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Working Paper Series

Coalgebraic Analysis of Subgame-perfect Equilibria in Infinite Games without Discounting

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Working Paper 12-17

October 2012

arXiv:1210.4537v1 [cs.GT] 16 Oct 2012

Coalgebraic Analysis of Subgame-perfect Equilibria in Infinite Games without Discounting

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October 17, 2012

Abstract

We present a novel coalgebraic formulation of infinite extensive games. We define both the game trees and the strategy profiles by possibly infinite systems of corecursive equations. Certain strategy profiles are proved to be subgame perfect equilibria using a novel proof principle of predicate coinduction which is shown to be sound by reducing it to Kozen's metric coinduction. We characterize all subgame perfect equilibria for the dollar auction game. The economically interesting feature is that in order to prove these results we do not need to rely on continuity assumptions on the payoffs which amount to discounting the future. In particular, we prove a form of one-deviation principle without any such assumptions. This suggests that coalgebra supports a more adequate treatment of infinite-horizon models in game theory and economics.

1 Introduction

Infinite structures turn up at many places in economic theory. Prominent examples are infinite games. The standard approach is to employ discounted dynamic programming to derive equilibria [2, 10]. In this paper we take a different approach. We shall use the methods of **coalgebra** [14, 5] to analyze two basic examples of infinite games without resorting to discounting and induction.

We use coalgebraic methods to define infinite extensive games and strategy profiles, and the notion of **subgame perfect equilibrium**. We then prove that certain profiles are subgame perfect equilibria. This leads us to formulate an (apparently) novel notion of predicate coinduction, which is shown to be sound by reducing it to Kozen's metric coinduction [7]. We give a complete characterization of the subgame perfect equilibria of the dollar auction game. We also prove a version of a standard game theoretical result, the **one deviation principle**, without needing future discounting assumptions.

Our work is inspired by recent work by Pierre Lescanne [8, 9], but it is explicitly coalgebraic, leading to a mathematically richer and more general approach. The coalgebraic approach we present is promising for further applications to economic modeling and we will discuss some directions for future work in the last section.

2 Games

We shall assume familiarity with the basic elements of coalgebra [14, 5].

We consider possibly infinite games of perfect information in extensive form and fix the following sets:

- A set $\mathcal{A}$ of **agents** or **players**. In our examples, we confine ourselves to two-player games, with $\mathcal{A} = \{A, B\}$.
- A set $\mathcal{C}$ of **choices**. We restrict our discussion to games where a player has a choice of two options, *left* or *right*, at every stage in the game where it is their turn to play. Thus $\mathcal{C} = \{\mathsf{l}, \mathsf{r}\}$.
- A set $\mathcal{U}$ of **utility functions**, which assign a utility to each player. We take $\mathcal{U} = \mathbb{R}^{\mathcal{A}}$.

The set of game trees is defined to be (the carrier of) the final coalgebra $(\mathcal{G}, \gamma)$ of the functor

$$F_{\mathsf{G}} : X \ \mapsto \ \mathcal{U} \, + \, \mathcal{A} \times X \times X.$$

on the category of **Set**. The game tree is a possibly infinite binary tree. The binary nodes are labelled with an agent whose turn it is to play at that stage in the game. The nodes have the form

$$\langle \alpha, g_{\mathsf{l}}, g_{\mathsf{r}} \rangle,$$

where α is the agent label, and g_{l}, g_{r} are the subgames corresponding to the left and right choices respectively. The leaf nodes are labelled with utility functions, representing the pay off for the game.

2.1 The 0/1 game

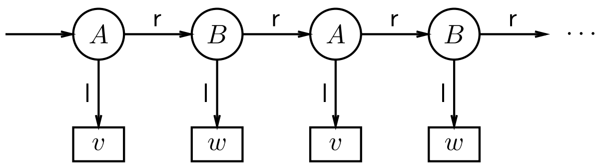
We define the utility functions to be

$$v := [A \mapsto 0, \ B \mapsto 1], \qquad w := [A \mapsto 1, \ B \mapsto 0].$$

The game is defined by the following pair of mutually corecursive equations:

$$\begin{aligned} G &= \langle A, v, H \rangle \\ H &= \langle B, w, G \rangle. \end{aligned}$$

We can depict the game tree as follows:



More formally, the above equations define a F_{G}-coalgebra $\alpha : \{G, H\} \to F_{\mathsf{G}}\{G, H\}$ on the set $\{G, H\}$. The 0/1 game is given by $[\![G]\!]$, the image of G under the unique coalgebra morphism

$$[\![\cdot]\!] : (\{G, H\}, \alpha) \longrightarrow (\mathcal{G}, \gamma)$$

from $(\{G, H\}, \alpha)$ to the final coalgebra.

Note that we can read the utility function v as 'A loses and B wins', while w corresponds to 'A wins and B loses'. Thus the player who chooses to stop first loses.

2.2 The dollar auction

The 0/1 game has only finite states; it can be represented by a finite system of equations. The dollar auction game, a well-known example in game theory, is an infinite-state refinement, where the utility functions change as we move down the tree.

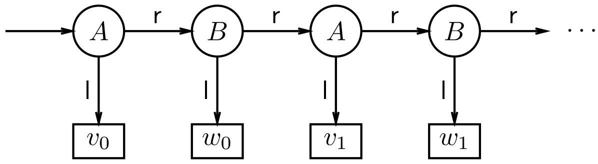
We fix a real number r, and define utility functions v_n, w_n for each stage $n \in \mathbb{N}$:

$$v_n := [A \mapsto -n, B \mapsto r - n], \qquad w_n := [A \mapsto r - (n+1), B \mapsto -n].$$

We define a game by simultaneous corecursion on the infinite set of variables $\{G_n, H_n \mid n \in \mathbb{N}\}$:

$$\begin{array}{rcl} G_n & = & \langle A, v_n, H_n \rangle \\ H_n & = & \langle B, w_n, G_{n+1} \rangle. \end{array}$$

The dollar auction game is again given by $[\![G_0]\!]$, the image of G_0 under the unique coalgebra morphism into the final coalgebra. We can depict the game tree as follows:



The motivation behind this game is as follows [15]. The value of the asset being bid for is r; in the original example, r is one dollar or 100 cents. The asset goes to the highest bidder, who is left with a profit of $r - b$, where b is the value of his highest bid. The loser must also pay the value of his highest bid, while getting nothing in return. The above definition describes the situation where each player raises their bid by one cent at each stage in the game. A player either gives up and finishes the game, conceding the auction to the other player and accepting their loss, or continues, hoping that the other player will give up. Both players have an incentive to continue playing well beyond the point where both will make a loss, in order to try to minimize their losses.

3 Strategy profiles

Intuitively, a strategy for a player P of a game G specifies a choice (left or right) for every node of G at which it is P's turn to move. A strategy profile specifies a strategy for every player. Following [8], we define the set of strategy profiles directly, as the final coalgebra $(\mathcal{S}, \sigma)$ of the functor

$$F_{\mathsf{SP}} : X \;\mapsto\; \mathcal{U} \,+\, \mathcal{A} \times \mathcal{C} \times X \times X.$$

There is an evident natural transformation $t : F_{\mathsf{SP}} \stackrel{\cdot}{\longrightarrow} F_{\mathsf{G}}$ defined by projection, which induces a functor from the category of F_{SP}-coalgebras to the category of F_{G}-coalgebras. It sends a strategy profile $s \in \mathcal{S}$ to the underlying game tree $\mathsf{game}(s) \in \mathcal{G}$. We say that s is a strategy profile for the game G if $G = \mathsf{game}(s)$.

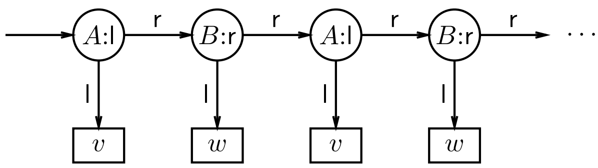
We now define a number of strategy profiles for the 0/1 and dollar auction games.

3.1 The 0/1 game

We define two strategy profiles for the 0/1 game where A always stops (plays left) and B always continues (plays right), by the following simultaneous corecursion:

$$\begin{array}{rcl} \mathsf{AsBc} & = & \langle A, \mathsf{l}, v, \mathsf{BcAs} \rangle \\ \mathsf{BcAs} & = & \langle B, \mathsf{r}, w, \mathsf{AsBc} \rangle \end{array}$$

We can depict this strategy profile as follows:



Formally, the strategy profile is given by $[\![\mathsf{AsBc}]\!]$, the image of the strategy profile under the morphism to the final coalgebra. The strategy profile $[\![\mathsf{AcBs}]\!]$, where A always continues and B always stops, is defined symmetrically.

3.2 The dollar auction

In a similar fashion, we define strategy profiles for the dollar auction by simultaneous corecursion on an infinite family of variables $\{\mathsf{AsBc}_n, \mathsf{BcAs}_n \mid n \in \mathbb{N}\}$:

$$\begin{aligned}\mathsf{AsBc}_n &= \langle A, \mathsf{l}, v_n, \mathsf{BcAs}_n \rangle \\ \mathsf{BcAs}_n &= \langle B, \mathsf{r}, w_n, \mathsf{AsBc}_{n+1} \rangle.\end{aligned}$$

Formally, $[\![\mathsf{AsBc}_0]\!]$ is a strategy profile for the dollar auction game in which A always stops and B always continues. The strategy profile $[\![\mathsf{AcBs}_0]\!]$ where A always continues and B always stops is defined symmetrically.

4 Subgame perfect equilibrium

We now show, following [8], how the game-theoretic notion of **subgame perfect equilibrium** can be defined coalgebraically. Firstly, we introduce two auxiliary notions.

4.1 Weak and strong convergence

We introduce two predicates on strategy profiles, of **weak** and **strong convergence**. A strategy profile is weakly convergent if following the choices that it specifies from the root eventually leads to a leaf; it is strongly convergent if this holds in every sub-profile.

We define weak convergence by:

$$s{\downarrow} \iff (s = U) \vee (s = \langle P, \mathsf{l}, s_\mathsf{l}, s_\mathsf{r} \rangle \wedge s_\mathsf{l}{\downarrow}) \vee (s = \langle P, \mathsf{r}, s_\mathsf{l}, s_\mathsf{r} \rangle \wedge s_\mathsf{r}{\downarrow}).$$

More formally, weak convergence is an element of the powerset $\mathcal{P}(\mathcal{S})$. It is defined as the least fixpoint of the monotone function

$$f_{\downarrow} : \mathcal{P}(\mathcal{S}) \longrightarrow \mathcal{P}(\mathcal{S}) \ :: \ S \mapsto \mathcal{U} \cup \{\langle P, \mathsf{l}, s_\mathsf{l}, s_\mathsf{r} \rangle \mid s_\mathsf{l} \in S\} \cup \{\langle P, \mathsf{r}, s_\mathsf{l}, s_\mathsf{r} \rangle \mid s_\mathsf{r} \in S\}.$$

Strong convergence is defined as follows:

$$s{\Downarrow} \iff (s = U) \vee (s = \langle P, c, s_\mathsf{l}, s_\mathsf{r} \rangle \wedge s{\downarrow} \wedge s_\mathsf{l}{\Downarrow} \wedge s_\mathsf{r}{\Downarrow}).$$

More formally, it is the greatest fixpoint of the monotone function $f_{\Downarrow}$ defined analogously to $f_{\downarrow}$

$$f_{\Downarrow} : \mathcal{P}(\mathcal{S}) \longrightarrow \mathcal{P}(\mathcal{S}) \ :: \ S \mapsto \mathcal{U} \cup \{\langle P, \mathsf{l}, s_\mathsf{l}, s_\mathsf{r} \rangle \mid s_\mathsf{l} \in S,\ s_\mathsf{r} \in S\}.$$

These fixpoints exist by the Knaster-Tarski fixed point theorem [6, 16].

Note that weak convergence is an **inductive** notion; the profile must specify a finite path from the root to a leaf. Thus it is defined as a least fixpoint. Strong convergence, by contrast, expresses a constraint on all subtrees of an infinite tree, and hence is defined **coinductively**, as a greatest fixpoint. The examples of the strategy profiles described in the previous section are all strongly convergent.

The relationship between the two notions can be nicely characterized in terms of coalgebraic modal logic [13, 12]. We can define an 'always' modality $\Box$ as an operator on $\mathcal{P}(\mathcal{S})$:

$$s \models \Box\phi \ \equiv \ s \models \phi \ \wedge \ s = \langle P, c, s_\mathsf{l}, s_\mathsf{r} \rangle \Rightarrow s_\mathsf{l} \models \Box\phi \ \wedge \ s_\mathsf{r} \models \Box\phi.$$

This operator is defined coinductively as a greatest fixpoint. This modality generalizes straighforwardly to any polynomial functor, and can in fact be defined in a much more general way in the context of coalgebraic modal logic [13, 12]. If we write WC for the weak convergence predicate, and SC for the strong convergence predicate, we have the following:

Proposition 4.1 $\mathsf{SC} = \Box\mathsf{WC}$.

4.2 Utility functions induced by strategy profiles

A strategy profile s induces a utility function $\hat{s}$. This function is defined as follows:

$$\hat{s} = \begin{cases} U, & s = U \\ \hat{s}_{\mathsf{l}}, & s = \langle P, \mathsf{l}, s_{\mathsf{l}}, s_{\mathsf{r}} \rangle \\ \hat{s}_{\mathsf{r}}, & s = \langle P, \mathsf{r}, s_{\mathsf{l}}, s_{\mathsf{r}} \rangle. \end{cases}$$

In general, this function may be partial; however, if s is weakly convergent, $\hat{s}$ is always a well-defined total function in $\mathcal{U}$.

4.3 Subgame perfect equilibria

We are now ready to define the notion of a strategy profile being a **subgame perfect equilibrium**. Firstly, we define a predicate PE on strategy profiles:

$$\mathsf{PE}(s) \iff s{\Downarrow} \wedge (s = \langle P, \mathsf{l}, s_{\mathsf{l}}, s_{\mathsf{r}} \rangle \Rightarrow \hat{s}_{\mathsf{l}}(P) \geq \hat{s}_{\mathsf{r}}(P)) \wedge (s = \langle P, \mathsf{r}, s_{\mathsf{l}}, s_{\mathsf{r}} \rangle \Rightarrow \hat{s}_{\mathsf{r}}(P) \geq \hat{s}_{\mathsf{l}}(P)).$$

This predicate is defined explicitly in terms of previous notions; neither induction nor coinduction is used. It says that the choice at the root for player P results in a better payoff for P than the other choice would have done. Note that this predicate implies in particular that strong convergence holds.

We now define the subgame perfect equilibrium predicate SPE on $\mathcal{S}$, coinductively as a greatest fixpoint.

$$\mathsf{SPE}(s) \iff \mathsf{PE}(s) \wedge (s = \langle P, c, s_{\mathsf{l}}, s_{\mathsf{r}} \rangle \Rightarrow \mathsf{SPE}(s_{\mathsf{l}}) \wedge \mathsf{SPE}(s_{\mathsf{r}})).$$

This says that the PE predicate holds at every node of the tree and we get the following analogue of Proposition 4.1.

Proposition 4.2 $\mathsf{SPE} = \Box\mathsf{PE}$.

The next step is to show that all the strategy profiles discussed in the previous section are in fact subgame perfect equilibria. In order to do this, we will need to have an appropriate proof principle in place. This is the topic of the next section.

5 Predicate coinduction

The main emphasis in coinductive proofs has been on proving equations; the main tool for this is provided by the notion of bisimulation. However, as emphasized by Kozen [7], the scope of coinductive methods is broader than this. In our case, we are interested in **predicates** (properties) rather than equations. In particular, we wish to show that various elements of the final coalgebra $\mathcal{S}$ satisfy the SPE predicate.

We formulate a proof principle which is adequate to carry out these proofs, and justify it in terms of Kozen's metric coinduction principle. It should be possible to give a much more general account; this is an interesting challenge, which is left to future work.

We formulate the principle in terms of an arbitrary polynomial functor T and apply it in the case $T = F_{\mathsf{SP}}$. The final coalgebra of T is denoted $(\mathcal{S}, \sigma)$.

Firstly, we need some notation. Suppose we have a T-coalgebra (X, α), which we think of as a corecursive system of equations on the set of variables X. We define a map

$$\bar{\alpha} : \mathcal{S}^X \longrightarrow \mathcal{S}^X :: \eta \mapsto [\, x \mapsto \sigma^{-1} \circ T\eta \circ \alpha(x) \,].$$

Here $\sigma^{-1} : T\mathcal{S} \longrightarrow \mathcal{S}$ is the inverse of σ, which is an isomorphism by the Lambek lemma.

The following proposition follows directly by unravelling the definitions and applying the final coalgebra property:

Proposition 5.1 *The map $\bar{\alpha}$ has a unique fixpoint $\bar{\alpha}^* \in \mathcal{S}^X$; moreover, $\bar{\alpha}^* = [\![\cdot]\!]$, the unique coalgebra morphism from (X, α) to the final coalgebra.*

Now let $\phi \subseteq \mathcal{S}$ be a predicate on $\mathcal{S}$. We lift this predicate to $\phi^X \subseteq \mathcal{S}^X$:

$$\eta \in \phi^X \iff \forall x \in X.\, \eta(x) \in \phi.$$

We say that ϕ is an **invariant** if $\phi = \Box\psi$ for some predicate ψ and formulate the predicate coinduction principle as a proof rule:

$$\frac{\phi \text{ invariant}, \ \phi \neq \varnothing, \quad \eta \in \phi^X \ \Rightarrow \ \bar{\alpha}(\eta) \in \phi^X}{\bar{\alpha}^* \in \phi^X}.$$

Using Proposition 5.1, we can restate the conclusion as follows:

$$\forall x \in X.\, [\![x]\!] \in \phi.$$

In order to show the soundness of this principle, we reduce it to the following metric coinduction principle due to Kozen [7][1]:

Proposition 5.2 *Let M be a complete metric space, $u : M \to M$ a contractive map, and $C \subseteq M$ a non-empty closed subset of M. We write u^* for the unique fixpoint of u, which exists by the Banach fixpoint theorem. Then the following proof rule is valid:*

$$\frac{x \in C \ \Rightarrow \ u(x) \in C}{u^* \in C}.$$

Proof We have $u^* = \lim_{n\to\infty} u^n(a)$ for any $a \in M$. Since C is non-empty, we can take $a \in C$. By the premise of the rule, $u^n(a) \in C$ for all n. Since C is closed, $u^* \in C$. $\Box$

The reduction is an immediate consequence of the following result.

Proposition 5.3 *We can define a distance function $d : \mathcal{S}^2 \to [0, 1]$ such that the following holds:*

1. *$(\mathcal{S}, d)$ is a complete ultrametric space.*
2. *An invariant $\phi \subseteq \mathcal{S}$ is closed in this space.*
3. *For any set X, $(\mathcal{S}^X, d^X)$ is a complete ultrametric space, where*

$$d^X(\eta, \mu) \ := \ \sup_{x \in X} d(\eta_x, \mu_x).$$

4. *For any coalgebra (X, α), the map $\bar{\alpha}$ is contractive.*

Proof This is essentially a variation of well-known results, see e.g. [1]. The basic point is that T is cocontinuous, and hence the final T-coalgebra is the limit of the ω^{op}-diagram

$$\mathbf{1} \longleftarrow T\mathbf{1} \longleftarrow T^2\mathbf{1} \longleftarrow \cdots \longleftarrow T^k\mathbf{1} \longleftarrow \cdots \tag{1}$$

where $\mathbf{1}$ is the terminal object in **Set**. The connecting maps are $T^k! : T^{k+1}\mathbf{1} \longrightarrow T^k\mathbf{1}$, where $! : T\mathbf{1} \longrightarrow \mathbf{1}$ is the unique map to the terminal object. Starting at the unique map π_0 to the terminal object, we define the maps $\pi_k : \mathcal{S} \to T^k\mathbf{1}$ inductively by

$$\pi_{k+1} \ := \ \mathcal{S} \xrightarrow{\sigma} T\mathcal{S} \xrightarrow{T\pi_k} TT^k\mathbf{1}.$$

Then $(\mathcal{S}, \{\pi_k\})$ is the limit cone of the diagram (1). This has the following consequences:

- For all $x, y \in \mathcal{S}$:

$$x = y \iff \forall k.\, \pi_k x = \pi_k y.$$

[1] Kozen states a more general principle, but this version will be sufficient for our purposes.

- Given a sequence $\{x_k\}$ with $x_k \in T^k\mathbf{1}$ and $x_k = T^k!(x_{k+1})$, there is a unique $x \in \mathcal{S}$ such that for all k, $x_k = \pi_k x$.

The ultrametric on $\mathcal{S}$ is given by

$$d(x, y) = \begin{cases} 0, & x = y \\ 2^{-k}, & \text{least } k \text{ such that } \pi_k x \neq \pi_k y,\ x \neq y. \end{cases}$$

The properties stated in the proposition follow readily from this description. For example, the completeness of d^X follows from the fact that Cauchy convergence in this metric implies that for each k, for some N, for all $x \in X$, $\pi_k(\eta_j(x))$ is fixed for all $j \geq N$. This implies a uniform mode of convergence, and thus justifies taking limits pointwise in $\mathcal{S}^X$. □

6 Proving subgame perfect equilibria

We now apply predicate coinduction to show that the strategy profiles defined in Section 3 are subgame perfect equilibria.

We begin with the strategy profiles for the 0/1-game.

Proposition 6.1 *The strategy profiles* $[\![\mathsf{AcBs}]\!]$ *and* $[\![\mathsf{AsBc}]\!]$ *are subgame perfect equilibria.*

Proof Applying predicate coinduction, to show that $[\![\mathsf{AsBc}]\!]$ is SPE, we must show that $\langle A, \mathsf{l}, v, \mathsf{BcAs}\rangle$ and $\langle B, \mathsf{r}, w, \mathsf{AsBc}\rangle$ are SPE, under the assumption that AsBc and BcAs are SPE. Using the coinduction hypothesis, we just need to show that these nodes are PE. Computing their induced payoff functions explicitly, this reduces to verifying the inequalities

$$v(A) = 0 \geq 0 = v(A), \quad v(B) = 1 \geq 0 = w(B).$$

The verification that $[\![\mathsf{AcBs}]\!]$ is SPE is similar. □

We now turn to the dollar auction. We recall the parameter r used to define the utility functions v_n, w_n.

Proposition 6.2 *If* $r \geq 1$*, the strategy profiles* $[\![\mathsf{AcBs}_n]\!]$ *and* $[\![\mathsf{AsBc}_n]\!]$ *are subgame perfect equilibria for all* n*.*

Proof The coinduction hypothesis is that AsBc_n and BcAs_n are SPE for all n. We must show that $\langle A, \mathsf{l}, v_n, \mathsf{BcAs}_n\rangle$ and $\langle B, \mathsf{r}, w_n, \mathsf{AsBc}_{n+1}\rangle$ are SPE for all n. In similar fashion to the previous result, this reduces to verifying the inequalities

$$v_n(A) = -n \geq -(n+1) = v_{n+1}(A), \quad v_{n+1}(B) = r - (n+1) \geq -n = w_n(B).$$

The latter inequalities are satisfied if and only if $r \geq 1$. □

7 The one-deviation principle

We now verify an important property of subgame perfect equilibria: a strategy profile is SPE if and only if it dominates any profile which differs from it in exactly one choice. In the standard game theoretical literature [3], this is proved for infinite games only under strong continuity assumptions on the payoffs, which amount to discounting beyond some finite horizon, thus allowing reduction to standard backwards induction reasoning. We need no such assumptions.

Given a strategy profile s of the form $\langle P, c, s_\mathsf{l}, s_\mathsf{r}\rangle$, we say that a profile t for the same game is a one-deviation from s if t has one of the following forms:

- $\langle P, c', s_\mathsf{l}, s_\mathsf{r}\rangle$, where $c' \neq c$.
- $\langle P, c, s'_\mathsf{l}, s_\mathsf{r}\rangle$, where s'_l is a one-deviation from s_l.

- $\langle P, c, s_\mathsf{l}, s'_\mathsf{r} \rangle$, where s'_r is a one-deviation from s_r.

This is an inductive definition. Given a strategy profile s we define a relation $s \succcurlyeq t$, where t is a one-deviation of s, inductively as follows:

- If $s = \langle P, \mathsf{l}, s_\mathsf{l}, s_\mathsf{r} \rangle$ and $t = \langle P, \mathsf{r}, s_\mathsf{l}, s_\mathsf{r} \rangle$, then $s \succcurlyeq t$ iff $\hat{s}_\mathsf{l}(P) \geq \hat{s}_\mathsf{r}(P))$.
- If $s = \langle P, \mathsf{r}, s_\mathsf{l}, s_\mathsf{r} \rangle$ and $t = \langle P, \mathsf{l}, s_\mathsf{l}, s_\mathsf{r} \rangle$, then $s \succcurlyeq t$ iff $\hat{s}_\mathsf{r}(P) \geq \hat{s}_\mathsf{l}(P))$.
- If $s = \langle P, c, s_\mathsf{l}, s_\mathsf{r} \rangle$ and $t = \langle P, c, s'_\mathsf{l}, s_\mathsf{r} \rangle$, then $s \succcurlyeq t$ iff $s_\mathsf{l} \succcurlyeq s'_\mathsf{l}$.
- If $s = \langle P, c, s_\mathsf{l}, s_\mathsf{r} \rangle$ and $t = \langle P, c, s_\mathsf{l}, s'_\mathsf{r} \rangle$, then $s \succcurlyeq t$ iff $s_\mathsf{r} \succcurlyeq s'_\mathsf{r}$.

Proposition 7.1 (The one-deviation principle) *A strongly convergent strategy profile s is* SPE *if and only if for every one-deviation t, $s \succcurlyeq t$.*

Proof Firstly, note that if $\neg(s \succcurlyeq t)$ for some one-deviation t, this means that some subprofile of s does not satisfy PE, and since SPE $= \Box$PE, this implies that s does not satisfy SPE. For the converse, note that if for some one-deviation t, $s \succcurlyeq t$, this implies that the sub-profile of s whose root is at the node where t differs from s satisfies PE. If this holds for all one-deviations, then all sub-profiles of s satisfy PE, and hence s satisfies SPE. $\Box$

8 Complete Characterization of SPE for the Dollar Auction

We can give a **complete characterization** of the subgame perfect equilibria for the dollar auction game.

Theorem 8.1 *A strategy profile for the dollar auction game is SPE if and only if it is of the form AsBc or A always stops and B always continues or symmetrically AcBs or B always stops and A always continues.*

Proof Firstly, note that if both players always continue from some point in the game, the profile will not be strongly convergent. Now suppose that both players choose to stop at some nodes of the game. Then there must be some node ν where player α chooses to stop, such that the next player choosing to stop is the other player. But then ν is not in PE, since α could improve his payoff from that node by choosing to continue. Thus one player must always continue in any SPE profile, while the other player α must stop infinitely often. Finally, α must in fact always choose to stop, since otherwise he could improve his payoff from any node where he chooses to continue. $\Box$

This analysis applies to any game sharing the following features of the dollar auction game:

1. At any point of the game, it is always better for a given player if the other player stops first.
2. At any point of the game, it is (strictly) better for the player who is the first to stop from that point to stop immediately rather than later.

As a final remark on the dollar auction game, we note that Lescanne proposes to use this form of analysis to explain the **rationality of infinite escalation** [8]. If we don't know which strategy the other player is following, we always have an incentive to continue! However, by our characterization result, after one round where both players choose to continue, **they both know they are not in an SPE** — and all bets are off! Thus it seems to us that a comprehensive analysis of escalation should use a refined model with an explicit representation of the beliefs of the players, as in Harsanyi type spaces [4] — which can also be modelled coalgebraically [11].

9 Further Directions

The present paper contains what should be regarded as some very preliminary results, presented in a manner which is mainly aimed at computer scientists and mathematicians rather than economists and game theorists. Nevertheless, in our view the general idea of applying coalgebraic and other structural methods which have been developed in computer science to economics and game theory is promising, and deserves further study and development. In particular, many topics in economics which refer to infinite horizons and reflexivity seem tailor-made for the use of coalgebraic methods. At the same time, they can suggest new challenges and technical directions for coalgebra.

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