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Quantile Maximizing Safety-First Investors: Separation, Performance Measurement and Capital Market Equilibrium

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QUANTILE MAXIMIZING SAFETY-FIRST INVESTORS: SEPARATION, PERFORMANCE MEASUREMENT AND CAPITAL MARKET EQUILIBRIUM

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We combine the safety-first principle of Telser (1955/56) and Arzac and Bawa (1977) with the principle of quantile maximization studied in Rostek (2010). While maintaining the shortfall constraint of the safety-first principle, we propose to maximize an upper quantile of the return distribution instead of maximizing its expected value. We study the implications of this new decision principle for portfolio selection and capital market equilibrium on one hand and for risk-adjusted performance measurement on the other hand.

KEY WORDS: Safety-first, separation, capital market equilibrium, performance ratio, RORAC

1. INTRODUCTION

Risk/value-models (see e.g. Sarin and Weber 1993; Mitchell and Gelles 2003) for evaluating decisions under risk serve as a standard approach for constructing optimal portfolios. Traditionally, following the seminal work of Markowitz (1952) and Tobin (1958), mean/variance (MV)-investors are considered in portfolio theory. MV-investors are using the expected value as a measure for value (or reward) and the variance resp. the standard deviation as a measure for risk.

The (risk-adjusted) performance ratio corresponding to this approach in the Sharpe ratio (Sharpe 1966, 1994). The CAPM (Sharpe 1964, Lintner 1965) is the corresponding model for capital market equilibrium. It is well known (see e.g. McNeil, Frey and Embrechts 2005, p. 247) that under the assumption of elliptical return distributions, the use of any positive-homogeneous and translation-invariant measure of risk to rank risks on one hand or to determine the optimal risk-minimizing portfolio under the condition that a certain expected return is attained on the other, is equivalent to using the variance resp. the standard deviation as the measure of risk. Alternative risk measures, such as value-at-risk (VaR) or conditional value-at-risk (CVaR) give different numerical values, but have no effect on the management of risk.

However, in a non-symmetrical world[1] this situation changes completely and alternative measures of risk and alternative performance ratios have attracted considerable interest[2].

There are many ways to construct risk/value-models alternative to the traditional MV-approach. One possible approach is to use a safety-first (SF) principle, which we will focus in the present contribution. The first version of the SF principle was advocated by Roy (1952), who suggested that investors have in mind some disaster level of returns and that they behave as to minimize the probability of disaster (nowadays usually called shortfall probability). In this version the SF principle is a pure risk-minimizing strategy. As there is an equivalence between shortfall probabilities and (lower) quantiles of the return distribution, Roy's (1952) approach is equivalent to a quantile minimizing approach resp., when the distribution of the loss $L = -R$ is considered instead of the return distribution, equivalent to a quantile maximizing approach. Later on, Telser (1955/56) proposed a different version of the SF principle, which possesses a non-compensating form, i.e. an arbitrary trade-off between risk and value is not possible. The basic idea of Telser was to introduce a shortfall constraint, i.e. to limit the shortfall probability (or equivalently to limit a corresponding quantile), in a first stage and then in a second stage to maximize the expected return, given a reduced set of actions with admissible risk. Chipman (1971) resp. Arzac and Bawa (1977) propose to use a lexicographic form of the SF principles of Roy (1952) resp. Telser (1955/56), because these original criteria fail to order risky assets, which are unambiguously ordered by the dominance principle (principle of absolute preference). Arzac and Bawa (1977), moreover, studied the implications of Telser's approach for portfolio selection and for capital market equilibrium. In the present contribution, we generalize Telser's resp. Arzac and Bawa's versions of the SF principle in that we combine it with the principle of quantile maximization recently studied in Rostek (2010). While maintaining the shortfall constraint of the SF-principle, we propose to maximize an upper quantile of the return distribution as a measure of value resp. reward instead of the expected value. Instead of "quantile maximization", which is a rather technical term, one could also speak of "maximizing the probable maximum return" to characterize the economic content of the corresponding behavior. After having introduced and characterized the new decision principle, which we call quantile maximization/safety-first (QSF) principle, we study

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1 According to Cont (2001, p. 224) a gain/loss asymmetry is a stylized fact of asset returns. Vast literature has documented the asymmetry in stock index return distributions, see e.g. Ekholm and Pasternack (2005). Moreover there are pronounced skewness effects in connection with financial positions containing options (Bookstaber and Clarke 1984) and in connection with alternative investments (Brooks and Kat 2002).

2 For recent overviews on risk measures alternative to the variance and performance ratios alternative to the Sharpe ratio, see e.g. Biglova et al. (2004) and Farinelli et al. (2008).

the implications of this decision principle for portfolio selection and capital market equilibrium on one hand (thereby generalizing the results of Arzac and Bawa 1977) and for risk-adjusted performance measurement on the other hand.

The present paper first contributes to the risk/value-approach for evaluating decisions under risk in that it proposes a new (non-compensating) risk/value-model. We introduce a new measure of risk (the quantile range) and a new measure of value resp. reward (the probable maximum return). Second, the paper contributes to the theory of portfolio selection in the presence of a riskless asset in providing a corresponding separation result. Third, the paper contributes to the field of capital market equilibrium models.

From an application perspective, the paper develops a new performance ratio (the Q ratio). This performance ratio possesses a sound decision theoretic basis, which is in contrast to most performance ratios introduced in literature, being mere ad hoc-modifications of the Sharpe ratio. Finally, again from an application perspective the paper develops a new measure of return on risk-adjusted capital (Q-RORAC), suitable for enterprise risk management.

The literature directly related to the present paper is limited. Beyond the already cited papers on SF principles our approach involves a quantile maximization and so there is a connection to this strand of literature. Chambers (2009) provides a characterization of quantiles on the basis of three axioms (ordinal covariance, weak monotonicity with respect to first-order stochastic dominance, and upper semicontinuity), but he does not consider quantiles in any decision theoretical context. Rostek (2010) takes this decision theoretical perspective and introduces the principle of quantile maximization. She gives a complete behavioral characterization of a decision maker, who when choosing between uncertain alternatives evaluates each alternative by a quantile of the distribution (given a fixed confidence level) and selects the one with the highest quantile payoff (resp., the highest quantile of an utility over outcomes). When applied to financial returns, this approach basically involves the maximization of an upper quantile resp. the minimization of a lower quantile of the return distribution, which is corresponding to Roy's approach. Rostek´s (2010) approach also contains maxmin and minmax choice rules. However, it does not involve any restriction of the risk of the considered decisions. In fact Rostek's (2010) approach is a choice model and does not try to model risk or value separately at all.

As we will see, our approach implicitly defines a risk measure (the quantile range). This implies that there is a connection to the extensive literature on risk measures and their axiomatization (see e.g. Artzner et al. 1999, Rockafellar et al. 2006, or Heyde et al. 2007). However, being a risk/value-model our model not only involves a quantification of risk, but moreover a tradeoff between risk and return. Among risk/value-models our approach belongs to[3] the class of non-compensating risk/value-models. With respect to the value component our model is non-standard as it is based on an upper quantile as a measure of value and not on the expected return. We are not aware of a similar decision model. To some extent, an exception is the paper of Bernardo and Ledoit (2000). However, their definition of gain and loss is different (being the positive resp. the negative part of an excess payoff relative to a benchmark) and as well the problem they are studying (asset pricing in incomplete markets) is different. However, the gain/loss-ratio Bernardo and Ledoit (2000) do introduce can be considered as a risk-adjusted performance ratio, although the authors evaluate this ratio with respect to a risk neutral measure and not with respect to a physical measure. This brings us to the connection of our contribution to the extensive literature on risk-adjusted performance ratios. We will elaborate on this connection further in section 4.1. of the present contribution. At this point we merely want to mention that with a few exceptions (which we will explicitly address) most of these performance ratios are mere ad hoc-modifications of the Sharpe ratio. In contrast, the performance ratio (Q ratio) proposed in the present contribution explicitly is based on a decision theoretic model.

With respect to the application in enterprise risk management, it is our impression that the RORAC-approach used in this field primarily is an ad hoc-approach, too, i.e. there is no decision theoretic model underlying this approach. The only exception we are aware of is the contribution of Stoughton und Zechner (2007), which is based on an agency theoretical approach. Stoughton und Zechner (2007) consider a related quantity (RAROC) and apply it to the capital allocation problem.

With respect to the resulting equations for capital market equilibrium in the presence of QSF-investors our paper contributes to the arguments put forward in Levy (2010), who gives evidence that the CAPM not only is valid in an expected utility framework. In the case of QSF-investors the CAPM is obtained from a linear approximation of the equilibrium equations (the

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[3] This feature is also the key difference to decision models of the mean-lower partial moment type (see e.g. Fishburn 1977 or Harlow and Rao 1989) which belong to the (standard) class of compensating risk/value-models.

result being exact for elliptical distributions). In a non-symmetrical world, however, the QSF equilibrium is different from the CAPM, which we will demonstrate on the basis of an explicit formula based on a quadratic approximation of the equilibrium equations.

2. THE DECISION PRINCIPLE

The intended primary application of the decision principle studied in the present distribution is the portfolio choice problem. Therefore we formulate the principle a priori in terms of one-period returns and take as given a set D of one-period returns $R \in D$. Throughout the paper, we will assume that the returns are random variables possessing a strictly positive density function. In consequence, all quantiles of the considered return distributions are unique. We specify two confidence levels $0 < \alpha < 0.5$ and $0 < \beta < 0.5$, with α and β being small numbers, and we specify a desired minimum return (target return, disaster level) z.

Given any two returns $R_1 \in D$ and $R_2 \in D$ with corresponding shortfall probabilities $p_1 = P(R_1 \le z)$ resp. $p_2 = P(R_2 \le z)$ and upper quantiles $Q_{1-\beta}(R_1)$ resp. $Q_{1-\beta}(R_2)$, we consider the following strict preference order:

$$\text{(2.1)} \qquad R_1 \prec R_2 \Leftrightarrow \begin{cases} Q_{1-\beta}(R_1) < Q_{1-\beta}(R_2)\ , & \text{if } \ p_1 \le \alpha \ \text{ and } \ p_2 \le \alpha \\ p_1 > \alpha \ \text{ and } \ p_2 \le \alpha & \\ p_2 < p_1\ , & \text{if } \ p_1 > \alpha \ \text{ and } \ p_2 > \alpha. \end{cases}$$

Returns $R_1 \in D$ and $R_2 \in D$ are indifferent $(R_1 \sim R_2)$, if $Q_{1-\beta}(R_1) = Q_{1-\beta}(R_2)$ in the case of $p_1 \le \alpha$ and $p_2 \le \alpha$ or if $p_1 = p_2$ in case of $p_1 > \alpha$ and $p_2 > \alpha$.

Given the disaster level z, the return with a smaller shortfall probability is preferred unless both returns possess a shortfall probability not exceeding the pre-specified confidence level α. In this case the return with a higher upper quantile $Q_{1-\beta}$ is preferred. As we have the property $P(R > Q_{1-\beta}) = \beta$, the quantity $Q_{1-\beta}(R)$ can intuitively be considered to be the probable maximum return at the confidence level $1-\beta$. The difference with respect to Arzac and Bawa's (1977) SF principle is that Arzac and Bawa take the expected value $E(R)$ instead of the probable maximum return $Q_{1-\beta}(R)$ as the quantity to be maximized. The confidence level

α constrains the shortfall probability and the confidence level $1-\beta$ is controlling the desired probable maximum return.

The decision principle (2.1) produces a complete ordering of all risky prospects possessing finite quantiles at the level $1-\beta$. Moreover[4], it satisfies the principle of absolute preference (dominance principle), as one sees as follows. Let there be given any two returns R_1 and R_2 with distribution functions $F_1(x)$ and $F_2(x)$. In case $F_2(x) \leq F_1(x)$ for all x, we obviously have $p_2 \leq p_1$. On the other hand, we also have[5] $Q_{1-\beta}(R_2) \geq Q_{1-\beta}(R_1)$. So $F_2(x) \leq F_1(x)$ for all x implies $R_1 \precsim R_2$.

We will call the decision principle (2.1) quantile-maximization/safety-first (QSF) principle. Correspondingly we will call investors who behave according to this principle quantile maximizing safety-first investors, or QSF-investors in short.

As the SF principle of Arzac and Bawa (1977) the QSF principle (2.1) is incompatible with the axiom of continuity and the axiom of independence, which can be proven using the same arguments as given in Arzac and Bawa (1977). In consequence the decision principle (2.1) is not compatible with expected utility theory[6]. The core of this incompatibility is, that the SF principle as well as the QSF principle strictly limit risk, and so there is no arbitrary trade-off possible between "risk" and "value" (where value is measured via the expected value by Arzac and Bawa and via an upper quantile in the present contribution). This feature may be not attractive from a decision theoretic point of view, but it precisely meets the restrictions to be kept in many important decision situations in business practice. Financial institutions, for instance banks and insurance companies[7], are regularly regulated in a way that the value-at-risk (which from a statistical perspective is nothing else than a quantile of the considered contribution) of the financial position of the company is strictly limited a priori, so there is no possi-

4 One can also show, following exactly the arguments of Arzac and Bawa (1977, p. 281) that the QSF-investors exhibit decreasing absolute risk aversion.

5 This corresponds to the well known monotonicity property of quantiles, see e.g. Chambers (2009, p. 337).

6 Consequently, the contributions in which the SF principle is studied from the perspective of expected utility theory, see e.g. Levy and Sarnat (1972) or Levy and Levy (2009), only relate to Roy's approach.

7 Consequently, in actuarial literature traditionally the restriction of the so called ruin probability (which is a dynamic version of a shortfall constraint) is considered as a stability criterion, see e.g. Bühlmann (1970, Chapter 6.2).

bility of an arbitrary trade-off between "risk" and "value". Limiting the value-at-risk corresponds exactly to the shortfall constraint involved in the SF resp. the QSF principle.

Arzac and Bawa (1977) very clearly state that the axioms of continuity and independence, which are discussed intensively in the decision theoretic literature, are not compelling restrictions on economic choices and they do not find sufficient a priori or empirical basis for preferring expected utility to safety-first or vice versa. Thus, it seems reasonable to consider both principles complementary, and to develop theories of financial markets which allow the existence of both type of investors.

In the standard situation (which regularly will be the case in the applications considered in the present contribution) where there is at least one return R in the decision set D which fulfills the shortfall constraint $P(R \le z) \le \alpha$, the optimal investment will be given on the basis of the following optimization problem:

(2.2a) $$Q_{1-\beta}(R) \to \max! \quad R \in D$$

(2.2b) $$P(R \le z) \le \alpha .$$

This generalizes the SF principle in the basic version of Telser (1955/56). Obviously $P(R \le z) \le \alpha$ is equivalent to

(2.2b') $$Q_\alpha(R) \ge z .$$

Therefore, an equivalent interpretation of the decision principle proposed is that we constrain the probable minimum return at confidence level α and maximize the probable maximum return at level $1-\beta$. Defining as usually the value-at-risk (on the return level) at confidence level α as[8] $VaR_\alpha(R) = Q_{1-\alpha}(-R) = -Q_\alpha(R)$, we in addition have another equivalent version of (2.2b), namely

(2.2b'') $$VaR_\alpha(R) \le -z ,$$

8 The last equation requires the existence of a density function of the return distribution.

In consequence, the shortfall constraint (2.2b) is equivalent to a constraint of the value-at-risk at confidence level α.

There is yet another interesting interpretation. Heyde et al. (2007) notice a connection between the risk measure "tail conditional median" and the risk measure "value at risk". In the context of the present contribution we have the (obvious) relations

(2.3a) $$Q_{0.5}[R \mid R > Q_{1-\beta}(R)] = Q_{1-\beta/2}(R)$$

and

(2.3b) $$Q_{0.5}[R \mid R \le Q_{\alpha}(R)] = Q_{\alpha/2}(R)\,.$$

This means, that the decision principle (2.2) can be interpreted to constrain the tail conditional median at confidence level 2α and to maximize the tail conditional median at confidence level $1-2\beta$.

As Heyde et al. (2007) argue, the risk measure tail conditional median is a more robust alternative to the risk measure tail conditional mean (which in turn is equivalent to the risk measure conditional value-at-risk for distributions possessing a density function).

3. OPTIMAL PORTFOLIOS: A SEPARATION RESULT

In our further analysis of the consequences of QSF-investors for portfolio selection we assume the standard model of one-period portfolio theory including a riskless asset. We have n risky one-period returns $R_i \ (i = 1,\dots,n)$ and a riskless interest rate r_0 at which arbitrary amounts can be invested or borrowed. The return R_a of a leverage portfolio based on investing a proportion a $(0 \le a < \infty)$ in a risky portfolio P with a return R_P and investing a proportion of $1-a \ (-\infty < 1-a \le 1)$ in the riskless security is given by

(3.1) $$R_a = aR_P + (1-a)\,r_0\,.$$

We assume[9] $z < r_0$ and in addition $E(R_P) - r_0 > 0$, i.e. a positive risk premium for risky portfolios P. We note that quantiles are linear functionals as long as the linear transformation is positive, that is we have for $a > 0$

$$(3.2) \qquad Q(aR_P + b) = aQ(R_P) + b$$

for any fixed confidence level. In consequence, we obtain

$$(3.3) \qquad Q_{1-\beta}(R_a) = r_0 + a[Q_{1-\beta}(R_P) - r_0].$$

Assuming sufficiently small confidence levels β so that $Q_{1-\beta}(R_P) \geq E(R_P)$, we see that the probable maximum return is (strictly) monotone increasing in the investment proportion a.

On the other hand we also have

$$(3.4) \qquad Q_\alpha(R_a) = a\,Q_\alpha(R_P) + (1-a)\,r_0 = r_0 - a[r_0 - Q_\alpha(R_P)].$$

Assuming small enough confidence levels so that $Q_\alpha(R_P) < r_0$, we see that $Q_\alpha(R_a)$ is (strictly) monotone decreasing in a. Now $Q_\alpha(R_a) \geq z$ is equivalent to

$$(3.5) \qquad a \leq \frac{r_0 - z}{r_0 - Q_\alpha(R_P)}.$$

As given our assumptions the right hand side is positive this means that the shortfall constraint (2.2b) can always be met for leverage portfolios involving an arbitrary risky portfolio P, if we choose the investment in P being "small enough".

As QSF-investors maximize (3.3), which is monotone increasing in a, they will realize a maximal amount of a. The optimal leverage portfolio involving the risky portfolio P therefore is characterized by the following choice of $a = a(P)$:

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9 In case $z \geq r_0$, the riskless security would not be admissible for QSF-investors.

(3.6) $$a(P) = \frac{r_0 - z}{r_0 - Q_\alpha(R_P)}.$$

Combining (3.6) with (3.3) we consequently have

(3.7) $$Q_{1-\beta}(R_a) = r_0 + (r_0 - z)\frac{Q_{1-\beta}(R_P) - r_0}{r_0 - Q_\alpha(R_P)}.$$

This in turn implies that for a given level of z, the quantity $Q_{1-\beta}(R_a)$ is maximized when we maximize the ratio

(3.8) $$QSF(R_P) = \frac{Q_{1-\beta}(R_P) - r_0}{r_0 - Q_\alpha(R_P)},$$

which we call the quantile safety-first ratio, or QSF ratio for short.

In consequence, we have demonstrated a separation property for QSF-investors. This separation result is generalizing the corresponding result of Arzac and Bawa (1977) for SF-investors. This property is in complete analogy to the separation property of MV-investors. In a first step QSF-investors determine the optimal risky portfolio P^* by maximizing the QSF-ratio (3.8). This corresponds to the case of MV-investors, where P^* is obtained by maximizing the Sharpe ratio.

The optimal risky portfolio P^* obviously is independent of the assumed minimum return z and only depends on the assumed confidence levels α and β. As from (3.2) we can easily obtain $QSF(R_a) = QSF(R_P)$, in turn all optimal leverage portfolios are characterized by having a maximal QSF-ratio (3.8). Summing up, this means that the optimal leverage portfolio of a QSF-investor can be obtained in the following two steps:

1. The investor determines the optimal (purely) risky portfolio maximizing the QSF ratio (3.8)
2. The investor determines the optimal leverage portfolio on the basis of equation (3.7), i.e. specifies a desired minimum return z. This in turn determines the chosen level of risk.

This separation result for QSF-investors contributes to the literature, which aims at generalizations of the separation property for MV-investors. Usually this is done within the framework of a (compensating) risk/value-model or within the framework of expected utility theory and either assumes specific families of utility functions (see e.g. Cass and Stiglitz, 1970, or Breuer and Gürtler, 2006) or specific families of distributions (see e.g. Owen and Rabinovitch, 1983). Alternatively, Breuer and Gürtler (2007) show that when the risk measure if fulfilling certain properties, a separation result also holds in a mean/risk-model.

4. PERFORMANCE MEASUREMENT

4.1. *Investment Management: Q Ratio*

The distinct feature of the Sharpe ratio

$$S(R) = \frac{E(R) - r_0}{\sigma(R)} \tag{4.1}$$

as a measure for risk-adjusted performance is based on the fact that for all MV-investors optimal leverage portfolios possess a maximal Sharpe ratio. In consequence, MV-investors have to maximize the Sharpe ratio to obtain an optimal portfolio. The basic risk measure for MV-investors is the standard deviation of returns and this basic risk measure also is central to the Sharpe ratio. As the standard deviation makes no distinction between positive and negative deviations from the mean, the standard deviation is a good risk measure only for distributions, which are (approximately) symmetric around the mean, which is the case for elliptical distributions. However, it is a stylized fact that asset returns often are of asymmetric nature[10]. Also in many areas of risk management loss distributions are highly skewed[11]. Consequently, there is a vast literature[12] on asymmetric risk measures and correspondingly on asymmetric performance measures where in (4.1) the standard deviation is replaced by an asymmetrical risk measure. However, most[13] of these alternative performance ratios are mere ad-hoc modifications of the Sharpe ratio, and they do not possess any decision theoretic foundation. As we

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10 See e.g. Cont (2001, p. 224).

11 See e.g. McNeil et al. (2005, p. 44).

12 For recent reviews see e.g. Biglova et al. (2004) and Farinelli et al. (2008).

13 Notable exceptions being the contributions of Breuer and Gürtler (2006, 2007), Favre and Galeano (2002) and Leland (1999).

have seen, for QSF-investors there is a separation property as in the case of the Sharpe ratio. This suggests to construct a risk-adjusted performance measure based on the separation property for QSF-investors. The risk measure implicitly contained in the QSF-ratio (3.8) is $r_0 - Q_\alpha(R)$, which measures the distance between the riskless interest rate and the α-quantile of the distribution. This risk measure, however, has a main drawback. It is location dependent, whereas the standard deviation is a location-independent risk measure. This drawback, however, can be removed based on the fact that we have

$$\frac{Q_{1-\beta}(R)-r_0}{r_0-Q_\alpha(R)} = \frac{Q_{1-\beta}(R)-r_0}{r_0-Q_{1-\beta}(R)+Q_{1-\beta}(R)-Q_\alpha(R)} = \frac{1}{\dfrac{Q_{1-\beta}(R)-Q_\alpha(R)}{Q_{1-\beta}(R)-r_0}-1}.$$

This means that the QSF-ratio (3.8) is maximized, iff $[Q_{1-\beta}(R)-Q_\alpha(R)]/[Q_{1-\beta}(R)-r_0]$ is minimized, which in turn is equivalent to the maximization of

$$Q(R) = \frac{Q_{1-\beta}(R)-r_0}{Q_{1-\beta}(R)-Q_\alpha(R)}. \tag{4.2}$$

In consequence, equivalent to (3.8) QSF-investors maximize the ratio (4.2), which we call the quantile ratio or Q ratio for short. The risk measure implicit in this ratio is the quantile range resp. quantile distance

$$\Delta Q_{\alpha,1-\beta}(R) = Q_{1-\beta}(R) - Q_\alpha(R), \tag{4.3}$$

or to put it equivalently, the distance between the probable maximum return at confidence level $1-\beta$ and the probable minimum return at confidence level α. Obviously $\Delta Q(R)$ is a location-independent risk measure, i.e. $\Delta Q(R) = \Delta Q[R - E(R)]$ and it is a non-negative risk measure, i.e. $\Delta Q_{\alpha,1-\beta}(R) \geq 0$, because from $0 < \alpha, \beta < 0.5$ we have $\alpha < 1-\beta$. From (3.2) we know, that the risk measure is positively homogeneous, i.e. $\Delta Q(aR) = a\Delta Q(R)$ for $a > 0$. Being the difference of two quantiles, the risk measure (4.3) shares a number of properties[14] with distribution quantiles resp. with value-at-risk. Beyond positive homogeneity we have law

[14] See e.g. Tasche (2002, Proposition 2.3).

invariance and comonotonic additivity. We do not, however, have translation invariance and sub-additivity (as quantiles are not sub-additive in general).

The risk measure ΔQ entering into (4.2) is not the only difference to the Sharpe ratio (4.1). Another distinctive feature is, that the expected return E(R) is replaced by the upper quantile $Q_{1-\beta}(R)$, the probable maximum return. In literature, usually only the risk measure is replaced in (4.1) to construct an alternative performance measure. Notable exceptions from this rule are the upside potential ratio advocated by Sortino et al. (1999), the omega measure studied by Keating and Shadwick (2002), the Farinelli/Tibiletti ratio proposed by Farinelli and Tibiletti (2008) and the Rachev ratio resp. the generalized Rachev ratio introduced in Biglova et al. (2004). The upside potential ratio and the omega measure are based on the upper partial moment $E[\max(R-r_0,0)]$ as a reward measure. Farinelli and Tibiletti use the measure $E[\max(R-r_0,0)^p]^{1/p}$ as a value measure. The Rachev ratio is based on the value measure $E[R-r_0 \mid R-r_0 > \mathrm{VaR}_\alpha(R-r_0)]$, the conditional value-at-risk of the excess return $R-r_0$. Finally, the generalized Rachev ratio is based on the value mea-sure $E[\max(R-r_0,0)^p \mid R-r_0 > -\mathrm{VaR}_{1-\beta}(R-r_0)]$, which the authors call a power CVaR.

From its structure the Q ratio (4.2) has further similarities with the Rachev ratio $E[R-r_0 \mid R-r_0 > -\mathrm{VaR}_{1-\beta}(R-r_0)] / E[r_0-R \mid R-r_0 \le -\mathrm{VaR}_\alpha(R-r_0)]$. Both ratios are based on two different confidence levels $1-\beta$ resp. α for the reward resp. the risk measure involved. Moreover, as already mentioned at the end of section 2, the quantiles Q_α and $Q_{1-\beta}$ can also be interpreted as a corresponding tail condition median and the tail conditional median can be seen as a robust alternative to the CVaR-measures contained in the nominator and in the denominator of the Rachev ratio.

As already noted, all the preceeding performance ratios are of an ad hoc-nature, they do not possess any decision theoretic basis. In contrast, the QSF ratio (4.2) is based on a separation property, i.e. it has a foundation comparable to the Sharpe ratio. Summing up, QSF-investors maximize the Q-ratio to obtain an optimal portfolio and the Q-ratio is based on $Q_{1-\beta}(R)-r_0$ as a reward measure and $\Delta Q_{\alpha,1-\beta}(R) = Q_{1-\beta}(R) - Q_\alpha(R)$ as a risk measure.

To obtain a standard example for this new performance measure, we look at the case of a normal distribution. In this case we have $Q_{1-\beta}(R) = E(R) + N_{1-\beta}\sigma(R)$ resp. $Q_{\alpha}(R) = E(R) - N_{1-\alpha}\sigma(R)$, where $N_{1-\beta}$ resp. $N_{1-\alpha}$ are the corresponding quantiles for the standard normal distribution. Altogether in case of normally distributed returns, i.e. $R \sim N(\mu, \sigma^2)$, we have with $H := N_{1-\beta} + N_{1-\alpha}$

(4.4)
$$\begin{aligned} Q(R) &= \frac{\mu + N_{1-\beta}\sigma - r_0}{H\sigma} \\ &= [S(R) + N_{1-\beta}]/H\ . \end{aligned}$$

This means although we not only have a different risk measure, but also a different reward measure, in case of the normal distribution the new performance ratio is a positive linear transformation of the Sharpe ratio and will therefore produce identical rankings in this case. As the quantiles of elliptical distributions possess a similar structure compared to the case of a normal distribution, i.e. they are of the form $E(R) \pm Z \cdot \sigma(R)$, this result can be generalized to all elliptical distributions. This is a reasonable result, because in the case of symmetrical distributions the validity of the Sharpe ratio is not put into question. Only in case of asymmetrical distributions we will have divergent results. For instance, if we look at the (translated) lognormal distribution, i.e. $\ln(1+R) \sim N(m, v^2)$, we have

(4.5a)
$$S(R) = \frac{\exp(m + \frac{1}{2}v^2) - (1 + r_0)}{[\exp(m + \frac{1}{2}v^2) - 1]\sqrt{\exp(v^2) - 1}}$$

for the Sharpe ratio and

(4.5b)
$$Q(R) = \frac{\exp(m + N_{1-\beta}v) - (1 + r_0)}{\exp(m + N_{1-\beta}v) - \exp(m - N_{1-\alpha}v)}$$

for the Q ratio. Obviously, there is no linear relationship between the Sharpe ratio and the Q ratio in this case. So these ratios will produce different rankings in general.

4.2. *Risk Management: Q-RORAC*

A second field of application of risk adjusted performance measures is enterprise risk management. Here the performance of a company or the performance of different business units of this company is usually measured using some sort of RORAC (return on risk-adjusted capital) approach[15], i.e. considering a ratio of the form

(4.6) $$\text{RORAC} = \frac{\text{expected profit}}{\text{risk capital}}.$$

To obtain a suitable RORAC-version being consistent with the QSF-principle we re-examine the QSF-ratio (3.8) in terms of absolute values instead of return quantities. Denoting v_0 as the (known) market value of a risky investment in $t = 0$ and V_1 as its (uncertain) market value in $t = 1$, we have the relations $R = \Delta V / v_0$ where $\Delta V = V_1 - v_0$ is the relevant profit variable in absolute terms. Expressing (3.8) in terms of ΔV , we in a first step obtain the ratio

(4.7) $$\frac{Q_{1-\beta}(\Delta V) - r_0 v_0}{r_0 v_0 - Q_\alpha(\Delta V)}.$$

Now, the value-at-risk at confidence level α, $\text{VaR}_\alpha(\Delta V)$, of the position ΔV corresponds to the quantile $Q_{1-\alpha}(-\Delta V)$, which in turn is equivalent[16] to the quantile $-Q_\alpha(\Delta V)$. So an equivalent form of the expression (4.7) is

(4.8) $$Q - \text{RORAC}(\Delta V) = \frac{Q_{1-\beta}(\Delta V - r_0 v_0)}{\text{VaR}_\alpha(\Delta V - r_0 v_0)},$$

which we will call the Q-RORAC. Interpreting the value-at-risk as a necessary risk capital, the denominator of (4.8) corresponds to the risk capital of the position $\Delta V - r_0 v_0$, i.e. the (uncertain) profit in excess of the riskless profit $r_0 v_0$. This riskless profit can be earned when the company invests the amount v_0 at the riskless interest rate. Similarly, the nominator in (4.8) corresponds to the upper quantile of this excess profit. Consistent with the generalized

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15 See e.g. McNeil et al. (2005, p. 256).

16 Assuming the existence of a density function of the distribution of ΔV.

safety-first principle not the expected profit is the basis for evaluating the value of the financial position, but the upper quantile of its gain/loss-distribution. Another insight of the Q-RORAC is, that not the profit ΔV itself is relevant, but only the profit $\Delta V - r_0 v_0$ in excess of the riskless profit. The denominator in (4.8) is location-dependent, which is reasonable, as the expected loss $E(-\Delta V)$ is a part of the total loss $L = -\Delta V$ and therefore a part of the necessary risk capital has to be calculated as a compensation for the expected loss. In case the risk capital is understood only to be a buffer for the unexpected loss[17] $UL = L - E(L) = E(\Delta V) - \Delta V$ the denominator of (4.8) should be replaced with $VaR_\alpha[\Delta V - r_0 v_0 - E(\Delta V - r_0 v_0)] = MVaR_\alpha(\Delta V)$, the mean value-at-risk of the profit position, which now is location-independent.

In practice, the RORAC typically is not maximized, but a minimum value r_H, the *hurdle rate*, is specified, which amounts to the relation

(4.9) $$RORAC \geq r_H\,.$$

In case of the Q-RORAC (4.8) the corresponding restriction is of the form

(4.10) $$Q_{1-\beta}(\Delta V) \geq r_0 v_0 + (\text{risk capital}) \cdot r_H\,.$$

This implies, that the company considered demands a profit surcharge in excess of the riskless profit $r_0 v_0$ and that this profit surcharge is proportional to the necessary risk capital.

5. CAPITAL MARKET EQUILIBRIUM

The existence of a separation property is the key to determine expected returns in capital market equilibrium. In case of MV-investors one has to maximize the Sharpe ratio in order to obtain the capital market line, one of the key results of the CAPM. The same method is used by Arzac and Bawa (1977) in their study of a capital market equilibrium for SF-investors and in the present contribution we will take the same avenue. To do this we assume a capital mar-

17 See e.g. McNeil et al. (2005, p. 412) for a corresponding approach. The necessary risk capital in this case is called *economic capital* by these authors.

ket with n risky assets with corresponding returns R_i $(i = 1,...,n)$ and one riskless asset with return r_0. All investors acting on this capital market are QSF-investors. We first define the Q-ratio (4.2) in terms of the portfolio weights $x_1,...,x_n$ of the risky assets

$$(5.1) \qquad Q(x_1,...,x_n) = \frac{Q_{1-\beta}(R) - r_0}{Q_{1-\beta}(R) - Q_\alpha(R)} = \frac{F(x_1,...,x_n)}{G(x_1,...,x_n)} .$$

The quantities $x_1,...,x_n$ are arbitrary real numbers. They do not have to sum up to one, because the difference to one corresponds to the amount of riskless investing resp. borrowing. With $R - r_0 = \Sigma\, x_i(R_i - r_0)$ we have

$$(5.2a) \qquad F(x_1,...,x_n) = Q_{1-\beta}[\Sigma\, x_i(R - r_0)]$$

and

$$(5.2b) \qquad \begin{aligned} G(x_1,...,x_n) &= Q_{1-\beta}(\Sigma x_i R_i) - Q_\alpha(\Sigma\, x_i R_i) \\ &= Q_{1-\beta}[\Sigma x_i(R_i - r_0)] - Q_\alpha[\Sigma x_i(R_i - r_0)] . \end{aligned}$$

The maximization of (5.1) leads to the first-order conditions $\partial Q / \partial x_j = 0$ for $j = 1,...n$, which is equivalent to

$$(5.3) \qquad \partial F / \partial x_j = \frac{F}{G} \cdot \partial G / \partial x_j \quad (j = 1,...,n) .$$

To calculate the partial derivaties of the functions F and G we have to use results in the context of quantile derivatives, as obtained e.g. by Gourieroux et al. (2000) and as well Martin and Wilde (2002). Adapting their results to the present situation, we first obtain the general result

$$(5.4) \qquad \partial Q_\alpha(R) / \partial x_j = E[R_j \,|\, R = Q_\alpha(R)] .$$

This result (only) assumes, that the vector $(R_1,...,R_n)^T$ of the returns of the risky securities possesses a (multivariate) density function $f(r_1,...,r_n) \geq 0$, which is a rather weak assumption. From a structural point of view the j-th quantile derivative corresponds to the best prediction (in terms of a minimum mean square error) of the return R_j given the information $R = Q_\alpha(R)$. Taking into consideration that in capital market equilibrium the optimal risky portfolio must agree with the market portfolio M and denoting the return of the market portfolio as R_M we obtain from (5.2a)

(5.5a)
$$\begin{aligned}\partial F / \partial x_j &= E[R_j - r_0 \mid R_M - r_0 = Q_{1-\beta}(R_M - r_0)] \\ &= E[R_j \mid R_M = Q_{1-\beta}(R_M)] - r_0\end{aligned}$$

And from (5.2b)

(5.5b)
$$\partial G / \partial x_j = \partial F / \partial x_j - \{E[R_j \mid R_M = Q_\alpha(R_M)] - r_0\}\ .$$

Summing up we obtain the following first-order conditions for capital market equilibrium.

(5.6)
$$\begin{aligned}&E[R_j \mid R_M = Q_{1-\beta}(R_M)] - r_0 \\ &= [Q_{1-\beta}(R_M) - r_0] \cdot \frac{E[R_j \mid R_M = Q_{1-\beta}(R_M)] - E[R_j \mid R_M = Q_\alpha(R_M)]}{Q_{1-\beta}(R_M) - Q_\alpha(R_M)}.\end{aligned}$$

This result first generalizes the result of Arzac and Bawa (1977) with respect to the equations for the capital market equilibrium of SF-investors. Moreover, Arzac and Bawa (1977) only work with the formal quantile derivatives $\partial Q_\alpha(x_1,\ldots,x_n)/\partial x_j$ and are not able to substantiate these quantities further. Using the result (5.4) enables us to give an additional probabilistic interpretation of the quantities $\partial Q_\alpha / \partial x_j$. We also note that the quantile function $Q_\alpha(x_1,\ldots,x_n) = Q_\alpha(\Sigma x_i R_i)$ is a positive homogeneous function and from Euler's Theorem we therefore have

$Q_\alpha(R_M - r_0) = Q_\alpha[\Sigma x_i^M (R_i - r_0)] = \Sigma x_j^M \partial Q_\alpha(R_M - r_0)$. Rewriting (5.6) in the form

(5.7)
$$\begin{aligned}&\frac{E[R_j \mid R_M = Q_{1-\beta}(R_M)] - r_0}{Q_{1-\beta}(R_M) - r_0} \\ &= \frac{E[R_j \mid R_M = Q_{1-\beta}(R_M)] - E[R_j \mid R_M = Q_\alpha(R_M)]}{Q_{1-\beta}(R_M) - Q_\alpha(R_M)}\end{aligned}$$

it can be seen, that the nominator on the left hand side in (5.7) corresponds to the contribution of security j to the value measure $Q_{1-\beta}(R_M) - r_0$ of the market portfolio. In addition, the nominator on the right hand side of (5.7) corresponds to the contribution of the j-th security to the risk measure $Q_{1-\beta}(R_M) - Q_\alpha(R_M)$ of the market portfolio. This is in complete analogy to the CAPM, where relation (28) would read $[E(R_j) - r_0]/[E(R_M) - r_0] = Cov(R_j, R_M)/Var(R_M)$.

Now (5.6) resp. (5.7) are structural relationships which are not easy to evaluate because of the conditional expectations involved. From an empirical point of view this is not a serious drawback, as one can perform a non-parametric estimation approach and estimate the conditional expectations involved in (5.6) resp. (5.7) on the basis of (the conditional version of the) Nadaraya-Watson kernel estimate. However, in the rest of this chapter we want to investigate how to obtain explicit versions of the capital market equilibrium equation (5.6). A first strategy would be to approximate the best prediction $E(Y \mid X)$ by the best linear prediction, i.e. the solution to the problem $E[(Y - a_0 - a_1X)^2)] \to \min!$ The solution is known to be

(5.8)
$$E(Y \mid X) \approx E(Y) + \beta(Y, X)[X - E(X)],$$

where $\beta(Y, X) = Cov(X, Y)/Var(X)$. Using this result, the left hand side of (5.6) reduces to $E(R_j) - r_0 + \beta(R_j, R_M)[Q_{1-\beta}(R_M) - r_0]$ and the right hand side of (5.6) reduces to $[Q_{1-\beta}(R_M) - r_0]\beta(R_j, R_M)$. Reducing this equation further we obtain $E(R_j) - r_0 = [E(R_M) - r_0]\beta(R_j, R_M)$, the capital market line equation of the CAPM! In this sense the CAPM can be understood to be a linear approximation to the equation (5.6), the QSF capital market line! In addition, it is well known since Kelker (1970) that for elliptical distributions the regressions $E(Y \mid X)$ are linear, i.e. that the best linear prediction is identical to the best prediction. In consequence, if we suppose $(R_1, ..., R_n)^T$ to follow a multivariate elliptical distribution, then $(R_j, R_M)^T$ follows a bivariate elliptical distribution and the expression (5.8) is exact for $E(R_j \mid R_M)$. Consequently, the QSF-capital market line is identical with the CAPM-capital market line for the family of elliptical distributions. This is a reasonable result, because only for asymmetrical distributions one is interested in an expression involving asymmetrical elements of risk and value. To achieve a result in this direction, which is not based on a particular distributional assumption, we consider the best quadratic approximation, i.e. the solution to the problem $E[(Y - a_0 - a_1X - a_2X^2)^2)] \to \min!$ instead of the best linear approximation. In this case the situation is slightly more involved. First, we have the standard result

(5.9a)
$$\begin{aligned} &E[R_j \mid R_M = Q_\gamma(R_M)] \\ &\quad = E(R_j) + a_1[Q_\gamma(R_M) - E(R_M)] + a_2[Q_\gamma(R_M)^2 - E(R_M^2)], \end{aligned}$$

where

(5.9b)
$$a_1 = \frac{1}{H}[\mathrm{Cov}(R_j, R_M)\mathrm{Var}(R_M^2) - \mathrm{Cov}(R_j, R_M^2)\mathrm{Cov}(R_M, R_M^2)]$$

(5.9c)
$$a_2 = \frac{1}{H}[\mathrm{Cov}(R_j, R_M^2)\mathrm{Var}(R_M) - \mathrm{Cov}(R_j, R_M)\mathrm{Cov}(R_M, R_M^2)]$$

and

(5.9d)
$$\begin{aligned} H &= \mathrm{Var}(R_M) \cdot \mathrm{Var}(R_M^2) - \mathrm{Cov}(R_M, R_M^2)^2 \\ &= \mathrm{Var}(R_M) \cdot \mathrm{Var}(R_M^2)[1 - \rho^2(R_M, R_M^2)] \,. \end{aligned}$$

With $\gamma = 1 - \beta$ resp. $\gamma = \alpha$ this result can be used to evaluate the conditional expectations involved in the general equation (5.6) for the QSF-capital market line. Realizing that

(5.10a)
$$\mathrm{Var}(R_M^2) = E(R_M^4) - (E(R_M^2))^2$$

(5.10b)
$$\mathrm{Cov}(R_M, R_M^2) = E(R_M^3) - E(R_M)\; E(R_M^2)\,,$$

it can be seen that using the quadratic approximation to (5.6), now third and fourth moments of the return of the market portfolio have to be taken into consideration as well. Another additional term compared to the linear approximation is $\mathrm{Cov}(R_j, R_M^2)$. This term is not very convenient and so we are interested in approximating this term, too. Expanding the function

$Z = f(Y)$ in a Taylor series about $E(Y)$ we obtain the approximation
$Z \approx f[E(Y)] + [Y - E(Y)]\, f'[E(Y)]$ and therefore

(5.11) $$\mathrm{Cov}(X, f(Y)) \approx f'[E(Y)]\, \mathrm{Cov}(X, Y)\,.$$

With $X = R_j$, $Y = R_M$ and $f(y) = y^2$ we therefore obtain the approximation

(5.12) $$\mathrm{Cov}(R_j, R_M^2) \approx 2\,\mathrm{Cov}(R_j, R_M)\, E(R_M)\,,$$

which makes our analysis complete.

6. CONNECTIONS TO THE SAFETY-FIRST PRINCIPLE

Within the framework of the traditional safety-first principle of Telser (1955/56) and Arzac and Bawa (1977) the expected return is maximized instead of an upper quantile as done in the present contribution. Coming closest to the traditional version would be to maximize the median of the return distribution – being a quantity which is more robust than the expected value. Only for return distributions for which the median is identical to the expected value, the traditional safety-first principle is a true special case of the present approach.

Looking at the traditional SF-principle a bit further and performing recalculations of the preceeding results using $E(R)$ instead of $Q_{1-\beta}(R)$, we obtain a number of corresponding results, which we will present in the following.

Corresponding to (3.8) SF-investors maximize the ratio $[E(R_P) - r_0]/[r_0 - Q_\alpha(R_P)]$ to obtain an optimal risky portfolio. This is the result obtained by Arzac and Bawa (1977) in connection with the separation property for SF-investors. Equivalently and corresponding to (4.2) they maximize the ratio $[E(R_P) - r_0]/[E(R_P) - Q_\alpha(R_P)]$. As for the mean value-at-risk we have $\mathrm{MVaR}_\alpha(R) = Q_{1-\alpha}(-R) + E(R) = E(R) - Q_\alpha(-R)$, equivalently SF-investors maximize the ratio $\mathrm{SF}(R) = [E(R) - r_0]/\mathrm{MVaR}_\alpha(R)$, which would be the relevant performance measure for SF-investors. The quantity SF(R) could be called safety-first ratio or reward-to-mean value-at-risk ratio and is a performance measure already known from literature[18]. It should be noted that Favre and Galeano (2002) use this risk measure, too, in connection with hedge funds. These authors combine the ratio with a Cornish-Fisher-Expansion and call the resulting quantity mean modified value-at-risk ratio. Moreover, Favre and Galeano (2002, pp. 22 – 24) explicitly address the fact that this ratio can be obtained on the basis of the separation result of Arzac and Bawa (1977). The quantity corresponding to (4.8) is $[E(\Delta V) - r_0 v_0]/[MVaR_\alpha(\Delta V)]$, which one could call the Safety-first RORAC. Finally, analyzing the first-order conditions (5.6) for SF-investors we obtain the expression

$$E(R_j) - r_0 = [E(R_M) - r_0] \cdot \frac{E(R_j) - E[R_j \mid R_M = Q_\alpha(R_M)]}{MVaR_\alpha(R_M)}, \tag{6.1}$$

which generalizes the corresponding result of Arzac and Bawa (1977) , who, however, are only able to express their results in terms of the formal quantities $\partial Q_\alpha(R_M)/\partial x_j$. Approximating $E[R_j \mid R_M = Q_\alpha(R_M)]$ by the corresponding best linear prediction again leads to the CAPM. This approximation is exact for elliptical return distributions, which generalizes the results obtained by Arzac and Bawa (1977), who note that this is the case for the normal distribution and for stable Paretian distributions (both of them belonging to the family of elliptical distributions).

7. CONCLUDING REMARKS

In the present paper we generalized the safety-first principle in the versions of Telser (1955/56) and Arzac and Bawa (1977) by combining it with the principle of quantile maximization of Rostek (2010). While maintaining the element of a shortfall constraint of the safety-first principle we proposed to maximize an upper quantile of the (return) distribution instead of maximizing its expected value. After having introduced and characterized the new decision principle, we have studied the implications of the new decision principle for portfolio selection and capital market equilibrium. We were able to derive a separation result and on this basis, using results in connection with quantile derivatives, we obtained a structural characterization of the equilibrium equations involving conditional expectations, i.e. best predictions.

[18] See e.g. Biglova et al. (2004, p. 106) and Farinelli et al. (2008, p. 2059), the ratio being called VaR-ratio by these authors.

Approximating these best predictions by best linear predictions resp. best quadratic predictions we were able to obtain explicit solutions for the capital market equilibrium. In case of the best linear approximation (which is exact for elliptical distributions) the CAPM is obtained. Moreover, we studied the consequences of the new decision principle for risk measurement and for risk-adjusted performance measurement. We introduced a new risk measure, the quantile range, a new performance ratio, the Q ratio and a new measure for the return on risk-adjusted capital, the Q-RORAC.

The present paper had its primary focus on the conceptual aspects of the problems studied. The safety-first principle in the versions of Telser (1955/56) resp. Arzac and Bawa (1977), however, has found a number of applications in the finance literature, too. These applications range from asset allocation (see e.g. Leibowitz and Kogelman 1991) to risk-adjusted performance measurement (see e.g. Favre and Galeano 2002). With respect to these applications the present paper therefore also paves the way to a number of subsequent empirical studies.

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