EINBETTEN Equation.2 }

And this game belongs to the class of shapley's game[4].

Robinson (1951) showed that every two-person zero-sum game has the *FP* property. Monderer and Shapley (1996) showed that every game with identical payoff functions[5] has the *FP* property. By these results and by proposition 3 we have the following multi-person games with the *FP* property :

Proposition 7: Let { EINBETTEN Equation.2 } be a compound, such that every { EINBETTEN Equation.2 }, is a zero sum game[6]. Then its reduced game { EINBETTEN Equation.2 } is also a zero sum game. Therefore { EINBETTEN Equation.2 } has the *FP* property.

Proposition 8: Let { EINBETTEN Equation.2 } be a compound game, such that every { EINBETTEN Equation.2 }, is a game with identical payoff functions[7]. Then its reduced game { EINBETTEN Equation.2 } is also a game with identical payoff functions. Therefore { EINBETTEN Equation.2 } has the *FP* property.

6. 2*xK* Reduced Games

Consider a compound game in which player 0 has only two possible (pure) strategies (say, low price and high price). Each of the other players has any finite number of (pure) strategies. The reduced game of such a compound game is a { EINBETTEN

[4] There is also a reversed example, in which { EINBETTEN Equation.2 }and { EINBETTEN Equation.2 } belong to the class of shapley's games, but the compound game of { EINBETTEN Equation.2 }and { EINBETTEN Equation.2 } has the *FP* property.
[5] A game with identical payoff functions is a game in which { EINBETTEN Equation.2 }for all { EINBETTEN Equation.2 }.
[6] Note that the compound game in this case is not necessarily a zero-sum game.
[7] Note that the compound game in this case is not necessarily a game with identical payoff functions.

Equation.2 } two player game. By proposition 3, a *FP* approaches equilibrium in the compound game { EINBETTEN Equation.2 } if and only if it approaches equilibrium in its { EINBETTEN Equation.2 } reduced game { EINBETTEN Equation.2 }. Miyasawa showed that every *FP* process approaches equilibrium in every 2x2 game. The case of { EINBETTEN Equation.2 } was proved by Monderer and Sela (1993) for the continuous (time) *FP* process. The general case ({ EINBETTEN Equation.2 }) is still unknown, although it seems that this class of games has the *FP* property. Here we identify a class of { EINBETTEN Equation.2 } games with the *FP* property and by this result we identify also a class of compound games with the *FP* property.

Let { EINBETTEN Equation.2 } be a compound game of { EINBETTEN Equation.2 } players, in which player 0 has only two (pure) strategies, { EINBETTEN Equation.2 }. The other players 1,2,…,n , have finite strategy sets { EINBETTEN Equation.2 }. Denote { EINBETTEN Equation.2 } and { EINBETTEN Equation.2 }.

For { EINBETTEN Equation.2 }, and for { EINBETTEN Equation.2 }denote by { EINBETTEN Equation.2 } the set of all mixed strategies { EINBETTEN Equation.2 }of player 0 such that { EINBETTEN Equation.2 } is a best reply to { EINBETTEN Equation.2 }. { EINBETTEN Equation.2 } can be identified with the closed segment : { EINBETTEN Equation.2 }.

For { EINBETTEN Equation.2 }, let { EINBETTEN Equation.2 } . Note that any two different intervals in { EINBETTEN Equation.2 } intersect at most one point. If { EINBETTEN Equation.2 }and { EINBETTEN Equation.2 }, { EINBETTEN Equation.2 }, belong to { EINBETTEN Equation.2 }and { EINBETTEN Equation.2 }, then { EINBETTEN Equation.2 }and { EINBETTEN Equation.2 } are adjacent intervals in { EINBETTEN Equation.2 } and the intersection point of these intervals is called an *overlapping point* of { EINBETTEN Equation.2 }.

There is a natural order on { EINBETTEN Equation.2 }: { EINBETTEN Equation.2 } if { EINBETTEN Equation.2 } and the left end point of { EINBETTEN Equation.2 }is greater or equal than the right end point of { EINBETTEN Equation.2 }.

For { EINBETTEN Equation.2 } we denote by { EINBETTEN Equation.2 }, and we denote { EINBETTEN Equation.2 }.

The concepts of adjacent intervals, overlapping points and order are naturally generalized from { EINBETTEN Equation.2 } to { EINBETTEN Equation.2 }.

For a subset { EINBETTEN Equation.2 } we say that { EINBETTEN Equation.2 } is a *dominant strategy* in { EINBETTEN Equation.2 } for { EINBETTEN Equation.2 } if { EINBETTEN Equation.2 }.

Two different strategies of player { EINBETTEN Equation.2 }are called *adjacent strategies* if they are dominant in adjacent intervals of { EINBETTEN Equation.2 }.

We say that { EINBETTEN Equation.2 } is a *generic compound game*, if there are no overlapping points for every { EINBETTEN Equation.2 } and { EINBETTEN Equation.2 }.{ EINBETTEN Equation.2 }

Now, we present another class of compound games with the *FP* property. We show that a compound game associated with a { EINBETTEN Equation.2 }weak weighted potential game, has the *FP* property.

Let { EINBETTEN Equation.2 } be a vector of positive numbers called *weights.*

A function { EINBETTEN Equation.2 } is a *weighted potential* for a game { EINBETTEN Equation.2 }, if for every { EINBETTEN Equation.2 } and for every { EINBETTEN Equation.2 } there exist :

{ **EINBETTEN Equation.2** }.

Where { EINBETTEN Equation.2 },{ EINBETTEN Equation.2 }, is the payoff function of player { EINBETTEN Equation.2 }.

In this case G is called a *weighted potential game*.

Monderer and Shapley (1996) proved that every weighted potential game has the *FP* property. It is easy to verify that if in a compound game { EINBETTEN Equation.2 } every { EINBETTEN Equation.2 } is a weighted potential, then its reduced game is not necessarily a weighted potential game. Thus, the above result of Monderer and Shapley (1996) is not applicable to our model.[8]

Let G be a { EINBETTEN Equation.2 } two-person game, where player 0 is the row player with payoff function { EINBETTEN Equation.2 } and player { EINBETTEN Equation.2 } is the column player with payoff function { EINBETTEN Equation.2 }.

A function { EINBETTEN Equation.2 } is a *weak weighted potential* for a game { EINBETTEN Equation.2 } if for every player { EINBETTEN Equation.2 }, there exist (positive) weights { EINBETTEN Equation.2 }such that :

{ EINBETTEN Equation.2 }.

[8] For more details concerning weighted potential games see Monderer and Shapley (1996).

{ EINBETTEN Equation.2 }, and for every two adjacent strategies of player { EINBETTEN Equation.2 }.

In this case { EINBETTEN Equation.2 } is called a *weak weighted potential game.*

We have the following result:

Proposition 10: Every weak weighted potential { EINBETTEN Equation.2 }game has the *FP* property.

Proof : See Appendix.

The implication of proposition 10 to our model is :

Proposition 11: Let { EINBETTEN Equation.2 } be a generic compound game, such that every { EINBETTEN Equation.2 } is a { EINBETTEN Equation.2 }weighted potential game[9]. Then its reduced game { EINBETTEN Equation.2 } is a weak weighted potential game, and therefore { EINBETTEN Equation.2 } has the *FP* property.

Proof : See Appendix.

6. Concluding Remarks

We study a set of multi-player games (*compound games*) which is restricted but hides important economic applications. We show that a *FP* process approaches equilibrium in some classes of compound games which are not included in any well known class of games with the *FP* property. We do so by associating with each such a game, a best response equivalent two player game, called the *reduced game*. The transformation from the set of compound games to the set of reduced games is a useful method to identify whether or not a given compound game has the *FP* property, and without this transformation the identification is intricate as was shown in Examples 4, 5 and 6. The mapping between the set of compound games and the associated set of reduced games is not a one-to-one mapping. Thus, any characterization of compound games by reduced games will not work. An analyzing along these lines is possible only if the transformation from compound games to reduced games preserves important structural properties such as in Propositions 7, 8 and 11.

[9] Note that the compound game in this case is not necessarily a weighted potential game.

The concepts of *IFP* and *JFP* coincide in our model. Usually, these two concepts do not coincide. Nevertheless, we do not know about formal convergence results that hold for one of these processes and do not hold for the other one.

7. Appendix

Proposition 10 : Every weak weighted potential { EINBETTEN Equation.2 }game has the *FP* property.

Proof : Let { EINBETTEN Equation.2 } be a { EINBETTEN Equation.2 } two player game. The players are denoted by 0 (row player) and 1 (column player). The strategy set of player { EINBETTEN Equation.2 }is denoted by { EINBETTEN Equation.2 }. The payoff function of player { EINBETTEN Equation.2 }is : { EINBETTEN Equation.2 }.

Denote by{ EINBETTEN Equation.2 } the set of all mixed strategies of player 0 in{ EINBETTEN Equation.2 } against which { EINBETTEN Equation.2 } is a best reply for player 1. Let { EINBETTEN Equation.2 }. Without loss of generality we assume that { EINBETTEN Equation.2 }, according to the natural order on { EINBETTEN Equation.2 }.

Let { EINBETTEN Equation.2 }, be the weights, and { EINBETTEN Equation.2 } is the weak weighted potential, such that :

1) { EINBETTEN Equation.2 }.

2) { EINBETTEN Equation.2 }.

A game which is best response equivalent in mixed strategies to a game with identical payoff functions is called a *game with identical interests.* As was shown by Monderer and Shapley (1996) every such game must have the *FP* property. Thus we proceed to show that { EINBETTEN Equation.2 } is a game with identical interests. In order to show that, it is enough to show that there exists a function { EINBETTEN Equation.2 } such that :

(1) { EINBETTEN Equation.2 },

{ EINBETTEN Equation.2 }.

(2) { EINBETTEN Equation.2 }

{ EINBETTEN Equation.2 }.

Where { EINBETTEN Equation.2 }, and { EINBETTEN Equation.2 }is defined similarly.

This function { EINBETTEN Equation.2 } is called the *potential* of the game. We will show that the weak weighted potential { EINBETTEN Equation.2 } is actually the potential of the game. That is, { EINBETTEN Equation.2 }satisfies the above conditions (1) + (2).

By the definition of { EINBETTEN Equation.2 } and the linearity of { EINBETTEN Equation.2 }, for all { EINBETTEN Equation.2 }:

{ EINBETTEN Equation.2 }.

Consequently condition (1) holds.

Suppose that { EINBETTEN Equation.2 } is a best response to { EINBETTEN Equation.2 }, that is, { EINBETTEN Equation.2 }.

We will show that { EINBETTEN Equation.2 }.

Given any { EINBETTEN Equation.2 }, by the definition of the weighted potential game, we obtain :

(3) { EINBETTEN Equation.2 }

Because the best response structure of { EINBETTEN Equation.2 } games, if { EINBETTEN Equation.2 } is a best response to { EINBETTEN Equation.2 }, then for every { EINBETTEN Equation.2 },[10] and { EINBETTEN Equation.2 },{ EINBETTEN Equation.2 }.

Hence, all the terms in the above sums of equation (3) are positive, and therefore { EINBETTEN Equation.2 }. That is, condition (2) holds.[11] ■

<u>Proposition 11</u>: Let { EINBETTEN Equation.2 }be a generic compound game, such that every { EINBETTEN Equation.2 } is a { EINBETTEN Equation.2 }weighted potential game. Then its reduced game { EINBETTEN Equation.2 } is a weak weighted potential game, and therefore { EINBETTEN Equation.2 } has the *FP* property.

---

[10] The same argument holds if { EINBETTEN Equation.2 }.
[11] The "only if" statement follows by the same argument.

Proof : Without loss of generality we denote by { EINBETTEN Equation.2 }, the weights of the game { EINBETTEN Equation.2 }, and { EINBETTEN Equation.2 } is the weighted potential of { EINBETTEN Equation.2 }.

Define a function { EINBETTEN Equation.2 }, where { EINBETTEN Equation.2 } and { EINBETTEN Equation.2 } are the strategy sets of player 0 and player{ EINBETTEN Equation.2 } respectively in { EINBETTEN Equation.2 }, such that : { EINBETTEN Equation.2 }.

We will show that { EINBETTEN Equation.2 } is a weak weighted potential of { EINBETTEN Equation.2 }.

By the definition of weighted potential we have for every { EINBETTEN Equation.2 } :

{ EINBETTEN Equation.2 }

On the other hand, we have for every { EINBETTEN Equation.2 } and for every two adjacent strategies

{ EINBETTEN Equation.2 }:

{ EINBETTEN Equation.2 }Since the game is generic, for any two adjacent strategies { EINBETTEN Equation.2 }, there exists a unique j such that { EINBETTEN Equation.2 } , and therefore we have :

{ EINBETTEN Equation.2 }where { EINBETTEN Equation.2 }.

We showed that the reduced game { EINBETTEN Equation.2 } has a weak weighted potential, and therefore it has the *FP* property. By proposition 3, its compound game { EINBETTEN Equation.2 } has also the *FP* property. ■

References

Brown, G. W. (1951). “Iterative Solution of Games by Fictitious Play”. *In Activity Analysis of Production,* Wiley, New York.

Krishna, V. (1992).“Learning in Games with Strategic Complementarities“. *Mimeo*.

Milgrom, P., and Roberts, J. (1991). “Adaptive and Sophisticated Learning in Normal Form Games’. *Games and Economic Behavior,* 3, 82-100.

Miyasawa, K. (1961). "On the Convergence of the Learning Process in 2x2 Non-Zero-sum Two-person Game". *Economic Research Program, Princeton University, Research memorandum No. 33.*

Monderer, D., and Sela, A. (1996). "A 2x2 Game Without the Fictitious Play Property". *Games and Economic Behavior*, 14, 144-148.

Monderer, D., and Sela, A. (1993) "Fictitious Play and No-Cycling Conditions". *Mimeo.*

Monderer, D., and Shapley, L. S. (1996). "Fictitious Play Property for Games with Identical Interests". *Journal of Economic Theory.*

Robinson, J. (1951). "An Iterative Method of Solving a Game". *Annals of Mathematics,* 54, 296-301.

Shapley, L. S. (1964). "Some Topics in Two-Person Games. In *Advances in Game Theory* eds. By Dresher, L.S. Shapley, and A.W Tucker, 1-28, Princeton U. Press.