Essays on Collateral Constraints

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Finally, I would like to turn to a concern that I have discussed a lot since studying economics. Given my great passion for the subject, I also want to cultivate some skepticism:
There are rarely definite answers to economic questions.
Our preliminary answers are both better and worse than no answers at all.
Better, if we are aware of their limitations; worse, if not.

Contents

Chapter 1

General Introduction

Your banker says that you can buy a house worth half a million, yet you have only one hundred thousand. He lends the remaining four hundred thousand to you and accepts the house as collateral. As everybody knows, this is the usual way to buy a house. It is also well known that the widespread practice of such collateralized borrowing can create enormous momentum at the aggregate level. Recently, the United States, Spain and Ireland experienced spectacular housing booms followed by busts with severe economic repercussions. At the heart of such dynamics lies a simple and powerful mechanism. With rising house prices, the wealth and borrowing capacity of households also rise. As a consequence, they can bid up prices even further, which reinforces the housing boom. The dynamics are even more dramatic when prices move the other way. Suppose prices start to fall. Then, homeowners can borrow less with their homes as collateral. Some are no longer able to renew their mortgages, thus they have to sell their houses. This reduces the price of housing further, which leads to more sales, ending in a crash.

Clearly, mortgages are the most prominent type of collateralized borrowing. However, houses are by no means the only assets used as collateral, and consumers are not the only economic agents that engage in collateralized borrowing. Producers use buildings, vehicles and machinery as collateral, while investors use bonds, stocks and commodities. In my dissertation, I model producers and investors who borrow on collateral. In both cases, I find that dynamics similar to the case of a housing bust may occur: A drop in the price of the collateral asset forces collateral constrained agents to sell these assets, which reduces the price further, thus forces additional sales, and thereby causes severe aggregate effects.

In *Chapter 2* of this dissertation, I study the impact of collateral constraints on *producers* and thereby on *aggregate output*. In a theoretical paper, Kiyotaki and Moore (1997) show that collateral constraints that restrict the investment decisions of producers can strongly amplify and propagate aggregate shocks. However, the subsequent quantitative literature tends to find rather weak and non-robust effects of collateral constraints. I try to improve on this by modeling the interaction between idiosyncratic risk and collateral constraints. To this aim, agents' productivities as workers and entrepreneurs are assumed to evolve stochastically. This leads to a perpetual mismatch between wealth and skills, which is the reason for collateralized borrowing in this economy. The advantage of this modeling strategy is threefold. First, the evolution of skills can be measured empirically. In contrast, the heterogeneity in patience that the previous literature assumes to excite collateralized borrowing is not even intended as a serious micro-foundation. Second, idiosyncratic risk creates a non-degenerate distribution of wealth. As a consequence, the percentage of constrained agents changes as shocks hit the economy. Among other things, this generates recessions that are much sharper than booms. Last but not least, the impact of collateral constraints turns out to be larger and more robust in the setup with idiosyncratic risk compared to models with heterogeneity in patience.

In the baseline calibration, an unanticipated shock that reduces total factor productivity (TFP) by one percent for only one year leads to a sharp and persistent drop in both output and the price of capital. One period after the shock, output is down by almost two percent and the price of capital by more than seven percent. Four years after the shock, the price of capital is still depressed by some four percent, and it takes about eight years until the economy roughly returns to its steady state. In contrast, without financing frictions the model economy is back at the steady state within one year. Thus, collateral constraints have a substantial impact on the aggregate economy. At the center of this impact is a reallocation of capital: When the shock hits, the wealth of constrained entrepreneurs falls by almost 30 percent, mainly due to the fall in the price of capital. This makes them reduce their capital holdings by nearly 25 percent. Conversely, unconstrained entrepreneurs increase their capital holdings to take advantage of the low prices. However, they operate at lower marginal returns. Thus, capital is now used less efficiently, which implies lower aggregate output. On top of that story and in contrast to the previous literature, the dynamics in my model are also driven by an increase in the number of agents that are constrained. It goes up by more than 35 percent on impact, and after four years it is still 10 percent above the steady state level.

In *Chapter 3*, which is joint work with Michael Grill, Felix Kübler and Karl Schmedders, we study the impact of collateral constraints on *investors* and thereby on *asset prices*. We consider an exchange economy with long-lived assets that pay risky dividends and can be used as collateral for short-term loans. Investors are heterogeneous with respect to risk-aversion. As a consequence, risk-tolerant agents invest in the risky assets and use them as collateral to borrow from the risk-averse. This investment behavior makes the risk-tolerant agents vulnerable to bad shocks in which case binding collateral constraints force them to sell some of their risky assets. The risk-averse buy these assets only after their price has dropped substantially. For this reason, collateral constraints increase the volatility of asset returns by 50 percent as compared to a benchmark where borrowing is not possible. We show that the high volatility does not substantially decrease when we allow for defaultable bonds. We also find that such bonds are only traded if the costs of default are small.
The main focus of our analysis is on the case of two risky assets, where the collateral requirement is determined endogenously for one, but exogenously for the other. In particular, we assume that a regulating agency sets an exogenous margin requirement for the second asset. We find that this regulation has a strong impact on the volatility of the first asset. In particular, a tightening of margin requirements for the regulated asset uniformly decreases volatility of the unregulated asset. For the regulated asset, tighter margins initially increase the return volatility, but then decrease it once margins become very large. Moreover, we find that regulation can substantially decrease volatility, if it is strict in booms and loose in recessions. This result holds true both for the model with a single asset as well as the two-asset economy and suggests a strong policy recommendation for counter-cyclical margin requirements.
In the calibration of the model, we allow for the possibility of disaster states as in Barro and Jin (2011). This leads to very large quantitative effects of collateral requirements and to realistic equity risk premia.

Summing up, Chapters 2 and 3 show within very different frameworks that collateral constraints can have a big impact on aggregate dynamics. In Table 1.1, I summarize the main properties of the two models and thus show the various differences.
What the two models have in common is that they require sophisticated numerical procedures to compute equilibria. In both cases, these procedures include a time iteration algorithm: The period-to-period equilibrium conditions are solved backwards in time, i.e. given today's state of the economy and also agents' expectations concerning what happens next period, the algorithm solves for their optimal choices today. To go back one more period in time, one takes the computed choices of this period as an input, which means that they need to be interpolated. At this stage, collateral constraints cause problems, as they typically imply that choices, also called policies, have kinks at locations that are a priori unknown. This is because a collateral constraint that becomes binding forces the marginal choice of agents to change abruptly. Where this happens, the policy function exhibits a kink, or mathematically speaking, a non-differentiability.

In *Chapter 4*, which is joint work with Michael Grill, we propose a method that overcomes the above described numerical problem caused by collateral constraints. It applies not only to collateral constraints, but to any type of occasionally binding constraint. Moreover, the method also works when policy functions need to be interpolated over several dimensions. The method uses the equilibrium conditions of the model to numerically locate the kinks in policy functions and then adds interpolation nodes exactly there. To handle the resulting non-uniform grid of interpolation nodes, it uses simplicial interpolation, which is piecewise-linear interpolation on efficiently chosen simplices (i.e. triangles in two dimensions, thetrahedra in three). Therefore, we call this method Adaptive Simplicial Interpolation (ASI). To evaluate its performance, we embed ASI into a time iteration algorithm. We then compute recursive equilibria in an infinite horizon endowment economy where heterogeneous agents trade in a bond and a stock subject to various trading constraints. A careful analysis shows that ASI computes equilibria accurately and outperforms standard interpolation schemes by far.

chapter	2	3
title	Collateral Constraints, Idiosyncratic Risk, and Aggregate Fluctuations	Collateral Requirements and Asset Prices
agents		
heterogeneity	heterogeneous skills: workers and entrepreneurs	heterogeneous preferences: risk-averse and risk-tolerant
who is borrowing	entrepreneurs	risk-tolerant investors
reason for borrowing	good business opportunities	risky investment opportunities
assets		
collateral asset(s)	one type of capital	two dividend paying assets
supply of collateral assets	positive and subject to adjustment costs (fixed in baseline)	positive and fixed
net supply of bonds	zero	zero
default in equilibrium	no default	default
interest rate	endogenous	endogenous
collateral constraints	endogenous	endogenous and exogenous
dynamics		
idiosyncratic risk	shocks to individual skills	none
aggregate risk	unanticipated shocks to TFP	anticipated shocks to output
main variables studied	prices and output	prices
reason why prices fall in bad times	constrained entrepreneurs with high marginal returns have to sell capital	constrained investors with low risk-aversion have to sell risky assets
impact of collateral constraints	strong and persistent	strong and persistent

Table 1.1: Overview and Comparison of the Models in Chapters 2 and 3

Chapter 2

Collateral Constraints, Idiosyncratic Risk, and Aggregate Fluctuations

2.1 Introduction

In an economy where loans have to be collateralized by productive assets, shocks can be amplified strongly through their impact on asset prices. In a nutshell, this works as follows. Suppose a shock reduces the productivity or the price of productive assets. This hits those entrepreneurs most who heavily borrowed using these assets as collateral. If they already hold as much debt as the value of their collateral allows, they are forced to reduce their debt and sell assets to those agents who are not collateral constrained. As the unconstrained operate at lower marginal returns, this implies lower aggregate output. It also leads to a further reduction in asset prices, thus reinforcing the initial impact. Moreover, as constrained entrepreneurs invest less today they are worse off in the future, which reduces future asset prices. This in turn reinforces the drop in today's prices.

Kiyotaki and Moore (1997) brilliantly present this mechanism in a very stylized model. There are two types of risk-neutral agents, patient and impatient. The impatient agents operate linear technologies and borrow from the patient ones as much as the collateral constraint allows. Subsequent papers move towards less stylized assumptions to be better able to quantify the impact of collateral constraints. In Kocherlakota (2000) entrepreneurs are risk-averse, operate concave technologies, and lend money from foreigners. This small open economy assumption implies a constant interest rate as in Kiyotaki and Moore (1997), where this is a consequence of risk-neutrality. In contrast, Cordoba and Ripoll (2004) assume that lenders are risk-averse which makes the interest rate truly endogenous. They also endogenize labor supply. In addition, Mendicino (2008) assumes inefficiencies in debt enforcement that imply tighter collateral constraints. Finally, Pintus (2011) considers agents that differ not only in patience but also in risk-aversion and their intertemporal elasticity of substitution. With respect to the quantitative impact of collateral constraints, Cordoba and Ripoll (2004) find that amplification is sizable only for extreme parameter values. Mendicino (2008) and Pintus (2011) provide slightly more promising results, but still need some strong assumptions. Overall, there is no consensus in the literature and the agnostic conclusion of Kocherlakota (2000) is still up to date: "This sets up a challenge for future work: to demonstrate, in a carefully calibrated model environment, that the amplification and propagation possible by credit constraints are quantitatively significant."

This paper presents a calibrated model that departs from the assumption that agents differ in discount factors. Instead, it assumes a continuum of agents who face idiosyncratic risk concerning their productivity as workers *and* entrepreneurs. In such a setup, the reason for borrowing is that idiosyncratic shocks create a perpetual mismatch between wealth and the skills needed to make productive use of it. Clearly, wealth and talent do not always go hand in hand in the real world. Moreover, from the perspective of economic modeling, replacing heterogeneous preferences by idiosyncratic risk has the following three advantages. First of all, the process driving the evolution of skills can be empirically measured. Specifically, I build on Quadrini (2000) and Bohácek (2006) when calibrating the process governing entrepreneurial skills. In contrast, *extreme* differences in patience are not even intended as a serious micro-foundation, as Kiyotaki and Moore (1997) admit: "A weakness of our model is that it provides no analysis of who becomes credit constrained, and when."
Second, thanks to a non-degenerate distribution of wealth, the percentage of constrained agents changes as shocks hit the economy. As a consequence, the relation between the magnitude of an impulse and the size of the response is markedly non-linear. In particular, a negative shock depresses the economy by more than a positive shock of the same magnitude boosts it. This may help explain why recessions are sharper than booms.
Third and most importantly, it turns out that amplification and propagation are stronger and more robust in this model than in models where two types of agents differ in discount

factors. The grounds for this surprising result are that in the latter models *all* borrowers are constrained and they are all *extremely* impatient. This necessitates a dramatic drop in the demand for credit in case of a negative shock, which leads to a sharp decline in the interest rate and thus mitigates the impact of collateral constraints. In contrast, the interest rate reacts only moderately in my model, as two counteracting effects are at work: Unconstrained agents take up additional credit and the constrained agents react to the shock by consuming much less, as they are not overly impatient.

In setting up the model, I follow three objectives. First, idiosyncratic risk shall be included in a reasonable way. Second, the model shall allow for an acceptable calibration. Finally, the model shall be as close to the previous literature on collateral constraints as the first two objectives admit. The details of the model are as follows. There is a continuum of infinitely lived agents with homogeneous preferences of the Greenwood-Hercowitz-Huffmann (GHH) type. They have heterogeneous skills which follow i.i.d. Markov chains, so that some agents are entrepreneurs, while others are workers, either low skilled or high skilled. Workers save due to a precautionary motive while entrepreneurs take up debt to buy capital. They employ their capital in agent-specific linear technologies to create differentiated intermediate goods, which are then combined with labor to produce the final output good. As the intermediate goods are imperfect substitutes for each other, the price of a good falls as its supply rises, and therefore entrepreneurs face decreasing returns to their investment in capital. As a consequence, poor entrepreneurs want to exploit their high marginal returns and borrow as much as the collateral constraint allows. The final good is used for consumption as well as investment in capital. To render movements in the price of capital possible, aggregate investment is assumed to be subject to convex adjustment costs. However, to stay in line with most of the previous literature, the baseline calibration has a fixed capital stock. This assumption seems to be an acceptable shortcut, since there is evidence that capital is supplied quite inelastically in the short run. Turning to financial markets, agents trade in two assets: Physical capital and risk-free debt, which is in zero net supply. Agents may only borrow if they hold enough capital as collateral. More precisely, the borrowing limit that each agent faces is proportionate to the current value of his capital holdings.

The numerical experiment consists of a one-time shock to total factor productivity (TFP), which is not anticipated. This is exactly the same as in the previous literature. Because of the idiosyncratic risk, I have to resort to global solution techniques as opposed to local methods like log-linearization. As a starting point, I solve for a stationary equilibrium, where individual variables are stochastic, but aggregates are constant. As markets are incomplete and capital is held by individual entrepreneurs, the interest rate is not equal to the marginal product of capital. Therefore, it does not suffice to determine the interest rate as in Aiyagari (1994), but output has to be computed at the same time.

When the unanticipated shock hits, the economy drops out of the stationary equilibrium, to which it subsequently converges back. To capture these dynamics, the transition paths of output, interest rate and price of capital have to be computed simultaneously. In the scenario where TFP falls by one percent for one period, output and the price of capital are reduced strongly and persistently. One period after the one-time shock, output is down by almost two percent and the price of capital by more than seven percent. Four years after the shock, the price of capital is still depressed by some four percent, and it takes about eight years until the economy roughly returns to its steady state. In contrast, without financing frictions the economy is back at the steady state within one year. Thus, collateral constraints have a substantial impact on the aggregate economy. I carefully quantify the mechanism creating this impact. When the shock hits, the wealth of constrained agents falls by almost 30 percent, which makes them reduce their capital holdings by nearly 25 percent. Consequently, capital is allocated less efficiently, which drags down output both directly and indirectly through its impact on labor supply.

In contrast to the previous literature, the dynamics are also driven by a change in the percentage of constrained agents. It goes up by more than 35 percent on impact, and after four years it is still 10 percent above the steady state level. This feature of the model implies that the relation between the size of a shock and its impact is far from linear. For instance, if the size of a negative shock is doubled, the number of constrained agents increases considerably. Thus, there are now more agents that are strongly affected by the shock which in addition is twice as severe. Consequently, the impact on the aggregate economy is much more than twice as large. This demonstrates that modeling idiosyncratic risk can reveal previously unexplored implications of collateral constraints.

In a series of papers, Moll (2010), Buera and Shin (2010), and Buera and Moll (2011) also analyze models where entrepreneurs are subject to productivity risk *and* collateral constraints. Following Angeletos (2007), they assume that entrepreneurs face constant returns to scale. As a consequence, each individual entrepreneur is either constrained or does not produce at all, depending only on his productivity. Thus, *all* agents that engage in collateralized borrowing are constrained—as in models with heterogeneous patience. While this is a drawback for quantitative analysis, it implies that individual decisions nicely aggregate, which makes strong theoretical results possible. One such result of Buera and Moll (2011) is that models with heterogeneous patience correspond to representative agents models which mainly feature investment wedges. Interestingly, Chari, Kehoe, and McGratten (2007) show that such investment wedges do not account for a large part of the aggregate fluctuations found in US data, but efficiency wedges do. This gives additional support to my paper, which generates and carefully measures endogenous movements in efficiency that are caused by collateral constraints.

The rest of this paper is organized as follows. Section 2.2 presents the model and derives some analytic results. Section 2.3 explains the numerical approach, describes the calibration, studies the steady state, analyzes the response to the shock, and finally carries out a sensitivity analysis. Section 2.4 concludes.

2.2 Model

Time is discrete and infinite. The economy is populated by a continuum of infinitely lived agents with homogeneous preferences and heterogeneous skills. While there is no aggregate risk, skills are subject to idiosyncratic risk. Depending on their skills, some agents are workers while others are entrepreneurs. The latter use capital to produce intermediate goods, which are then combined with labor to produce the final output good. This good is used for consumption as well as investment in capital, which is subject to convex adjustment costs. Financial markets are incomplete: agents trade in capital and debt subject to collateral constraints. The details are as follows.

2.2.1 Preferences and Skills

Agents have homogeneous preferences: the discount factor, β, the risk aversion parameter, γ, and the elasticity of labor supply, $1/\theta$, are all equal across agents. Each agent $i \in [0,1]$ maximizes expected lifetime utility given by

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \frac{1}{1-\gamma}\left(\left(c_{i,t} - \frac{{h_{i,t}}^{1+\theta}}{1+\theta}\right)^{1-\gamma} - 1\right)\right],$$

where $c_{i,t}$ and $h_{i,t}$ are consumption and hours worked of agent i at time t. Preferences are assumed to be of the above GHH type, because this excludes any wealth effect on the choice of hours worked (see Greenwood, Hercowitz, and Huffman (1988)). As a consequence, aggregate labor supply does not depend on the distribution of wealth among agents (see Lemma 2.1).

Agents differ with respect to their labor productivity, $a^w_{i,t} \in A^w$, as well as their entrepreneurial productivity, $a^e_{i,t} \in A^e$. Agent's types, $s_{i,t} = \left(a^w_{i,t}, a^e_{i,t}\right) \in S$, follow Markov processes that are independent and identically distributed across agents.[1] Their transition probabilities are denoted by $M(s_{i,t+1}, s_{i,t})$. The distribution over types is assumed to be stationary, and the respective measure over types is denoted by μ. At time t agent i

[1] As pointed out by Judd (1985), a continuum of independent random variables causes problems for Lebesgue integration. Therefore, I follow Uhlig (1996) and use the L^2-Riemann integral when integrating over a continuum of agents who face idiosyncratic risk.

supplies $a^w_{i,t}h_{i,t}$ units of labor to the market. If $a^e_{i,t} > 0$, then agent i also runs a business as explained below. Therefore, he is called an entrepreneur. All other agents are called worker.

2.2.2 Production

Each entrepreneur may invest to produce a differentiated intermediate good. The amount produced,

$$y_{i,t} = f(a^e_{i,t-1}, a^e_{i,t})k_{i,t},$$

is linear in the capital invested, $k_{i,t}$, which is chosen in period $t-1$. Through the function f, production also depends on the entrepreneurial productivity in the period of investment as well as the period of production. I assume that $f(0,\cdot) = 0$, i.e. workers, which have $a^e_{i,t-1} = 0$, do not have an investment opportunity in $t-1$.
Final output, Y_t, is produced competitively from intermediate goods and labor[2]:

$$Y_t = A_t \left(\int y^{\phi}_{i,t} di \right)^{\alpha/\phi} L^{1-\alpha}_t,$$

where A_t is total factor productivity (TFP) and L_t is aggregate labor given by

$$L_t = \int a^w_{i,t} h_{i,t} di.$$

The elasticity of substitution between intermediate goods in final good production is $1/(1-\phi)$. I assume $0 < \phi < 1$, i.e. intermediate goods are imperfect substitutes. Final output is used for consumption and investment:

$$Y_t = C_t + I_t.$$

Apart from labor, capital is the only productive factor in this model. This is in contrast to Kiyotaki and Moore (1997) and Kocherlakota (2000) who make the stark distinction

[2]Labor enters final good production rather than intermediate goods production, because this ensures that the labor share in aggregate output does not depend on the distribution of capital among entrepreneurs. This property is needed to establish Lemma 2.1.

between non-depreciating land and fully depreciating capital. Omitting this distinction but assuming non-trivial depreciation, the model is closer to standard quantitative macro models. However, for the collateral constraint mechanism to work, the price of collateral, i.e. of capital, has to fall in bad times. The most standard assumption that generates this reaction is convex adjustment costs as in Hayashi (1982). In particular, I assume quadratic costs as in Lorenzoni and Walentin (2007). There is a competitive capital production sector which produces K_{t+1} units of new capital from combining K_t units of old capital with an amount of investment given by[3]

$$I_t(K_t, K_{t+1}) = K_{t+1} - (1-\delta)K_t + \frac{\xi}{2}\frac{(K_{t+1}-K_t)^2}{K_t},$$

where $\delta \in [0,1]$ is depreciation and $\xi \in [0,+\infty]$ parameterizes adjustment costs.[4] This specification includes two important special cases: For $\xi = 0$, standard neoclassical capital accumulation; for $\xi = +\infty$, a fixed capital stock.

2.2.3 Markets

The consumption good is traded at the normalized price of one. The price of the intermediate good produced by agent i at time t is denoted by $\pi_{i,t}$. The price of old capital, k_t, sold to the capital production sector at t is p_t, the price charged for new capital, k_{t+1}, is q_t. Agents may take up debt, $d_{i,t+1}$, in which case they have to repay $d_{i,t+1}R_{t+1}$ at $t+1$. Thus, the budget constraint (BC) faced by each agent is given by:

$$\text{BC:} \qquad c_t + q_t k_{t+1} - d_{t+1} \leq \pi_t k_t f(a^e_{t-1}, a^e_t) + p_t k_t - d_t R_t + w_t h_t a^w_t.$$

[3]Alternatively, one could make the adjustment costs proportional to $(K_{t+1} - K_t(1-\delta))^2$ rather than $(K_{t+1} - K_t)^2$. While both specifications are intuitive, the latter is technically more convenient. First, it makes the steady state independent of the parameter ξ. Second, the extreme case of $\xi = +\infty$ corresponds to a fixed capital stock, rather than a capital stock that decays by the rate of depreciation.

[4]In Lorenzoni and Walentin (2007) each individual entrepreneur faces an adjustment costs function as specified above. They also assume that old capital may be traded among entrepreneurs before investments are made. With this assumption, all entrepreneurs choose the same ratio of old capital to new capital and thus face the same shadow cost of new capital. For this reason, the specification in Lorenzoni and Walentin (2007) is equivalent to my assumption, i.e. a competitive capital production sector which buys all old capital and sells new capital. I choose to assume convex adjustment costs at the *aggregate* level, because empirical studies show that convex adjustment costs are consistent with aggregate investment date, but not with firm level-data (see Bloom, Bond, and Van Reenen (2007)).

Debt is in zero net supply,

$$D_{t+1} = \int d_{i,t+1} di = 0,$$

i.e. some agents hold negative debt, thus lending money to other agents. However, lenders cannot force borrowers to repay their debts unless these debts are secured by collateral assets. Therefore, they impose a collateral constraint on borrowers. More precisely, the repayment obligation of a borrower may not exceed a fraction $\kappa \in [0, 1]$ of next period's value of the capital acquired today:

$$d_{t+1} R_{t+1} \leq \kappa p_{t+1} k_{t+1}.$$

This collateral constraint might for example follow from the fact that liquidation of the collateralized assets is inefficient, i.e. in case of default lenders can recover only a fraction κ of the collateral value. Knowing this, borrowers can renegotiate and reduce the repayment obligation, if it is higher than the liquidation value. Thus, the lender does not lend more than this value in the first place. Alternatively, the collateral constraint might arise because the entrepreneur is able to divert a fraction $(1 - \kappa)$ of the capital invested. Anticipating this, lenders will not allow borrowers to take up a repayment obligation that exceeds the value of the non-divertible part of capital.

As an alternative to the above collateral constraint, I also consider the following modification:

$$\text{CC:} \qquad d_{t+1} R_{t+1} \leq \kappa p_t k_{t+1}.$$

Here, lenders (or regulators) use the current price, p_t, to assess the collateral value and set the borrowing limit accordingly. In what follows, I concentrate on this specification, but consider the first one in Section 2.3.5 as a robustness analysis.

Finally, agents face a short-sale constraint (SC) on capital:

$$\text{SC:} \qquad 0 \leq k_{t+1}.$$

2.2.4 Equilibrium

I now define equilibrium for the economy described above. In this definition, agents' policies do not depend on i, but only on time t and individual characteristics, $x_t \equiv (d_t, k_t, a_t^w, a_t^e, a_{t-1}^e)$. This is exactly the kind of equilibrium I compute in Section 2.3. Consequently, the quantities ($c_t, d_{t+1}, h_t, k_{t+1}, y_t$, and π_t) in the below definition are functions that map the state of an agent, x_t, into a real number. In contrast, prices (p_t, q_t, R_{t+1} and w_t) are just real numbers. The distribution over individual characteristics is denoted by Φ_t. Recall that there is no aggregate risk, i.e. expectations are over idiosyncratic shocks only.

Definition 2.1. (Competitive Equilibrium)
A competitive equilibrium of the economy $\langle \beta, \gamma, \theta, S, M, f, \{A_t\}, \alpha, \phi, \delta, \xi, \kappa, \Phi_0 \rangle$ is a sequence of quantities, prices and distributions[5]

$$\{c_t, d_{t+1}, h_t, k_{t+1}, y_t, Y_t, p_t, q_t, R_{t+1}, w_t, \pi_t, \Phi_t\}_{t \in \mathbb{N}}$$

such that:

- *Given prices $\{p_t, q_t, R_{t+1}, w_t, \pi_t\}_{t \in \mathbb{N}}$,*
  agents choose quantities $\{c_t, d_{t+1}, h_t, k_{t+1}\}_{t \in \mathbb{N}}$ to maximize

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \frac{1}{1-\gamma}\left(\left(c_t - \frac{h_t^{1+\theta}}{1+\theta}\right)^{1-\gamma} - 1\right)\right],$$

  subject to BC, CC, and SC.

- *For all $t \in \mathbb{N}$, given prices (π_t, w_t), the final output firm chooses inputs (y_t, L_t) to maximize profits*

$$A_t \left(\int y_t(x)^\phi d\Phi_t(x)\right)^{\alpha/\phi} L_t^{1-\alpha} - \int \pi_t(x) y_t(x) d\Phi_t(x) - w_t L_t.$$

- *For all $t \in \mathbb{N}$, given prices (q_t, p_t), the capital production firm chooses inputs (I_t, K_t)*

[5] To streamline the exposition, I omit to state requirements on the measurability of the policy functions. Note also that I use the definition $\mathbb{N} = \{0, 1, \dots\}$.

to maximize profits

$$q_t K_{t+1}(I_t, K_t) - I_t - p_t K_t,$$

where $K_{t+1}(I_t, K_t)$ *is given by* $I_t = K_{t+1} - (1-\delta)K_t + \frac{\xi}{2}\frac{(K_{t+1}-K_t)^2}{K_t}$.

- *For all* $t \in \mathbb{N}$, *all markets clear*[6]*:*

$$\begin{aligned}
&L_t = \int a_t^w(x) h_t(x) d\Phi_t(x), \;\; Y_t = C_t + I_t, \\
&K_t = \int k_t(x) d\Phi_t(x), \;\; K_{t+1} = \int k_{t+1}(x) d\Phi_t(x), \;\; 0 = \int d_{t+1}(x) d\Phi_t(x), \\
&\forall s: \quad y_t(x) = f(a_{t-1}^e(x), a_t^e(x)) k_t(x).
\end{aligned}$$

- *For all* $t \in \mathbb{N}$, Φ_{t+1} *is generated from* Φ_t *by the exogenous Markov processes for skills and from individual policies* d_{t+1}, k_{t+1}.

In Section 2.3, I will numerically solve for competitive equilibria. To facilitate the computations, some analytical results have to be established. First of all, I make use of GHH preferences to derive labor supply and the wage as functions of aggregate output only. For this purpose, I assume that the distribution of labor productivity satisfies the following normalization:[7]

$$\int (a_{i,t}^w)^{1+1/\theta} di = 1.$$

Lemma 2.1. (Wage and Labor Supply)
Labor supply and the wage satisfy: $L_t = ((1-\alpha) Y_t)^{\frac{1}{1+\theta}}$, $w_t = ((1-\alpha) Y_t)^{\frac{\theta}{1+\theta}}$.

Proof. From the FOCs with respect to h_t and c_t:
$u_h(c_t, h_t) + u_c(c_t, h_t) w a_t^w = 0 \Leftrightarrow \left(c_t - \frac{h_t^{1+\theta}}{1+\theta}\right)^{-\gamma} (h_t^\theta - w a_t^w) = 0 \Leftrightarrow h_t = (w a_t^w)^{1/\theta}$. Conse-

[6]Note that K_{t+1} denotes new capital in period t and also old capital in period $t+1$. Hence, it has to satisfy the two respective market clearing conditions. Note also that I use Walras' Law to omit the market clearing condition for the consumption good.

[7]This normalization ensures that aggregate labor supply is as in an economy where all agents have homogeneous labor productivity of one (see proof to Lemma 2.1).

quently, aggregate efficiency units of labor are

$$L_t = \int a^w_{i,t}(w_t a^w_{i,t})^{1/\theta} di = w_t^{1/\theta} \int (a^w_{i,t})^{1+1/\theta} di = w_t^{1/\theta},$$

where the final step employs the normalization from above. Using this and the FOC for final good production, i.e. $w_t = (1-\alpha)\, Y_t L_t^{-1}$, the above results follow. □

Lemma 2.2. (Price of Intermediate Goods)
The price of intermediate goods $y(x_t)$ is given by: $\pi(x_t) = Z y_t(x_t)^{\phi-1}$,
where $Z \equiv \alpha\,(1-\alpha)^{\frac{(1-\alpha)\phi}{\alpha(1+\theta)}} A^{\phi/\alpha} Y^{\frac{\alpha-\phi}{\alpha}+\frac{(1-\alpha)\phi}{\alpha(1+\theta)}}$.

Proof. The first order condition of the final output firm with respect to input from firms with characteristics x_t provides:

$$\begin{aligned}
\pi(x_t) \quad &= \frac{\alpha A L^{1-\alpha}}{\phi} \left(\int y_t(x)^{\phi} d\Phi_t(x) \right)^{\frac{\alpha}{\phi}-1} \phi y_t(x_t)^{\phi-1} \\
&= \alpha A^{\phi/\alpha} Y^{\frac{\alpha-\phi}{\alpha}} L^{\frac{(1-\alpha)\phi}{\alpha}} y_t(x_t)^{\phi-1} \\
&= \alpha\,(1-\alpha)^{\frac{(1-\alpha)\phi}{\alpha(1+\theta)}} A^{\phi/\alpha} Y^{\frac{\alpha-\phi}{\alpha}+\frac{(1-\alpha)\phi}{\alpha(1+\theta)}} y_t(x_t)^{\phi-1} \\
&= Z y_t(x_t)^{\phi-1},
\end{aligned}$$

where Lemma 2.1 and the definition of Z are used. □

Lemma 2.3. (Price of Capital)
The prices of old and new capital are given by:

$$p_t = (1-\delta) + \frac{\xi}{2}\left(\left(\frac{K_{t+1}}{K_t} \right)^2 - 1 \right), \quad q_t = 1 + \xi \left(\frac{K_{t+1}}{K_t} - 1 \right).$$

Proof. These prices follow from differentiating

$$I_t(K_{t+1}, K_t) = K_{t+1} - (1-\delta)K_t + \frac{\xi}{2}\frac{(K_{t+1}-K_t)^2}{K_t}.$$

The price of old capital is given by the marginal reduction in investment due to a marginal increase of old capital, while the price of new capital is given by the marginal increase in investment needed to produce an additional marginal unit of new capital:

$$p_t = -\left(\frac{\partial I_t}{\partial K_t}\right), \quad q_t = \frac{\partial I_t}{\partial K_{t+1}}.$$

□

In Section 2.3, I consider the case of a fixed capital stock. To make this assumption a special case of the economy with convex adjustment costs, the limit for ξ going to infinity has to be considered. Being slightly imprecise, I refer to this limit as $\xi = +\infty$. In this situation, the prices of old and new capital differ exactly by δ.

Lemma 2.4. (Price of Fixed Capital)
As ξ goes to infinity, the prices of old and new capital satisfy:

$$p_t = q_t - \delta.$$

Proof. First use Lemma 2.3, then simplify, and finally take the limit:

$$\begin{aligned} p_t &= (1-\delta) + \frac{\xi}{2}\left(\left(\frac{q_t - 1}{\xi} + 1\right)^2 - 1\right) = q_t - \delta + \frac{(q_t-1)^2}{\xi}, \\ \lim_{\xi \to +\infty} p_t &= q_t - \delta + \lim_{\xi \to +\infty} \frac{(q_t-1)^2}{\xi} = q_t - \delta. \end{aligned}$$

□

Lemma 2.5. (FOCs to Individual Problem)
The FOCs to the individual optimization problem are for all $t \in \mathbb{N}$:

$$\begin{aligned} \textit{BC:} \quad & 0 = \pi_t k_t f(a^e_{t-1}, a^e_t) + p_t k_t - d_t R_t + w_t h_t a^w_t - c_t - q_t k_{t+1} + d_{t+1} \\ \textit{SC:} \quad & 0 \le k_{t+1} \ \wedge \ 0 \le \nu_t \ \wedge \ 0 = k_{t+1}\nu_t \\ \textit{CC:} \quad & 0 \le \kappa p_t k_{t+1} - d_{t+1} R_{t+t} \ \wedge \ 0 \le \mu_t \ \wedge \ 0 = (\kappa p_t k_{t+1} - d_{t+1} R_{t+1})\,\mu_t \\ k_{t+1}\textit{:} \quad & 0 = -u_c(c_t, h_t) q_t + \nu_t + \mu_t \kappa p_t + \beta \mathbb{E}\left[u_c(c_{t+1}, h_{t+1})\left(Z f^\phi k^{\phi-1}_{t+1} + p_{t+1}\right)\right] \\ d_{t+1}\textit{:} \quad & 0 = -u_c(c_t, h_t) + \mu_t R_{t+1} + \beta R_{t+1} \mathbb{E}\left[u_c(c_{t+1}, h_{t+1})\right] \\ h_t\textit{:} \quad & 0 = u_h(c_t, h_t) + u_c(c_t, h_t) w a^w_t \end{aligned}$$

Proof. Follows from differentiating the Lagrangian to the individual optimization problem and substituting the consumption Euler equation into the other equations. For the capital Euler equation, Lemma 2.2 is used. □

For the computational exercises presented later, the natural starting point is a stationary equilibrium, which is defined as follows.

Definition 2.2. (Stationary Competitive Equilibrium)
A stationary competitive equilibrium of the economy $\langle \beta, \gamma, \theta, S, M, f, \{A_t\}, \alpha, \phi, \delta, \xi, \kappa, \Phi_0 \rangle$ *is s a competitive equilibrium*

$$\{c_t, d_{t+1}, h_t, k_{t+1}, y_t, Y_t, p_t, q_t, R_{t+1}, w_t, \pi_t, \Phi_t\}_{t\in\mathbb{N}}$$

with all components being constant over time.

2.3 Results

This section presents a quantitative analysis of the model described above. The main exercise is a one-time unanticipated shock to aggregate TFP as in the previous literature on collateral constraints. Thus, the economy starts from a steady state, i.e. a stationary competitive equilibrium as in Definition 2.2. Then TFP drops for one period, which is not anticipated. Starting from that period, the economy is on a transition path back to the steady state. Along the transition path the economy is in a non-stationary competitive equilibrium as in Definition 2.1. In this section, I first describe the numerical procedures used to compute the steady state and the transition path. Then I present the baseline calibration of the model. A brief description of the steady state precedes a thorough study of the impact generated by a shock. Finally, I check how robust the results are to changes in crucial parameters.

2.3.1 Numerical Solution

The numerical procedure used to solve for the steady state builds on the pioneering work of Aiyagari (1994). With a neoclassical production function and exogenous labor supply, the only aggregate variable that Aiyagari (1994) has to determine numerically is the interest rate—because output, capital stock and the wage are all determined by the interest rate through the firm's FOCs. This is different in my model, where the distribution of capital among entrepreneurs matters for output. Consequently, I have to compute both the interest rate and output, which is done by Algorithm 1.[8] As this algorithm solves for a stationary equilibrium, there are no time-indexes in its description.

[8]To understand why Algorithm 1 iterates over the interest rate and output, consider its second step. In this step individual policy functions have to be computed given aggregate variables. From the FOCs to the individual problem (see Lemma 2.5) it is clear that the endogenous aggregate variables that influence individual choice are: $Y, p, q, R,$ and w. By Lemma 2.1, w follows from Y. From Lemma 2.3, it is obvious that the steady state prices of capital are simply: $p_t = 1 - \delta,\ \ q_t = 1$. Hence, it indeed suffices to guess R and Y in order to compute individual policies.

Algorithm 1. (Solve for Steady State)

1. *Guess aggregate variables* $\{R, Y\}$.
2. *Given* $\{R, Y\}$, *solve for individual policy functions* $\{d, k\}$.
3. *Given* $\{R, Y\}$ *and* $\{d, k\}$, *find the stationary distribution* Φ.
4. *From* $\{d, k\}$ *and* Φ, *calculate the implied* $\{\hat{D}, \hat{Y}\}$.
5. *Using* $\{\hat{D}, \hat{Y}\}$, *update the guess for* $\{R, Y\}$ *and go back to step 2.*

Concerning the implementation of Algorithm 1, a few remarks are in place. Step 2 is carried out by iterating on policy functions. To reduce the number of continuous dimensions to the individual problem, I define financial wealth:

$$\omega_t \equiv \pi_t k_t f(a_{t-1}^e, a_t^e) + p_t k_t - d_t R_t.$$

To interpolate policy functions along this dimension, I use piece-wise linear interpolation. The grid of interpolation nodes is finer at lower levels of financial wealth and it is automatically refined near the kink induced by the collateral constraint (see Brumm and Grill (2010)). To find the invariant distribution in step 3, I discretize the transition (between individual states (ω, a^w, a^e)), which is implied by $\{d, k\}$. For this purpose, I use a transition grid that is ten times finer than the interpolation grid, which results in a large transition matrix. Using standard numerical procedures, it is nevertheless possible to find its non-negative eigenvector with eigenvalue one. If the transition grid is fine enough, this eigenvector provides a good approximation to the true invariant distribution over individual states (which is continuous). In step 4, the policies from step 2 are evaluated over the distribution from step 3 to get the implied level of output and net aggregate debt. Finally, step 4 employs a linear regression approach to update the guesses, which is explained in Appendix 2.A.

When it comes to the transition path, the computational burden increases substantially for two reasons: First, as aggregate variables change along the transition path, sequences rather than steady state levels of aggregate variables have to be computed. Second, in addition to output and the interest rate, one additional aggregate variable has to be determined numerically, as the prices of capital are no longer determined by steady state conditions. I choose to guess the price of new capital, which implies the price of old capital and the evolution of the aggregate capital stock through Lemma 2.3. Accordingly, the transition path is computed as in Algorithm 2.

Algorithm 2. (Solve for Transition Path)

1. *Choose a time horizon T and guess the transition path* $\{R_t, Y_t, q_t\}_{t\in\{0,\dots,T\}}$
2. *Given* $\{R_t, Y_t, q_t\}_{t\in\{0,\dots,T\}}$*,*
   solve backwards for individual policy functions $\{d_{t+1}, k_{t+1}\}_{t\in\{0,\dots,T\}}$*.*
3. *Given* $\{R_t, Y_t, q_t\}_{t\in\{0,\dots,T\}}$ *and* $\{d_{t+1}, k_{t+1}\}_{t\in\{0,\dots,T\}}$*,*
   solve forwards for the distributions $\{\Phi_t\}_{t\in\{0,\dots,T\}}$*.*
4. *From* $\{d_{t+1}, k_{t+1}\}_{t\in\{0,\dots,T\}}$ *and* $\{\Phi_t\}_{t\in\{0,\dots,T\}}$*,*
   calculate the implied $\{\hat{D}_t, \hat{Y}_t, \hat{K}_t\}_{t\in\{1,\dots,T\}}$*.*
5. *Using* $\{\hat{D}_t, \hat{Y}_t, \hat{K}_t\}_{t\in\{1,\dots,T\}}$*,*
   update the guess for $\{R_t, Y_t, q_t\}_{t\in\{0,\dots,T\}}$ *and go back to step 2.*

Concerning step 1, T has to be chosen such that the economy indeed converges to the steady state within T years (up to the desired numerical precision). If this is not the case, T has to be increased. For steps 2, 3 and 4, the procedures used are similar to the ones used in Algorithm 1. It is step 5, that causes most problems. The guess for each of the $T \times 3$ variables (e.g. q_t) influences not only the implied value for the corresponding variable (e.g. $\hat{K}_t$), but also other concurrent variables (e.g $\hat{D}_t$), future variables (e.g $\hat{Y}_{t+1}$), and past variables (e.g $\hat{q}_{t-1}$). Because of this, it is difficult to update the $T \times 3$ guesses in a way that makes Algorithm 2 converge. The procedure I use to achieve this is explained in Appendix 2.A.

preferences:		
discount factor	β	.96
coefficient of risk aversion	γ	2
Frisch elasticity	$1/\theta$	3
skills:		
productivity types	s^l	$(0.91, 0)$
	s^h	$(1.06, 0)$
	s^e	$(1.06, 2.48)$
transition matrix	M	$\begin{pmatrix} .756 & .233 & .011 \\ .233 & .756 & .011 \\ .050 & .050 & .900 \end{pmatrix}$
production:		
capital share	α	.36
elasticity of substitution between intermediate goods	$1/(1-\phi)$	4
depreciation	δ	.08
capital adjustment costs	ξ	$+\infty$
capital markets:		
collateralizability of capital	κ	.86

Table 2.1: Baseline Calibration

2.3.2 Calibration

I calibrate the model to annual U.S. data.[9] The parameters of the model, which are reported in Table 2.1, fall into four classes, relating to preferences, skills, production, and capital markets. Concerning preferences, I pick parameter values that are standard in the literature. The discount factor, β, is set to 0.96, which matches a real interest rate of 2.0%.[10] The parameter γ is set equal to 2, which is a moderate level of risk aversion. Finally, θ is taken to be 1/3, which implies a comparatively high Frisch elasticity of 3, as in Prescott (2004). A high Frisch elasticity is helpful to generate a realistic co-movement of labor and output without assuming frictions in the labor market. It turns out that the high Frisch elasticity reinforces propagation generated by credit constraints (see Sections

[9] One important reason for choosing the period length to be one year rather than three months is as follows. The length of a period fixes the duration of debt contracts, which clearly plays a role in models with collateral constraints. A duration of one year is a much better approximation to the actual maturity structure of corporate debt than a duration of one quarter—Barclay and Smith (1995) report that more than 70% of outstanding corporate debt is due in more than one year.

[10] As debt is risk-free in this model, the model interest rate should be close to the actual risk-free rate. On the other hand, debt is the only investment opportunity for non-entrepreneurs in this economy, which speaks in favor of setting it equal to the (weighted) average of the return on risk-free and risky investments. To strike a compromise between these two lines of reasoning, I choose 2.0%, which is in between the return on risk-free and risky investments, but closer to the risk-free rate.

2.3.4 and 2.3.5), while it does not generate propagation on its own (see Appendix 2.B). With regard to skills, I assume that the Markov process for labor productivity and entrepreneurial productivity, $s = (a^w, a^e)$, has a support of three states only:

$$s^l = (a_l^w, 0), \quad s^h = (a_h^w, 0), \quad s^e = (a_h^w, a_h^e).$$

Agents with these skill levels are called: low skilled workers, high skilled workers and entrepreneurs.[11] The two levels for labor productivity, a_l^w and a_h^w, are determined by a two-state approximation to the first-order autoregression of (log of) individual labor income reported in Heaton and Lucas (1996).[12] Concerning entrepreneurs, I assume that they make up 10% of the population and that the yearly exit rate from the state of being an entrepreneur is 10%. These choices are compromises between the respective values used by Quadrini (2000) and Bohácek (2006).[13] If an entrepreneur becomes a worker, the chances of being low skilled or high skilled are equal. Likewise, the probability of becoming an entrepreneur is the same for both types of workers. Combined, all these properties uniquely determine the transition matrix between individual states, M, which is given in Table 2.1.

Concerning intermediate good production, I assume

$$f(a_{i,t-1}^e, a_{i,t}^e) = a_{i,t}^e 1_{\{a_{i,t-1}^e > 0\}} \text{ , i.e. } \quad y_{i,t} = \begin{cases} a_{i,t}^e k_{i,t} & \text{for} \quad a_{i,t-1}^e > 0 \\ 0 & \text{for} \quad a_{i,t-1}^e = 0 \text{ .} \end{cases}$$

[11] The assumption that entrepreneurs have high labor productivity is innocuous. Assuming low labor productivity for entrepreneurs does not change the results to any significant amount.

[12] Consider a discrete Markov chain for the log of labor income with states $\{-\epsilon, +\epsilon\}$ and transition probabilities $\mathbb{P}(\epsilon|\epsilon) = \mathbb{P}(-\epsilon|-\epsilon)$. I want this process to match the variance and autocovariance of the autoregressive process from Heaton and Lucas (1996), which has persistence $\rho = 0.529$ and an error term with standard deviation $\sigma = 0.251$. Thus, the discrete Markov chain has to satisfy, $\epsilon^2 = \sigma^2/(1-\rho^2)$, and $\rho\sigma^2/(1-\rho^2) = (2\mathbb{P}(\epsilon|\epsilon) - 1)\epsilon^2$, which implies $\epsilon = \sqrt{\sigma^2/(1-\rho^2)}$ and $\mathbb{P}(\epsilon|\epsilon) = (1+\rho)/2$. The states $\{-\epsilon, +\epsilon\}$ are normalized values for the log of labor income (normalized such that the two states sum up to zero). To find the corresponding values for labor productivity, (a_h^w, a_l^w), I use that individual labor income is given by $wh_t a_t^w = (wa_t^w)^{1+1/\theta}$ (see proof of Lemma 2.1). Consequently, (a_h^w, a_l^w) have to satisfy: $(a_h^w)^{1+1/\theta}/(a_l^w)^{1+1/\theta} = e^{+\epsilon}/e^{-\epsilon}$. Combining this condition with the normalizing assumption about the distribution of a^w from Section 2.2, which now reads $(a_h^w)^{1+1/\theta}/2 + (a_l^w)^{1+1/\theta}/2 = 1$, the parameters (a_h^w, a_l^w) are pinned down.

[13] In Quadrini (2000) the percentage of entrepreneurs is 12% and the yearly exit rate 18%. In Bohácek (2006) the percentage of entrepreneurs is 9% and the yearly exit rate 4.5%.

Similar to Quadrini (2000), this entrepreneurial production function implies a high correlation between output and investment opportunities, e.g. $a_t^e = 0$ implies that current output is zero and that there are also no entrepreneurial investment opportunities. As a normalization, I set $a_h^e = M(s^e|s^e)^{-1/\phi}\mu(s^e)^{(\phi-1)/\phi}$. This ensures that in the complete markets benchmark aggregate output is as in an economy where all agents are entrepreneurs with entrepreneurial productivity equal to one, and the production function collapses to $Y_t = A_t K_t L_t$ (see Appendix 2.B).

Concerning aggregate TFP and depreciation, I set $A = 1$ and $\delta = 0.08$, which matches a capital-to-output ratio of 3.0. I assume that the capital share, α, equals 0.36, as in Aiyagari (1994). The elasticity of substitution between intermediate goods is set equal to 4 (i.e. $\phi = .75$), which is well within the range of estimates from the trade and industrial organization literature (see Hsieh and Klenow (2009)).

Concerning adjustment costs, I make an extreme assumption for the baseline calibration, namely $\xi = +\infty$. This results in a fixed capital stock, as assumed in Cordoba and Ripoll (2004), Mendicino (2008) and also the basic model of Kiyotaki and Moore (1997) (where the fixed input is called land). Thus, choosing $\xi = +\infty$ makes my results comparable to many papers from the literature on collateral constraints. Additionally, from an empirical perspective, one could argue that capital is supplied almost inelastically in the short-run. The reason is that it takes on average much more than one year from the decision to invest until the completion of the investment (see Kydland and Prescott (1982)). Given that modeling this time-to-build lag is computationally burdensome, assuming infinite adjustment costs seems to be an acceptable shortcut for the question considered. In Section 2.3.5, I analyze the sensitivity of the results with respect to the adjustment cost parameter.

The only parameter of the model that relates to capital markets is κ. I follow Mendicino (2008) and set $\kappa = 0.86$. This parameter value is based on a measure of the efficiency of debt enforcement in the US which is constructed by Djankov, Hart, McLiesh, and Shleifer (2008).

2.3.3 Steady State

Before turning to the dynamics induced by a shock, I briefly describe the steady state. While individual variables move in the steady state, their distribution does not vary. Figure 2.1 plots the distribution over financial wealth for each type of agent, and also for the total

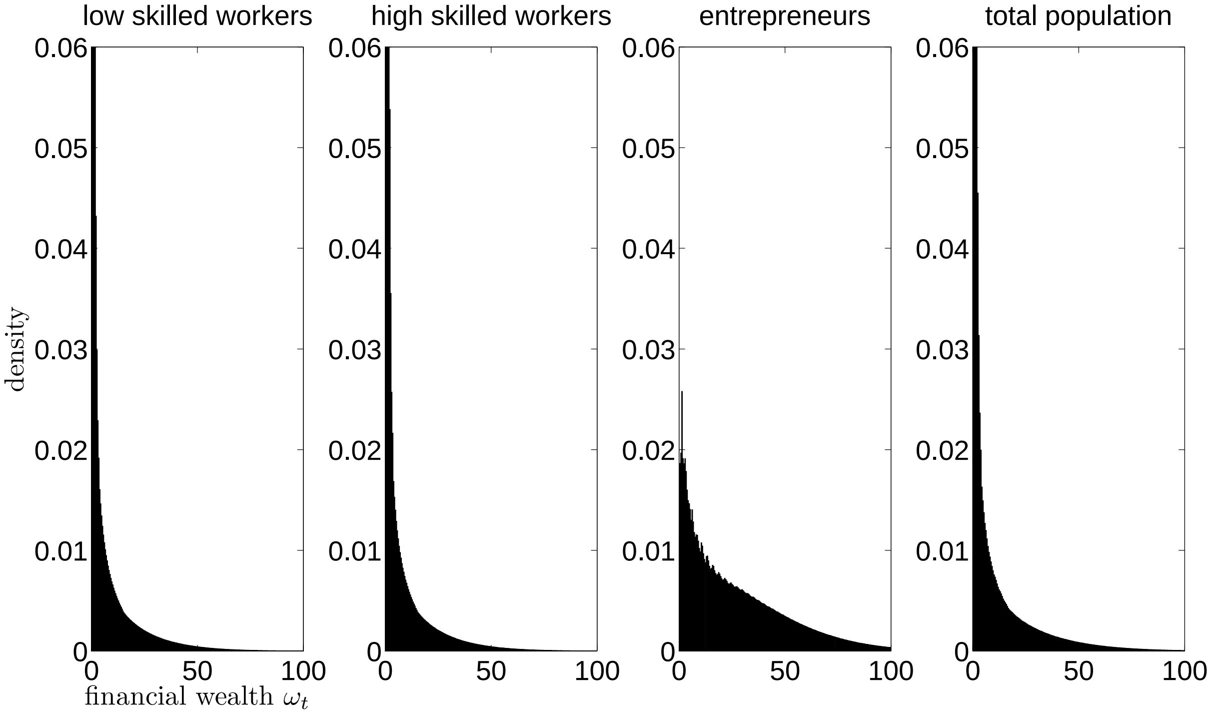

Figure 2.1: Steady State Distribution of Wealth

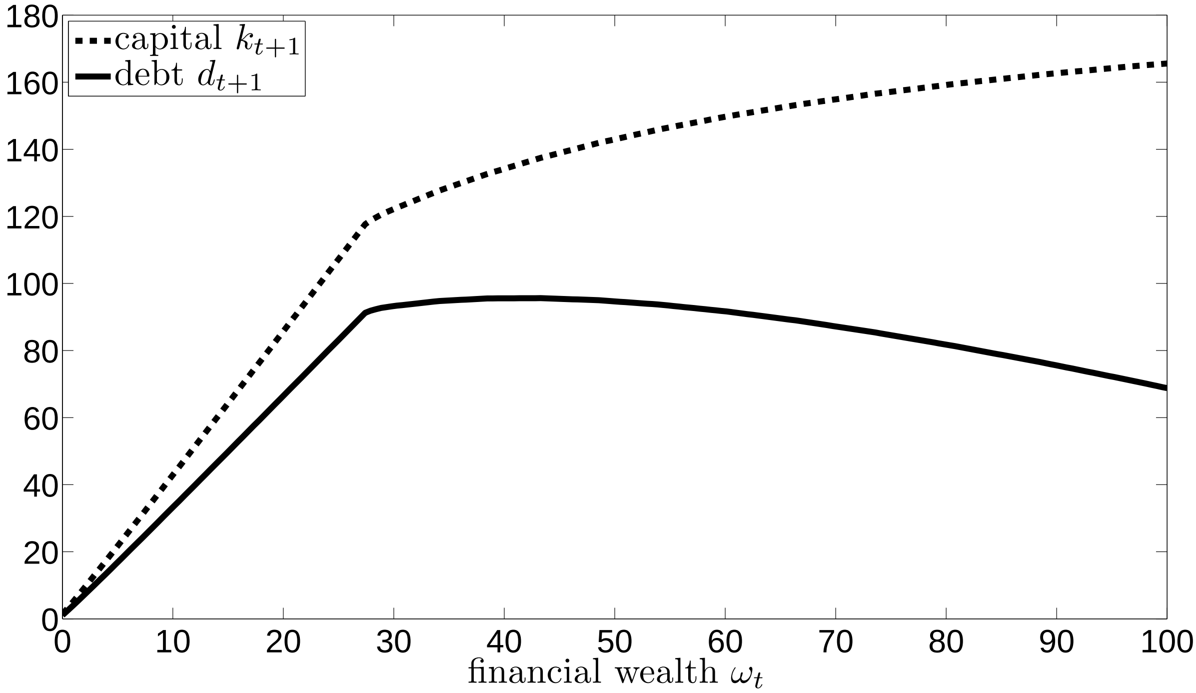

Figure 2.2: Policy Functions of Entrepreneurs

population.[14] High skilled workers are just a little richer than low skilled, but entrepreneurs are much richer than workers: The average entrepreneur has financial wealth 26.7, while the average worker has only 6.7. There are two important reasons for this large gap in wealth. First, entrepreneurs make business profits on top of labor earnings. Second, they have a stronger incentive to save, because they have high-return investment opportunities. To make use of these, they take up debt but also have to inject their own wealth as equity. This is nicely displayed in Figure 2.2, which plots the policy functions of entrepreneurs. For levels of financial wealth below 27.4, which applies to 61% of all entrepreneurs, the collateral contraint is binding. In this situation, debt is proportional to capital:

$$d_{t+1} = \frac{\kappa p_t}{R_{t+1}} k_{t+1}.$$

At the point where the collateral constraint ceases to be binding, there are kinks in the policy functions. From that point onwards, investment in capital moderately increases further, while the demand for debt soon starts to decrease. The reason is that very rich entrepreneurs finance a large part of their investments out of their own pockets.

2.3.4 Response to TFP Shock

I now analyze the response of the model economy to an unanticipated shock to aggregate TFP. The shock occurs in period $t = 0$. From $t = 1$ onwards, TFP is assumed to be back at its steady state level. With a shock like that, output would also return to its steady state level in period $t = 1$ already, if financing frictions were absent (see Appendix 2.B). In contrast, the model with collateral constraints exhibits sizable propagation. This is displayed in Figure 2.3, which plots the response of key aggregate variables to the one-time shock in TFP. The variable that exhibits the strongest reaction to the shock is the price of capital.[15] It initially drops by 8%, and it is still depressed by about 4% after three years. The interest rate reacts only slightly in period zero, but it is moderately above the steady

[14] Note that the scales of all plots are the same and that they do not entirely cover all four densities. Concerning entrepreneurs, the richest 1% have financial wealth above 100, but below 150. When it comes to workers, the density near zero is above 0.06, in fact reaching levels about 0.1.

[15] I choose to plot the price of new capital, q_t, rather than p_t. Lemma 2.4 shows that $p_t = q_t - \delta$, if the capital stock is fixed.

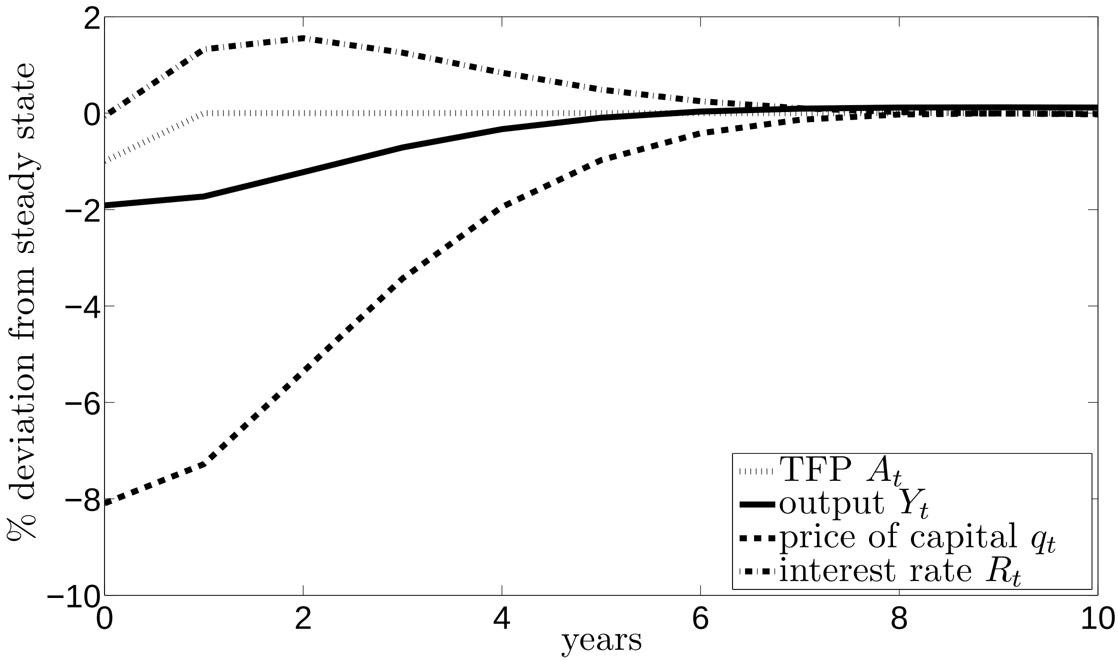

Figure 2.3: Response of Output and Prices

state level as the economy recovers.[16] Most importantly, the drop in output is sizable and persistent. It takes about eight years until the economy is roughly back at the steady state. In what follows, I analyze why there is so much persistence in this economy.

In a model with collateral constraints, output depends not only on the amount of labor and capital in the economy, but also on the allocation of capital among entrepreneurs. Poor entrepreneurs operate at a less than efficient scale, because they get less than optimal financing. To quantify how (in-)efficient capital is allocated among entrepreneurs, define *capital efficiency*,

$$E_t \equiv \frac{1}{K_t}\left(\int y_{i,t}^{\phi} di\right)^{1/\phi},$$

and observe how E_t enters aggregate output:

$$Y_t = A_t E_t^{\alpha} K_t^{\alpha} L_t^{1-\alpha}.$$

[16]This is in line with King and Watson (1996), who report that the correlation between the real interest rate and output is small but positive.

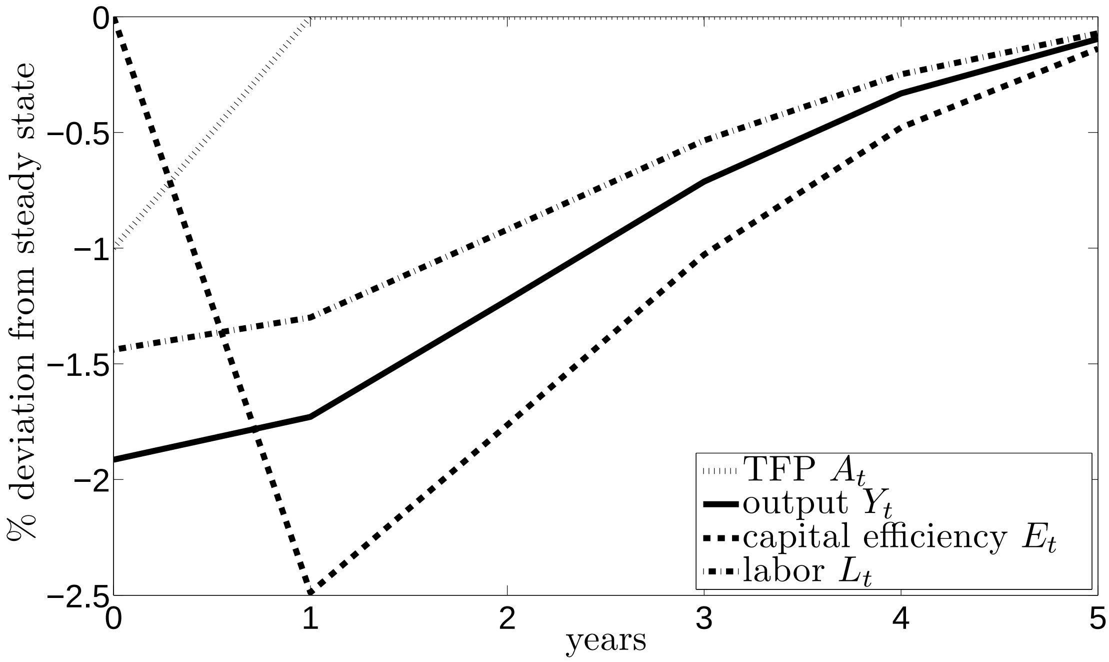

Figure 2.4: Decomposing the Reaction of Output

With $\tilde{X}$ denoting the log-deviation of the variable X from its steady state value, the output response may be decomposed as follows:

$$\tilde{Y}_t = \tilde{A}_t + \alpha \tilde{E}_t + \alpha \tilde{K}_t + (1-\alpha)\tilde{L}_t.$$

Remember that capital is fixed in the baseline calibration, thus $\tilde{K}_t = 0$ throughout. Note also that $\tilde{E}_0 = 0$, because the allocation of capital is determined one period ahead. Consequently, the initial drop in output is due only to the drop in TFP and the associated reaction in labor supply. This drop would be of the same magnitude without financial frictions. However, from period $t = 1$ onwards, output is diminished for a quite different reason, as can be seen from Figure 2.4. In period $t = 1$, TFP has already recovered, but capital efficiency is now down by 2.5%. The direct effect of this is a reduction in output of about $2.5\% \times \alpha = 0.9\%$. On top of that, a lower capital efficiency implies a reduced marginal product of labor, which drags down labor supply by 1.4% resulting in an additional drop in output of $1.4\% \times (1-\alpha) = 0.9\%$. Summing up, a one-time negative shock to TFP causes a persistent drop in capital efficiency which in turn depresses labor and output for several years.

The drop in capital efficiency is caused by the collateral constraints. The basic mechanism at work is illustrated in Figure 2.5, which is similar to Figure 1 in Kiyotaki and Moore (1997). The key role is played by the constrained entrepreneurs, who have low financial wealth, are highly leveraged, and operate at high marginal returns. In contrast to Kiyotaki and Moore (1997) the aggregate shock does not only reduce the wealth of constrained agents but also increases their number. Consider the intratemporal effects first. In period $t = 0$, TFP is low, which implies low returns on entrepreneurial investment. However, the repayment obligations that entrepreneurs face remain unaffected. Thus, the financial wealth of highly leveraged agents is reduced sharply. For agents who are collateral constrained—and their number is rising due to the shock—lower financial wealth necessitates lower investment in capital. In the aggregate, the reduced capital demand from the constrained entrepreneurs has to be offset by increased capital demand from rich entrepreneurs. However, to make them invest in spite of their low marginal returns, the price of capital has to fall. But a falling price of capital further reduces the financial wealth of constrained entrepreneurs, which constitutes a powerful intratemporal feedback effect. Now consider the intertemporal effects. First of all, as constrained entrepreneurs have to forgo profitable investment opportunities in $t = 0$, they have less financial wealth in $t = 1$, which lowers the demand for capital and its price in that period. A lower price in $t = 1$ reduces the payoff to investments made in $t = 0$, thus depressing the price in $t = 0$ further. In principle, these intertemporal effects are effective up until $t = \infty$.

The quantitative significance of the collateral constraint mechanism is documented in Figure 2.6. It shows the response of constrained agents' financial wealth and capital demand. In period $t = 0$ their financial wealth falls by 28%, which causes a reduction of their capital holding by 23%. The reduced demand in turn causes the price of capital to drop by 8%, which accounts for most of the total loss in financial wealth of constrained agents. Thus, the intratemporal feedback effect described above has quantitative bite. That the combined impact of the intratemporal and the intertemporal effects is substantial can be inferred from the fact that all three plotted variables remain depressed for several years. This is not least because the number of constrained agents goes up as the shock hits. Figure 2.7 plots the response of the percentage of constrained agents; it rises by some 36%

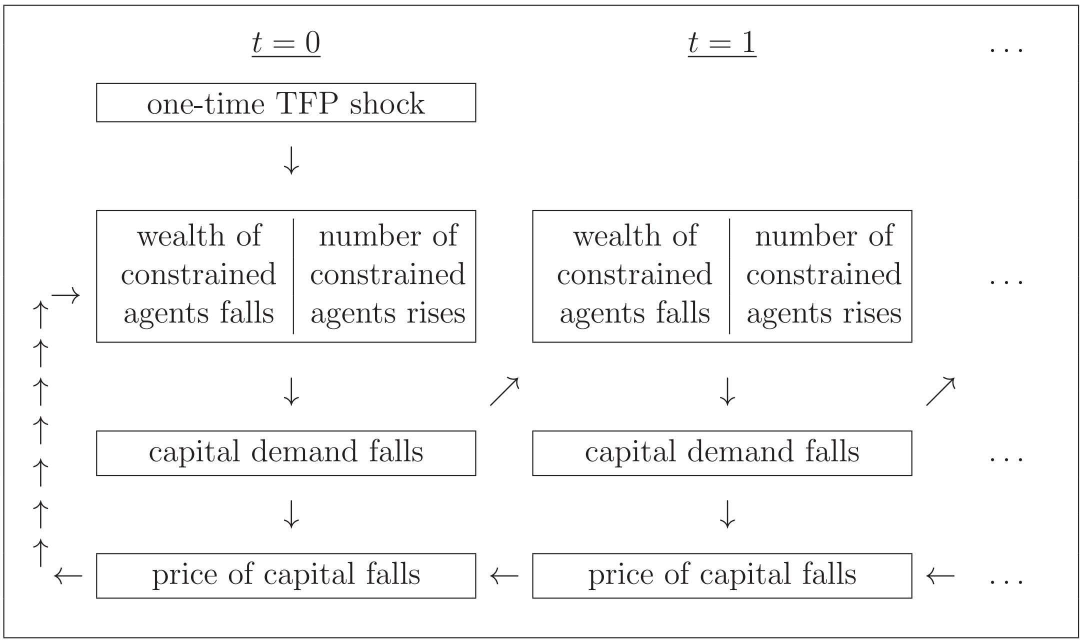

Figure 2.5: Collateral Constraint Mechanism

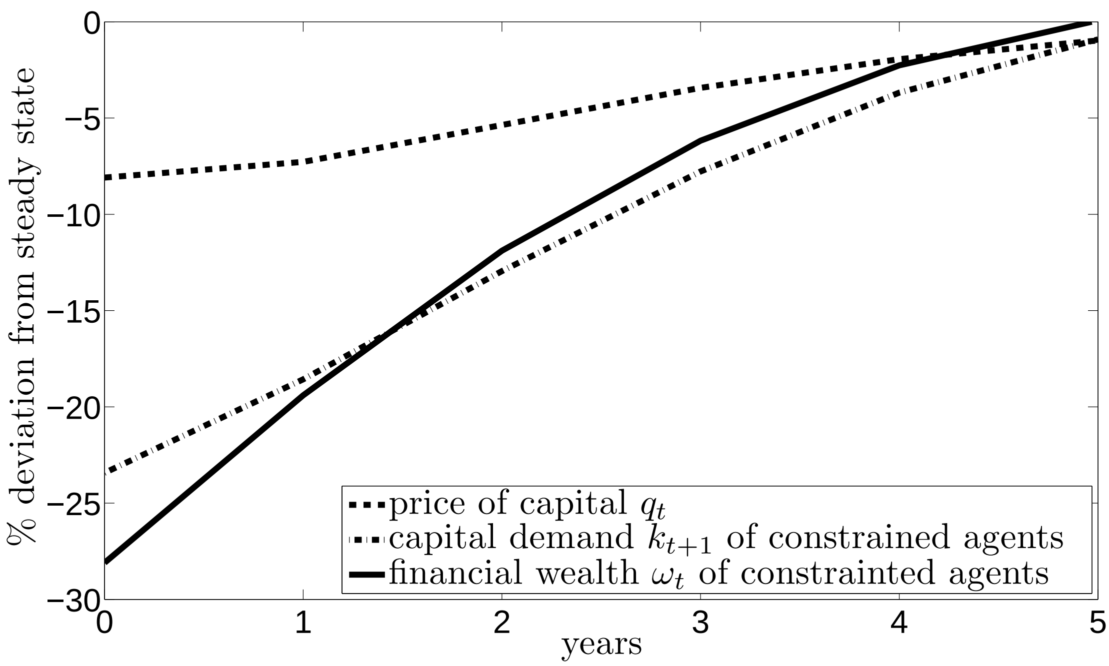

Figure 2.6: Quantifying the Collateral Constraint Mechanism

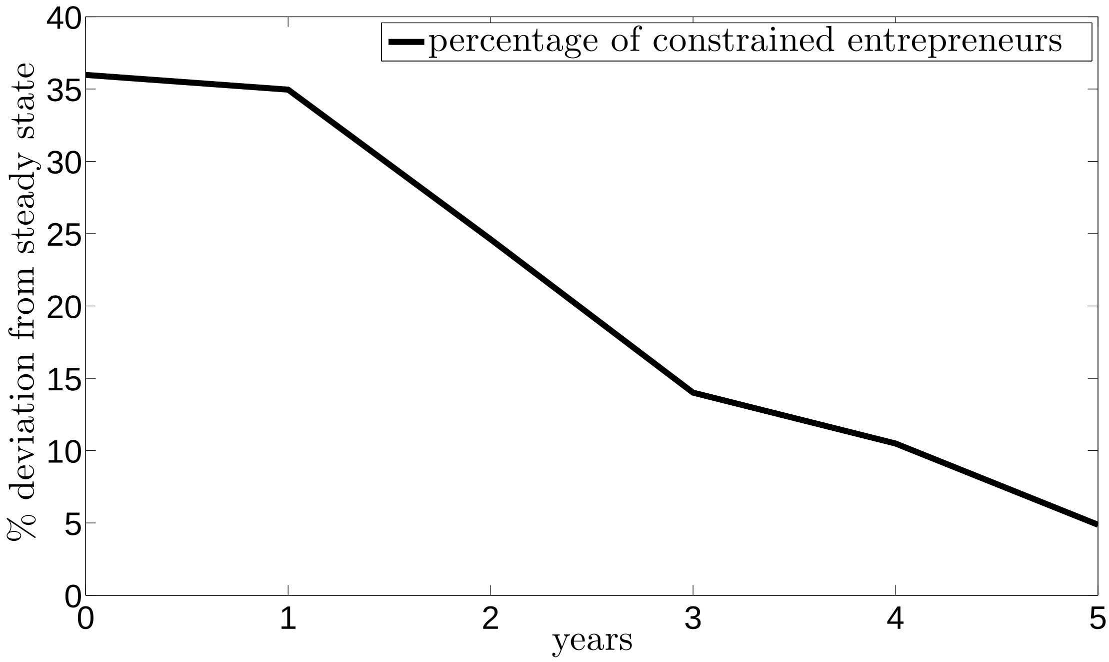

Figure 2.7: Percentage of Constrained Entrepreneurs

and takes several years to return to its steady state level. This constitutes an extensive margin to the collateral constraint mechanism which is neither present in Kiyotaki and Moore (1997), nor in any of the papers that quantify their mechanism.[17]

The feature that the number of constrained agents reacts to shocks has an interesting implication: The size of the response to a shock is a markedly non-linear function of its magnitude.[18] In particular, a negative shock drags down the economy by more than a positive shock of the same magnitude boosts the economy. This may help explain why recessions tend to be sharper than booms, as econometric studies like Hamilton (1989) find. To quantify the non-linearity, I define two measures for the response in output. As collateral constraints have no impact on the reaction of output in $t = 0$, I follow

[17]To the best of my knowledge, in none of the papers that quantify Kiyotaki and Moore (1997) does the percentage of constrained agents change non-trivially as the economy is hit by a shock.

[18]Kocherlakota (2000) presents a knife-edge example of this non-linearity. In the steady state of his model, the collateral constraints of all agents are *just* binding. As a consequence, all agents are constrained and there is positive amplification in case of a bad shock. In contrast, nobody is constrained in case of a good shock and there is no amplification. By including more heterogeneity among agents and using non-linear solution techniques, I go beyond this artificial case.

size of TFP shock $\Delta \equiv \left(A_0 - \bar{A}\right)/\bar{A}$	Amplification $\left(\left(Y_1 - \bar{Y}\right)/\bar{Y}\right)/\Delta$	Persistence $\frac{\sum_{i=1}^{5}\left(Y_i - \bar{Y}\right)}{5\left(Y_1 - \bar{Y}\right)}$	Constrained entrepreneurs in $t = 1$
+1.0%	1.08	0.42	43%
+0.5%	1.14	0.43	50%
−0.5%	1.42	0.46	69%
−1.0%	1.74	0.47	80%

Table 2.2: Impact of Differently Sized Shocks

Kocherlakota (2000) and measure amplification as the deviation of output in $t = 1$ relative to the shock in $t = 0$, i.e. *amplification* is defined as

$$\left(\left(Y_1 - \bar{Y}\right)/\bar{Y}\right)/\Delta, \text{ where } \Delta \equiv \left(A_0 - \bar{A}\right)/\bar{A}.$$

If measured like that, amplification is zero in the complete markets benchmark. Thus, any positive value signifies an impact of collateral constraints. To measure the persistence of the impact on output, I divide the average impact over five years by the impact in year $t = 1$, i.e. *persistence* is defined as

$$\frac{\frac{1}{5}\sum_{i=1}^{5}\left(Y_i - \bar{Y}\right)}{\left(Y_1 - \bar{Y}\right)}.$$

Table 2.2 reports both amplification and persistence for differently sized shocks. These measures would be equal across shocks, if the response was a linear function of the impulse. This is clearly not the case. For instance, there is 61% more amplification in case of a −1% shock as opposed to a +1% shock. The last column in Table 2.2 reports the percentage of agents that are constrained right after the respective shock hits. The variation from 43% to 80% suggests that the percentage of constrained agents is indeed the driving force for the non-linearity in the relation between impulse and response.

While a changing number of constrained agents is a nice feature of this model, it cannot entirely explain why there is more propagation than in previous quantitative studies on collateral constraints, like Cordoba and Ripoll (2004) or Mendicino (2008). Such models assume that there are two types of entrepreneurs, impatient ones which are collateral constrained, and patient ones which lend money to the impatient. By assuming strongly heterogeneous preferences—e.g. yearly discount rates of 0.96 versus 0.66—these papers can generate huge differences in marginal productivities between entrepreneurs. Within this context, reallocation of capital can have large effects on output. However, these models do not create enough reallocation of capital in the first place, and to the extent that they do, reallocation is only short-lived. The reason is as follows. As the shock reduces borrowers' financial wealth, their demand for debt falls dramatically, because two potential counteracting effects are shut down almost by assumption. First, borrowing could be increased by reducing any slackness in the collateral constraint. This is not possible as *all* borrowers are already constrained before the shock hits. Second, borrowing could be increased by reducing consumption and buying collateral instead. However, being *very* impatient, agents are quite unwilling to do that. The resulting reduction in the demand for debt is not met by a corresponding reduction in its supply, as lenders are patient and still want to save. Thus, the interest rate falls sharply,[19] which mitigates the collateral constraint mechanism for the following two reasons. First, a lower interest rate loosens the collateral constraint, implying less capital reallocation. Second, cheap loans help the wealth of constrained agents to recover quickly, which makes the reallocation of capital short-lived. In contrast, capital reallocation is large and persistent in my model (see Figure 2.6) as there is no sizable drop in the interest rate. This is because the above rationale for a sharp drop in the demand for loans does not apply: Borrowers are not impatient relative to lenders, and not all borrowers are constrained. In fact, constrained entrepreneurs cut back consumption substantially, while unconstrained entrepreneurs take up additional debt and invest in capital in order to profit from the expected rise in the price of capital. Therefore, the interest rate does not fall substantially, and the collateral constraint mechanism is not mitigated.

2.3.5 Sensitivity Analysis

The results in Section 2.3.4 show that collateral constraints can generate large amplification and persistence. I now analyze how sensitive this finding is to the value of crucial parameters, namely to risk aversion, γ, the Frisch elasticity, $1/\theta$, the adjustment cost parameter,

[19] See Figure 7 in Cordoba and Ripoll (2004) and Figure 2.a in Mendicino (2008). Note that Cordoba and Ripoll (2004) consider positive shocks, thus their plots show the reverse reactions to the ones described here.

Sensitivity w.r.t.	Parameter value	Amplification	Persistence
Risk aversion γ	2.5	2.00	0.55
	2	1.74	0.47
	1.5	1.33	0.41
Frisch elasticity $1/\theta$	3	1.74	0.47
	1	0.91	0.37
	1/3	0.53	0.32
Adjustment costs ξ	$+\infty$	1.74	0.47
	100	1.43	0.49
	10	0.73	0.55
	1	0.31	0.79
	0	0.24	0.91
Collateralizability κ	1	1.27	0.37
	0.86	1.74	0.47
	0.85	1.79	0.47

Table 2.3: Sensitivity Analysis

ξ, and the collateralizability of capital, κ. I also consider a different specification for the collateral constraint. Overall, it turns out that such changes have a substantial impact on the dynamics of the model, but they do not reduce the impact of collateral constraints to a negligible size. This robustness stands in contrast to models with heterogeneous preferences, where Cordoba and Ripoll (2004) find that substantial amplification is a knife-edge result only.

First, consider risk aversion. It turns out that higher risk aversion leads to higher amplification *and* persistence. This is in line with Pintus (2011), who shows that the trade-off between amplification and persistence found by Kiyotaki and Moore (1997) does not necessarily arise in models with collateral constraints.

Next, consider different values for the Frisch elasticity. To put them into perspective, note that in a recent meta analysis of existing evidence Chetty, Guren, Manoli, and Weber (2011) report a micro estimate of 0.82 and a macro estimate of 2.84. Clearly, a lower Frisch elasticity reduces amplification and persistence, as labor supply reacts less to changes in the efficiency of capital allocation. However, even with a Frisch elasticity of 1/3, which is even lower than most micro estimates, there is still substantial amplification: 0.53% compared to 0% with complete markets.

When the adjustment cost parameter ξ is reduced, amplification goes down, but persistence goes up. Without adjustment costs, the price of capital does not move at all and the collateral constraint mechanism cannot work. Instead, investment reacts to shocks, which changes the capital stock. The impact on output that this change in the capital stock has is far less pronounced than the one generated by the collateral constraint mechanism in the case of fixed capital. However, it persists for longer. For intermediate values of ξ both effects are at work. For $\xi = 10$, which is close to the $\xi = 8.5$ assumed by Lorenzoni and Walentin (2007), amplification is still large but much lower than with fixed capital.

Another crucial parameter of the model is the collateralizability of capital, κ. For $\kappa = 1$, amplification and persistence are lower than for $\kappa = 0.86$. This is qualitatively in line with the results of Mendicino (2008). However, it does not confirm her finding that a value below one is needed to get any sizable effect of collateral constraints.

Finally, consider the alternative specification for the collateral constraint presented in Section 2.2.3, which is $d_{t+1}R_{t+1} \leq \kappa p_{t+1}k_{t+1}$. For this specification amplification and persistence are 1.38 and 0.31 respectively. This is still large but considerably smaller than in the baseline specification, where p_t rather than p_{t+1} enters. The reason is as follows: As the shock is entirely non-persistent, the price of capital increases from period $t = 0$ to $t = 1$. Therefore, the collateral constraint is looser, if next period's price rather than this period's price enters. A looser constraint in turn leads to lower amplification and persistence.

2.4 Conclusion

In this paper, I have constructed and calibrated a model featuring collateral constraints and idiosyncratic risk. Within this framework, I analyzed the impact of an unanticipated one-time shock to TFP under the assumption that aggregate capital is fixed, which is exactly the exercise carried out in many previous papers. The propagation of shocks through collateral constraints turns out to be larger and also more robust than in models assuming differences in patience. The reason is that in these papers all borrowers are constrained and very impatient. This assumption, which is precisely made to make collateral constraints effective, indeed mitigates their effect through its implausibly strong impact on the interest rate. Apart from avoiding this problem, the model with idiosyncratic risk exhibits interesting dynamics. For instance, the response to a shock is far from being a linear function of its magnitude: If the size of a negative shock is doubled, the impact on output is much more than doubled; also, a negative shock depresses the economy by much more than a positive shock of the same size boosts it. The main reason for this non-linearity is that the number of agents who become collateral constrained depends heavily on the size of a shock.

There are many interesting ways to extend the current framework in future research. Concerning agents' skills, one could introduce a finer discretization and make the choice of occupation endogenous. When it comes to the dynamics of the aggregate capital stock, it seems promising to explore assumptions different from convex adjustment costs, e.g. a time-to-build lag. With respect to financing, allowing for defaultable bonds and costs of default has the potential to strengthen propagation. However, a word of caution is in place: Such extensions further increase the computational burden which is already substantial.

Appendix

2.A Details of Numerical Solution

This Appendix provides details of the numerical solution procedure presented in Section 2.3.1. Specifically, it explains the update steps of both algorithms.

In step 5 of Algorithm 1, I regress the deviations from equilibrium $\{\hat{Y}-Y,\hat{D}\}$ from the previous iterations of the algorithm on the respective guesses $\{Y,R\}$ used in these iterations. Then I use the coefficients of this regression to choose new guesses. These are determined such that there would be no deviations from equilibrium, if the relation between guesses and deviations were linear. Clearly, it is not linear, nevertheless the procedure converges very fast—it turned out to be much faster than a nested bisection method.

In step 5 of Algorithm 2, the guesses for all $T\times 3$ variables that represent the transition path are updated according to the following principles. First, the new guess is given by the old guess plus a measure of the error in the old guess. For instance, as a measure of the error in the interest rate the implied aggregate net supply of debt is used. Second, the size of the update step is governed by an update-factor which is equal across all $T\times 3$ variables (except that it is scaled up by $\bar{Y}$ in case of output). Third, the update-factor is itself updated depending on the speed of convergence. The relevant speed is the one of the variable which converges fastest. If this speed is very low, then the update-factor is increased. If it is too high, then the update-factor is reduced in order to avoid oscillating behavior. The update procedure implied by these three principles is stated below. The index n denotes iterations of the algorithm, and v_n denotes the update-factor. The decreasing function ϕ

governs the update of v_n.

$$
\begin{aligned}
R_{t,n+1} &= R_{t,n} + \hat{D}_{t,n} v_n, \\
Y_{t,n+1} &= Y_{t,n} + \left(\hat{Y}_{t,n} - Y_{t,n}\right) v_n \bar{Y}, \\
q_{t,n+1} &= q_{t,n} + \left(\hat{K}_{t,n} - K_{t,n}\right) v_n, \\
v_{n+1} &= v_n \cdot \phi\left(\min_t \left\{ \min \left\{ \frac{\hat{D}_{t,n}}{\hat{D}_{t,n-1}}, \frac{\hat{K}_{t,n} - K_{t,n}}{\hat{K}_{t,n-1} - K_{t,n-1}}, \frac{\hat{Y}_{t,n} - Y_{t,n}}{\hat{Y}_{t,n-1} - Y_{t,n-1}} \right\}\right\}\right), \\
\text{where} \quad K_{t,n} &= \begin{cases} \bar{K} & \text{if} \quad \xi = +\infty \text{ or } t = 0 \\ \left(\frac{q_{t,n-1}}{\xi} + 1\right) K_{t-1,n} & \text{if} \quad \xi < +\infty \text{ and } t > 0. \end{cases}
\end{aligned}
$$

2.B Complete Markets Benchmark

This Appendix analyzes a complete markets version of the model presented in Section 2.2. In particular, it verifies the statements made in Section 2.3.2 and 2.3.4 about the relation between the two model versions.

Suppose that markets are complete. Thus, agents may write (and enforce) contracts that are contingent on individual entrepreneurial output. As there is no aggregate risk, the expected marginal return on all assets is equalized in equilibrium. Consequently, all entrepreneurs operate at the same expected marginal return. In the baseline calibration with only one type of entrepreneur, this implies that all agents who are entrepreneurs in $t-1$ invest $K_t/\mu(s^e)$ units of capital. Among these agents only the ones who still have positive entrepreneurial productivity in period t are productive. Thus, aggregate output is given by

$$
Y_t = A_t \left(\int y_{i,t}^{\phi} di\right)^{\alpha/\phi} L_t^{1-\alpha} = A_t \left(\mu(s^e) M(s^e|s^e)(a_h^e)^{\phi} \left(\frac{K_t}{\mu(s^e)}\right)^{\phi}\right)^{\frac{\alpha}{\phi}} L_t^{1-\alpha}.
$$

Finally, using the normalization from Section 2.3.2, which is

$$a_h^e = M(s^e|s^e)^{-1/\phi}\mu(s^e)^{(\phi-1)/\phi},$$

the aggregate production function turns out to be as claimed in Section 2.3.2:

$$Y_t = A_t\left(\mu(s^e)M(s^e|s^e)M(s^e|s^e)^{-1}\mu(s^e)^{(\phi-1)}\left(\frac{K_t}{\mu(s^e)}\right)^{\phi}\right)^{\frac{\alpha}{\phi}} L_t^{1-\alpha} = A_t K_t^{\alpha} L_t^{1-\alpha}.$$

Building on this result, it is straightforward to analyze the transition path for the complete markets economy. Note that TFP and capital (which is assumed to be fixed) are both at their steady state levels from period $t = 1$ onwards:

$$A_t = \bar{A}, K_t = \bar{K} \;\; \forall t \geq 1.$$

Using the aggregate production function just derived, it follows that:

$$Y_t = \bar{A}\bar{K}^{\alpha}L_t^{1-\alpha} \;\; \forall t > 1.$$

In addition, Lemma 2.1 implies:

$$L_t = \left((1-\alpha)\,Y_t\right)^{\frac{1}{1+\theta}} \quad \forall t.$$

Combined, these equations imply that $\forall t > 1 : Y_t = \bar{Y}, L_t = \bar{L}$. Thus, the economy returns to the steady state right after being hit by the shock, which verifies the claim made in Section 2.3.4.

Chapter 3

Collateral Requirements and Asset Prices

3.1 Introduction

The vast majority of debt, especially if it extends over a long period of time, is guaranteed by tangible assets called collateral. For example, residential homes serve as collateral for short- and long-term loans to households, and investors can borrow money to establish a position in stocks, using these as collateral. The margin requirement dictates how much collateral one has to hold in order to borrow one dollar. Clearly these margin requirements will have important implications for the price of collateral. In the recent financial crisis it was argued that excessively low margin requirements were part of the cause of the crisis. In this paper, we conduct a quantitative study on the effect of margins requirements on asset prices.

Many previous papers have formalized the idea that borrowing on collateral might give rise to cyclical fluctuations in real activity and enhance volatility of prices (see e.g. Geanakoplos (1997), Kiyotaki and Moore (1997) and Aiyagari and Gertler (1999)). In these models, it is possible to have substantial departures of the market price from the corresponding price under frictionless markets. These results have led researchers to suggest that by managing leverage (or the amount of collateralized borrowing), a central bank can reduce aggregate fluctuations (see e.g. Ashcraft, Gârleanu, and Pedersen (2010) or Geanakoplos (2009)). However, establishing the quantitative importance of collateral requirements as a source of excess volatility has been a challenge in the literature (see Kocherlakota (2000) or Cordoba and Ripoll (2004)). Moreover, so far, there have been few quantitative studies that take into account that a household can use several different assets as collateral, and that regulated margin requirements for loans on one asset might have important effects on the volatility of other assets in the economy.

In this paper we consider a Lucas (1978) style exchange economy with heterogeneous agents and collateral constraints. We assume that agents can only take short positions if they hold an infinitely-lived asset (a Lucas tree) as a long position. This model was first analyzed by Kubler and Schmedders (2003) and subsequently used by Cao (2010) and Brumm and Grill (2010). As in Kubler and Schmedders (2003) we assume that agents can default on a negative bond position at any time without any utility penalties or loss of reputation. Financial securities are therefore only traded if the promises associated with these securities are backed by collateral. Our main focus is on an economy with two trees which can be used as collateral for short-term loans. For the first tree the collateral requirement is determined endogenously while the collateral requirement for loans on the second tree is exogenously regulated. We show that the presence of collateral constraints and the endogenous margin requirements for the first tree lead to large excess price-volatility of the second tree. Changes in the regulated margin requirements for the second tree have large effects on the volatility of both trees. While tightening margins for loans on the second tree always decreases the price volatility of the first tree, price volatility of the second tree might very well increase with this change. In our calibration we allow for the possibility of disaster states. This leads to very large quantitative effects of collateral requirements and to realistic equity risk premia.

Margin requirements are a crucial feature of our model. They determine with how much leverage agents can invest in risky assets. Following Geanakoplos (1997) and Geanakoplos and Zame (2002), we endogenize the margin requirements by introducing a menu of financial securities. All securities promise the same payoff, but they distinguish themselves by their respective margin requirement. In equilibrium only some of them are traded, thereby determining an endogenous margin requirement. This implies, of course, that for many bonds and many next period's shocks, the face value of the debt falls below the value of the collateral. As a result there is default in equilibrium. However, in an extension of the model we allow for costly default by introducing a real cost to the lender. We examine the impact of such default costs on equilibrium trading volume and prices. As an alternative to endogenous margin requirements, we also consider regulated margin requirements. In particular, our two-tree economy allows us to compare a tree with endogenous margins to a tree with regulated margins.[1]

In our calibration of the model there are two heterogeneous agents with Epstein-Zin utility. They have identical elasticities of substitution (IES) but distinguish themselves by their risk-aversion (RA). The agent with the low risk aversion is the natural buyer of risky assets and takes on leverage to finance these investments. The agent with the high risk aversion has a strong insurance motive against bad shocks and, therefore, is a natural buyer of safe bonds and a natural seller of risky assets. The idea behind this model setup is as follows. When the economy is hit with a negative shock, the collateral constraint forces the leveraged agent to reduce consumption or to even sell risky assets to the risk-averse agent, thereby resulting in substantial changes in the wealth distribution which in turn affect agents' portfolios and asset prices.

We start our analysis with an economy with a single Lucas tree that can be used as collateral. In this baseline model we exogenously assume that collateral requirements are set to the lowest possible level that still ensures that there is never default in equilibrium. To obtain a sizable market price of risk, we follow the specification in Barro and Jin (2011) and introduce the possibility of 'disaster shocks' into the otherwise standard calibration. In this model, the effect of scarce collateral on the volatility of the tree is quantitatively large. We then allow agents to choose from a menu of bonds with different margin requirements which are determined in equilibrium. Agents do trade bonds that have a positive probability of default. However, as soon as we introduce moderate default cost, trade in these default bonds is shut down.

The main contribution of the paper is the analysis of an economy with two trees which have identical cash-flows but distinguish themselves by their 'collateralizability'. We first

[1] Depending on the asset that is used as collateral, market forces might play an important role in establishing margin requirements. For stocks the situation is not obvious: The Federal Reserve Board sets minimum margin requirements for broker-dealer loans, using what is called Regulation T. In fact, until 1974, the Fed considered initial margin percentages as an active component of monetary policy and changed them fairly often (see Willen and Kubler (2006)). In the US housing market, there are no such regulations and margins can be arbitrarily small.

analyze a specification of the model in which only the first tree can be used as collateral. In this specification, the return volatility of the collateralizable tree is significantly smaller than that of the single tree in the baseline model. However, the volatility of the second tree, which cannot be used as collateral, is comparable. A possible interpretation of these findings is to identify the collateralizable tree with housing and the non-collateralizable tree with the aggregate stock market. Using stocks as collateral is subject to many regulations and often very costly, while individuals can easily use houses. Volatility and excess returns for houses is much smaller than for stocks, which is in line with our findings.

We then relax the assumption of the non-collaterizability of the second tree. We assume that a regulating agency sets an exogenous margin requirement for this tree. We find that regulation of the second tree has a strong impact on the volatility of the first tree. In particular, a tightening of margin requirements for the regulated tree uniformly decreases volatility of the unregulated tree. For the regulated tree, tighter margins initially increase the price volatility but then decrease it once margins become very large. We further show how the regulation of margin requirements only in times when the economy exhibits strong growth can substantially decrease volatility compared to the case of uniform regulation of margin requirements. This result holds true both for the baseline model with a single tree as well as the two-tree economy and suggests a strong policy recommendation for counter-cyclical margin requirements.

Finally, we conduct a thorough sensitivity analysis and show that our qualitative results are robust to the actual parametrization of the economy. In particular, we document that the key effects for the two-tree economy are robust to changes in the magnitude of the disaster shocks.

The remainder of this paper is organized as follows. We introduce the model in Section 3.2. In Section 3.3 we discuss results for economies with a single tree. Section 3.4 focuses on economies with two trees. In Section 3.5 we consider extensions and sensitivity analysis. Section 3.6 concludes.

3.2 The Economic Model

We examine a model of an exchange economy that extends over an infinite time horizon and is populated by infinitely-lived heterogeneous agents.

3.2.1 Infinite-Horizon Economy

This section describes the details of the infinite-horizon economy.

The Physical Economy

Time is indexed by $t = 0, 1, 2, \ldots$. A time-homogeneous Markov chain of exogenous shocks (s_t) takes values in the finite set $\mathcal{S} = \{1, \ldots, S\}$. The $S \times S$ Markov transition matrix is denoted by π. We represent the evolution of time and shocks in the economy by a countably infinite event tree Σ. The root node of the tree represents the initial shock s_0. Each node of the tree, $\sigma \in \Sigma$, describes a finite history of shocks $\sigma = s^t = (s_0, s_1, \ldots, s_t)$ and is also called date-event. We use the symbols σ and s^t interchangeably. To indicate that $s^{t'}$ is a successor of s^t (or s^t itself) we write $s^{t'} \succeq s^t$. We use the notation s^{-1} to refer to the initial conditions of the economy prior to $t = 0$.
At each date-event $\sigma \in \Sigma$ there is a single perishable consumption good. The economy is populated by H agents, $h \in \mathcal{H} = \{1, 2, \ldots, H\}$. Agent h receives an individual endowment in the consumption good, $e^h(\sigma) > 0$, at each node. In addition, at $t = 0$ the agent owns shares in Lucas trees. We interpret these Lucas trees to be physical assets such as firms, machines, land or houses. There are A different such assets, $a \in \mathcal{A} = \{1, 2, \ldots, A\}$. At the beginning of period 0, each agent h owns initial holdings $\theta_a^h(s^{-1}) \geq 0$ of tree a. We normalize aggregate holdings in each Lucas tree, that is, $\sum_{h \in \mathcal{H}} \theta_a^h(s^{-1}) = 1$ for all $a \in \mathcal{A}$. At date-event σ, we denote agent h's (end-of-period) holding of Lucas tree a by $\theta_a^h(\sigma)$.
The Lucas trees pay positive dividends $d_a(\sigma)$ in units of the consumption good at all date-events. We denote aggregate endowments in the economy by

$$\bar{e}(\sigma) = \sum_{h \in \mathcal{H}} e^h(\sigma) + \sum_{a \in \mathcal{A}} d_a(\sigma).$$

The agents have preferences over consumption streams representable by the following recursive utility function, see Epstein and Zin (1989),

$$U^h(c, s^t) = \left\{ \left[c^h(s^t)\right]^{\rho^h} + \beta \left[\sum_{s_{t+1}} \pi(s_{t+1}|s_t) \left(U^h(c, s^{t+1})\right)^{\alpha^h}\right]^{\frac{\rho^h}{\alpha^h}} \right\}^{\frac{1}{\rho^h}},$$

where $\frac{1}{1-\rho^h}$ is the intertemporal elasticity of substitution (IES) and $1 - \alpha^h$ is the relative risk aversion of the agent.

Security Markets

At each date-event agents can engage in security trading. Agent h can buy $\theta_a^h(\sigma) \geq 0$ shares of tree a at node σ for a price $q_a(\sigma)$. Agents cannot assume short positions of the Lucas trees. Therefore, the agents make no promises of future payments when they trade shares of physical assets and thus there is no possibility of default.

In addition to the physical assets, there are J financial securities, $j \in \mathcal{J} = \{1, 2, \ldots, J\}$, available for trade. These assets are one-period securities in zero-net supply. Security j traded at node s^t promises a payoff of one unit of the consumption good at each immediate successor node s^{t+1}. We denote agent h's (end-of-period) portfolio of financial securities at date-event σ by $\phi^h(\sigma) \in \mathbb{R}^J$ and denote the price of security j at this date-event by $p_j(\sigma)$. Whenever an agent assumes a short position in a financial security j, $\phi_j^h(\sigma) < 0$, she promises a payment in the next period. In our economy such promises must be backed up by collateral holdings.

Collateral and Default

At each node σ, we associate with each financial security $j \in \mathcal{J}$ a tree $a(j) \in \mathcal{A}$ and a collateral requirement $k_{a(j)}^j(\sigma) > 0$. If an agent sells one unit of security j, then she is required to hold $k_{a(j)}^j(\sigma)$ units of tree $a(j)$ as collateral. If an asset a can be used as collateral for different financial securities, the agent is required to buy $k_{a(j)}^j(\sigma)$ shares for each security $j \in \mathcal{J}_a$, where $\mathcal{J}_a \subset \mathcal{J}$ denotes the set of financial securities collateralized by the same tree a. In the next period, the agent can default on her earlier promise. In this case the agent loses the collateral she had to put up. In turn, the buyer of the financial

security receives this collateral associated with the initial promise.[2]
Since there are no penalties for default, a seller of security j at date-event s^{t-1} defaults on her promise at a successor node s^t whenever the initial promise exceeds the current value of the collateral, that is, whenever

$$1 > k^j_{a(j)}(s^{t-1})\left(q_{a(j)}(s^t) + d_{a(j)}(s^t)\right).$$

The payment by a borrower of security j at node s^t is, therefore, always given by

$$f_j(s^t) = \min\left\{1, k^j_{a(j)}(s^{t-1})\left(q_{a(j)}(s^t) + d_{a(j)}(s^t)\right)\right\}.$$

Our model includes the possibility of costly default. This feature of the model is meant to capture default costs such as legal cost or the physical deterioration of the collateral asset. For example, it is well known that housing properties in foreclosure deteriorate because of moral hazard, destruction, or simple neglect. We model such costs by assuming that part of the collateral value is lost and thus the payment received by the lender is smaller than the value of the borrower's collateral. Specifically, the loss is proportional to the difference between the face value of the debt and the value of collateral, that is, the loss is

$$l_j(s^t) = \lambda\left(1 - k^j_{a(j)}(s^{t-1})\left(q_{a(j)}(s^t) + d_{a(j)}(s^t)\right)\right)$$

for some parameter $\lambda \geq 0$. The resulting payment to the lender of the loan in security j when $f_j(s^t) < 1$ is thus given by

$$r_j(s^t) = \max\left\{0, f_j(s^t) - l_j(s^t)\right\} = \max\left\{0, (1+\lambda)k^j_{a(j)}(s^{t-1})(q_{a(j)}(s^t) + d_{a(j)}(s^t)) - \lambda\right\}.$$

[2]Following Geanakoplos and Zame (2002) we make the strong assumption that an agent can default on individual promises without declaring personal bankruptcy and giving up all the assets he owns. There are no penalties for default and a borrower always defaults once the value of the debt is above the value of the collateral. Since this implies that the decision to default on a promise is independent of the debtor, we do not need to consider pooling of contracts as in Dubey, Geanakoplos, and Shubik (2000), even though there may be default in equilibrium. This treatment of default is somewhat unconvincing since default does not affect a household's ability to borrow in the future and it does not lead to any direct reduction in consumption at the time of default. Moreover, declaring personal bankruptcy typically results in a loss of all assets, and it is rarely possible to default on some loans while keeping the collateral for others. However, there do exist laws for collateralizable borrowing where default is possible without declaring bankruptcy. Examples include pawn shops and the housing market in many US states, in which households are allowed to default on their mortgages without defaulting on other debt. It is certainly true that the recent 2008 housing crises makes this assumption look much better.

If $f_j(s^t) = 1$ then $r_j(s^t) = f_j(s^t) = 1$. This repayment function does not capture all costs associated with default. For example, it does not allow for fixed costs which are independent of how much the collateral value falls short of the repayment obligation. However, our functional form offers the advantage that the resulting model remains tractable since the repayment function is continuous in the value of the collateral.

The specification of the collateral requirements $k_a^j(s^t)$ for bond j, tree a and across date-events s^t has important implications for equilibrium prices and allocations. The collateral levels $k_a^j(s^t)$ are endogenously determined in equilibrium. In this paper we examine two different rules for the endogenous determination of collateral levels. The first rule determines endogenous collateral requirements along the lines of Geanakoplos and Zame (2002). The second rule assumes exogenously regulated capital-to-value ratios which in turn lead to endogenous collateral requirements.

Default and Endogenous Collateral Requirements

One of the contributions of this paper is to endogenize collateral requirements in an infinite-horizon dynamic general equilibrium model. For this purpose, our first collateral rule follows Geanakoplos (1997) and Geanakoplos and Zame (2002) who suggest a simple and tractable way to endogenize collateral requirements. They assume that, in principle, financial securities with any collateral requirement could be traded in equilibrium. Only the scarcity of available collateral leads to equilibrium trade in only a small number of such securities. Our first rule follows this approach.

Recall that the S direct successors of a node s^t are denoted $(s^t, 1), \ldots, (s^t, S)$ and that $\mathcal{J}_a$ denotes the set of bonds collateralized by the same tree a. We define endogenous margin requirements for bonds $j \in \mathcal{J}_a$ collateralized by the same tree $a \in \mathcal{A}$ as follows. For each shock next period, $s' \in \mathcal{S}$, there is at at least one bond which satisfies $k_{a(j)}^j(s^t)\left(q_{a(j)}(s^t, s') + d_{a(j)}(s^t, s')\right) = 1$. For each bond in the set $\mathcal{J}_a$ the promised payoff is equal to the collateral in (generically) exactly a single state. Generically the set $\mathcal{J}_a$ thus contain exactly S bonds, however the bond with the lowest collateral requirement is redundant in our model because its payoff vector is collinear with the tree's dividend vector. (Therefore, we consider only models with at most $S - 1$ bonds in our numerical analysis of the model.) The arguments in Araújo, Kubler, and Schommer (2010) show that adding additional bonds with other collateral requirements (also only using tree a as collateral) do not change the equilibrium allocation. In the presence of S bonds as specified above, any bond with an intermediate collateral requirement can be replicated by holding a portfolio of the existing bonds using the same amount of collateral.

We begin our model examinations always with economies with a single bond, $J = 1$, on which agents cannot default. That is, the collateral requirements are endogenously set to the lowest possible value which still ensures no default in the subsequent period (this specification is similar to the collateral requirements in Kiyotaki and Moore (1997)). Formally, the resulting condition for the collateral requirement $k^1_{a(1)}(s^t)$ of this bond is

$$k^1_{a(1)}(s^t)\left(\min_{s^{t+1}\succ s^t}\left(q_{a(1)}(s^{t+1}) + d_{a(1)}(s^{t+1})\right)\right) = 1.$$

We refer to this bond as the 'risk-free' or 'no-default' bond.

To simplify the discussion of models with several bonds, it is useful to refer to the different bonds by the number of states in which they default, respectively. In our model specifications below, the set $\mathcal{J}_a$ always contains a no-default bond. In models with several bonds, the second bond defaults in precisely one state, the third bond in precisely two states, and so on. Hence we refer to these additional bonds as the 1-default bond, the 2-default bond etc. In the absence of default costs, some of these bonds will typically be traded in equilibrium. However, we see below that, in our calibration, rather moderate default costs generally suffice to shut down trade in these bonds.

Financial Markets Equilibrium with Collateral

We are now in the position to formally define the notion of a financial markets equilibrium. To simplify the statement of the definition, we assume that for a set of trees $\hat{\mathcal{A}} \subset \mathcal{A}$ collateral requirements are endogenous, that is for each $\hat{a} \in \hat{\mathcal{A}}$, there exist a set $\mathcal{J}_{\hat{a}}$ of S bonds for which this tree can be used as collateral. It is helpful to define the terms $[\phi^h_j]^+ = \max(0, \phi^h_j)$ and $[\phi^h_j]^- = \min(0, \phi^h_j)$. We denote equilibrium values of a variable x by $\bar{x}$.

Definition 3.1. *A financial markets equilibrium for an economy with initial tree holdings* $(\theta^h(s^{-1}))_{h\in\mathcal{H}}$ *and initial shock* s_0 *is a collection of agents' portfolio holdings and consumption allocations as well as security prices and collateral requirements for all trees* $\hat{a} \in \hat{\mathcal{A}} \subset \mathcal{A}$

$$\left(\left(\bar{\theta}^h(\sigma), \bar{\phi}^h(\sigma), \bar{c}^h(\sigma)\right)_{h\in\mathcal{H}} ; \left(\bar{q}_a(\sigma)\right)_{a\in\mathcal{A}}, \left(\bar{p}_1(\sigma)\right)_{j\in\mathcal{J}}, \left(\bar{k}^j_{\hat{a}}(\sigma)\right)_{j\in\mathcal{J}_{\hat{a}}, \hat{a}\in\hat{\mathcal{A}}} \right)_{\sigma\in\Sigma}$$

satisfying the following conditions:

(1) Markets clear:

$$\sum_{h\in\mathcal{H}} \bar{\theta}^h(\sigma) = 1 \quad \text{and} \quad \sum_{h\in\mathcal{H}} \bar{\phi}^h(\sigma) = 0 \quad \text{for all } \sigma \in \Sigma.$$

(2) For each agent h, the choices $\left(\bar{\theta}^h(\sigma), \bar{\phi}^h(\sigma), \bar{c}^h(\sigma)\right)$ *solve the agent's utility maximization problem,*

$$\begin{aligned}
\max_{\theta\geq 0, \phi, c\geq 0} U_h(c) \quad & s.t. \quad \text{for all } s^t \in \Sigma \\
c(s^t) \quad &= \quad e^h(s^t) + \sum_{j\in\mathcal{J}} \left([\phi_j(s^{t-1})]^+ r_j(s^t) + [\phi_j(s^{t-1})]^- f_j(s^t) \right) + \\
& \qquad \theta^h(s^{t-1}) \cdot \left(\bar{q}(s^t) + d(s^t)\right) - \theta^h(s^t) \cdot \bar{q}(s^t) - \phi^h(s^t) \cdot \bar{p}(s^t) \\
0 \quad &\leq \quad \theta^h_{\hat{a}}(s^t) + \sum_{j\in\mathcal{J}_{\hat{a}}} \bar{k}^j_{\hat{a}}(s^t) [\phi^h_j(s^t)]^-, \qquad \text{for all } \hat{a} \in \hat{\mathcal{A}}.
\end{aligned}$$

(3) For all s^t *and for each* $\hat{a} \in \hat{\mathcal{A}}$*, there exists for each state* $s' \in \mathcal{S}$ *a financial security j such that* $\hat{a} = a(j)$ *and*

$$\bar{k}^j_{\hat{a}}(s^t) \left(\bar{q}_{\hat{a}}(s^t, s') + d_{\hat{a}}(s^t, s')\right) = 1.$$

The approach in Kubler and Schmedders (2003) can be used to prove existence. The only non-standard part—besides the assumption of recursive utility, which can be handled easily—is the assumption of default costs. Note, however, that our specification of these costs still leaves us with a convex problem and standard arguments for continuity of best responses go through.

To approximate equilibrium numerically, we use the algorithm in Brumm and Grill (2010). In Appendix 3.A, we describe the computations and the numerical error analysis in detail. For the interpretation of the results to follow it is useful to understand the recursive formulation of the model. The natural endogenous state-space of this economy consists of all agents' beginning of period financial wealth as a fraction of total financial wealth (i.e. value of the trees cum dividends) in the economy. That is, we keep track of the current shock s_t and of

$$\omega^h(s^t) = \frac{\sum_{j \in \mathcal{J}} \left([\phi_j^h(s^{t-1})]^+ r_j(s^t) + [\phi_j^h(s^{t-1})]^- f_j(s^t) \right) + \theta^h(s^{t-1}) \cdot (\bar{q}(s^t) + d(s^t))}{\sum_{a \in \mathcal{A}} q_a(s^t) + d(s^t)},$$

across all agents $h \in \mathcal{H}$. As in Kubler and Schmedders (2003) we assume that a recursive equilibrium on this state space exists and compute prices, portfolios and individual consumptions as a function of the exogenous shock and the distribution of financial wealth. In our calibration we assume that shocks are iid and that these shocks only affect the aggregate growth rate. In this case, policy- and pricing functions are independent of the exogenous shock, thus depend on the wealth distribution only, and our results can be easily interpreted in terms of these functions.

Regulated Collateral Requirements

The second rule for setting collateral requirements relies on regulated capital-to-value ratios. An agent selling one unit of bond j with price $p_j(s^t)$ must hold collateral with a value of at least $k^j_{a(j)}(s^t)q_{a(j)}(s^t)$. We can interpret the difference between the value of the collateral holding and the debt as the amount of capital an agent must put up to obtain the loan in form of a short position in the financial security. A (not further modeled) regulating agency now requires debtors to hold a certain minimal amount of capital relative to the value of the collateral they hold. Put differently, the regulator imposes a lower bound $m^j_{a(j)}(s^t)$ on this capital-to-value ratio,

$$m^j_{a(j)}(s^t) = \frac{k^j_{a(j)}(s^t)q_{a(j)}(s^t) - p_j(s^t)}{k^j_{a(j)}(s^t)q_{a(j)}(s^t)}.$$

Using language from financial markets we also call these bounds margin requirements. If the margin requirement is regulated to be $m^j_{a(j)}(s)$ in shock $s \in \mathcal{S}$ and constant over time, then the collateral requirement at each node s^t is

$$k^j_{a(j)}(s^t) = \frac{p_j(s^t)}{q_{a(j)}(s^t)(1 - m^j_{a(j)}(s_t))}.$$

Note that, contrary to the exogenous margin requirement, the resulting collateral requirement is endogenous since it depends on equilibrium prices. For economies with regulated margins, condition (3) of the definition of a financial markets equilibrium must be replaced by the following condition.

(3') For all s^t and for each $\hat{a} \in \hat{\mathcal{A}}$, the collateral requirement $\bar{k}^j_{\hat{a}}(s^t)$ of the unique bond j with $\hat{a} = a(j)$ and the given margin requirement $m^j_{\hat{a}}(s_t)$ satisfies

$$\bar{k}^j_{\hat{a}}(s^t) = \frac{\bar{p}_j(s^t)}{\bar{q}_{\hat{a}}(s^t)(1 - m^j_{\hat{a}}(s_t))}.$$

Sometimes people use the term margin requirement for the capital-to-loan ratio,

$$\frac{k^j_a q_a(s^t) - p_j(s^t)}{p_j(s^t)},$$

which does not have a natural normalization and can be larger than one. On the contrary, the margin requirement $m^j_{a(j)}(s_t)$ as defined above has a natural normalization since it is bounded above by one.

3.2.2 Calibration

This section discusses the calibration of the model's exogenous parameters. We calibrate our model to yearly data.

Growth Rates

We consider a growth economy with stochastic growth rates. The aggregate endowment at date-event s^t grows at the stochastic rate $g(s_{t+1})$ which (if no default cost are incurred) only depends on the new shock $s_{t+1} \in \mathcal{S}$, that is, if either $\lambda = 0$ or $f_j(s_{t+1}) = 1$ for all $j \in \mathcal{J}$, then

$$\frac{\bar{e}(s^{t+1})}{\bar{e}(s^t)} = g(s_{t+1})$$

for all date-events $s^t \in \Sigma$. If there is default in s_{t+1}, then the endowment $\bar{e}(s_{t+1})$ is reduced by the costs of default and the growth rate is reduced respectively.
There are $S = 6$ exogenous shocks. We declare the first three of them, $s = 1, 2, 3$, to be "disasters". We calibrate the disaster shocks to match the first three moments of the distribution of disasters in Barro and Jin (2011). Also following Barro and Jin, we choose transition probabilities such that the six exogenous shocks are i.i.d. The non-disaster shocks, $s = 4, 5, 6$, are then calibrated such that their standard deviation matches "normal" business cycle fluctuations with a standard deviation of 2 percent and an average growth rate of 2.5 percent, which results in an overall average growth rate of about 2 percent. We sometimes find it convenient to call shock $s = 4$ a "recession" since $g(4) = 0.966$ indicates a moderate decrease in aggregate endowments. Table 3.1 provides the resulting growth rates and probability distribution for the six exogenous shocks of the economy.

Shock s	1	2	3	4	5	6
$g(s)$	0.566	0.717	0.867	0.966	1.025	1.089
$\pi(s)$	0.005	0.005	0.024	0.065	0.836	0.065

Table 3.1: Growth Rates and Distribution of Exogenous Shocks

In our results sections below we report that collateral requirements have quantitatively strong effects on equilibrium prices. Obviously, the question arises what portion of these effects is due to the large magnitude of the disaster shocks. We address this issue in the discussion of our results. In addition, Section 3.5 examines the equilibrium effects of collateral requirements for an economy with less severe disaster shocks.

Endowments and Dividends

There are $H = 2$ types of agents in the economy, the first type, $h = 1$, being less risk-averse than the second. Each agent h receives a fixed share of aggregate endowments as individual endowments, that is, $e^h(s^t) = \eta^h \bar{e}(s^t)$. We assume that $\eta^1 = 0.092$, $\eta^2 = 0.828$. Agent 1 receives 10 percent of all individual endowments, and agent 2 receives the remaining 90 percent of all individual endowments. The remaining part of aggregate endowments enters the economy as dividends of Lucas trees, that is, $d_a(s^t) = \delta_a(s_t)\bar{e}(s^t)$ and $\sum_a \delta_a(s) = 0.08$ for all $s \in \mathcal{S}$.

Several comments on the distribution of the aggregate endowment are in order. First, we abstract from idiosyncratic income shocks because it is difficult to disentangle idiosyncratic and aggregate shocks for a model with two types of agents. We conjecture that our effects would likely be larger if we considered a model with a continuum of agents receiving i.i.d. idiosyncratic shocks. Second, a dividend share of 8 percent may appear a little too low if one interprets the tree as consisting of both the aggregate stock market as well as housing wealth. However, this number is in line with Chien and Lustig (2010) who base their calibration on NIPA data. We conduct some sensitivity analysis below and, in particular, report results for the case $\sum_a \delta_a(s) = 0.15$ and thus $\eta^1 = 0.085$, $\eta^2 = 0.765$. Third, for simplicity we do not model trees' and other assets' dividends to have different stochastic characteristics as aggregate consumption. Fourth, in Section 3.4 we examine an economy with two Lucas trees. For such economies, we want to interpret the first tree as aggregate housing and its dividends as housing services while we interpret the second tree as the aggregate stock market. Following Cecchetti, Stephen, and Mark (1993), we calibrate dividends to be 4 percent of aggregate consumption which leaves housing services to be of the same size. In order to focus on the effects of collateral and margin requirements, we assume that the two trees have the exact same dividend payments, that is, in the absence of collateral constraints these two trees would be identical assets. Therefore, this calibration allows for a careful examination of the impact of different collateral properties of the two trees.

Utility Parameters

The choice of an appropriate value for the IES is rather difficult. On the one hand, several studies that rely on micro-data find values of about 0.2 – 0.8, see, for example, Attanasio and Weber (1993). On the other hand, Vissing-Jørgensen and Attanasio (2003) use data on stock owners only and conclude that the IES for such investors is likely to be above one. Barro (2009) finds that for a successful calibration of a representative-agent asset-pricing model the IES needs to be larger than one.

In our benchmark calibration both agents have identical IES of 1.5, that is, $\rho^1 = \rho^2 = 1/3$. In our sensitivity analysis we also consider the case of both agents having an IES of 0.5. For this specification the quantitative results slightly change compared to the benchmark calibration, but the qualitative insights remain intact.

Agent 1 has a risk aversion of 0.5 and so $\alpha^1 = 0.5$ while agent 2's risk aversion is 6 and thus $\alpha^2 = -5$. Recall the weights for the two agents in the benchmark calibration, $\eta^1 = 0.092$ and $\eta^2 = 0.828$. The majority of the population is therefore very risk-averse, while 10 percent of households have low risk aversion. This heterogeneity of the risk aversion among the agents is the main driving force for volatility in the model. (Agent 1 wants to hold the risky assets in the economy and leverages to do so. In a bad shock, his de-leveraging leads to excess volatility.) In the equilibria of our model, the risky assets are mostly held by agent 1, but there are extended periods of time where also agent 2 holds part of the asset. Loosely speaking, we therefore choose the fraction of very risk-averse agents to match observed stock-market participation.

Finally, we set $\beta^h = 0.95$ for both $h = 1, 2$, which turns out to give us a good match for the annual risk-free rate.

3.3 Economies with a Single Lucas Tree

We first consider economies with a single Lucas tree available as collateral. We show that scarce collateral has a large effect on the price volatility of this tree and examine how the magnitude of this effect depends on the specification of margin requirements. This section sets the stage for our analysis of economies with two trees in Section 3.4.

3.3.1 Collateral and Volatility with a Single Risk-Free Bond

The starting point of our analysis is an economy with a single Lucas tree and a single bond. We assume that the collateral requirements on the single bond ensure that there is no default in equilibrium and so the bond is risk-free. We calibrate this baseline model according to the parameters presented above.

For an evaluation of the quantitative effects of scarce collateral, we benchmark our results against those for two much simpler models. The model *B1: No bonds* is an economy with a single tree and no bond. Thus, agents in this economy cannot borrow. The model *B2: Unconstrained* is an economy in which agents can use their entire endowment as collateral. This model is equivalent to a model with natural borrowing constraints. Table 3.2 reports four statistics for each of the three economies. (See Appendix 3.A for a description of the estimation procedure.) Throughout the paper we measure tree-price volatility by the average standard deviation of tree returns over a long horizon. Another meaningful measure is the average one-period-ahead conditional price volatility. These two measures are closely correlated for our models. In Table 3.2 we report both measures but omit the second one in the remainder of the paper. We also report average interest rates and equity premia. While our paper does not focus on an analysis of these measures, we do check them because we want to ensure that our calibration delivers reasonable values for these measures.

Recall that in our calibration agents of type 1 are much less risk averse than type 2 agents. And, therefore, in the long run agent 1 holds the entire Lucas tree in model *B1* with no borrowing and agent 2 effectively lives in autarchy. As a result the tree price is determined entirely by the Euler equation of agent 1, and so the price volatility is as low as in the model with a representative agent whose preferences exhibit very low risk aversion. The wealth

Model	Std returns	1-period price vol.	Risk-free rate	EP
B1: No bonds	5.33	4.98	n/a	n/a
B2: Unconstrained	5.38	5.05	5.88	0.55
Scarce Collateral	8.14	7.54	1.10	3.86

Table 3.2: Three Economies with a Single Tree (all Figures in Percent)

distribution remains constant across all date-events. In the second benchmark model *B2* the less risk-averse agent 1 holds the entire tree during the vast majority of time periods. A bad shock to the economy leads to shifts in the wealth distribution and a decrease of the tree price. However, since in our calibration shocks are iid, these shifts in the wealth distribution have generally small effects on prices (except in the very low-probability case of several consecutive disaster shocks). The resulting price volatility in model *B2* is of similar magnitude as the volatility in *B1*. Moreover, in the model *B2* the risk-free rate is high and the equity premium is very low. Despite the presence of disaster shocks, the market price of risk is low because it is borne almost entirely by agent 1 who has very low risk aversion.

Table 3.2 shows that both first and second moments show substantial differences when we compare models without collateral requirements to a model with tight collateral constraints. The perhaps most striking result reported in Table 3.2 is that volatility in our baseline economy is about 50 percent larger than in the two benchmark models without borrowing (*B1: No bonds*) and with natural borrowing constraints (*B2: Unconstrained*), respectively. The standard deviation of returns is 8.14 percent in the baseline economy but only 5.33 percent and 5.38 percent for the benchmark models *B1* and *B2*, respectively.[3] Collateral constraints drastically increase the volatility in the standard incomplete markets model. Figure 3.1 shows the typical behavior of four variables in the long run during a

[3]The stock return volatility in our baseline economy is considerably smaller than the volatility in U.S. data. For comparison, Lettau and Uhlig (2002) report that the quarterly standard deviation of returns of S&P-500 stocks in post-war US data is about 7.5 percent. Similarly, Fei, Ding, and Deng (2010) report an annual volatility of about 14.8 percent for the period January 1987 to May 2008. However, it is important to note that we want to interpret the aggregate tree as a mix of stocks and housing assets. The volatility of housing prices is U.S. data is much lower. Fei, Ding, and Deng (2010) report an annual volatility of the Case/Shiller housing price index of less than 3 percent (for January 1987 to May 2008). A similar comment applies to the equity premium. While the average risk-free rate roughly matches U.S. data, the equity premium is substantially lower than in the data. We discuss this point in more detail in Section 3.4 for an economy with two trees.

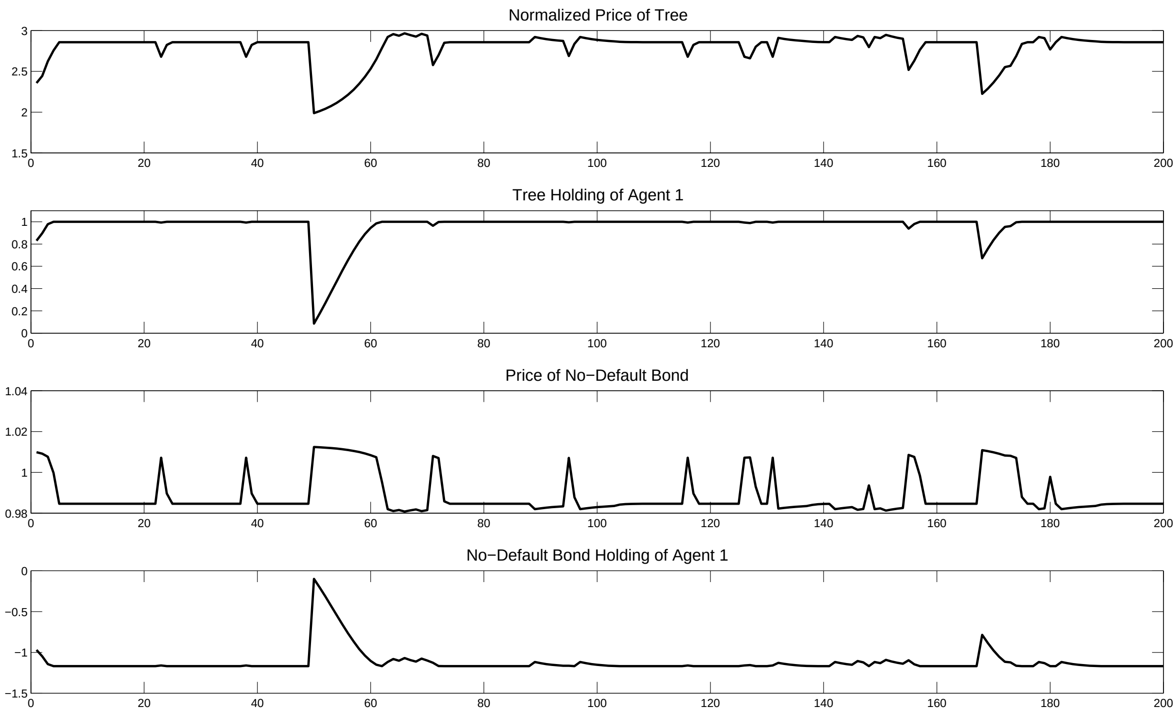

Figure 3.1: Snapshot from a Simulation of the Baseline Model

simulation for a time window of 200 periods. The first graph displays agent 1's holding of the Lucas tree. The second graph shows the normalized tree price, that is, the equilibrium price of the tree divided by aggregate consumption in the economy. The last two graphs show the price and agent 1's holding of the risk-free bond, respectively. In the sample displayed in Figure 3.1, the disaster shock $s = 3$ (smallest disaster with a drop of aggregate consumption of 13.3 percent) occurs in periods 71 and 155 while disaster shock 2 occurs in period 168 and disaster shock 1 (worst disaster) hits the economy in period 50.

When a bad shock occurs, both the current dividend and the expected net present value of all future dividends of the tree decrease. As a result the price of the tree drops, but in the absence of further effects, the normalized price should remain the same since shocks are iid. (That's exactly what happens in the benchmark model *B1*.) In our baseline economy with collateral constraints, however, additional effects occur in equilibrium. First, note that agent 1 is typically leveraged, that is, when a bad shock occurs his beginning-of-period financial wealth falls relative to the financial wealth of agent 2. This effect is the strongest when the worst disaster shock 1 occurs. If agent 1 was fully leveraged in the previous period then her wealth decreases to zero because shock 1 always determines the collateral requirement $k^j_{a(j)}$.

High leverage leads to large changes in the wealth distribution when bad shocks occur. The fact that collateral is scarce in our economy now implies that these changes in the wealth distribution strongly affect equilibrium portfolios and prices. Since agent 1 cannot borrow against her future labor income, she can only afford to buy a small portion of the tree if her financial wealth is low. In equilibrium, therefore, the price has to be sufficiently low to induce the much more risk-averse agent 2 to buy a substantial portion of the tree. On top of that within-period effect, there is a dynamic effect at work. As agent 1 is poorer today, she will also be poorer tomorrow (at least in shocks 2-6) implying that the price of the tree tomorrow is depressed as well. This further reduces the price that agent 2 is willing to pay for the tree today. Clearly, this dynamic effect is active not only for one but for several periods ahead, which is displayed in Figure 3.1 by the slow recovery of the normalized price of the tree after bad shocks. Figure 3.1 shows that the total impact of the above described effects is very strong for shock $s = 1$ but also large for shock 2. Note that the prices are normalized prices, so the drop of the actual tree price is much larger than displayed in the figure. In disaster shock 1, agent 1 is forced to sell almost the entire tree and the normalized price drops by almost 30 percent (the actual price drops by approximately 60 percent). In shock 2 she sells less than half of the tree but the price effect is still substantial. In shock 3 the effect is still clearly visible, although the agent has to sell only very little of her tree.

While the effects of collateral and leverage on volatility are very large, it is important to note that in the baseline specification of our model with a single tree and a single bond there is no *financial accelerator*. Kiyotaki and Moore (1997), Aiyagari and Gertler (1999) and others highlight the idea that in the presence of collateral constraint the fact that the price of collateral decreases might make it more difficult for the borrower to maintain his debt position because collateral requirements increase in anticipation of a value of the collateral in the next period which is now lower than if the shock had not happend. In the baseline case, this effect is absent for two reasons. First, whenever agent 1 is constrained, the collateral requirement $k^j_{a(j)}$ is independent of today's price of the collateral, it is in fact constant. This is because the collateral requirement is determined by tomorrow's tree price (plus dividend) in case of the worst shock. If this shock occurs and agent 1 is constrained today, he has to use his entire tree holding to repay his debt. Hence, no matter how large agent 1's tree holding is today, he ends up with zero financial wealth tomorrow. This implies a specific price for the tree tomorrow in shock 1 which is independent of today's price (as long as agent 1 is constrained) and consequently a specific collateral requirement today[4]. Second, an examination of the bond price in Figure 3.1 reveals an important general equilibrium effect in our economy that counteracts an increase of the margin requirement. When a bad shock occurs and the share of financial wealth of agent 1 decreases, then the demand of the now relatively richer agent 2 for the risk-free asset increases the bond price substantially. In fact, occasionally the interest rate even becomes negative. As a result of the constant collateral requirement, the increase in the bond price and the decrease in the tree price the equilibrium margin requirement actually decreases substantially in a bad shock.

In sum, scarce collateral plays an important role for the volatility of the tree price because it leads to large price drops in bad shocks since agent 1 cannot borrow against future labor income. As we would expect, this effect depends on the amount of available collateral in the economy. Figure 3.2 illustrates this point. The figure depicts the tree's average return volatility and the fraction of times the collateral constraint is binding for agent 1 (i.e. the probability of constraint being binding) as a function of the dividend share δ in the economy.

For very small values of δ, there is only little collateral in the economy and so the collateral constraint is almost always binding. However, the stock is so small that agent 1 does not have to sell the stock even if the economy is hit by an extremely bad aggregate shock. The resulting return volatility is relatively small. As δ increases the probability of the collateral constraint being binding decreases rapidly but the effects of it being binding become larger. There is an interior maximum for the stock-return volatility around $\delta = 0.07$. Although the constraint is much less often binding than for a smaller tree, the trade-off between agent 1 being forced to sell the tree and agent 1 getting into this situation leads to maximal volatility. As δ increases further, the constraint becomes binding much less frequently and eventually at $\delta = 1$ the stock return volatility is very low, simply because the collateral constraint never binds and so collateral plays no role. This situation is identical to the case

[4]If we assume that the tree's dividends cannot be used as collateral, this argument is no longer correct. However, for our calibration the effects of this assumption are quantitatively negligable.

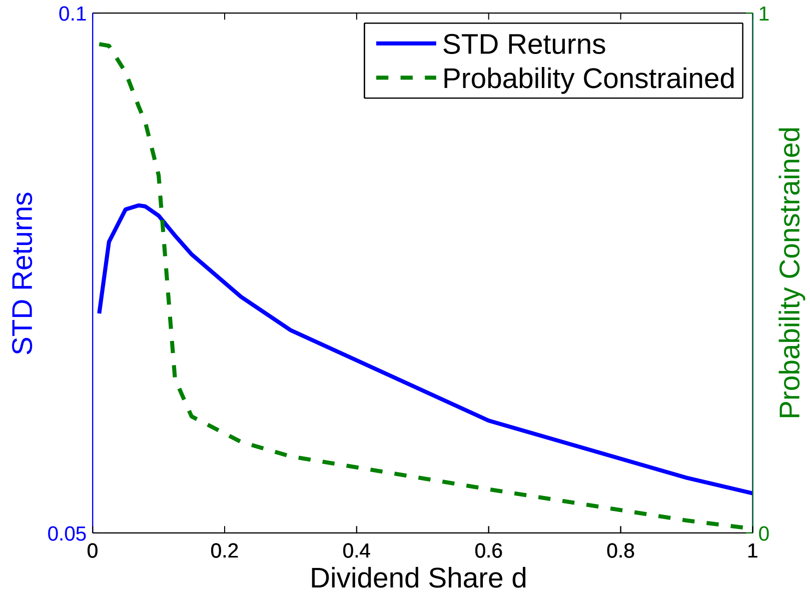

Figure 3.2: Volatility as a Function of the Dividend Share

of natural borrowing constraints where a binding constraint would imply zero consumption for the borrower.

3.3.2 Collateral and Several Bonds

In the economy with a single tree and a single bond, equilibrium margin requirements are sufficiently high to ensure that there is no default. The bond is risk-free and always pays its face value. We now examine whether the observed results are just a consequence of this restrictive assumption. In the enhanced model a menu of bonds is available for trade and the accompanying collateral requirements are endogenously determined in equilibrium.

Full Set of Bonds without Costly Default

Our calibrated model with $S = 6$ exogenous states allows the analysis of economies with five bonds. As explained above, these bonds are characterized by the number of shocks in which they default and so we call them no-default bond, 1-default bond, 2-default bond, etc. Figure 3.3 shows the portfolio holdings of agent 1 as well as the normalized tree price along the same simulated series of shocks as in Figure 3.1 above.

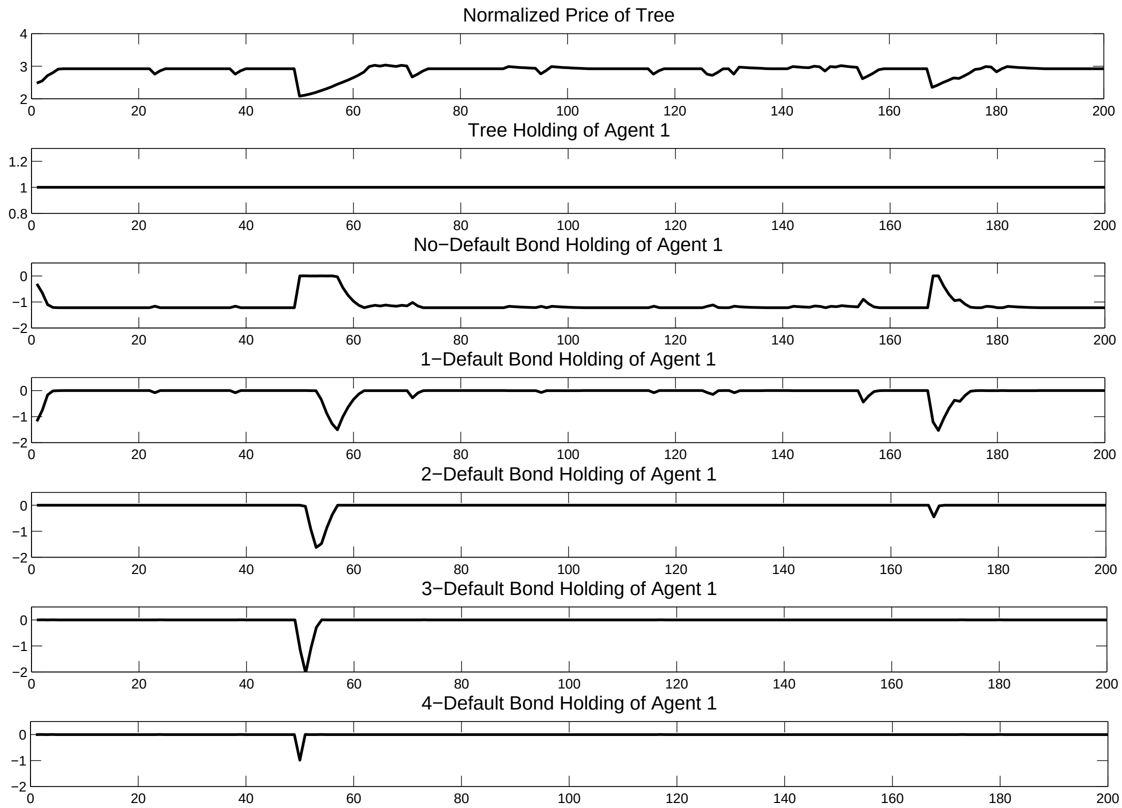

Figure 3.3: Snapshot from a Simulation of the Model with 1 Tree and 5 Bonds

During "normal times" (that is, if the last disaster shock occurred sufficiently long ago) only the no-default bond is traded in equilibrium. (There is a tiny amount of trade in the 1-default bond in recessions, shock 4, which is quantitatively negligible.) In normal times the agents' portfolios resemble those in an economy with a single risk-free bond. The risk-averse agent 2 holds the risk-free bond while agent 1 holds the risky tree and is short in the bond.

Disaster shocks are the only reason for equilibrium trade in default bonds. In our economy, the risk-averse agent 2 always seeks to buy an asset that insures him against bad aggregate shocks — only the risk-free bond can play this role. However, the risky default bonds play an important role once a disaster shock occurs. Agent 1 no longer needs to sell the stock but is now able to raise additional funds by selling default bonds to agent 2. Such a trade shifts some of the tree's risk to agent 2 who demands a high interest rate for assuming such risk. But the default bonds are still less risky than the tree and thus preferred by the risk-averse agent. In fact, the presence of the default bonds enables agent 1 to always hold the entire tree. Figure 3.3 shows that after an occurrence of the worst disaster shock 1, which happens in period 50, agent 1 is able to hold on to the entire tree and to sell the 4-default and the 3-default bond to agent 2. As the economy recovers, agent 1 sells the 1-default bond to agent 2 and holds a short-position in this bond for approximately 10 periods until her wealth has recovered sufficiently so that she is able to leverage exclusively in the default free bond.

Despite the fact that the leveraged agent 1 no longer has to sell the tree after bad shocks, such shocks continue to have a strong impact on asset prices. Figure 3.3 shows that the normalized tree price decreases in all three disaster shocks as well as in recessions, just as in an economy with a single risk-free bond, see Figure 3.1. By selling the default bonds to the risk-averse agent 2, agent 1 shifts the tree's (tail) risk to agent 2. This circumstance must be reflected in the equilibrium price. This reasoning becomes clear if we considered the case of identical dividends in shocks 5 and 6. Under this scenario, the tree and the 4-default bond have identical payoffs and hence it should be irrelevant for the price of the tree who holds it, that is, whether agent 1 holds it financed by a short position in the 4-default bond or agent 2 holds it directly.

Moreover, unlike in the previous model with one bond, the financial accelerator now plays a role. A lower tree holding of agent 1 in this period reduces the price of the tree in the next period in shocks 2-6 and hence makes it more difficult for agent 1 to hold default bonds.

Table 3.3 reports the tree-return volatility for economies with $1, 2, \ldots, 5$ bonds, respectively. The presence of a bond that defaults only in shock 1 (when the economy shrinks by 43.4 percent) leads to a decrease in the volatility of the tree price. A third bond that defaults in shocks 1 and 2 leads to an additional small reduction of volatility. The impact of additional bonds is negligible. This fact is not surprising since we observed that these bonds are rarely traded.

Unfortunately, the fact that investors only trade bonds with a high probability of default during bad times seems counterfactual. Several features of our model may lead to this

	One bond	Two bonds	Three bonds	Four bonds	All bonds
Std returns	8.14	7.87	7.84	7.84	7.84

Table 3.3: The Effect of Endogenous Margins on Return Volatility

	$\lambda = 0$	$\lambda = 0.01$	$\lambda = 0.05$	$\lambda = 0.10$	$\lambda = 0.2$	$\lambda = 0.25$
Std dev tree return	7.84	7.87	7.98	8.12	8.15	8.14
Total trading	1.260	1.236	1.183	1.161	1.126	1.123
No-default bond	1.110	1.099	1.076	1.076	1.099	1.123
1-default bond	0.084	0.080	0.075	0.085	0.027	0
2-default bond	0.034	0.034	0.032	0	0	0
3-default bond	0.026	0.023	0	0	0	0
4-default bond	0.006	0	0	0	0	0

Table 3.4: The Effect of Default Costs on Tree-Return Volatility and Bond Trading Volume

result. Clearly bad times are often persistent and not iid as in our calibration. More importantly, default is typically costly. We next show that fairly small default costs eliminate trade in default bonds.

Costly Default

Until now our treatment of default is somewhat unsatisfactory since it neglects both private and social costs of default. We now introduce default costs as described in Section 3.2.1 above. Table 3.4 shows how the trading volume of the default bonds changes as a function of the cost parameter λ. The reported trading volume is the average absolute bond holding of agent 1 (which is the same as that of agent 2) over the simulation path.

In the absence of default costs ($\lambda = 0$), the average trading volume of all bonds is nonzero. As we observed in the previous section, it is substantial for the no-default and 1-default bond and rather small for the remaining bonds. Proportional default cost of as low as 10 percent ($\lambda = 0.1$) result in zero trade for the bonds defaulting in two or more states. For default costs of 25 percent, trade in any type of default bond ceases to exist. Only the risk-free bond is traded and the resulting equilibrium prices and allocations are identical to our baseline economy above.

Recall from the description in Section 3.2.1 that the cost is proportional to the difference of the face value of the bond and the value of the underlying collateral. Therefore, a proportional cost of 25 percent means a much smaller cost as a fraction of the underlying collateral. Campbell, Giglio, and Pathak (2011) find an average 'foreclosure discount' of 27 percent for foreclosures in Massachusetts from 1988 until 2008. This discount is measured as a percentage of the total value of the house. As a percentage of the difference between the house value and face value of the debt this figure would be substantially larger. A value of $\lambda = 0.25$, therefore, seems certainly realistic and is, if anything, too small when we compare it to figures from the U.S. housing market.

Table 3.4 also reveals that the trading volume of the 1-default bond remains stable up to default costs of around 10 percent when other default bonds are no longer traded. The 1-default bond remains an attractive asset in this economy even for moderate default costs. It is traded when agent 1 is poor. Compared to the no-default bond, it allows to take on more debt for a given amount of collateral. Compared to bonds that default in more states, the expected default costs are much lower. For these reasons, the 1-default bond is the preferred choice in this situation.

Table 3.4 also shows that the volatility of the tree return increases as cost of default increases, and for sufficiently high default cost the economy is the same as the baseline economy with a single risk-free bond. It appears that an economy with default costs of 20 percent and trade in the 1-default bond exhibits slightly higher return volatility than the baseline economy. This feature is due to the fact that default implies real losses in our economy which make the economic impact of the worst disaster shock even worse since default leads to a further drop in aggregate endowment.

3.3.3 Volatility with Regulated Margin Requirements

As a final step in the analysis of economies with a single collateralizable tree, we consider the case of regulated collateral requirements as described in Section 3.2.1. We assume that there is a regulatory agency setting minimal margin requirements (just as in stock markets). We first consider margin requirements that are constant across all shocks, so $m^j_{a(j)}(s^t)$ does not depend on the current date-event s^t. As margin requirements become larger, we observe two opposing effects. On the one hand, the amount of leverage decreases in equilibrium which leads to less de-leveraging in disaster shocks which in turn leads to smaller price changes. On the other hand, the collateral constraint is more likely to become

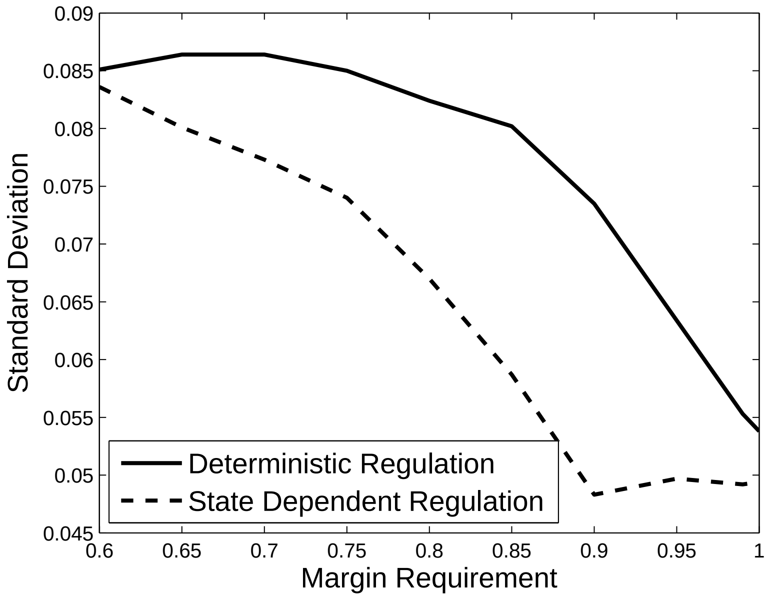

Figure 3.4: Volatility as a Function of the Margin Requirement

binding in equilibrium which increases the probability of de-leveraging episodes which in turn should lead to a higher volatility of the tree return. The solid line in Figure 3.4 displays the resulting tree return volatility.

Initially, volatility increases as margin requirements increase. At a margin level of about 70 percent, the volatility reaches its maximum. A further tightening of margins then decreases volatility substantially. Of course, as the margin level approaches one the economy approaches the benchmark model *(B1: No bonds)* without borrowing and so volatility becomes very small.

At a margin level of 60 percent, the implied collateral requirement uniformly exceeds the corresponding varying levels for the no-default bond under the rule of endogenous collateral requirements in our baseline economy analyzed above. Therefore, the regulated bond is default-free for all possible values of $m^j_{a(j)}$ in Figure 3.4. Interestingly, for values of the margin level between 60 and 80 percent, the regulated bond leads to higher tree return volatility than the no-default bond under the rule of endogenous collateral requirements.

As a last exercise, we examine an economy in which margins are only regulated in booms while in recessions and disasters they are left to the market. In particular, we assume that in shocks 1 through 4 collateral requirements are endogenously determined at the level of the risk-free bond as in our baseline economy, while a regulating agency sets margin requirements in the shocks with positive growth. We assume that the margin levels are set to the same level in both shocks 5 and 6. The dashed line in Figure 3.4 shows the resulting tree return volatility.

It is readily apparent that limiting the regulation of margin requirements to boom times reduces the tree return volatility substantially if margin levels are sufficiently high. For example, boom-time margin levels of 80 percent lead to a return volatility of 6.5 percent as compared to values exceeding 8 percent when collateral requirements are determined endogenously or margin regulation is state-independent.

Why is state-dependent regulation so much better in reducing volatility? As with state-independent margins, agent 1 holds less leverage in good times, which leaves him with more financial wealth if a bad shock hits. In addition, collateral constraints are now looser in case of a bad shock and agent 1 may retain an even larger portion of the tree. In the extreme, if margin requirements in booms are well above 80 percent, agent 1 even increases its tree holding in case of a bad shock. This increases the relative price of the tree and thus dampens the drop in the absolute price. All in all, setting conservative margins in good times turns out to be a powerful tool to dampen the negative impact of bad shocks.

3.4 Two Trees

Up to this point our analysis focused on an economy with a single tree representing aggregate collateralizable wealth in the economy. However, households trade in various assets and durable goods. Some of them, e.g. houses, can be used as collateral very easily and at comparatively low interest rates, others assets, e.g. stocks, can only be used as collateral for loans with high margin requirements and typically very high interest rates (see Willen and Kubler (2006)), and still others, like works of art, cannot be used as collateral at all. These observations motivate us to examine a model with two Lucas trees. For simplicity, we assume that the two trees have identical cash-flows and distinguish themselves only by the extent to which they can be used as collateral. This model feature allows for a clean analysis of the effect of collateral. We consider two different cases. First, we assume that tree 1 can be used as collateral with endogenous margin requirements, while tree 2 cannot be used as collateral. We then allow the second tree to serve as collateral, but we assume that the collateral requirements on loans backed by tree 2 are exogenously regulated. In both cases we find that the two assets' price dynamics are substantially different, despite the fact that they have identical cash-flows. Furthermore, we show that tightening the margin requirements on the regulated tree has a strong impact on the return volatility of the non-regulated tree. This effect proves to be quantitatively important. Our analysis suggests that this effect should be carefully considered in any policy discussion on the regulation of margin requirements.

3.4.1 Only one Tree can be Used as Collateral

We first consider the case where the second tree cannot be used as collateral. As before in an economy with a single tree, default costs of $\lambda = 0.25$ suffice to shut down all trade in default bonds. We therefore restrict attention to an economy in which only the no-default bond is traded. We conclude the analysis in this section below with a brief discussion of an economy with costless default and argue that it produces similar quantitative results.

Table 3.5 reports moments of the two trees' returns as well as the interest rate and aggregate moments. Observe that the two trees exhibit substantially different returns despite the fact that the two trees have identical cash-flows. The tree that can be used as collateral, tree 1, now exhibits much lower return volatility and a slightly lower expected excess return than

	Std returns	EP agg	Std returns agg	Risk-free rate	Equity-premium
Tree 1	6.64	3.69	7.04	0.38	4.50
Tree 2	8.05	6.31			

Table 3.5: Moments of Trees' Returns (only Tree 1 Collateralizable)

the single tree in the baseline economy in Section 3.3. The standard deviation of returns of the second tree is much higher than that of tree 1. In fact, it is comparable to the corresponding value (8.14) of the single tree in the baseline economy. Turning to equity premia, the excess return of tree 2 — the tree that cannot be used as collateral — is now almost twice as large as it is for the single tree in the baseline economy and is similar to figures observed in the data.

To understand the price dynamics of the two trees, we consider the analogue of Figure 3.1. Figure 3.5 shows the time series of eight variables along 'our' sample path. The first two graphs show the (normalized) price and the first agent's holding of tree 1, respectively. The next two graphs display the corresponding values for tree 2. The fifth and sixth graph show the corresponding values for the no-default bond. The price and holding graphs reveal three features of the equilibrium. First, the price volatility of tree 1 is much lower than that of the single tree in the baseline economy. Secondly, the price volatility for tree 2 is larger than for tree 1 and its average price is much smaller. Lastly, agent 1 holds tree 1 the entire time (except for a tiny blip in disaster shock 1) but frequently sells tree 2. The second-to-last graph in the figure shows the endogenous margin requirement and the last graph depicts the *collateral premium* for tree 1. This quantity is the difference between the actual price of the tree and next period's payoff, normalized with agent 1's marginal utilities. Whenever agent 1 is unconstrained then this value is zero. However, when agent 1 becomes constrained, the collateral premium is significant.

Our observations lead us to a simple explanation of the first moments for the two tree prices. Tree 1 is more valuable to agent 1 because of its collateral value — when agent 1 is fully leveraged the value of the tree exceeds next period's discounted (with agent 1's state prices) cash-flows since it provides value for agent 1 as collateral. Since both trees have identical cash-flows, an agent can only be induced to hold tree 2 if it pays a higher average return. The specific magnitude of the difference between the two tree prices is, of course, a quantitative issue. In our calibration with a reasonable market price of risk, the effect

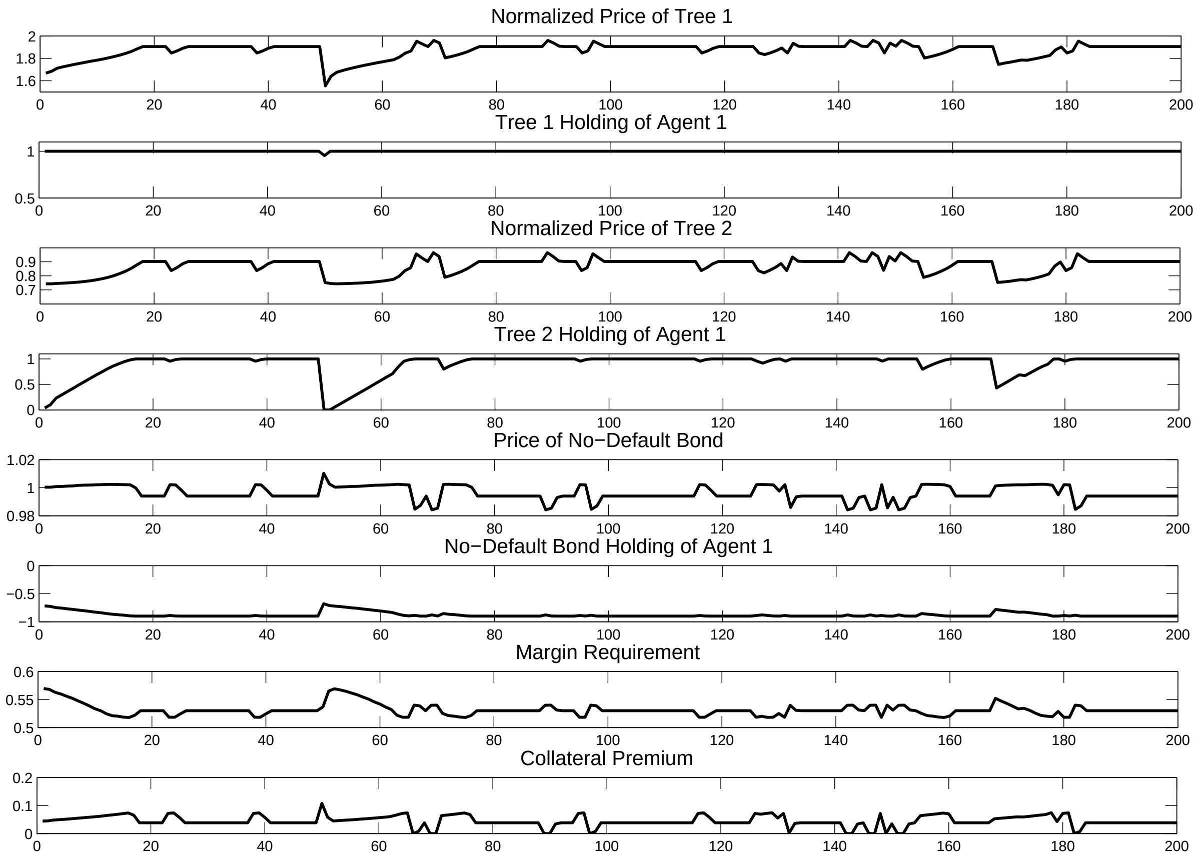

Figure 3.5: Snapshot from a Simulation of the Model with 2 Trees and 5 Bonds

is indeed large — the average excess return of the second tree is now comparable to that observed in U.S. stock market data.

There are several key factors that play a role for asset price volatility in the two-tree economy. For a discussion of these factors it is helpful to consider the policy and price functions in Figure 3.6. When faced with financial difficulties after a bad shock, agent 1 holds on to tree 1 for as long as possible, because this tree allows her to hold a short-position in the bond. (In fact, as the bond-holding function of agent 1 in Figure 3.6 shows, agent 2 never goes short in the bond. Therefore, the collateral value is one of the reasons why tree 1 is much more valuable to agent 1.) So, after suffering a reduction in financial wealth, agent 1 first sells tree 2. In fact, in our calibration agent 1 only sells a portion of tree 1 after she sold off the entire tree 2. In our sample path this happens only after the

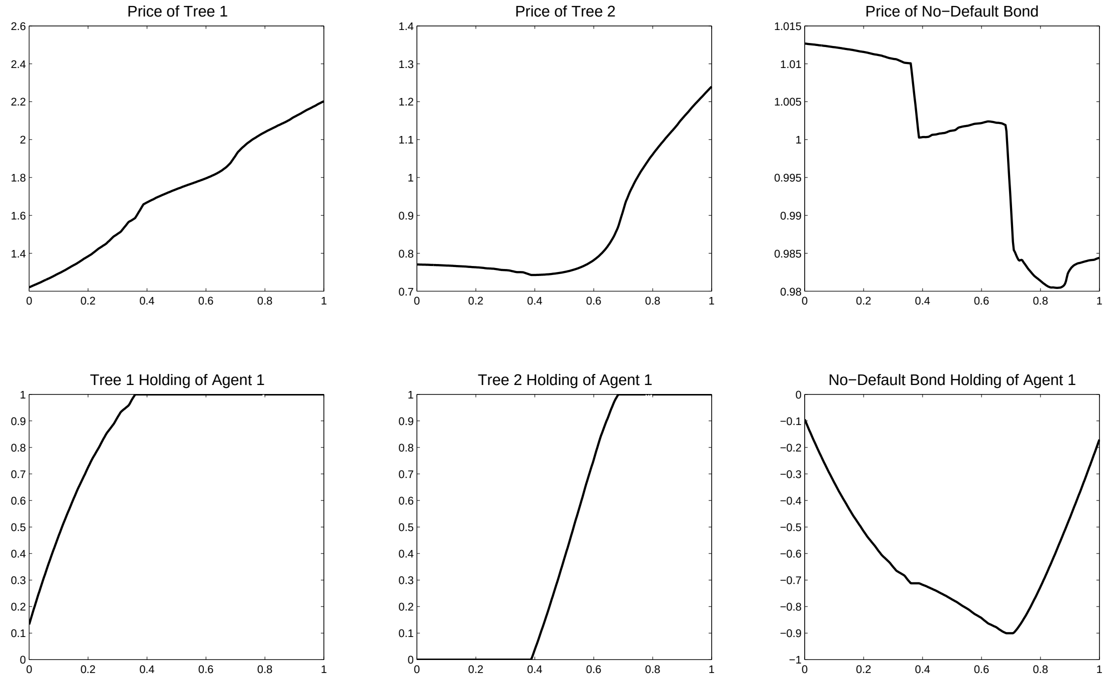

Figure 3.6: Price and Policy Functions of the Model with 2 Trees and 5 Bonds

worst disaster shock in period 50. (Of course, the policy functions in Figure 3.6 show that it would happen in a more pronounced way after two or more consecutive disaster shocks but such a sequence has extremely low probability.) Whenever agent 1 sells a portion of a risky tree to agent 2 its price must fall, just as in the single-tree baseline economy. And so one key factor contributing to the different volatility levels of the two trees is that tree 2 is traded much more often and in larger quantities than tree 1.

Furthermore, since tree 2 is not collateralizable, only half of the aggregate tree can be used as collateral. This constraint limits the ability of agent 1 to leverage and consequently makes her less vulnerable to negative aggregate shocks. This factor reduces the return volatility of both trees.

If agent 1 holds both trees and then becomes poorer after a bad shock, the prices of both trees fall. But since the agent first sells tree 2, the price of tree 2 falls much faster than the price of tree 1. In fact, the price drop for tree 1 is dampened by the onset of the collateral premium. This effect also contributes to the difference in the return volatilities of the two trees.

Finally, there is another key effect that was not present in the one-tree baseline economy. Now the financial accelerator plays an important role! In 'normal times' agent 1 holds both trees but is fully leveraged. In a bad shock, agent 1 must sell part of tree 2 which makes him poorer in the subsequent period. This in turn increases the collateral requirement this period, leading to an increase in the margin requirement despite the fact that the interest rate decreases. This effect is clearly visible in the second-to-last graph of Figure 3.5. Whenever a bad shock occurs the margin requirement increases.

In sum, the fact that only 4 percent of aggregate output are collateralizable in this economy leads to a decrease in leverage and to much smaller movements in the wealth distribution than in the baseline economy. This effect reduces the return volatility of tree 1. For tree 2 such a reduction effect is strongly counteracted through two channels. First, the price of tree 2 is not stabilized by a collateral premium since this tree cannot be used as collateral. Secondly, a decrease in the holdings of tree 2 leads to an increase in the margin requirements for loans on tree 1 which in terms forces agent 1 to sell more of tree 2 (recall that initially he does not sell tree 1, since only this tree can be used as collateral).

While we do not want to push the interpretation of our results too far, it is worthwhile to note that a natural interpretation of the two trees is the aggregate stock market versus the aggregate housing market. As Willen and Kubler (2006) report, it is much more difficult to use stocks instead of a house as collateral. The data clearly shows that volatility in the stock market is much higher than in the housing market, see Fei, Ding, and Deng (2010). This interpretation clearly should be taken with some caution, since we do not really have a good model of the housing market — such a model would need to include transaction costs, non-divisibilities, and certainly different cash-flow dynamics. Nevertheless it is worthwhile to point out that the equity premium for tree 2 is similar to what can be observed in the data for stock returns. Moreover, volatility of "housing returns" (tree 1) is much smaller than that of stock returns.

We complete our discussion of the economy in which the second tree cannot be used as collateral with a robustness check and consider the case of costless default, $\lambda = 0$. Just as in the economy with a single tree, the default bonds are traded if the economy experiences a disaster shock. However, trade in these bonds is typically much smaller because agent 1's financial wealth remains larger, as we discussed above. Overall, zero default costs lead to very small changes in the first and second moments. Without default costs, the standard deviation of tree 1's return drops from 6.64% to 6.56% while the standard deviation of tree 2's return drops from 8.05% to 7.98%.

3.4.2 One Tree is Regulated

Until now we have assumed that tree 2 cannot be used as collateral. This assumption is rather restrictive if not unrealistic. Stocks can be used as collateral, however, margins are regulated and large, and interest rates are much higher than mortgage rates. Therefore, we assume now that margins for tree 2 are set exogenously while collateral requirements for tree 1 are endogenous. Throughout this section, we assume default costs of $\lambda = 0.25$ which suffice to shut down all trade in default bonds.

State-Independent Regulation of Tree 2

As before, we first consider margin requirements that are constant across states. The effect of an exogenous margin requirement is obvious in the limit as the requirement m_2 for tree 2 approaches one. In this case the resulting collateral requirement k_2 diverges to infinity and so the model tends to the economy of Section 3.4.1 in which this tree cannot be used as collateral. Figure 3.7 display the volatility of both trees' returns as a function of the margin m_2 set for tree 2. Observe that as m_2 tends to one, the return volatilities for the two trees approach the values from Table 3.5, namely 6.64% and 8.05%, respectively.
Figure 3.7 shows the return volatilities for values of m_2 between 0.6 and 1. The lowest value of 0.6 of the margin requirement exceeds the endogenously determined (unregulated) margin requirement of tree 1 in all states. As a result, the return volatility of tree 2 is higher that that of tree 1. If margin requirements on tree 2 are now increased, the volatility of this tree's return initially increases, while the volatility of the freely collateralizable tree 1 substantially decreases. The volatility of tree 2 is largest when its exogenous margin requirement is quite high (about 75 percent). After this peak, the volatility of tree 2 decreases until the boundary value of one has been reached. At this point tree 2 can no longer be used as collateral. The quantitatively most interesting case is a regulated margin requirement of 75 percent. At this point, the volatility of tree 2 is above 8.6 percent while the volatility of tree 2 is below 7.5 percent. Aggregated volatility is still high, but it is readily apparent that the regulation of tree 2 has substantial effects on its own volatility

as well as on the volatility of the other, unregulated, tree 1.

For an interpretation of the observed volatility variation, note that an increase of the margin requirement m_2 of tree 2 has two immediate effects. This tree becomes less attractive as collateral and the agents' (aggregated) ability to leverage decreases. These two effects influence agent 1's portfolio decisions after a bad shock occurs. First, when agent 1 must de-leverage her position, then she first sells tree 2. In equilibrium, this effect occurs more often as m_2 increases. Initially this effect leads to an increase in the return volatility of tree 2. The second effect, a reduced ability to leverage, decreases the return volatility of tree 1. Similar to the effect we observed in the one-tree economy in Section 3.3, the return volatility of tree 1 decreases as agent 1's ability to leverage decreases. The reason for this effect is the increased probability with which she can hold onto the tree after a bad shock. Observe that the two described effects counteract each other for tree 2. For small increases of m_2 above 0.6, the first effect dominates the second and the tree's return volatility increases. As m_2 increases further, the second effect eventually dominates and the return volatility of tree 2 starts to decrease. Moreover, as the margin requirement on tree 2 becomes large, price effects as a result of agent 1 de-leveraging her positions become smaller. Recall that whenever agent 1 is collateral constrained, then the price of the underlying collateralized tree reflects a collateral premium. Since agent 2 never enters leveraged positions, this price impact is never present when agent 2 holds tree 2. As a result the collateral premium affects the price volatility of tree 2. This effect is greatly diminished as m_2 becomes sufficiently large. Put differently, the impact of the collateral premium on the return volatility fades as m_2 gets large.

To support this interpretation of our results, it is interesting to consider the excess returns of the two trees as a function of the margin requirement m_2 on tree 2. Figure 3.8 shows that the relation between excess return and m_2 is monotone for tree 2. As its margin requirement increases, the collateral premium and the price of the tree decrease and the average return increases. For tree 1, average excess returns remain more or less constant. They initially decrease slightly, then increase slightly. Aggregate excess returns increase, but clearly the quantitatively most striking effect is on the returns of tree 2. Collateral constraints and regulated margins clearly have a quantitatively significant impact on asset prices in this economy.

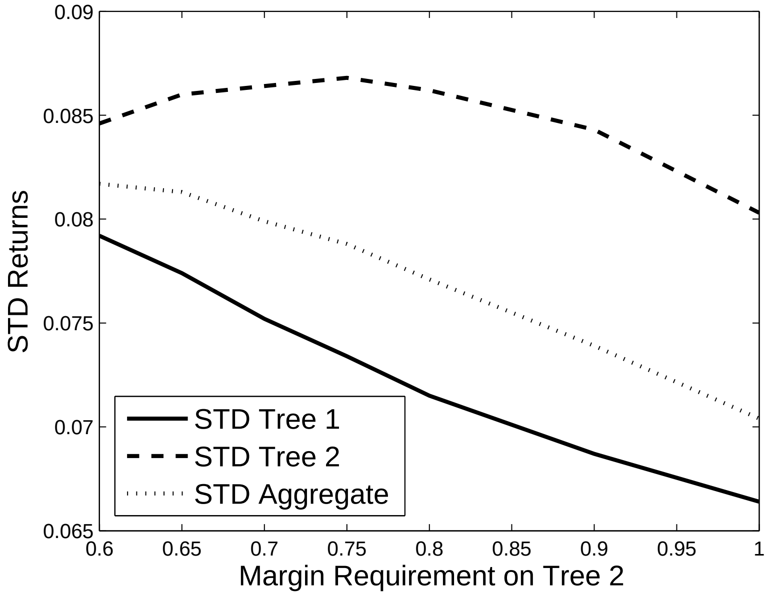

Figure 3.7: Volatility of Tree 1 and 2 as a Function of the Margin Requirement on Tree 2

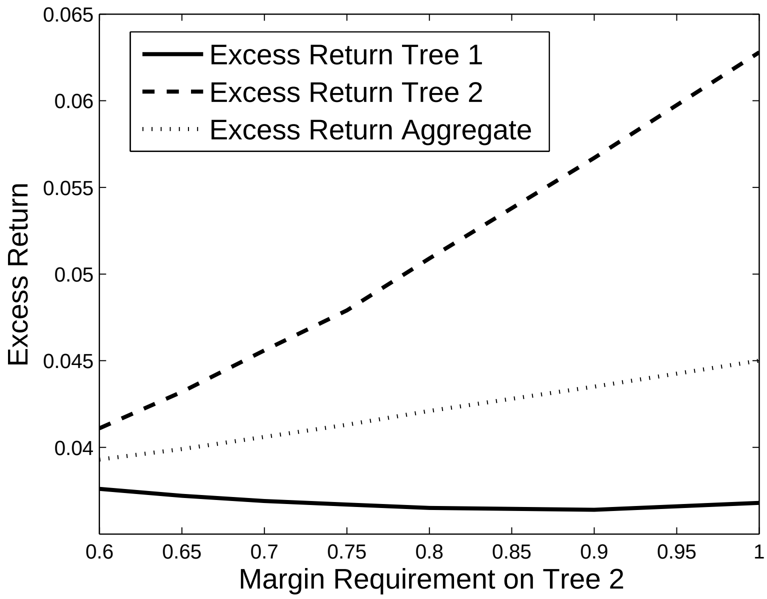

Figure 3.8: Excess Returns of Tree 1 and 2 as a Function of the Requirement on Tree 2

State-Dependent Regulation of Tree 2

Our results so far have shown that, for moderate margin requirements between 0.6 and 0.75, it is impossible to reduce volatility for both trees by adjusting the regulated margin of tree 2. A small change of the margin requirement always reduces volatility of one tree at the expense of the other. We now analyze whether a state-dependent regulation of the second tree can solve this dilemma.

We examine an economy in which the margins of tree 2 are only regulated for positive-growth shocks 5 and 6 while they are endogenously determined for the remaining four shocks. Figure 3.9 shows that the return volatilities of both trees are monotonically decreasing in the margin requirement imposed on the regulated tree 2 in good shocks. Not only does increasing the regulated margin now reduce the volatilities of both assets, but, in fact, it does reduce aggregate market volatility much more than in the economy with state-independent regulation. For instance, an increase of state-dependent margin requirements from 0.6 to 0.7 on tree 2 decreases aggregate volatility by about 4.5% (see Figure 3.9), while such an increase would bring about a reduction of only 2% in the case of state-independent regulation (see Figure 3.7). Therefore, concerning the regulation of margin requirements, the result from the single-tree economy is strongly confirmed by the analysis of the two-tree economy: regulation is much more efficient at reducing price volatility, if it is state-dependent.

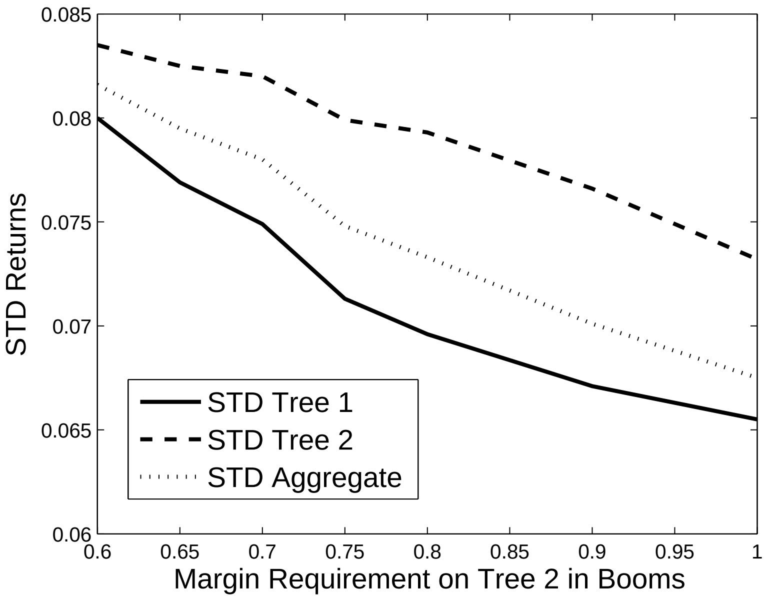

Figure 3.9: Volatility as a Function of the Margin Requirement on Tree 2 in Booms

3.5 Sensitivity Analysis and Extensions

As in any quantitative study, our results above hinge on the parametrization of the economy. In this section, we first discuss how our results change with other preference parameters. Then we highlight the important role of the disaster shocks for our quantitative results. Finally, we present an example which has less severe disaster shocks but nevertheless exhibits strong quantitative effects of collateral constraints.

3.5.1 Different Preferences in the Baseline Model

As a robustness check for the results in our baseline model (with one tree and one bond) from Section 3, we consider different specifications for the IES, the coefficients of risk aversion, and the discount factor, β. Obviously, changes in the IES and the risk aversion coefficients affect the risk-free rate. For these cases, we also examine specifications with an adjusted β so that the risk-free rates remain comparable. Table 3.6 reports asset-price moments for several different combinations of these parameters. For convenience, we repeat the results for our baseline model, $(IES, RA, \beta) = ((1.5, 1.5), (0.5, 6), (0.95, 0.95))$, and report them as the case (P1). For each model specification, we also report the standard deviation of returns for the benchmark case *B1: No bonds.*
In case (P2), a model in which both agents have an IES of 0.5, the tree return volatility is considerably lower than in the baseline case (P1). However, it is still much higher than in an economy with the same preferences but without borrowing, see column B1 of (P2). We checked this result for other values of the IES below 1.5 and always observed

$(IES^1, IES^2), (RA^1, RA^2), (\beta^1, \beta^2)$	Std returns	Risk-free rate	EP	Std in B1
(P1): (1.5,1.5),(0.5,6),(0.95,0.95)	8.14	1.10	3.86	5.33
(P2): (0.5,0.5),(0.5,6),(0.95,0.95)	7.20	1.75	4.18	5.33
(P3): (1.5,1.5),(0.5,6),(0.92,0.92)	7.70	4.07	3.77	5.51
(P4): (1.5,1.5),(0.5,6),(0.98,0.98)	8.57	-1.17	3.95	5.23
(P5): (1.5,1.5),(0.5,10),(0.95,0.95)	10.79	-8.58	12.55	5.34
(P6): (1.5,1.5),(0.5,10),(0.81,0.81)	8.50	1.25	13.36	6.24
(P7): (1.5,1.5),(0.5,4),(0.95,0.95)	6.58	1.59	4.22	5.34
(P8): (1.5,1.5),(0.5,4),(0.98,0.98)	6.97	1.18	1.73	5.22

Table 3.6: Sensitivity Analysis for Preferences (all Reported Figures in Percent)

the same phenomenon: Volatility effects are qualitatively similar but quantitatively less pronounced.[5]

Next we consider a change in the discount factor β. For the benchmark case *B1*, a higher β decreases return volatility simply because it decreases levels of returns and we report absolute volatility as opposed to the coefficient of variation. The effects in our model with one tree and one bond are quite different. As β increases from 0.95 in our baseline case (P1) to 0.98 in (P4), the return volatility increases from 8.14 to 8.47. The reason for this increase is simple. As β increases and the stock becomes more expensive, it is more difficult for agent 1 to buy a significant portion of the stock when he is in financial difficulties. This fact depresses the price of the stock when agent 1 is poor. Changes in the wealth distribution are large when agent 1 is fully leveraged and lead now to larger swings in the tree price.

In light of the intuition that we developed for the baseline case in Section 3.3, we expect an increase in the risk aversion of agent 2 to lead to both a higher price volatility and a higher equity premium. This intuition is strongly confirmed by the comparison of (P1) and (P5). However, the increase in the second agent's risk aversion also leads to a large reduction of the interest rate to unrealistically low levels. In (P6) we recalibrate the model to obtain a positive interest rate and we find that the previously described effect of a smaller β dampens the impact of a higher risk aversion. But still, overall volatility increases substantially once the risk aversion and β are changed simultaneously: For risk aversions of 4, 6, and 10, (cases (P8), (P1) and (P6)) the return volatility is 6.97, 8.14, and 8.50 respectively.

3.5.2 Endowments

As we have seen repeatedly in our analysis, our model produces asset pricing moments that are comparable to observed values in the data. Clearly, this nice feature of our model depends on the magnitude of the disaster shocks. We now report results for models with less severe disaster shocks and demonstrate that the results remain qualitatively the same. We conduct two different types of sensitivity analysis for our shock process. First, in the

[5] For low values of the IES, there is an additional unwanted effect. As one agent holds most of the wealth (that is, as the other agent becomes poor), asset prices increase because of the desire of the rich agent to save. This effect on the boundary of the state space is absent when the IES is set to 1.5 which we, therefore, do for the remainder of our analysis.

	Std returns	Risk-free rate	Equity-premium	Std in B1
Case (E1)	5.95	3.44	2.17	4.15
Case (E2)	3.92	5.97	0.36	3.51

Table 3.7: Sensitivity Analysis for Endowments (all Figures are in Percent)

case (E1) we hold the magnitude of the disaster shocks constant, but reduce the overall probability of a disaster by 50 percent. Instead of setting the probabilities of shocks 1, 2, and 3 to 0.005, 0.005, and 0.024, respectively, we set them at 0.0025, 0.0025, and 0.012, respectively, and increase the probability of shock 5 accordingly. Secondly, in the case (E2) we leave the probabilities of the shocks unchanged but shift their support. In particular, we replace the growth rates in shocks 1, 2, and 3 of 0.566, 0.717, and 0.867, respectively, by the new values of 0.783, 0.8585, and 0.9335, respectively. Table 3.7 shows the analogue of Table 3.6 for these two cases. The table shows that a decrease in the probability of disaster has a relatively small effect on volatility while a change in the support has quite a large effect. As we explained above, the disaster states play two roles in our model. First, they lead to high excess returns of the tree, in particular whenever the risk-averse agent 2 must hold the tree. Secondly, they lead to endogenously high margin requirements. As we decrease the probability of disaster, the second effect remains unchanged. In contrast, the change in the support of the disaster shocks mitigates both effects above.

3.5.3 Large Effects with Smaller Shocks

The results for the case (E2) above show that the quantitative impact of collateral constraints depends heavily on the size of the disaster shocks. However, we now demonstrate that even with halved disaster shocks as in (E2), there are still substantial effects. For this purpose, we consider the model with two trees where tree 1 is collateralizable and tree 2 is not, and assume that agent 1's risk aversion is 10. We recalibrate the discount factor β to be 0.98, which results in a risk-free rate of 1.94. Table 3.8 shows that aggregate volatility with collateral constraints is now 48% higher than in the benchmark B1. This increase is of similar magnitude as in the baseline model. The high aggregate volatility is mostly driven by the volatility of tree 2, which increases by 95% compared to this benchmark.

	Std returns	EP	Std returns agg	Risk-free rate	EP agg	Std in B1
Tree 1	4.41	0.77	5.05	1.94	1.02	3.42
Tree 2	6.68	1.65				

Table 3.8: Moments of Trees' Returns (Tree 1 Collateralizable, Tree 2 not)

3.6 Conclusion

In this paper we show that collateral and margin requirements play a quantitatively important role for prices of long-lived assets. This is true even for assets that cannot be used as collateral. In fact, somewhat surprisingly, we show that the presence of collateral constraints has a larger effect on the volatility of non-collateralizable assets than on the underlying collateral.

The recent financial crisis has lead researchers to suggest that central banks should regulate collateral requirements, see, for example, Ashcraft, Gârleanu, and Pedersen (2010) or Geanakoplos (2009). We show that tightening margins uniformly over the business cycle can increase the price volatility of the underlying collateral but typically decreases price volatility of other long-lived assets in the economy that are not directly affected by the regulation. The only policy to achieve a decrease of the price volatility of all assets is to tighten margins only in boom times but leave them to market forces in recessions or crises. Our calibration assumes the presence of disaster shocks as in Barro (2009). We provide alternative parameterizations of preferences and endowments under which our main qualitative results continue to hold.

Appendix

3.A Details on Computations

The algorithm used to solve all versions of the model is based on Brumm and Grill (2010). Equilibrium policy functions are computed by iterating on the period-to-period equilibrium conditions, which are transformed into a system of equations. We use KNITRO to solve this system of equations for each grid point. Policy functions are approximated by piecewise linear functions. By using fractions of financial wealth as the endogenous state variables, the dimension of the state space is equal to the number of agents minus one. Hence with two agents, the model has an endogenous state space of one dimension only. This makes computations much easier than in Brumm and Grill (2010), where two and three dimensional problems are solved. In particular, in one dimension reasonable accuracy may be achieved without adapting the grid to the kinks. For the reported results we used 320 or 640 grid points depending on the complexity of the version of the model, which results in average (relative) Euler errors with order of magnitude 10^{-4}, while maximal errors are about ten times higher. If the number of gridpoints is increased to a few thousands, then Euler errors fall about one order of magnitude. However, the considered moments only change by about 0.1 percent. Hence, using 320 or 640 points provides a solution which is precise enough for our purposes. Compared to other models the ratio of Euler errors to the number of grid points used might seem large. However, note that due to the number of assets and inequality constraints our model is numerically much harder to handle than standard models. For example, in the version with one tree and five bonds, eleven assets are needed (as long and short positions in bonds have to the treated as separate assets) and we have to impose eleven inequality constraints per agent.

3.B Equilibrium Conditions

We state the equilibrium equations as we implemented them in Matlab for economies with a single tree and a single bond. For our computation of financial markets equilibria we normalized all variables by the aggregate endowment $\bar{e}$. To simplify the notation, we drop the dependence on the date-event s^t and, in an abuse of notation, denote the normalized parameters and variables by e_t, d_t and c_t, q_t, p_t, r_t, f_t, respectively. Similarly, we normalize both the objective function and the budget constraint of agents' utility maximization problem. The resulting maximization problem is then as follows (index h is dropped).

$$\begin{aligned} \max \quad u_t(c_t) = \quad & \left\{(c_t)^\rho + \beta\left[E\left(u_{t+1}g_{t+1}\right)^\alpha\right]^{\frac{\rho}{\alpha}}\right\}^{\frac{1}{\rho}} \\ \text{s.t.} \qquad 0 = \quad & c_t + \phi_t p_t + \theta_t q_t - e_t - [\phi_{t-1}]^+ \frac{r_t}{g_t} + [\phi_{t-1}]^- \frac{f_t}{g_t} - \theta_{t-1}\left(q_t + d_t\right) \\ 0 \leq \quad & \theta_t + k_t[\phi_t]^-, \quad 0 \leq [\phi_t]^+, \quad [\phi_t]^- \leq 0, \end{aligned}$$

The latter two inequalities are imposed because, for the computations, we treat the long and short position in the bond, $[\phi_t]^+$ and $[\phi_t]^-$, as separate assets. Note that $\phi_t = [\phi_t]^+ + [\phi_t]^-$. Let λ_t denote the Lagrange multiplier on the budget constraint. The first-order condition with respect to c_t is as follows,

$$0 = (u_t)^{1-\rho}(c_t)^{\rho-1} - \lambda_t.$$

Next we state the first-order condition with respect to c_{t+1}.

$$0 = \beta u_t^{1-\rho}\left[E\left(u_{t+1}g_{t+1}\right)^\alpha\right]^{\frac{\rho-\alpha}{\alpha}}\left(u_{t+1}g_{t+1}\right)^{\alpha-1} g_{t+1}(u_{t+1})^{1-\rho}(c_{t+1})^{\rho-1} - \lambda_{t+1}.$$

Below we need the ratio of the Lagrange multipliers,

$$\frac{\lambda_{t+1}}{\lambda_t} = \beta\left[E\left(u_{t+1}g_{t+1}\right)^\alpha\right]^{\frac{\rho-\alpha}{\alpha}}(u_{t+1})^{\alpha-\rho}(g_{t+1})^\alpha\left(\frac{c_{t+1}}{c_t}\right)^{\rho-1}$$

Let μ_t denote the multiplier for the collateral constraint and let $\hat{\mu}_t = \frac{\mu_t}{\lambda_t}$. We divide the first-order condition with respect to θ_t,

$$0 = -\lambda_t q_t + \mu_t + E\left(\lambda_{t+1}\left(q_{t+1} + d_{t+1}\right)\right)$$

by λ_t and obtain the equation

$$0 = -q_t + \hat{\mu}_t + \beta\left[E\left(u_{t+1}g_{t+1}\right)^\alpha\right]^{\frac{\rho-\alpha}{\alpha}} E\left((u_{t+1})^{\alpha-\rho}(g_{t+1})^\alpha\left(\frac{c_{t+1}}{c_t}\right)^{\rho-1}(q_{t+1} + d_{t+1})\right)$$

Similarly, the first-order conditions for $[\phi_t]^+$ and $[\phi_t]^-$ are as follows,

$$\begin{aligned} 0 &= -p_t + \nu_t^+ + \beta \left[E\left(u_{t+1}g_{t+1}\right)^\alpha\right]^{\frac{\rho-\alpha}{\alpha}} E\left((u_{t+1})^{\alpha-\rho}(g_{t+1})^\alpha \left(\frac{c_{t+1}}{c_t}\right)^{\rho-1}\left(\frac{r_{t+1}}{g_{t+1}}\right)\right) \\ 0 &= -p_t + \hat{\mu}_t k_t - \nu_t^- + \beta \left[E\left(u_{t+1}g_{t+1}\right)^\alpha\right]^{\frac{\rho-\alpha}{\alpha}} E\left((u_{t+1})^{\alpha-\rho}(g_{t+1})^\alpha \left(\frac{c_{t+1}}{c_t}\right)^{\rho-1}\left(\frac{f_{t+1}}{g_{t+1}}\right)\right), \end{aligned}$$

where ν_t^+ and ν_t^- denote the multipliers on $0 \leq [\phi_t]^+$ and $[\phi_t]^- \leq 0$, respectively.

Chapter 4

Computing Equilibria in Models with Occasionally Binding Constraints

4.1 Introduction

In many applications of dynamic stochastic (general) equilibrium models, it is a natural modeling choice to include constraints that are occasionally binding. Examples are models with borrowing constraints, liquidity constraints, a zero bound on the nominal interest rate, or irreversible investments. These constraints induce non-differentiabilities in the policy functions, which make it challenging to compute equilibria. In particular, standard interpolation techniques using non-adaptive grids perform poorly both in terms of accuracy and shape of the computed policy function (see, e.g. Judd, Kubler, and Schmedders (2003), pp.270-1). This paper proposes a method that overcomes these problems, even for models with several continuous state variables. We call this method Adaptive Simplicial Interpolation (ASI). Its working principle is to locate the non-differentiabilities that are induced by occasionally binding constraints, and to put additional interpolation nodes there.

We present our algorithm in the setting of a dynamic endowment economy where three or four (types of) agents face aggregate and idiosyncratic risk. To explain the main features of ASI we first compute equilibria in a simple two period version where agents trade in a bond subject to an ad-hoc borrowing constraint. Second, we embed ASI into a time iteration algorithm to solve an infinite horizon version of the model. Finally, we add a Lucas tree-type stock, which is subject to a short sale constraint, and we replace the ad-hoc

borrowing constraint by a collateral constraint. Consequently, short positions in the bond need to be collateralized by stock holdings, while the stock may not be shorted.
Compared to earlier papers using a similar setup, such as Heaton and Lucas (1996), den Haan (2001) or Kubler and Schmedders (2003), the models we consider differ in two respects, which both make it harder to compute equilibria: First, we solve models with more agents, which results in a continuous state space of higher dimension. As the kinks[1] naturally form hypersurfaces in the state space, they are of higher dimension as well. Second, in our extension, the trading constraints that agents face depend on tomorrow's equilibrium price of the stock, which is endogenously determined. Consequently, it is much harder to locate the kink and ad hoc methods fail.

Figure 4.1 illustrates the working principle of ASI. The dashed line displays a simple one-dimensional policy function with a kink. Suppose this function is approximated by linear interpolation between equidistant grid points. The resulting interpolated policy is displayed as a solid line in the left hand side of Figure 1. Clearly, the approximation error is comparatively large around the kink, and this is just because there is no interpolation node near the kink. If we knew the location of the kink and put a node there, then the approximation would be much better, as the right hand side of Figure 4.1 shows. This is the motivation for ASI, which directly addresses the problem of kinks in policy functions by placing additional grid points, called *adapted points*, at these non-differentiabilities. Clearly, in higher dimensional state spaces and with complex constraints, this approach is not as simple as Figure 4.1 suggests. Hence, we need a flexible interpolation technique and a systematic adaptation procedure.
To be able to place grid points wherever needed, we use *Delaunay interpolation*, which consists of two steps. First, the convex hull of the set of grid points is covered with simplices, which results in a so-called *tessellation*. Then we linearly interpolate locally on each simplex.[2]
We adapt the grid as follows: First, we solve the system of equilibrium conditions on an

[1] In our terminology, a *kink* associated with a certain constraint is the set of points at which the policy function fails to be differentiable because the constraint is *just* binding, i.e. the constraint is binding *and* the associated multiplier is zero.

[2] Clearly, linear simplicial interpolation is only C^0 at the boundaries. For our purposes, this is desirable, because it provides a better fit at the kinks, and it ensures stability of the time iteration algorithm.

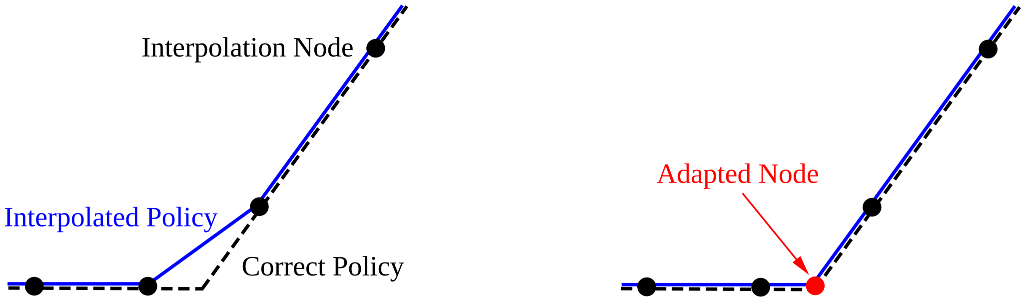

Figure 4.1: Non-Adaptive (lhs) and Adaptive (rhs) Linear Interpolation in 1D

initial grid. Second, we use these solutions to determine which edges of the tessellation cross kinks. Third, on each of these edges, we solve a modified system of equilibrium conditions to determine the point of intersection with the kink. Finally, we place a new grid point there. Using this procedure with state spaces of more than one dimension, we get several adapted grid points for each kink. Delaunay tessellation connects these points by edges, such that the kinks are matched very accurately.
To solve the above described infinite horizon models, we embed adaptive simplicial interpolation in a standard time iteration algorithm (see, e.g. Judd (1998)). To assess the accuracy of the computed equilibria, we follow Judd (1992) in calculating relative errors in Euler equations, subsequently called *Euler errors*. Concerning the measured Euler errors, we find that our method accurately computes equilibria for the two economies considered, both for reasonable and extreme calibrations of our model. Furthermore, we assess the relative performance of the adaptive grid scheme by comparing it to a standard equidistant grid scheme using the same interpolation technique. We find that the adaptive grid scheme dominates by far: One needs to increase the number of equidistant grid points, and thereby CPU time, by more than two orders of magnitude in order to reach the high accuracy of the adaptive grid scheme. Finally, we demonstrate that ad hoc update procedures that place additional points near the kinks are much less efficient than ASI.

In the literature, many algorithms have been applied to dynamic models with occasionally binding constraints. However, none of the existing algorithms addresses the problems of non-differentiabilities directly. Christiano and Fisher (2000) compare how several algorithms compute equilibria in a one sector growth model with irreversible investment, which has only one continuous state variable. None of the applied algorithms uses an adaptive grid scheme. A grid structure which is not adaptive, but endogenous, is proposed by Carrol (2006) and extended by Barillas and Fernández-Villaverde (2007), Rendahl (2007), and Hintermaier and Koeniger (2010). This so called endogenous grid method defines a grid on tomorrow's variables, resulting in an endogenous grid on today's variables. Its major advantage is that it avoids the root finding step. However, as it exploits the specific mapping from next period's variables to today's variables, the applicability as well as the concrete implementation of this method depends very much on details of the model. Maybe most related to our paper, Grune and Semmler (2004) propose an adaptive grid scheme for solving dynamic programming problems. However, this method is designed for value function iteration, it interpolates on rectangular elements, and uses estimated local errors of the value function to update the grid. Along all these dimensions their method is orthogonal to our algorithm. The sparse grid Smolyak (1963) algorithm is a well known approach to high-dimensional interpolation in economics. Krueger and Kubler (2004) use it to compute equilibria in OLG models with state spaces that have up to 30 dimensions. Certainly, this cannot be achieved in feasible time with our algorithm. However, the Smolyak algorithm requires policy functions to be smooth, which is not the case in models with occasionally binding constraints.

Section 4.2 presents adaptive simplicial interpolation, which consists of two components: Delaunay interpolation and an adaptive grid scheme. The example used to explain ASI is a two period exchange economy where several types of agents trade in a bond subject to ad-hoc borrowing constraints. Section 4.3 shows how the infinite horizon version of this economy is solved by embedding ASI in a time iteration setup. In Section 4.4, ASI is applied to a model where trade in a bond and a stock is subject to collateral constraints and short-selling constraints. Sections 4.3 and 4.4 examine carefully the computational performance of ASI as to the respective models. Section 4.5 concludes.

4.2 Adaptive Simplicial Interpolation

The main innovation of this paper is ASI, which is tailor-made for interpolating policy functions in models with occasionally binding constraints. Section 4.2.1 gives a simple example of such a model: An exchange economy where heterogeneous agents trade in a one-period bond subject to ad-hoc borrowing constraints. Section 4.2.2 provides a formal characterization of the problems we are considering. Section 4.2.3 outlines the adaptive simplicial interpolation algorithm we propose, while Sections 4.2.4 and 4.2.5 describe the two essential ingredients of the method: a simplicial interpolation technique based on Delaunay tessellation, and an adaptive grid scheme. Finally, Section 4.2.6 illustrates the workings of ASI with the help of the simple example from Section 4.2.1.

4.2.1 Simple Example: Borrowing Constraints

The Bond Economy

The economy is populated by H types of agents h $\in \mathbb{H} = \{1, \ldots, H\}$ living for T periods. Agents have identical preferences[3], but differ with respect to endowment realizations. They maximize expected time-separable lifetime utility

$$\mathbb{E}\left[\sum_{t=1}^{T} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}\right],$$

where c_t denotes consumption at t, β is the time discount factor, and γ is the coefficient of relative risk aversion.

Uncertainty is captured by a first-order Markov process with domain $X = \{1, ..., K\}$. Aggregate endowment of the single consumption good is given by a time invariant function $\bar{e} : X \to \mathbb{R}^{++}$, which depends on the current shock only. Similarly, agent h's individual endowment is given by $e^h : X \to \mathbb{R}^{++}$.

Each period, agents trade in a one-period bond, which is in zero net supply. Hence, agents face the following budget constraints:

$$c_t^h + b_t^h p_t \leq e_t^h + b_{t-1}^h \;\; \forall t = 1, \ldots, T \;\; \forall h \in \mathbb{H},$$

[3]Allowing for heterogeneous preferences does not impede using ASI.

where b_t^h denotes the bond holding that agent h acquires at time t, and p_t denotes the respective price. Moreover, agents face an ad-hoc borrowing constraint:

$$b_t \geq \underline{b} \;\; \forall t = 1, \ldots, T,$$

where $\underline{b} \in \mathbb{R}^-$.

State Space

The state of the economy at the beginning of a period is characterized by the exogenous shock and the asset distribution among agents. Because of bond market clearing, we may use the bond holdings of $H - 1$ agents as the endogenous state variable:

$$y_t = \left(b_{t-1}^1, \ldots, b_{t-1}^{H-1}\right).$$

Assuming that last period's constraints of all agents were satisfied, agent h enters period t with bond holding restricted by

$$b_{t-1}^h \in \left[\underline{b}, -(H-1)\underline{b}\right].$$

Hence, we take the endogenous state space to be

$$Y \equiv \left\{ y \in \left[\underline{b}, -(H-1)\underline{b}\right]^{H-1} \middle| \sum_{i=1}^{H-1} y_i \in \left[\underline{b}, -(H-1)\underline{b}\right] \right\}.$$

The whole state space S is then given by the product of the exogenous part and the endogenous part, i.e.

$$S = X \times Y.$$

Equilibrium Conditions

The endogenous choices and prices in period t are:

$$z_t \equiv \left(\left\{ c_t^h, b_t^h \right\}_{h \in \mathbb{H}}, p_t \right).$$

We call the collection of these endogenous variables *policies*, and denote the space of policies by Z.

The definition of competitive equilibrium is standard and given in Appendix 4.A, where we also derive the first order necessary conditions for equilibrium. Here, we just state these conditions. Along an equilibrium path, policies satisfy market clearing in the bond market, budget constraints, Euler equations, borrowing constraints and complementary slackness conditions[4]:

$$\begin{aligned}
&\sum_{h \in \mathbb{H}} b_t^h = 0, \\
&c_t^h + b_t^h p_t - e_t^h - b_{t-1}^h = 0 \;\; \forall h \in \mathbb{H}, \\
&-u'(c_t^h) p + \mu_t^h + \mathbb{E}\left[\beta u'(c_{t+1}^h) \right] = 0 \;\; \forall h \in \mathbb{H}, \\
&0 \leq b_t^h - \underline{b} \;\perp\; \mu_t^h \geq 0 \;\; \forall h \in \mathbb{H},
\end{aligned}$$

where μ^h denotes the Kuhn-Tucker multiplier on the borrowing constraint of agent h.

[4]The sign $\perp$ denotes orthogonality of two vectors. Hence, for $a, b \in \mathbb{R}^n$:

$$a \perp b :\Leftrightarrow \sum_{k=1}^{n} a_k b_k = 0.$$

If $a, b \geq 0$, then $a \perp b$ implies that for each coordinate $k = 1, \ldots, n$ either $a_k = 0$, $b_k = 0$, or both. Hence, $0 \leq a \;\perp\; b \geq 0$ is equivalent to

$$\forall k = 1, \ldots, n: \quad 0 \leq a_k \;\wedge\; 0 \leq b_k \;\wedge\; (\; a_k = 0 \;\vee\; b_k = 0 \;).$$

Two Period Version

Now consider the simplest dynamic setting: $T = 2$. In this case there is no trade in the second period and agents simply consume all their funds:

$$c_2^h = e_2^h + b_1^h.$$

Consequently, in period one, equilibrium conditions for given initial bond holdings $\{b_0^h\}_{h\in\mathbb{H}}$ simplify to:

$$\begin{aligned}
&\sum_{h\in\mathbb{H}} b_1^h = 0,\\
&c_1^h + b_1^h p_1 - e_1^h - b_0^h = 0 \ \ \forall h \in \mathbb{H},\\
&-u'(c_t^h)p_1 + \mu_1^h + \mathbb{E}\left[\beta u'(e_2^h + b_1^h)\right] = 0 \ \ \forall h \in \mathbb{H},\\
&0 \leq b_1^h - \underline{b} \ \perp \ \mu_1^h \geq 0 \ \ \forall h \in \mathbb{H}.
\end{aligned}$$

4.2.2 The General Problem

The above problem of finding an equilibrium policy for the two period bond economy with given initial bond holdings has the following structure:

Equilibrium Problem:

Given a state $s \in S$, and functions

$\phi : S \times \mathbb{R}^{m+n} \to \mathbb{R}^m, \ \ \psi : S \times \mathbb{R}^m \to \mathbb{R}^n,$

find policies and multipliers $(z, \mu) \in \mathbb{R}^m \times \mathbb{R}^n$,

s.t. $\phi(s, z, \mu) = 0, \quad 0 \leq \psi(s, z) \ \perp \ \mu \geq 0.$

In the case of our example, the equations $\phi = 0$ contain market clearing, budget constraints, and Euler equations. The inequalities $0 \leq \psi$ contain the borrowing constrains, and μ contains the respective Kuhn-Tucker multipliers. To solve such a problem for a given state s, there are many well established procedures. Either one applies solvers that accept complementarity conditions, or one transforms these conditions into equations—as explained in Appendix 4.C—and applies standard non-linear equation solvers.

However, things get more involved, if one is interested in the mapping from the state of the economy, s, into choices and prices, $f(s)$. Then, one faces a parametric problem, with the state of the economy, s, being the parameter.

Parametric Equilibrium Problem:

Given $\phi : S \times \mathbb{R}^{m+n} \to \mathbb{R}^m,\ \ \psi : S \times \mathbb{R}^m \to \mathbb{R}^n,$

find $f : S \to \mathbb{R}^m,\ \ \mu : S \to \mathbb{R}^n,$

s.t. $\forall s \in S$: $\phi(s, f(s), \mu(s)) = 0, \quad 0 \leq \psi(s, f(s)) \ \perp\ \mu(s) \geq 0.$

One way to compute functions (f, μ) that approximately satisfy these conditions is collocation (see, e.g. Judd (1998)): choose a finite grid $G \subset S$, on which the above conditions have to be satisfied precisely, i.e. require

$$\forall g \in G: \quad \phi(g, f(g), \mu(g)) = 0, \quad 0 \leq \psi(g, f(g)) \ \perp\ \mu(g) \geq 0.$$

For each point on the grid, $g \in G$, the solution $f(g)$ is determined by solving a complementarity problem. Aside from the grid G, collocation determines f by interpolating the solutions $\{f(g)\}_{g\in G}$ found on the grid. Clearly, this does not result in a perfect fit, and more importantly, the quality of the fit depends crucially on the location of the grid points $g \in G$. In particular, if there are kinks in the function f, it is desirable to put grid points there, because any method that interpolates over the kink has no chance to match it exactly.

In general, f is non-differentiable at the points k where for some j both $\psi_j(k, f(k))$ and $\mu_j(k)$ are equal to zero. The reason is as follows: $\psi_j(k, f(k)) = 0$ means that this constraint is binding, and $\mu_j(k) = 0$ means that the associated multiplier is zero though. Loosely speaking, the constraint is binding at one side and non-binding at the other side of the point. In general, this implies that the optimal solution is determined by different sets of equations on the two sides of the point, resulting in different slopes of the policy function. All in all, the above reasoning suggests that we should put interpolation nodes at points where constraints are just binding. We achieve this using the algorithm presented in Sections 4.2.3 to 4.2.5.

4.2.3 The Algorithm

To solve the parametric equilibrium problem presented above, we propose Adaptive Simplicial Interpolation. An overview of this procedure is given below. Steps two and three are black boxes for now. Sections 4.2.4 and 4.2.5 explain these steps in detail. We will explain Delaunay interpolation first, as it includes the concept of tessellation, which we use in the grid adaptation procedure.

Adaptive Simplicial Interpolation:

1. *Initialization:*
   Start with an initial grid G_{init} and solve for the solutions $\{f(g)\}_{g \in G_{init}}$ using standard numerical procedures.

2. *Grid Adaptation:*
   Use the solutions $\{f(g)\}_{g \in G_{init}}$, as explained in *Section 4.2.5*, to solve jointly for adapted grid points G_{adapt} that lie directly on the kinks and for the solutions $\{f(k)\}_{k \in G_{adapt}}$ at these points.

3. *Simplicial Interpolation:*
   Interpolate f on $G = G_{init} \cup G_{adapt}$. To interpolate on a grid with such an irregular shape, use simplicial interpolation, namely Delaunay interpolation, which is explained in *Section 4.2.4*.

4.2.4 Delaunay Interpolation

To get as much flexibility as possible in adapting the collocation grid, we need to have a method that is able to interpolate between points from any arbitrary set of scattered points. In addition, we require the method to work in arbitrary dimensions. Delaunay interpolation fulfills both criteria. This interpolation technique consists of two main steps: First, the state space is divided into simplices, which is done by Delaunay tessellation. Second, simplicial interpolation interpolates locally on these simplices.

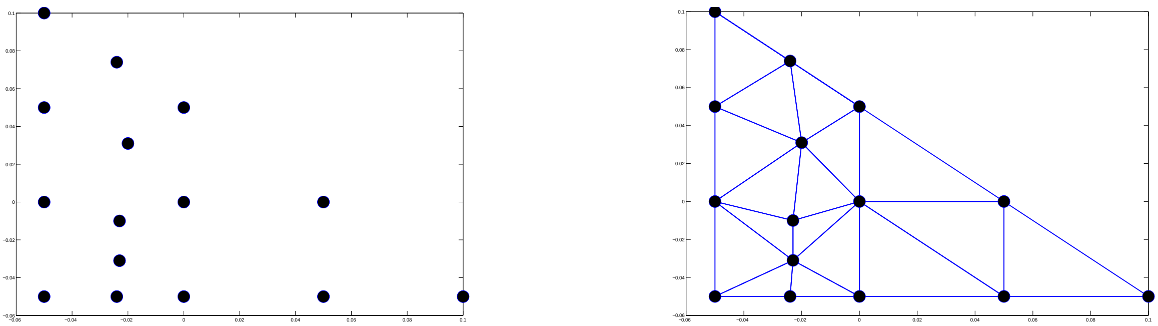

Figure 4.2: Set of Grid Points (lhs) and its Delaunay Triangulation (rhs)

Delaunay Tessellation

In computational geometry, Delaunay tessellation[5] is a well established method to cover the convex hull of an arbitrary set of points with simplices. For the sake of simplicity, we explain Delaunay tessellation for the two dimensional case. In this case, the simplices are just triangles and the method is called triangulation. In Figure 4.2, the left hand side picture shows a set of scattered grid points. The right hand side picture shows the Delaunay triangulation of this set of grid points. Delaunay triangulation is just one possible way to triangulate a set of grid points. However, it imposes discipline on the triangulation by satisfying the following property: inside the circumcircle of any triangle there is no point from the set of points. To make sense of this requirement, note that: by definition, the vertices of a triangle lie on its circumcircle, and in a Delaunay triangulation other points might as well lie on this circumcircle but not inside. Simpson (1978) shows that this procedure maximizes the minimum angle among all angles within the triangulation. Hence, it avoids pointed triangles. From a numerical perspective, this is a convenient property, since it implies that the information used to interpolate at a particular point stems from points that are relatively nearby. For a more extensive discussion of Delaunay Tessellation, see de Berg, Cheong, van Kreveld, and Overmars (2008).

[5] Delaunay tessellation was introduced by Delaunay (1934) and is well known in engineering. However, up to our knowledge, it has never been used to compute equilibria in dynamic models.

Simplicial Interpolation

Having the tessellation of a set of points at hand, linear simplicial interpolation is straightforward: For any arbitrary point, find the simplex it is contained in. Then calculate its barycentric coordinates within this simplex. Finally, the interpolation value is just a linear combination of the values at the corners of the simplex. The weights are given by the barycentric coordinates of the corners.

4.2.5 An Adaptive Grid Scheme

Let us now turn to the process of adapting the grid. Our aim is to detect kinks and place points on these kinks in order to match them precisely. In terms of the notation of Section 4.2.2, we want to determine points that lie on

$$K = \{s| \ \exists j \ \ \psi_j(s, f(s)) = 0 \text{ and } \mu_j(s) = 0\}.$$

Hence, we are looking for points where a constraint holds with equality but the respective multiplier is zero, i.e. where the constraint is just binding. To determine such points we proceed as follows.

How to Determine Which Edges Cross Kinks

To determine the location of kinks, we use the solutions $\{f(g)\}$ computed on the initial grid G_{init}. Clearly, if $\psi_j(g, f(g)) = 0$, we know that this constraint, which we call constraint j, is binding at g. Otherwise it is not binding. Furthermore, we make use of the tessellation of the initial grid. We consider each edge of the tessellation and check whether constraint j is binding at one corner and non-binding at the other corner of this edge. If this is the case, we conclude that the associated kink, which we call kink j, crosses this edge. In this way, we find sets of edges $\{\mathcal{E}_j\}$ crossing the kinks $j = 1, \ldots, m$.

How to Put Points Exactly on the Kink

Given the sets of edges $\{\mathcal{E}_j\}$ crossing the kinks $j = 1, \ldots, m$, we need to determine where exactly to put points on these edges. For each individual edge $E \in \mathcal{E}_j$ this is done by solving a modified version of the equation system that characterizes equilibrium. The key conceptional difference is that we let the state variable vary on the edge and do not solve the equation system at a given point in the state space. To pin down the one point that lies on the kink, we force that both ψ_j and μ_j are equal to zero. Hence, we solve jointly for the equilibrium solution *and* for a point in the state space on which the equilibrium solution fulfills a certain requirement, namely that the considered constraint is just binding. More formally, we solve for the point k, policies z, and multipliers μ such that:

$$\begin{aligned}
&\phi(k, z, \mu) = 0, \quad 0 \leq \psi_{-j}(k, z) \perp \mu_{-j} \geq 0, \\
&\psi_j(k, z) = 0, \quad \mu_j = 0, \\
&k \in E.
\end{aligned}$$

By demanding $\psi_j(k, z) = 0$, $\mu_j = 0$ instead of $0 \leq \psi_j(k, z) \perp \mu_j \geq 0$, we reduce the degrees of freedom by one. But letting the state variable k vary on the one-dimensional object E, in contrast to fixing a point in the state space, increases the degrees of freedom by one. Hence, the modified equation system has a (locally unique) solution (k, z, μ), if (z, μ) is a (locally unique) solution to the original equation system at k. This solution does not only provide the point k that lies on the kink, but at the same time it provides the optimal policy at this point, namely $f(k) = z$.

In this way—for all edges E in all sets $\mathcal{E}_j$—we compute points k and policies $f(k)$. We call these points adaptive, and denote the set containing them by G_{adapt}. Finally, we add them to the initial points to generate the adapted grid: $G = G_{init} \cup G_{adapt}$.[6]

4.2.6 ASI at Work

Figure 4.3 visualizes the working principle of ASI. The left hand side displays an initial grid for a given exogenous state of the 2-period bond economy. On the x-axis we have wealth of agent 1, on the y-axis wealth of agent 2—remember that the wealth of agent 3 is given by market clearing. We place 15 equidistant grid points on this state space, and we solve the equilibrium problem on this initial grid. Knowing the optimal policies at these points, we now consider each constraint at a time. We start with the borrowing

[6] Instead of this fine tuned adaptation procedure, one could also use a rather mechanical update of the grid. Instead of locating the kink exactly, one could just add arbitrary points into the triangles of interest, e.g. the center point of the triangle or say 5 randomly distributed points. This is easier to program, but comes at the cost of a less accurate result.

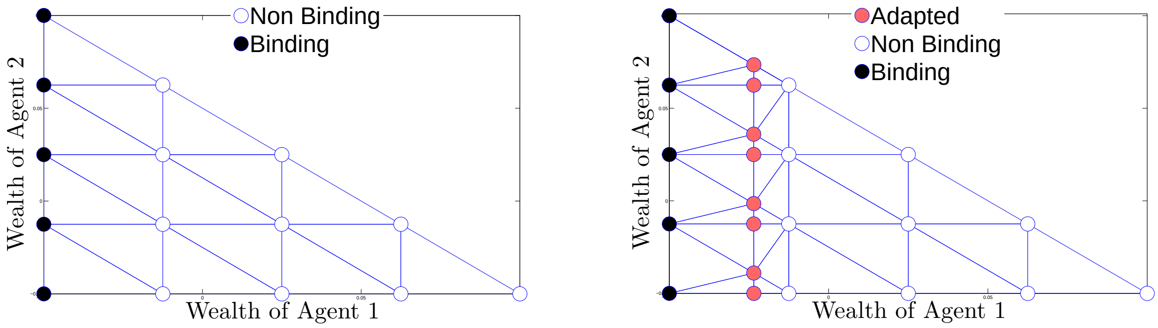

Figure 4.3: Initial Grid (lhs) and Adapted Grid (rhs) Using ASI in 2D

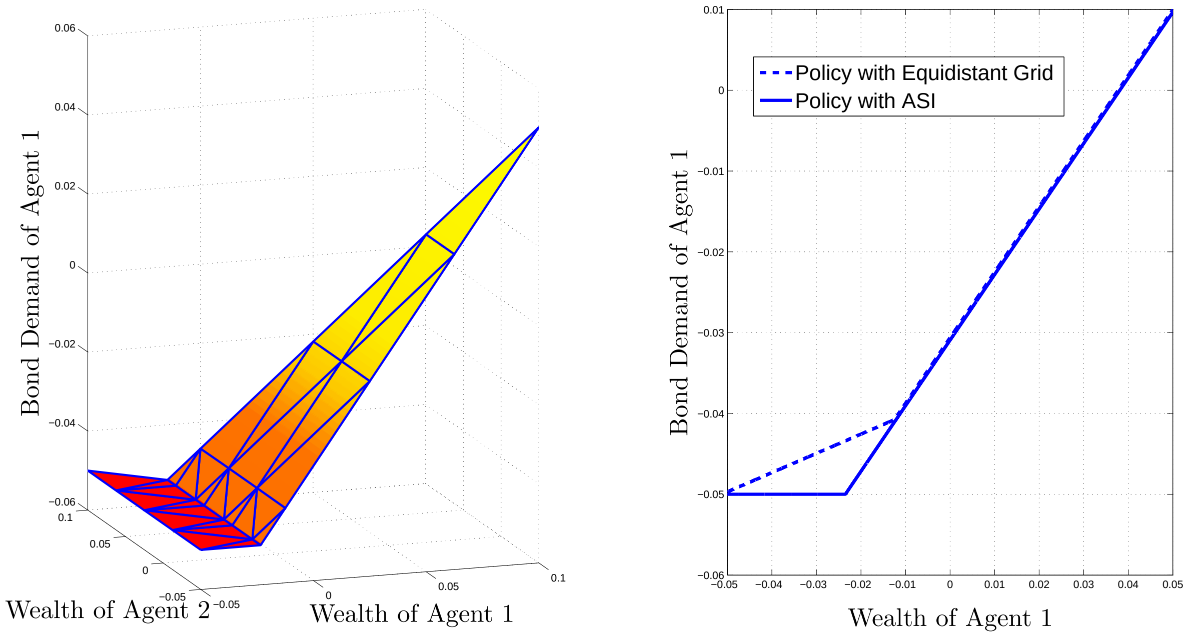

Figure 4.4: 2D Policy with ASI (lhs) and 1D Slice with and without ASI (rhs)

constraint of agent 1. In the left picture, black dots indicate that the constraint of agent 1 is binding, while white dots indicate that it is not binding. Hence, we know on which edges of the triangulation the constraint change from binding to non-binding. On these edges, we apply the second part of our adaptation scheme: we solve the modified equation system that allows us to find the particular point on the edge where the constraint is just binding (e.g. where the kink crosses the edge). Doing this for all relevant edges, we end up with 8 adapted points in this example, which are displayed in the right picture in Figure 4.3. Finally, a new triangulation is computed for the set of all grid points, initial and adapted. After this, we consider the next constraint. However, all other constraints are always non binding in this simple example. Hence, there are no further points to be added. Note that the new triangulation connects the adapted points by edges, thus kinks are matched very accurately. This can also be seen in Figure 4.4, where the left graph shows the equilibrium bond demand function of agent 1. The range where agent 1 is constrained by the borrowing limit is displayed by the dark shaded area. The kink induced by the inequality constraint is well approximated by the adapted points. The solid line in the right graph displays a slice of the bond demand function of agent 1. The dashed line represents the policy one gets if an equidistant grid is used. Clearly, this policy is quite inaccurate at the kink.

4.3 Time Iteration with ASI

We now consider the infinite horizon version of the bond economy from section 4.2.1. Section 4.3.1 characterizes recursive equilibrium policies for this model. Section 4.3.2 shows how such policies may be computed by embedding ASI into a standard time iteration setup. Details of how we implement this algorithm are given in Section 4.3.3. Finally, Section 4.3.4 analyzes the computational performance of time iteration with ASI.

4.3.1 The Infinite Horizon Bond Economy

Consider the bond economy of Section 4.2.1 with $T = \infty$. We want to describe equilibrium in terms of policy functions that map the current state into current policies:

$$f_t : S \to Z,\;\; f_t : \left(x_t, \left(b^1_{t-1}, \ldots, b^{H-1}_{t-1}\right)\right) \mapsto \left(\left\{c^h_t, b^h_t\right\}_{h\in\mathbb{H}}, p_t\right).$$

For the components of the policy function, we use the same notation as for their values, hence

$$f_t = \left(\left\{c^h_t, b^h_t\right\}_{h\in\mathbb{H}}, p_t\right).$$

For all states, these functions $\{f_t\}$ have to satisfy the period-to-period first order equilibrium conditions (see Appendix 4.A):

$$\begin{aligned}
\forall s: \qquad & \sum_{h\in\mathbb{H}} b^h(s) = 0, \\
& c^h_t(s) + b^h_t(s)p_t(s) - e^h_t(s) - b^h_{t-1}(s) = 0,\;\; \forall h \in \mathbb{H}, \\
& -u'(c^h_t(s))p_t(s) + \mu^h_t(s) + \mathbb{E}\left[\beta u'\left(c^h_{t+1}\left(s_{t+1}\right)\right)\right] = 0,\;\; \forall h \in \mathbb{H}, \\
& 0 \leq b^h_t(s) - \underline{b} \;\perp\; \mu^h_t(s) \geq 0,\;\; \forall h \in \mathbb{H},
\end{aligned}$$

where $s_{t+1} = \left(x_{t+1}, \left(b^1_t, \ldots, b^{H-1}_t\right)\right)$.

A recursive equilibrium policy function of this economy is a time invariant policy function f that satisfies these conditions, i.e. the sequence $\{f_t\}$ with $f_t = f \;\forall t$ satisfies the above conditions.

4.3.2 The Algorithm

The above period-to-period equilibrium conditions have the following structure:

$$\forall s: \ \ \phi[f^{next}]\,(s, f(s), \mu(s)) = 0, \quad 0 \leq \psi\,(s, f(s)) \ \perp \ \mu(s) \geq 0,$$

where time t variables have no index, and the policy in $t+1$ is denoted by f^{next}. The equations $\phi[f^{next}] = 0$, which depend on f^{next}, contain market clearing, budget constraints and Euler equations. Only the latter depend on f^{next}—in this case on the consumption policies only. The inequalities $0 \leq \psi$ contain the borrowing constraints, and μ contains the respective Kuhn-Tucker multipliers. A recursive equilibrium policy function f satisfies:

$$\forall s: \ \ \phi[f]\,(s, f(s), \mu(s)) = 0, \quad 0 \leq \psi\,(s, f(s)) \ \perp \ \mu(s) \geq 0.$$

The problem of finding a policy function that (approximately) satisfies this condition is very hard to address directly. In a time iteration procedure, the recursive equilibrium policy function is approximated iteratively: in each step, a simpler problem is solved, where next period's policy, f^{next}, is taken as given. This brings us back to the period-to-period equilibrium conditions:

$$\forall s: \ \ \phi[f^{next}]\,(s, f(s), \mu(s)) = 0, \quad 0 \leq \psi\,(s, f(s)) \ \perp \ \mu(s) \geq 0.$$

This problem takes exactly the form of the parametric equilibrium problem discussed in Section 4.2.2. Hence, we may use adaptive simplicial interpolation for this essential step in the time iteration algorithm. The formal structure of the full algorithm is given below. We deviate from a standard time iteration procedure only with regard to the interpolation procedure, which is contained in the inner box.

Time Iteration with Adaptive Simplicial Interpolation:

1. Select a grid G_{init}, an initial policy function f^{init}, and an error tolerance ϵ. Set $f^{next} \equiv f^{init}$.

2. Make one time iteration step: For all $g \in G_{init}$, find $f(g)$ that solves

$$\phi[f^{next}]\,(s, f(s), \mu(s)) = 0, \quad 0 \leq \psi\,(s, f(s)) \;\perp\; \mu(s) \geq 0.$$

   Interpolate f by *adaptive simplicial interpolation*:

   First, use the solutions $\{f(g)\}_{g \in G_{init}}$ to solve jointly for adapted points G_{adapt} that lie directly on kinks and for the optimal policy $\{f(g)\}_{g \in G_{adapt}}$ at these points.

   Second, use solutions at all grid points $G = G_{init} \cup G_{adapt}$ to interpolate f by *simplicial interpolation*.

   If $\|f - f^{next}\|_\infty < \epsilon$, go to step 3.
   Else set $f^{next} \equiv f$ and repeat step 2.

3. Set the numerical solution to the infinite horizon optimization problem: $\tilde{f} = f$.

4.3.3 Implementation of the Algorithm

To demonstrate that our algorithm works well with standard equipment, we use Matlab on an Intel Core 2 Duo 2.40 GHz computer to implement our algorithm.

Solving the System of Equilibrium Conditions

To solve the complementarity problem at each grid point, one could use a solver that directly applies to complementarity problems. However, we prefer to transform the complementarity problem into a system of equations (see Appendix 4.C) and then apply a standard non-linear equation solver, e.g. Matlab's fsolve or Ziena's Knitro. We are able to solve our models with both solvers. However, we find that the more equations the equlibrium system involves the better the performance of Knitro compared to fsolve.

Adaptive Simplicial Interpolation

Our method of choice for interpolation is Delaunay interpolation as described in Section 4.2.4. Delaunay Interpolation is widely used in many areas, and hence code in several languages like C++ or Fortran is available on the web. In Matlab, routines for computing Delaunay tessellations and simplicial interpolation come with the standard version.

Time Iteration

For the computation exercise presented below we set the error tolerance $\epsilon = 10^{-5}$. We set the initial policy function f^{init} such that agents consume all their wealth and the price of all assets is equal to zero. Hence, f^{init} corresponds to the policy function in the final period of a finite horizon economy. This is not an efficient starting guess, but it makes the computing times of our examples comparable. As a starting guess for solving the equilibrium problem at a given point, we use the solution from the previous iteration. In case the solver cannot find a root we use the solution from neighboring points as new starting guesses. In this way we always find solutions that satisfy the error tolerance.

To decrease CPU time, we start the time iteration procedure with a relatively coarse equidistant grid, and increase the density of the grid as the error in $\|f - f^{next}\|_\infty$ falls below $\epsilon \cdot 10$. We repeat this several times until we reach a grid of certain predefined size. In the comparison studies below, this refinement of the equidistant grid is done in exactly the same way for the adaptive grid method and the equidistant benchmark.

To further decrease CPU time, we do not use adaptive simplicial interpolation at each iteration step. The first step is not done until all refinements of the equidistant grid are carried out and the error in $\|f - f^{next}\|_\infty$ falls below $\epsilon \cdot 10$ again. Note that kinks in policy functions change their location along the time iteration procedure. Hence, it is important to use a sufficient number of adaptation steps. Furthermore, note that at each adaptation step, we compute new adapted nodes and do not use the adapted nodes from the last step any more.

4.3.4 Computational Performance

To evaluate the computational performance of *time iteration with adaptive simplicial interpolation*, we first report the accuracy of the computed equilibria for various examples.

Second, we compare time iteration with ASI to two other grid structures: an equidistant grid, and an ad hoc update scheme that places additional grid points randomly into simplices that are cut by kinks.

Measuring Accuracy

Following Judd (1992) we evaluate the accuracy of a computed equilibrium by calculating *relative errors in Euler equations* (EEs). An EE measures the error that an agent would make in terms of his period-to-period consumption decision, if he used the computed policy function. The unit of measure is the relative deviation of computed (i.e. interpolated) consumption, c_t^{int}, from the one that is optimal, c_t^{opt}, given next periods interpolated consumption, c_{t+1}^{int}. To derive c_t^{opt} from c_{t+1}^{int} one uses an Euler equation. For instance, in the Bond economy of Section 4.2.1 the Euler error $EE^h(\cdot)$ for agent h at a particular point s in the state space is given by

$$EE^h(s) = \left| \frac{c_t^{opt}}{c_t^{int}} - 1 \right| = \left| \frac{u'^{-1}\left(\beta \mathbb{E}_t \left[\frac{u'(c_{t+1}^{int})}{p_{int}}\right]\right)}{c_t^{int}(s)} - 1 \right|,$$

where p_{int} is the interpolated price of the bond today. However, it is possible to back out c_t^{opt} from c_{t+1}^{int} only if the Kuhn-Tucker multiplier entering the Euler equation is zero, i.e. if the respective constraint is non-binding. If it is binding, we set the Euler error equal to zero. Because of this problem with computing Euler errors when constraints are occasionally binding we also report an alternative error measure in Appendix 4.D.

To evaluate the accuracy of computed equilibria, we calculate the Euler errors of all agents at many points in the state space. Concerning the choice of points, we make two alternative choices. First, we draw 10.000 random points from a uniform distribution over the whole state space (EE state space), and compute Euler errors for all agents at these points. Second, we take the points reached along the equilibrium path, when the economy is simulated for 5.000 periods (EE equilibrium path). In both cases, we report both the maximum over all agents and points (max EE) as well as the average across points of the maximum across agents (∅ EE). This results in four different statistics, which we all report in $\log_{10}$ scale.

The examples that we consider have three or four agents and a borrowing limit of $\underline{b} = 0.1$

or 1.0, i.e. borrowing is restricted to 10% or 100% of average individual yearly income. Concerning all other parameters, we choose values that are considered standard in the literature, which we report in Appendix 4.E. Tables 4.1 and 4.2 report the accuracy measures for the three and four agent examples respectively. Maximal Euler errors over the state space range from −3 (for three agents and $\underline{b} = 0.1$) to −1.7 (for four agents and $\underline{b} = 1.0$). All errors are reasonably low, but could be improved much further by increasing the number of initial grid points, which would in turn also increase the number of adapted points. Generally speaking, a looser borrowing limit $\underline{b}$ and/or a greater number of agents—which both enlarge the state space—result in higher Euler errors. In the case of four agents, we are dealing with a three-dimensional state space, and kinks become two dimensional objects. This is illustrated in Figure 4.5, which displays a three dimensional grid that is adapted to a kink that lies approximately orthogonal to the horizontal axis.

Bond Economy with Three Agents

			EE state space		EE equilibrium path	
$\underline{b}$	points	time(min)	max EE	∅ EE	max EE	∅ EE
0.1	**40**(45)	**0.5**(0.4)	**-3.0**(−1.2)	**-3.8**(−2.1)	**-2.4**(−1.2)	**-4.4**(−2.1)
0.1	**113**(120)	**1.1**(1.0)	**-3.2**(−1.6)	**-4.2**(−2.8)	**-3.2**(−1.6)	**-4.8**(−3, 4)
1.0	**185**(190)	**6.5**(4.5)	**-2.1**(−1.1)	**-3.1**(−2.6)	**-2.2**(−1.1)	**-3.1**(−1.8)
1.0	**941**(946)	**13**(11)	**-3.2**(−1.2)	**-4.2**(−2.9)	**-3.2**(−1.6)	**-4.8**(−3.4)

Table 4.1: Accuracy of Adaptive Grid (Equidistant Grid in Brackets)

Bond Economy with Four Agents

			EE state space		EE equilibrium path	
$\underline{b}$	points	time(min)	max EE	∅ EE	max EE	∅ EE
0.1	**112**(120)	**4.5**(4)	**-2.7**(−1.3)	**-3.3**(−2.0)	**-2.7**(−1.3)	**-3.9**(−1.7)
1.0	**914**(969)	**60**(51)	**-1.7**(−1.1)	**-2.6**(−2.4)	**-1.8**(−1.1)	**-2.6**(−3.9)

Table 4.2: Accuracy of Adaptive Grid (Equidistant Grid in Brackets)

Comparison to Equidistant Grid

In order to assess the relative performance of ASI, we also compute equilibria on a standard equidistant grid, but still use Delaunay interpolation. To assess the gains from using an adaptive grid scheme, we ask the following questions: First, how do solutions on equidistant

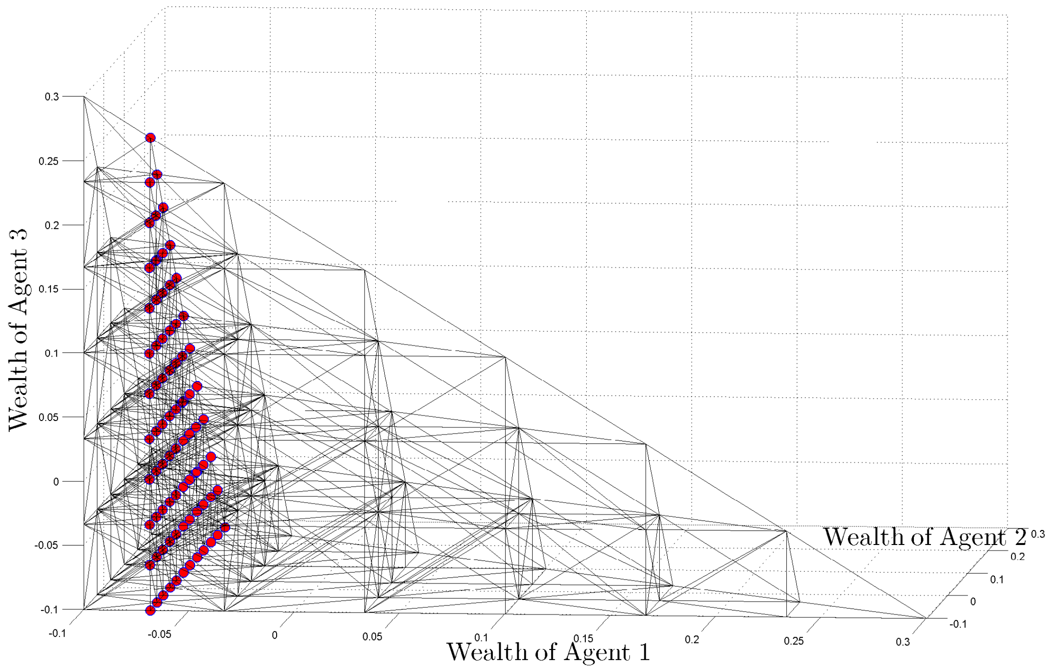

Figure 4.5: Adapted Grid with Three Continuous State Variables

grids compare to solutions on adaptive grids, if the same number of grid points is used? Second, how many equidistant grid points are needed to match the accuracy of ASI? When using the same number or slightly more points, the equidistant grid scheme is slightly faster. However, the difference is quite small, reinforcing our claim that adapting the grid takes very little time compared to overall computing time. More importantly, in all examples our algorithm outperforms the standard grid scheme between one and two orders of magnitude in terms of maximum Euler errors. This holds both for Euler errors drawn over the whole state space and along the equilibrium path. In the first example of Table 4.1, where we compare our results to an equidistant grid with about the same number of points, the adaptive grid yields maximum Euler errors that are about 70 times lower both on the state space and along the equilibrium path. Regarding the average Euler error, these factors are slightly lower but still substantial. We get these lower factors for average Euler errors, because the adaptive grid scheme rather targets the maximum Euler error by placing grid points on kinks, and not elsewhere in the state space. However, for two reasons the impact on average errors is also quite substantial. First, errors at the kinks are lowered dramatically, having a sizable effect on the average error. And second, even at

Bond Economy with Three Agents: Match Accuracy

			EE state space		EE equilibrium path	
$\underline{b}$	points	time(min)	max EE	∅ EE	max EE	∅ EE
0.1	**40**(20301)	**0.5**(79)	**-3.0**(−2.8)	**-4.1**(−5.3)	**-3.0**(−3.1)	**-4.4**(−6.2)
1.0	**185**(21945)	**6.5**(300)	**-2.1**(−1.9)	**-3.1**(−4.4)	**-2.2**(−1.9)	**-3.1**(−4, 6)

Table 4.3: Accuracy of Adaptive Grid (Equidistant Grid in Brackets)

Bond Economy with Four Agents: Match Accuracy

			EE state space		EE equilibrium path	
$\underline{b}$	points	time(min)	max EE	∅ EE	max EE	∅ EE
0.1	**112**(20825)	**4.5**(895)	**-2.7**(−2.0)	**-3.3**(−3.6)	**-2.7**(−2.1)	**-3.9**(−4.0)
1.0	**914**(20825)	**90**(3655)	**-1.7**(−1.1)	**-2.6**(−3.0)	**-1.8**(−1.1)	**-2.6**(−1.9)

Table 4.4: Accuracy of Adaptive Grid (Equidistant Grid in Brackets)

a point located elsewhere, kinks may still play a role, because agents potentially end up near a kink tomorrow.[7]

As a second exercise, we ask how many equidistant grid points are needed to get the same maximum Euler error as with a given adapted grid. Instead of targeting the number of grid points as above, we therefore target the maximum Euler error over the state space. For the first example with $\underline{b} = 0.1$, we increase the grid size by a factor of 500. Interestingly, adaptive simplicial interpolation still outperforms the equidistant grid in terms of maximum Euler as reported in Table 4.3. Obviously, in terms of average Euler errors, taking 500 times more points makes a big difference, resulting in a lower error for the equidistant grid. For $\underline{b} = 1.0$, due to memory constraints, we cannot multiply the number of grid points by 500. We therefore increase the grid size by a factor of 120, which yields maximum errors that are still higher than with adaptive simplicial interpolation.

When it comes to four agents, we also find that ASI outperforms equidistant grid points by far, as the results in Table 4.4 suggest. Trying to match the maximum Euler Error from the ASI example, we increase the amount of grid points by a factor of 200 for $\underline{b} = 0.1$ and 20 for $\underline{b} = 1.0$. For both cases we find that the maximum Euler Error on the equidistant grid is still far higher.

[7] At these points, the equilibrium policy functions also exhibit non-differentiabilities. These are induced by the kinks in next period's policy. An extension of ASI could identify these non-differentiablities as well.

Comparison to ad hoc Update

Finally we compare the accuracy of equilibria computed with ASI to the accuracy of equilibria computed with an ad hoc update scheme. Using the solution from the initial grid this scheme detects which simplices are cut by a kink. Instead of adding points exactly on the kink as done by ASI, the ad hoc update randomly places additional grid points into these simplices. To compare this ad hoc update scheme with ASI we now compute equilibria for the examples considered above using the same initial grid as with ASI. As the results in Table 4.5 and 4.6 suggest ASI outperforms such an ad hoc update, even if we use up to 200 times more grid points.

Bond Economy with Three Agents: Comparison to ad hoc Update

			EE state space		EE equilibrium path	
$\underline{b}$	points	time(min)	max EE	∅ EE	max EE	∅ EE
0.1	**40**(8000)	**0.5**(40)	**-3.0**(-1.3)	**-4.1**(-2.7)	**-3.0**(-2.7)	**-4.4**(-4.5)
1.0	**185**(8000)	**6.5**(122)	**-2.1**(-2.0)	**-3.1**($-3,1$)	**-2.2**(-1.9)	**-3.1**($-4,6$)

Table 4.5: Accuracy of Adaptive grid (Grid with ad hoc Update in Brackets)

Bond Economy with Four Agents: Comparison to ad hoc Update

			EE state space		EE equilibrium path	
$\underline{b}$	points	time(min)	max EE	∅ EE	max EE	∅ EE
0.1	**112**(8000)	**4.5**(333)	**-2.7**(-2.3)	**-3.3**(-3.9)	**-2.7**(-2.4)	**-3.9**(-4.3)
1.0	**914**(20825)	**90**(3655)	**-1.7**(-1.3)	**-2.6**(-3.0)	**-1.8**(-1.1)	**-2.6**(-1.9)

Table 4.6: Accuracy of Adaptive Grid (Grid with ad hoc Update in Brackets)

4.4 Extension: Endogenous Collateral Constraints

4.4.1 The Bond and Stock Economy

Setup

We extend the bond economy of Section 4.2.1 by introducing a Lucas tree-type stock which is in unit net supply. It pays out a fixed fraction δ of aggregate endowment each period, i.e. stock holders receive dividends $d(x) = \delta \cdot \bar{e}(x)$ per unit of the stock. Hence, aggregate endowment is given by the sum of individual endowments and dividends, i.e.

$$\bar{e}(x) = \sum_{h \in \mathbb{H}} e^h(x) + d(x) \ \ \forall x \in X.$$

The Lucas tree is traded each period after dividends are paid. Each agent h buys l^h shares of the stock at a price q. Hence, agents face the following budget constraints:

$$c_t^h + b_t^h p_t + l_t^h q_t \leq e_t^h + b_{t-1}^h + l_{t-1}^h \left(q_t + d_t\right) \ \ \forall t = 1, \ldots, T \ \ \forall h \in \mathbb{H}.$$

Moreover, trade in the bond and the stock is subject to constraints. First, we impose a *short-selling constraint* on the stock, i.e.

$$l_t^h \geq 0 \ \ \forall t = 1, \ldots, T \ \ \forall h \in \mathbb{H}.$$

In contrast to the stock, the bond may be shorted. However, only if the stock is used as collateral. More precisely, the short position in the bond may not exceed the minimal value—in terms of resale value plus dividends—that the stock has next period:

$$-b_t^h \leq \min_{x_{t+1} \in X} \left\{ l_t^h \left(q(s_{t+1}) + d(x_{t+1}) \right) \right\}, \forall t = 1, \ldots, T \ \ \forall h \in \mathbb{H},$$

where tomorrow's state is $s_{t+1} = (x_{t+1}, y_{t+1})$. The endogenous part of the state, y_{t+1}, will be specified below. This constraint is motivated by a bankruptcy law which makes it possible to seize an agents' stock holding, but not his income. To put it differently, all future income is exempted. As there is no further punishment for default, an agent will default on his asset position, if and only if his portfolio has a negative value. As this behavior is anticipated—and we assume that default premia may not be charged—no agent will be allowed to acquire such a portfolio, which imposes the above constraint.

State Space

With the above collateral constraint, financial wealth,

$$w_t^h \equiv l_{t-1}^h \left(q(s_t) + d(x_t)\right) + b_{t-1}^h,$$

cannot go below zero. Hence, the fraction of total financial wealth that an agent holds,

$$y^h = \frac{w^h}{\sum_{j\in\mathbb{H}} w^j},$$

is bounded between zero and one. By market clearing, we may use the fractions of financial wealth of the first $H-1$ agents as the endogenous state space:

$$y = \left(y^1, \ldots, y^{H-1}\right) \in Y \equiv \left\{ y \in \mathbb{R}_+^{H-1} \middle| \sum_{i=1}^{H-1} y^i \leq 1 \right\} \subset \mathbb{R}_+^{H-1}.$$

Finally, we define the whole state space S as the product of the exogenous part and the endogenous part, i.e.

$$S = X \times Y.$$

With this definition of the state space, reconsider the collateral constraint above, and note that: Todays choice of any agent, through its impact on tomorrows state, influences tomorrows price of the stock, and hence today's collateral constraint of agent h. In this sense, the collateral constraint is endogenous, which complicates the model considerably.

Equilibrium Conditions

The endogenous choices and prices in period t are

$$z_t \equiv \left(\left(c_t^h, b_t^h, l_t^h\right)_{h\in\mathbb{H}}, p_t, q_t \right).$$

In Appendix 4.B we define competitive equilibrium and derive the first-order equilibrium conditions of this model. Along an equilibrium path, policies have to satisfy market clearing on both asset markets, budget constraints, Euler equations for both assets, and complementary slackness conditions for both kinds of multipliers:

$$
\begin{aligned}
&\sum_{h \in \mathbb{H}} b_t^h = 0, \quad \sum_{h \in \mathbb{H}} l_t^h = 1, \\
&c_t^h + b_t^h p_t + l_t^h q_t - e_t^h - b_{t-1}^h - l_{t-1}^h \left(q_t + d_t\right) = 0, \;\; \forall h \in \mathbb{H}, \\
&-u'(c_t^h) p_t + \mu^h + \mathbb{E}\left[\beta u'(c_{t+1}^h)\right] = 0, \;\; \forall h \in \mathbb{H}, \\
&-u'(c_t^h) q_t + \mu^h \min_{x_{t+1} \in X} \left\{q(s_{t+1}) + d(x_{t+1})\right\} + \nu_t^h + \mathbb{E}\left[\beta u'(c_{t+1}^h)\left(q_{t+1} + d_{t+1}\right)\right] = 0, \;\; \forall h \in \mathbb{H}, \\
&0 \leq \min_{x_{t+1} \in X} \left\{l_t^h \left(q(s_{t+1}) + d(x_{t+1})\right) + b_t^h\right\} \;\perp\; \mu_t^h \geq 0, \;\; \forall h \in \mathbb{H}, \\
&0 \leq l_t^h \;\perp\; \nu_t^h \geq 0, \;\; \forall h \in \mathbb{H},
\end{aligned}
$$

where μ^h and ν^h denote the Kuhn-Tucker multipliers on the collateral and the short-selling constraint of agent h.

4.4.2 Computational Performance

Before we look at errors in Euler equations, we first discuss how the kinks induced by the short selling and collateral constraints are located within the state space. Figure 4.6 shows the adapted grid for an exogenous state where the first agent is hit by a bad idiosyncratic shock. To clearly visualize the kinks, we highlight the edges that connect adapted points. The short selling constraint of the first agent induces a kink which has two components, the one which lies almost on the y-axis and the curved one to the very right. Furthermore, each of the collateral constraints induces one kink, where the kink from the first agent's constraint runs approximately parallel to the y-axis at about 0.08 fraction of wealth of agent 1. In Figure 4.7 one can see how these kinks shape equilibrium an equilibrium policy function. The left hand picture displays the stock demand over the full state space, whereas the picture on the right hand side displays a slice at 0.1 wealth fraction of agent 2. The distinct peak at 0.08 wealth fraction of agent 1 corresponds to the kink induced by his collateral constraint. To the left, the collateral constraint is binding. At higher levels of wealth his demand for the stock goes down until the short selling constraint becomes binding again.

As in Section 4.3.4, we evaluate the performance of our algorithm by computing relative errors in Euler equations. In Table 4.7, we show results for equilbria computed with ASI using two different values for the dividend parameter δ. For all other parameters, we use the same calibration as for the Bond economy (see Appendix 4.E). Obviously, as the figures above suggest, more points are needed than in the bond model to bring Euler errors down to reasonable values. Comparing the results from ASI with results on equidistant grids, we find that for the same number of grid points, ASI outperforms equidistant grids by approximately one order of magnitude in terms of maximum Euler Error. Again, we ask how many points are needed to match the accuracy of ASI. Increasing the number of points up to a factor of 20 yields almost the same maximum Euler Error, as the results in Table 4.8 show. This factor is still substantial, however, not as high as for the Bond model. The reason are non-linearities away from the kink, as can be seen in Figure 4.7. We have developed an adaptation scheme that adapts the grid to non-linearities, which further improves the relative performance of our algorithm. However, as this is not the focus of this paper, we do not elaborate more on this.

Bond and Stock Economy

			EE state space		EE equilibrium path	
δ	points	time(min)	max EE	∅ EE	max EE	∅ EE
0.10	**1235**(1250)	**310**(260)	**-2.5**(−1.4)	**-3.8**(−3.2)	**-3.1**(−1.6)	**-4.1**(−3.5)
0.25	**1160**(1225)	**302**(251)	**-2.2**(−1.4)	**-3.3**(−2.9)	**-2.2**(−1.4)	**-3.4**(−2.7)

Table 4.7: Accuracy of Adaptive Grid (Equidistant Grid in Brackets)

Bond and Stock Economy: Match Accuracy

			EE state space		EE equilibrium path	
δ	points	time(min)	max EE	∅ EE	max EE	∅ EE
0.10	**1235**(25425)	**310**(4500)	**-2.5**(−2.4)	**-3.8**(−4.0)	**-3.1**(−2.6)	**-4.1**(−4.2)
0.25	**1160**(25425)	**302**(4812)	**-2.2**(−2.1)	**-3.3**(−4.2)	**-2.2**(−2.3)	**-3.4**(−4.4)

Table 4.8: Accuracy of Adaptive Grid (Equidistant Grid in Brackets)

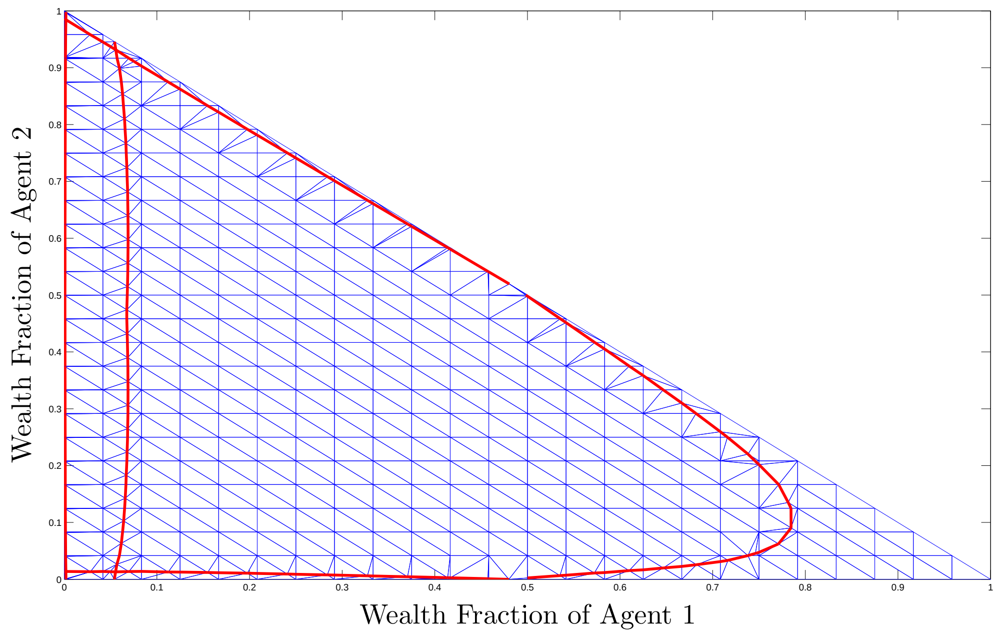

Figure 4.6: Bond and Stock Economy: Adapted Grid with Several Identified Kinks

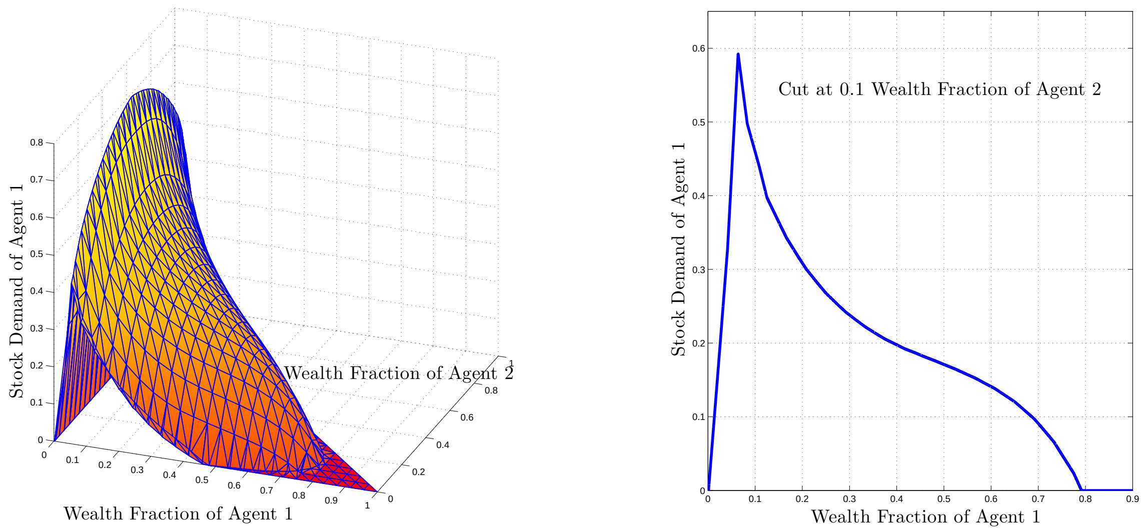

Figure 4.7: Bond and Stock Economy: 2D Stock Demand (lhs) and 1D Slice (rhs)

4.5 Conclusion

This paper presents an algorithm that is tailor-made for computing equilibria in dynamic models with occasionally binding constraints. To directly address the problem of kinks in such models, we develop a new interpolation technique based on adaptive grids and simplicial interpolation. We show that Adaptive Simplicial Interpolation accurately computes equilibria in dynamic models with several continuous state variables and various inequality constraints. Comparison studies show that our method outperforms standard grid techniques by up to two orders of magnitude in terms of maximum errors in Euler equations. Clearly, occasionally binding constraints become more and more important in quantitative economics, e.g. in modeling financial frictions. We hope that ASI will help economists in solving such models.

Appendix

4.A Details Bond Economy

In this appendix, we define competitive equilibrium and derive first-order equilibrium conditions for the bond economy presented in Section 4.2.1. For this purpose, some additional notation is needed. We denote the shock at time t by x_t, but the history of shocks that occurred up to period t by x^t. The set of histories up to period t is denoted by X^t, and the set of all possible histories by $\mathbb{X} \equiv \bigcup_{t=1}^{T} X^t$. For x^{t+1} being a possible successor of x^t we write $x^{t+1} \geq x^t$. Finally, the probability of history x^t is denoted by $\pi(x^t)$ and the conditional transition probability by $\pi(x^{t+1}\,|x^t)$

Competitive Equilibrium

A competitive equilibrium for an economy with agents' initial bond holdings

$$\left(b_0^h\right)_{h\in\mathbb{H}}$$

is a collection

$$\{z(x^t)\}_{x^t\in\mathbb{X}} \equiv \left\{\left(c^h(x^t), b^h(x^t)\right)_{h\in\mathbb{H}}, p(x^t)\right\}_{x^t\in\mathbb{X}}$$

of consumption allocations, bond holdings, and bond prices that satisfy the following conditions:

1. Markets clear[8]:

$$\sum_{h\in\mathbb{H}} b^h(x^t) = 0 \quad \forall x^t \in \mathbb{X}.$$

2. Given prices $(p(x^t))_{x^t\in\mathbb{X}}$, each agent chooses

$$\left(c^h(x^t), b^h(x^t))\right)_{x^t\in\mathbb{X}}$$

[8]By Walras' Law market clearing in the asset market(s) implies market clearing in the consumption goods market.

to maximize lifetime utility such that $\forall x^t \in \mathbb{X}$ the following constraints hold:

$$\begin{aligned} \textit{budget constraint} \qquad & c^h(x^t) + b^h(x^t)p(x^t) \leq e^h(x^t) + b^h(x^{t-1}), \\ \textit{borrowing constraint} \qquad & b^h(x^t) \geq \underline{b}. \end{aligned}$$

First-Order Equilibrium Conditions

Each individual agent faces the following optimization problem:

$$\begin{aligned} & \max_{(c(x^t),b(x^t))_{x^t \in \mathbb{X}}} \quad \mathbb{E}\left[\sum_{t=1}^{T} \beta^t u(c(x^t))\right] \\ & \text{s.t. } \forall x^t \in \mathbb{X}: \\ \textit{budget constraint} \qquad & c^h(x^t) + b^h(x^t)p(x^t) \leq e^h(x^t) + b^h(x^{t-1}), \\ \textit{borrowing constraint} \qquad & b^h(x^t) \geq \underline{b}. \end{aligned}$$

Denote the multiplier associated with these constraints by $\lambda(x^t)$ and $\mu(x^t)$. Differentiating the Lagrangian with respect to the different choice variables gives

$$\begin{aligned} c(x^t): \qquad & \pi(x^t)\beta^t u'(c(x^t)) - \lambda(x^t) = 0 \\ c(x^{t+1}): \qquad & \pi(x^{t+1})\beta^{t+1} u'(c(x^{t+1})) - \lambda(x^{t+1}) = 0 \\ b(x^t): \qquad & -\lambda(x^t)p(x^t) + \mu(x^t) + \sum_{x^{t+1} \geq x^t} \left(\lambda(x^{t+1})\right) = 0 \end{aligned}$$

Substituting the first two FOCs into the last one, we get the following Euler equation for the bond:

$$-u'(c(x^t))p(x^t) + \mu(x^t) + \sum_{x^{t+1} \geq x^t} \beta\pi(x^{t+1}|x^t)u'(c(x^{t+1})) = 0.$$

In addition, the Kuhn-Tucker FOCs include the following complementarity condition:

$$0 \leq b(x^t) - \underline{b} \ \perp \ \mu(x^t) \geq 0.$$

Combined with market clearing conditions and budget constraints, these are the equilibrium conditions stated in Section 4.2.1.

4.B Details Bond and Stock Economy

In this appendix, we define competitive equilibrium and derive first-order equilibrium conditions for the economy presented in Section 4.4. The notation is as introduced in the beginning of Appendix 4.A.

Competitive Equilibrium

A competitive equilibrium for an economy with agents' initial portfolios

$$\left(b_0^h, l_0^h\right)_{h\in\mathbb{H}}$$

is a collection

$$\{z(x^t)\}_{x^t\in\mathbb{X}} \equiv \left\{\left(c^h(x^t), b^h(x^t), l^h(x^t)\right)_{h\in\mathbb{H}}, p(x^t), q(x^t)\right\}_{x^t\in\mathbb{X}}$$

of consumption allocations, bond and stock holdings, and prices that satisfy the following conditions:

1. Markets clear:

$$\sum_{h\in\mathbb{H}} b^h(x^t) = 0, \quad \sum_{h\in\mathbb{H}} l^h(x^t) = 1 \quad \forall x^t \in \mathbb{X}.$$

2. Given prices $(p(x^t), q(x^t))_{x^t\in\mathbb{X}}$, each agent chooses

$$\left(c^h(x^t), b^h(x^t), l^h(x^t))\right)_{x^t\in\mathbb{X}}$$

to maximize lifetime utility such that $\forall x^t \in \mathbb{X}$ the following constraints hold:

$$\begin{aligned}
&\textit{budget constraint} && c^h(x^t) + b^h(x^t)p(x^t) + l^h(x^t)q(x^t) \leq \\
& && e^h(x^t) + b^h(x^{t-1}) + l^h(x^{t-1})\left(q_t(x^t) + d_t(x^t)\right), \\
&\textit{short selling constraint} && l^h(x^t) \geq 0 \text{ and} \\
&\textit{collateral constraints} && \min_{x^{t+1}\geq x^t}\left\{l^h(x^t)\left(q(x^{t+1}) + d(x^{t+1})\right) + b^h(x^t)\right\} \geq 0.
\end{aligned}$$

First-Order Equilibrium Conditions

Each individual agent faces the following optimization problem:

$$
\begin{aligned}
& \max_{(c(x^t),b(x^t),l(x^t))_{x^t\in\mathbb{X}}} \quad \mathbb{E}\left[\sum_{t=1}^{T}\beta^t u(c(x^t))\right] \\
& \text{s.t. } \forall x^t \in \mathbb{X}: \\
\textit{budget constraint} \quad & c^h(x^t) + b^h(x^t)p(x^t) + l^h(x^t)q(x^t) \leq \\
& e^h(x^t) + b^h(x^{t-1}) + l^h(x^{t-1})\left(q_t(x^t) + d_t(x^t)\right), \\
\textit{short selling constraint} \quad & l^h(x^t) \geq 0 \;\text{ and} \\
\textit{collateral constraints} \quad & \min_{x^{t+1}\geq x^t}\left\{l^h(x^t)\left(q(x^{t+1}) + d(x^{t+1})\right) + b^h(x^t)\right\} \geq 0.
\end{aligned}
$$

Denote the multipliers associated with these constraints by $\lambda(x^t)$, $\nu(x^t)$, and $\mu(x^t)$. Differentiating the Lagrangian gives

$$
\begin{aligned}
c(x^t): \quad & \pi(x^t)\beta^t u'(c(x^t)) - \lambda(x^t) = 0 \\
c(x^{t+1}): \quad & \pi(x^{t+1})\beta^{t+1} u'(c(x^{t+1})) - \lambda(x^{t+1}) = 0 \\
b(x^t): \quad & -\lambda(x^t)p(x^t) + \mu(x^t) + \sum_{x^{t+1}\geq x^t}\left(\lambda(x^{t+1})\right) = 0 \\
l(x^t): \quad & \nu(x^t) - \lambda(x^t)q(x^t) + \mu(x^t)\min_{x^{t+1}\geq x^t}\left\{q(x^{t+1}) + d(x^{t+1})\right\} \\
& + \sum_{x^{t+1}\geq x^t}\left(\lambda(x^{t+1})\right)\left(q(x^{t+1}) + d(x^{t+1})\right) = 0.
\end{aligned}
$$

Substituting the first two FOCs into the last two, we get the following Euler equations for the bond and the stock:

$$
\begin{aligned}
& -u'(c(x^t))p(x^t) + \mu(x^t) + \sum_{x^t+\geq x^t}\left(\beta\pi(x^{t+1}|x^t)u'(c(x^{t+1}))\right) = 0, \\
& \nu(x^t) - u'(c(x^t))q(x^t) + \mu(x^t)\min_{x^{t+1}\geq x^t}\left\{q(x^{t+1}) + d(x^{t+1})\right\} \\
& + \sum_{x^{t+1}\geq x^t}\left(\beta\pi(x^{t+1}|x^t)u'(c(x^{t+1}))\right)\left(q(x^{t+1}) + d(x^{t+1})\right) = 0.
\end{aligned}
$$

In addition, the Kuhn-Tucker FOCs include the following complementarity conditions:

$$
\begin{aligned}
&0 \leq \min_{x^{t+1} \geq x^t} \left\{ l^h(x^t) \left(q(x^{t+1}) + d(x^{t+1}) \right) + b^h(x^t) \right\} \perp \mu(x^t) \geq 0 \\
&0 \leq l(x^t) \perp \nu(x^t) \geq 0.
\end{aligned}
$$

Combined with market clearing conditions and budget constraints, these are the equilibrium conditions stated in Section 4.4.

4.C Transforming Complementarities into Equations

At the initial gridpoints, ASI solves the follwing complementarity problem:

$$
\begin{aligned}
&\text{Given a state } s \in S \text{ , and functions} \\
&\phi : S \times \mathbb{R}^{m+n} \to \mathbb{R}^m, \;\; \psi : S \times \mathbb{R}^m \to \mathbb{R}^n, \\
&\text{find policies and multipliers } (z, \mu) \in \mathbb{R}^m \times \mathbb{R}^n, \\
&\;\text{s.t. } \phi(s, z, \mu) = 0, \quad 0 \leq \psi(s, z) \perp \mu \geq 0.
\end{aligned}
$$

Following Garcia and Zangwill (1981), we transform this complementarity problem into a system of equations, to be able to apply a standard non-linear equation solver. Key to the transformation are the following definitions:

$$
\alpha \equiv \begin{cases} \mu & \text{for} \quad \mu \geq 0, \;\; \psi(s,z) = 0 \\ -\psi(s,z) & \text{for} \quad \mu = 0, \;\; \psi(s,z) > 0 \end{cases}
$$

and

$$
\begin{aligned}
\alpha^+ &= (\max(0, \alpha))^k \\
\alpha^- &= (\max(0, -\alpha))^k,
\end{aligned}
$$

where $k \in \mathbb{N}^+$. Using these definitions, the problem reads:

$$\begin{aligned}
&\text{Given a state } s \in S \text{ , and functions}\\
&\phi : S \times \mathbb{R}^{m+n} \to \mathbb{R}^m, \;\; \psi : \mathbb{R}^m \to S \times \mathbb{R}^n,\\
&\text{find policies and alphas } (z, \alpha) \in \mathbb{R}^m \times \mathbb{R}^n,\\
&\text{ s.t. } \phi(s, z, \alpha^+) = 0, \;\; \psi(s, z) - \alpha^- = 0.
\end{aligned}$$

4.D Alternative Error Measure

As explained in Section 4.3.4, measuring accuracy in models with occasionally binding constraints using EEs is not unproblematic. We therefore suggest an alternative error measure. To apply this measure we need to solve the equilibrium system (given the solution from the time iteration algorithm as tomorrow's policies) at the point in the state space where we want to measure accuracy. From that solution, we get consumption values c_{opt} for all agents. Then, we compare these values to the interpolated consumption values c_{int}. In spirit of the Euler Error we compute the relative deviation of the interpolated policy from the optimal solution. Hence, the error is given by $E = \left|\frac{c_{int}}{c_{opt}} - 1\right|$. In the tables below we report the maximum and average errors over the state space and along the equilibrium path for the same examples as in Section 4.3.4. With respect to the alternative error measure, ASI still outperforms standard equidistant grid schemes by far. However, the difference in accuracy is not as extreme as with EEs.

Bond Economy with Three Agents

			E state space		E equilibrium path	
$\underline{b}$	points	time(min)	max E	∅ E	max E	∅ E
0.1	**40**(45)	**0.5**(0.4)	**-3.2**(−2.6)	**-4.1**(−3.5)	**-3.2**(−2.6)	**-4.7**(−4.0)
0.1	**113**(120)	**1.1**(1.0)	**-3.3**(−2.5)	**-4.5**(−3.6)	**-3.3**(−2.5)	**-5.1**(−4.2)
1.0	**185**(190)	**6.5**(4.5)	**-2.3**(−1.7)	**-3.2**(−3.0)	**-2.5**(−1.7)	**-3.2**(−2.7)
1.0	**941**(946)	**13**(11)	**-3.3**(−2.3)	**-4.5**(−3.6)	**-3.3**(−2.5)	**-5.1**(−3.3)

Table 4.9: Accuracy of Adaptive Grid (Equidistant Grid in Brackets)

Bond Economy with Four Agents

			E state space		E equilibrium path	
$\underline{b}$	points	time(min)	max E	∅ E	max E	∅ E
0.1	**112**(120)	**4.5**(4)	**-2.9**(−1.9)	**-3.5**(−2.7)	**-2.9**(−2.0)	**-4.0**(−3.0)
1.0	**914**(969)	**60**(51)	**-1.9**(−1.5)	**-2.7**(−2.6)	**-1.9**(−1.5)	**-2.8**(−3.4)

Table 4.10: Accuracy of Adaptive Grid (Equidistant Grid in Brackets)

Bond and Stock Economy

			E state space		E equilibrium path	
δ	points	time(min)	max E	∅ E	max E	∅ E
0.10	**1235**(1250)	**310**(260)	**-2.1**(−1.8)	**-3.5**(−3.4)	**-2.1**(−1.8)	**-3.6**(−3.4)
0.25	**1160**(1225)	**302**(251)	**-2.2**(−1.8)	**-3.3**(−2.2)	**-2.2**(−1.9)	**-3.4**(−3.2)

Table 4.11: Accuracy of Adaptive Grid (Equidistant Grid in Brackets)

4.E Parameterization

We set the discount factor $\beta = 0.95$ and the risk aversion parameter $\gamma = 1.5$ for all agents. Concerning the exogenous shock process, we make the following choices: We assume that agents may either receive a good or a bad idiosyncratic shock. One agent always gets the bad shock and all others get the good one. This results in three or four states per aggregate shock, depending on the number of agents. Allowing for two aggregate shocks the exogenous part of the state space comprises six or eight states respectively. We denote the ratios of good to bad idiosyncratic and aggregate shocks by ν_{idio} and ν_{agg}. We finally denote the persistence of idiosyncratic and aggregate shocks by ρ_{idio} and ρ_{agg}. We compute equilibria for two values of the borrowing limit $\underline{b}$, namely $\underline{b} = 0.1$ and 1, i.e. borrowing up to 10% or 100% of average individual yearly income. All parameter values can be found in Table 4.12.

γ	ν_{idio}	ν_{agg}	ρ_{idio}	ρ_{agg}	β	$\underline{b}$
1.5	1.6	1.06	0.9	0.65	0.95	0.1/1.0

Table 4.12: Parameter Values

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