Symmetries in covariant quantum mechanics

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Dirk Saller

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Dekan: Prof. Dr. Guido Moerkotte, Universität Mannheim
Referent: Prof. Dr. Ernst Binz, Universität Mannheim
Korreferent: Prof. Dr. Marco Modugno, Universität Florenz, Italien

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In deep gratitude to my family

Deutsche Zusammenfassung: "Symmetries in covariant quantum mechanics"

Die Arbeit besteht aus zwei Teilen.

Im ersten Teil der Arbeit präsentiere ich eine aktuelle Version der Theorie der kovarianten Quantenmechanik und des klassischen Hintergrunds. Diese Theorie wurde ursprünglich von Jadczyk und Modugno vorgeschlagen und dann in Kooperation mit anderen Personen weiterentwickelt. Sie ist ein geometrisches Modell für die Quantenmechanik eines skalaren oder spin-tragenden Teilchens in einer gekrümmten Raumzeit mit absoluter Zeit mit gegebenem klassischem gravitativen und elektromagnetischem Feld. Von ein paar Axiomen ausgehend, liefert die Theorie auf kovariantem Weg, über einen globalen Lagrange Formalismus, die Schrödinger Gleichung und die Quantenoperatoren, welche klassischen quantisierbaren Funktionen über die Klassifizierung ausgezeichneter Quantenvektorfelder zugeordnet sind.

Im zweiten Teil der Arbeit untersuche ich systematisch die infinitesimalen Symmetrien der klassischen und der quantistischen geometrischen Struktur. Dieser Teil handelt von den wesentlichen originellen Beiträgen der Arbeit. Einige Ergebnisse spiegeln wohlbekannte Tatsachen aus anderen mathematischen Modellen der Quantenmechanik wider, werden aber mit neuartigen Techniken hergeleitet. Andere Ergebnisse sind vollständig neu. Die wichtigsten originellen Ergebnisse können mit Hilfe der folgenden Aussagen kurz beschrieben werden.

Die Quantensymmetrien sind Lie Algebren, welche natürlich isomorph zu Unteralgebren der neuen klassischen Lie Algebra der quantisierbaren Funktionen sind. Ich möchte betonen, dass diese Algebra keine Poisson-Unteralgebra ist.

Auf der anderen Seite kann man durch Verwendung des Quanten-Lagrange Formalismus zu jedem Element einer ausgezeichneten klassischen Unteralgebra einen erhaltenen Quantenstrom zuordnen. Diese Ergebnisse deuten auf einen Prequanten Ursprung der Wahrscheinlichkeitsinterpretation der Quantenmechanik.

CONTENTS

The methods of progress in theoretical physics have undergone a vast change during the present century. The classical tradition has been to consider the world to be an association of observable objects (particles, fluids, fields, etc.) moving about according to definite laws of force, so that one could form a mental picture in space and time of the whole scheme. This led to a physics whose aim was to make assumptions about the mechanism and forces connecting these observable objects, to account for their behaviour in the simplest possible way. It has become increasingly evident in recent times, however, that nature works on a different plan. Her fundamental laws do not govern the world as it appears in our mental picture in any very direct way, but instead they control a substratum of which we cannot form a mental picture without introducing irrelevancies. The formulation of these laws requires the use of the mathematics of transformations. The important things in the world appear as the invariants (or more generally the nearly invariants, or quantities with simple transformation properties) of these transformations. The things we are immediately aware of are the relations of these nearly invariants to a certain frame of reference, usually one chosen so as to introduce special simplifying features which are unimportant from the point of view of general theory. ... There is the symbolic method, which deals directly in an abstract way with the quantities of fundamental importance (the invariants, etc., of the transformations) and there is the method of coordinates or representations, ... The second of these has usually been used for the presentation of quantum mechanics ...

P. A. M. Dirac, Principles of Quantum Mechanics (1958)

INTRODUCTION

Our fundamental picture of the physical world is due to the theory of general relativity and to the quantum field theory, which got great theoretical and experimental success.

The well established historical steps in classical theory have been: non relativistic theory, special relativity, general relativity. Analogously, the well established steps in quantum theory have been: non relativistic quantum mechanics, special relativistic quantum field theory. For instance, well known monographs on this subject are [29, 40] for non relativistic quantum mechanics, [16, 137] for quantum field theory and [123] for relativity.

Unfortunately, these theories deal with different objects, use partially incompatible mathematical methods and fulfill different requirements of covariance. In particular, the standard formulation of quantum theories is highly based on concepts and methods strictly related to a flat spacetime and inertial observers, which conflict with general covariance on a curved spacetime.

So, a still open problem is a consistent formulation of quantum field theories and general relativity. The problem has at least two faces:

- general relativistic covariant formulation of quantum theories in a curved spacetime,
- quantum theory of gravitational field.

The model of *covariant quantum mechanics* discussed in this paper is aimed at contributing to the first face of the problem, by means of new ideas and methods [55, 56, 57, 20, 59, 60, 64, 130, 64, 161, 162, 67, 163, 164, 62, 128, 140]. Namely, we study a general relativistic covariant formulation of quantum mechanics on a classical background constituted by a curved spacetime fibred over absolute time and equipped with given spacelike Riemannian metric, and gravitational and electromagnetic fields. Thus, we restrict our investigation just to fundamental fields of classical and quantum mechanics, because we believe that this is an arena which could possibly suggest us good ideas for unifying deeper fundamental theories of physics.

The framework of our model is allowed by the possible general relativistic formulation of classical physics in a curved spacetime with absolute time. This theory is well established in the literature [31, 32, 33, 34, 35, 36, 38, 51, 81, 82, 83, 84, 85, 87, 91, 100, 110, 141, 151, 152, 153, 154], even if it is much less popular than the Einstein theory of relativity. This theory is rigorous and self–consistent from a mathematical viewpoint and describes the phenomena of classical physics by an approximation which is intermediate between the classical theory and the Einstein theory of relativity.

Our model can be regarded as an intermediate step between the standard non relativistic quantum mechanics and a possible fully general relativistic quantum theory. This framework allows us to focus our attention on the general relativistic covariance and the curved spacetime, detaching them from the difficulties due to the Lorentz metric. Actually, our choice seems to be quite fruitful.

The main new methods and achievements can be summarised as follows.

First of all, our basic guide is the *covariance* (even more, the manifest covariance) of the theory as heuristic requirement. Nowadays, the concept of "covariance" has been formulated in a rigorous mathematical way through the geometric concept of "naturality" [75]. According to the covariance of the theory, *time* is not just a parameter, but a fundamental object of the theory; accordingly, the main objects of the theory are not assumed to be split into time and space components. As classical *phase space* we take the first jet space of spacetime and not its tangent space; indeed, this minimal choice allows us to skip anholonomic constraints. Another consequence of our choices is that classical mechanics is ruled not by a symplectic structure, but by a *cosymplectic structure*[3, 19, 103, 114]; actually, we do get a symplectic structure, but this describes only the spacelike aspects of classical theory and is insufficient to account for classical dynamics. An achievement of our theory is the Lie algebra of "*special quadratic functions*" (different from the Poisson algebra), which allows us to treat energy, momentum and spacetime functions on the same footing. We emphasize the fact that classical mechanics can be formulated in a covariant way by a *Lagrangian approach*, but not by a Hamiltonian approach, because the Hamiltonian depends essentially on an observer.

As far as quantum mechanics is concerned, all objects are derived, in a covariant way, from three minimal objects. Here, we have some novelties. The quantum bundle lives *on spacetime* and not on the phase space and the quantum connection is "*universal*". These assumptions allow us to skip all problems of polarisations [168]. Indeed, we replace the problematic search for such inclusions with a *method of projectability*, which turns out to be our implementation of covariance in the quantum theory. Another new assumption concerns the *Hermitian metric* of the quantum bundle, which takes its values in the space of complexified spacelike volume forms. This assumption allows us to skip the problems related to half–densities. The Schrödinger equation is obtained, in a covariant way, through a *Lagrangian approach* and not through the standard non covariant Hamiltonian approach. Indeed, we exhibit an *explicit expression* of the Schrödinger equation for any quantum system. The quantum operators arise automatically, in a covariant way, from the *classification* of distinguished first and second order differential operators of the quantum framework and not from a quantisation requirement of a classical system. The seat for the covariant probabilistic interpretation of quantum mechanics is a *Hilbert bundle*, naturally yielded by the quantum bundle, and not just a Hilbert space. Our theory provides explicit expressions of all objects for any *accelerated observer* and yields, at the same time, an interpretation in terms of gravitational field, according to the principle of equivalence.

In a few words, we start with really minimal geometric structures representing physical fields and proceed along a thread naturally imposed by the only requirement of general covariance. We take the well established *results* of classical and quantum mechanics as touchstone of our model. On the other hand, according to the aims of our theory, we disregard those standard *methods*, which are incompatible with general covariance. Indeed, in the flat case, the results of our model reduce to the results of the standard classical and quantum mechanics.

The standard term "relativistic theory" links the special or general covariance with the Minkowski or Lorentz metric. This usage is clearly motivated by the historical developments of the Einstein theory. However, it would be more appropriate to refer the word "relativistic theory" only to its semantic meaning related to covariance. Indeed, the standard usage would be highly misleading in our context. In fact, our model is general relativistic, in the sense of covariance, but is not Minkowskian or Lorentzian.

Clearly, the Minkowski or Lorentz metric is physically related to the distinguished constant c. Actually, in our model this constant does not occur. The classical limit of Einstein general relativity for $c \to \infty$ is quite delicate, if we wish to understand the limit of the geometric structures of the model and not only the limit of some measurements. In a sense, our model could be regarded as the "true" classical limit of Einstein general relativity.

In this paper, we deal just with a given gravitational and electromagnetic field; this is sufficient as classical background of our covariant model of quantum mechanics. On the other hand, our classical model can be completed by adding, in a covariant way, the equations linking the gravitational and electromagnetic fields to their mass and charge sources [57]. These equations are a spacelike version of the Einstein and Maxwell equations. In fact, due to the spacelike nature of the metric, there is no way to couple the timelike components of the gravitational and electromagnetic fields with the timelike components of their sources. It is just this the main point which makes the Einstein model physically much more complete than ours.

The reader might be puzzled by the fact that we do not mention explicitly the representations of the (finite dimensional and infinite dimensional) groups involved in our theory. In fact, our natural geometric constructions provide these representations automatically. This is an outproduct of our explicitly covariant approach.

In our model we never make an essential use of the fact that the dimension of spacetime is $n = 1 + 3$; We just need $n \geq 1 + 2$. In fact we have applied our machinery to the quantisation of a rigid body, whose configuration space has dimension $n = 1+3+3$ [163]. However, within this thesis I have set $n = 1 + 3$ for simplicity. All my results can be generalized to $n > 1 + 3$.

Even more, in our model we never make an essential use of the fact that the spacelike metric of spacetime is Riemannian; we just need that it is non degenerate on each fibre. So, we could, for instance, apply our machinery to a model of dimension 5, with a fibring on an extra parameter, whose fibres are four dimensional Lorentzian manifolds. Such a model would work pretty well mathematically, but we do not know any interesting physical interpretation.

The scheme developed for covariant quantum mechanics of a scalar particle can be easily and nicely extended to the case of a spin particle [20].

In spite of the differences of the starting scheme of spacetime, several steps of the above methodology appeared to be usefully translatable to the *Einstein case*. In particular, so far, we have been able to apply to the Einstein case the methods concerning the classical phase space, the algebra of quantisable functions and the algebra of pre–quantum operators [59, 63, 64, 61, 68, 65].

We hope that the new methods arising in our model could yield fruitful hints for a possible generally covariant formulation of quantum field theory in an Einstein framework.

Now, let us come to the contents of this thesis. I would like to stress that in the following systematic analysis of *symmetries in covariant quantum mechanics*, I have been influenced strongly by the spirits of Marco Modugno and Ernst Binz. On the other hand, the literature which I have studied during this research work was extensive. I have collected it in the bibliography. Beside the works of J. Janyška, M. Modugno and R. Vitolo, I have spent the most time with [2, 3, 19, 25, 29, 75, 94, 102, 103, 104, 105, 107, 108, 111, 113, 114, 115, 118, 139, 145, 146, 155, 168].

The rest of this introduction is a summary of the contents of this thesis[70] and serves as a leading guide in order to get a first, rough impression. The first section sketches the the classical background and the quantum theory within our framework. It is basically an up to date rearrangement of known results. The second section presents the main original contributions of this thesis to the theory. More precisely, I have studied systematically the symmetries of classical and quantum theory and, additionally, their interplay.

0.1 Covariant quantum mechanics

0.1.1 Classical background

We start by sketching our covariant model of classical curved spacetime fibred over absolute time, and the related formulation of classical mechanics. We recall the basic elements of the model and present new results, as well.

According to [57, 54], we postulate:

(C.1) a *classical spacetime* $\boldsymbol{E}$, which is an oriented four dimensional manifold;

(C.2) the *absolute time* $\boldsymbol{T}$, which is an oriented one dimensional affine space, associated with the vector space $\bar{\mathbb{T}}$;

(C.3) a *time fibring* $t : \boldsymbol{E} \to \boldsymbol{T}$, which is a surjective map of rank 1;

(C.4) a "scaled" *spacelike metric* g, which is a "scaled" Riemannian metric on the fibres of spacetime;

(C.5) a *gravitational field* $K^{\natural}$, which is a linear connection of spacetime (i.e. of the fibring $T\boldsymbol{E} \to \boldsymbol{E}$), which preserves the time fibring and the spacelike metric and whose curvature fulfills the typical symmetry of Riemannian connections;

(C.6) a "scaled" *electromagnetic field* f, which is a "scaled" closed 2–form of spacetime.

Here, the word "scaled" used for the spacelike metric and the electromagnetic field means that these objects are tensorialised by a suitable scale factor which accounts for the appropriate units of measurement.

A *time unit of measurement* will be denoted by $u^0 \in \bar{\mathbb{T}}$ and its dual by $u_0 \in \bar{\mathbb{T}}^*$.

We refer to charts of spacetime $(x^\lambda) = (x^0, x^i)$ adapted to the time fibring, to the affine structure of time and to a time unit of measurement $u_0 \in \bar{\mathbb{T}}$.

With reference to a given particle of mass m and charge q, in order to get rid of any choice of length and mass units of measurement, it is convenient to "normalise" the spacelike metric and the electromagnetic field, by considering the Planck constant $\hbar$.

Thus, we consider the "re–scaled" spacelike metric $G := \frac{m}{\hbar}\, g$, which takes its values in $\bar{\mathbb{T}}$. Its coordinate expression is

$$G = G^0_{ij}\, u_0 \otimes \check{d}^i \otimes \check{d}^j\,,$$

where $\check{d}^i$ is the spacelike differential of the coordinate x^i.

Analogously, we consider the "re–scaled" electromagnetic field $F := \frac{q}{\hbar}\, f$, which is a true form.

Accordingly, all objects derived from G and F will be re–scaled and will include the mass and the charge of the particle, and the Planck constant as well.

As *phase space* for the classical particle we take the first order jet space $J_1\boldsymbol{E}$ of the spacetime fibring [75]. We recall that $J_1\boldsymbol{E}$ can be naturally identified with the affine subspace of $\bar{\mathbb{T}}^* \otimes T\boldsymbol{E}$, whose elements v are normalised according to the condition $v^0_0 = 1$ (which is independent from the choice of a unit of measurement of time). The chart naturally induced on the phase space by a spacetime chart is denoted by (x^0, x^i, x^i_0).

We have assumed a projection of spacetime over time, but, according to the principle of general relativity, not a distinguished splitting of spacetime into space and time. In other words, for each spacetime vector X, we obtain, in a covariant way, its projection on time $X^0\, u_0$, but not a timelike and a spacelike component.

On the other hand, an *observer* is defined to be a section $o : \boldsymbol{E} \to J_1\boldsymbol{E}$. The coordinate expression of an observer o is of the type $o = u^0 \otimes (\partial_0 + o^i_0\, \partial_i)$. An observer o yields a splitting of each spacetime vector X into its *observed timelike and spacelike components* $v = v^0(\partial_0 + o^i_0 \partial_i) + (v^i - v^0\, o^i_0)\, \partial_i$"]

A spacetime chart is said to be adapted to an observer if $o^i_0 = 0$; conversely, each spacetime chart determines an observer.

According to the principle of general relativity, we do not assume distinguished observers.

The above objects C.1, ... , C.6 yield in a covariant way [57, 54]:
- the *time form* $dt : \boldsymbol{E} \to \bar{\mathbb{T}} \otimes T^*\boldsymbol{E}$ on spacetime;
- a *spacelike volume form* η and a *spacetime volume form* υ on spacetime;
- a *2–form* $\Omega^\natural : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1\boldsymbol{E}$ on the phase space;
- a dt–vertical *2–vector* $\Lambda^\natural : J_1\boldsymbol{E} \to \Lambda^2 V J_1\boldsymbol{E}$ of the phase space,
- a second order connection $\gamma^\natural : J_1\boldsymbol{E} \to \bar{\mathbb{T}}^* \otimes T J_1\boldsymbol{E}$ of spacetime,

Here, we have used the symbol $\natural$ to label objects derived from the gravitational field. We obtain the following identities

$$i(\gamma^\natural)\, dt = 1\,, \qquad i(\gamma^\natural)\, \Omega^\natural = 0\,, \qquad dt \wedge \Omega^\natural \wedge \Omega^\natural \wedge \Omega^\natural \not\equiv 0\,,$$
$$d\Omega^\natural = 0\,, \qquad L[\gamma^\natural]\, \Lambda^\natural = 0\,, \qquad [\Lambda^\natural, \Lambda^\natural] = 0\,.$$

Hence, the pair $(dt, \Omega^\natural)$ turns out to be a *scaled cosymplectic structure* of the phase space.

Moreover, $\Lambda^\natural$ and $\Omega^\natural$ yield inverse linear isomorphisms between the vector spaces of dt–vertical vectors and $\gamma^\natural$–horizontal forms of the phase space.

The Lie derivative of the spacelike metric G and of the spacelike volume form η with respect to a vector field of $\boldsymbol{E}$ is well defined provided that the vector field is projectable on $\boldsymbol{T}$.

If X is a vector field of $\boldsymbol{E}$ projectable on $\boldsymbol{T}$, then we define its *spacelike divergence* by means of the equality $\operatorname{div}_\eta X = L[X]\, \eta$. We have the coordinate expression

$$\operatorname{div}_\eta X = X^0\, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} + \frac{\partial_i (X^i \sqrt{|g|})}{\sqrt{|g|}}\,.$$

It is convenient to add an electromagnetic term to the gravitational field, in a covariant way [57, 54], according to the formula

$$K = K^\natural + K^\mathfrak{e} := K^\natural + \tfrac{1}{2}\, (dt \otimes \hat{F} + \hat{F} \otimes dt)\,,$$

i.e., in coordinates,

$$K_h{}^i{}_k = K^\natural{}_h{}^i{}_k\,, \qquad K_0{}^i{}_k = K^\natural{}_0{}^i{}_k + \tfrac{1}{2}\, G^{ij}_0\, F_{jk}\,, \qquad K_0{}^i{}_0 = K^\natural{}_0{}^i{}_0 + G^{ij}_0\, F_{j0}\,,$$

where $\hat{F} := G^{ij}_0\, F_{j\lambda}\, d^0 \otimes \partial_i \otimes d^\lambda$.

Then, "total" object K turns out to be a connection of spacetime, which fulfills the same properties postulated in (C.5). Moreover, all main formulas in classical and quantum mechanics concerning the given particle and involving the gravitational and electromagnetic fields can be expressed through the "total" K and its derived objects, without the need of splitting it into its gravitational and electromagnetic components.

Proceeding with the total spacetime connection K as before, we obtain the "total" second order connection, 2–form and 2–vector

$$\gamma = \gamma^\natural + \gamma^\mathfrak{e}\,, \qquad \Omega = \Omega^\natural + \Omega^\mathfrak{e}\,, \qquad \Lambda = \Lambda^\natural + \Lambda^\mathfrak{e}\,,$$

where the electromagnetic terms $\gamma^\mathfrak{e}$, $\Omega^\mathfrak{e}$ and $\Lambda^\mathfrak{e}$ turn out to be, respectively, the Lorentz force, $\frac{1}{2}$ the re-scaled electromagnetic field and $\frac{1}{2}$ the re-scaled contravariant spacelike electromagnetic field (i.e. the magnetic field).

These total objects fulfill all properties fulfilled by the gravitational objects as above.

The total *cosymplectic 2–form* Ω encodes the full structure of spacetime (metric, gravitational field and electromagnetic field), hence it plays a central role in the theory.

We have the following coordinate expressions

$$\begin{aligned}
\gamma_{00}{}^{i} &= K_h{}^i{}_k\, x_0^h x_0^k + 2\, K_0{}^i{}_k\, x_0^k + K_0{}^i{}_0 \\
\Omega &= G^0_{ij}\, \Big(d^i_0 - \gamma_{00}{}^{i}\, d^0 - (K_h{}^i{}_0 + K_h{}^i{}_k\, x^k_0)\,(d^h - x^h_0\, d^0)\Big) \wedge (d^j - x^j_0\, d^0) \\
\Lambda &= G^{ij}_0\, \Big(\partial_i + (K_i{}^h{}_0 + K_i{}^h{}_k\, x^k_0)\, \partial^0_h\Big) \wedge \partial^0_j\,.
\end{aligned}$$

The classical mechanics can be achieved as follows.

The second order connection γ yields, in a covariant way, the *generalised Newton law* $\nabla j_1 s = 0$, for a motion $s : \boldsymbol{T} \to \boldsymbol{E}$. Clearly, this equation splits into its gravitational and electromagnetic components as $\nabla^\natural j_1 s = \gamma^{\mathfrak{e}} \circ j_1 s$.

Moreover, the classical dynamics can be derived from Ω, by a Lagrangian formalism, in the following covariant way [130, 54].

0.1.1. Proposition. The closed 2–form Ω admits locally *horizontal potentials* $\Theta : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$, which are defined up to closed 1–forms of spacetime.

The horizontal potentials Θ have coordinate expression of the type

$$\Theta = -(\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 - A_0)\, d^0 + (G^0_{ij}\, x^j_0 + A_j)\, d^i\,, \qquad \text{with} \qquad A \in \mathrm{Sec}(\boldsymbol{E},\, T^*\boldsymbol{E})\,. \square$$

A horizontal potential Θ and observer o yield the *classical potential* $A := o^*\Theta : \boldsymbol{E} \to T^*\boldsymbol{E}$, which is defined locally up to a closed form and depends on the observer.

0.1.2. Proposition. Let us consider a given horizontal potential Θ; if o and $\bar{o} = o+v$ are two observers, then the associated potentials A and $\bar{A}$ are related, in a chart adapted to o, by the formula

$$\bar{A} = A - \tfrac{1}{2}\, G^0_{ij}\, v^i_0\, v^j_0\, d^0 + G^0_{ij}\, v^j_0\, d^i\,.$$

Therefore, each horizontal potential Θ determines a distinguished observer; in fact, there is a unique observer o, such that the spacelike component of the associated potential A vanishes. $\square$

An observer o yields the *observed 2–form* $\Phi := 2\, o^*\Omega : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E}$.

0.1.3. Proposition. We have $\Phi_{\lambda\mu} = \partial_\lambda A_\mu - \partial_\mu A_\lambda$.

We obtain also $\Phi_{0k} := -G^0_{kj}\, K_0{}^j{}_0$ and $\Phi_{hk} := G^0_{hj}\, K_k{}^j{}_0 - G^0_{kj}\, K_h{}^j{}_0$.

0.1.4. Proposition. A horizontal potential Θ yields, in a covariant way, the *classical Lagrangian* $\mathcal{L} : J_1\boldsymbol{E} \to T^*\boldsymbol{T}$, with coordinate expression

$$\mathcal{L} = (\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 + x^i_0\, A_i + A_0)\, d^0\,,$$

where A_λ are the components of the potential A observed by the observer o associated with the chart. The Lagrangian is defined locally and up to a gauge, but does not depend on any observer. The Poincaré–Cartan form associated with the Lagrangian $\mathcal{L}$ turns out to be just Θ.

The Euler–Lagrange equation associated with $\mathcal{L}$ turns out to coincide with the generalised Newton law. □

0.1.5. Proposition. A horizontal potential Θ and an observer o yield the *classical Hamiltonian* $\mathcal{H} : J_1\boldsymbol{E} \to T^*\boldsymbol{T}$ and the *classical momentum* $\mathcal{P} : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$, defined as the negative of the o–horizontal component and the o–vertical components of Θ, respectively. Thus, we can write

$$\Theta = -\mathcal{H} + \mathcal{P} .$$

In an adapted chart, we have the coordinate expressions

$$\mathcal{H} = (\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 - A_0)\, d^0 , \qquad \mathcal{P} = (G^0_{ij}\, x^j_0 + A_i)\, d^i .$$

They are defined locally and up to a gauge, and depend on the choice of the observer. □

The Newton law can be achieved also through $\mathcal{H}$ and $\mathcal{P}$, by means of a Hamiltonian formalism; but this procedure is non covariant, as it depends on the choice of an observer.

Additionally, our structures yield further results on Lie algebras of functions and lifts of functions.

First of all, we obtain the *Poisson Lie bracket* $\{f,\, g\} := \Lambda^\sharp(df \wedge dg)$ for the functions of phase space.

A function f of phase space is *conserved* along the solutions of the Newton law if and only if $\gamma.f = 0$. We denote the space of conserved functions by $\mathrm{Con}(J_1\boldsymbol{E},\, I\!\!R)$. This space turns out to be a subalgebra of the Poisson algebra.

The time fibring and the spacelike metric yield, in a covariant way, a distinguished subset of the set of functions of phase space [57]. Namely, we define a *special quadratic function* to be a function of phase space, whose second fibre derivative (with respect to the affine fibres of phase space over spacetime) is proportional to the spacelike metric. In other words, the special quadratic functions are the functions of the type

$$f = \tfrac{1}{2}\, f^0\, G^0_{ij}\, x^i_0\, x^j_0 + f^i\, G^0_{ij}\, x^j_0 + \overset{o}{f} , \qquad \text{with} \qquad f^0, f^i, \overset{o}{f} \in \mathrm{Map}(\boldsymbol{E},\, I\!\!R) .$$

The *time component* of a special quadratic function f as above is defined to be the (coordinate independent) map $f'' := f^0\, u_0 : \boldsymbol{E} \to \bar{\mathbb{T}}$.

0.1.6. Proposition. The space of special quadratic functions $\mathrm{Spec}(J_1\boldsymbol{E},\, I\!\!R)$ turns out to be a Lie algebra through the *special* Lie bracket

$$[\![\, f,\, g \,]\!] := \{f,\, g\} + \gamma(f'').g - \gamma(g'').f ,$$

with coordinate expression

$$
\begin{aligned}
[\![f, g]\!]^{0} &= f^0 \partial_0 g^0 - g^0 \partial_0 f^0 - f^h \partial_h g^0 + g^h \partial_h f^0 \\
[\![f, g]\!]^{i} &= f^0 \partial_0 g^i - g^0 \partial_0 f^i - f^h \partial_h g^i + g^h \partial_h f^i \\
[\![\overset{o}{f}, \overset{o}{g}]\!] &= f^0 \partial_0 \overset{o}{g} - g^0 \partial_0 \overset{o}{f} - f^h \partial_h \overset{o}{g} + g^h \partial_h \overset{o}{f} - (f^0\, g^k - g^0\, f^k)\, \Phi_{0k} + f^h g^k\, \Phi_{hk} \,. \square
\end{aligned}
$$

0.1.7. Corollary. We have the following distinguished subalgebras of the special Lie algebra:

- the subalgebra $\mathrm{Quan}(J_1\boldsymbol{E},\, I\!\!R) \subset \mathrm{Spec}(J_1\boldsymbol{E},\, I\!\!R)$ of *quantisable functions* f, whose time components f'' depend only on time;

- the subalgebra $\mathrm{Time}(J_1\boldsymbol{E},\, I\!\!R) \subset \mathrm{Quan}(J_1\boldsymbol{E},\, I\!\!R)$ of *time functions* f, whose time components f'' are constant;

- the subalgebra $\mathrm{Aff}(J_1\boldsymbol{E},\, I\!\!R) \subset \mathrm{Time}(J_1\boldsymbol{E},\, I\!\!R)$ of *affine functions* f, whose time components f'' vanish;

- the subalgebra $\mathrm{Map}(\boldsymbol{E},\, I\!\!R) \subset \mathrm{Aff}(J_1\boldsymbol{E},\, I\!\!R)$ of *spacetime functions*. $\square$

0.1.8. Example. We obtain

$$
\mathcal{L}_0, \mathcal{H}_0 \in \mathrm{Time}(J_1\boldsymbol{E},\, I\!\!R)\,, \qquad \mathcal{P}_i \in \mathrm{Aff}(J_1\boldsymbol{E},\, I\!\!R)\,, \qquad x^\lambda \in \mathrm{Map}(\boldsymbol{E},\, I\!\!R)\,.
$$

Clearly, the special bracket and the Poisson bracket coincide on $\mathrm{Aff}(J_1\boldsymbol{E},\, I\!\!R)$. $\square$

We have distinguished lifts of special quadratic functions to vector fields of spacetime and of phase space. Let us denote by $\mathrm{Pro}(\boldsymbol{E},\, T\boldsymbol{E}) \subset \mathrm{Sec}(\boldsymbol{E},\, T\boldsymbol{E})$ the Lie subalgebra of vector fields of $\boldsymbol{E}$ which are projectable on $\boldsymbol{T}$.

0.1.9. Proposition. The time fibring and the spacelike metric yield, in a covariant way, for each $f \in \mathrm{Spec}(J_1\boldsymbol{E},\, I\!\!R)$, the *tangent lift* $X[f] : \boldsymbol{E} \to T\boldsymbol{E}$, whose coordinate expression is

$$
X[f] = f^0\, \partial_0 - f^i\, \partial_i \,.
$$

The lift $\mathrm{Spec}(J_1\boldsymbol{E},\, I\!\!R) \to \mathrm{Pro}(\boldsymbol{E}, T\boldsymbol{E}) : f \mapsto X[f]$ turns out to be a Lie algebra morphism (with respect to the special bracket and the standard Lie bracket, respectively); its kernel is $\mathrm{Map}(\boldsymbol{E},\, I\!\!R)$. $\square$

0.1.10. Example. We obtain

$$
X[\mathcal{L}_0] = \partial_0 - A^i_0\, \partial_i\,, \qquad X[\mathcal{H}_0] = \partial_0\,, \qquad X[\mathcal{P}_i] = -\partial_i\,, \qquad X[x^\lambda] = 0\,,
$$

where $A^i_0 := G^{ij}_0\, A_j$.

We observe that $X[\mathcal{L}] = u^0 \otimes X[\mathcal{L}_0]$ turns out to be the unique observer for which the spacelike component of the observed potential A vanishes.

Moreover, $X[\mathcal{H}] = u^0 \otimes X[\mathcal{H}_0]$ turns out to be just the observer by which we have defined the Hamiltonian. $\square$

0.1.11. Proposition. For each vector field X of $\boldsymbol{E}$ projectable on $\boldsymbol{T}$, the spacetime fibring yields, in a covariant way [75], the *holonomic prolongation*

$$X^{\uparrow}_{\text{hol}} := X_{(1)} : J_1\boldsymbol{E} \to TJ_1\boldsymbol{E}\,,$$

whose coordinate expression is

$$X^{\uparrow}_{\text{hol}} = X^{\lambda}\,\partial_{\lambda} + (\partial_0 X^i + \partial_j X^i\,x^j_0 - \partial_0 X^0\,x^i_0)\,\partial^0_i\,.$$

This prolongation turns out to be an injective Lie algebra morphism. □

0.1.12. Corollary. For each $f \in \mathrm{Quan}(J_1\boldsymbol{E},\, I\!\!R)$, the time fibring yields, in a covariant way, the *holonomic lift*

$$X^{\uparrow}_{\text{hol}}[f] := \big(X[f]\big)_{(1)} : J_1\boldsymbol{E} \to TJ_1\boldsymbol{E}\,,$$

whose coordinate expression is

$$X^{\uparrow}_{\text{hol}}[f] = f^0\,\partial_0 - f^i\,\partial_i - (\partial_0 f^i + \partial_j f^i\,x^j_0 + \partial_0 f^0\,x^i_0)\,\partial^0_i\,.$$

This lift turns out to be a Lie algebra morphism (with respect to the special bracket and the standard Lie bracket, respectively); its kernel is $\mathrm{Map}(\boldsymbol{E},\, I\!\!R)$. □

0.1.13. Example. We obtain

$$X^{\uparrow}_{\text{hol}}[\mathcal{L}_0] = \partial_0 - A^i_0\,\partial_i - (\partial_0 A^i_0 + \partial_j A^i_0\,x^j_0)\,\partial^0_i\,,$$
$$X^{\uparrow}_{\text{hol}}[\mathcal{H}_0] = \partial_0\,, \qquad X^{\uparrow}_{\text{hol}}[\mathcal{P}_i] = -\partial_i\,, \qquad X^{\uparrow}_{\text{hol}}[x^{\lambda}] = 0\,. \; \square$$

0.1.14. Proposition. For each $f \in \mathrm{Spec}(J_1\boldsymbol{E},\, I\!\!R)$, the cosymplectic structure yields, in a covariant way, the *Hamiltonian lift*

$$X^{\uparrow}_{\text{Ham}}[f] := \gamma(f'') + \Lambda^{\sharp}(df) : J_1\boldsymbol{E} \to TJ_1\boldsymbol{E}\,,$$

whose coordinate expression is

$$X^{\uparrow}_{\text{Ham}}[f] = f^0\,\partial_0 - f^i\,\partial_i + X^i_0\,\partial^0_i\,,$$

where

$$X^i_0 = G^{ij}_0\,\Big(\tfrac{1}{2}\,\partial_j f^0 G^0_{hk} x^h_0 x^k_0 + (-f^0\partial_0 G^0_{jh} + f^k\,\partial_k G^0_{jh} + G^0_{hk}\,\partial_j f^k)\,x^h_0 + \partial_j \overset{o}{f} + \Phi_{hj} f^h + f^0 \Phi_{j0}\Big)\,.$$

The lift $\mathrm{Quan}(J_1\boldsymbol{E}, I\!\!R) \to \mathrm{Sec}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E}) : f \mapsto X^{\uparrow}_{\text{Ham}}[f]$ turns out to be a Lie algebra morphism (with respect to the special bracket and the standard Lie bracket, respectively); its kernel is $\mathrm{Map}(\boldsymbol{T},\, I\!\!R)$. □

0.1.15. Example. We obtain

$$X^{\uparrow}_{\mathrm{Ham}}[\mathcal{H}_0] = \partial_0 - G^{ij}_0 \, \partial_0 \mathcal{P}_j \, \partial^0_i \, ,$$
$$X^{\uparrow}_{\mathrm{Ham}}[\mathcal{P}_i] = -\partial_i + G^{hj}_0 \, \partial_i \mathcal{P}_h \, \partial^0_j \, , \qquad X^{\uparrow}_{\mathrm{Ham}}[x^0] = 0 \, , \qquad X^{\uparrow}_{\mathrm{Ham}}[x^i] = G^{ij}_0 \, \partial^0_j \, . \square$$

The above definition of Hamiltonian lift is motivated by the following result concerning the projectability, which will play an important role in quantum mechanics.

For each function f of phase space, we obtain, in a covariant way, the dt–vertical *Hamiltonian lift* $\Lambda^\sharp(df) : J_1\boldsymbol{E} \to V J_1\boldsymbol{E}$.

More generally, for each function f of phase space and for each *time scale* $\tau : J_1\boldsymbol{E} \to \bar{\mathbb{T}}$, we obtain the *$\tau$–Hamiltonian lift* $\gamma(\tau) + \Lambda^\sharp(df) : J_1\boldsymbol{E} \to T J_1\boldsymbol{E}$.

0.1.16. Proposition. [57] The τ-Hamiltonian lift of a function f of phase space is projectable on a vector field of spacetime if and only if $f \in \mathrm{Spec}(J_1\boldsymbol{E}, I\!R)$ and $\tau = f''$.

Moreover, if these conditions are fulfilled, then the τ-Hamiltonian lift projects on the tangent lift of f. $\square$

0.1.2 Covariant quantum mechanics

We proceed by sketching our covariant model of quantum mechanics on a curved spacetime fibred over absolute time. We recall the basic elements of the model and present new results, as well.

According to [57, 54], for quantum mechanics of a charged spinless particle in the above classical background (including the given gravitational and electromagnetic external fields), we postulate:

(Q.1) a *quantum bundle* $\boldsymbol{Q} \to \boldsymbol{E}$, which is a one dimensional complex vector bundle over spacetime;

(Q.2) a *Hermitian metric* $\mathfrak{h} : \boldsymbol{E} \to (\boldsymbol{Q}^* \otimes_{\boldsymbol{E}} \boldsymbol{Q}^*) \otimes_{\boldsymbol{E}} \Lambda^3 V^*\boldsymbol{E}$ of the quantum bundle, with values in the space of spacelike volume forms of spacetime.

Locally, we shall refer to a scaled complex *quantum basis* $(\mathfrak{b})$ normalized by the condition $h(\mathfrak{b}, \mathfrak{b}) = \eta$. The associated scaled complex chart is denoted by (z). Then, we obtain the scaled real basis $(\mathfrak{b}_1, \, \mathfrak{b}_2) := (\mathfrak{b}, \, \mathrm{i}\, \mathfrak{b})$ and the associated scaled real chart (w^1, w^2).

If $\Psi \in \mathrm{Sec}(\boldsymbol{E}, \, \boldsymbol{Q})$ then we write $\Psi = \Psi^1 \, \mathfrak{b}_1 + \Psi^2 \, \mathfrak{b}_2 = \psi \, \mathfrak{b}$, where Ψ^1, Ψ^2 and ψ are, respectively, the scaled real and complex components of Ψ.

Moreover, we consider the *extended quantum bundle*, $\boldsymbol{Q}^\uparrow \to J_1\boldsymbol{E}$, obtained by extending the base space of the quantum bundle to the classical phase space, which here plays the role of space of classical observers.

Each system of connections $\{\overset{o}{Ч}\}$ of the quantum bundle parametrised by the classical observers induces, in a covariant way, a connection $Ч$ of the extended quantum bundle, which is said to be *universal* [44, 54]. The universal connections are characterised in coordinates by the condition $\text{Ч}_i^0 = 0$.

Then, we postulate:

(Q.3) a *quantum connection* Ч on the extended quantum bundle, which is Hermitian, universal and with curvature proportional to Ω.

We recall that Ω incorporates the mass m of the particle and the Planck constant $\hbar$.

0.1.17. Proposition. The coordinate expression of the quantum connection, with respect to a quantum basis and a spacetime chart, turns out to be locally of the type

$$\text{Ч}_0 = -\mathrm{i}\,\mathcal{H}_0\,, \qquad \text{Ч}_i = \mathrm{i}\,\mathcal{P}_i\,, \qquad \text{Ч}_i^0 = 0\,.\,\square$$

The above classical Hamiltonian $\mathcal{H}$ and momentum $\mathcal{P}$ are referred to the observer o associated with the spacetime chart (x^λ) and to a classical horizontal potential Θ of Ω, which is locally determined by the quantum connection Ч and the quantum basis b.

Then, the gauge of the classical potential $A := o^*\Theta$ is determined by the quantum connection and the quantum basis. Moreover, we recall that A includes both the gravitational and the electromagnetic potential.

These minimal geometric objects Q.1, ... , Q.3 constitute the only source, in a covariant way, of all further objects of quantum mechanics.

Actually, the quantum connection lives on the extended quantum bundle, whose base space is the phase space; on the other hand, the covariance of the theory requires that the significant physical objects be independent from observers. This fact suggests a method of projectability, in order to get rid of the observers encoded in the phase space. Actually, we have already used this method in the classical theory, just in view of these developments of quantum mechanics. Indeed, this method turns out to be fruitful.

The *quantum dynamics* can be obtained in the following way.

The method of projectability yields, in a covariant way, a distinguished quantum Lagrangian (hence, the generalised Schrödinger equation, the quantum momentum and the probability current) [57, 54].

Even more, the covariance implies the essential uniqueness of the above Lagrangian and of the Schrödinger equation [60, 61, 62].

0.1.18. Proposition. The coordinate expression of the quantum Lagrangian is

$$\begin{aligned} \mathsf{L}[\Psi] = \tfrac{1}{2}\Big(&\mathrm{i}\,(\bar\psi\,\partial_0\psi - \psi\,\partial_0\bar\psi) + 2\,A_0\,\bar\psi\,\psi \\ &- G_0^{ij}\,(\partial_i\bar\psi\,\partial_j\psi + A_i\,A_j\,\bar\psi\,\psi) - \mathrm{i}\,G_0^{ij}\,A_j\,(\bar\psi\,\partial_i\psi - \psi\,\partial_i\bar\psi) + k\,\rho_0\,\bar\psi\,\psi\Big) \\ &\sqrt{|g|}\,d^0\wedge d^1\wedge d^2\wedge d^3\,, \end{aligned}$$

where ρ is the scalar curvature of the fibres of spacetime determined by the spacelike metric and $k \in I\!R$ is a real constant (which is not determined by the covariance). $\square$

0.1.19. Corollary. The coordinate expression of the generalised Schrödinger equation turns out to be

$$\Big(\partial_0 - \mathrm{i}\, A_0 + \tfrac{1}{2}\, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} - \tfrac{1}{2}\, \mathrm{i}\, (\overset{o}{\Delta}_0 + k\, \rho_0)\Big)\, \psi = 0\,,$$

where

$$\overset{o}{\Delta}_0 := G^{hk}_0\, (\partial_h - \mathrm{i}\, A_h)\, (\partial_k - \mathrm{i}\, A_k) + \frac{\partial_h (G^{hk}_0 \sqrt{|g|})}{\sqrt{|g|}}\, (\partial_k - \mathrm{i}\, A_k)$$

is the spacelike *quantum Laplacian.* □

0.1.20. Corollary. We obtain in the standard way the conserved *probability current* with coordinate expression

$$j_1 \Psi^* \big(\mathsf{J}[\mathsf{L}]\big) = (\bar{\psi}\, \psi)\, \upsilon^0_0 - G^{hk}_0\, \Big(\mathrm{i}\, \tfrac{1}{2}\, (\bar{\psi}\, \partial_h \psi - \psi\, \partial_h \bar{\psi}) + A_h\, \bar{\psi}\, \psi\Big)\, \upsilon^0_k\,,$$

where $\upsilon^0_\lambda := i(\partial_\lambda)\, \sqrt{|g|}\, d^0 \wedge d^1 \wedge d^2 \wedge d^3$ .END

We obtain distinguished operators acting on the sections of the quantum bundle in the following covariant way.

First of all, we have a distinguished family of *second order pre–quantum operators*.

0.1.21. Proposition. The Schrödinger operator yields, for each time scale $\tau : \boldsymbol{E} \to \bar{\mathbb{T}}$, the second order linear operator $\mathsf{S}(\tau) : J_2 \boldsymbol{Q} \to \boldsymbol{Q}$, which acts on the sections Ψ of the quantum bundle, according to the coordinate expression

$$\mathsf{S}(\tau)[\Psi] = \mathrm{i}\, \tau^0\, \Big(\partial_0 - \mathrm{i}\, A_0 + \tfrac{1}{2}\, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} - \tfrac{1}{2}\, \mathrm{i}\, (\overset{o}{\Delta}_0 + k\, \rho_0)\Big)\, \psi\, \mathsf{b}\,.$$

In particular, each $f \in \mathrm{Spec}(J_1 \boldsymbol{E}, I\!R)$ yields, in a covariant way, the second order pre–quantum operator $\mathsf{S}[f] := \mathsf{S}(f'')$. □

Then, we obtain a distinguished family of first order operators, by classifying the vector fields of the quantum bundle which preserve the Hermitian metric.

A vector field Y of $\boldsymbol{Q}$ is said to be *Hermitian* if it is projectable on $\boldsymbol{E}$ and on $\boldsymbol{T}$, is real linear over its projection on $\boldsymbol{E}$ and $L[Y]\, \mathsf{h} = 0$.

We denote the space of Hermitian vector fields of $\boldsymbol{Q}$ by $\mathrm{Her}(\boldsymbol{Q}, T\boldsymbol{Q})$.

0.1.22. Proposition. A vector field Y of $\boldsymbol{Q}$ is Hermitian if and only if its coordinate expression is of the type

$$Y \;\equiv\; Y[f] = f^0\, \partial_0 - f^i\, \partial_i + \Big(\mathrm{i}\, (\overset{o}{f} + A_0\, f^0 - A_i\, f^i) - \tfrac{1}{2}\, \mathrm{div}_\eta\, X[f]\Big)\, \mathbb{I}\,,$$

where $f \in \mathrm{Quan}(J_1\boldsymbol{E}, I\!R)$ and where $\mathbb{I} = (w^1\,\partial w_1 + w^2\,\partial w_2)$ denotes the identity vertical vector field of the quantum bundle.

The space of Hermitian vector fields $\mathrm{Her}(\boldsymbol{E}, T\boldsymbol{Q})$ is closed with respect to the Lie bracket. Moreover, the map

$$\mathrm{Quan}(J_1\boldsymbol{E}, I\!R) \to \mathrm{Her}(\boldsymbol{Q}, T\boldsymbol{Q}) : f \mapsto Y[f]$$

is independent of the choice of coordinates and is an isomorphism of Lie algebras (with respect to the special bracket and the standard Lie bracket, respectively).

Furthermore, the map $\mathrm{Her}(\boldsymbol{Q}, T\boldsymbol{Q}) \to \mathrm{Pro}(\boldsymbol{E}, T\boldsymbol{E}) : Y[f] \mapsto X[f]$ turns out to be a central extension of Lie algebras by $\mathrm{Map}(\boldsymbol{E}, \mathrm{i}\,I\!R) \otimes \mathbb{I}$. $\square$

For each $f \in \mathrm{Quan}(J_1\boldsymbol{E}, I\!R)$, the vector field $Y[f] : \boldsymbol{Q} \to T\boldsymbol{Q}$ is said to be the *quantum lift* of f.

0.1.23. Example. We obtain

$$Y[\mathcal{L}_0] = \partial_0 - A^i_0\,\partial_i - \Big(\mathrm{i}\,A_i\,A^i_0 + \tfrac{1}{2}\,(\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(A^i_0\sqrt{|g|})}{\sqrt{|g|}})\Big)\,\mathbb{I}\,,$$

$$Y[\mathcal{H}_0] = \partial_0 - \tfrac{1}{2}\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\,\mathbb{I}\,, \qquad Y[\mathcal{P}_i] = -\partial_i + \tfrac{1}{2}\,\frac{\partial_i\sqrt{|g|}}{\sqrt{|g|}}\,\mathbb{I}\,, \qquad Y[x^\lambda] = \mathrm{i}\,x^\lambda\,\mathbb{I}\,. \square$$

0.1.24. Corollary. Each quantisable function f yields, in a covariant way, the first order operator acting on the sections of the quantum bundle

$$Z[f] := \mathrm{i}\,L\Big[Y[f]\Big]\,,$$

whose coordinate expression is, for each $\Psi \in \mathrm{Sec}(\boldsymbol{E}, \boldsymbol{Q})$,

$$Z[f].\Psi = \mathrm{i}\,\Big(f^0\,\partial_0\psi - f^i\,\partial_i\psi - \Big(\mathrm{i}\,(\overset{o}{f} + A_0\,f^0 - A_i\,f^i) - \tfrac{1}{2}\,\mathrm{div}_\eta\,X[f]\Big)\,\psi\Big)\,\mathsf{b}\,. \square$$

For each quantisable function f, we say $Z[f]$ to be the associated *first order pre–quantum operator*. We denote the space of the first order pre–quantum operators by $\mathrm{Oper}_1(\boldsymbol{Q})$.

0.1.25. Proposition. The space $\mathrm{Oper}_1(\boldsymbol{Q})$ turns out to be a Lie algebra through the bracket

$$\Big[Z[f], Z[g]\Big] := -\mathrm{i}\,\Big(Z[f]\circ Z[g] - Z[g]\circ Z[f]\Big)\,.$$

Moreover, the map $\mathrm{Quan}(J_1\boldsymbol{E}, I\!R) \to \mathrm{Oper}_1(\boldsymbol{Q}) : f \mapsto Z[f]$ turns out to be an isomorphism of Lie algebras (with respect to the special bracket and the above Lie bracket, respectively). $\square$

0.1.26. Example. We obtain

$$Z[\mathcal{H}_0].\Psi = \mathrm{i}\,\Big(\partial_0\psi + \tfrac{1}{2}\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\,\psi\Big)\,\mathsf{b}$$

$$Z[\mathcal{P}_i].\Psi = -\mathrm{i}\,\Big(\partial_i\psi + \tfrac{1}{2}\,\frac{\partial_i\sqrt{|g|}}{\sqrt{|g|}}\,\psi\Big)\,\mathsf{b}$$

$$Z[x^\lambda].\Psi = x^\lambda\,\psi\,\mathsf{b}\,.\,\square$$

The above results appear to be a covariant "correspondence principle" yielding *pre–quantum operators* associated with quantisable functions.

However, we still need to introduce the Hilbert stuff carrying the standard probabilistic interpretation of quantum mechanics. It can be done in the following covariant way [57, 54].

We consider the infinite dimensional *functional quantum bundle* $\boldsymbol{H}_c \to \boldsymbol{T}$, whose fibres are constituted by the compact support smooth sections, at fixed time, of the quantum bundle ("regular sections"). The Hermitian metric h equips this bundle with a pre–Hilbert product $\langle\,,\,\rangle$. Then, a true Hilbert bundle $\boldsymbol{H} \to \boldsymbol{T}$ can be obtained by a completion procedure. This bundle has no distinguished splittings into time and type Hilbert fibre; such a splitting can be obtained by choosing a classical observer.

Each regular section Ψ of the quantum bundle can be regarded as a section $\widehat{\Psi}$ of the functional quantum bundle. Accordingly, each "regular" operator O acting on sections of the quantum bundle can be regarded as an operator $\widehat{\mathsf{O}}$ acting on the sections of the functional quantum bundle.

Our previous results yield, for each quantisable function f, two distinguished operators acting on the sections of the functional quantum bundle, namely $\widehat{Z[f]}$ and $\widehat{\mathsf{S}[f]}$. Actually, in general, both operators do not act on the fibres of the functional bundle (at fixed time), because they involve the partial derivative ∂_0.

On the other hand, we have the following results [57, 54, 128].

0.1.27. Proposition. Let $f \in \mathrm{Quan}(J_1\boldsymbol{E},\, I\!R)$. Then, the combination

$$\widehat{f} := \widehat{Z[f]} - \widehat{\mathsf{S}[f]}$$

acts on the fibres of the functional bundle. We have the following coordinate expression

$$\widehat{f}(\widehat{\Psi}) = \Big(-\tfrac{1}{2}\,f^0\,(\overset{o}{\Delta}_0 + k\,\rho_0) - \mathrm{i}\,f^j\,(\partial_j - \mathrm{i}\,A_j) + \overset{o}{f} - \mathrm{i}\,\tfrac{1}{2}\,\frac{\partial_j(f^j\sqrt{|g|})}{\sqrt{|g|}}\Big)\,\psi\,\widehat{\mathsf{b}}\,.$$

Moreover, $\widehat{f}$ is symmetric with respect to the Hermitian metric $\langle\,,\,\rangle$. $\square$

For the self–adjointness of $\widehat{f}$ further global conditions on f are needed.

For each $f \in \mathrm{Quan}(J_1\boldsymbol{E},\, I\!R)$, we say $\widehat{f}$ to be the *quantum operator* associated with f.

0.1.28. Example. We obtain the following distinguished quantum operators

$$\widehat{\mathcal{H}_0}(\widehat{\Psi}) = -(\tfrac{1}{2}\,\overset{o}{\Delta}_0 + \tfrac{1}{2}\,k\,\rho_0 - A_0)\,\psi\,\widehat{\mathfrak{b}}\,,$$

$$\widehat{\mathcal{P}_j}(\widehat{\Psi}) = -\mathrm{i}\left(\partial_j + \tfrac{1}{2}\,\frac{\partial_j\sqrt{|g|}}{\sqrt{|g|}}\right)\psi\,\widehat{\mathfrak{b}}\,,$$

$$\widehat{x^\lambda}(\widehat{\Psi}) = x^\lambda\,\psi\,\widehat{\mathfrak{b}}\,.\square$$

The space of the fibre preserving maps of the functional quantum bundle into itself becomes a Lie algebra through the bracket $[h, k] := -\mathrm{i}\,(h\circ k - k\circ h)$.

0.1.29. Proposition. For each $f, g \in \mathrm{Quan}(J_1\boldsymbol{E},\, I\!R)$, we obtain

$$[\widehat{f},\, \widehat{g}] = \widehat{[\![\, f,\, g\,]\!]} - \mathrm{i}\left[S[f],\, Z[g]\right] + \mathrm{i}\left[S[g],\, Z[f]\right].$$

In particular, for each $f, g \in \mathrm{Aff}(J_1\boldsymbol{E},\, I\!R)$, we obtain

$$[\widehat{f},\, \widehat{g}] = \widehat{[\![\, f,\, g\,]\!]} = \widehat{\{f,\, g\}}\,.\square$$

Thus, the above results suggest our covariant "equivalence principle".

The Feynmann path integral approach can be nicely formulated in our framework [57]. In fact, the quantum connection Ч yields, in a covariant way, a non linear connection of the extended quantum bundle over time; moreover, this connection allows us to interpret the Feynmann amplitudes through the parallel transport of this connection. However, unfortunately, our theory does not contribute so far to the hard problem of the measure arising in the Feynmann theory.

The case of a spin particle (generalised Pauli equation) can be approached in an analogous way, by considering a further quantum bundle of dimension two, with the only additional postulate of a suitable soldering form [20].

0.2 Symmetries

Next, we classify the infinitesimal symmetries of the classical and quantum structures. We show that these symmetries are controlled by the Lie algebra of quantisable functions and its distinguished subalgebras. Moreover, we discuss the strict relations between classical and quantum symmetries.

0.2.1 Classical symmetries

We start by discussing the main results concerning symmetries of the classical structure.

A vector field $X^{\uparrow}$ of $J_1\boldsymbol{E}$ is said to be a *symmetry of the classical structure*, if it is projectable on $\boldsymbol{E}$, $\boldsymbol{T}$ and fulfills

$$L[X^{\uparrow}]\, dt = 0 \qquad L[X^{\uparrow}]\, \Omega = 0.$$

We denote the space of symmetries of the classical structure by $\mathrm{Clas}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$.

0.2.1. Proposition. We have the following distinguished subalgebras of the algebra of quantisable functions:

– the subalgebra $\mathrm{Hol}(J_1\boldsymbol{E},\, I\!\!R) \subset \mathrm{Quan}(J_1\boldsymbol{E},\, I\!\!R)$, which is constituted by the functions f such that $X^{\uparrow}_{\mathrm{hol}}[f] = X^{\uparrow}_{\mathrm{Ham}}[f]$;

– the subalgebra $\mathrm{Unim}(J_1\boldsymbol{E},\, I\!\!R) \subset \mathrm{Quan}(J_1\boldsymbol{E},\, I\!\!R)$, which is constituted by the functions f such that $\mathrm{div}_\eta X[f] = 0$;

– the subalgebra $\mathrm{Self}(J_1\boldsymbol{E},\, I\!\!R) \subset \mathrm{Quan}(J_1\boldsymbol{E},\, I\!\!R)$, which is constituted by the functions f such that $i(X^{\uparrow}_{\mathrm{hol}}[f])\, \Omega = df$. $\square$

If $f \in \mathrm{Hol}(J_1\boldsymbol{E},\, I\!\!R)$, then we set

$$X^{\uparrow}[f] := X^{\uparrow}_{\mathrm{Ham}}[f] = X^{\uparrow}_{\mathrm{hol}}[f]\,.$$

0.2.2. Theorem. *We have*

$$\begin{aligned}
&\mathrm{Time}(J_1\boldsymbol{E}, I\!\!R) \cap \mathrm{Con}(J_1\boldsymbol{E}, I\!\!R) = \mathrm{Time}(J_1\boldsymbol{E},\, I\!\!R) \cap \mathrm{Self}(J_1\boldsymbol{E},\, I\!\!R)\\
&\mathrm{Time}(J_1\boldsymbol{E}, I\!\!R) \cap \mathrm{Con}(J_1\boldsymbol{E}, I\!\!R) \subset \mathrm{Hol}(J_1\boldsymbol{E},\, I\!\!R)\\
&\mathrm{Time}(J_1\boldsymbol{E}, I\!\!R) \cap \mathrm{Con}(J_1\boldsymbol{E}, I\!\!R) \subset \mathrm{Unim}(J_1\boldsymbol{E},\, I\!\!R)\,. \square
\end{aligned}$$

We set

$$\mathrm{Clas}(J_1\boldsymbol{E}, I\!\!R) := \mathrm{Time}(J_1\boldsymbol{E}, I\!\!R) \cap \mathrm{Con}(J_1\boldsymbol{E}, I\!\!R) = \mathrm{Time}(J_1\boldsymbol{E},\, I\!\!R) \cap \mathrm{Self}(J_1\boldsymbol{E},\, I\!\!R)$$

and denote the space of the tangent lifts of elements of $\mathrm{Clas}(J_1\boldsymbol{E}, I\!\!R)$ by

$$\mathrm{Clas}(\boldsymbol{E}, T\boldsymbol{E}) \subset \mathrm{Pro}(\boldsymbol{E}, T\boldsymbol{E})\,.$$

0.2.3. Proposition. The special and the Poisson brackets coincide in $\mathrm{Clas}(J_1\boldsymbol{E},\, I\!\!R)$. Hence, this space turns out to be a subalgebra of the Poisson and of the special algebras.

Moreover, $\mathrm{Clas}(\boldsymbol{E}, T\boldsymbol{E})$ turns out to be closed with respect to the standard Lie bracket. $\square$

We call the elements of $\mathrm{Clas}(J_1\boldsymbol{E}, I\!\!R)$ *classical generators*. This name will be justified by Proposition 0.2.4, Corollary 0.2.5 and Corollary 0.2.6.

0.2.4. Theorem. *[140] A vector field $X^{\uparrow}$ of $J_1\boldsymbol{E}$ projectable on $\boldsymbol{E}$ fulfills $L[X^{\uparrow}]\, dt = 0$ and $L[X^{\uparrow}]\, \Omega = 0$ if and only if, locally,*

$$X^{\uparrow} = X^{\uparrow}_{\mathrm{Ham}}[f]\,, \qquad \textit{with} \qquad f \in \mathrm{Clas}(J_1\boldsymbol{E},\, I\!\!R)\,,$$

where f is defined up to a real constant. $\square$

0.2.5. Corollary. If $f \in \mathrm{Clas}(J_1\boldsymbol{E}, I\!R)$, then we obtain

$$L\big[X[f]\big]\, G = 0\,, \qquad L\big[X[f]\big]\, \eta = 0\,, \qquad L\big[X^{\uparrow}[f]\big]\, \gamma = 0\,, \qquad L\big[X^{\uparrow}[f]\big]\, K = 0\,.\square$$

0.2.6. Corollary. If X is a vector field of $\boldsymbol{E}$ projectable on $\boldsymbol{T}$, such that $L[X^{\uparrow}_{\mathrm{hol}}]\,\mathcal{L} = 0$, then we obtain locally

$$X = X[f]\,, \qquad X^{\uparrow}_{\mathrm{hol}} = X^{\uparrow}[f]\,, \qquad \text{with} \qquad f \in \mathrm{Clas}(J_1\boldsymbol{E}, I\!R)\,,$$

where f is defined up to a real constant. $\square$

0.2.2 Quantum symmetries

Eventually, we classify the vector fields of the extended quantum bundle which preserve the full quantum structure: all fibrings (on quantum bundle, on phase space, on spacetime, on time), the Hermitian metric, the quantum connection. Moreover, we compare the symmetries of the quantum structure with the symmetries of the quantum Lagrangian.

A vector field $Y^{\uparrow}$ of $\boldsymbol{Q}^{\uparrow}$ is said to be a *symmetry of the quantum structure* if it is projectable on $\boldsymbol{Q}$, $J_1\boldsymbol{E}$, $\boldsymbol{E}$, $\boldsymbol{T}$, is real linear over $J_1\boldsymbol{E}$ and fulfills

$$L[Y^{\uparrow}]\, dt = 0\,, \qquad L[Y^{\uparrow}]\, \mathrm{h} = 0\,, \qquad L[Y^{\uparrow}]\, \text{Ч} = 0\,.$$

We denote the space of the symmetries of the quantum structure by $\mathrm{Quan}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow})$.

For each $f \in \mathrm{Hol}(J_1\boldsymbol{E}, I\!R)$, we define its *extended quantum lift* to be the vector field of the extended quantum bundle

$$Y^{\uparrow}[f] := \text{Ч}\big(X^{\uparrow}[f]\big) + \big(\mathrm{i}\, f - \tfrac{1}{2}\, \mathrm{div}_{\eta}\, X[f]\big)\, \mathbb{I}\,.$$

0.2.7. Theorem. *A vector field $Y^{\uparrow}$ of $\boldsymbol{Q}^{\uparrow}$ is a symmetry of the quantum structure if and only if it is of the type*

$$Y^{\uparrow} = Y^{\uparrow}[f]\,, \qquad \textit{with} \qquad f \in \mathrm{Clas}(J_1\boldsymbol{E}, I\!R)\,.$$

The space $\mathrm{Quan}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow})$ *is closed with respect to the Lie bracket. Moreover, the map* $\mathrm{Clas}(J_1\boldsymbol{E}, I\!R) \to \mathrm{Quan}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow}) : f \mapsto Y^{\uparrow}[f]$ *is an isomorphism of Lie algebras (with respect to the special bracket and the standard Lie bracket, respectively).*

Furthermore, the map $\mathrm{Quan}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow}) \to \mathrm{Clas}(\boldsymbol{E}, T\boldsymbol{E}) : Y^{\uparrow}[f] \mapsto X[f]$ *turns out to be a central extension of Lie algebras by* $\mathrm{i}\, I\!R \otimes \mathbb{I}$. $\square$

Next, we compare the symmetries of the quantum connection and the symmetries of the quantum Lagrangian.

0.2.8. Proposition. For each $f \in \mathrm{Quan}(J_1\boldsymbol{E},\, I\!\!R)$, we obtain, in a covariant way, the *holonomic quantum lift* of f, defined as the holonomic prolongation [75]

$$Y_{\mathrm{hol}}[f] := \big(Y[f]\big)_{(1)} : J_1\boldsymbol{Q} \to TJ_1\boldsymbol{Q}$$

of the quantum lift $Y[f]$, whose coordinate expression is

$$\begin{aligned}
&Y_{\mathrm{hol}}[f] = f^0\,\partial_0 - f^i\,\partial_i \\
&- \tfrac{1}{2}\Big(f^0\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(f^i\sqrt{|g|})}{\sqrt{|g|}}\Big)\,(w^1\,\partial_1 + w^2\,\partial_2) + (f^0\,A_0 - f^i\,A_i + \overset{o}{f})\,(w^1\,\partial_2 - w^2\,\partial_1) \\
&- \tfrac{1}{2}\,\partial_\lambda\Big(f^0\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(f^i\sqrt{|g|})}{\sqrt{|g|}}\Big)\,(w^1\,\partial^\lambda_1 + w^2\,\partial^\lambda_2) + \partial_\lambda(f^0\,A_0 - f^i\,A_i + \overset{o}{f})\,(w^1\,\partial^\lambda_2 - w^2\,\partial^\lambda_1) \\
&- \tfrac{1}{2}\Big(f^0\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(f^i\sqrt{|g|})}{\sqrt{|g|}}\Big)\,(w^1_\lambda\,\partial^\lambda_1 + w^2_\lambda\,\partial^\lambda_2) + (f^0\,A_0 - f^i\,A_i + \overset{o}{f})\,(w^1_\lambda\,\partial^\lambda_2 - w^2_\lambda\,\partial^\lambda_1) \\
&- \partial_0 f^0\,(w^1_0\,\partial^0_1 + w^2_0\,\partial^0_2) + \partial_0 f^i\,(w^1_i\,\partial^0_1 + w^2_i\,\partial^0_2) + \partial_h f^i\,(w^1_i\,\partial^h_1 + w^2_i\,\partial^h_2)\,.\,\square
\end{aligned}$$

0.2.9. Theorem. *Let $f \in \mathrm{Time}(J_1\boldsymbol{E}, I\!\!R)$. Then, the following conditions are equivalent:*

$$\begin{aligned}
L\Big[Y^{\uparrow}[f]\Big]\,\mathsf{Ч} &= 0 \\
L\Big[Y_{\mathrm{hol}}[f]\Big]\,\mathsf{L} &= 0 \\
i(X^{\uparrow}_{\mathrm{hol}}[f])\,\Omega &= df \\
\gamma.f &= 0 \\
f \in \mathrm{Clas}&(J_1\boldsymbol{E},\, I\!\!R)\,.\,\square
\end{aligned}$$

We can also show, in analogy with the symmetries of classical and quantum strucutre, that any vector field of $J_1\boldsymbol{Q}$, which is projectable over $\boldsymbol{Q}$ and $\boldsymbol{T}$, complex linear over its projection on $\boldsymbol{Q}$ and preserves the quantum Lagrangian, is necessarily of holonomic type.

Eventually, we consider the conserved currents associated with symmetries of the quantum Lagrangian, according to the standard Nöther theorem. Additionally, our results allow us to associate such currents with classical quantisable functions.

For each $f \in \mathrm{Quan}(J_1\boldsymbol{E},\, I\!\!R)$, we define the associated *quantum current* to be the 3–form

$$\mathsf{j}[f] := -i\big(Y[f]\big)\,\Pi : J_1\boldsymbol{Q} \to \Lambda^3 T^*\boldsymbol{Q}\,,$$

where Π is the Poincaré–Cartan form [130] associated with the quantum Lagrangian.

0.2.10. Corollary. For each $f \in \mathrm{Clas}(J_1\boldsymbol{E},\, I\!\!R)$, the current $\mathfrak{j}[f]$ is conserved along the solutions $\Psi : \boldsymbol{E} \to \boldsymbol{Q}$ of the Schrödinger equation. □

0.2.11. Example. The current associated with the constant function $1 \in \mathrm{Clas}(J_1\boldsymbol{E}, I\!\!R)$ is just the conserved probability current. □

Moreover, for each affine function and quantum section, we obtain, in a covariant way, a spacelike 3–form (which can be integrated on the fibres of spacetime), according to the following result.

0.2.12. Proposition. Let $f \in \mathrm{Aff}(J_1\boldsymbol{E},\, I\!\!R)$. Then, for each $\Psi \in \mathrm{Sec}(\boldsymbol{E},\, \boldsymbol{Q})$ we obtain

$$\Big(\Psi^*(\mathfrak{j}[f])\Big)^{\vee} = \tfrac{1}{2}\,\Big(\mathfrak{h}(Z[f].\Psi,\, \Psi) - \mathfrak{h}(\Psi,\, Z[f].\Psi)\Big)$$

where $^{\vee}$ denotes the vertical restriction. We have the coordinate expression

$$\Big(\Psi^*(\mathfrak{j}[f])\Big)^{\vee} = \Big(f^i\,(\Psi^1\,\partial_i\Psi^2 - \Psi^2\,\partial_i\Psi^1) + (\overset{o}{f} - f^i A_i)\,(\Psi^1\,\Psi^1 + \Psi^2\,\Psi^2)\Big)\,\sqrt{|g|}\,\check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3\,. \square$$

CHAPTER 1

PRELIMINARIES

1.1 Basic notation

All manifolds and maps between manifolds are smooth.
If $\boldsymbol{M}$ and $\boldsymbol{N}$ are manifolds, then we denote the sheaf of local maps $f : \boldsymbol{M} \to \boldsymbol{N}$ by

$$\mathrm{Map}(\boldsymbol{M}, \boldsymbol{N}) := \{f : \boldsymbol{M} \to \boldsymbol{N}\} \,.$$

If $\boldsymbol{F} \to \boldsymbol{B}$ and $\bar{\boldsymbol{F}} \to \bar{\boldsymbol{B}}$ are fibred manifolds, then we denote the sheaf of local fibred morphisms $f : \boldsymbol{F} \to \bar{\boldsymbol{F}}$ by

$$\mathrm{Fib}(\boldsymbol{F}, \bar{\boldsymbol{F}}) \subset \mathrm{Map}(\boldsymbol{F}, \bar{\boldsymbol{F}}) \,.$$

If $\boldsymbol{F} \to \boldsymbol{B}$ is a fibred manifold, then we denote the sheaf of local sections $s : \boldsymbol{B} \to \boldsymbol{F}$ by

$$\mathrm{Sec}(\boldsymbol{B}, \boldsymbol{F}) \subset \mathrm{Map}(\boldsymbol{B}, \boldsymbol{F}) \,.$$

Accordingly, if $\boldsymbol{M}$ is a manifold, then the sheaf of local vector fields $X : \boldsymbol{M} \to T\boldsymbol{M}$ is denoted by

$$\mathrm{Sec}(\boldsymbol{M}, T\boldsymbol{M}) \subset \mathrm{Map}(\boldsymbol{M}, T\boldsymbol{M}) \,.$$

Moreover, if $\boldsymbol{F} \to \boldsymbol{B}$ is a fibred manifold, then the subsheaf of projectable local vector fields is denoted by

$$\mathrm{Pro}(\boldsymbol{F}, T\boldsymbol{F}) \subset \mathrm{Sec}(\boldsymbol{F}, T\boldsymbol{F}) \,.$$

If $\boldsymbol{F} \to \boldsymbol{B}$ is an affine bundle associated with the vector bundle $\bar{\boldsymbol{F}} \to \boldsymbol{B}$, then we can write

$$V\boldsymbol{F} \simeq \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \bar{\boldsymbol{F}} \,.$$

In particular, if $\boldsymbol{F} \to \boldsymbol{B}$ is a vector bundle, then we can write

$$V\boldsymbol{F} \simeq \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} .$$

If $\boldsymbol{F} \to \boldsymbol{B}$ is a vector bundle, then we define the *Liouville vector field* to be the vertical vector field

$$\mathbb{I} : \boldsymbol{F} \to V\boldsymbol{F} \simeq \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} : f \mapsto (f,\, f) ,$$

which can be identified with

$$\mathbf{1} : \boldsymbol{B} \to \boldsymbol{F}^* \underset{\boldsymbol{B}}{\otimes} \boldsymbol{F} \qquad \text{and} \qquad \mathrm{id} : \boldsymbol{F} \to \boldsymbol{F} .$$

1.2 General connections

Here we recall a few basic results on general connections, which will be needed in the following.

Let us consider a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ and refer to a fibred chart (x^λ, y^i).
A *(general) connection* is defined to be a tangent valued 1–form

$$c : \boldsymbol{F} \to T^*\boldsymbol{B} \underset{\boldsymbol{F}}{\otimes} T\boldsymbol{F} ,$$

which projects on $\mathbf{1} : \boldsymbol{B} \to T^*\boldsymbol{B} \underset{\boldsymbol{B}}{\otimes} T\boldsymbol{B}$.
Let us consider a general connection c.
Its coordinate expression is of the type

$$c = d^\lambda \otimes (\partial_\lambda + c^i_\lambda\, \partial_i) , \qquad \text{with} \qquad c^i_\lambda \in \mathrm{Map}(\boldsymbol{B},\, I\!R) .$$

The *curvature* of c is defined to be the vertical valued 2–form

$$R[c] := \tfrac{1}{2}\, [c,\, c] ,$$

where $[\,,\,]$ is the Frölicher–Nijenhuis bracket.
We have the coordinate expression

$$R[c] = (\partial_\lambda c^i_\mu + c^j_\lambda\, \partial_j c^i_\mu)\, d^\lambda \wedge d^\mu \otimes \partial_i .$$

1.2.1. Lemma. Let $X, \bar{X} \in \mathrm{Sec}(\boldsymbol{B}, T\boldsymbol{B})$. Then, we have

$$[c(X),\, c(\bar{X})] = c([X, \bar{X}]) + i(\bar{X})\, i(X)\, R[c] .$$

PROOF. We have

$$
\begin{aligned}
[c(X),\, c(\bar X)] &= \big[X^\lambda\,(\partial_\lambda + c^i_\lambda\,\partial_i),\ \bar X^\mu\,(\partial_\mu + c^j_\mu\,\partial_j)\big] \\
&= (X^\lambda\,\partial_\lambda \bar X^\mu - \bar X^\lambda\,\partial_\lambda X^\mu)\,(\partial_\mu + c^i_\mu\,\partial_i) + (X^\lambda\,\bar X^\mu - \bar X^\lambda\,X^\mu)\,(\partial_\lambda c^j_\mu + c^i_\lambda\,\partial_i c^j_\mu)\,\partial_j\,.\ \text{QED}
\end{aligned}
$$

1.2.2. Lemma. Let $X \in \mathrm{Sec}(\boldsymbol{B},\, T\boldsymbol{B})$. Then, we have

$$L[c(X)]\,c = i_X\,R[c]\,.$$

PROOF. It can be proved in coordinates. QED

1.2.3. Lemma. Let $\boldsymbol{M}$ be a manifold and let $\phi \in \mathrm{Sec}(\boldsymbol{M},\ \Lambda^r T^*\boldsymbol{M} \underset{\boldsymbol{M}}{\otimes} T\boldsymbol{M})$ and $Y \in \mathrm{Sec}(\boldsymbol{M},\, T\boldsymbol{M})$.

Then, we have

$$L[Y]\,\phi = [Y,\,\phi]\,,$$

where the bracket in the right hand side is the Frölicher–Nijenhuis bracket.

PROOF. By recalling the general expression of the Frölicher–Nijenhuis bracket and the Leibniz rule for the Lie derivative, we obtain, for each $Z_1,\dots,Z_r \in \mathrm{Sec}(\boldsymbol{M},\, T\boldsymbol{M})$,

$$
\begin{aligned}
[\phi,\,Y](Z_1,\dots,Z_r) &= \tfrac{1}{r!}\sum_{\sigma\in\mathfrak{S}(r+1)} |\sigma|\,\big(\big[\phi(Z_{\sigma(1)},\dots,Z_{\sigma(r)}),\ Y\big] - r\,\phi(Z_{\sigma(1)},\dots,Z_{\sigma(r)},\,[Z_{\sigma(r)},\,Y])\big) \\
&= \tfrac{1}{r!}\sum_{\sigma\in\mathfrak{S}(r+1)} |\sigma|\,\big(- (L[Y]\,\phi)(Z_{\sigma(1)},\dots,Z_{\sigma(r)}) \\
&\quad - \phi(L[Y]Z_{\sigma(1)},\dots,Z_{\sigma(r)}) - \dots - \phi(Z_{\sigma(1)},\dots,L[Y]Z_{\sigma(r)}) \\
&\quad - r\,\phi(Z_{\sigma(1)},\dots,Z_{\sigma(r)},\,[Z_{\sigma(r)},\,Y])\big) \\
&= \tfrac{1}{r!}\sum_{\sigma\in\mathfrak{S}(r+1)} |\sigma|\,\big(- (L[Y]\,\phi)(Z_{\sigma(1)},\dots,Z_{\sigma(r)}) \\
&= -(L[Y]\,\phi)(Z_1,\dots,Z_r)\big)\,.\ \text{QED}
\end{aligned}
$$

1.2.4. Proposition. Let $Y \in \mathrm{Sec}(\boldsymbol{F},\, T\boldsymbol{F})$. Then, we have

$$L[Y]c = -d[c]Y \qquad \text{and} \qquad L[Y]\,(R[c]) = -d^2[c]Y\,,$$

where we have set $d[c]Y := [c, Y]$.

Proof. The first equality is a particular case of the above Lemma.
The second equality follows from

$$L[Y]\,(R[c]) = -[Y,\, R[c]] = -d^2[c]Y\,.\,\mathrm{QED}$$

1.2.5. Corollary. Let $Y \in \mathrm{Sec}(\boldsymbol{F},\, T\boldsymbol{F})$. Then,

$$L[Y]c = 0 \qquad \Rightarrow \qquad L[Y]\,(R[c]) = 0\,.\,\square$$

1.2.6. Proposition. Let $Y \in \mathrm{Sec}(\boldsymbol{F},\, T\boldsymbol{F})$.
Then, the following conditions are equivalent:

1) $$L[Y]c = 0\,;$$

2) Y is projectable on a vector field, $X \in \mathrm{Sec}(\boldsymbol{B},\, T\boldsymbol{B})$, i.e.

$$\partial_j X^\mu \;\equiv\; \partial_j Y^\mu = 0\,,$$

and

$$\partial_\lambda Y^i - X^\mu\,\partial_\mu c^i_\lambda - c^i_\mu\,\partial_\lambda X^\mu + c^j_\lambda\,\partial_j Y^i - Y^j\,\partial_j c^i_\lambda = 0\,.$$

Proof. We have the following expression

$$\begin{aligned} L[Y]\,(d^\lambda \otimes \partial_\lambda + c^i_\lambda\, d^\lambda \otimes \partial_i) &= \partial_\mu Y^\lambda\, d^\mu \otimes \partial_\lambda + \partial_j Y^\lambda\, d^j \otimes \partial_\lambda - \partial_\lambda Y^\mu\, d^\lambda \otimes \partial_\mu - \partial_\lambda Y^i\, d^\lambda \otimes \partial_i \\ &\quad + Y^\mu\,\partial_\mu c^i_\lambda\, d^\lambda \otimes \partial_i + Y^j\,\partial_j c^i_\lambda\, d^\lambda \otimes \partial_i + c^i_\lambda\,\partial_\mu Y^\lambda\, d^\mu \otimes \partial_i + c^i_\lambda\,\partial_j Y^\lambda\, d^j \otimes \partial_i \\ &\quad - c^i_\lambda\,\partial_i Y^\mu\, d^\lambda \otimes \partial_\mu - c^i_\lambda\,\partial_i Y^j\, d^\lambda \otimes \partial_j \end{aligned}$$

Hence, $L[Y]c = 0$ if and only if

$$\begin{aligned} \partial_j Y^\lambda &= 0 \\ \partial_\lambda Y^i + c^j_\lambda\,\partial_j\, Y^i - c^i_\mu\,\partial_\lambda X^\mu - X^\mu \partial_\mu c^i_\lambda\, - Y^j\,\partial_j c^i_\lambda &= 0\,.\,\mathrm{QED} \end{aligned}$$

Next, let us suppose that $\boldsymbol{F}$ be a vector bundle and c a linear connection.

1.2.7. Lemma. Let $\phi \in \mathrm{Map}(\boldsymbol{F}, I\!R)$. Then, we have

$$L[\phi\,\mathbb{I}]\, c = -d\phi \otimes \mathbb{I}\,.$$

Proof. We have

$$\begin{aligned} L[\phi\,\mathbb{I}]\,(d^\lambda \otimes \partial_\lambda + c_\lambda{}^i{}_j\, y^j\, d^\lambda \otimes \partial_i) &= -\partial_\lambda \phi\, y^i\, d^\lambda \otimes \partial_i + \phi\, y^j\, c_\lambda{}^i{}_j\, d^\lambda \otimes \partial_i - \phi\, c_\lambda{}^i\, d^\lambda \otimes \partial_i \\ &= -\partial_\lambda \phi\, y^i\, d^\lambda \otimes \partial_i \\ &= -d\phi \otimes \mathbb{I}\,.\,\mathrm{QED} \end{aligned}$$

1.2.8. Lemma. Let $\phi \in \mathrm{Map}(\boldsymbol{B}, I\!R)$, $X \in \mathrm{Sec}(\boldsymbol{B}, T\boldsymbol{B})$. Then, we have

$$[c(X),\, \phi\,\mathbb{I}] = X.\phi\,\mathbb{I}\,.$$

Proof. We have

$$\begin{aligned}
[c(X),\, \phi\,\mathbb{I}] &= \big[X^\lambda\,(\partial_\lambda + c_\lambda{}^i{}_j\, y^j\,\partial_i),\ \phi\, y^i\,\partial_i\big] \\
&= X^\lambda\,\partial_\lambda\phi\, y^i\,\partial_i + X^\lambda\,\phi\, c_\lambda{}^i{}_j\, y^j\,\partial_i - X^\lambda\,\phi\, c_\lambda{}^i j\, y^j\,\partial_i \\
&= X^\lambda\,\partial_\lambda\phi\, y^i\,\partial_i \\
&= X.\phi\,\mathbb{I}\,.\ \mathrm{QED}
\end{aligned}$$

1.2.9. Lemma. Let $\phi, \bar\phi \in \mathrm{Map}(\boldsymbol{B}, I\!R)$. Then, we have

$$[\phi\,\mathbb{I},\, \bar\phi\,\mathbb{I}] = 0\,.$$

Proof. We have

$$\begin{aligned}
[\phi\,\mathbb{I},\, \bar\phi\,\mathbb{I}] &= \big[\phi\, y^i\,\partial_i,\ \bar\phi\, y^j\,\partial_j\big] \\
&= \phi\,\bar\phi\,(y^i\,\partial_i - y^j\,\partial_j) \\
&= 0\,.\ \mathrm{QED}
\end{aligned}$$

1.3 Lie derivatives of sections

Let us consider a vector bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ and refer to a linear fibred chart (x^λ, y^i) and to the associated local basis (b_i).

1.3.1. Lemma. A section $s \in \mathrm{Sec}(\boldsymbol{B},\, \boldsymbol{F})$ yields the vertical vector field

$$\tilde s := (\mathrm{id}[\boldsymbol{F}],\, s \circ p) \in \mathrm{Sec}(\boldsymbol{F},\, V\boldsymbol{F}) \,\simeq\, \mathrm{Sec}(\boldsymbol{F},\, \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \boldsymbol{F})\,,$$

according to the following commutative diagram

$$\begin{tikzcd}
\boldsymbol{B} \arrow[r, "s"] & \boldsymbol{F} \\
\boldsymbol{F} \arrow[u, "p"] \arrow[r, "\tilde s"] \arrow[dr, "\mathrm{id}[\boldsymbol{F}]"'] & \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \arrow[u, "\mathrm{pro}_2"'] \arrow[d, "\mathrm{pro}_1"] \\
 & \boldsymbol{F}
\end{tikzcd}$$

Actually, the map

$$s \mapsto \tilde{s}$$

is a natural bijection between the sheaf of sections $\boldsymbol{B} \to \boldsymbol{F}$ and the sheaf of vertical vector fields $\boldsymbol{F} \to V\boldsymbol{F}$ whose second projection $\boldsymbol{F} \to \boldsymbol{F}$ factorises through a section $\boldsymbol{B} \to \boldsymbol{F}$.

The coordinate expression of $\tilde{s}$ is

$$\tilde{s} = s^i \partial_i \qquad s^i := y^i \circ s \in \mathrm{Map}(\boldsymbol{B}, I\!R)\,. \square$$

1.3.2. Lemma. The coordinate expression of a vector field $Y \in \mathrm{Pro}(\boldsymbol{F}, T\boldsymbol{F})$, which is linear over its projection is of the type

$$Y = Y^\lambda \, \partial_\lambda + Y^i_j \, y^j \, \partial_i \,, \qquad Y^\lambda, Y^i_j \in \mathrm{Map}(\boldsymbol{B}, I\!R)\,.$$

1.3.3. Lemma. Let us consider a vector field $Y \in \mathrm{Pro}(\boldsymbol{F}, T\boldsymbol{F})$, which is linear over its projection, and a section $s \in \mathrm{Sec}(\boldsymbol{B}, \boldsymbol{F})$. Then, the Lie bracket

$$[Y, \tilde{s}] \in \mathrm{Sec}(\boldsymbol{F}, V\boldsymbol{F})$$

can be naturally regarded as a section

$$Y.s \in \mathrm{Sec}(\boldsymbol{B}, \boldsymbol{F})\,.$$

We have the coordinate expression

$$Y.s = (Y^\lambda \, \partial_\lambda s^i - Y^i_j \, s^j)\, \mathfrak{b}_i\,.$$

Proof. In fact we have

$$\tilde{s} = s^i \, \partial_i \,,$$

hence

$$[Y, \tilde{s}] = (Y^\lambda \, \partial_\lambda s^i - Y^i_j \, s^j)\, \partial_i$$

and

$$\partial_h (Y^\lambda \, \partial_\lambda s^i - Y^i_j \, s^j) = 0\,. \text{QED}$$

Let us analyze two complementary cases of the Lie derivative of a section with respect to a horizontal and a vertical vector field.

1.3.4. Lemma. Let us consider a linear connection $c : \boldsymbol{F} \to T^*\boldsymbol{B} \underset{\boldsymbol{B}}{\otimes} T\boldsymbol{F}$, a section $s \in \mathrm{Sec}(\boldsymbol{B}, \boldsymbol{F})$, a vector field $X \in \mathrm{Sec}(\boldsymbol{B}, T\boldsymbol{B})$ and the induced vector field $Y := c(X) \in \mathrm{Sec}(\boldsymbol{F}, T\boldsymbol{F})$.

Then, we obtain

$$Y.s = \nabla_X s\,.$$

PROOF. We obtain the coordinate expression

$$Y = X^\lambda\,(\partial_\lambda + c_\lambda{}^i{}_j\, y^j\,\partial_i)\,,$$

hence

$$Y.s = X^\lambda\,(\partial_\lambda s^i - c_\lambda{}^i{}_j\, s^j)\mathfrak{b}_i = \nabla_X s\,.\,\text{QED}$$

1.3.5. Lemma. Let us consider a section $s \in \mathrm{Sec}(\boldsymbol{B},\, \boldsymbol{F})$ and a linear vertical vector field $Y \in \mathrm{Sec}(\boldsymbol{F},\, V\boldsymbol{F})$.

Then, we obtain

$$Y.s = -Y \circ s\,,$$

where we have identified Y with the associated linear fibred endomorphism $\boldsymbol{F} \to \boldsymbol{F}$.

PROOF. We obtain the coordinate expression

$$Y = Y^i_j\, y^j\,\mathfrak{b}_i\,, \qquad Y^i_j \in \mathrm{Map}(\boldsymbol{B},\, I\!R)\,,$$

hence

$$Y.s = -Y^i_j\, s^j\,\mathfrak{b}_i = -Y \circ s\,.\,\text{QED}$$

Thus, we obtain the following general case.

1.3.6. Proposition. Let us consider a linear connection $c : \boldsymbol{F} \to T^*\boldsymbol{B} \underset{\boldsymbol{B}}{\otimes} T\boldsymbol{F}$, a section $s \in \mathrm{Sec}(\boldsymbol{B},\, \boldsymbol{F})$, and a vector field $Y \in \mathrm{Sec}(\boldsymbol{F},\, T\boldsymbol{F})$, which is projectable over a vector field $X \in \mathrm{Sec}(\boldsymbol{B},\, T\boldsymbol{B})$ and linear over X. Then, we can split Y into its horizontal and vertical components

$$Y = c(X) + \nu[c](Y)\,,$$

and obtain

$$Y.s = \nabla_X s - \nu[c](Y) \circ s\,.\,\square$$

1.4 Infinitesimal symmetries

Throughout the paper we shall be involved with the following concept.

1.4.1. Definition. Let $\boldsymbol{M}$ be a manifold and

$$\phi \in \mathrm{Sec}(\boldsymbol{M}, \, (\otimes^r T^*\boldsymbol{M}) \underset{\boldsymbol{M}}{\otimes} (\otimes^s T\boldsymbol{M}))$$

a tensor field. Then, we define an *infinitesimal symmetry* (*i.s.* for short) of ϕ to be a vector field $X \in \mathrm{Sec}(\boldsymbol{M}, T\boldsymbol{M})$, such that

$$L[X] \, \phi = 0 \, . \square$$

We denote the sheaf of i.s. of $\phi \in \mathrm{Sec}(\boldsymbol{M}, \, (\otimes^r T^*\boldsymbol{M}) \underset{\boldsymbol{M}}{\otimes} (\otimes^s T\boldsymbol{M}))$ by

$$\mathrm{Sym}_\phi(\boldsymbol{M}, T\boldsymbol{M}) \, .$$

1.4.2. Proposition. Let $\boldsymbol{M}$ be a manifold and

$$\phi \in \mathrm{Sec}(\boldsymbol{M}, \, (\otimes^r T^*\boldsymbol{M}) \underset{\boldsymbol{M}}{\otimes} (\otimes^s T\boldsymbol{M}))$$

a tensor field.

Then,

$$\mathrm{Sym}_\phi(\boldsymbol{M}, \, T\boldsymbol{M}) \subset \mathrm{Sec}(\boldsymbol{M}, \, T\boldsymbol{M})$$

is a Lie subalgebra.

Proof. In fact, for each vector fields $X, Y \in \mathrm{Sec}(\boldsymbol{M}, \, T\boldsymbol{M})$ and for tensor fields ϕ of degree 0 and 1, we have

$$L\big[[X, \, Y]\big] \, \phi = \big[L[X], L[Y]\big] \, \phi \, .$$

Moreover, the above equality can be extended to tensor fields ϕ of any degree by means of the Leibnitz rule. QED

1.4.3. Proposition. Let $\boldsymbol{F} \to \boldsymbol{B}$ be a fibred manifold.

If

$$\phi \in \mathrm{Sec}(\boldsymbol{F}, \, \otimes^r V^*\boldsymbol{F}) \qquad \text{and} \qquad X \in \mathrm{Pro}(\boldsymbol{F}, \, T\boldsymbol{F})$$

then the Lie derivative

$$L[X] \, \phi \in \mathrm{Sec}(\boldsymbol{F}, \, \otimes^r V^*\boldsymbol{F}),$$

where $L[X] \, \phi$ is the vertical restriction of the standard Lie derivative of an arbitrary extension of ϕ, is well defined, i.e. does not depend on the extension of ϕ.

With reference to a fibred chart (x^λ, y^i), we have the coordinate expression

$$L[X]\phi = (X^\mu \, \partial_\mu \phi_{i_1 \ldots i_r} + X^j \, \partial_j \phi_{i_1 \ldots i_r} + \partial_{i_1} X^j \, \phi_{j i_2 \ldots i_r} + \cdots + \partial_{i_r} X^j \, \phi_{i_1 \ldots i_{r-1} j}) \, \check{d}^{i_1} \otimes \check{d}^{i_2} \otimes \ldots \otimes \check{d}^{i_r} . \square$$

Accordingly, we give the following definition.

1.4.4. Definition. An *infinitesimal symmetry* (*i.s.* for short) of $\phi \in \mathrm{Sec}(\boldsymbol{F}, \otimes^r V^*\boldsymbol{F})$ is defined to be a projectable vector field $X \in \mathrm{Pro}(\boldsymbol{F}, T\boldsymbol{F})$, such that

$$L[X]\,\phi = 0\,.$$

We denote the sheaf of i.s. of $\phi \in \mathrm{Sec}(\boldsymbol{F}, \otimes^r V^*\boldsymbol{F})$ by

$$\mathrm{Sym}_\phi(\boldsymbol{F}, T\boldsymbol{F})\,.$$

1.4.5. Proposition. Let $\boldsymbol{F} \to \boldsymbol{B}$ be a fibred manifold and

$$\phi \in \mathrm{Sec}(\boldsymbol{F}, \otimes^r V^*\boldsymbol{F})$$

a vertical tensor field.

Then,

$$\mathrm{Sym}_\phi(\boldsymbol{F}, T\boldsymbol{F}) \subset \mathrm{Pro}(\boldsymbol{F}, T\boldsymbol{F})$$

is a Lie subalgebra. $\square$

1.5 Jets

Let us consider a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ and refer to a fibred chart (x^λ, y^i).
Then, we denote the k–jet prolongation of $\boldsymbol{F}$ by $p^k : J_k\boldsymbol{F} \to \boldsymbol{B}$.
For each $0 \leq h < k$, we have the natural projection $p^k_h : J_k\boldsymbol{F} \to \boldsymbol{J}_h\boldsymbol{F}$.
The fibred charts induced on $J_1\boldsymbol{F}$ and $J_2\boldsymbol{F}$ are denoted by

$$(x^\lambda, y^i, y^i_\lambda) \qquad \text{and} \qquad (x^\lambda, y^i, y^i_\lambda, y^i_{\lambda\mu})\,, \text{ with } \lambda \leq \mu\,.$$

1.5.1. Lemma. The fibred manifold $p^k_{k-1} : J_k\boldsymbol{F} \to \boldsymbol{J}_{k-1}\boldsymbol{F}$ turns out to be naturally an affine bundle associated with the vector bundle $S^kT^*\boldsymbol{B} \underset{\boldsymbol{F}}{\otimes} V\boldsymbol{F}$. $\square$

1.5.2. Lemma. Let us consider two fibred manifolds $p : \boldsymbol{F} \to \boldsymbol{B}$ and $q : \boldsymbol{G} \to \boldsymbol{C}$ and refer to fibred charts (x^λ, y^i) and (w^α, z^r).

For each fibred morphism $\Phi : \boldsymbol{F} \to \boldsymbol{G}$ projectable on a diffeomorphism $\underline{\Phi} : \boldsymbol{B} \to \boldsymbol{C}$ there is a unique fibred morphism $J_1\Phi : J_1\boldsymbol{F} \to J_1\boldsymbol{G}$, which makes the following diagram commutative for each $s \in \mathrm{Sec}(\boldsymbol{B}, \boldsymbol{F})$

$$\begin{tikzcd} J_1\boldsymbol{F} \arrow[r, "J_1\Phi"] & J_1G \\ B \arrow[u, "j_1 s"] \arrow[r, "\Phi"'] & C \arrow[u, "j_1(\Phi \circ s \circ \underline{\Phi}^{-1})"'] \end{tikzcd}$$

We have the coordinate expression

$$(w^\mu, z^j, z^j_\mu) \circ J_1\Phi = \Big(\Phi^\mu,\ \Phi^j,\ (\partial_\alpha \Phi^j + \partial_r \Phi^j\, y^r_\alpha)\,(\partial_\mu (\underline{\Phi}^{-1})^\alpha) \circ \underline{\Phi}\Big)\,.$$

The map J_1 turns out to be a covariant functor. □

In particular, we have the following result.

1.5.3. Lemma. Let us consider two fibred manifolds $p : \boldsymbol{F} \to \boldsymbol{B}$ and $q : \boldsymbol{G} \to \boldsymbol{B}$ and refer to fibred charts (x^λ, y^i) and (x^λ, z^r).

For each fibred morphism $\Phi : \boldsymbol{F} \to \boldsymbol{G}$ on $\mathrm{id}[\boldsymbol{B}]$, there is a unique fibred morphism $J_1\Phi : J_1\boldsymbol{F} \to J_1\boldsymbol{G}$, which makes the following diagram commutative for each $s \in \mathrm{Sec}(\boldsymbol{B}, \boldsymbol{F})$

$$\begin{tikzcd} J_1\boldsymbol{F} \arrow[r, "J_1\Phi"] & J_1 G \\ B \arrow[u, "j_1 s"] \arrow[r, "\mathrm{id}"'] & B \arrow[u, "j_1(\Phi\circ s)"'] \end{tikzcd}$$

We have the coordinate expression

$$(x^\lambda, z^j, z^j_\lambda) \circ J_1\Phi = \Big(\Phi^\lambda,\ \Phi^j,\ (\partial_\lambda \Phi^j + \partial_r \Phi^j\, y^r_\lambda)\Big)\,. \square$$

1.6 Holonomic prolongation

Let $\boldsymbol{F} \to \boldsymbol{B}$ be a fibred manifold and refer to a fibred chart (x^λ, y^i).

1.6.1. Lemma. Let $X \in \mathrm{Pro}(\boldsymbol{F}, T\boldsymbol{F})$ and denote the flow of X by $\Phi \in \mathrm{Fib}(I\!R \times \boldsymbol{F},\, \boldsymbol{F})$.

Then, we obtain the vector field

$$X_{(1)} := \partial J_1 \Phi \in \mathrm{Pro}(J_1\boldsymbol{F},\, T J_1\boldsymbol{F})\,,$$

which projects on X, where J_1 denotes the jet prolongation at constant parameter and ∂ denotes the tangent prolongation with respect to the parameter evaluated at $0 \in I\!R$.

We have the coordinate expression

$$X_{(1)} = X^\lambda\, \partial_\lambda + X^i\, \partial_i + (\partial_\lambda X^i + \partial_j X^i\, y^j_\lambda - \partial_\lambda X^\mu\, y^i_\mu)\, \partial^\lambda_i\,.$$

Proof. We have

$$y^i_\lambda \circ J_1\Phi_t = (\partial_\lambda \Phi^i_t + \partial_j \Phi^i_t\, y^j_\mu)\, \partial_\lambda (\underline{\Phi}_t^{-1})^\mu \circ \underline{\Phi}_t\,,$$

hence

$$\partial (y^i_\lambda \circ J_1\Phi_t) = (\partial_\lambda X^i + \partial_j X^i\, y^j_\lambda - \partial_\lambda X^\mu\, y^i_\mu)\,. \text{QED}$$

1.6.2. Definition. We define the vector field

$$X_{(1)} := \partial J_1\Phi \in \mathrm{Pro}(J_1\boldsymbol{F},\, TJ_1\boldsymbol{F})$$

to be the *(first) holonomic prolongation* of $X \in \mathrm{Pro}(\boldsymbol{F},\, T\boldsymbol{F})$. $\square$

1.6.3. Proposition. The map

$$\mathrm{Pro}(\boldsymbol{F},\, T\boldsymbol{F}) \to \mathrm{Pro}(J_1\boldsymbol{F},\, J_1T\boldsymbol{F}) : X \mapsto X_{(1)}$$

is a morphism of Lie algebras.

Proof. If $X, Y \in \mathrm{Pro}(\boldsymbol{F},\, T\boldsymbol{F})$, then we obtain

$$[X_{(1)}, Y_{(1)}]^\lambda = X^\alpha\, \partial_\alpha Y^\lambda + X^h\, \partial_h Y^\lambda - Y^\alpha\, \partial_\alpha X^\lambda - Y^h\, \partial_h X^\lambda = ([X,Y]_{(1)})^\lambda$$

and

$$[X_{(1)}, Y_{(1)}]^i = X^\alpha\, \partial_\alpha Y^i + X^h\, \partial_h Y^i - Y^\alpha\, \partial_\alpha X^i - Y^h\, \partial_h X^i = ([X,Y]_{(1)})^i$$

Moreover, we obtain

$$\begin{aligned}
[X_{(1)}, Y_{(1)}]^i_\lambda &= X^\alpha\, \partial_\alpha(\partial_\lambda Y^i + \partial_j Y^i\, y^j_\lambda - \partial_\lambda Y^\mu\, y^i_\mu) \\
&\quad + X^h\, \partial_h(\partial_\lambda Y^i + \partial_j Y^i\, y^j_\lambda - \partial_\lambda Y^\mu\, y^i_\mu) \\
&\quad - Y^\alpha\, \partial_\alpha(\partial_\lambda X^i + \partial_j X^i\, y^j_\lambda - \partial_\lambda X^\mu\, y^i_\mu) \\
&\quad - Y^h\, \partial_h(\partial_\lambda X^i + \partial_j X^i\, y^j_\lambda - \partial_\lambda X^\mu\, y^i_\mu) \\
&\quad + (\partial_\lambda X^h + \partial_j X^h\, y^j_\lambda - \partial_\lambda X^\mu\, y^h_\mu)\, \partial_h Y^i \\
&\quad - (\partial_\alpha X^i + \partial_j X^i\, y^j_\alpha - \partial_\alpha X^\mu\, y^i_\mu)\, \partial_\lambda Y^\alpha \\
&\quad - (\partial_\lambda Y^h + \partial_j Y^h\, y^j_\lambda - \partial_\lambda Y^\mu\, y^h_\mu)\, \partial_h X^i \\
&\quad + (\partial_\alpha Y^i + \partial_j Y^i\, y^j_\alpha - \partial_\alpha Y^\mu\, y^i_\mu)\, \partial_\lambda X^\alpha
\end{aligned}$$

i.e.

$$\begin{aligned}
[X_{(1)}, Y_{(1)}]^i_\lambda &= X^\alpha\, \partial_\alpha (\partial_\lambda Y^i + \partial_j Y^i\, y^j_\lambda - \partial_\lambda Y^\mu\, y^i_\mu) \\
&+ X^h\, \partial_h (\partial_\lambda Y^i + \partial_j Y^i\, y^j_\lambda) \\
&\\
&- Y^\alpha\, \partial_\alpha (\partial_\lambda X^i + \partial_j X^i\, y^j_\lambda - \partial_\lambda X^\mu\, y^i_\mu) \\
&- Y^h\, \partial_h (\partial_\lambda X^i + \partial_j X^i\, y^j_\lambda) \\
&\\
&+ (\partial_\lambda X^h + \partial_j X^h\, y^j_\lambda - \partial_\lambda X^\mu\, y^h_\mu)\, \partial_h Y^i \\
&- (\partial_\alpha X^i + \partial_j X^i\, y^j_\alpha - \partial_\alpha X^\mu\, y^i_\mu)\, \partial_\lambda Y^\alpha \\
&\\
&- (\partial_\lambda Y^h + \partial_j Y^h\, y^j_\lambda - \partial_\lambda Y^\mu\, y^h_\mu)\, \partial_h X^i \\
&+ (\partial_\alpha Y^i + \partial_j Y^i\, y^j_\alpha - \partial_\alpha Y^\mu\, y^i_\mu)\, \partial_\lambda X^\alpha
\end{aligned}$$

i.e.

$$\begin{aligned}
[X_{(1)}, Y_{(1)}]^i_\lambda &= X^\alpha\, \partial_\alpha (\partial_\lambda Y^i + \partial_j Y^i\, y^j_\lambda - \partial_\lambda Y^\mu\, y^i_\mu) \\
&+ X^h\, \partial_h (\partial_\lambda Y^i + \partial_j Y^i\, y^j_\lambda) \\
&\\
&- Y^\alpha\, \partial_\alpha (\partial_\lambda X^i + \partial_j X^i\, y^j_\lambda - \partial_\lambda X^\mu\, y^i_\mu) \\
&- Y^h\, \partial_h (\partial_\lambda X^i + \partial_j X^i\, y^j_\lambda) \\
&\\
&+ (\partial_\lambda X^h - \partial_\lambda X^\mu\, y^h_\mu)\, \partial_h Y^i \\
&- (\partial_\alpha X^i - \partial_\alpha X^\mu\, y^i_\mu)\, \partial_\lambda Y^\alpha \\
&\\
&- (\partial_\lambda Y^h + \partial_j Y^h\, y^j_\lambda)\, \partial_h X^i \\
&+ (\partial_\alpha Y^i - \partial_\alpha Y^\mu\, y^i_\mu)\, \partial_\lambda X^\alpha
\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}
([X, Y]_{(1)})^i_\lambda &= \partial_\lambda (X^\alpha\, \partial_\alpha Y^i + X^h\, \partial_h Y^i - Y^\alpha\, \partial_\alpha X^i - Y^h\, \partial_h X^i) \\
&+ \partial_j (X^\alpha\, \partial_\alpha Y^i + X^h\, \partial_h Y^i - Y^\alpha\, \partial_\alpha X^i - Y^h\, \partial_h X^i)\, y^j_\lambda \\
&- \partial_\lambda (X^\alpha\, \partial_\alpha Y^\mu - Y^\alpha\, \partial_\alpha X^\mu)\, y^i_\mu
\end{aligned}$$

i.e.

$$\begin{aligned}
([X, Y]_{(1)})^i_\lambda &= \partial_\lambda X^\alpha\, \partial_\alpha Y^i + \partial_\lambda X^h\, \partial_h Y^i - \partial_\lambda Y^\alpha\, \partial_\alpha X^i - \partial_\lambda Y^h\, \partial_h X^i \\
&+ (\partial_j X^h\, \partial_h Y^i - \partial_j Y^h\, \partial_h X^i)\, y^j_\lambda \\
&- (\partial_\lambda X^\alpha\, \partial_\alpha Y^\mu - \partial_\lambda Y^\alpha\, \partial_\alpha X^\mu)\, y^i_\mu \\
&\\
&+ (X^\alpha\, \partial_\lambda \partial_\alpha Y^i + X^h\, \partial_\lambda \partial_h Y^i - Y^\alpha\, \partial_\lambda \partial_\alpha X^i - Y^h\, \partial_\lambda \partial_h X^i) \\
&+ (X^\alpha\, \partial_j \partial_\alpha Y^i + X^h\, \partial_j \partial_h Y^i - Y^\alpha\, \partial_j \partial_\alpha X^i - Y^h\, \partial_j \partial_h X^i)\, y^j_\lambda \\
&- (X^\alpha\, \partial_\lambda \partial_\alpha Y^\mu - Y^\alpha\, \partial_\lambda \partial_\alpha X^\mu)\, y^i_\mu
\end{aligned}$$

hence

$$[X_{(1)}, Y_{(1)}]^i_\lambda = ([X, Y]_{(1)})^i_\lambda \,. \mathrm{QED}$$

We can extend in a natural way the above holonomic prolongation to non projectable vector fields, but this is not relevant for this thesis.

CHAPTER 2

CLASSICAL THEORY

In the first section, I recall the basic facts of the *CCG* theory. For more details, see for instance [56, 54]. Moreover, I sketch the results of [140] about symmetries in the *CCG* theory, which are necessary for understanding the symmetries of quantum theory. Using these results we can classify the vector fields which preserve the classical structure by means of algebras of functions of classical phase space, which are subalgebras of the algebra of special functions. The algebra of special functions is not a Poisson subalgebra. In the case, when the vector fields preserve the full classical structure, they are generated by functions of the special subalgebra $\mathrm{Clas}(J_1\boldsymbol{E}, I\!R)$, called classical generators, which is also a Poisson subalgebra. This subalgebra plays an essential role in the quantum theory.

The covariance of the theory includes also independence from the choice of units of measurements. For this reason, a rigorous treatment of this feature is needed.

Therefore, we assume the following "positive 1-dimensional semi-vector spaces" over $I\!R^+$ as fundamental unit spaces (roughly speaking, they have the same algebraic structure as $I\!R^+$, but no distinguished generator over $I\!R^+$): the space of $\mathbb{T}$ *time intervals*, the space $\mathbb{L}$ of *lengths* and the space $\mathbb{M}$ of *masses*.

Moreover, we assume the *Planck constant* to be an element $\hbar : \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \mathbb{M}$.

We refer to particles with mass $m \in \mathbb{M}$ and charge $q \in \mathbb{T}^* \otimes \mathbb{L}^{\frac{3}{2}} \otimes \mathbb{M}^{\frac{1}{2}}$.

Moreover, we refer to a time unit $u_0 \in \mathbb{T}$ or to its dual $u^0 \in \mathbb{T}^*$.

2.1 Classical framework

The classical background is introduced starting from minimal axioms. A curved spacetime fibred over an absolut time, a spacelike scaled Riemannian metric and a compatible gravitational and electromagnetic field. This structure yields in a covariant way the full classical structure. In particular, it yields the covariant Newton law, a scaled cosymplectic structure of classical phase space and a classical Lagrangian formalism.

Choosing an observer, it yields a classical Hamiltonian formalism. Hence, the Hamiltonian formalism is not covariant.

2.1.1 Spacetime

We assume *(absolute) time* to be an affine space $\boldsymbol{T}$ associated with the vector space $\bar{\mathbb{T}} := \mathbb{T} \otimes I\!\!R$. We assume *spacetime* to be an oriented $(n+1)$-dimensional manifold $\boldsymbol{E}$ fibred over time by the *absolute time map*

$$t : \boldsymbol{E} \to \boldsymbol{T} .$$

Thus, the time fibring yields the *time form*

$$dt : \boldsymbol{E} \to \mathbb{T} \otimes T^* \boldsymbol{E} .$$

We refer to charts of spacetime $(x^\lambda) = (x^0, x^i)$ adapted to the time fibring, to the affine structure of time and to a time unit of measurement u_0.

The induced local bases of $T\boldsymbol{E}$, $V\boldsymbol{E}$, $T^*\boldsymbol{E}$ and $V^*\boldsymbol{E}$ are, respectively, (∂_λ), (∂_i), (d^λ) and $(\check{d}^i)$.

We have the coordinate expression

$$dt = u_0 \otimes d^0 .$$

An *observer* is defined to be a section $o \in \mathrm{Sec}(\boldsymbol{E}, \mathbb{T}^* \otimes T\boldsymbol{E})$, which projects on $\mathbf{1} \in \mathbb{T}^* \otimes \mathbb{T}$. Hence, an observer is a connection of the spacetime fibring. In a chart, we have the expression $o = d^0 \otimes (\partial_0 + o_0^i \partial_i)$, where $o_0^i \in \mathrm{Map}(\boldsymbol{E}, I\!\!R)$. The charts (x^λ) for which $o_0^i = 0$ are said to be *adapted* to o. Each chart (x^λ) determines the observer $o := u^0 \otimes \partial_0$.

Each observer o yields the splitting

$$T\boldsymbol{E} = (\boldsymbol{E} \times \bar{\mathbb{T}}) \underset{\boldsymbol{E}}{\oplus} V\boldsymbol{E} : X \mapsto (dt(X)) + (X - o(dt(X)) .$$

We shall be involved with the Lie subalgebras

$$\mathrm{Pro}(\boldsymbol{E}, T\boldsymbol{E}) \subset \mathrm{Sec}(\boldsymbol{E}, T\boldsymbol{E}) \qquad \text{and} \qquad \mathrm{Time}(\boldsymbol{E}, T\boldsymbol{E}) \subset \mathrm{Pro}(\boldsymbol{E}, T\boldsymbol{E})$$

of vector fields of $\boldsymbol{E}$ which are projectable on $\boldsymbol{T}$ and whose time component is constant, respectively.

2.1.2 Metric field

Assumption 0.1. We assume *spacelike metric* to be a scaled Riemannian metric of the fibres of spacetime

$$g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (V^* \boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^* \boldsymbol{E}) . \square$$

2.1.1. Definition. Given a particle of mass m, we define the *re–scaled spacelike metric*

$$G := \tfrac{m}{\hbar} g : \boldsymbol{E} \to \mathbb{T} \otimes (V^* \boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^* \boldsymbol{E}) \,.$$

We denote the contravariant spacelike metric and the contravariant re–scaled spacelike metric by

$$\bar{g} : \boldsymbol{E} \to \mathbb{L}^{2*} \otimes (V\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E})$$
$$\bar{G} := \tfrac{\hbar}{m} \bar{g} : \boldsymbol{E} \to \mathbb{T}^* \otimes (V\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E}) \,.$$

2.1.2. Proposition. We have the coordinate expressions

$$G = G^0_{ij}\, u^0 \otimes \check{d}^i \otimes \check{d}^j \,, \qquad \text{with} \qquad G^0_{ij} = \tfrac{m}{\hbar_0}\, g_{ij} \in \mathrm{Map}(\boldsymbol{E},\ I\!R) \,.$$
$$\bar{G} = G^{ij}_0\, u_0 \otimes \partial_i \otimes \partial_j \,, \qquad \text{with} \qquad G^{ij}_0 = \tfrac{\hbar_0}{m}\, g^{ij} \in \mathrm{Map}(\boldsymbol{E},\ I\!R) \,. \square$$

2.1.3. Proposition. The spacelike metric g and the spacetime orientation naturally yield a scaled *spacelike volume form*

$$\eta : \boldsymbol{E} \to \mathbb{L}^n \otimes \Lambda^n V^* \boldsymbol{E} \,,$$

with coordinate expression

$$\eta = \sqrt{|g|}\, \check{d}^1 \wedge \ldots \wedge \check{d}^n \,.$$

Moreover, the time form and the spacelike volume form yield the *spacetime volume form*

$$\upsilon := dt \wedge \eta : \boldsymbol{E} \to (\mathbb{T} \otimes \mathbb{L}^n) \otimes \Lambda^{n+1} T^* \boldsymbol{E} \,,$$

with coordinate expression

$$\upsilon \equiv u_0 \otimes \upsilon^0 \,,$$

where we have set

$$\upsilon^0 := \sqrt{|g|}\, u_0 \otimes d^0 \wedge d^1 \wedge \ldots \wedge d^n \,. \square$$

2.1.4. Proposition. The spacelike metric g (or, equivalently, the re–scaled spacelike metric G) naturally yield the fibre–wise Riemannian connection

$$\varkappa : V\boldsymbol{E} \to V^* \boldsymbol{E} \underset{V\boldsymbol{E}}{\otimes} VV\boldsymbol{E} \,,$$

with coordinate expression

$$\varkappa_i{}^h{}_j = -\tfrac{1}{2}\, g^{hk}\, (\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij})\,.\square$$

2.1.5. Corollary. We obtain the fibre–wise curvature tensor of $\varkappa$

$$R[\varkappa] := \tfrac{1}{2}\,[\varkappa,\, \varkappa] : V\boldsymbol{E} \to \Lambda^2 V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E}\,,$$

where $[\,,\,]$ denotes the fibre–wise Frölicher–Nijenhuis bracket, with coordinate expression

$$R[\varkappa] = (\partial_i \varkappa_j{}^h{}_k + \varkappa_i{}^p{}_k\, \varkappa_j{}^h{}_p)\, \dot{x}^k\, (\check{d}^i \wedge \check{d}^j \otimes \partial_h)\,.\square$$

2.1.6. Corollary. We obtain the fibre–wise Ricci tensor

$$R_{\mathrm{Ricci}} : \boldsymbol{E} \to V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^*\boldsymbol{E}$$

and the fibre–wise scalar curvature of $\varkappa$

$$\rho[\varkappa] := \langle \bar{G},\, R_{\mathrm{Ricci}} \rangle : \boldsymbol{E} \to \mathbb{T}^* \otimes I\!\!R\,.\square$$

Lie derivatives of the metric

2.1.7. Proposition. For each $X \in \mathrm{Sec}(\boldsymbol{E},\, T\boldsymbol{E})$, we obtain the Lie derivative

$$L[X]\,\bar{G} \in \mathrm{Sec}(\boldsymbol{E},\, \mathbb{T}^* \otimes V\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E})\,,$$

with coordinate expression

$$L[X]\,\bar{G} = (X^\lambda\, \partial_\lambda G^{ij}_0 - G^{hj}_0\, \partial_h X^i - G^{ih}_0\, \partial_h X^j)\, u^0 \otimes \partial_i \otimes \partial_j\,.$$

Moreover, for each $X \in \mathrm{Pro}(\boldsymbol{E},\, T\boldsymbol{E})$, we obtain the Lie derivative

$$L[X]\, G \in \mathrm{Sec}(\boldsymbol{E},\, \mathbb{T} \otimes V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^*\boldsymbol{E})\,,$$

with coordinate expression

$$L[X]\, G = (X^\lambda\, \partial_\lambda G^0_{ij} + G^0_{hj}\, \partial_i X^h + G^0_{ih}\, \partial_j X^h)\, u_0 \otimes \check{d}^i \otimes \check{d}^j\,.$$

2.1.8. Corollary. Let $X \in \mathrm{Pro}(\boldsymbol{E},\, T\boldsymbol{E})$ and $f \in \mathrm{Map}(\boldsymbol{E},\, I\!\!R)$. Then the following conditions are equivalent:

$$L[X]\,\bar{G} = f\,\bar{G}$$
$$L[X]\, G = -f\, G\,.$$

PROOF. The equality

$$X^\lambda \, \partial_\lambda G_0^{ij} - G_0^{hj} \, \partial_h X^i - G_0^{ih} \, \partial_h X^j = f \, G_0^{ij}$$

is equivalent to the equality

$$G^0_{pi} \, G^0_{qj} \, (X^\lambda \, \partial_\lambda G_0^{ij} - G_0^{hj} \, \partial_h X^i - G_0^{ih} \, \partial_h X^j) = G^0_{pi} \, G^0_{qj} \, (f \, G_0^{ij})$$

i.e. to the equality

$$-X^\lambda \, \partial_\lambda G^0_{pq} - G^0_{hq} \, \partial_p X^h - G^0_{ph} \, \partial_q X^h = f \, G^0_{pq} \, . \square$$

2.1.9. Definition. For each $X \in \mathrm{Sec}(\boldsymbol{E}, T\boldsymbol{E})$, we define the *spacetime divergence*

$$\mathrm{div}_\upsilon X \in \mathrm{Map}(\boldsymbol{E}, I\!R)$$

by the equality

$$L[X] \, \upsilon = (\mathrm{div}_\upsilon X) \, \upsilon$$

and the *timelike divergence*

$$\mathrm{div}_{\mathrm{dt}} X \in \mathrm{Map}(\boldsymbol{E}, I\!R)$$

by the equality

$$L[X] \, dt = (\mathrm{div}_{\mathrm{dt}} X) \, dt \, . \square$$

2.1.10. Definition. For each $X \in \mathrm{Pro}(\boldsymbol{E}, T\boldsymbol{E})$, we define the *spacelike divergence*

$$\mathrm{div}_\eta X \in \mathrm{Map}(\boldsymbol{E}, I\!R)$$

by the equality

$$L[X] \, \eta = (\mathrm{div}_\eta X) \, \eta \, . \square$$

2.1.11. Proposition. For each $X \in \mathrm{Sec}(\boldsymbol{E}, T\boldsymbol{E})$, we have the coordinate expressions

$$\mathrm{div}_\upsilon X = \frac{\partial_0 (X^0 \sqrt{|g|})}{\sqrt{|g|}} + \frac{\partial_j (X^j \sqrt{|g|})}{\sqrt{|g|}}$$
$$\mathrm{div}_{\mathrm{dt}} X = \partial_0 X^0 \, .$$

Moreover, for each $X \in \mathrm{Pro}(\boldsymbol{E}, T\boldsymbol{E})$, we have the coordinate expression

$$\mathrm{div}_\eta X = X^0 \, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} + \frac{\partial_j (X^j \sqrt{|g|})}{\sqrt{|g|}} \, . \square$$

2.1.12. Corollary. For each $X \in \mathrm{Pro}(\boldsymbol{E}, T\boldsymbol{E})$, we obtain

$$\mathrm{div}_\upsilon X = \mathrm{div}_{\mathrm{dt}} X + \mathrm{div}_\eta X \,.$$

Hence, for each $X \in \mathrm{Time}(\boldsymbol{E}, T\boldsymbol{E})$, we obtain

$$\mathrm{div}_\upsilon X = \mathrm{div}_\eta X \,. \square$$

2.1.13. Corollary. For each $X \in \mathrm{Pro}(\boldsymbol{E}, T\boldsymbol{E})$, we obtain

$$\mathrm{div}_\eta X = \tfrac{1}{2} \langle \bar{G},\, L[X]\, G\rangle \,. \square$$

2.1.14. Definition. A vector field $X \in \mathrm{Pro}(\boldsymbol{E}, T\boldsymbol{E})$ is called *conformal unimodular*, or *unimodular*, if we have, respectively,

$$d(\mathrm{div}_\eta X) = 0 \,, \qquad \text{or} \qquad \mathrm{div}_\eta X = 0 \,. \square$$

We denote the sheaves of conformal unimodular and unimodular vector fields by

$$\widetilde{\mathrm{Unim}}(\boldsymbol{E}, T\boldsymbol{E}) \subset \mathrm{Pro}(\boldsymbol{E}, T\boldsymbol{E}) \qquad \text{and} \qquad \mathrm{Unim}(\boldsymbol{E}, T\boldsymbol{E}) \subset \widetilde{\mathrm{Unim}}(\boldsymbol{E}, T\boldsymbol{E}) \,.$$

2.1.15. Lemma. For each $X, \bar{X} \in \mathrm{Pro}(\boldsymbol{E}, T\boldsymbol{E})$, we have

$$\mathrm{div}_\eta([X, \bar{X}]) = X.\, \mathrm{div}_\eta \bar{X} - \bar{X}.\, \mathrm{div}_\eta X \,. \square$$

2.1.16. Proposition. The sheaves $\widetilde{\mathrm{Unim}}(\boldsymbol{E}, T\boldsymbol{E})$ and $\mathrm{Unim}(\boldsymbol{E}, T\boldsymbol{E})$ are closed with respect to the Lie bracket. $\square$

The spacelike metric g yields a Riemannian connection of the fibres of $\boldsymbol{E}$ and the scalar curvature ρ.

2.1.3 Gravitational and electromagnetic fields

We assume *gravitational field* to be a torsion free linear connection of the vector bundle $T\boldsymbol{E} \to \boldsymbol{E}$

$$K^\natural : T\boldsymbol{E} \to T^*\boldsymbol{E} \underset{T\boldsymbol{E}}{\otimes} TT\boldsymbol{E} \,,$$

such that

$$\nabla^\natural dt = 0 \,, \qquad \nabla^\natural g = 0$$

and such that the curvature tensor $R^\natural$ fulfills the condition

$$R^{\natural j}_{0\lambda}{}^{i}{}_{\mu} = R^{\natural i}_{0\mu}{}^{j}{}_{\lambda} \,.$$

We assume *electromagnetic field* to be a closed scaled 2-form

$$f : \boldsymbol{E} \to (\mathbb{L}^{\frac{1}{2}} \otimes \mathbb{M}^{\frac{1}{2}}) \otimes \Lambda^2 T^* \boldsymbol{E} \,,$$

Given a particle of a charge q, it is convenient to consider the re–scaled electromagnetic field

$$F := \tfrac{q}{\hbar} f : \boldsymbol{E} \to \Lambda^2 T^* \boldsymbol{E} \,.$$

2.1.17. Proposition. The electromagnetic field F can be "added", in a covariant way, to the gravitational connection yielding a *(total) spacetime connection*

$$K = K^{\natural} + \tfrac{1}{2} \, (dt \otimes \hat{F} + \hat{F} \otimes dt) \,,$$

where $\hat{F} = G^{ih}_0 \, F_{hj} \, u^0 \otimes \partial_i \otimes d^j$.

The total K still fulfills the properties that we have assumed for $K^{\natural}$.

2.1.18. Proposition. [JadMod93] The coordinate expression of K is

$$\begin{aligned} K_{k}{}^{0}_{j}{}_{h} = K_{h}{}^{0}_{j}{}_{k} &= -\tfrac{1}{2} \, (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk}) \\ K_{0}{}^{0}_{j}{}_{h} = K_{h}{}^{0}_{j}{}_{0} &= -\tfrac{1}{2} \, (\partial_h A_j - \partial_j A_h + \partial_0 G^0_{hj}) \\ K_{0}{}^{0}_{j}{}_{0} &= -(\partial_0 A_j - \partial_j A_0) \,, \end{aligned}$$

where $A[o] = A_0 \, d^0 + A_j \, d^j$ is any local potential of the closed 2–form

$$\Phi[o] := \mathrm{Ant}(G^{\flat}(\nabla o)) \in \mathrm{Sec}(\boldsymbol{E}, \, \Lambda^2 T^* \boldsymbol{E}) \,,$$

where o is the observer associated with the chosen spacetime chart. $\square$

2.1.4 Examples of spacetime

As a particular example of spacetime we shall consider a *special spacetime*, constituted by an affine spacetime $\boldsymbol{E}$, equipped with an affine time map t and the flat gravitational field $K^{\natural}$ induced by the affine structure of $\boldsymbol{E}$.

2.1.5 Classical phase space

The *phase space* is defined to be the first jet space $t^1 : J_1\boldsymbol{E} \to \boldsymbol{T}$ of sections of spacetime [139]. We denote fibred charts of the phase space by (x^0, x^i, x^i_0).

We recall that $J_1\boldsymbol{E} \to \boldsymbol{E}$ is an affine bundle associated with the vector bundle $\mathbb{T}^* \otimes V\boldsymbol{E}$.

Hence, the vertical space of $J_1\boldsymbol{E}$ with respect to $\boldsymbol{E}$ turns out to be

$$V_{\boldsymbol{E}} J_1 \boldsymbol{E} = \mathbb{T}^* \otimes V \boldsymbol{E} \,.$$

Moreover, we obtain the natural tensor

$$\nu : J_1\boldsymbol{E} \to \mathbb{T} \otimes (V^*\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} V J_1\boldsymbol{E}) \,,$$

with coordinate expression

$$\nu = u_0 \otimes \check{d}^i \otimes \partial_i^0 \,.$$

We recall the natural *contact maps*

$$д : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E} \qquad \text{and} \qquad \theta : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E} \,,$$

with coordinate expressions

$$д = u^0 \otimes (\partial_0 + x_0^i\,\partial_i) \qquad \text{and} \qquad \theta = \partial_i \otimes (d^i - x_0^i\,d^0) \,.$$

We shall be involved with the Lie subalgebras

$$\begin{aligned} \mathrm{Pro}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E}) &\subset \mathrm{Sec}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E}) \,, \\ \mathrm{Time}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E}) &\subset \mathrm{Pro}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E}) \end{aligned}$$

of vector fields of $J_1\boldsymbol{E}$ which are projectable on $\boldsymbol{T}$ and whose time component is constant, respectively.

2.1.19. Proposition. For each $X \in \mathrm{Pro}(\boldsymbol{E},\, T\boldsymbol{E})$, we have the *holonomic prolongation*

$$X_{(1)} \in \mathrm{Pro}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E}) \,,$$

with coordinate expression

$$X_{(1)} = X^0\,\partial_0 + X^i\,\partial_i + (\partial_0 X^i + \partial_j X^i\,x_0^j - \partial_0 X^0\,x_0^i)\,\partial_i^0 \,.$$

Moreover, the map $\mathrm{Pro}(\boldsymbol{E},\, T\boldsymbol{E}) \to \mathrm{Pro}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$ is an injective Lie algebra morphism. □

For each $X^\uparrow \in \mathrm{Pro}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$, we define the *spacelike divergence*

$$\mathrm{div}\, X^\uparrow \in \mathrm{Map}(J_1\boldsymbol{E}, I\!R)$$

by the equality $L[X^\uparrow]\,\eta = \mathrm{div}\, X^\uparrow\,\eta$.

2.1.20. Proposition. We have the coordinate expression

$$\mathrm{div}\, X^\uparrow = X^0\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} + \frac{\partial_j(X^j\sqrt{|g|})}{\sqrt{|g|}} \,. \,\square$$

A vector field $X^{\uparrow} \in \mathrm{Pro}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ is called *conformal unimodular*, or *unimodular*, if we have, respectively,

$$d(\mathrm{div}\, X^{\uparrow}) = 0\,, \qquad \text{or} \qquad \mathrm{div}\, X^{\uparrow} = 0\,.$$

We denote the sheaves of conformal unimodular and unimodular vector fields by

$$\begin{aligned} &\tilde{\mathrm{Unim}}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) \subset \mathrm{Pro}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) \\ &\mathrm{Unim}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) \subset \tilde{\mathrm{Unim}}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})\,. \end{aligned}$$

2.1.21. Lemma. For each $X^{\uparrow}, \bar{X}^{\uparrow} \in \mathrm{Pro}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$, we have

$$\mathrm{div}([X^{\uparrow}, \bar{X}^{\uparrow}]) = X^{\uparrow}.\,\mathrm{div}\,\bar{X}^{\uparrow} - \bar{X}^{\uparrow}.\,\mathrm{div}\,X^{\uparrow}\,. \square$$

2.1.22. Proposition. The sheaves $\tilde{\mathrm{Unim}}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ and $\mathrm{Unim}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ are closed with respect to the Lie bracket. $\square$

2.1.6 Distinguished phase fields

2.1.23. Proposition. [56] The spacetime connection K yields in a covariant way a torsion free affine connection of the affine bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$, called *phase connection*,

$$\Gamma : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} TJ_1\boldsymbol{E}\,,$$

with coordinate expression

$$\Gamma_{\lambda}{}^{i}_{0} := \Gamma_{\lambda}{}^{i}_{0}{}^{0}_{j}\, x^{j}_{0} + \Gamma_{\lambda}{}^{i}_{0}{}^{0}_{0}\,, \qquad \text{with} \qquad \Gamma_{\lambda}{}^{i}_{0}{}^{0}_{\mu} = K_{\lambda}{}^{i}{}_{\mu}\,.$$

Conversely, the phase connection Γ characterizes the total spacetime connection K. $\square$

We have the following useful equalities.

2.1.24. Proposition. We have the following coordinate expressions

$$\begin{aligned} \Gamma^{i\,j}_{0\,0} - \Gamma^{j\,i}_{0\,0} &= -G^{ih}_{0} G^{jk}_{0} \left((\partial_h G^0_{kl} - \partial_k G^0_{hl})\, x^l_0 + \Phi_{hk} \right), \\ \Gamma_{ij} - \Gamma_{ji} &= -(\partial_i G^0_{jh} - \partial_j G^0_{ih})\, x^h_0 - \Phi_{ij} \end{aligned}$$

where

$$\Gamma^{hk}_{00} := G^{hl}_0\, \Gamma_{l}{}^{k}_{0}\,, \qquad \Gamma_{hk} := G^0_{kl}\, \Gamma_{h}{}^{l}_{0}\,. \square$$

2.1.25. Proposition. [56] The phase connection yields in a covariant way the *second order connection*

$$\gamma := д \lrcorner \Gamma : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E}\,,$$

with coordinate expression

$$\gamma = u^0 \otimes (\partial_0 + x^i_0\, \partial_i + \gamma_{00}{}^{i}\, \partial^0_i)\,,$$

where

$$\begin{aligned}\gamma_{00}{}^{i} &= \Gamma_{h0k}{}^{i0}\, x^h_0 x^k_0 + 2\,\Gamma_{h00}{}^{i0}\, x^h_0 + \Gamma_{000}{}^{i0} \\ &= -\, G^{ij}_0 \Big(\tfrac{1}{2}\, (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk})\, x^h_0\, x^k_0 + (\partial_0 G^0_{hj} + \Phi_{hj})\, x^h_0 + \Phi_{0j} \Big)\,.\end{aligned}$$

Conversely, the second order connection γ characterizes the phase connection Γ. □

2.1.26. Proposition. [56] The phase connection Γ and the spacelike metric G yield in a covariant way the *phase 2–form*

$$\Omega := \nu[\Gamma] \bar{\wedge} \theta : J_1\boldsymbol{E} \to \Lambda^2 J_1 T\boldsymbol{E}\,,$$

with coordinate expression

$$\Omega = G^0_{ij}\, (d^i_0 - \gamma_{00}{}^{i} d^0 - \Gamma_{h0}{}^{i}\, \theta^h) \wedge \theta^j\,,$$

where $\nu[\Gamma]$ is the vertical valued form associated with Γ and $\bar{\wedge}$ is the wedge product followed by a contraction through G.

Conversely, the phase 2-form Ω characterizes the spacelike metric G and the phase connection Γ. □

2.1.27. Proposition. For each observer o, we obtain

$$\Phi[o] = 2\, o^* \Omega\,.\ \square$$

2.1.28. Proposition. The phase connection Γ and the spacelike metric G yield in a covariant way the 2–vector

$$\Lambda[G, \Gamma] := \check{\Gamma}\, \bar{\wedge}\, \nu : J_1\boldsymbol{E} \to \Lambda^2 T J_1\boldsymbol{E}\,,$$

with coordinate expression

$$\Lambda[G, \Gamma] = G^{ij}_0\, (\partial_i + \Gamma_{i0}{}^{h}\, \partial^0_h) \wedge \partial^0_j\,,$$

where $\check{\Gamma} : J_1\boldsymbol{E} \to V^*\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} V J_1\boldsymbol{E}$ is the vertical restriction of Γ. □

2.1.29. Proposition. [56, 66] We have

$$dt \wedge \Omega^n \not\equiv 0\,, \qquad i(\gamma)\, \Omega = 0\,, \qquad d\Omega = 0\,, \qquad L[\gamma]\, \Lambda = 0\,, \qquad [\Lambda, \Lambda] = 0\,.\ \square$$

Additionally, we have that $i(\gamma)\, dt = 1$. Hence, $(J_1\boldsymbol{E},\, dt, \Omega)$ turns out to be a (scaled) cosymplectic manifold, γ the associated (scaled) Reeb vector field and Λ the associated 2–vector field.

2.1.7 Classical kinematics

A *motion* is defined to be a section $s \in \mathrm{Sec}(\boldsymbol{T}, \boldsymbol{E})$. The *absolute velocity* of a motion s is defined to be its first jet prolongation $j_1 s \in \mathrm{Sec}(\boldsymbol{T}, J_1\boldsymbol{E})$.

An observer can be regarded as a section $o \in \mathrm{Sec}(\boldsymbol{E}, J_1\boldsymbol{E})$.
An observer o yields the fibred morphism

$$\nabla[o] \in \mathrm{Fib}(J_1\boldsymbol{E},\, \mathbb{T}^* \otimes V\boldsymbol{E})$$

with coordinate expression

$$\nabla[o] = (x^i_0 - o^i_0)\, u^0 \otimes \partial_i\,.$$

Then, for each motion s, we obtain the *observed velocity*

$$\nabla[o]s := j_1 s - o \circ s \in \mathrm{Sec}(\boldsymbol{T},\, \mathbb{T}^* \otimes V\boldsymbol{E})\,.$$

We define the *kinetic momentum* and the *kinetic energy* to be the maps

$$\begin{aligned}
\mathcal{Q}[o] &:= G^\flat \circ \nabla[o] : J_1\boldsymbol{E} \to V^*\boldsymbol{E} \\
\mathcal{K}[o] &:= \frac{1}{2} G \circ (\nabla[o], \nabla[o]) : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{R}\,,
\end{aligned}$$

with expressions, in adapted coordinates,

$$\mathcal{Q}[o] = G^0_{ij} x^i_0 \check{d}^j \qquad \text{and} \qquad \mathcal{K}[o] = \tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0\, d^0\,.$$

2.1.8 Classical mechanics

We assume the *generalised Newton's equation*

$$\nabla[\gamma] j_1 s = 0$$

as equation of motion for classical dynamics.

A function $f \in \mathrm{Map}(J_1\boldsymbol{E},\, \mathbb{R})$ such that $\gamma.f = 0$ is said to be *conserved*. We denote the subsheaf of conserved functions by

$$\mathrm{Con}(J_1\boldsymbol{E},\, \mathbb{R}) \subset \mathrm{Map}(J_1\boldsymbol{E},\, \mathbb{R})\,.$$

We can also obtain the classical dynamics by a Lagrangian formalism according to a cohomological procedure in the following way [130].

2.1.30. Proposition. The phase 2-form Ω admits locally *horizontal potentials*

$$\Theta \in \mathrm{Sec}(J_1\boldsymbol{E}, T^*\boldsymbol{E})$$

(defined up to a closed form of the type $\alpha \in \mathrm{Sec}(\boldsymbol{E}, T^*\boldsymbol{E})$).

We have a first natural splitting of each horizontal potential induced by the contact structure of $J_1\boldsymbol{E}$.

2.1.31. Proposition. Each horizontal potential Θ splits, in a covariant way as,

$$\Theta = \mathcal{L}[\Theta] + \mathcal{P}[\Theta]\,,$$

through the t–horizontal component and the д–vertical component

$$\mathcal{L}[\Theta] := д \lrcorner \Theta \qquad \text{and} \qquad \mathcal{P}[\Theta] := \theta \lrcorner \Theta\,,$$

called *Lagrangian* and *momentum*, respectively.

Moreover, we obtain

$$D\mathcal{L}[\Theta] = \check{\mathcal{P}}[\Theta]\,.$$

Hence, each potential Θ turns out to be the *Poincaré–Cartan form* of the associated Lagrangian $\mathcal{L}[\Theta]$. □

On the other hand, given an observer o, we obtain a further splitting of each horizontal potential Θ.

2.1.32. Proposition. Let us consider an observer o.

Then, each horizontal potential Θ splits as

$$\Theta = -\mathcal{H}[\Theta, o] + \mathcal{V}[\Theta, o]\,,$$

through the t–horizontal component and the o–vertical component

$$\mathcal{H}[\Theta, o] := -o \lrcorner \Theta \qquad \text{and} \qquad \mathcal{V}[\Theta, o] := \nu[o] \lrcorner \Theta\,,$$

called observed *Hamiltonian* and observed *momentum*, respectively.

Moreover, we obtain

$$\mathcal{L}[\Theta] = -\mathcal{H}[\Theta, o] + д \lrcorner \mathcal{V}[\Theta, o]\,. \,\square$$

2.1.33. Definition. Let us consider an observer o.

We define the local *observed potential* associated with a horizontal potential Θ to be the form

$$A[\Theta, o] := o^*(\Theta) \in \mathrm{Sec}(\boldsymbol{E}, T^*\boldsymbol{E})\,. \,\square$$

2.1.34. Proposition. For each horizontal potential Θ and observer o, we obtain

$$A[\Theta, o] = \Theta \circ o\,.$$

Proof. It follows from the fact that Θ is a horizontal form. QED

2.1.35. Proposition. Let us consider an observer o.

Then, for each horizontal potential Θ, $A[\Theta, o]$ is a potential of the closed 2–form $\Phi[o] = 2\, o^*(\Omega) \in \mathrm{Sec}(\boldsymbol{E},\, \Lambda^2 T^*\boldsymbol{E})$, according to

$$2\, dA[\Theta, o] = \Phi[o]\,.$$

PROOF. We obtain

$$2\, dA[\Theta, o] := 2\, d(o^*\Theta) = 2\, o^*(d\Theta) = 2\, o^*\Omega := \Phi[o]\,.\ \text{QED}$$

2.1.36. Proposition. Let us consider an observer o and a horizontal potential Θ.

Then, we obtain

$$\begin{aligned} \mathcal{L}[\Theta] &= \mathcal{K}[o] + д \lrcorner A[\Theta, o] \\ \mathcal{H}[\Theta, o] &= \mathcal{K}[o] - o \lrcorner A[\Theta, o] \\ \mathcal{V}[\Theta, o] &= \nu[o] \lrcorner \Big(\mathcal{Q}[o] + \check{A}[\Theta, o] \Big)\,. \square \end{aligned}$$

2.1.37. Proposition. Let us consider an observer o, an adapted chart and a horizontal potential Θ.

We have the following coordinate expressions

$$\Theta = (-\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 + A_0)\, d^0 + (G^0_{ij}\, x^j_0 + A_i)\, d^i$$

$$\mathcal{L}[\Theta] = (\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 + A_i\, x^i_0 + A_0)\, d^0\,, \qquad \mathcal{P}[\Theta] = (G^0_{ij}\, x^j_0 + A_i)\, (d^i - x^i_0\, d^0)$$

and, in a chart adapted to o,

$$\mathcal{H}[\Theta, o] = (\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 - A_0)\, d^0\,, \qquad \mathcal{V}[\Theta, o] = (G^0_{ij}\, x^j_0 + A_i)\, d^i\,. \square$$

2.1.38. Proposition. Let us consider two observers o and $\bar{o} = o + v$ and a horizontal potential Θ.

Then, we obtain

$$A[\Theta, \bar{o}] = A[\Theta, o] - \tfrac{1}{2}\, G9v, v) + \nu[o] \lrcorner G^\flat(v)$$

i.e., in a chart adapted to o,

$$A[\Theta, \bar{o}] = A[\Theta, o] - \tfrac{1}{2}\, G^0_{ij}\, v^i_0\, v^j_0\, d^0 + G^0_{ij}\, v^j_0\, d^i\,.$$

PROOF. In a chart adapted to o, we have

$$\begin{aligned} A[\Theta, \bar{o}] &= \Theta \circ \bar{o} \\ &= \big((-\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 + A_0)\, d^0 + (G^0_{ij}\, x^j_0 + A_i)\, d^i \big) \circ \bar{o} \\ &= A_\lambda\, d^\lambda - \tfrac{1}{2}\, G^0_{ij}\, v^i_0\, v^j_0\, d^0 + G^0_{ij}\, v^j_0\, d^i\,.\ \text{QED} \end{aligned}$$

2.1.39. Proposition. The "t–horizontal" component of Ω turns out to be the fibred morphism

$$\mathcal{E} = G^{\flat}(\nabla[\gamma]) : J_2\boldsymbol{E} \to \mathbb{T}^* \otimes V^*\boldsymbol{E}\,.$$

Moreover, $\mathcal{E}$ turns out to be the Euler-Lagrange operator associated with the Lagrangian $\mathcal{L}(\Theta)$, for each horizontal potential Θ. $\square$

2.2 Hamiltonian stuff

The minimal axioms yield in a covariant way a distinguished lift of functions of classical phase space to vector fields of phase space, called the Hamiltonian lift. The vector field of phase space which preserve the cosymplectic structure turn out to be locally of the type of a Hamiltonian lift.

2.2.1 Musical morphisms

2.2.1. Lemma. We have the natural dual splittings

$$TJ_1\boldsymbol{E} = T_\gamma J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\oplus} VJ_1\boldsymbol{E} \qquad \text{and} \qquad T^*J_1\boldsymbol{E} = H^*J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\oplus} T^*_\gamma J_1\boldsymbol{E}\,,$$

given by

$$X^\uparrow = dt(X^\uparrow)\,\gamma + (X^\uparrow - dt(X^\uparrow)\,\gamma) \qquad \text{and} \qquad \phi^\uparrow = \phi^\uparrow(\gamma)\,dt + (\phi^\uparrow - \phi^\uparrow(\gamma)\,dt)\,,$$

where
- $VJ_1\boldsymbol{E} \subset TJ_1\boldsymbol{E}$ is the vertical subbundle (with respect to dt),
- $H^*J_1\boldsymbol{E} \subset T^*J_1\boldsymbol{E}$ is the horizontal subbundle generated by dt,
- $T_\gamma J_1\boldsymbol{E} \subset TJ_1\boldsymbol{E}$ is the subbundle generated by γ,
- $T^*_\gamma J_1\boldsymbol{E} \subset T^*J_1\boldsymbol{E}$ is the subbundle of forms which kill γ. $\square$

We define the *musical morphisms* to be the linear fibred morphisms

$$\Omega^\flat : \mathrm{Sec}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E}) \to \mathrm{Sec}(J_1\boldsymbol{E},\, T^*J_1\boldsymbol{E}) : X^\uparrow \mapsto i(X^\uparrow)\,\Omega$$
$$\Lambda^\sharp : \mathrm{Sec}(J_1\boldsymbol{E},\, T^*J_1\boldsymbol{E}) \to \mathrm{Sec}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E}) : \phi^\uparrow \mapsto i(\phi^\uparrow)\,\Lambda\,.$$

2.2.2. Lemma. The musical morphisms restrict to the mutually inverse linear fibred isomorphisms

$$\Omega^\flat : \mathrm{Sec}(J_1\boldsymbol{E},\, VJ_1\boldsymbol{E}) \to \mathrm{Sec}(J_1\boldsymbol{E},\, T^*_\gamma J_1\boldsymbol{E}) : X^\uparrow \mapsto i(X^\uparrow)\,\Omega$$
$$\Lambda^\sharp : \mathrm{Sec}(J_1\boldsymbol{E},\, T^*_\gamma J_1\boldsymbol{E}) \to \mathrm{Sec}(J_1\boldsymbol{E},\, VJ_1\boldsymbol{E}) : \phi^\uparrow \mapsto i(\phi^\uparrow)\,\Lambda\,. \square$$

2.2.3. Lemma. For each $X^\uparrow \in \mathrm{Sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ and $\phi^\uparrow \in \mathrm{Sec}(J_1\boldsymbol{E}, T^*J_1\boldsymbol{E})$, we obtain

$$(\Lambda^\sharp \circ \Omega^\flat)(X^\uparrow) = X^\uparrow - dt(X^\uparrow)\,\gamma \qquad \text{and} \qquad (\Omega^\flat \circ \Lambda^\sharp)(\phi^\uparrow) = \phi^\uparrow - \phi^\uparrow(\gamma)\,dt$$
$$\Lambda^\sharp(\phi^\uparrow) = (\Omega^\flat)^{-1}(\phi^\uparrow - \phi^\uparrow(\gamma)\,dt) \qquad \text{and} \qquad \Omega^\flat(X^\uparrow) = (\Lambda^\sharp)^{-1}(X^\uparrow - dt(X^\uparrow)\,\gamma)\,.\square$$

2.2.4. Theorem. *The natural dual splittings*

$$TJ_1\boldsymbol{E} = T_\gamma J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\oplus} VJ_1\boldsymbol{E} \qquad \textit{and} \qquad T^*J_1\boldsymbol{E} = H^*J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\oplus} T^*_\gamma J_1\boldsymbol{E}\,,$$

are given by

$$X^\uparrow = dt(X^\uparrow)\,\gamma + (\Lambda^\sharp \circ \Omega^\flat)\,(X^\uparrow) \qquad \textit{and} \qquad \phi^\uparrow = \phi^\uparrow(\gamma)\,dt + (\Omega^\flat \circ \Lambda^\sharp)\,(\phi^\uparrow)\,.\square$$

Given a *time scale* $\tau \in \mathrm{Map}(J_1\boldsymbol{E}, \bar{\mathbb{T}})$, we define the subbundle

$$T_\tau J_1\boldsymbol{E} \subset TJ_1\boldsymbol{E}$$

consisting of vectors whose time component is given by τ.

2.2.5. Lemma. Given a *time scale* $\tau \in \mathrm{Map}(J_1\boldsymbol{E}, \bar{\mathbb{T}})$, we obtain the mutually inverse affine fibred isomorphism

$$\Omega^\flat_\tau : \mathrm{Sec}(J_1\boldsymbol{E}, T_\tau J_1\boldsymbol{E}) \to \mathrm{Sec}(J_1\boldsymbol{E}, T^*_\gamma J_1\boldsymbol{E}) : X^\uparrow \mapsto i(X^\uparrow)\,\Omega$$
$$\Lambda^\sharp_\tau : \mathrm{Sec}(J_1\boldsymbol{E}, T^*_\gamma J_1\boldsymbol{E}) \to \mathrm{Sec}(J_1\boldsymbol{E}, T_\tau J_1\boldsymbol{E}) : \phi^\uparrow \mapsto \gamma(\tau) + i(\phi^\uparrow)\,\Lambda\,.\square$$

2.2.6. Lemma. Let $X^\uparrow, \bar{X}^\uparrow \in \mathrm{Sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ and $\phi^\uparrow, \bar{\phi}^\uparrow \in \mathrm{Sec}(J_1\boldsymbol{E}, T^*J_1\boldsymbol{E})$. Then, we have the following equivalences:

$$X^\uparrow = \bar{X}^\uparrow \quad \Leftrightarrow \quad dt(X^\uparrow) = dt(\bar{X}^\uparrow)\,, \quad \Omega^\flat(X^\uparrow) = \Omega^\flat(\bar{X}^\uparrow)$$
$$\phi^\uparrow = \bar{\phi}^\uparrow \quad \Leftrightarrow \quad \phi^\uparrow(\gamma) = \phi^\uparrow(\bar{\gamma})\,, \quad \Lambda^\sharp(\phi^\uparrow) = \Lambda^\sharp(\bar{\phi}^\uparrow)\,.$$

Proof. If

$$dt(X^\uparrow) = dt(\bar{X}^\uparrow)\,, \quad \Omega^\flat(X^\uparrow) = \Omega^\flat(\bar{X}^\uparrow)\,,$$

then

$$0 = \Lambda^\sharp(\Omega^\flat(X^\uparrow)) - \Lambda^\sharp(\Omega^\flat(\bar{X}^\uparrow)) = X^\uparrow - dt(X)\,\gamma - \bar{X}^\uparrow + dt(\bar{X})\,\gamma = X^\uparrow - \bar{X}^\uparrow\,.$$

If

$$\phi^\uparrow(\gamma) = \bar{\phi}^\uparrow(\gamma)\,, \quad \Lambda^\sharp(\phi^\uparrow) = \Lambda^\sharp(\bar{\phi}^\uparrow)\,,$$

then

$$0 = \Omega^\flat(\Lambda^\sharp(\phi^\uparrow)) - \Omega^\flat(\Lambda^\sharp(\phi^\uparrow)) = \phi^\uparrow - \phi^\uparrow(\gamma)\,dt - \bar{\phi}^\uparrow + \bar{\phi}^\uparrow(\gamma)\,dt = \phi^\uparrow - \bar{\phi}^\uparrow\,.\square$$

2.2.2 Hamiltonian lift of functions

2.2.7. Definition. The *vertical Hamiltonian lift* of a function $f \in \mathrm{Map}(J_1\boldsymbol{E}, I\!R)$ is defined to be the vector field

$$\Lambda^\sharp(df) \in \mathrm{Sec}(J_1\boldsymbol{E}, V J_1\boldsymbol{E}) .$$

2.2.8. Proposition. For each $f \in \mathrm{Map}(J_1\boldsymbol{E}, I\!R)$, we have the coordinate expression

$$\Lambda^\sharp(df) = -G^{ij}_0 \, \partial^0_j f \, \partial_i + \Big(G^{ij}_0 \, \partial_j f + (\Gamma^{i\,j}_{00} - \Gamma^{j\,i}_{00}) \, \partial^0_j f \Big) \, \partial^0_i \, . \square$$

2.2.9. Definition. Given a *time scale* $\tau \in \mathrm{Map}(J_1\boldsymbol{E}, \bar{\mathbb{T}})$, we define the *$\tau$–Hamiltonian lift* of a function $f \in \mathrm{Map}(J_1\boldsymbol{E}, I\!R)$ to be the vector field

$$X^\uparrow_{\mathrm{Ham}}[\tau, f] := \gamma(\tau) + \Lambda^\sharp(df) \in \mathrm{Sec}(J_1\boldsymbol{E}, T J_1\boldsymbol{E}) \, . \square$$

2.2.10. Proposition. For each $\tau \in \mathrm{Map}(J_1\boldsymbol{E}, \bar{\mathbb{T}})$ and $f \in \mathrm{Map}(J_1\boldsymbol{E}, I\!R)$, we have the coordinate expression

$$X^\uparrow_{\mathrm{Ham}}[\tau, f] = \tau^0 \, \partial_0 + (\tau^0 \, x^i_0 - G^{ij}_0 \, \partial^0_j f) \, \partial_i + \Big(\tau^0 \, \gamma_{00}{}^i + G^{ij}_0 \, \partial_j f + (\Gamma^{i\,j}_{00} - \Gamma^{j\,i}_{00}) \, \partial^0_j f \Big) \, \partial^0_i \, . \square$$

2.2.11. Lemma. For every $f \in \mathrm{Map}(J_1\boldsymbol{E}, I\!R)$, we have the distinguished time scale, called the *time component* of f,

$$f'' := \tfrac{1}{n} \, \langle \bar{G} \, , \, D^2 f \rangle \in \mathrm{Map}(J_1\boldsymbol{E}, \bar{\mathbb{T}}) \, ,$$

where

$$D^2 f \in \mathrm{Fib}(J_1\boldsymbol{E}, \;\; \mathbb{T} \otimes \mathbb{T} \otimes V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^*\boldsymbol{E})$$

is the second fibre derivative of f with respect to the affine fibre of the bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$. Thus, we have the coordinate expression

$$f'' \;\equiv\; f^0 \, u_0 = \tfrac{1}{n} \, G^{ij}_0 \, \partial^0_i \partial^0_j f \, u_0 \, . \square$$

2.2.12. Definition. We define the *Hamiltonian lift* of $f \in \mathrm{Map}(J_1\boldsymbol{E}, I\!R)$ to be the vector field

$$X^\uparrow_{\mathrm{Ham}}[f] := X^\uparrow_{\mathrm{Ham}}[f'', f] = \gamma(f'') + \Lambda^\sharp(df) \in \mathrm{Sec}(J_1\boldsymbol{E}, T J_1\boldsymbol{E}) \, . \square$$

2.2.13. Example. If $f \in \mathrm{Map}(\boldsymbol{E}, I\!R)$, then we obtain

$$X^\uparrow_{\mathrm{Ham}}[f] = \mathrm{grad}\, f := G^\sharp(\check{d} f) \in \mathrm{Sec}(\boldsymbol{E}, \, \mathbb{T}^* \otimes V\boldsymbol{E}) \subset \mathrm{Sec}(J_1\boldsymbol{E}, \; V_0 J_1\boldsymbol{E}) \, . \square$$

We denote the subsheaf of Hamiltonian lifts of functions by

$$\mathrm{Ham}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) \subset \mathrm{Sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) .$$

2.2.3 Classical symmetries

We classify the vector fields of the phase space which are infinitesimal symmetries of the time 1–form and of the phase 2–form.

2.2.14. Proposition. The i.s.'s

$$X \in \mathrm{Sec}(\boldsymbol{E}, T\boldsymbol{E}) \qquad \text{and} \qquad X^{\uparrow} \in \mathrm{Sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$$

of dt are the vector fields with constant time component

$$dt(X) \in \bar{\mathbb{T}} \qquad \text{and} \qquad dt(X^{\uparrow}) \in \bar{\mathbb{T}} ,$$

respectively

Proof. In fact, we have

$$L[X]\, dt = d\, i(X)\, dt \qquad \text{and} \qquad L[X^{\uparrow}]\, dt = d\, i(X^{\uparrow})\, dt . \text{QED}$$

2.2.15. Theorem. *The i.s.'s*

$$X^{\uparrow} \in \mathrm{Sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$$

of Ω are the vector fields of the local type

$$X^{\uparrow} = X^{\uparrow}_{\mathrm{Ham}}[\tau, f] , \qquad \textit{with} \qquad \tau \in \mathrm{Map}(J_1\boldsymbol{E}, \bar{\mathbb{T}}) , \quad f \in \mathrm{Con}(J_1\boldsymbol{E}, I\!\!R) ,$$

where, for each $X^{\uparrow}$, the time component is given by $\tau := dt(X^{\uparrow})$ and the function f is defined up to a constant.

Proof. Let us consider any $X^{\uparrow} \in \mathrm{Sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ and set $\tau := dt(X^{\uparrow}) \in \mathrm{Map}(J_1\boldsymbol{E}, \bar{\mathbb{T}})$. Then, $X^{\uparrow}$ can be uniquely written as

$$X^{\uparrow} = \gamma(\tau) + \bar{X}^{\uparrow} , \qquad \text{with} \qquad \bar{X}^{\uparrow} \in \mathrm{Sec}(J_1\boldsymbol{E}, VJ_1\boldsymbol{E}) .$$

Moreover, we have

$$L[\gamma(\tau)]\, \Omega = 0 .$$

Furthermore,

$$L[\bar{X}^{\uparrow}]\, \Omega = 0$$

if and only if

$$di(\bar{X}^{\uparrow})\,\Omega = 0\,,$$

i.e. if and only if locally

$$i(\bar{X}^{\uparrow})\,\Omega = df, \qquad \text{with} \qquad \gamma.f = 0\,,$$

i.e. if and only if locally

$$\bar{X}^{\uparrow} = \Lambda^{\sharp}(df)\,, \qquad \text{with} \qquad \gamma.f = 0\,.\,\text{QED}$$

2.2.16. Corollary. The i.s.'s

$$X^{\uparrow} \in \mathrm{Sec}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$$

of dt and Ω are the vector fields of the local type

$$X^{\uparrow} = X^{\uparrow}_{\mathrm{Ham}}[\tau, f]\,, \qquad \text{with} \qquad \tau \in \bar{\mathbb{T}}\,, \quad f \in \mathrm{Con}(J_1\boldsymbol{E},\, I\!\!R)\,,$$

where, for each $X^{\uparrow}$, the time component is given by $\tau := dt(X^{\uparrow})$ and the function f is defined up to a constant. □

2.2.17. Proposition. The subsheaf

$$\mathrm{Sym}_{dt}(\boldsymbol{E},\, T\boldsymbol{E}) \subset \mathrm{Sec}(\boldsymbol{E},\, T\boldsymbol{E})$$

is a Lie subalgebra.

The subsheaves

$$\begin{aligned} \mathrm{Sym}_{dt}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E}) &\subset \mathrm{Sec}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})\,, \\ \mathrm{Sym}_{\Omega}((J_1\boldsymbol{E},\, TJ_1\boldsymbol{E}) &\subset \mathrm{Sec}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})\,, \\ \mathrm{Sym}_{(dt,\Omega)}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E}) &\subset \mathrm{Sec}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E}) \end{aligned}$$

are Lie subalgebras.

Proof. They are particular cases of the general Proposition 1.4.2. QED

2.2.4 Poisson Lie algebra

2.2.18. Definition. We define the *Poisson bracket* to be the map

$$\mathrm{Map}(J_1\boldsymbol{E},\, I\!\!R) \times \mathrm{Map}(J_1\boldsymbol{E},\, I\!\!R) \to \mathrm{Map}(J_1\boldsymbol{E},\, I\!\!R) : (f, g) \mapsto \{f,\, g\} := i(df \wedge dg)\,\Lambda\,. \,\square$$

2.2.19. Proposition. For each $f, g \in \mathrm{Map}(J_1\boldsymbol{E}, I\!\!R)$, we have the coordinate expression

$$
\begin{aligned}
\{f, g\} &= G^{ij}_0\,(\partial_i f\,\partial^0_j g - \partial_i g\,\partial^0_j f) - (\Gamma^{i\,j}_{00} - \Gamma^{j\,i}_{00})\,\partial^0_i f\,\partial^0_j g \\
&= G^{ij}_0\,(\partial_i f\,\partial^0_j g - \partial_i g\,\partial^0_j f) + G^{ih}_0 G^{jk}_0 \Big((\partial_h G^0_{kl} - \partial_k G^0_{hl})\,x^l_0 + \Phi_{hk}\Big)\,\partial^0_i f\,\partial^0_j g\,.\,\square
\end{aligned}
$$

2.2.20. Lemma. For each $f, g \in \mathrm{Map}(J_1\boldsymbol{E}, I\!\!R)$, we have

$$\gamma.\{f, g\} = \{\gamma.f, g\} + \{f, \gamma.g\}\,.$$

Proof. The equality

$$L[\gamma]\Lambda \;\equiv\; [\gamma, \Lambda] = 0$$

yields

$$
\begin{aligned}
L[\gamma]\,\{f,\, g\} &:= L[\gamma]\,\big(\Lambda^\sharp(df,\, dg)\big) = \Lambda^\sharp(L[\gamma]\,df,\, dg) + \Lambda^\sharp(df,\, L[\gamma]\,dg) \\
&= \Lambda^\sharp(dL[\gamma]\,f,\, dg) + \Lambda^\sharp(df,\, dL[\gamma]\,g) = \{\gamma.f, g\} + \{f, \gamma.g\}\,.\ \mathrm{QED}
\end{aligned}
$$

2.2.21. Proposition. The sheaf $\mathrm{Map}(J_1\boldsymbol{E}, I\!\!R)$ turns out to be an $I\!\!R$–Lie algebra through the Poisson bracket. Moreover, the map

$$\mathrm{Map}(J_1\boldsymbol{E},\, I\!\!R) \to \mathrm{Sec}(J_1\boldsymbol{E},\, V J_1\boldsymbol{E}) : f \to \Lambda^\sharp(df)$$

turns out to be a morphism of Lie algebras. Its kernel is $\mathrm{Map}(\boldsymbol{T}, I\!\!R)$.

Proof. Clearly, $\mathrm{Map}(J_1\boldsymbol{E}, I\!\!R)$ is an algebra.

First, let us prove that the vertical Hamiltonian lift is a morphism of algebras. For this, it is sufficient to prove that, for each $f, g \in \mathrm{Map}(J_1\boldsymbol{E}, I\!\!R)$,

$$\Omega^\flat\big(\Lambda^\sharp(d\{f, g\})\big) = \Omega^\flat\big([\Lambda^\sharp(df),\, \Lambda^\sharp(dg)]\big)\,.$$

In fact, we have

$$\Omega^\flat\big(\Lambda^\sharp(d\{f,\, g\})\big) = d\{f,\, g\} - \gamma.\{f,\, g\}$$

and

$$
\begin{aligned}
\Omega^\flat\big([\Lambda^\sharp(df),\, \Lambda^\sharp(dg)]\big) &= L[\Lambda^\sharp(df)]\, i(\Lambda^\sharp(dg))\,\Omega - i(\Lambda^\sharp(dg))\, L[\Lambda^\sharp(df)]\,\Omega \\
&= L[\Lambda^\sharp(df)]\,(dg - \gamma.g) - i(\Lambda^\sharp(dg))\, d(df - \gamma.f) \\
&= di(\Lambda^\sharp(df))\, dg - i(\Lambda^\sharp(df))\, d\gamma.g + i(\Lambda^\sharp(dg))\, d\gamma.f \\
&= d\{f, g\} - \{f,\, \gamma.g\} + \{g,\, \gamma.f\} \\
&= d\{f, g\} - \{f, \gamma.g\} - \{\gamma.f,\, g\} \\
&= d\{f, g\} - \gamma.\{f, g\}\,.
\end{aligned}
$$

Then, let us prove that the Poisson bracket fulfills the Jacobi property.

In fact, for each $f, g \in \mathrm{Map}(J_1\boldsymbol{E},\, I\!R)$, we have

$$\begin{aligned}
\{\{f,g\},h\} &= \Lambda^\sharp(d\{f,g\}).h \\
&= [\Lambda^\sharp(df), \Lambda^\sharp(dg)].h \\
&= \Lambda^\sharp(df).(\Lambda^\sharp(dg).h) - \Lambda^\sharp(dg).(\Lambda^\sharp(df).h) \\
&= \{f,\{g,h\}\} - \{g,\{f,h\}\} \\
&= -\{\{h,f\},g\} - \{\{g,h\},f\}\,.
\end{aligned}$$

It follows from the coordinate expression that the kernel is $\mathrm{Map}(\boldsymbol{T}, I\!R)$. QED

2.3 Lie algebra of special quadratic functions

The minimal axioms yield in a covariant way a distinguished algebra of functions of phase space, called the algebra of special functions, which is not a Poisson subalgebra. It turns out that the Hamiltonian lifts which project over a vector field of spacetime are necessarily of the type of a Hamiltonian lift of a special function. Moreover, the map which associates with a special function the projectable Hamiltonian lift is a morphism of Lie algebras.

2.3.1 Special quadratic functions

2.3.1. Definition. A *special quadratic function* is defined to be a function

$$f \in \mathrm{Map}(J_1\boldsymbol{E},\, I\!R)\,,$$

such that

$$D^2 f = \tau \otimes G\,, \qquad \text{with} \qquad \tau \in \mathrm{Map}(\boldsymbol{E},\, \bar{\mathbb{T}})\,. \square$$

Clearly, if $f \in \mathrm{Map}(J_1\boldsymbol{E},\, I\!R)$ is a special quadratic function, then we obtain $\tau = f''$, hence

$$D^2 f = f'' \otimes G\,, \qquad \text{with} \qquad f'' \in \mathrm{Map}(\boldsymbol{E},\, \bar{\mathbb{T}})\,.$$

The subsheaf of special quadratic functions is denoted by

$$\mathrm{Spec}(J_1\boldsymbol{E},\, I\!R) \subset \mathrm{Map}(J_1\boldsymbol{E},\, I\!R)\,.$$

We stress that the definition of special quadratic function involves only on the time fibring and the spacelike metric; indeed, it does not involve Ω.

Moreover, we shall be involved with the distinguished subsheaves related to the affine structure of the bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$.

2.3.2. Definition. We define the following subsheaves:

- the sheaf

$$\mathrm{Quan}(J_1\boldsymbol{E},\,I\!\!R) \subset \mathrm{Spec}(J_1\boldsymbol{E},\,I\!\!R)$$

consisting of functions, called *quantisable*, whose time component $f'' \in \mathrm{Map}(\boldsymbol{T},\,\bar{\mathbb{T}})$ depends only on $\boldsymbol{T}$;

- the sheaf

$$\mathrm{Time}(J_1\boldsymbol{E},\,I\!\!R) \subset \mathrm{Quan}(J_1\boldsymbol{E},\,I\!\!R)$$

consisting of quantisable functions whose time component $f'' \in \bar{\mathbb{T}}$ is constant;

- the sheaf

$$\mathrm{Aff}(J_1\boldsymbol{E},\,I\!\!R) \subset \mathrm{Time}(J_1\boldsymbol{E},\,I\!\!R)$$

consisting of quantisable functions, called *affine*, whose time component $f'' = 0$ vanishes, i.e. the subsheaf of affine functions with respect to the affine fibres of the bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$;

- the sheaf

$$\mathrm{Map}(\boldsymbol{E},\,I\!\!R) \subset \mathrm{Aff}(J_1\boldsymbol{E},\,I\!\!R)$$

consisting of affine functions such that $Df = 0$, i.e. the subsheaf of functions which depend only on $\boldsymbol{E}$. □

2.3.3. Proposition. With reference to any observer o and to an adapted chart, the special quadratic functions are the functions $f \in \mathrm{Map}(J_1\boldsymbol{E},\,I\!\!R)$ of the type

$$f = \langle \mathcal{K}[o], f'' \rangle + \langle f'[o],\, \nabla[o] \rangle + f[o]\,,$$

i.e. of the type

$$f = \tfrac{1}{2}\, f^0 G^0_{ij}\, x^i_0\, x^j_0 + f^0_i\, x^i_0 + \overset{o}{f}\,,$$

where

$$f'' \in \mathrm{Map}(\boldsymbol{E},\,\bar{\mathbb{T}})\,, \qquad f'[o] \in \mathrm{Sec}(\boldsymbol{E},\,\mathbb{T} \otimes V^*\boldsymbol{E})\,, \qquad f[o] \in \mathrm{Map}(\boldsymbol{E}, I\!\!R)\,,$$
$$f^0, f^0_i, \overset{o}{f} \in \mathrm{Map}(\boldsymbol{E},\,I\!\!R)\,,$$
$$f'' = f^0\, u_0\,, \qquad f'[o] = f^0_i\, u_0 \otimes \check{d}^i\,, \qquad f[o] = \overset{o}{f}\,.$$

Moreover, if $f \in \mathrm{Map}(J_1\boldsymbol{E},\,I\!\!R)$ is a special quadratic function, then we obtain

$$f'' = \tfrac{1}{n}\, \langle D^2 f, \bar{G} \rangle\,, \qquad f'[o] = Df \circ o\,, \qquad f[o] = f \circ o\,. \; \square$$

2.3.4. Corollary. If $f \in \mathrm{Quan}(J_1\boldsymbol{E},\, I\!R)$ and $\bar{o}, o$ are two observers, with $\bar{o} = o + v$, then we obtain

$$f'[\bar{o}] = f'[o] + G^\flat(v)\, f''\,, \qquad f[\bar{o}] = f[o] + f'[o]\,(v) + \tfrac{1}{2}\, G(v,v)\, f''\,.$$

Proof. We have

$$f'[\bar{o}] = Df \circ \bar{o} = Df \circ (o + v) = Df \circ o + D^2 f(v) = f'[o] + f''\, G^\flat(v)$$
$$f[\bar{o}] = f \circ \bar{o} = f \circ (o + v) = f \circ o + (Df \circ o)(v) + \tfrac{1}{2}\, D^2 f \circ (v,v) = f[o] + f'[o](v) + \tfrac{1}{2}\, G(v,v)\, f''\,.\ \mathrm{QED}$$

2.3.5. Theorem. [56] *Let* $\tau \in \mathrm{Map}(J_1\boldsymbol{E},\, \bar{\mathbb{T}})$ *and* $f \in \mathrm{Map}(J_1\boldsymbol{E},\, I\!R)$*. Then, the following conditions are equivalent:*

1) $X^\uparrow_{\mathrm{Ham}}[\tau, f] \in \mathrm{Sec}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$ *is projectable on* $X[\tau, f] \in \mathrm{Sec}(\boldsymbol{E},\, T\boldsymbol{E})$*,*

2) $f \in \mathrm{Spec}(J_1\boldsymbol{E},\, I\!R)$ *and* $\tau = f''$*.*

Thus, if the above conditions are fulfilled, then we obtain

$$X^\uparrow_{\mathrm{Ham}}[\tau, f] = X^\uparrow_{\mathrm{Ham}}[f] := \gamma(f'') + \Lambda^\sharp(df)\,.$$

Proof. The vector field

$$X^\uparrow_{\mathrm{Ham}}[\tau, f] = \tau^0\, \partial_0 + (\tau^0\, x^i_0 - G^{ij}_0\, \partial^0_j f)\, \partial_i + \big(\tau^0\, \gamma_0{}^i_0 + G^{ij}_0\, \partial_j f + (\Gamma^{i\,j}_{00} - \Gamma^{j\,i}_{00})\, \partial^0_j f\big)\, \partial^0_i$$

is projectable if and only if

$$\tau^0 \in \mathrm{Map}(\boldsymbol{E},\, I\!R)\,, \qquad \tau^0\, G^0_{ij}\, x^i_0 - \partial^0_j f \in \mathrm{Map}(\boldsymbol{E},\, I\!R)\,.$$

Moreover, by recalling the affine structure of the bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$, the functions f which fulfill the equation

$$\tau^0\, G^0_{ij}\, x^i_0 - \partial^0_j f \;\equiv\; f^0_j \in \mathrm{Map}(\boldsymbol{E},\, I\!R)\,.$$

are of the type

$$f = \tfrac{1}{2}\, \tau^0\, G^0_{ij}\, x^i_0\, x^j_0 + f^0_i\, x^i_0 + \overset{o}{f}\,, \qquad \text{with} \qquad \overset{o}{f} \in \mathrm{Map}(\boldsymbol{E},\, I\!R)\,.\ \mathrm{QED}$$

2.3.6. Definition. If $f \in \mathrm{Spec}(J_1\boldsymbol{E},\, I\!R)$, then the vector field

$$X[f] := X[f'', f]$$

is called the *tangent lift* of f. □

We stress that the tangent lift of special quadratic functions turns out to depend only on the time fibring and the spacelike metric; indeed, it does not involve Ω.

2.3.2 Classification of classical symmetries

2.3.7. Corollary. The i.s.'s

$$X^{\uparrow} \in \mathrm{Sec}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$$

of Ω projectable on $\boldsymbol{E}$ are the vector fields of the local type

$$X^{\uparrow} = X^{\uparrow}_{\mathrm{Ham}}[f]\,, \qquad \text{with} \qquad f \in \mathrm{Con}(J_1\boldsymbol{E},\, I\!\!R) \cap \mathrm{Spec}(J_1\boldsymbol{E},\, I\!\!R)\,,$$

where, for each $X^{\uparrow}$, the function f is defined up to a constant.

Proof. It follows from Proposition 2.3.5. QED

2.3.8. Corollary. The i.s.'s

$$X^{\uparrow} \in \mathrm{Sec}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$$

of dt and Ω projectable on $\boldsymbol{E}$ are the vector fields of the local type

$$X^{\uparrow} = X^{\uparrow}_{\mathrm{Ham}}[f]\,, \qquad \text{with} \qquad f \in \mathrm{Con}(J_1\boldsymbol{E},\, I\!\!R) \cap \mathrm{Time}(J_1\boldsymbol{E},\, I\!\!R)\,,$$

where, for each $X^{\uparrow}$, the function f is defined up to a constant. □

2.3.9. Corollary. Let us consider a vector field $X \in \mathrm{Pro}(\boldsymbol{E},\, T\boldsymbol{E})$.

If its holonomic prolongation $X_{(1)} \in \mathrm{Sec}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$ is an i.s. of Ω, then we obtain locally

$$X = X[f] \qquad \text{and} \qquad X_{(1)} := X^{\uparrow}_{\mathrm{hol}}[f] = X^{\uparrow}_{\mathrm{Ham}}[f]\,,$$

where

$$f \in \mathrm{Spec}(J_1\boldsymbol{E},\, I\!\!R)\,. \,\square$$

2.3.10. Corollary. Let us consider a vector field $X \in \mathrm{Pro}(\boldsymbol{E},\, T\boldsymbol{E})$.

If its holonomic prolongation $X_{(1)} \in \mathrm{Sec}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$ is an i.s. of dt and Ω, then we obtain locally

$$X = X[f] \qquad \text{and} \qquad X_{(1)} := X^{\uparrow}_{\mathrm{hol}}[f] = X^{\uparrow}_{\mathrm{Ham}}[f]\,,$$

where

$$f \in \mathrm{Time}(J_1\boldsymbol{E},\, I\!\!R)\,. \,\square$$

2.3.11. Definition. A *symmetry of the classical structure* is defined to be a vector field $X^{\uparrow} \in \mathrm{Sec}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$, which is an i.s. of dt and Ω and which is projectable on $\boldsymbol{E}$. □

We denote the subalgebra of i.s.'s of the classical structure by

$$\mathrm{Clas}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E}) \subset \mathrm{Ham}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})\,.$$

Moreover, we set

$$\mathrm{Clas}(J_1\boldsymbol{E},\, I\!R) := \mathrm{Con}(J_1\boldsymbol{E},\, I\!R) \cap \mathrm{Time}(J_1\boldsymbol{E},\, I\!R)\,.$$

2.3.12. Lemma. If $X \in \mathrm{Time}(\boldsymbol{E}, T\boldsymbol{E})$, then

$$L[X_{(1)}]\,\theta = 0\,,$$

where we have regarded θ as a vertical valued form

$$\vartheta : J_1\boldsymbol{E} \to \mathbb{T} \otimes (T^*\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} VJ_1\boldsymbol{E})\,,$$

through the isomorphism $VJ_1\boldsymbol{E} \simeq \mathbb{T}^* \otimes V\boldsymbol{E}$.

Proof. We have

$$\begin{aligned} L[X_{(1)}]\,\theta^0 &:= L[X_{(1)}]\,\big((d^i - x^i_0\,d^0) \otimes \partial^0_i\big) \\ &= \big(\partial_0 X^i\,d^0 + \partial_j X^i\,d^j - (\partial_0 X^i + \partial_j X^i\,x^j_0 - \partial_0 X^0\,x^i_0)\,d^0\big) \otimes \partial^0_i - x^i_0\,(\partial_0 X^0\,d^0 + \partial_j X^0\,d^j) \otimes \partial^0_i \\ &\quad - \partial_i X^h\,(d^i - x^i_0\,d^0) \otimes \partial^0_h + \partial_0 X^0\,(d^i - x^i_0\,d^0) \otimes \partial^0_i \\ &= 0\,.\ \mathrm{QED} \end{aligned}$$

2.3.13. Lemma. If $X \in \mathrm{Time}(\boldsymbol{E}, T\boldsymbol{E})$, then, $L[X_{(1)}]\,\nu[\Gamma]$, where $\nu[\Gamma]$ is the vertical projector associated with the phase connection Γ, is a tensor of the type

$$L[X_{(1)}]\,\nu[\Gamma] \in \mathrm{Sec}(J_1\boldsymbol{E}, T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E}).$$

We have the expression

$$L[X_{(1)}]\,\nu[\Gamma] = (L[X_{(1)}]\,\Gamma_{\mu 0}{}^{i})\,d^\mu \otimes \partial_i.\,\square$$

2.3.14. Remark. I want to observe that we have also proved in[140], that ,for a vector field X in $\mathrm{Time}(\boldsymbol{E}, T\boldsymbol{E})$, the following conditions are equivalent

1. X is a holonomic symmetry of the phase connection Γ;
2. X is a holonomic symmetry of the spacetime connection K; here, holonomic denotes the natural prolongation of X to a vector field of $T\boldsymbol{E}$;
3. X is a holonomic symmetry of the dynamical connection γ. holonomic symmetry of the phase connection.

2.3.15. Proposition. [140] If $X \in \mathrm{Time}(\boldsymbol{E}, T\boldsymbol{E})$, then

$$L[X_{(1)}]\,\Omega = 0 \qquad \Leftrightarrow \qquad L[X_{(1)}]\,\Gamma = 0\,, \quad L[X]\,G = 0. \square$$

PROOF. 1) We have

$$L[X_{(1)}]\,\Omega = (L[X]\,G)\,(\nu[\Gamma] \wedge \theta) + G\,\big((L[X]\,\nu[\Gamma]) \wedge \theta\big)\,.$$

Hence, if $L[X_{(1)}]\,\Gamma = 0$ and $L[X]\,G = 0$, then $L[X_{(1)}]\,\Omega = 0$.
2) By Lemma 2.3.13, we have

$$L[X_{(1)}]\,\Omega = (L[X]\,G)^0_{ij}\,(d^i_0 - \Gamma_{\lambda}{}^{i}{}_{0}\,d^\lambda) \wedge \theta^j) + G^0_{ij}\,\big((L[X_{(1)}]\,\Gamma_{\mu}{}^{i}{}_{0})\,d^\mu \wedge \theta^j\big)$$

Hence, if $L[X_{(1)}]\,\Omega = 0$, then $L[X_{(1)}]\,\Gamma = 0$ and $L[X]\,G = 0$. QED

2.3.16. Lemma. Let $X \in \mathrm{Pro}(\boldsymbol{E},\, T\boldsymbol{E})$.
Then, we have the following implications:

$$\begin{array}{llcl} 1. & L[X]\,G = 0 & \Leftrightarrow & L[X]\,\bar{G} = 0\,, \\ 2. & L[X]\,G = 0 & \Rightarrow & \operatorname{div} X = 0\,, \\ 3. & L[X]\,G = 0 & \Rightarrow & L[X]\,\rho = 0\,, \end{array}$$

where $\bar{G}$ is the contravariant spacelike metric and ρ the scalar curvature of G.

PROOF. (1) We have $0 = L[X]\mathbf{1} = G^\sharp(L[X]G) + G^\flat(L[X]\bar{G})$.
(2) We have $\operatorname{div} X = \frac{1}{2}\,\langle \bar{G},\, L[X]G\rangle$.
(3) It follows from the functorial construction of ρ through G. QED

2.3.17. Proposition. If $X \in \mathrm{Time}(\boldsymbol{E},\, T\boldsymbol{E})$ then,

$$L[X_{(1)}]\,\Omega = 0 \qquad \Rightarrow \qquad \operatorname{div} X = 0\,.$$

PROOF. In virtue of Proposition 2.3.15, $L[X_{(1)}]\,\Omega = 0$ implies $L[X]\,G = 0$. QED

2.3.3 Special Lie bracket

2.3.18. Proposition. If $f, g \in \operatorname{Spec}(J_1\boldsymbol{E}, I\!R)$, then, the coordinate expression of their Poisson bracket is

$$
\begin{aligned}
\{f,\, g\} &= \tfrac{1}{2}\,(\partial_i f^0\, g^0 - \partial_i g^0\, f^0)\, G^0_{hk}\, x^i_0 x^h_0 x^k_0 \\
&+ (\partial_h f^0_k\, g^0 - \partial_h g^0_k\, f^0)\, x^h_0 x^k_0 \\
&+ G^{ij}_0\, (\partial_i f^0\, g^0_j - \partial_i g^0\, f^0_j)\, \tfrac{1}{2}\, G^0_{hk}\, x^h_0 x^k_0 \\
&+ G^{ij}_0\, (f^0\, g^0_j - g^0\, f^0_j)\, \tfrac{1}{2}\, \partial_i G^0_{hk}\, x^h_0 x^k_0 \\
&+ G^{ih}_0 G^{jk}_0\, (\partial_h G^0_{kl} - \partial_k G^0_{hl})\, (G^0_{im}\, f^0 g^0_j + G^0_{jm}\, g^0 f^0_i)\, x^l_0 x^m_0 \\
&+ G^{ij}_0\, (\partial_i f^0_h\, g^0_j - \partial_i g^0_h\, f^0_j)\, x^h_0 \\
&+ G^{ij}_0\, (\partial_i \overset{o}{f}\, g^0 - \partial_i \overset{o}{g}\, f^0)\, G^0_{jh}\, x^h_0 \\
&+ G^{ih}_0 G^{jk}_0\, \Phi_{hk} (G^0_{im}\, f^0 g^0_j + G^0_{jm}\, g^0 f^0_i)\, x^m_0 \\
&+ G^{ih}_0 G^{jk}_0\, (\partial_h G^0_{kl} - \partial_k G^0_{hl})\, f^0_i\, g^0_j\, x^l_0 \\
&+ G^{ij}_0\, (\partial_i \overset{o}{f}\, g^0_j - \partial_i \overset{o}{g}\, f^0_j) \\
&+ G^{ih}_0 G^{jk}_0\, \Phi_{hk}\, f^0_i\, g^0_j\,.
\end{aligned}
$$

Hence, the sheaf $\operatorname{Spec}(J_1\boldsymbol{E}, I\!R)$ is not closed under the Poisson bracket.

PROOF. We have

$$
\begin{aligned}
\{f,\, g\} &= G^{ij}_0\, (\partial_i f^0\, g^0 - \partial_i g^0\, f^0)\, \tfrac{1}{2}\, G^0_{hk}\, G^0_{jl}\, x^h_0 x^k_0 x^l_0 \\
&+ G^{ij}_0\, (\partial_i f^0_h\, g^0 - \partial_i g^0_h\, f^0)\, G^0_{jk}\, x^h_0 x^k_0 \\
&+ G^{ij}_0\, (\partial_i f^0\, g^0_j - \partial_i g^0\, f^0_j)\, \tfrac{1}{2}\, G^0_{hk}\, x^h_0 x^k_0 \\
&+ G^{ij}_0\, (f^0\, g^0_j - g^0\, f^0_j)\, \tfrac{1}{2}\, \partial_i G^0_{hk}\, x^h_0 x^k_0 \\
&+ G^{ij}_0\, (\partial_i f^0_h\, g^0_j - \partial_i g^0_h\, f^0_j)\, x^h_0 \\
&+ (\partial_i \overset{o}{f}\, g^0 - \partial_i \overset{o}{g}\, f^0)\, x^i_0 \\
&+ G^{ij}_0\, (\partial_i \overset{o}{f}\, g^0_j - \partial_i \overset{o}{g}\, f^0_j) \\
&+ G^{ih}_0 G^{jk}_0\, \big((\partial_h G^0_{kl} - \partial_k G^0_{hl})\, x^l_0 + \Phi_{hk}\big)(G^0_{im}\, f^0 g^0_j + G^0_{jm}\, g^0 f^0_i)\, x^m_0 \\
&+ G^{ih}_0 G^{jk}_0\, \big((\partial_h G^0_{kl} - \partial_k G^0_{hl})\, x^l_0 + \Phi_{hk}\big) f^0_i\, g^0_j\,. \text{QED}
\end{aligned}
$$

On the other hand, we have the following result.

2.3.19. Definition. We define the *special bracket* to be the map

$$
\begin{aligned}
&\operatorname{Spec}(J_1\boldsymbol{E}, I\!R) \times \operatorname{Spec}(J_1\boldsymbol{E}, I\!R) \to \operatorname{Map}(J_1\boldsymbol{E}, I\!R) : \\
&\qquad\qquad (f,\, g) \mapsto [\![\, f, g \,]\!] := \{f, g\} + \gamma(f'').g - \gamma(g'').f \,. \square
\end{aligned}
$$

2.3.20. Proposition. For each $f, g \in \mathrm{Spec}(J_1\boldsymbol{E},\ I\!R)$, we have the coordinate expression

$$[\![f, g]\!] = [\![f, g]\!]^{0}\, \tfrac{1}{2}\, G^{0}_{ij}\, x^{i}_{0}\, x^{j}_{0} + [\![f, g]\!]^{0}_{i}\, x^{i}_{0} + [\![f, \overset{o}{g}]\!] \,,$$

with

$$\begin{aligned}
[\![f, g]\!]^{0} &= f^0\partial_0 g^0 - g^0\partial_0 f^0 - f^h\partial_h g^0 + g^h\partial_h f^0 \\
[\![f, g]\!]^{0}_{i} &= G^0_{ij}(f^0\partial_0 g^j - g^0\partial_0 f^j - f^h\partial_h g^j + g^h\partial_h f^j) \\
[\![f, \overset{o}{g}]\!] &= f^0\partial_0\overset{o}{g} - g^0\partial_0\overset{o}{f} - f^h\partial_h\overset{o}{g} + g^h\partial_h\overset{o}{f} + (f^0\, g^h - g^0\, f^h)\, \Phi_{h0} + f^h g^k\, \Phi_{hk} \,.
\end{aligned}$$

Hence, the sheaf $\mathrm{Spec}(J_1\boldsymbol{E},\ I\!R)$ is closed under the special bracket. □

The following theorem will be proved in two different ways. For the second proof, I need following Lemma

2.3.21. Lemma. Let $f, g, h \in \mathrm{Spec}(J_1\boldsymbol{E}, I\!R)$. Then, we have the following coordinate expressions

$$\begin{aligned}
&1)\quad \{\gamma(f'').g, h\} = f^0\{\gamma_0.g, h\} + \gamma_0.g\{f^0, h\} \\
&2)\quad \gamma([\![f, g]\!]'').h - \gamma(f'').\gamma(g'').h + \gamma(g'').\gamma(f'').h \\
&\qquad\qquad = (g^h\partial_h f^0 - f^h\partial_h g^0 - f^0\partial_i g^0 x^i_0 + g^0\partial_i f^0 x^i_0)\gamma_0.h \,\square
\end{aligned}$$

2.3.22. Theorem. [56] *The sheaf* $\mathrm{Spec}(J_1\boldsymbol{E}, I\!R)$ *turns out to be an* $I\!R$*-Lie algebra through the special bracket.*

Proof. It is sufficient to prove the Jacobi property of the bracket.
In fact, for each $f, g, h \in \mathrm{Spec}(J_1\boldsymbol{E},\ I\!R)$, we have

$$[\![\, [\![f \,,\, \overset{o}{g}]\!] \,,\, h]\!] + [\![\, [\![h \,,\, \overset{o}{f}]\!] \,,\, g]\!] + [\![\, [\![g \,,\, \overset{o}{h}]\!] \,,\, f]\!] = 0 \,.$$

Actually, we have the following coordinate expression

$$\begin{aligned}
&[\![\, [\![f \,,\, \overset{o}{g}]\!] \,,\, h]\!] = \\
&= \partial_0\overset{o}{h}\,(f^0\,\partial_0 g^0 - g^0\,\partial_0 f^0 - f^j\,\partial_j g^0 + g^j\,\partial_j f^0) \\
&\quad - h^0\,\partial_0\big(f^0\,\partial_0\overset{o}{g} - g^0\,\partial_0\overset{o}{f} - f^h\,\partial_h\overset{o}{g} + g^h\,\partial_h\overset{o}{f} + (f^0 g^h - g^0 f^h)\,\Phi_{h0} + f^h g^k\,\Phi_{hk}\big) \\
&\quad - \partial_l\overset{o}{h}\,(f^0\,\partial_0 g^l - g^0\,\partial_0 f^l - f^j\,\partial_j g^l + g^j\,\partial_j f^l) \\
&\quad + h^l\partial_l\big(f^0\,\partial_0\overset{o}{g} - g^0\,\partial_0\overset{o}{f} - f^h\,\partial_h\overset{o}{g} + g^h\,\partial_h\overset{o}{f} + (f^0 g^h - g^0 f^h)\,\Phi_{h0} + f^h g^k\,\Phi_{hk}\big) \\
&\quad + h^l\,(f^0\,\partial_0 g^0 - g^0\,\partial_0 f^0 - f^j\,\partial_j g^0 + g^j\,\partial_j f^0)\,\Phi_{l0} \\
&\quad - h^0\,(f^0\,\partial_0 g^l - g^0\,\partial_0 f^l - f^j\,\partial_j g^l + g^j\,\partial_j f^l)\,\Phi_{l0} \\
&\quad + h^k\,(f^0\,\partial_0 g^h - g^0\,\partial_0 f^h - f^j\,\partial_j g^h + g^j\,\partial_j f^h)\,\Phi_{hk} \,.
\end{aligned}$$

hence

$$
\begin{aligned}
&[\![\, [\![\, f \, , \overset{o}{g} \,]\!] \, , h \,]\!] = \\
= \; & \partial_0 \overset{o}{h} \, (f^0 \, \partial_0 g^0 - g^0 \, \partial_0 f^0 - f^j \, \partial_j g^0 + g^j \, \partial_j f^0) \\
& - h^0 \big(\partial_0 f^0 \, \partial_0 \overset{o}{g} - \partial_0 g^0 \, \partial_0 \overset{o}{f} - \partial_0 f^h \, \partial_h \overset{o}{g} + \partial_0 g^h \, \partial_h \overset{o}{f} + (\partial_0 f^0 g^h - \partial_0 g^0 f^h) \, \Phi_{h0} + \partial_0 f^h g^k \, \Phi_{hk} \big) \\
& - h^0 \big(f^0 \, \partial_0 \partial_0 \overset{o}{g} - g^0 \, \partial_0 \partial_0 \overset{o}{f} - f^h \, \partial_0 \partial_h \overset{o}{g} + g^h \, \partial_0 \partial_h \overset{o}{f} + (f^0 \partial_0 g^h - g^0 \partial_0 f^h) \, \Phi_{h0} + f^h \partial_0 g^k \, \Phi_{hk} \big) \\
& - h^0 \big((f^0 g^h - g^0 f^h) \, \partial_0 \Phi_{h0} + f^h g^k \, \partial_0 \Phi_{hk} \big) \\
& - \partial_l \overset{o}{h} \, (f^0 \, \partial_0 g^l - g^0 \, \partial_0 f^l - f^j \, \partial_j g^l + g^j \, \partial_j f^l) \\
& + h^l \big(\partial_l f^0 \, \partial_0 \overset{o}{g} - \partial_l g^0 \, \partial_0 \overset{o}{f} - \partial_l f^h \, \partial_h \overset{o}{g} + \partial_l g^h \, \partial_h \overset{o}{f} + (\partial_l f^0 g^h - \partial_l g^0 f^h) \, \Phi_{h0} + \partial_l f^h g^k \, \Phi_{hk} \big) \\
& + h^l \big(f^0 \, \partial_l \partial_0 \overset{o}{g} - g^0 \, \partial_l \partial_0 \overset{o}{f} - f^h \, \partial_l \partial_h \overset{o}{g} + g^h \, \partial_l \partial_h \overset{o}{f} + (f^0 \partial_l g^h - g^0 \partial_l f^h) \, \Phi_{h0} + f^h \partial_l g^k \, \Phi_{hk} \big) \\
& + h^l \big((f^0 g^h - g^0 f^h) \, \partial_l \Phi_{h0} + f^h g^k \, \partial_l \Phi_{hk} \big) \\
& + h^l \, (f^0 \, \partial_0 g^0 - g^0 \, \partial_0 f^0 - f^j \, \partial_j g^0 + g^j \, \partial_j f^0) \, \Phi_{l0} \\
& - h^0 \, (f^0 \, \partial_0 g^l - g^0 \, \partial_0 f^l - f^j \, \partial_j g^l + g^j \, \partial_j f^l) \, \Phi_{l0} \\
& + h^k \, (f^0 \, \partial_0 g^h - g^0 \, \partial_0 f^h - f^j \, \partial_j g^h + g^j \, \partial_j f^h) \, \Phi_{hk} \, .
\end{aligned}
$$

On the other hand, we obtain the following equalities (here the symbol $\sum$ denotes the sum of three terms obtained by circular permutation of f, g, h)

$$
\begin{aligned}
0 &= \sum (-h^0 g^h \, \partial_0 f^0 + h^0 f^h \, \partial_0 g^0 - h^h f^0 \partial_0 g^0 + h^h g^0 \, \partial_0 f^0) \, \Phi_{h0} \\
0 &= \sum (h^0 f^0 \, \partial_0 g^h - h^0 g^0 \, \partial_0 f^h) \, \Phi_{h0}
\end{aligned}
$$

$$
\begin{aligned}
0 &= \sum (-h^0 g^k \, \partial_0 f^h - h^0 f^h \, \partial_0 g^k + h^k f^0 \, \partial_0 g^h - h^k g^0 \, \partial_0 f^h) \, \Phi_{hk} \\
0 &= \sum (h^k f^j \, \partial_j g^h - h^k g^j \, \partial_j f^h - h^j g^k \, \partial_j f^h - h^j f^h \, \partial_j g^k) \, \Phi_{hk}
\end{aligned}
$$

$$
\begin{aligned}
0 &= \sum -h^0 f^h g^k \, \partial_0 \Phi_{hk} + h^l \, (f^0 g^h - g^0 f^h) \, \partial_l \Phi_{h0} \\
0 &= \sum h^l f^h g^k \, \partial_l \Phi_{hk}
\end{aligned}
$$

$$
\begin{aligned}
0 = \sum \big(& h^l \, (\partial_l f^0 g^h - \partial_l g^0 f^h) + h^l \, (f^0 \partial_l g^h - g^0 \partial_l f^h) \\
& + h^h \, (f^l \partial_l g^0 - g^l \partial_l f^0) - h^0 \, (f^l \partial_l g^h - f^l \partial_l g^h) \big) \, \Phi_{h0} \\
0 = \sum \, & h^0 \, (f^0 \partial_0 g^h - g^0 \partial_0 f^h) \, \Phi_{h0}
\end{aligned}
$$

$$
0 = \sum h^0 \, (f^0 g^h - g^0 f^h) \, \partial_0 \Phi_{h0}
$$

$$0 = \sum \partial_0 \overset{o}{h} \, (f^0 \partial_0 g^0 - g^0 \partial_0 f^0) - h^0 \, (\partial_0 f^0 \partial_0 \overset{o}{g} - \partial_0 g^0 \partial_0 \overset{o}{f})$$
$$0 = \sum \partial_0 \overset{o}{h} \, (f^j \partial_j g^0 - g^j \partial_j f^0) - h^j \, (\partial_j f^0 \partial_0 \overset{o}{g} - \partial_j g^0 \partial_0 \overset{o}{f})$$

$$0 = \sum h^0 \, (\partial_0 f^h \partial_h \overset{o}{g} - \partial_0 g^h \partial_h \overset{o}{f}) - \partial_h \overset{o}{h} \, (f^0 \partial_0 \overset{o}{g} - g^0 \partial_0 \overset{o}{f})$$
$$0 = \sum h^0 \, (f^h \partial_{0h} \overset{o}{g} - g^h \partial_{0h} \overset{o}{f}) + h^h \, (f^0 \partial_{0h} \overset{o}{g} - g^0 \partial_{0h} \overset{o}{f})$$

$$0 = \sum h^0 \, (f^0 \partial_{00} \overset{o}{g} - g^0 \partial_{00} \overset{o}{f})$$
$$0 = \sum h^h \, (f^k \partial_{hk} \overset{o}{g} - g^k \partial_{hk} \overset{o}{f})$$

$$0 = \sum \partial_h \overset{o}{h} \, (f^k \partial_k g^h - g^k \partial_k h^h) - h^k \, (\partial_k f^h \partial_h \overset{o}{g} - \partial_k g^h \partial_h \overset{o}{f}) \,. \text{QED}$$

In the following, I sketch another proof of the previous theorem.

Proof. * The result follows by Lemma 2.2.21, Proposition 2.2.20 and the coordinate expressions of Lemma 2.3.21. QED

2.3.23. Proposition. For each $f, g \in \mathrm{Quan}(J_1\boldsymbol{E}, I\!R)$, we have

$$[\![f, g]\!]'' = [f'', g''] \,,$$

where the right bracket is the standard Lie bracket of the vector fields $f'', g'' \in \mathrm{Sec}(\boldsymbol{T}, T\boldsymbol{T})$.

Proof. It follows immediately from the coordinate expression of the special bracket. QED

2.3.24. Proposition. The subsheaves

$$\begin{aligned} \mathrm{Quan}(J_1\boldsymbol{E}, I\!R) &\subset \mathrm{Spec}(J_1\boldsymbol{E}, I\!R) \\ \mathrm{Time}(J_1\boldsymbol{E}, I\!R) &\subset \mathrm{Quan}(J_1\boldsymbol{E}, I\!R) \\ \mathrm{Aff}(J_1\boldsymbol{E}, I\!R) &\subset \mathrm{Time}(J_1\boldsymbol{E}, I\!R) \\ \mathrm{Map}(\boldsymbol{E}, I\!R) &\subset \mathrm{Aff}(J_1\boldsymbol{E}, I\!R) \end{aligned}$$

are subsheaves of subalgebras.

Moreover, the subsheaf

$$\mathrm{Map}(\boldsymbol{E}, I\!R) \subset \mathrm{Aff}(J_1\boldsymbol{E}, I\!R)$$

is a subsheaf of ideals.

Proof. It follows easily from the coordinate expression of the special bracket. QED

2.3.25. Proposition. The quotient sheaf $\mathrm{Spec}(J_1\boldsymbol{E}, I\!R)/\mathrm{Map}(\boldsymbol{E}, I\!R)$ turns out to be an $I\!R$–Lie algebra through the quotient special bracket.

For each $f, g \in \mathrm{Spec}(J_1\boldsymbol{E}, I\!R)/\mathrm{Map}(\boldsymbol{E}, I\!R)$, we have the coordinate expression

$$\begin{aligned} [\![f, g]\!]^0 &= f^0 \partial_0 g^0 - g^0 \partial_0 f^0 - f^h \partial_h g^0 + g^h \partial_h f^0 \\ [\![f, g]\!]^0_i &= G^0_{ij}(f^0 \partial_0 g^j - g^0 \partial_0 f^j - f^h \partial_h g^j + g^h \partial_h f^j) \,. \square \end{aligned}$$

We stress that the above quotient bracket turns out to depend only on the time fibring and the spacelike metric; indeed, it does not involve Ω.

2.3.4 Tangent lift of special quadratic functions

2.3.26. Proposition. [56] For each $f \in \mathrm{Spec}(J_1\boldsymbol{E}, I\!R)$, we have the coordinate expression

$$X[f] = f^0\, \partial_0 - f^i\, \partial_i \,. \square$$

2.3.27. Proposition. For each $f, g \in \mathrm{Spec}(J_1\boldsymbol{E}, I\!R)$, we have

$$[\![f, g]\!]'' = Tt \circ [X[f],\, X[g]] \,.$$

Proof. It follows from the coordinate expression of the special bracket. QED

2.3.28. Lemma. For each $f, g \in \mathrm{Spec}(J_1\boldsymbol{E}, I\!R)$, we have the coordinate expression

$$\begin{aligned} [\![f,\, g]\!]^0 &= \Big[X[f],\, X[g]\Big]^0 \\ [\![f,\, g]\!]^i &= \Big[X[f],\, X[g]\Big]^i \,. \end{aligned}$$

Proof. It follows from the coordinate expression of the special bracket. QED

2.3.29. Lemma. Let us consider an observer o. Then, for each $f, g \in \mathrm{Spec}(J_1\boldsymbol{E}, I\!R)$, we obtain

$$[\![f \overset{o}{,}\, g]\!] = X[f].\overset{o}{g} - X[g].\overset{o}{f} + \frac{1}{2}\, i(X[g])\, i(X[f])\, \Phi[o] \,.$$

Proof. It follows from the coordinate expression of the special bracket. QED

2.3.30. Proposition. For each $f, g \in \mathrm{Spec}(J_1\boldsymbol{E}, I\!R)$, we have

$$\Big[X[f],\, X[g]\Big] = X\Big[\, [\![f,\, g]\!] \,\Big] \,.$$

Hence, the map

$$\mathrm{Spec}(J_1\boldsymbol{E},\, I\!R) \to \mathrm{Sec}(\boldsymbol{E},\, T\boldsymbol{E}) : f \mapsto X[f]$$

turns out to be a morphism of Lie algebras, with respect to the special bracket and the standard bracket, respectively. Its kernel is the ideal $\mathrm{Map}(\boldsymbol{E}, I\!R) \subset \mathrm{Spec}(J_1\boldsymbol{E}, I\!R)$.

Proof. It follows from a computation in coordinates. QED

2.3.31. Corollary. We have a natural isomorphism of Lie algebras

$$\mathrm{Sec}(\boldsymbol{E},\, T\boldsymbol{E}) \to \mathrm{Spec}(J_1\boldsymbol{E},\, I\!R)/\,\mathrm{Map}(\boldsymbol{E}, I\!R)\,,$$

whose coordinate expression is

$$X^0\,\partial_0 + X^i\,\partial_i \mapsto [X^0\,\tfrac{1}{2}\,G^0_{ij}\,x^i_0\,x^j_0 - G^0_{ij}\,X^j\,x^i_0]\,.\,\square$$

We stress that this isomorphism turns out to involve only the time fibring and the spacelike metric; indeed, it does not involve Ω.

2.3.5 Further expression of the special bracket

In this section we prove a further expression of the special bracket in terms of arbitrary prolongations of the tangent lift of the special functions.

This formula will be used in the quantum theory for the study of Hermitian vector fields.

2.3.32. Lemma. If $f, \bar f \in \mathrm{Spec}(J_1\boldsymbol{E},\, I\!R)$ and $X^\uparrow, \bar X^\uparrow \in \mathrm{Pro}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$ are prolongations of $X[f]$ and $X[\bar f]$, then we obtain

$$\begin{aligned} X^\uparrow.\bar f - \bar X^\uparrow.f = {} & [\![\, f, \bar f \,]\!] \\ & + (f^0\,\bar f^i - \bar f^0\,f^i)\,(\partial_i\tfrac{1}{2}\,G^0_{hk}\,x^h_0\,x^k_0 + \partial_0 G^0_{ih}\,x^h_0 + \Phi_{0i}) \\ & - (f^i\,\bar f^j - \bar f^i\,f^j)\,(\partial_i G^0_{jh}\,x^h_0 + \tfrac{1}{2}\,\Phi_{ij}) \\ & + (X^i_0\,\bar f^0 - \bar X^i_0\,f^0)\,G^0_{ih}\,x^h_0 + (X^i_0\,\bar f^j - \bar X^i_0\,f^j)\,G^0_{ij}\,. \end{aligned}$$

PROOF. We have

$$
\begin{aligned}
X^{\uparrow}.g - \bar{X}^{\uparrow}.f &= f^0\,\partial_0\bar{f} - \bar{f}^0\,\partial_0 f - (f^i\,\partial_i\bar{f} - \bar{f}^i\,\partial_i f) + X_0^i\,\partial_i^0\bar{f} - \bar{X}_0^i\,\partial_i^0 f \\[2ex]
&= (f^0\,\partial_0\bar{f}^0 - \bar{f}^0\,\partial_0 f^0)\,\tfrac{1}{2}\,G^0_{hk}\,x_0^h\,x_0^k + (f^0\,\partial_0\bar{f}^j - \bar{f}^0\,\partial_0 f^j)\,G^0_{jh}\,x_0^h + (f^0\,\partial_0\overset{o}{\bar{f}} - \bar{f}^0\,\partial_0\overset{o}{f}) \\
&\quad - (f^i\,\partial_i\bar{f}^0 - \bar{f}^i\,\partial_i f^0)\,\tfrac{1}{2}\,G^0_{hk}\,x_0^h\,x_0^k - (f^i\,\partial_i\bar{f}^j - \bar{f}^i\,\partial_i f^j)\,G^0_{jh}\,x_0^h - (f^i\,\partial_i\overset{o}{\bar{f}} - \bar{f}^i\,\partial_i\overset{o}{f}) \\
&\quad + (f^0\,\bar{f}^0 - \bar{f}^0\,f^0)\,\partial_0\tfrac{1}{2}\,G^0_{hk}\,x_0^h\,x_0^k + (f^0\,\bar{f}^i - \bar{f}^0\,f^i)\,\partial_0 G^0_{ih}\,x_0^h \\
&\quad - (f^i\,\bar{f}^0 - \bar{f}^i\,f^0)\,\partial_i\tfrac{1}{2}\,G^0_{hk}\,x_0^h\,x_0^k - (f^i\,\bar{f}^j - \bar{f}^i\,f^j)\,\partial_i G^0_{jh}\,x_0^h \\
&\quad + (X_0^i\,\bar{f}^0 - \bar{X}_0^i\,f^0)\,G^0_{ih}\,x_0^h + (X_0^i\,\bar{f}^j - \bar{X}_0^i\,f^j)\,G^0_{ij} \\[2ex]
&= [\![f,\,\bar{f}]\!]^{\,0}\,\tfrac{1}{2}\,G^0_{ij}\,x_0^i\,x_0^j + [\![f,\,\bar{f}]\!]^{\,i}\,G^0_{ij}\,x_0^j + [\![\overset{o}{f,\bar{f}}]\!] \\
&\quad - (f^0\,\bar{f}^i - \bar{f}^0\,f^i)\,\Phi_{i0} - \tfrac{1}{2}\,(f^i\,\bar{f}^j - \bar{f}^i\,f^j)\,\Phi_{ij} \\
&\quad + (f^0\,\bar{f}^i - \bar{f}^0\,f^i)\,\partial_0 G^0_{ih}\,x_0^h \\
&\quad - (f^i\,\bar{f}^0 - \bar{f}^i\,f^0)\,\partial_i\tfrac{1}{2}\,G^0_{hk}\,x_0^h\,x_0^k - (f^i\,\bar{f}^j - \bar{f}^i\,f^j)\,\partial_i G^0_{jh}\,x_0^h \\
&\quad + (X_0^i\,\bar{f}^0 - \bar{X}_0^i\,f^0)\,G^0_{ih}\,x_0^h + (X_0^i\,\bar{f}^j - \bar{X}_0^i\,f^j)\,G^0_{ij} \\[2ex]
&= [\![f,\bar{f}]\!] \\
&\quad + (f^0\,\bar{f}^i - \bar{f}^0\,f^i)\,(\partial_i\tfrac{1}{2}\,G^0_{hk}\,x_0^h\,x_0^k + \partial_0 G^0_{ih}\,x_0^h + \Phi_{0i}) \\
&\quad - (f^i\,\bar{f}^j - \bar{f}^i\,f^j)\,(\partial_i G^0_{jh}\,x_0^h + \tfrac{1}{2}\,\Phi_{ij}) \\
&\quad + (X_0^i\,\bar{f}^0 - \bar{X}_0^i\,f^0)\,G^0_{ih}\,x_0^h + (X_0^i\,\bar{f}^j - \bar{X}_0^i\,f^j)\,G^0_{ij}\,. \text{ QED}
\end{aligned}
$$

2.3.33. Lemma. If $f, \bar{f} \in \mathrm{Spec}(J_1\boldsymbol{E},\, I\!R)$ and $X^{\uparrow}, \bar{X}^{\uparrow} \in \mathrm{Pro}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$ are prolongations of $X[f]$ and $X[\bar{f}]$, then we obtain

$$
\begin{aligned}
i(\bar{X}^{\uparrow})\,i(X^{\uparrow})\,\Omega = &-(f^0\,\bar{f}^j - \bar{f}^0\,f^j)\,\Big(\tfrac{1}{2}\,\partial_j G^0_{hk}\,x_0^h\,x_0^k + \partial_0 G^0_{jh}\,x_0^h + \Phi_{0j}\Big) \\
&- \tfrac{1}{2}\,(f^j\,\bar{f}^h - \bar{f}^j\,f^h)\,\Big(\Phi_{hj} + (\partial_h G^0_{jk} - \partial_j G^0_{hk})\,x_0^k\Big) \\
&- (X_0^i\,\bar{f}^0 - \bar{X}_0^i\,f^0)\,G^0_{ih}\,x_0^h - (X_0^i\,\bar{f}^j - \bar{X}_0^i\,f^j)\,G^0_{ij}\,.
\end{aligned}
$$

PROOF. We have

$$
\Omega = G^0_{ij}\,(d_0^i - \gamma_0{}^i_0\,d^0 - \Gamma_h{}^i_0\,\theta^h)\,\theta^j\,,
$$

hence

$$
\begin{aligned}
i(X^{\uparrow})\,\Omega = &\;X_0^i\,G^0_{ij}\theta^j - f^0\,G^0_{ij}\,\gamma_0{}^i_0\,\theta^j + f^0\,G^0_{ij}\,\Gamma_h{}^i_0\,x_0^h\,\theta^j + f^0\,G^0_{ij}\,(d_0^i - \gamma_0{}^i_0\,d^0 - \Gamma_h{}^i_0\,\theta^h)\,x_0^j \\
&+ f^h\,G^0_{ij}\,\Gamma_h{}^i_0\,\theta^j + G^0_{ij}\,(d_0^i - \gamma_0{}^i_0\,d^0 - \Gamma_h{}^i_0\,\theta^h)\,f^j\,,
\end{aligned}
$$

hence

$$\begin{aligned}
i(\bar X^{\uparrow})\, i(X^{\uparrow})\,\Omega &= -X^i_0\, G^0_{ij}\,(\bar f^j + \bar f^0\, x^j_0) + f^0\, G^0_{ij}\, \gamma_0{}^i_0\,(\bar f^j + \bar f^0\, x^j_0) - f^0\, G^0_{ij}\,\Gamma_h{}^i_0\, x^h_0\,(\bar f^j + \bar f^0\, x^j_0)\\
&+ f^0\, G^0_{ij}\,\big(\bar X^i_0 - \gamma_0{}^i_0\,\bar f^0 + \Gamma_h{}^i_0\,(\bar f^h + \bar f^0\, x^h_0)\big)\, x^j_0 - f^h\, G^0_{ij}\,\Gamma_h{}^i_0\,(\bar f^j + \bar f^0\, x^j_0)\\
&+ G^0_{ij}\,\big(\bar X^i_0 - \gamma_0{}^i_0\,\bar f^0 + \Gamma_h{}^i_0\,(\bar f^h + \bar f^0\, x^h_0)\big)\, f^j\\
&\\
&= f^0\, G^0_{ij}\,\gamma_0{}^i_0\,\bar f^j - f^0\, G^0_{ij}\,\Gamma_h{}^i_0\, x^h_0\,\bar f^j\\
&+ f^0\, G^0_{ij}\,\Gamma_h{}^i_0\,\bar f^h\, x^j_0 - f^h\, G^0_{ij}\,\Gamma_h{}^i_0\,(\bar f^j + \bar f^0\, x^j_0)\\
&+ G^0_{ij}\,\big(-\gamma_0{}^i_0\,\bar f^0 + \Gamma_h{}^i_0\,(\bar f^h + \bar f^0\, x^h_0)\big)\, f^j\\
&- (X^i_0\,\bar f^0 - \bar X^i_0\, f^0)\, G^0_{ih}\, x^h_0 - (X^i_0\,\bar f^j - \bar X^i_0\, f^j)\, G^0_{ij}\\
&\\
&= (f^0\,\bar f^j - \bar f^0\, f^j)\, G^0_{ij}\,(\gamma_0{}^i_0 - \Gamma_h{}^i_0\, x^h_0) + (f^0\,\bar f^h - \bar f^0\, f^h)\, G^0_{ij}\,\Gamma_h{}^i_0\, x^j_0\\
&- (f^h\,\bar f^j - \bar f^h\, f^j)\, G^0_{ij}\,\Gamma_h{}^i_0\\
&- (X^i_0\,\bar f^0 - \bar X^i_0\, f^0)\, G^0_{ih}\, x^h_0 - (X^i_0\,\bar f^j - \bar X^i_0\, f^j)\, G^0_{ij}\\
&\\
&= (f^0\,\bar f^j - \bar f^0\, f^j)\,(\gamma_{0j} + (\Gamma_{jh} - \Gamma_{hj})\, x^h_0) - (f^h\,\bar f^j - \bar f^h\, f^j)\,\Gamma_{hj}\\
&- (X^i_0\,\bar f^0 - \bar X^i_0\, f^0)\, G^0_{ih}\, x^h_0 - (X^i_0\,\bar f^j - \bar X^i_0\, f^j)\, G^0_{ij}\,,
\end{aligned}$$

hence, by recalling the equalities

$$\begin{aligned}
\Gamma_{hj} &= \Gamma_h{}_j{}^0_0 + \Gamma_h{}_j{}^0_k\, x^h_0\\
\gamma_{0i} &= \Gamma_h{}_i{}^0_k\, x^h_0\, x^k_0 + 2\,\Gamma_h{}_i{}^0_0\, x^h_0 + \Gamma_0{}_i{}^0_0\,,
\end{aligned}$$

we obtain

$$\begin{aligned}
i(\bar X^{\uparrow})\, i(X^{\uparrow})\,\Omega &= (f^0\,\bar f^h - \bar f^0\, f^h)\,\big(\Gamma_j{}_h{}^0_k\, x^h_0\, x^k_0 + (\Gamma_h{}_j{}^0_0 + \Gamma_j{}_h{}^0_0)\, x^h_0) + \Gamma_0{}_j{}^0_0\big)\\
&+ (f^j\,\bar f^h - \bar f^j\, f^h)\,(\Gamma_h{}_j{}^0_0 + \Gamma_h{}_j{}^0_k\, x^k_0)\\
&- (X^i_0\,\bar f^0 - \bar X^i_0\, f^0)\, G^0_{ih}\, x^h_0 - (X^i_0\,\bar f^j - \bar X^i_0\, f^j)\, G^0_{ij}\,,
\end{aligned}$$

hence, by recalling the equalities

$$\begin{aligned}
\Gamma_h{}_j{}^0_k &= -\tfrac{1}{2}\,(\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk})\\
\Gamma_h{}_j{}^0_0 &= -\tfrac{1}{2}\,(\Phi_{hj} + \partial_0 G^0_{hj})\\
\Gamma_0{}_j{}^0_0 &= -\Phi_{0j}\,,
\end{aligned}$$

we obtain

$$\begin{aligned}
i(\bar X^{\uparrow})\, i(X^{\uparrow})\,\Omega &= -(f^0\,\bar f^j - \bar f^0\, f^j)\,\big(\tfrac{1}{2}\,\partial_j G^0_{hk}\, x^h_0\, x^k_0 + \partial_0 G^0_{jh}\, x^h_0 + \Phi_{0j}\big)\\
&- \tfrac{1}{2}\,(f^j\,\bar f^h - \bar f^j\, f^h)\,\big(\Phi_{hj} + \partial_0 G^0_{hj} + (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk})\, x^k_0\big)\\
&- (X^i_0\,\bar f^0 - \bar X^i_0\, f^0)\, G^0_{ih}\, x^h_0 - (X^i_0\,\bar f^j - \bar X^i_0\, f^j)\, G^0_{ij}\\
&\\
&= -(f^0\,\bar f^j - \bar f^0\, f^j)\,\big(\tfrac{1}{2}\,\partial_j G^0_{hk}\, x^h_0\, x^k_0 + \partial_0 G^0_{jh}\, x^h_0 + \Phi_{0j}\big)\\
&- \tfrac{1}{2}\,(f^j\,\bar f^h - \bar f^j\, f^h)\,\big(\Phi_{hj} + (\partial_h G^0_{jk} - \partial_j G^0_{hk})\, x^k_0\big)\\
&- (X^i_0\,\bar f^0 - \bar X^i_0\, f^0)\, G^0_{ih}\, x^h_0 - (X^i_0\,\bar f^j - \bar X^i_0\, f^j)\, G^0_{ij}\,.\ \text{QED}
\end{aligned}$$

2.3.34. Proposition. If $f, \bar{f} \in \mathrm{Spec}(J_1\boldsymbol{E}, I\!\!R)$ and $X^\uparrow, \bar{X}^\uparrow \in \mathrm{Pro}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ are prolongations of $X[f]$ and $X[\bar{f}]$, then we obtain

$$X^\uparrow.\bar{f} - \bar{X}^\uparrow.f + i(\bar{X}^\uparrow)\, i(X^\uparrow)\, \Omega = [\![\, f,\, \bar{f}\,]\!] \,.$$

In particular,

$$X^\uparrow.\bar{f} - \bar{X}^\uparrow.f + i(\bar{X}^\uparrow)\, i(X^\uparrow)\, \Omega \in \mathrm{Spec}(J_1\boldsymbol{E}, I\!\!R)$$

depends only on f and $\bar{f}$ and not on the prolongations $X^\uparrow$ and $\bar{X}^\uparrow$ of $X[f]$ and $X[\bar{f}]$.

Proof. It follows immediately from the two Lemmas above. QED

2.3.6 Hamiltonian lift of special quadratic functions

We denote the sheaves of Hamiltonian lifts of the subalgebras

$$\mathrm{Aff}(J_1\boldsymbol{E}, I\!\!R) \subset \mathrm{Time}(J_1\boldsymbol{E}, I\!\!R) \subset \mathrm{Quan}(J_1\boldsymbol{E}, I\!\!R) \subset \mathrm{Spec}(J_1\boldsymbol{E}, I\!\!R) \subset \mathrm{Map}(J_1\boldsymbol{E}, I\!\!R)$$

respectively by

$$\mathrm{Aff}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) \subset \mathrm{Time}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) \subset \mathrm{Quan}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) \subset \mathrm{Spec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) \subset \mathrm{Ham}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})\,.$$

2.3.35. Proposition. [56] For each $f \in \mathrm{Spec}(J_1\boldsymbol{E}, I\!\!R)$, we have the coordinate expression

$$X^\uparrow_{\mathrm{Ham}}[f] = f^0\, \partial_0 - f^i\, \partial_i + X^i_0\, \partial^0_i\,,$$

with

$$X^i_0 = G^{ij}_0 \Big(\tfrac{1}{2}\, \partial_j f^0 G^0_{hk} x^h_0 x^k_0 + (\partial_j f^0_h + f^k\, (\partial_k G^0_{jh} - \partial_j G^0_{kh}) - f^0 \partial_0 G^0_{jh})\, x^h_0 + \partial_j \overset{o}{f} + \Phi_{hj} f^h + f^0 \Phi_{j0} \Big)\,. \square$$

2.3.36. Corollary. For each $f \in \mathrm{Quan}(J_1\boldsymbol{E}, I\!\!R)$, we have the coordinate expression

$$X^\uparrow_{\mathrm{Ham}}[f] = f^0\, \partial_0 - f^i\, \partial_i + X^i_0\, \partial^0_i\,,$$

with

$$X^i_0 = G^{ij}_0 \Big((\partial_j f^0_h + f^k\, (\partial_k G^0_{jh} - \partial_j G^0_{kh}) - f^0 \partial_0 G^0_{jh})\, x^h_0 + \partial_j \overset{o}{f} + \Phi_{hj} f^h + f^0 \Phi_{j0} \Big)\,. \square$$

2.3.37. Corollary. For each $f \in \mathrm{Aff}(J_1\boldsymbol{E},\, I\!R)$, we have the coordinate expression

$$X^{\uparrow}_{\mathrm{Ham}}[f] = -f^i\,\partial_i + X^i_0\,\partial^0_i\,,$$

with

$$X^i_0 = G^{ij}_0\,\Big((\partial_j f^0_h + f^k\,(\partial_k G^0_{jh} - \partial_j G^0_{kh}))\,x^h_0 + \partial_j \overset{o}{f} + \Phi_{hj} f^h\Big)\,.\,\square$$

2.3.38. Corollary. For each $f \in \mathrm{Map}(\boldsymbol{E},\, I\!R)$, we have the coordinate expression

$$X^{\uparrow}_{\mathrm{Ham}}[f] = G^{ij}_0\,\partial_j f\,\partial^0_i\,.\,\square$$

2.3.39. Lemma. For each $f'', g'' \in \mathrm{Map}(\boldsymbol{E},\, \bar{\mathbb{T}})$, we have

$$[\gamma(f''),\, \gamma(g'')] = \gamma(f'')\,\gamma.g'' - \gamma(g'')\,\gamma.f''\,.\,\square$$

2.3.40. Lemma. For each $f'' \in \mathrm{Map}(\boldsymbol{E},\, \bar{\mathbb{T}})$, we have

$$L[\gamma(f'')]\,\Lambda = -\gamma \wedge \big(\Lambda^{\sharp}(df'')\big)\,.$$

Proof. For each $X, Y \in \mathrm{Sec}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$, we obtain

$$\begin{aligned} L[\gamma(f'')]\,(X \wedge Y) &= (L[\gamma(f'')]\,X) \wedge Y + X \wedge (L[\gamma(f'')]\,Y) \\ &= f''\,\big((L[\gamma]\,X) \wedge Y + X \wedge (L[\gamma]\,Y)\big) - X.f''\,\gamma \wedge Y - Y.f''\,X \wedge \gamma \\ &= f''\,L[\gamma]\,(X \wedge Y) - \gamma \wedge (X \wedge Y)^{\sharp}(df'')\,. \end{aligned}$$

Hence, we obtain the result by recalling the equality $L[\gamma]\,\Lambda = 0$. QED

2.3.41. Lemma. For each $f'' \in \mathrm{Map}(\boldsymbol{E},\, \bar{\mathbb{T}})$ and $g \in \mathrm{Spec}(J_1\boldsymbol{E}, I\!R)$, we have

$$\Big[\gamma(f''),\, \Lambda^{\sharp}(dg)\Big] = \gamma\Big(\Lambda^{\sharp}(df'' \wedge dg)\Big) + \Lambda^{\sharp}\Big(d(\gamma(f'').g) - (\gamma.g)(df'')\Big)\,.$$

Proof. We have

$$\begin{aligned} \big[\gamma(f''),\, \Lambda^{\sharp}(dg)\big] &= \big(L[\gamma(f'')]\,\Lambda\big)^{\sharp}(dg) + \Lambda^{\sharp}\big(L[\gamma(f'')]\,dg\big) \\ &= -\big(\gamma \wedge (\Lambda^{\sharp}(df''))\big)^{\sharp}(dg) + \Lambda^{\sharp}\big(d(\gamma(f'').g)\big) \\ &= -\gamma.g\,\Lambda^{\sharp}(df'') + \gamma\big(\Lambda^{\sharp}(df'' \wedge dg)\big) + \Lambda^{\sharp}\big(d(\gamma(f'').g)\big) \\ &= \gamma\big(\Lambda^{\sharp}(df'' \wedge dg)\big) + \Lambda^{\sharp}\big(d(\gamma(f'').g) - (\gamma.g)(df'')\big)\,. \text{ QED} \end{aligned}$$

2.3.42. Proposition. For each $f, g \in \mathrm{Spec}(J_1\boldsymbol{E},\, I\!R)$, we have

$$\begin{aligned} \Big[X^{\uparrow}_{\mathrm{Ham}}[f],\, X^{\uparrow}_{\mathrm{Ham}}[g]\Big] &= \gamma\Big(f''\,\gamma.g'' - g''\,\gamma.f'' + \{g'', f\} - \{f'', g\}\Big) \\ &\quad + \Big[\Lambda^{\sharp}(df),\, \Lambda^{\sharp}(dg)\Big] \\ &\quad + \Lambda^{\sharp}\Big(d(\gamma(f'').g - \gamma(g'').f)\Big) + \Lambda^{\sharp}\Big((\gamma.f)(dg'') - (\gamma.g)(df'')\Big)\,. \end{aligned}$$

PROOF. We have

$$\begin{aligned}
\big[X^{\uparrow}_{\mathrm{Ham}}[f],\, X^{\uparrow}_{\mathrm{Ham}}[g]\big] &= \big[\gamma(f'') + \Lambda^{\sharp}(df),\, \gamma(g'') + \Lambda^{\sharp}(dg)\big] \\
&= [\gamma(f''),\, \gamma(g'')] + \big[\gamma(f''),\, \Lambda^{\sharp}(dg)\big] - \big[\gamma(g''),\, \Lambda^{\sharp}(df)\big] + \big[\Lambda^{\sharp}(df),\, \Lambda^{\sharp}(dg)\big] \\
&= \gamma\big(f''\,\gamma.g'' - g''\,\gamma.f''\big) \\
&\quad + \quad \gamma\big(\Lambda^{\sharp}(df'' \wedge dg)\big) + \Lambda^{\sharp}\big(d(\gamma(f'').g) - (\gamma.g)(df'')\big) \\
&\quad - \gamma\big(\Lambda^{\sharp}(dg'' \wedge df)\big) - \Lambda^{\sharp}\big(d(\gamma(g'').f) - (\gamma.f)(dg'')\big) \\
&\quad + \big[\Lambda^{\sharp}(df),\, \Lambda^{\sharp}(dg)\big] \\
&= \gamma(f''\,\gamma.g'' - g''\,\gamma.f'') + \gamma\big(\Lambda^{\sharp}(dg'' \wedge df - df'' \wedge dg)\big) \\
&\quad + \big[\Lambda^{\sharp}(df),\, \Lambda^{\sharp}(dg)\big] \\
&\quad + \Lambda^{\sharp}\big(d(\gamma(f'').g - \gamma(g'').f)\big) + \Lambda^{\sharp}\big((\gamma.f)(dg'') - (\gamma.g)(df'')\big)\,.\ \mathrm{QED}
\end{aligned}$$

2.3.43. Proposition. For each $f, g \in \mathrm{Spec}(J_1\boldsymbol{E},\, I\!\!R)$, we have

$$X^{\uparrow}_{\mathrm{Ham}}\Big[\,[\![\, f, g \,]\!]\,\Big] = \gamma\Big(\,[\![\, f, g \,]\!]\,''\Big) + [\Lambda^{\sharp}(df), \Lambda^{\sharp}(df)] + \Lambda^{\sharp}\Big(d(\gamma(f'').g - \gamma(g'').f)\Big)\,.$$

PROOF. We have

$$\begin{aligned}
X^{\uparrow}_{\mathrm{Ham}}\big[\,[\![f, g]\!]\,\big] &= \gamma\big(\,[\![f, g]\!]\,''\big) + \Lambda^{\sharp}\big(d\,[\![f, g]\!]\,\big) \\
&= \gamma\big(\,[\![f, g]\!]\,''\big) + \Lambda^{\sharp}\big(d\{f, g\}\big) + \Lambda^{\sharp}\big(d(\gamma(f'').g - \gamma(g'').f)\big) \\
&= \gamma\big(\,[\![f, g]\!]\,''\big) + [\Lambda^{\sharp}(df), \Lambda^{\sharp}(df)] + \Lambda^{\sharp}\big(d(\gamma(f'').g - \gamma(g'').f)\big)\,.\ \mathrm{QED}
\end{aligned}$$

2.3.44. Theorem. *For each $f, g \in \mathrm{Spec}(J_1\boldsymbol{E},\, I\!\!R)$, we have*

$$\begin{gathered}
X^{\uparrow}_{\mathrm{Ham}}\Big[\,[\![\, f, g \,]\!]\,\Big] - \Big[X^{\uparrow}_{\mathrm{Ham}}[f],\, X^{\uparrow}_{\mathrm{Ham}}[g]\Big] \\
= \gamma\Big(\,[\![\, f, g \,]\!]\,'' - f''\,\gamma.g'' + g''\,\gamma.f'' - \{g'', f\} + \{f'', g\}\Big) - \Lambda^{\sharp}\Big((\gamma.f)(dg'') - (\gamma.g)(df'')\Big)\,.\ \square
\end{gathered}$$

2.3.45. Corollary. The sheaf $\mathrm{Spec}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$ is *not* closed under the Lie bracket.

PROOF. In fact, in general, $f''\,\gamma.g'' - g''\,\gamma.f'' + \{g'', f\} - \{f'', g\} \notin \mathrm{Map}(\boldsymbol{E},\, \bar{\mathbb{T}})$, hence it is not the time component of a special quadratic function. QED

2.3.46. Example. Let us consider an affine spacetime with vanishing electromagnetic field and refer to cartesian coordinates.

Let us consider the two special quadratic functions

$$f = f^0\,\tfrac{1}{2}\,G^0_{ij}\,x^i_0\,x^j_0 \qquad \text{and} \qquad g^0 = g^0\,\tfrac{1}{2}\,G^0_{hk}\,x^h_0\,x^k_0\,.$$

Then, we obtain

$$\begin{aligned}
X^{\uparrow}_{\mathrm{Ham}}[f] &= f^0\,\partial_0 + \partial_j f^0\,G^{ij}_0\,\tfrac{1}{2}\,G^0_{hk}\,x^h_0\,x^k_0\,\partial^0_i \\
X^{\uparrow}_{\mathrm{Ham}}[g] &= g^0\,\partial_0 + \partial_l g^0\,G^{hl}_0\,\tfrac{1}{2}\,G^0_{rs}\,x^r_0\,x^s_0\,\partial^0_i\,,
\end{aligned}$$

hence

$$\Big[X^{\uparrow}_{\mathrm{Ham}}[f],\; X^{\uparrow}_{\mathrm{Ham}}[g]\Big] = (g^0\,\partial_0 f^0 - f^0\,\partial_0 g^0)\,\partial_0 + \tfrac{1}{2}\Big((\partial_k g^0\,\partial_j f^0 - \partial_k f^0\,\partial_j g^0)\,G^{ij}_0\,G^0_{rs}\,x^k_0\,x^r_0\,x^s_0\Big)\,\partial^0_i\,.$$

Indeed, the above vector field is not the Hamiltonian lift of a special quadratic function, because the vertical component is a polynomial of degree 3. □

2.3.47. Corollary. The map

$$\mathrm{Spec}(J_1\boldsymbol{E},\, I\!R) \to \mathrm{Ham}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E}) : f \mapsto X^{\uparrow}_{\mathrm{Ham}}[f]$$

is *not* a morphism of Lie algebras, with respect to the special bracket and to the Lie bracket, respectively. □

2.3.48. Example. Let us consider an affine spacetime with vanishing electromagnetic field and refer to cartesian coordinates.

Let us consider the two special quadratic functions

$$f = f^0\,\tfrac{1}{2}\,G^0_{ij}\,x^i_0\,x^j_0\,, \qquad \text{with} \qquad \partial_0 f^0 = 0\,,$$
$$g = g^0\,\tfrac{1}{2}\,G^0_{ij}\,x^i_0\,x^j_0\,, \qquad \text{with} \qquad \partial_h g^0 = 0\,.$$

Then, we obtain

$$X^{\uparrow}[f] = f^0\,\partial_0 + G^{hk}_0\,\partial_k f^0\,\tfrac{1}{2}\,G^0_{ij}\,x^i_0\,x^j_0\,\partial^0_h\,, \qquad X^{\uparrow}[g] = g^0\,\partial_0$$

and

$$[X^{\uparrow}[f],\, X^{\uparrow}[g]] = f^0\,\partial_0 g^0\,\partial_0\,.$$

On the other hand, we obtain

$$[\![\, f,\, g\,]\!] = f^0\,\partial_0 g^0\,\tfrac{1}{2}\,G^0_{ij}\,x^i_0\,x^j_0$$

and

$$X^{\uparrow}[\,[\![\, f,\, g\,]\!]\,] = f^0\,\partial_0 g^0\,\partial_0 + G^{hk}_0\,\partial_k(f^0\,\partial_0 g^0)\,\tfrac{1}{2}\,G^0_{ij}\,x^i_0\,x^j_0\,\partial^0_h\,.$$

Hence,

$$[X^{\uparrow}[f],\, X^{\uparrow}[g]] \neq X^{\uparrow}[\,[\![\, f,\, g\,]\!]\,]\,. \square$$

On the other hand, we have the following result.

2.3.49. Theorem. *For each* $f, g \in \mathrm{Quan}(J_1\boldsymbol{E}, I\!R)$. *we have*

$$[X^{\uparrow}_{\mathrm{Ham}}[f],\, X^{\uparrow}_{\mathrm{Ham}}[g]] = X^{\uparrow}_{\mathrm{Ham}}[\,[\![\, f,\, g\,]\!]\,]\,.$$

Hence, the sheaf $\mathrm{Quan}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$ *is closed with respect to the Lie bracket. Moreover, the map*

$$\mathrm{Quan}(J_1\boldsymbol{E}, I\!R) \to \mathrm{Quan}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E}) : f \mapsto X^{\uparrow}_{\mathrm{Ham}}[f]$$

is a morphism of Lie algebras, with respect to the special bracket and to the Lie bracket, respectively. Its kernel is

$$\mathrm{Map}(\boldsymbol{T}, I\!R) \subset \mathrm{Map}(\boldsymbol{E}, I\!R)\,.$$

Proof. The formula of Theorem 2.3.44

$$X^{\uparrow}_{\mathrm{Ham}}\big[\,[\![\, f, g \,]\!]\,\big] - \big[X^{\uparrow}_{\mathrm{Ham}}[f],\, X^{\uparrow}_{\mathrm{Ham}}[g]\big]$$
$$= \gamma\big(\,[\![\, f, g \,]\!]'' - f''\,\gamma.g'' + g''\,\gamma.f'' - \{g'', f\} + \{f'', g\}\big) - \Lambda^{\sharp}\big((\gamma.f)(dg'') - (\gamma.g)(df'')\big)$$

reduces to

$$X^{\uparrow}_{\mathrm{Ham}}\big[\,[\![\, f, g \,]\!]\,\big] - \big[X^{\uparrow}_{\mathrm{Ham}}[f],\, X^{\uparrow}_{\mathrm{Ham}}[g]\big] = 0\,,$$

because

$$[\![\, f, g \,]\!]'' = f''\,\gamma.g'' + g''\,\gamma.f''\,, \qquad \{g'', f\} = 0 = \{f'', g\}$$

and

$$\Lambda^{\sharp}(dg'') = 0 = \Lambda^{\sharp}(df'')\,.\ \mathrm{QED}$$

2.4 Subalgebras of the algebra of special quadratic functions

We classify the projectable vector fields of phase space which are distinguished by the classical structure. It turns out that those v.fs which preserve the full classical structure are locally generated by a special subalgebra, called the classical generators, which is also a Poisson subalgebra.

2.4.1 Subalgebra of constants of motion

Further, we consider the subalgebra of special quadratic functions which are also constants of motion.

Let us consider the subsheaf

$$\mathrm{Con}(J_1\boldsymbol{E},\, I\!R) \cap \mathrm{Spec}(J_1\boldsymbol{E},\, I\!R) \subset \mathrm{Spec}(J_1\boldsymbol{E},\, I\!R)\,.$$

2.4.1. Proposition. The sheaf $\mathrm{Con}(J_1\boldsymbol{E},\, I\!R) \cap \mathrm{Spec}(J_1\boldsymbol{E},\, I\!R)$ is closed with respect to the special bracket.

PROOF. If $f, g \in \mathrm{Con}(J_1\boldsymbol{E}, I\!R) \cap \mathrm{Spec}(J_1\boldsymbol{E}, I\!R)$, then we obtain

$$\gamma.[\![f, g]\!] = \gamma.\{f, g\} = \{\gamma.f, g\} + \{f, \gamma.g\} = 0\,.\ \text{QED}$$

2.4.2. Proposition. The sheaf $\mathrm{Con}(J_1\boldsymbol{E}, I\!R) \cap \mathrm{Spec}(J_1\boldsymbol{E}, I\!R)$ is constituted by the special quadratic functions such that

$$\begin{aligned}
\partial_j f^0\, G^0_{hk} + \partial_k f^0\, G^0_{jh} + \partial_h f^0\, G^0_{kj} &= 0\\
\partial_0 f^0\, G^0_{hk} - f^0\, \partial_0 G^0_{hk} + \partial_k f^0_h + \partial_h f^0_k - f^j\,(\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk}) &= 0\\
\partial_0 f^0_i + \partial_i \overset{o}{f} - f^0\,\Phi_{0i} - f^j\,(\partial_0 G^0_{ij} + \Phi_{ij}) &= 0\\
\partial_0 \overset{o}{f} - \Phi_{0i}\, f^i &= 0\,.
\end{aligned}$$

PROOF. Taking into account the coordinate expressions

$$\begin{aligned}
\gamma_0 &= \partial_0 + x^i_0\,\partial_i - G^{il}_0\big(\tfrac{1}{2}\,(\partial_h G^0_{lk} + \partial_k G^0_{lh} - \partial_l G^0_{hk})\,x^h_0\,x^k_0 + (\partial_0 G^0_{hl} + \Phi_{hl})\,x^h_0 + \Phi_{0l}\big)\,\partial^0_i\,,\\
f &= \tfrac{1}{2}\,f^0\,G^0_{hk}\,x^h_0\,x^k_0 + f^0_h\,x^h_0 + \overset{o}{f}\,,
\end{aligned}$$

we obtain

$$\begin{aligned}
\gamma_0.f = {} & \tfrac{1}{2}\big(\partial_j f^0\, G^0_{hk} + f^0\,\partial_j G^0_{hk} - f^0\,(\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk})\big)\,x^j_0\,x^h_0\,x^k_0\\
& + \tfrac{1}{2}\big(\partial_0 f^0\, G^0_{hk} + f^0\,\partial_0 G^0_{hk} - 2\,f^0\,(\partial_0 G^0_{hk} + \Phi_{hk}) + 2\,\partial_k f^0_h - f^j\,(\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk})\big)\,x^h_0\,x^k_0\\
& + \big(\partial_0 f^0_h + \partial_h \overset{o}{f} - f^0\,\Phi_{0h} - f^j\,(\partial_0 G^0_{hj} + \Phi_{hj})\big)\,x^h_0\\
& + \partial_0 \overset{o}{f} - \Phi_{0i}\,f^i
\end{aligned}$$

i.e.

$$\begin{aligned}
\gamma_0.f = {} & \tfrac{1}{6}\,(\partial_j f^0\, G^0_{hk} + \partial_k f^0\, G^0_{jh} + \partial_h f^0\, G^0_{kj})\,x^j_0\,x^h_0\,x^k_0\\
& + \tfrac{1}{2}\big(\partial_0 f^0\, G^0_{hk} - f^0\,\partial_0 G^0_{hk} + \partial_k f^0_h + \partial_h f^0_k - f^j\,(\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk})\big)\,x^h_0\,x^k_0\\
& + \big(\partial_0 f^0_h + \partial_h \overset{o}{f} - f^0\,\Phi_{0h} - f^j\,(\partial_0 G^0_{hj} + \Phi_{hj})\big)\,x^h_0\\
& + \partial_0 \overset{o}{f} - \Phi_{0i}\,f^i\,.\ \text{QED}
\end{aligned}$$

2.4.3. Corollary. The sheaf $\mathrm{Con}(J_1\boldsymbol{E}, I\!R) \cap \mathrm{Quan}(J_1\boldsymbol{E}, I\!R)$ is constituted by the quantisable functions such that

$$\begin{aligned}
\partial_0 f^0\, G^0_{hk} - f^0\, \partial_0 G^0_{hk} + \partial_k f^0_h + \partial_h f^0_k - f^j\,(\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk}) &= 0\\
\partial_0 f^0_i + \partial_i \overset{o}{f} - f^0\,\Phi_{0i} - f^j\,(\partial_0 G^0_{ij} + \Phi_{ij}) &= 0\\
\partial_0 \overset{o}{f} - \Phi_{0i}\, f^i &= 0\,.\ \square
\end{aligned}$$

2.4.4. Corollary. The sheaf $\mathrm{Con}(J_1\boldsymbol{E}, I\!\!R) \cap \mathrm{Time}(J_1\boldsymbol{E}, I\!\!R)$ is constituted by the quantisable functions, with constant time component, such that

$$\begin{aligned} -f^0\, \partial_0 G^0_{hk} + \partial_k f^0_h + \partial_h f^0_k - f^j\, (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk}) &= 0 \\ \partial_0 f^0_i + \partial_i \overset{o}{f} - f^0\, \Phi_{0i} - f^j\, (\partial_0 G^0_{ij} + \Phi_{ij}) &= 0 \\ \partial_0 \overset{o}{f} - \Phi_{0i}\, f^i &= 0\,. \square \end{aligned}$$

2.4.5. Corollary. The sheaf $\mathrm{Con}(J_1\boldsymbol{E}, I\!\!R) \cap \mathrm{Aff}(J_1\boldsymbol{E}, I\!\!R)$ is constituted by the affine functions such that

$$\begin{aligned} \partial_k f^0_h + \partial_h f^0_k - f^j\, (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk}) &= 0 \\ \partial_0 f^0_i + \partial_i \overset{o}{f} - f^j\, (\partial_0 G^0_{ij} + \Phi_{ij}) &= 0 \\ \partial_0 \overset{o}{f} - \Phi_{0i}\, f^i &= 0\,. \square \end{aligned}$$

2.4.6. Corollary. The sheaf $\mathrm{Con}(J_1\boldsymbol{E}, I\!\!R) \cap \mathrm{Map}(\boldsymbol{E}, I\!\!R)$ is constituted by the spacetime functions such that

$$\partial_\lambda f = 0\,.$$

Thus,

$$\mathrm{Con}(J_1\boldsymbol{E}, I\!\!R) \cap \mathrm{Map}(\boldsymbol{E}, I\!\!R) = I\!\!R\,. \square$$

The systems of first order linear partial differential equations in the above Proposition and Corollaries would deserve an analysis which is beyond the scope of the present paper. The above sheaves might reduce to $I\!\!R$, unless strong conditions are imposed to spacetime. Here, we just show non constant examples of special quadratic functions which are also constant of motion in the simplest case of spacetime.

2.4.7. Example. Let us consider an affine spacetime with vanishing electromagnetic field and refer to an inertial observer o and to an adapted cartesian chart.

Then, the sheaf $\mathrm{Con}(J_1\boldsymbol{E}, I\!\!R) \cap \mathrm{Spec}(J_1\boldsymbol{E}, I\!\!R)$ is constituted by the special quadratic functions such that

$$\begin{gathered} \partial_h f^0 = 0 \\ \partial_1 f^0_1 = \dots = \partial_n f^0_n = -\tfrac{1}{2}\, \partial_0 f^0\,, \qquad \partial_i f^0_j + \partial_j f^0_i = 0\,, \quad i \neq j \\ \partial_i \overset{o}{f} = -\partial_0 f^0_i\,, \qquad \partial_0 \overset{o}{f} = 0\,. \end{gathered}$$

In particular, we obtain

$$\mathrm{Con}(J_1\boldsymbol{E}, I\!\!R) \cap \mathrm{Spec}(J_1\boldsymbol{E}, I\!\!R) \subset \mathrm{Quan}(J_1\boldsymbol{E}, I\!\!R)\,.$$

For instance,

$$\mathcal{K}_0\,,\ \mathcal{Q}_i\,,\ \epsilon_{i_1 i_2 \ldots i_n}\, x^{i_1}\, \mathcal{Q}_0^{i_2}\ \in\ \mathrm{Con}(J_1\boldsymbol{E},\ I\!R) \cap \mathrm{Spec}(J_1\boldsymbol{E},\ I\!R)\,.$$

PROOF. $\mathrm{Con}(J_1\boldsymbol{E},\ I\!R) \cap \mathrm{Spec}(J_1\boldsymbol{E},\ I\!R)$ is constituted by the special quadratic functions such that

$$\begin{aligned}
\partial_j f^0\, G^0_{hk} + \partial_k f^0\, G^0_{jh} + \partial_h f^0\, G^0_{kj} &= 0 \\
\partial_0 f^0\, G^0_{hk} + \partial_k f^0_h + \partial_h f^0_k &= 0 \\
\partial_0 f^0_h + \partial_h \overset{o}{f} &= 0 \\
\partial_0 \overset{o}{f} &= 0\,.\ \text{QED}
\end{aligned}$$

2.4.2 Holonomic subalgebra

We can compare the holonomic and Hamiltonian lifts of a quantisable function. The quantisable functions whose holonomic and Hamiltonian lifts coincide constitute a subalgebra.

2.4.8. Definition. The *holonomic lift* of a quantisable function $f \in \mathrm{Quan}(J_1\boldsymbol{E},\ I\!R)$ is defined to be the holonomic prolongation of the tangent lift of f

$$X^{\uparrow}_{\mathrm{hol}}[f] := (X[f])_{(1)}\,.\ \square$$

2.4.9. Proposition. Let $f \in \mathrm{Quan}(J_1\boldsymbol{E},\ I\!R)$. Then, we have the coordinate expression

$$X^{\uparrow}_{\mathrm{hol}}[f] = f^0\, \partial_0 - f^i\, \partial_i - (\partial_0 f^i + \partial_j f^i\, x^j_0 + \partial_0 f^0\, x^i_0)\, \partial^0_i\,.\ \square$$

2.4.10. Theorem. *If* $f, g \in \mathrm{Quan}(J_1\boldsymbol{E},\ I\!R)$*, then we obtain*

$$\Big[X^{\uparrow}_{\mathrm{hol}}[f],\ X^{\uparrow}_{\mathrm{hol}}[g]\Big] = X^{\uparrow}_{\mathrm{hol}}\Big[\,[\![f, g]\!]\,\Big]\,.$$

Hence, the map

$$\mathrm{Quan}(J_1\boldsymbol{E},\ I\!R) \to \mathrm{Sec}(J_1\boldsymbol{E},\ TJ_1\boldsymbol{E}) : f \mapsto X^{\uparrow}_{\mathrm{hol}}[f]$$

is a morphism of Lie algebras. Its kernel is $\mathrm{Map}(\boldsymbol{E},\ I\!R)$.

PROOF. If $f, g \in \mathrm{Quan}(J_1\boldsymbol{E},\ I\!R)$, then, in virtue of Proposition 1.6.3 and Proposition 2.3.30, we obtain

$$\begin{aligned}
[X^{\uparrow}_{\mathrm{hol}}[f],\ X^{\uparrow}_{\mathrm{hol}}[g]] &:= \big[(X[f])_{(1)},\ (X[g])_{(1)}\big] = \big([X[f], X[g]]\big)_{(1)} = \big(X\,[\,[\![f, g]\!]\,]\big)_{(1)} \\
&:= X^{\uparrow}_{\mathrm{hol}}\,[\,[\![f, g]\!]\,]\,.\ \text{QED}
\end{aligned}$$

2.4.11. Definition. A quantisable function f is said to be *holonomic* if

$$X^{\uparrow}_{\mathrm{hol}}[f] = X^{\uparrow}_{\mathrm{Ham}}[f] \,. \square$$

We denote the sheaf of holonomic functions by

$$\mathrm{Hol}(J_1\boldsymbol{E},\, I\!R) \subset \mathrm{Quan}(J_1\boldsymbol{E},\, I\!R) \,.$$

2.4.12. Proposition. The sheaf $\mathrm{Hol}(J_1\boldsymbol{E},\, I\!R)$ is closed with respect to the special bracket.

PROOF. If $f, g \in \mathrm{Hol}(J_1\boldsymbol{E},\, I\!R)$, then, in virtue of Proposition 1.6.3, Proposition 2.3.30 and Theorem 2.3.49, we obtain

$$\begin{aligned}
X^{\uparrow}_{\mathrm{Ham}}[\,[\![f, g]\!]\,] &= [X^{\uparrow}_{\mathrm{Ham}}[f],\, X^{\uparrow}_{\mathrm{Ham}}[g]] \\
&= [X^{\uparrow}_{\mathrm{hol}}[f],\, X^{\uparrow}_{\mathrm{hol}}[g]] := [(X[f])_{(1)},\, (X[g])_{(1)}] \\
&= \big([X[f],\, X[g]]\big)_{(1)} \\
&= \big(X[\,[\![f, g]\!]\,]\big)_{(1)} := X^{\uparrow}_{\mathrm{hol}}[\,[\![f, g]\!]\,] \,. \mathrm{QED}
\end{aligned}$$

2.4.13. Proposition. The sheaf $\mathrm{Hol}(J_1\boldsymbol{E},\, I\!R)$ is constituted by the quantisable functions such that

$$\begin{aligned}
\partial_0 f^0 &= -\partial_i f^i - G^{ij}_0\, (\partial_j f^0_i + f^k\, (\partial_k G^0_{ij} - \partial_j G^0_{ik}) - f^0\, \partial_0 G^0_{ij}) \,, \qquad \text{no sum on } i \\
0 &= -\partial_j f^i - G^{ih}_0\, (\partial_h f^0_j + f^k\, (\partial_k G^0_{jh} - \partial_h G^0_{jk}) - f^0\, \partial_0 G^0_{jh}) \,, \quad i \neq j \\
\partial_0 f^i &= -G^{ij}_0\, (\partial_j \overset{o}{f} + \Phi_{hj} f^h + f^0 \Phi_{j0}) \,.
\end{aligned}$$

PROOF. The equality

$$\begin{aligned}
G^{ij}_0\, (\partial_j f^0_h + f^k\, (\partial_k G^0_{jh} - \partial_j G^0_{kh}) - f^0 \partial_0 G^0_{jh})\, x^h_0 + \partial_j \overset{o}{f} + \Phi_{hj} f^h + f^0 \Phi_{j0}) = \\
= -(\partial_0 f^i + \partial_j f^i\, x^j_0 + \partial_0 f^0\, x^i_0)
\end{aligned}$$

is equivalent to the system

$$\begin{aligned}
G^{ij}_0\, (\partial_j f^0_h + f^k\, (\partial_k G^0_{jh} - \partial_j G^0_{kh}) - f^0 \partial_0 G^0_{jh})\, x^h_0 &= -(\partial_h f^i\, x^h_0 + \partial_0 f^0\, x^i_0) \\
\partial_0 f^i &= -G^{ij}_0\, (\partial_j \overset{o}{f} + \Phi_{hj} f^h + f^0 \Phi_{j0}) \,. \mathrm{QED}
\end{aligned}$$

2.4.14. Corollary. The sheaf $\mathrm{Hol}(J_1\boldsymbol{E},\, I\!R) \cap \mathrm{Time}(J_1\boldsymbol{E},\, I\!R)$ is constituted by the special quadratic functions, with constant time component, such that

$$\begin{aligned}
0 &= -\partial_i f^i - G^{ij}_0\, (\partial_j f^0_i + f^k\, (\partial_k G^0_{ij} - \partial_j G^0_{ik}) - f^0\, \partial_0 G^0_{ij}) \,, \qquad \text{no sum on } i \\
0 &= -\partial_j f^i - G^{ih}_0\, (\partial_h f^0_j + f^k\, (\partial_k G^0_{jh} - \partial_h G^0_{jk}) - f^0\, \partial_0 G^0_{jh}) \,, \quad i \neq j \\
\partial_0 f^i &= -G^{ij}_0\, (\partial_j \overset{o}{f} + \Phi_{hj} f^h + f^0 \Phi_{j0}) \,. \square
\end{aligned}$$

2.4.15. Corollary. The sheaf $\mathrm{Hol}(J_1\boldsymbol{E}, I\!R) \cap \mathrm{Aff}(J_1\boldsymbol{E}, I\!R)$ is constituted by the affine functions such that

$$\begin{aligned}
0 &= -\partial_i f^i - G^{ij}_0 \, (\partial_j f^0_i + f^k \, (\partial_k G^0_{ij} - \partial_j G^0_{ik}))\,, \qquad \text{no sum on } i \\
0 &= -\partial_j f^i - G^{ih}_0 \, (\partial_h f^0_j + f^k \, (\partial_k G^0_{jh} - \partial_h G^0_{jk}))\,, \quad i \neq j \\
\partial_0 f^i &= -G^{ij}_0 \, (\partial_j \overset{o}{f} + \Phi_{hj} f^h)\,. \square
\end{aligned}$$

2.4.16. Corollary. We have

$$\mathrm{Hol}(J_1\boldsymbol{E}, I\!R) \cap \mathrm{Map}(\boldsymbol{E}, I\!R) = \mathrm{Map}(\boldsymbol{T}, I\!R)\,. \square$$

The systems of first order linear partial differential equations in the above Proposition would deserve an analysis which is beyond the scope of the present paper. The above sheaves might reduce to $I\!R$, unless strong conditions are imposed to spacetime. Here, we just show non constant examples of special quadratic functions which are also constant of motion in the simplest case of spacetime.

2.4.17. Example. Let us consider an affine spacetime with vanishing electromagnetic field and refer to a cartesian chart.

Then, $f \in \mathrm{Hol}(J_1\boldsymbol{E}, I\!R)$ if and only if

$$\partial_\lambda f^0 = -2\,\partial_1 f^0_1 = \cdots = -2\,\partial_n f^0_n\,, \qquad \partial_j f^0_i = -\partial_i f^0_j\,, \qquad \partial_0 f^0_i = -\partial_i \overset{o}{f}\,.$$

2.4.3 Self–holonomic subalgebra

2.4.18. Definition. A function $f \in \mathrm{Quan}(J_1\boldsymbol{E}, I\!R)$ is said to be *self–holonomic* if

$$i\Big(X^{\uparrow}_{\mathrm{hol}}[f]\Big)\,\Omega = df\,. \square$$

The subsheaf of self–holonomic functions is denoted by

$$\mathrm{Self}(J_1\boldsymbol{E}, I\!R) \subset \mathrm{Quan}(J_1\boldsymbol{E}, I\!R)\,.$$

2.4.19. Lemma. If $f \in \mathrm{Self}(J_1\boldsymbol{E}, I\!R)$, then we have

$$\gamma.f = 0\,,$$

hence

$$\mathrm{Self}(J_1\boldsymbol{E}, I\!R) \subset \mathrm{Con}(J_1\boldsymbol{E}, I\!R) \cap \mathrm{Quan}(J_1\boldsymbol{E}, I\!R)\,.$$

PROOF. We have

$$\gamma.f = i(\gamma)\,df = i(\gamma)\,i(X^{\uparrow}_{\mathrm{hol}}[f])\,\Omega = -i(X^{\uparrow}_{\mathrm{hol}}[f])\,i(\gamma)\,\Omega = 0\,. \mathrm{QED}$$

2.4.20. Lemma. If $f \in \operatorname{Self}(J_1\boldsymbol{E},\ I\!R)$, then we have

$$X^{\uparrow}_{\mathrm{hol}}[f] = X^{\uparrow}_{\mathrm{Ham}}[f]\,,$$

hence

$$\operatorname{Self}(J_1\boldsymbol{E},\ I\!R) \subset \operatorname{Hol}(J_1\boldsymbol{E},\ I\!R)\,.$$

Proof. The equality

$$i\big(X^{\uparrow}_{\mathrm{hol}}[f]\big)\,\Omega = df$$

yields

$$\begin{aligned} X^{\uparrow}_{\mathrm{Ham}}[f] &:= \gamma(f'') + \Lambda^{\sharp}(df) \\ &= \gamma(f'') + \Lambda^{\sharp}(\Omega^{\flat}(X^{\uparrow}_{\mathrm{hol}}[f])) \\ &= \gamma(f'') + X^{\uparrow}_{\mathrm{hol}}[f] - \gamma(X^{\uparrow}_{\mathrm{hol}}[f]) \\ &= \gamma(f'') + X^{\uparrow}_{\mathrm{hol}}[f] - \gamma(X[f]) \\ &= \gamma(f'') + X^{\uparrow}_{\mathrm{hol}}[f] - \gamma(f'') \\ &= X^{\uparrow}_{\mathrm{hol}}[f]\,.\ \mathrm{QED} \end{aligned}$$

2.4.21. Proposition. For each $f, g \in \operatorname{Self}(J_1\boldsymbol{E}, I\!R)$, we obtain

$$\Omega^{\flat}\big(X^{\uparrow}_{\mathrm{hol}}\big[\,[\![f, g]\!]\,]\big) = d\,[\![f, g]\!]\ .$$

Hence, the sheaf $\operatorname{Self}(J_1\boldsymbol{E},\ I\!R)$ is closed with respect to the special bracket.

Proof. If $f, g \in \operatorname{Self}(J_1\boldsymbol{E},\ I\!R)$, then, by recalling that $\operatorname{Self}(J_1\boldsymbol{E},\ I\!R) \subset \operatorname{Hol}(J_1\boldsymbol{E},\ I\!R)$, we obtain

$$\begin{aligned} \Lambda^{\sharp}\big(\Omega^{\flat}(X^{\uparrow}_{\mathrm{hol}}[\,[\![f, g]\!]\,])\big) &= X^{\uparrow}_{\mathrm{hol}}[\,[\![f, g]\!]\,] - \gamma(X^{\uparrow}_{\mathrm{hol}}[\,[\![f, g]\!]\,]) \\ &= X^{\uparrow}_{\mathrm{hol}}[\,[\![f, g]\!]\,] - \gamma(X[\,[\![f, g]\!]\,]) \\ &= X^{\uparrow}_{\mathrm{hol}}[\,[\![f, g]\!]\,] - \gamma(\,[\![f, g]\!]'') \\ &= X^{\uparrow}_{\mathrm{Ham}}[\,[\![f, g]\!]\,] - \gamma(\,[\![f, g]\!]'') \\ &= \Lambda^{\sharp}\big(d\,[\![f, g]\!]\big) \end{aligned}$$

and, by recalling that $\operatorname{Self}(J_1\boldsymbol{E},\ I\!R) \subset \operatorname{Con}(J_1\boldsymbol{E},\ I\!R)$, we obtain

$$i(\gamma)\,\Omega^{\flat}(X^{\uparrow}_{\mathrm{hol}}[\,[\![f, g]\!]\,]) = 0$$

and

$$\begin{aligned} i(\gamma)\,d\,[\![f, g]\!] &= i(\gamma)\,d\big(\{f, g\} + \gamma(f'').g - \gamma(g'').f\big) \\ &= \{\gamma.f, g\} + \{f, \gamma.g\} + i(\gamma)\,d\big(\gamma(f'').g - \gamma(g'').f\big) = 0\,. \end{aligned}$$

Hence,

$$\Omega^{\flat}(X^{\uparrow}_{\mathrm{hol}}[\,[\![f, g]\!]\,]) = d\,[\![f, g]\!]\ .\ \square$$

Now, we state the conditions aimed at classifying the fine functions.

2.4.22. Lemma. If $f \in \mathrm{Quan}(J_1\boldsymbol{E}, I\!R)$, then we obtain

$$\begin{aligned} i\Big(X^{\uparrow}_{\mathrm{hol}}[f]\Big)\,\Omega = \Big(& f^0\,(\tfrac{1}{2}\,\partial_i G^0_{hk}\,x^i_0\,x^h_0\,x^k_0 + \partial_0 G^0_{hk}\,x^h_0\,x^k_0 + \Phi_{0i}\,x^i_0) \\ &+ f^j\,(\tfrac{1}{2}\,(\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk})\,x^h_0\,x^k_0 + (\partial_0 G^0_{hj} + \Phi_{hj})\,x^h_0 + \Phi_{0j})\Big)\,d^0 \\ &+ \Big((\tfrac{1}{2}\,f^0\,\partial_i G^0_{hk}\,x^h_0\,x^k_0 \\ &\quad + (f^0\,\partial_0 G^0_{hi} + f^j\,(\partial_i G^0_{jh} - \partial_j G^0_{ih}) - G^0_{ih}\,\partial_0 f^0 - G^0_{ij}\,\partial_h f^j)\,x^h_0 \\ &\quad + f^j\,\Phi_{ij} - G^0_{ij}\,\partial_0 f^j + f^0\,\Phi_{0i}\Big)\,\theta^i \\ &+ G^0_{ij}\,(f^0\,x^j_0 + f^j)\,d^i_0\,. \end{aligned}$$

Proof. We have

$$\begin{aligned} i(X^{\uparrow}_{\mathrm{hol}}[f])\,\Omega &= G^0_{ij}\,\big(f^0\,(-\gamma_0{}^i{}_0\,\theta^j + \Gamma_h{}^i{}_0\,x^h_0)\,\theta^j + f^0\,x^j_0(d^i_0 - \gamma_0{}^i{}_0\,d^0 - \Gamma_h{}^i{}_0\,\theta^h) \\ &\qquad + f^h\,\Gamma_h{}^i{}_0\,\theta^j + f^j\,(d^i_0 - \gamma_0{}^i{}_0\,d^0 - \Gamma_h{}^i{}_0\,\theta^h) \\ &\qquad - (\partial_0 f^i + \partial_j f^i\,x^j_0 + \partial_0 f^0\,x^i_0)\big) \\ &= -\gamma_{0i}\,(f^0\,x^i_0 + f^i)\,d^0 \\ &\qquad + \big((\Gamma_{ji} - \Gamma_{ij})(f^0\,x^j_0 + f^j) - G^0_{ij}\,(\partial_0 + x^h_0\,\partial_h)(f^0\,x^j_0 + f^j) - f^0\,\gamma_{0i}\big)\,\theta^i \\ &\qquad + G^0_{ij}\,(f^0\,x^j_0 + f^j)\,d^i_0\,.\ \mathrm{QED} \end{aligned}$$

2.4.23. Lemma. If $f \in \mathrm{Quan}(J_1\boldsymbol{E}, I\!R)$, then we obtain

$$\begin{aligned} df = & \\ = \Big(& f^0\,\tfrac{1}{2}\,\partial_i G^0_{hk}\,x^i_0\,x^h_0\,x^k_0 + \tfrac{1}{2}\,(\partial_0 f^0 G^0_{hk} + f^0\partial_0 G^0_{hk} + \partial_h f^0_k + \partial_k f^0_h)\,x^h_0\,x^k_0 \\ &\qquad + (\partial_0 f^0_h + \partial_h \overset{o}{f})\,x^h_0 + \partial_0 \overset{o}{f}\Big)\,d^0 \\ &+ \Big(f^0\,\tfrac{1}{2}\,\partial_i G^0_{hk}\,x^h_0\,x^k_0 + \partial_i f^0_h\,x^h_0 + \partial_i \overset{o}{f}\Big)\,\theta^i \\ &+ G^0_{ih}\,(f^0\,x^h_0 + f^h)\,d^i_0\,.\ \square \end{aligned}$$

2.4.24. Proposition. The sheaf $\mathrm{Self}(J_1\boldsymbol{E}, I\!R)$ is constituted by the quantisable functions such that

$$\begin{aligned} f^0\,\partial_0 G^0_{hk} - \partial_0 f^0\,G^0_{hk} - f^i\,\partial_i G^0_{hk} - G^0_{ih}\,\partial_k f^i - G^0_{ik}\,\partial_h f^i &= 0 \\ \partial_0 \overset{o}{f} - f^i\,\Phi_{0i} &= 0 \\ \partial_j \overset{o}{f} + G^0_{ij}\partial_0 f^i - f^0\,\Phi_{0j} + f^i\,\Phi_{ij} &= 0\,. \end{aligned}$$

PROOF. The above equality is fulfilled if and only if

$$\big(f^0\,\partial_0 G^0_{hk} + f^j\,(\tfrac12\,(\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk})\big)\,x^h_0\,x^k_0 = \tfrac12\,(\partial_0 f^0 G^0_{hk} + f^0\partial_0 G^0_{hk} + \partial_h f^0_k + \partial_k f^0_h)\,x^h_0\,x^k_0$$
$$(\partial_j \overset{o}{f} - f^0\,\Phi_{0j} + f^h\,\Phi_{hj} + G^0_{jh}\,\partial_0 f^h)\,x^j = 0$$
$$f^i\,\Phi_{0i} - \partial_0 \overset{o}{f} = 0$$

i.e. if and only if

$$\begin{aligned}
\partial_0 f^0\,G^0_{hk} - f^0\,\partial_0 G^0_{hk} + f^j\,\partial_j G^0_{hk} + G^0_{ki}\,\partial_h f^j + G^0_{hi}\,\partial_k f^j &= 0\\
\partial_j \overset{o}{f} - f^0\,\Phi_{0j} + f^h\,\Phi_{hj} + G^0_{jh}\,\partial_0 f^h &= 0\\
f^i\,\Phi_{0i} - \partial_0 \overset{o}{f} &= 0\\
(f^0\,\partial_0 G^0_{hi} + f^j\,(\partial_i G^0_{jh} - \partial_j G^0_{ih}) - G^0_{ih}\,\partial_0 f^0 - G^0_{ij}\,\partial_h f^j)\,x^h_0 &= \partial_i f^0_h\,x^h\\
f^j\,\Phi_{ij} - G^0_{ij}\,\partial_0 f^j + f^0\,\Phi_{0i} &= \partial_i \overset{o}{f}\,.
\end{aligned}$$

On the other hand, in the above system, the 4-th equation is consequence of the 1-st equation and the 5-th equation is a consequence of the 2-nd equation. QED

The above result can be re–expressed in the following way through the tangent lift of the quantisable function.

2.4.25. Proposition. The sheaf $\mathrm{Self}(J_1\boldsymbol{E},\,I\!\!R)$ is constituted by the quantisable functions such that

$$\begin{aligned}
\partial_\mu G^0_{hk}\,X[f]^\mu + G^0_{ih}\,\partial_k(X[f]^i) + G^0_{ik}\,\partial_h(X[f]^i) &= \partial_0 X[f]^0\,G^0_{hk}\\
\partial_0 \overset{o}{f} + X[f]^i\,\Phi_{0i} &= 0\\
\partial_j \overset{o}{f} - G^0_{ij}\partial_0(X[f]^i) &= X[f]^\mu\,\Phi_{\mu j}\,,
\end{aligned}$$

i.e. , such that

$$\begin{aligned}
\partial_0 G^0_{hk}\,f^0 - \partial_i G^0_{hk}\,f^i - G^0_{ih}\,\partial_k f^i - G^0_{ik}\,\partial_h f^i &= \partial_0 f^0\,G^0_{hk}\\
\partial_0 \overset{o}{f} - f^i\,\Phi_{0i} &= 0\\
\partial_j \overset{o}{f} + G^0_{ij}\partial_0 f^i &= f^0\,\Phi_{0j} - f^i\,\Phi_{ij}\,.
\end{aligned}$$

2.4.26. Remark. Let $f \in \mathrm{Quan}(J_1\boldsymbol{E},\,I\!\!R)$. Then, the equality

$$\partial_\mu G^0_{hk}\,X[f]^\mu + G^0_{ih}\,\partial_k(X[f]^i) + G^0_{ik}\,\partial_h(X[f]^i) \quad = \partial_0 f^0\,G^0_{hk}$$

is the coordinate expression of

$$L\Big[X[f]\Big]\,G = df''\,G\,.\,\square$$

2.4.27. Corollary. The sheaf $\mathrm{Self}(J_1\boldsymbol{E}, I\!\!R) \cap \mathrm{Time}(J_1\boldsymbol{E}, I\!\!R)$ is constituted by the quantisable functions such that

$$\begin{aligned} f^i\,\partial_i G^0_{hk} + G^0_{ih}\,\partial_k f^i + G^0_{ik}\,\partial_h f^i - f^0\,\partial_0 G^0_{hk} &= 0 \\ \partial_0 \overset{o}{f} - f^i\,\Phi_{0i} &= 0 \\ \partial_j \overset{o}{f} + G^0_{ij}\partial_0 f^i - f^0\,\Phi_{0j} + f^i\,\Phi_{ij} &= 0\,. \end{aligned}$$

2.4.28. Corollary. The sheaf $\mathrm{Self}(J_1\boldsymbol{E}, I\!\!R) \cap \mathrm{Aff}(J_1\boldsymbol{E}, I\!\!R)$ is constituted by the quantisable functions such that

$$\begin{aligned} f^i\,\partial_i G^0_{hk} + G^0_{ih}\,\partial_k f^i + G^0_{ik}\,\partial_h f^i &= 0 \\ \partial_0 \overset{o}{f} - f^i\,\Phi_{0i} &= 0 \\ \partial_j \overset{o}{f} + G^0_{ij}\partial_0 f^i + f^i\,\Phi_{ij} &= 0\,. \end{aligned}$$

2.4.29. Corollary. We have

$$\mathrm{Self}(J_1\boldsymbol{E}, I\!\!R) \cap \mathrm{Map}(\boldsymbol{E}, I\!\!R) = I\!\!R\,.\,\square$$

2.4.30. Example. Let us consider an affine spacetime with vanishing electromagnetic field and refer to an inertial observer o and to an adapted cartesian chart.

Then, the sheaf $\mathrm{Self}(J_1\boldsymbol{E}, I\!\!R)$ is constituted by the quantisable functions such that

$$\begin{gathered} \partial_0 f^0 = -2\,\partial_1 f^1 = \dots = -2\,\partial_n f^n \\ \partial_i f^0_j + \partial_j f^0_i = 0 \\ \partial_0 \overset{o}{f} = 0 \\ \partial_i \overset{o}{f} = -\partial_0 f^0_i\,. \end{gathered}$$

For instance,

$$\mathcal{K}_0\,,\ \mathcal{Q}_i\,,\ \epsilon_{i_1 i_2 \dots i_n}\, x^{i_1}\, \mathcal{Q}^{i_2}_0 \ \in\ \mathrm{Self}(J_1\boldsymbol{E}, I\!\!R)\,.\,\square$$

2.4.31. Proposition. Let $f \in \mathrm{Quan}(J_1\boldsymbol{E}, I\!\!R)$. Then, the following conditions are equivalent:

1) $\quad L\Big[X^{\uparrow}_{\mathrm{hol}}[f]\Big]\,\Omega = 0 \qquad \text{with} \qquad f \in \mathrm{Con}(J_1\boldsymbol{E}, I\!\!R)\,;$

2) $\quad X^{\uparrow}_{\mathrm{hol}}[f] = X^{\uparrow}_{\mathrm{Ham}}[f]\,, \qquad \text{with} \qquad f \in \mathrm{Con}(J_1\boldsymbol{E}, I\!\!R)\,;$

3) $$i\Big[X^{\uparrow}_{\mathrm{hol}}[f]\Big]\,\Omega = df\,.$$

PROOF. 1) $\Rightarrow$ 2). If $L\big[X^{\uparrow}_{\mathrm{hol}}[f]\big]\,\Omega = 0$, then

$$di\big(X^{\uparrow}_{\mathrm{hol}}[f]\big)\,\Omega = 0\,,$$

hence, locally,

$$i\big(X^{\uparrow}_{\mathrm{hol}}[f]\big)\,\Omega = dg\,, \qquad \text{with} \qquad g \in \mathrm{Con}(J_1\boldsymbol{E},\, I\!R)\,,$$

hence, in virtue of Proposition 2.2.3,

$$X^{\uparrow}_{\mathrm{hol}}[f] = \gamma(f'') + \Lambda^{\sharp}(dg) := X^{\uparrow}_{\mathrm{Ham}}[f'', g]\,.$$

Moreover, being $X^{\uparrow}_{\mathrm{hol}}[f]$ projectable on $\boldsymbol{E}$, in virtue of Theorem 2.3.5,

$$g \in \mathrm{Spec}(J_1\boldsymbol{E}, I\!R) \qquad \text{and} \qquad g'' = f''\,.$$

Hence, we obtain

$$X^{\uparrow}_{\mathrm{hol}}[f] = X^{\uparrow}_{\mathrm{Ham}}[g]\,,$$

which yields

$$X[f] = X[g]\,,$$

hence

$$f = g + h\,, \qquad \text{with} \qquad h \in \mathrm{Map}(\boldsymbol{E}, I\!R)\,.$$

On the other hand, $f, g \in \mathrm{Con}(J_1\boldsymbol{E},\, I\!R)$ implies $h \in \mathrm{Map}(\boldsymbol{E}, I\!R) \cap \mathrm{Con}(J_1\boldsymbol{E},\, I\!R) = I\!R$.
Therefore, we obtain

$$X^{\uparrow}_{\mathrm{hol}}[f] = X^{\uparrow}_{\mathrm{Ham}}[f]$$

in the domain of definition of g. But, if the above equality holds locally, then it holds in the domain of definition of f.

2) $\Rightarrow$ 3). If $f \in \mathrm{Con}(J_1\boldsymbol{E},\, I\!R)$, then, in virtue of Proposition 2.2.3, we obtain

$$i\big(X^{\uparrow}_{\mathrm{Ham}}[f]\big)\,\Omega = i\big(\gamma(f'') + \Lambda^{\sharp}(df)\big)\,\Omega = df\,.$$

Hence, $X^{\uparrow}_{\mathrm{hol}}[f] = X^{\uparrow}_{\mathrm{Ham}}[f]$ implies

$$i\big(X^{\uparrow}_{\mathrm{hol}}[f]\big)\,\Omega = df\,.$$

3) $\Rightarrow$ 1). If $i\big(X^{\uparrow}_{\mathrm{hol}}[f]\big)\,\Omega = df$, then

$$L\big(X^{\uparrow}_{\mathrm{hol}}[f]\big)\,\Omega = di\big(X^{\uparrow}_{\mathrm{hol}}[f]\big)\,\Omega = 0\,.\ \mathrm{QED}$$

2.4.32. Proposition. Let $f \in \text{Quan}(J_1\boldsymbol{E}, I\!R)$. Then, the following conditions are equivalent:

1) $$L\Big[X^{\uparrow}_{\text{hol}}[f]\Big]\,\Omega = 0 \qquad \text{with} \qquad f \in \text{Con}(J_1\boldsymbol{E}, I\!R)\,;$$

2) $$X^{\uparrow}_{\text{hol}}[f] = X^{\uparrow}_{\text{Ham}}[f]\,, \qquad \text{with} \qquad f \in \text{Con}(J_1\boldsymbol{E}, I\!R)\,;$$

3) $$i\Big[X^{\uparrow}_{\text{hol}}[f]\Big]\,\Omega = df\,.$$

Proof. 1) $\Rightarrow$ 2). If $L\big[X^{\uparrow}_{\text{hol}}[f]\big]\,\Omega = 0$, then

$$di\big(X^{\uparrow}_{\text{hol}}[f]\big)\,\Omega = 0\,,$$

hence, locally,

$$i\big(X^{\uparrow}_{\text{hol}}[f]\big)\,\Omega = dg\,, \qquad \text{with} \qquad g \in \text{Con}(J_1\boldsymbol{E}, I\!R)\,,$$

hence, in virtue of Proposition 2.2.3,

$$X^{\uparrow}_{\text{hol}}[f] = \gamma(f'') + \Lambda^{\sharp}(dg) := X^{\uparrow}_{\text{Ham}}[f'', g]\,.$$

Moreover, being $X^{\uparrow}_{\text{hol}}[f]$ projectable on $\boldsymbol{E}$, in virtue of Theorem 2.3.5,

$$g \in \text{Spec}(J_1\boldsymbol{E}, I\!R) \qquad \text{and} \qquad g'' = f''\,.$$

Hence, we obtain

$$X^{\uparrow}_{\text{hol}}[f] = X^{\uparrow}_{\text{Ham}}[g]\,,$$

which yields

$$X[f] = X[g]\,,$$

hence

$$f = g + h\,, \qquad \text{with} \qquad h \in \text{Map}(\boldsymbol{E}, I\!R)\,.$$

On the other hand, $f, g \in \text{Con}(J_1\boldsymbol{E}, I\!R)$ implies $h \in \text{Map}(\boldsymbol{E}, I\!R) \cap \text{Con}(J_1\boldsymbol{E}, I\!R) = I\!R$.
Therefore, we obtain

$$X^{\uparrow}_{\text{hol}}[f] = X^{\uparrow}_{\text{Ham}}[f]$$

in the domain of definition of g. But, if the above equality holds locally, then it holds in the domain of definition of f.

2) $\Rightarrow$ 3). If $f \in \text{Con}(J_1\boldsymbol{E}, I\!R)$, then, in virtue of Proposition 2.2.3, we obtain

$$i\big(X^{\uparrow}_{\text{Ham}}[f]\big)\,\Omega = i\big(\gamma(f'') + \Lambda^{\sharp}(df)\big)\,\Omega = df\,.$$

Hence, $X^{\uparrow}_{\text{hol}}[f] = X^{\uparrow}_{\text{Ham}}[f]$ implies

$$i\big(X^{\uparrow}_{\text{hol}}[f]\big)\,\Omega = df\,.$$

3) $\Rightarrow$ 1). If $i\big(X^{\uparrow}_{\text{hol}}[f]\big)\,\Omega = df$, then

$$L\big(X^{\uparrow}_{\text{hol}}[f]\big)\,\Omega = di\big(X^{\uparrow}_{\text{hol}}[f]\big)\,\Omega = 0\,.\ \text{QED}$$

2.4.33. Theorem. *We have*

$$\begin{aligned}\operatorname{Self}(J_1\boldsymbol{E},\, I\!R) &= \operatorname{Con}(J_1\boldsymbol{E},\, I\!R) \cap \operatorname{Quan}(J_1\boldsymbol{E},\, I\!R)\\ \operatorname{Self}(J_1\boldsymbol{E},\, I\!R) &= \operatorname{Con}(J_1\boldsymbol{E},\, I\!R) \cap \operatorname{Hol}(J_1\boldsymbol{E},\, I\!R)\,.\end{aligned}$$

Proof. 1) The classifying system of Proposition 2.4.25 and the classifying system of Corollary 2.4.3 coincide. Hence

$$\operatorname{Self}(J_1\boldsymbol{E},\, I\!R) = \operatorname{Con}(J_1\boldsymbol{E},\, I\!R) \cap \operatorname{Quan}(J_1\boldsymbol{E},\, I\!R)\,.$$

2) The equality

$$\operatorname{Self}(J_1\boldsymbol{E},\, I\!R) = \operatorname{Con}(J_1\boldsymbol{E},\, I\!R) \cap \operatorname{Hol}(J_1\boldsymbol{E},\, I\!R)$$

follows from Proposition 2.4.32. QED

2.4.4 Unimodular and conformal unimodular subalgebras

Next, we consider the subalgebras of the algebra of quantisable functions related to the divergence of the tangent lift.

The subsheaves of the sheaf of quantisable functions f, whose tangent lifts fulfill the properties

$$\operatorname{div} X[f] = 0 \qquad \text{and} \qquad d(\operatorname{div} X[f]) = 0\,,$$

are denoted by

$$\operatorname{Unim}(J_1\boldsymbol{E},\, I\!R) \subset \operatorname{Quan}(J_1\boldsymbol{E},\, I\!R) \qquad \text{and} \qquad \widetilde{\operatorname{Unim}}(J_1\boldsymbol{E},\, I\!R) \subset \operatorname{Quan}(J_1\boldsymbol{E},\, I\!R)$$

and called *unimodular* and *conformal unimodular*, respectively.

2.4.34. Proposition. The sheaves $\operatorname{Unim}(J_1\boldsymbol{E},\, I\!R)$ and $\widetilde{\operatorname{Unim}}(J_1\boldsymbol{E},\, I\!R)$ are closed with respect to the special bracket.

Proof. If $f, g \in \operatorname{Quan}(J_1\boldsymbol{E},\, I\!R)$, then we obtain

$$\operatorname{div}(X\big[[f,\, g]\big]) = \operatorname{div}\big(\big[X[f],\, X[g]\big]\big) = X[g].\operatorname{div}(X[f]) - X[f].\operatorname{div}(X[g])\,.\ \text{QED}$$

2.4.35. Proposition. If $f \in \mathrm{Self}(J_1\boldsymbol{E}, I\!\!R) \cap \mathrm{Time}(J_1\boldsymbol{E}, I\!\!R)$, then we obtain

$$\mathrm{div}(X[f]) = 0\,,$$

hence

$$\mathrm{Self}(J_1\boldsymbol{E}, I\!\!R) \cap \mathrm{Time}(J_1\boldsymbol{E}, I\!\!R) = \mathrm{Con}(J_1\boldsymbol{E}, I\!\!R) \cap \mathrm{Time}(J_1\boldsymbol{E}, I\!\!R) \subset \mathrm{Unim}(J_1\boldsymbol{E}, I\!\!R)\,.$$

Proof. The equality

$$i(X^{\uparrow}_{\mathrm{hol}}[f])\,\Omega = df$$

yields

$$L\big[X^{\uparrow}_{\mathrm{hol}}[f]\big]\,\Omega = 0\,,$$

hence, in virtue of Proposition 2.3.15 and Lemma 2.3.16,

$$\mathrm{div}(X[f]) = 0\,.\ \mathrm{QED}$$

2.4.5 Classical subalgebra

Eventually, summarizing several results of the above sections, we consider the subalgebra of the algebra of special functions which generate the infinitesimal symmetries of the full classical structure.

2.4.36. Definition. A function $f \in \mathrm{Con}(J_1\boldsymbol{E}, I\!\!R) \cap \mathrm{Time}(J_1\boldsymbol{E}, I\!\!R)$ is said to be a *classical generator* .□

We set

$$\mathrm{Clas}(J_1\boldsymbol{E}, I\!\!R) := \mathrm{Con}(J_1\boldsymbol{E}, I\!\!R) \cap \mathrm{Time}(J_1\boldsymbol{E}, I\!\!R)\,.$$

2.4.37. Theorem. *We have*

$$\begin{aligned}\mathrm{Clas}(J_1\boldsymbol{E}, I\!\!R) &:= \mathrm{Con}(J_1\boldsymbol{E}, I\!\!R) \cap \mathrm{Time}(J_1\boldsymbol{E}, I\!\!R)\\ &= \mathrm{Con}(J_1\boldsymbol{E}, I\!\!R) \cap \mathrm{Time}(J_1\boldsymbol{E}, I\!\!R)\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Clas}(J_1\boldsymbol{E}, I\!\!R) \subset \mathrm{Hol}(J_1\boldsymbol{E}, I\!\!R)\\ &\mathrm{Clas}(J_1\boldsymbol{E}, I\!\!R) \subset \mathrm{Unim}(J_1\boldsymbol{E}, I\!\!R)\,.\end{aligned}$$

Proof. It follows immediately from Theorem 2.4.33, Lemma 2.4.20 and Proposition 2.4.35. QED

2.4.38. Example. Let us consider an affine spacetime with vanishing electromagnetic field and refer to an inertial observer o and to an adapted cartesian chart.

Then, the sheaf $\mathrm{Clas}(J_1\boldsymbol{E}, I\!R)$ is constituted by the quantisable functions such that

$$\begin{gathered} -2\,\partial_1 f^1 = \cdots = -2\,\partial_n f^n = 0 \\ \partial_i f^0_j + \partial_j f^0_i = 0 \\ \partial_0 \overset{o}{f} = 0 \\ \partial_i \overset{o}{f} = -\partial_0 f^0_i \,. \end{gathered}$$

For instance,

$$\mathcal{K}_0\,,\; \mathcal{Q}_i\,,\; \epsilon_{i_1 i_2 \ldots i_n}\, x^{i_1}\, \mathcal{Q}^{i_2}_0 \;\in\; \mathrm{Clas}(J_1\boldsymbol{E},\, I\!R)\,.\square$$

2.5 Nöther Symmetries

The results on classical symmetries are applied to the classical Lagrangian formalism. I recall some of our results from [140].

Let us consider a local Lagrangian $\mathcal{L}$ and the corresponding Poincaré–Cartan form Θ for the phase 2–form Ω. Clearly, any infinitesimal symmetry $X^{\uparrow} : J_1\boldsymbol{E} \to TJ_1\boldsymbol{E}$ of Θ is an infinitesimal symmetry of Ω. In fact, if $L_{X^{\uparrow}}\,\Theta = 0$, then $0 = dL[X^{\uparrow}]\,\Theta = L[X^{\uparrow}]\,d\Theta = L[X^{\uparrow}]\,\Omega$.

Now, we can formulate the following (Nöther) theorem which relates holonomic infinitesimal symmetries of Θ to conserved quantities.

2.5.1. Theorem. *Let $X_{(1)}$ be a holonomic infinitesimal symmetry of Θ. Then, on the domain of Θ, $i_{X_{(1)}}\,\Omega$ is exact and $f := -i_X\,\Theta \in \mathrm{Self}(J_1\boldsymbol{E}, I\!R) \subset \mathrm{Con}(J_1\boldsymbol{E}, I\!R)$ is a potential.*

Moreover, $X_{(1)} = X^{\uparrow}_{\mathrm{hol}}[f] = X^{\uparrow}_{\mathrm{Ham}}[f]$.

Proof. The equation

$$0 = L[X_{(1)}]\,\Theta = (d\,i_{X_{(1)}} + i_{X_{(1)}} d)\,\Theta = d\,i_{X_{(1)}}\,\Theta + i_{X_{(1)}}\,\Omega$$

is equivalent to

$$i_{X_{(1)}}\,\Omega = -d\,i_{X_{(1)}}\,\Theta = -d\,i_X\,\Theta.$$

Let us define, on the domain of Θ, $f := -i_X\,\Theta$.

Then, by Corollary 2.3.7, on the whole domain of Θ,

$$X_{(1)} = X^{\uparrow}_{\mathrm{Ham}}[f] = X[f]_{(1)},$$

with $f \in \mathrm{Self}(J_1\boldsymbol{E}, I\!R)$. QED

2.5.2. Corollary. Let X be an infinitesimal spacetime symmetry which, additionally, is a holonomic infinitesimal symmetry of Θ. Then, $f := -i_X\,\Theta \in \mathrm{Clas}(J_1\boldsymbol{E}, I\!R)$.

2.5.3. Remark. In particular, if an observer o is a (scaled) infinitesimal symmetry of Θ, then the Hamiltonian $\mathcal{H}[o]$ turns out to be the associated classical generator. In particular, $\mathcal{H}[o]$ is a conserved function.

Let $\mathcal{L}$ be the Lagrangian corresponding to a Poincaré-Cartan form Θ and let $\mathcal{P}$ be the corresponding momentum. In [140], we have proved the following theorem

2.5.4. Theorem. *Let X be an infinitesimal spacetime symmetry. Then, the following equivalence holds*

$$1)\ L_{X_{(1)}} \Theta = 0 \qquad \Leftrightarrow \qquad 2)\ L_{X_{(1)}} \mathcal{L} = 0$$

Theorem 2.5.4 yields immediately another formulation of the (Nöther) Theorem 2.5.1. This version may be more popular to the physicist.

2.5.5. Corollary. Let X be an infinitesimal spacetime symmetry which, additionally, is a holonomic infinitesimal symmetry of $\mathcal{L}$. Then, on the domain of Θ, a classical generator $f \in \mathrm{Clas}(J_1\boldsymbol{E}, I\!\!R)$ is given by

$$f := -(X \lrcorner \mathcal{P} + \underline{X} \lrcorner \mathcal{L})\,.\,\square$$

2.6 Covariant momentum map

We define a covariant momentum map for classical symmetries. The components of a momentum map turn out to be classical generators.

Let us suppose a closed dynamical phase 2–form Ω and a left action $\hat{\Phi} : \boldsymbol{G} \times J_1\boldsymbol{E} \to J_1\boldsymbol{E}$ of a group $\boldsymbol{G}$ of symmetries of the cosymplectic structure $(J_1\boldsymbol{E}, \Omega, dt)$, i.e. $\hat{\Phi}^*_g \Omega = \Omega$ and $\hat{\Phi}^*_g dt = dt$. Let $\mathfrak{g}$ be the associated Lie algebra. Hence, $L_{\partial\hat{\Phi}(\xi)} \Omega = 0$ and $L_{\partial\hat{\Phi}(\xi)}\, dt = 0$ for all $\xi \in \mathfrak{g}$.

We would like to define a *momentum map* in our (covariant) setting by analogy with the standard symplectic and cosymplectic literature [3, 98, 113, 118] and ref. therein.

Let $X^\uparrow \in \mathrm{Sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ an infinitesimal symmetry of Ω and dt. Then, by the Propositions 2.2.15 and 2.2.14, $X^\uparrow$ is of local type $X^\uparrow_{\mathrm{Ham}}[\tau, f]$, with $\tau := dt(X^\uparrow) : J_1\boldsymbol{E} \to \mathbb{T}$ and $f \in \mathrm{Con}(J_1\boldsymbol{E}, I\!\!R)$, where f is determined up to an additive constant $c \in I\!\!R$.

Hence, we can locally associate with any infinitesimal symmetry $X^\uparrow$ of Ω and dt of the type $X^\uparrow = \partial\hat{\Phi}(\xi)$ locally a pair (f_ξ, τ_ξ), where τ_ξ is the constant $dt(\partial\hat{\Phi}(\xi))$ and f_ξ is a potential function of $i_{\partial\hat{\Phi}(\xi)}\, \Omega$.

2.6.1. Definition. A (local) map J

$$J : \mathfrak{g} \to \mathrm{Con}(J_1\boldsymbol{E}) \times \bar{\mathbb{T}} : \xi \mapsto (J_\xi, \tau_\xi)\,,$$

where J_ξ is a potential of $i_{\partial\hat{\Phi}(\xi)}\, \Omega$ and $\tau_\xi := i_{\partial\hat{\Phi}(\xi)}\, dt$ for all $\xi \in \mathfrak{g}$, is said to be a *momentum map* for the action $\hat{\Phi}$. $\square$

We stress that a momentum map is not unique, since the functions J_ξ are defined only up to a real additive constant. However, the time scales τ_ξ are uniquely determined.

2.6.2. Remark. In general, a momentum map J is defined locally. But if we assume suitable hypotheses on spacetime or on the Lie algebra $\mathfrak{g}$, then we can find a global momentum map. Of course, a global J always exists if $H^1(\boldsymbol{E}) = \{0\}$. A detailed list of other hypotheses under which J is globally defined is given in [**?**]; they are the same as in our case. □

Now, let us suppose that we have given a momentum map J for a group of symmetries Φ of the cosymplectic structure. Let $\tau, \sigma \in \mathbb{T}$ be constant time scales and $f, g \in \mathrm{Con}(J_1\boldsymbol{E}, I\!R)$ be conserved quantities. We say $\{(f, \tau), (g, \sigma)\} := ([\tau, \sigma], \{f, g\}) = (0, \{f, g\})$, where $[\tau, \sigma]$ is the Lie bracket and $\{f, g\}$ is the Poisson bracket, to be the *Poisson bracket for pairs.*

2.6.3. Proposition. The map

$$(\tau_\xi, J_\xi) \mapsto \partial\hat{\Phi}(\xi),$$

which associates with any component (τ_ξ, J_ξ) of J its infinitesimal generator turns out to be a morphism of Lie algebras. Its kernel $\mathrm{Map}(\boldsymbol{T}, I\!R)$.

Proof. The equality

$$i_{[H_\tau[f], H_\sigma[g]]}\, dt = [L_{H_\tau[f]}, i_{H_\sigma[g]}]\, dt = L_{H_\tau[f]}\, \sigma - i_{H_\sigma[g]}\, d\tau = 0$$

shows that the bracket $[H_\tau[f], H_\sigma[g]]$ is a vertical Hamiltonian lift. Hence, Proposition 2.2.21 yields the result. QED

Now, let us suppose, additionally, that the left action $\hat{\Phi}$ of $\boldsymbol{G}$ is projectable on a left action $\Phi : \boldsymbol{G} \times \boldsymbol{E} \to \boldsymbol{E}$, that is, $\hat{\Phi}$ is an action of classical symmetries. Theorem 2.3.5 yields the following result.

2.6.4. Proposition. The components of a momentum map are pairs (J_ξ, τ_ξ), where $J_\xi \in \mathrm{Clas}(J_1\boldsymbol{E}, I\!R) \subset \mathrm{Hol}(J_1\boldsymbol{E}, I\!R)$ with the second fiber derivative $D^2 J_\xi$ of J_ξ equal to the constant time scale $\tau_\xi = dt(\hat{\Phi}(\xi))$.

Hence, in the case of the projectable actions, each function J_ξ encodes all information of the pair (J_ξ, τ_ξ). □

Consequently, we call the map $J : \mathfrak{g} \to \mathrm{Clas}(J_1\boldsymbol{E}) : \xi \to J(\xi) := J_\xi$ momentum map, denoted by the same symbol J.

Now, let us consider a Poincaré–Cartan form Θ and an action $\hat{\Phi}$ of classical symmetries which, additionally, preserves Θ.

2.6.5. Proposition. There exists a momentum map on the domain of Θ. Namely, the map

$$J_\xi = \partial\Phi(\xi) \lrcorner \mathcal{P} + \partial\Phi(\xi) \lrcorner \mathcal{L}\,. \tag{2.6.1}$$

Moreover, Let (e_p) be a basis of $\mathfrak{g}$, and $\xi = \xi^p e_p$. Then, the coordinate expression is

$$J_\xi = \xi^p \Big((\partial_p \phi^i - x_0^i \partial_p \phi^0)\, \partial_i^0 L + \partial_p \phi^0\, L \Big)\,,$$

Given an observer o, the momentum map can be expressed in terms of the observed Hamiltonian $\mathcal{H}[o]$ and the observed momentum $\mathcal{P}[o]$ by

$$J_\xi = \partial\Phi(\xi) \lrcorner \mathcal{P}[o] + \partial\Phi(\xi) \lrcorner \mathcal{H}[o]\,. \tag{2.6.2}$$

Proof. The first expression follows simply from the contact splitting of Θ and Theorem 2.5.1. The observer dependent expression of J follows simply from the splitting of Θ through the observer.

The coordinate expression

$$\partial\phi(\xi) = \xi^p \partial_p \phi^0 \partial_0 + \xi^p \partial_p \phi^i \partial_i$$

with respect to a basis (e_p) yields the second expression. QED

2.6.6. Remark. There is a connection between the momentum of a Lagrangian and the momentum map. In fact, let $\boldsymbol{G}$ be a group of vertical holonomic symmetries of Θ, i.e. $i_{\partial\Phi(\xi)}\, dt = 0$. Then we have the expression

$$J_\xi = \partial\phi(\xi) \lrcorner \mathcal{P} \equiv \mathcal{P}(\partial\phi(\xi))\,,$$

so the momentum map coincides with the momentum of the Lagrangian. □

Now, we apply the machinery developed in the above subsection to analyze three groups of symmetries acting in simple cases.

2.6.7. Example. We suppose the spacetime $\boldsymbol{E}$ to be an affine space with affine projection t. In this case $V\boldsymbol{E} \simeq \boldsymbol{E} \times \boldsymbol{S}$, where $\boldsymbol{S} := \ker Dt$. So, we assume an Euclidean scaled metric g on $\boldsymbol{S}$.

Let us consider the natural vertical action

$$\boldsymbol{S} \times \boldsymbol{E} \to \boldsymbol{E} : (v, e_0) \mapsto (e_0 + v)\,;$$

Let $K^\natural$ be the natural flat connection on $\boldsymbol{E}$ and $F = 0$. Then, any Poincaré–Cartan form exists globally and $\boldsymbol{S}$ is a group of symmetries of a Θ. The momentum map J is just the standard linear momentum.

In fact, Θ is invariant with respect to spacelike translations. Of course, the Lie algebra of $\boldsymbol{S}$ is $\boldsymbol{S}$ and we have the momentum map

$$J : \boldsymbol{S}_0 \to \mathrm{Clas}(J_1\boldsymbol{E}, I\!R) \cap \mathrm{Aff}(J_1\boldsymbol{E}, I\!R) : v \mapsto J(v) \equiv \mathcal{P}(v)\,.$$

We have the coordinate expression $\mathcal{P}(v) = v^i\, G_{ij} x_0^j$ (see remark 2.6.6). □

2.6.8. Example. Assume the same spacetime and fields as in the above example, and assume additionally that $\boldsymbol{E} \simeq \boldsymbol{T} \times \boldsymbol{P}$, i.e. , assume a complete observer o. Then, we can consider the natural action

$$\overline{\mathbb{T}} \times (\boldsymbol{T} \times \boldsymbol{P}) \to \boldsymbol{T} \times \boldsymbol{P} : (v, (\tau, \boldsymbol{p})) \mapsto (v + \tau, \boldsymbol{p})\,.$$

It turns out that $\overline{\mathbb{T}}$ is a group of symmetries of Θ, i.e. o is a (scaled) infinitesimal symmetry of Θ, and the momentum map J is just the (observed) kinetic energy $\mathcal{H}[o]$.

In fact, Θ is as in the above example, hence it is invariant with respect to time translations because the metric does not depend on time. Of course, the Lie algebra of $\overline{\mathbb{T}}$ is $\overline{\mathbb{T}}$ and we have the momentum map

$$J : \overline{\mathbb{T}} \to \mathrm{Clas}(J_1\boldsymbol{E},\, I\!R) : \xi \mapsto J(\tau) \equiv \tau \lrcorner (o \lrcorner \Theta)\,.$$

Obviously, $J = \mathcal{H}[o]$. □

2.6.9. Example. Now, we suppose our spacetime to be $\boldsymbol{T} \times SO(g)$, where g is the metric of the above spacetime. The manifold $\boldsymbol{T} \times SO(g)$ is interpreted as the configuration space for the relative configurations of a rigid body with respect to the center of mass (see [118, 127] for a more detailed account).

We assume the *inertia tensor* I as the scaled vertical metric. Consider the action

$$SO(g) \times (\mathbb{T} \times SO(g)) \to \mathbb{T} \times SO(g) : (A, (\tau, B)) \mapsto (\tau, AB)\,.$$

Let $K^\natural$ be the natural flat connection on $\boldsymbol{T} \times SO(g)$ and $F = 0$. Then, $SO(g)$ is a group of symmetries of Θ and a momentum map J is just the angular momentum.

In fact, as in the previous examples, Θ reduces to the kinetic energy of particles with respect to the center of mass. This is obviously invariant with respect orthogonal transformations [127]. We have the momentum map

$$J : so(g_a) \to \mathrm{Clas}(\boldsymbol{T} \times \mathbb{T} \otimes TSO(g),\, I\!R) \cap \mathrm{Aff}(\boldsymbol{T} \times \mathbb{T} \otimes TSO(g),\, I\!R) : \omega \mapsto J_\omega \equiv \omega^* \lrcorner \mathcal{P}\,,$$

where, by definition, $\mathcal{P} = V_{\boldsymbol{E}}\mathcal{L}$, with the coordinate expression $\mathcal{P} = I_{ij} x^j_0 \check{d}^i$. A simple computation shows that

– $\omega^* : SO(g) \to TSO(g) : r \mapsto \omega(r)$;

– $\omega^* \lrcorner \mathcal{P}(v) = I(\omega(r), v) = \omega(r \times v)$.

The Lie algebra of $SO(g_a)$ is $so(g_a)$, but the Hodge star isomorphism yields a natural Lie algebra isomorphism $so(g_a) \simeq \mathbb{L}^{-1} \otimes \boldsymbol{S}_a$. The isomorphism carries the Lie bracket of $so(g_a)$ into the cross product. In this way, if $\omega \in so(g_a)$ and $\bar{\omega} \in \mathbb{L}^{-1} \otimes \boldsymbol{S}_a$ is the corresponding element, we can equivalently write

$$J : \mathbb{L}^{-1} \otimes \boldsymbol{S}_a \to \mathrm{Clas}(\boldsymbol{T} \times \mathbb{T} \otimes TSO(g)) : \bar{\omega} \mapsto J_{\bar{\omega}} \equiv I(r \times v, \omega)\,,$$

where $v \in \mathbb{T}^* \otimes T\boldsymbol{R}_a \equiv J_1(\boldsymbol{T} \times \boldsymbol{R}_a)$. This proves the last part of the statement. □

CHAPTER 3

QUANTUM THEORY

The Galilei covariant quantum mechanics (CQM) provides a framework for a charged scalar quantum particle on a curved spacetime with absolute time interacting with given gravitational and electromagnetic fields.

I recall a few basic facts of the Galilei covariant quantum theory according to [**?**] starting from minimal axioms. The minimal axioms yield, in a covariant way, the full quantum structure. In particular, it yields the covariant Schrödinger equation and a (canonical) quantum Lagrangian formalsim.

I study systematically the v.fs of the quantum theory which preserve the quantum structure. It turns out that these vector fields are generated by subalgebras of the algebra of special functions (of classical phase space). In particular, the v.fs which preserve the Hermitian structure are generated by the quantisable functions and the vector fields which preserve the full quantum structure are generated by the classical generators. The map which associates with a classical function the corresponding quantum symmetry turns out to be an isomorphism of Lie algebras.

On the other hand, I apply these results to the quantum Lagrangian formalism. It suggests me to define quantum currents associated with a quantisable function. For classical generators, the associated quantum currents are conserved along the solutions of the covariant Schrödinger equation.

3.1 Quantum framework

We recall the minimal axioms of the quantum theory. These axioms yield in a covariant way the full quantum structure. In particular, it yields a (canonical) quantum Lagrangian, the covariant Schrödinger equation, a Hilbert bundle the related stuff.

3.1.1 Quantum bundle

We refer to the classical background structure $(J_1\boldsymbol{E}, \Omega, dt)$ described in the previous chapter.

3.1.1. Definition. A *quantum bundle* is defined to be a one–dimensional complex bundle over spacetime

$$\pi : \boldsymbol{Q} \to \boldsymbol{E}$$

equipped with a Hermitian fibred metric

$$\mathtt{h} : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \mathbb{C} \otimes \Lambda^n V^* \boldsymbol{E}$$

with values in the bundle of complexified spacelike volume forms. □

We shall refer to a complex basis

$$\mathtt{b} \in \mathrm{Sec}(\boldsymbol{E}, \mathbb{L}^{\frac{n}{2}} \otimes \boldsymbol{Q}),$$

normalised by

$$h(\mathtt{b}, \mathtt{b}) = \eta,$$

to the associated complex dual basis

$$z \in \mathrm{Map}(\boldsymbol{Q}, \mathbb{L}^{*\frac{n}{2}} \otimes \mathbb{C}),$$

to the associated real basis

$$\mathtt{b}_1, \mathtt{b}_2 \in \mathrm{Sec}(\boldsymbol{E}, \mathbb{L}^{\frac{n}{2}} \otimes \boldsymbol{Q})$$

and to the real dual basis

$$w^1, w^2 \in \mathrm{Map}(\boldsymbol{Q}, \mathbb{L}^{*\frac{n}{2}} \otimes I\!R),$$

where

$$\mathtt{b}_1 := \mathtt{b}, \quad \mathtt{b}_2 := \mathrm{i}\,\mathtt{b} \qquad \text{and} \qquad z = w^1 + \mathrm{i}\, w^2 .$$

We refer to scaled real fibred charts $(x^\lambda, w^{\mathrm{a}})$ of $\boldsymbol{Q}$ and to the induced basis of vector fields and forms

$$(\partial_\lambda, \partial_{\mathrm{a}}) \qquad \text{and} \qquad (d^\lambda, d^{\mathrm{a}}) .$$

We can identify ∂_{a} with $\mathtt{b}_{\mathrm{a}}$.

3.1.2. Proposition. We obtain the following expressions

$$\mathrm{i} = w^1 \otimes \mathtt{b}_2 - w^2 \otimes \mathtt{b}_1 = \mathrm{i}\, z \otimes \mathtt{b},$$
$$\mathtt{h} = \big((w^1 \otimes w^1 + w^2 \otimes w^2) + \mathrm{i}\,(w^1 \otimes w^2 - w^2 \otimes w^1)\big) \otimes \eta = \bar{z} \otimes z\, \eta,$$
$$\mathbb{I} = w^1\, \mathtt{b}_1 + w^2\, \mathtt{b}_2 = z\, \mathtt{b} . \,□$$

3.1.3. Proposition. Let $Y \in \mathrm{Sec}(\boldsymbol{Q}, T\boldsymbol{Q})$.
Then, the following conditions are equivalent:
1) Y is projectable on a vector field $X \in \mathrm{Sec}(\boldsymbol{E}, T\boldsymbol{E})$ and real linear over X;
2) the coordinate expression of Y is of the type

$$Y = X^\lambda\, \partial_\lambda + Y^{\mathrm{a}}_{\mathrm{b}}\, w^{\mathrm{b}}\, \partial_{\mathrm{a}}\,,$$

with

$$X^\lambda,\, Y^{\mathrm{a}}_{\mathrm{b}} \in \mathrm{Map}(\boldsymbol{E}, I\!R)\,.$$

Moreover, the following conditions are equivalent:
1) Y is projectable on a vector field $X \in \mathrm{Sec}(\boldsymbol{E}, T\boldsymbol{E})$ and complex linear over X;
2) the coordinate expression of Y is of the type

$$\begin{aligned} Y &= X^\lambda\, \partial_\lambda + (r + \mathrm{i}\, s)\, w^{\mathrm{a}}\, \partial_{\mathrm{a}}\,,\\ &= X^\lambda\, \partial_\lambda + r\,(w^1\, \partial_1 + w^2\, \partial_2) + s\,(w^1\, \partial_2 + w^2\, \partial_1)\,, \end{aligned}$$

with

$$\begin{aligned} X^\lambda &\in \mathrm{Map}(\boldsymbol{E}, I\!R)\,,\\ r \equiv Y^1_1 = Y^1_1 &\in \mathrm{Map}(\boldsymbol{E}, I\!R)\,,\\ s \equiv Y^2_1 - Y^1_2 &\in \mathrm{Map}(\boldsymbol{E}, I\!R)\,. \square \end{aligned}$$

3.1.4. Proposition. If $\bar{\mathsf{b}}$ is another quantum basis, then we have locally

$$\bar{\mathsf{b}} = \exp(\mathrm{i}\,\phi)\,\mathsf{b}\,, \qquad \text{with} \qquad \phi \in \mathrm{Map}(\boldsymbol{E}, I\!R)\,.$$

Accordingly, we obtain

$$\bar{z} = \exp(-\mathrm{i}\,\phi)\, z\,,$$

and

$$\begin{aligned} \bar{w}^1 &= \cos\phi\, w^1 + \sin\phi\, w^2\,,\\ \bar{w}^2 &= -\sin\phi\, w^1 + \cos\phi\, w^2\,. \square \end{aligned}$$

The history of a quantum particle is described by a *quantum section* $\Psi \in \mathrm{Sec}(\boldsymbol{E}, \boldsymbol{Q})$. We write $\Psi = \psi\,\mathsf{b}$, with $\psi \in \mathrm{Map}(\boldsymbol{E}, I\!R)$.

We shall be involved with the first jet space $J_1\boldsymbol{Q}$ of $\boldsymbol{Q} \to \boldsymbol{E}$. We denote the fibred charts of $J_1\boldsymbol{Q}$ by $(x^\lambda, w^{\mathrm{a}}, w^{\mathrm{a}}_\lambda)$.

We have the natural contact maps

$$Д : J_1\boldsymbol{Q} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} T\boldsymbol{Q} \qquad \text{and} \qquad \vartheta : J_1\boldsymbol{Q} \to T^*\boldsymbol{Q} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q}\,,$$

with coordinate expressions

$$Д = d^\lambda \otimes (\partial_\lambda + w^{\mathrm{a}}_\lambda \,\partial_{\mathrm{a}}) \qquad \text{and} \qquad \vartheta = \partial_{\mathrm{a}} \otimes (d^{\mathrm{a}} - w^{\mathrm{a}}_\lambda \, d^\lambda)\,.$$

3.1.5. Proposition. Let $Y \in \mathrm{Sec}(J_1\boldsymbol{Q},\, TJ_1\boldsymbol{Q})$.
Then, the following conditions are equivalent:
1) Y is projectable on a vector field $X \in \mathrm{Sec}(\boldsymbol{E},\, T\boldsymbol{E})$ and real linear over X;
2) the coordinate expression of Y is of the type

$$Y = X^\lambda \,\partial_\lambda + Y^{\mathrm{a}}_{\mathrm{b}}\, w^{\mathrm{b}}\, \partial_{\mathrm{a}} + Y^{\mathrm{a}\mu}{}_{\mathrm{b}}\, w^{\mathrm{b}}_\mu\, \partial_{\mathrm{a}} + Y^{\mathrm{a}}_{\lambda\,\mathrm{b}}\, w^{\mathrm{b}}\, \partial^\lambda_{\mathrm{a}} + Y^{\mathrm{a}\mu}_{\lambda\,\mathrm{b}}\, w^{\mathrm{b}}_\mu\, \partial^\lambda_{\mathrm{a}}\,,$$

with

$$X^\lambda,\, Y^{\mathrm{a}}_{\mathrm{b}},\, Y^{\mathrm{a}\mu}{}_{\mathrm{b}},\, Y^{\mathrm{a}}_{\lambda\,\mathrm{b}},\, Y^{\mathrm{a}\mu}_{\lambda\,\mathrm{b}} \in \mathrm{Map}(\boldsymbol{E}, I\!R)\,.$$

Moreover, the following conditions are equivalent:
1) Y is projectable on a vector field $X \in \mathrm{Sec}(\boldsymbol{E},\, T\boldsymbol{E})$ and complex linear over X;
2) the coordinate expression of Y is of the type

$$Y = X^\lambda\,\partial_\lambda + (r + \mathrm{i}\, s)\, w^{\mathrm{a}}\,\partial_{\mathrm{a}} + (r^\mu + \mathrm{i}\, s^\mu)\, w^{\mathrm{a}}_\mu\,\partial_{\mathrm{a}} + (r_\lambda + \mathrm{i}\, s_\lambda)\, w^{\mathrm{a}}\,\partial^\lambda_{\mathrm{a}} + (r^\mu_\lambda + \mathrm{i}\, s^\mu_\lambda)\, w^{\mathrm{a}}_\mu\,\partial^\lambda_{\mathrm{a}}\,,$$

$$\begin{aligned} &= X^\lambda\,\partial_\lambda \\ &+ r\,(w^1\,\partial_1 + w^2\,\partial_2) + s\,(w^1\,\partial_2 - w^2\,\partial_1) + r^\mu\,(w^1_\mu\,\partial_1 + w^2_\mu\,\partial_2) + s^\mu\,(w^1_\mu\,\partial_2 - w^2_\mu\,\partial_1) \\ &+ r_\lambda\,(w^1\,\partial^\lambda_1 + w^2\,\partial^\lambda_2) + s_\lambda\,(w^1\,\partial^\lambda_2 - w^2\,\partial^\lambda_1) + r^\mu_\lambda\,(w^1_\mu\,\partial^\lambda_1 + w^2_\mu\,\partial^\lambda_2) + s^\mu_\lambda\,(w^1_\mu\,\partial^\lambda_2 - w^2_\mu\,\partial^\lambda_1) \end{aligned}$$

with

$$\begin{aligned} X^\lambda &\in \mathrm{Map}(\boldsymbol{E},\, I\!R)\,, \\ r \;\equiv\; Y^1_1 = Y^1_1 &\in \mathrm{Map}(\boldsymbol{E},\, I\!R)\,, \\ s \;\equiv\; Y^2_1 - Y^1_2 &\in \mathrm{Map}(\boldsymbol{E},\, I\!R)\,, \\ r^\mu \;\equiv\; Y^{1\mu}{}_1 = Y^{1\mu}{}_1 &\in \mathrm{Map}(\boldsymbol{E},\, I\!R)\,, \\ s^\mu \;\equiv\; Y^{2\mu}{}_1 - Y^{1\mu}{}_2 &\in \mathrm{Map}(\boldsymbol{E},\, I\!R)\,, \\ r_\lambda \;\equiv\; Y^1_{\lambda\,1} = Y^1_{\lambda\,1} &\in \mathrm{Map}(\boldsymbol{E},\, I\!R)\,, \\ s_\lambda \;\equiv\; Y^2_{\lambda\,1} - Y^1_{\lambda\,2} &\in \mathrm{Map}(\boldsymbol{E},\, I\!R)\,, \\ r^\mu_\lambda \;\equiv\; Y^{1\mu}_{\lambda\,1} = Y^{1\mu}_{\lambda\,1} &\in \mathrm{Map}(\boldsymbol{E}, I\!R)\,, \\ s^\mu_\lambda \;\equiv\; Y^{2\mu}_{\lambda\,1} - Y^{1\mu}_{\lambda\,2} &\in \mathrm{Map}(\boldsymbol{E},\, I\!R)\,.\,\square \end{aligned}$$

3.1.2 Extended quantum bundle

3.1.6. Definition. We define the *extended quantum bundle*

$$\pi^{\uparrow} : \boldsymbol{Q}^{\uparrow} := J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to J_1\boldsymbol{E}\,,$$

by taking the pullback of $\pi : \boldsymbol{Q} \to \boldsymbol{E}$, with respect to the map $t^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}$. □

We refer to the fibred charts $(x^\lambda, x^i_0, w^{\mathrm{a}})$ of $\boldsymbol{Q}^{\uparrow}$ and to the induced bases of vector fields and forms $(\partial_\lambda, \partial^0_i, \partial_{\mathrm{a}})$ and $(d^\lambda, d^i_0, d^{\mathrm{a}})$.

3.1.7. Remark. We have

$$T\boldsymbol{Q}^{\uparrow} = TJ_1\boldsymbol{E} \underset{T\boldsymbol{E}}{\times} T\boldsymbol{Q}\,.$$

In particular, we obtain the natural linear fibred inclusion over $\boldsymbol{Q}$

$$V_{\boldsymbol{E}}J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \subset T\boldsymbol{Q}^{\uparrow}$$

by considering the $\boldsymbol{E}$–vertical subbundle $V_{\boldsymbol{E}}J_1\boldsymbol{E} \subset TJ_1\boldsymbol{E}$ and the "zero subbundle" $\boldsymbol{Q} \subset T\boldsymbol{Q}$, which project on the "zero subbundle" $\boldsymbol{E} \subset T\boldsymbol{E}$. We stress that the above natural inclusion is independent from any connection of the extended quantum bundle. □

3.1.8. Proposition. Let $Y^{\uparrow} \in \mathrm{Sec}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow})$.

If $Y^{\uparrow}$ is projectable on a vector field $Y \in \mathrm{Sec}(\boldsymbol{Q}, T\boldsymbol{Q})$, then $Y^{\uparrow}$ is projectable on a vector field $X \in \mathrm{Sec}(\boldsymbol{E}, T\boldsymbol{E})$.

If $Y^{\uparrow}$ is projectable on a vector field $X^{\uparrow} \in \mathrm{Sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ and on a vector field $X \in \mathrm{Sec}(\boldsymbol{E}, T\boldsymbol{E})$, then $Y^{\uparrow}$ is projectable on a vector field $Y \in \mathrm{Sec}(\boldsymbol{Q}, T\boldsymbol{Q})$.

Proof. If $Y^{\uparrow}$ is projectable on Y, then

$$\partial^0_j Y^\lambda = 0 \qquad \text{and} \qquad \partial^0_j Y^{\mathrm{a}} = 0\,,$$

hence, in particular,

$$\partial^0_j Y^\lambda = 0\,.$$

If $Y^{\uparrow}$ is projectable on $X^{\uparrow}$ and X, then

$$\partial^0_j Y^\lambda = 0 \qquad \text{and} \qquad \partial^0_j Y^{\mathrm{a}} = 0\,,$$

hence, $Y^{\uparrow}$ is projectable on Y. QED

3.1.9. Proposition. Let $Y^{\uparrow} \in \mathrm{Sec}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow})$.

Then, the following conditions are equivalent:

1) $Y^{\uparrow}$ is projectable on a vector field $X^{\uparrow} \in \mathrm{Sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ and real linear over $X^{\uparrow}$;

2) the coordinate expression of $Y^{\uparrow}$ is of the type

$$Y^{\uparrow} = X^\lambda\,\partial_\lambda + X^i_0\,\partial^0_i + Y^{\mathrm{a}}_{\mathrm{b}}\,w^{\mathrm{b}}\,\partial_{\mathrm{a}}\,,$$

with

$$X^\lambda,\ X^i_0,\ Y^{\mathrm{a}}_{\mathrm{b}} \in \mathrm{Map}(J_1\boldsymbol{E}, I\!R)\,.$$

Moreover, the following conditions are equivalent:

1) $Y^\uparrow$ is projectable on a vector field $X^\uparrow \in \mathrm{Sec}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$ and complex linear over $X^\uparrow$;

2) the coordinate expression of $Y^\uparrow$ is of the type

$$\begin{aligned} Y^\uparrow &= X^\lambda\,\partial_\lambda + X^i_0\,\partial^0_i + (r + \mathrm{i}\,s)\,w^{\mathrm{a}}\,\partial_{\mathrm{a}}\,, \\ &= X^\lambda\,\partial_\lambda + X^i_0\,\partial^0_i + r\,(w^1\,\partial_1 + w^2\,\partial_2) + s\,(w^1\,\partial_2 + w^2\,\partial_1) \end{aligned}$$

with

$$\begin{aligned} X^\lambda &\in \mathrm{Map}(J_1\boldsymbol{E},\, I\!R)\,, \\ r \;\equiv\; Y^1_1 = Y^1_1 &\in \mathrm{Map}(J_1\boldsymbol{E},\, I\!R)\,, \\ s \;\equiv\; Y^2_1 - Y^1_2 &\in \mathrm{Map}(J_1\boldsymbol{E},\, I\!R)\,.\,\square \end{aligned}$$

We shall be involved with the first jet space $J_1\boldsymbol{Q}^\uparrow$ of $\boldsymbol{Q}^\uparrow \to J_1\boldsymbol{E}$. We refer to the fibred charts $(x^\lambda, x^i_0, w^{\mathrm{a}}, w^{\mathrm{a}}_\lambda, w^{\mathrm{a}0}_{\ i})$ of $J_1\boldsymbol{Q}^\uparrow$ and to the induced bases of vector fields and forms $(\partial_\lambda, \partial^0_i, \partial_{\mathrm{a}}, \partial^\lambda_{\mathrm{a}}, \partial^{\ i}_{\mathrm{a}0})$, $(d^\lambda, d^i_0, d^{\mathrm{a}}, d^{\mathrm{a}}_\lambda, d^{\mathrm{a}0}_{\ i})$.

The Hermitian metric $\mathfrak{h}$ of $\boldsymbol{Q}$ yields, by pullback, a Hermitian metric $\mathfrak{h}^\uparrow$ of $\boldsymbol{Q}^\uparrow$.

3.1.3 Quantum connection

3.1.10. Definition. A linear connection C of $\boldsymbol{Q} \to \boldsymbol{E}$, or Ψ of $\boldsymbol{Q}^\uparrow \to J_1\boldsymbol{E}$, is said to be *Hermitian* if

$$\nabla[K, C]\,\mathfrak{h} = 0\,, \qquad \text{or} \qquad \nabla[K, \Psi]\,\mathfrak{h}^\uparrow = 0\,,$$

respectively. $\square$

3.1.11. Proposition. A Hemitian connection C of $\boldsymbol{Q}$ has coordinate expression of the type

$$C = d^\lambda \otimes \partial_\lambda + \mathrm{i}\,c_\lambda\,d^\lambda \otimes \mathbb{I}\,, \qquad \text{with} \qquad c_\lambda \in \mathrm{Map}(\boldsymbol{E}, I\!R)\,.$$

A Hemitian connection Ψ of $\boldsymbol{Q}^\uparrow$ has coordinate expression of the type

$$\Psi = d^\lambda \otimes \partial_\lambda + d^i_0 \otimes \partial^0_i + \mathrm{i}\,(\psi_\lambda\,d^\lambda + \psi^0_i\,d^i_0) \otimes \mathbb{I}\,, \quad \text{with} \quad \psi_\lambda, \psi^0_i \in \mathrm{Map}(J_1\boldsymbol{E},\, I\!R)\,.\,\square$$

3.1.12. Proposition. [44, 56, 54] A system of connections $\{\overset{o}{\Psi}\}$ of $\boldsymbol{Q}$ parametrised by observers $o \in \mathrm{Sec}(\boldsymbol{E}, J_1\boldsymbol{E})$ induces, in a covariant way, a connection Ψ of $\boldsymbol{Q}^\uparrow$, called *universal* [56], whose symbols ψ^0_i vanish. Conversely, the connections of the system can be recovered from the universal connection through the pullback $\overset{o}{Ч} = o^*(Ч)$. Analogously, the curvature of the connections of the system can be recovered from the curvature of the universal connection through the pullback $R[\overset{o}{Ч}] = o^*(R[Ч])$. □

3.1.13. Definition. A *quantum connection* is defined to be a connection $Ч$ of $\boldsymbol{Q}^\uparrow$, which is Hermitian, universal and whose curvature is

$$R[Ч] = \mathrm{i}\,\Omega \otimes \mathbb{I}\,. \square$$

We stress that $\frac{1}{\hbar}$ has been incorporated in Ω through the re-scaled metric G.

Here, the closure of Ω turns out to be a necessary integrability condition because of the Bianchi identity of $R[Ч]$.

Thus, quantum connections are associated with the classical background through Ω.

3.1.14. Theorem. [56] *Given a quantum basis* $\mathfrak{b}$*, a quantum connection* $Ч$ *turns out to be locally of the type*

$$Ч = \chi^{\uparrow\|}[\mathfrak{b}] + \mathrm{i}\,ч[\mathfrak{b}] \otimes \mathbb{I}\,,$$

where $\chi^{\uparrow\|}[\mathfrak{b}]$ *is the flat connection of* $\boldsymbol{Q}^\uparrow$ *associated with* $\mathfrak{b}$ *and* $ч[\mathfrak{b}] \equiv \Theta$ *is a horizontal potential (determined by* $Ч$ *and* $\mathfrak{b}$*) of* Ω*.*

Given a quantum basis $\mathfrak{b}$*, the connections of the system associated with a quantum connection* $Ч$ *turn out to be locally of the type*

$$\overset{o}{Ч} = \chi^{\|}[\mathfrak{b}] + \mathrm{i}\,A[\mathfrak{b}, o] \otimes \mathbb{I}\,,$$

where $\chi^{\|}[\mathfrak{b}]$ *is the flat connection of* $\boldsymbol{Q}$ *associated with* $\mathfrak{b}$ *and* $A[\mathfrak{b}, o] := o^*(ч[\mathfrak{b}])$ *is a potential (determined by* $Ч$*,* $\mathfrak{b}$ *and* o*) of* $\Phi[o] = 2\,o^*\Omega$*.*

Thus, the coordinate expression of $Ч$ *is locally of the type*

$$Ч = d^\lambda \otimes \partial_\lambda + d^i_0 \otimes \partial^0_i + \mathrm{i}\Big(- (\tfrac{1}{2}\,G^0_{ij}\,x^i_0\,x^j_0 - A_0)\,d^0 + (G^0_{ij}\,x^j_0 + A_i)\,d^i\Big) \otimes \mathbb{I}\,. \square$$

We observe that the classical horizontal potentials Θ are defined up to a gauge; on the other hand, the quantum potential $ч[\mathfrak{b}]$ is determined by the quantum connection $Ч$ and the quantum basis $\mathfrak{b}$.

We observe that quantum connections exist locally because Ω admits horizontal potentials.

3.1.15. Lemma. Let $\bar{\mathfrak{b}}$ and $\mathfrak{b}$ be two quantum bases related by the transition map $\bar{\mathfrak{b}} = \exp(\mathrm{i}\,\varphi)\,\mathfrak{b}$.

Then, we have

$$\bar{\chi}^{\|} = \chi^{\|} + \mathrm{i}\,d\phi \otimes \mathbb{I} \qquad \text{and} \qquad \bar{\chi}^{\uparrow\|} = \chi^{\uparrow\|} + \mathrm{i}\,d\phi \otimes \mathbb{I}\,.$$

Proof. In a chart adapted to $\mathfrak{b}$ and $\bar{\mathfrak{b}}$, we have, respectively,

$$\chi^{\|}[\mathfrak{b}] = d^\lambda \otimes \partial_\lambda \qquad \text{and} \qquad \chi^{\|}[\bar{\mathfrak{b}}] = \bar{d}^\lambda \otimes \bar{\partial}_\lambda$$
$$\chi^{\uparrow\|}[\mathfrak{b}] = d^\lambda \otimes \partial_\lambda + d^i_0 \otimes \partial^0_i \qquad \text{and} \qquad \chi^{\uparrow\|}[\bar{\mathfrak{b}}] = \bar{d}^\lambda \otimes \bar{\partial}_\lambda + \bar{d}^i_0 \otimes \bar{\partial}^0_i$$

Then, we obtain

$$\begin{aligned}
\chi^{\|}[\bar{\mathfrak{b}}] &= \bar{d}^{\lambda} \otimes \bar{\partial}_{\lambda} \\
&= (\partial_{\mu}\bar{x}^{\lambda}\, d^{\mu}) \otimes (\bar{\partial}_{\lambda} x^{\mu}\, \partial_{\mu} + \bar{\partial}_{\lambda} w^{\mathrm{a}}\, \partial_{\mathrm{a}}) \\
&= \partial_{\mu} \otimes d^{\mu} + (\bar{\partial}_{\lambda}\phi)\, (\partial_{\mu}\bar{x}^{\lambda}\, d^{\mu}) \otimes \big((-\sin\phi\, \bar{w}^{1} - \cos\phi\, \bar{w}^{2})\, \partial w_{1} + (\cos\phi\, \bar{w}^{1} - \sin\phi\, \bar{w}^{2})\, \partial w_{2}\big) \\
&= \partial_{\mu} \otimes d^{\mu} + d\phi \otimes (w^{1}\, \partial w_{2} - w^{2}\, \partial w_{1}) \\
&= \partial_{\mu} \otimes d^{\mu} + \mathrm{i}\, d\phi \otimes \mathbb{I}
\end{aligned}$$

and

$$\begin{aligned}
\chi^{\uparrow\|}[\bar{\mathfrak{b}}] &= \bar{d}^{\lambda} \otimes \bar{\partial}_{\lambda} + \bar{d}^{i}_{0} \otimes \bar{\partial}^{0}_{i} \\
&= (\partial_{\mu}\bar{x}^{\lambda}\, d^{\mu}) \otimes (\bar{\partial}_{\lambda} x^{\mu}\, \partial_{\mu} + \bar{\partial}_{\lambda} w^{\mathrm{a}}\, \partial_{\mathrm{a}}) + (\partial_{\mu}\bar{x}^{i}_{0}\, d^{\mu} + \partial^{0}_{j}\bar{x}^{i}_{0}\, d^{j}_{0}) \otimes \bar{\partial}^{0}_{i} x^{j}_{0}\, \partial^{0}_{j} \\
&= \partial_{\mu} \otimes d^{\mu} + \bar{d}^{i}_{0} \otimes \bar{\partial}^{0}_{i} + (\bar{\partial}_{\lambda}\phi)\, (\partial_{\mu}\bar{x}^{\lambda}\, d^{\mu}) \otimes \big((-\sin\phi\, \bar{w}^{1} - \cos\phi\, \bar{w}^{2})\, \partial w_{1} + (\cos\phi\, \bar{w}^{1} - \sin\phi\, \bar{w}^{2})\, \partial w_{2}\big) \\
&= \partial_{\mu} \otimes d^{\mu} + \bar{d}^{i}_{0} \otimes \bar{\partial}^{0}_{i} + d\phi \otimes (w^{1}\, \partial w_{2} - w^{2}\, \partial w_{1}) \\
&= \partial_{\mu} \otimes d^{\mu} + \bar{d}^{i}_{0} \otimes \bar{\partial}^{0}_{i} + \mathrm{i}\, d\phi \otimes \mathbb{I}\,. \text{ QED}
\end{aligned}$$

3.1.16. Proposition. Let us consider a quantum connection Ч, two quantum bases $\mathfrak{b}$ and $\bar{\mathfrak{b}} = \exp(\mathrm{i}\,\varphi)\,\mathfrak{b}$ and two observers o and $\bar{o} = o + v$.

Then, we have

$$ч[\bar{\mathfrak{b}}] = ч[\mathfrak{b}] - d\varphi$$

and

$$\begin{aligned}
A[\bar{\mathfrak{b}}, o] &= A[\mathfrak{b}, o] - d\varphi\,, \\
A[\mathfrak{b}, \bar{o}] &= A[\mathfrak{b}, o] - \tfrac{1}{2} G(v, v) + \nu[o] \lrcorner\, G^{\flat}(v)\,, \\
A[\bar{\mathfrak{b}}, \bar{o}] &= A[\bar{\mathfrak{b}}, o] - \tfrac{1}{2} G(v, v) + \nu[o] \lrcorner\, G^{\flat}(v)\,, \\
A[\bar{\mathfrak{b}}, \bar{o}] &= A[\mathfrak{b}, o] - \tfrac{1}{2} G(v, v) + \nu[o] \lrcorner\, G^{\flat}(v) - d\varphi\,. \square
\end{aligned}$$

Proof. It follows from Theorem 3.1.14, Lemma 3.1.15 and Proposition 2.1.38. QED

3.1.17. Corollary. Let us consider a quantum connection Ч, a quantum basis $\mathfrak{b}$ and two observers o and $\bar{o} = o + v$.

Then, we have

$$\breve{A}[\mathfrak{b}, \bar{o}] = \breve{A}[\mathfrak{b}, o] + G^{\flat}(v)\,. \square$$

3.1.18. Corollary. Let Ч be a quantum connection and $\mathfrak{b}$ a quantum basis. Then, there exist a unique observer

$$o[\mathfrak{b}] \in \mathcal{S}(\boldsymbol{E},\, \mathbb{T}^{*} \otimes T\boldsymbol{E})$$

such that the vertical restriction of the induced potential vanishes, i.e. such that

$$\check{A}\Big[\mathfrak{b}, o[\mathfrak{b}]\Big] = 0 .$$

Indeed, if o is any observer, then we obtain

$$o[\mathfrak{b}] = o - G^{\sharp}(\check{A}[\mathfrak{b}, o]) .$$

Moreover, if $\bar{\mathfrak{b}}$ is another quantum basis, with $\bar{\mathfrak{b}} = \exp(\mathrm{i}\,\varphi)\,\mathfrak{b}$, then we obtain

$$o[\bar{\mathfrak{b}}] = o - G^{\sharp}(\check{A}[\mathfrak{b}, o] - \check{d}\varphi) . \square$$

3.1.19. Theorem. [129] *A quantum connection exists globally if and only if the cohomology class of Ω is integer; the equivalence classes of quantum bundles equipped with a quantum connection are classified by the cohomology group $H^1(\boldsymbol{E}, U(1))$.* $\square$

We suppose that the classical structure fulfills the above chomological condition for the existence of quantum structures and assume a *quantum structure* $(\boldsymbol{Q}, \text{Ч})$.

This is our only assumption for the quantum theory. We derive all other quantum objects from it (including the quantum dynamics and quantum operators) by means of covariant procedures.

Our approach has evident analogies with geometric quantisation, but there are important differences. In particular, we stress that our quantum bundle lives on spacetime and that the quantum connection is universal. In this way we avoid the intricate problems related to polarisations. Actually, we replace the difficult search for the inclusion of polarisations with an easier search for projectable objects.

3.1.4 Quantum differentials

The quantum structure yields all further quantum objects in a covariant way. Actually, the quantum connection lives on the extended quantum bundle $\boldsymbol{Q}^{\uparrow}$; on the other hand, we implement the covariance of the quantum theory by requiring that the physically significant objects live on the quantum bundle $\boldsymbol{Q}$ through a *method of projectabilty* [54]. In this way, we get the quantum Lagrangian, the quantum Schrödinger operator and the quantum operators.

For each $\Psi \in \mathrm{Sec}(\boldsymbol{E}, \boldsymbol{Q})$, we obtain the *quantum covariant differential*

$$\nabla\Psi := \nabla\Psi^{\uparrow} \in \mathrm{Sec}(J_1\boldsymbol{E},\, T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q}) ,$$

with respect to Ч, by considering the pullback $\Psi^{\uparrow} \in \mathrm{Sec}(J_1\boldsymbol{E},\, \boldsymbol{Q}^{\uparrow})$ of Ψ.

We observe that $\nabla\Psi$ turns out to be valued just in $T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q}$, because of the universality of Ч.

We have the coordinate expression

$$\nabla\Psi = (\partial_\lambda - \mathrm{i}\,\text{ч}_\lambda)\,\psi\, d^\lambda \otimes \mathsf{b} = \Big((\partial_0 + \mathrm{i}\,\mathcal{H}_0)\,\psi\, d^0 + (\partial_j - \mathrm{i}\,\mathcal{P}_j)\,\psi\, d^j\Big) \otimes \mathsf{b}\,.$$

Moreover, we define the *time-like* and *space-like differentials* to be the maps

$$\widehat{\nabla}\Psi := \text{Д} \lrcorner \nabla\Psi \in \mathrm{Fib}(J_1\boldsymbol{E},\, \mathbb{T}^* \otimes \boldsymbol{Q}) \qquad \text{and} \qquad \overset{\vee}{\nabla}\Psi \in \mathrm{Fib}(J_1\boldsymbol{E},\, V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q})\,,$$

with coordinate expressions

$$\widehat{\nabla}\Psi = (\partial_0 + x^j_0\,\partial_j - \mathrm{i}\,\mathcal{L}_0)\,\psi\, d^0 \otimes \mathsf{b} \qquad \text{and} \qquad \overset{\vee}{\nabla}\Psi = (\partial_j - \mathrm{i}\,\mathcal{P}_j)\,\psi\, \check{d}^j \otimes \mathsf{b}\,.$$

3.1.5 Quantum Lagrangian

3.1.20. Proposition. [56, 54] The quantum structure yields, in a covariant way, the distinguished fibred morphisms over $\boldsymbol{E}$,

$$\mathsf{L} \in \mathrm{Fib}(J_1\boldsymbol{Q}, \Lambda^{1+n}T^*\boldsymbol{E})\,,$$

which, for any $\Psi \in \mathrm{Sec}(\boldsymbol{E},\, \boldsymbol{Q})$ is expressed by

$$\mathsf{L}[\Psi] = dt \wedge \Big(\mathsf{h}(\Psi,\, \mathrm{i}\,\widehat{\nabla}\Psi) + \mathsf{h}(\mathrm{i}\,\widehat{\nabla}\Psi,\, \Psi) - (\bar{G} \otimes \mathsf{h}(\overset{\vee}{\nabla}\Psi,\, \overset{\vee}{\nabla}\Psi) + k\rho\,(\Psi,\, \Psi)\Big)\,,$$

with $k \in I\!R$.

Proof. We have

$$\begin{aligned}
\widehat{\mathsf{L}}[\Psi] \;&\equiv\; dt \wedge \big(\mathsf{h}(\Psi,\, \mathrm{i}\,\widehat{\nabla}\Psi) + \mathsf{h}(\mathrm{i}\,\widehat{\nabla}\Psi,\, \Psi)\big) \in \mathrm{Fib}(J_1\boldsymbol{E},\, \Lambda^{1+n}T^*\boldsymbol{E})\\
\overset{\vee}{\mathsf{L}}[\Psi] \;&\equiv\; dt \wedge \big((\bar{G} \otimes \mathsf{h})\,(\overset{\vee}{\nabla}\Psi,\, \overset{\vee}{\nabla}\Psi)\big) \in \mathrm{Fib}(J_1\boldsymbol{E},\, \Lambda^{1+n}T^*\boldsymbol{E})\\
\mathsf{L}_{(0)}[\Psi] \;&\equiv\; \rho\,\mathsf{h}(\Psi,\, \Psi) \in \mathrm{Sec}(\boldsymbol{E},\, \Lambda^{1+n}T^*\boldsymbol{E})\,.
\end{aligned}$$

However, the coordinate expression of the above terms, shows that

$$\widehat{\mathsf{L}}[\Psi] - \overset{\vee}{\mathsf{L}}[\Psi] \in \mathrm{Sec}(\boldsymbol{E},\, \Lambda^{1+n}T^*\boldsymbol{E})\,.\,\mathrm{QED}$$

3.1.21. Proposition. We have the coordinate expression

$$\begin{aligned}
\mathsf{L}[\Psi] = \tfrac{1}{2}\Big(\mathrm{i}\,(\bar{\psi}\,\partial_0\psi - \psi\,\partial_0\bar{\psi}) - G^{ij}_0\,(\partial_i\bar{\psi}\,\partial_j\psi + A_i\,A_j\,\bar{\psi}\,\psi) - \mathrm{i}\,A^i_0\,(\bar{\psi}\,\partial_i\psi - \psi\,\partial_i\bar{\psi})\\
+ 2\,A_0\,\bar{\psi}\,\psi + k\,\rho_0\,\bar{\psi}\,\psi\Big)\sqrt{|g|}\,d^0 \wedge d^1 \wedge \ldots \wedge d^n\,.
\end{aligned}$$

Hence, we obtain the coordinate expression

$$\mathsf{L} \;\equiv\; \mathfrak{l}_0\,\upsilon^0$$

where, in complex coordinates,

$$\mathfrak{l}_0 = \tfrac{1}{2}\Big(\mathrm{i}\,(\bar z^{\mathrm{A}}\, z^{\mathrm{A}}_0 - z^{\mathrm{A}}\,\bar z^{\mathrm{A}}_0) + 2\,A_0\,\bar z^{\mathrm{A}}\, z^{\mathrm{A}} \\ - G^{ij}_0\,(\bar z^{\mathrm{A}}_i\, z^{\mathrm{A}}_j + A_i\,A_j\,\bar z^{\mathrm{A}}\, z^{\mathrm{A}}) - \mathrm{i}\,A^i_0\,(\bar z^{\mathrm{A}}\, z^{\mathrm{A}}_i - z^{\mathrm{A}}\,\bar z^{\mathrm{A}}_i) + k\,\rho_0\,\bar z^{\mathrm{A}}\, z^{\mathrm{A}}\Big)\,,$$

and, in real coordinates,

$$\begin{aligned}\mathfrak{l}_0 =& (w^2\,w^1_0 - w^1\,w^2_0) + A_0\,(w^1\,w^1 + w^2\,w^2) \\ &- \tfrac{1}{2}\,G^{ij}_0\,(w^1_i\,w^1_j + w^2_i\,w^2_j) - \tfrac{1}{2}\,A^i_0\,A_i\,(w^1\,w^1 + w^2\,w^2) - A^i_0\,(w^2\,w^1_i - w^1\,w^2_i) \\ &+ \tfrac{1}{2}\,k\,\rho_0\,(w^1\,w^1 + w^2\,w^2)\,.\square\end{aligned}$$

3.1.22. Definition. For each $k \in I\!R$, the fibred morphism

$$\mathsf{L} \in \mathrm{Fib}(J_1\boldsymbol{Q}, \Lambda^{1+n}T^*\boldsymbol{E})\,,$$

as above is called a *canonical quantum Lagrangian*. $\square$

3.1.23. Proposition. [**?**, 61, 62] All covariant fibred morphisms

$$\mathsf{L}' \in \mathrm{Fib}(J_1\boldsymbol{Q}, \Lambda^{1+n}T^*\boldsymbol{E})$$

are proportional to a canonical quantum Lagrangian L. $\square$

Thus, the requirement of covariance is insufficient to determine the constant k. A theoretical determination of this constant needs further requirements.

According to the above results, we assume a canonical quantum Lagrangian L as the source of quantum dynamics (leaving k undetermined).

3.1.24. Definition. We define the *quantum momentum* to be the momentum associated with the quantum Lagrangian, i.e. the fibred morphism over $\boldsymbol{E}$

$$V_{\boldsymbol{Q}}\mathsf{L} \in \mathrm{Fib}(J_1\boldsymbol{Q},\ \boldsymbol{Q}^* \underset{\boldsymbol{E}}{\otimes} T\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \Lambda^{1+n}T^*\boldsymbol{E})\,.$$

Moreover, we consider the fibred morphism over $\boldsymbol{E}$

$$\mathsf{P} := \mathrm{i}\,h^\sharp(V_{\boldsymbol{Q}}\mathsf{L}) \in \mathrm{Fib}(J_1\boldsymbol{Q},\ \mathbb{T}^* \otimes (\boldsymbol{Q} \underset{\boldsymbol{E}}{\otimes} T\boldsymbol{E}))\,.\square$$

3.1.25. Remark. In the above Definition, we have identified $V^*J_1\boldsymbol{Q}$ with $\boldsymbol{Q}^* \underset{\boldsymbol{E}}{\otimes} T\boldsymbol{E}$ by means of a natural isomorphism of vector bundles. In the case of classical phase space we have denoted the analogous isomorphism by the tensor ν.

On the other hand, we have the natural isomorphism $\lrcorner : T\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \Lambda^{n+1}T^*\boldsymbol{E} \to \Lambda^n T^*\boldsymbol{E}$. Hence, we can naturally regard $V_{\boldsymbol{Q}}\mathsf{L}$ as a morphism $J_1\boldsymbol{Q} \to \boldsymbol{Q}^* \underset{\boldsymbol{E}}{\otimes} \Lambda^n T^*\boldsymbol{E}$.

3.1.26. Proposition. For each $\Psi \in \mathrm{Sec}(\boldsymbol{E}, \boldsymbol{Q})$, we have the coordinate expression

$$\mathsf{P}[\Psi] = u^0 \otimes \Big(\psi\, \partial_0 - \mathrm{i}\, G_0^{hk}\, (\partial_h - \mathrm{i}\, A_h)\, \psi\, \partial_k\Big)\, \mathfrak{b}\,. \square$$

3.1.27. Definition. We define the *quantum Poincaré–Cartan form* to be the Poincaré–Cartan form associated with the quantum Lagrangian, i.e. the fibred morphism over $\boldsymbol{Q}$

$$\Pi := \mathsf{L} + \vartheta \bar{\wedge} V_{\boldsymbol{Q}} \mathsf{L} \in \mathrm{Fib}(J_1\boldsymbol{Q},\ \Lambda^{1+n} T^*\boldsymbol{Q})\,. \square$$

3.1.28. Proposition. We have the coordinate expression

$$\begin{aligned}\Pi =& \Big((A_0 - \tfrac{1}{2}\, A_0^i\, A_i + \tfrac{1}{2}\, k\, \rho_0)\, (w^1\, w^1 + w^2\, w^2) + \tfrac{1}{2}\, G_0^{ij}\, (w_i^1\, w_j^1 + w_i^2\, w_j^2)\Big)\, \upsilon^0 \\ &+ (w^2\, dw^1 - w^1\, dw^2) \wedge \upsilon_0^0 \\ &- \Big(G_0^{ij}\, (w_j^1\, dw^1 + w_j^2\, dw^2) + A_0^i\, (w^2\, dw^1 - w^1\, dw^2\Big) \wedge \upsilon_i^0\,.\end{aligned}$$

Proof. We have the coordinate expression

$$\Pi = \mathsf{L} + \partial_{\mathrm{a}}^{\lambda}\, \mathfrak{l}_0\, \vartheta^{\mathrm{a}} \wedge \upsilon_{\lambda}^0\,,$$

where

$$\vartheta^{\mathrm{a}} = d^{\mathrm{a}} - w_{\lambda}^{\mathrm{a}}\, d^{\lambda}$$

and

$$\upsilon_{\lambda}^0 = \sqrt{|g|}\, \imath(\partial_\lambda)\, (d^0 \wedge d^1 \ldots \wedge d^n)\,.$$

Moreover, we obtain

$$\begin{aligned}\partial_1^0 \mathfrak{l}_0 &= w^2 \\ \partial_2^0 \mathfrak{l}_0 &= -w^1 \\ \partial_1^i \mathfrak{l}_0 &= -G_0^{ij}\, w_j^1 - A_0^i\, w^2 \\ \partial_2^i \mathfrak{l}_0 &= -G_0^{ij}\, w_j^2 + A_0^i\, w^1\,,\end{aligned}$$

which yields

$$\begin{aligned}\partial_{\mathrm{a}}^{\lambda}\, \mathfrak{l}_0\, \vartheta^{\mathrm{a}} \wedge \upsilon_{\lambda}^0 =& (w^2\, dw^1 - w^1\, dw^2) \wedge \upsilon_0^0 - \big(G_0^{ij}\, (w_j^1\, dw^1 - w_j^2\, dw^2) + A_0^i\, (w^2\, dw^1 - w^1\, dw^2\big) \wedge \upsilon_i^0 \\ &+ \big(- (w^2\, w_0^1 - w^1\, w_0^2) + G_0^{ij}\, (w_i^1\, w_j^1 + w_i^2\, w_j^2) + A_0^i\, (w^2\, w_i^1 - w^1\, w_i^2)\big)\, \upsilon^0\,. \square\end{aligned}$$

3.1.29. Definition. We define the *quantum Euler–Lagrange operator* to be the Euler–Lagrange fibred morphism associated with the quantum Lagrangian

$$\mathsf{E} \in \mathrm{Sec}(J_1\boldsymbol{Q},\ \Lambda^{1+n} T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q}^*)\,.$$

Moreover, we define the *Schrödinger operator* to be the fibred morphism

$$\mathsf{S} := \tfrac{1}{2}\,\mathrm{i}\,(\mathrm{re}\,\mathsf{h})^\sharp(\mathsf{E}) \in \mathrm{Sec}(J_2\boldsymbol{Q},\ \mathbb{T}^* \otimes \boldsymbol{Q})\,.\,\square$$

3.1.30. Proposition. For each $\Psi \in \mathrm{Sec}(\boldsymbol{E},\,\boldsymbol{Q})$, we have the coordinate expression

$$\mathsf{S}[\Psi] = \Big((\overset{o}{\nabla}_0 + \tfrac{1}{2}\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}})\,\psi - \tfrac{1}{2}\,\mathrm{i}\,(\overset{o}{\Delta}_0 + k\,\rho_0)\,\psi)\Big)\,u^0 \otimes \mathsf{b}\,,$$

where

$$\begin{aligned}
\overset{o}{\nabla}_0\psi &:= (\partial_0 - \mathrm{i}\,A_0)\,\psi \\
\overset{o}{\Delta}_0\psi &:= \Big(G^{hk}_0\,(\partial_h - \mathrm{i}\,A_h)\,(\partial_k - \mathrm{i}\,A_k) + \frac{\partial_h(G^{hk}_0\sqrt{|g|})}{\sqrt{|g|}}\,(\partial_k - \mathrm{i}\,A_k)\Big)\,\psi\,.\,\square
\end{aligned}$$

We stress that $\frac{m}{\hbar}$ has been incorporated in G.
Thus, we assume the generalised Schrödinger equation

$$\mathsf{S}[\Psi] = 0$$

as *quantum dynamical equation*.

3.2 Symmetries of the quantum framework

In the following I study the v.fs which preserve the quantum structure. The special subalgebras play an essential role. I classify systematically the vector fields of the quantum bundle and the extended quantum bundle. We find that the vector fields which preserve the full quantum structure are generated by functions of $\mathrm{Clas}(J_1\boldsymbol{E}, I\!\!R) \subset \mathrm{Quan}(J_1\boldsymbol{E}, I\!\!R)$. Moreover, the map which associates with a function f the resp. symmetry of the quantum structure is an isomorphism of Lie algebras.

3.2.1 Symmetries of the complex linear structure structure

3.2.1. Proposition. Let us consider a vector field $Y \in \mathrm{Sec}(\boldsymbol{Q},\,T\boldsymbol{Q})$ projectable on $X \in \mathrm{Sec}(\boldsymbol{E},\,T\boldsymbol{E})$. Then, Y is (real) linear over X if and only if its coordinate expression is of the type

$$Y = X^\lambda\,\partial_\lambda + Y^{\mathrm{a}}_{\mathrm{b}}\,w^{\mathrm{b}}\,\partial_{\mathrm{a}}\,, \qquad \text{with} \qquad X^\lambda, Y^{\mathrm{a}}_{\mathrm{b}} \in \mathrm{Map}(\boldsymbol{E}, I\!\!R)\,.\,\square$$

3.2.2. Lemma. Let $f \in \mathrm{Fib}(\boldsymbol{Q}, \boldsymbol{Q})$ be (real) linear. Then, the following conditions are equivalent:

1) f is complex linear,

2)
$$f \circ \mathrm{i} = \mathrm{i} \circ f ,$$

3) the coordinate expression of f is of the type
$$f = f^{\mathrm{a}}_{\mathrm{b}} \, w^{\mathrm{b}} \otimes \mathsf{b}_{\mathrm{a}} , \qquad \text{with} \qquad f^1_1 = f^2_2 , \quad f^1_2 = -f^2_1 ,$$

4) the coordinate expression of f is of the type
$$f = (f^1_1 + \mathrm{i}\, f^2_1) \, (w^1 \otimes \mathsf{b}_1 + w^2 \otimes \mathsf{b}_2) , \qquad \text{with} \qquad f^1_1 = f^2_2 , \quad f^1_2 = -f^2_1 ,$$

5)
$$f = (f^1_1 + \mathrm{i}\, f^2_1) \, \mathbb{I} .$$

PROOF. The equivalence 1) $\Leftrightarrow$ 2) follows immediately from the definition of complex linearity. Moreover, we have
$$f \circ \mathrm{i} = \mathrm{i} \circ f ,$$
if and only if
$$f^1_1 - f^2_2 = 0 , \qquad f^1_2 + f^2_1 = 0 . \text{QED}$$

3.2.3. Proposition. Let $Y \in \mathrm{Sec}(\boldsymbol{Q}, T\boldsymbol{Q})$ projectable on $X \in \mathrm{Sec}(\boldsymbol{E}, T\boldsymbol{E})$ and (real) linear over X. Then, the following conditions are equivalent:

1) Y is complex linear,

2)
$$L[Y] \, (\mathrm{i}\, \mathbb{I}) = 0 ,$$

3) the coordinate expression of Y is of the type
$$Y = X^\lambda \, \partial_\lambda + Y^{\mathrm{a}}_{\mathrm{b}} \, w^{\mathrm{b}} \, \partial_{\mathrm{a}} , \qquad \text{with} \qquad X^\lambda, Y^{\mathrm{a}}_{\mathrm{b}} \in \mathrm{Map}(\boldsymbol{E}, I\!R) \quad Y^1_1 = Y^2_2 \;\; Y^1_2 = -Y^2_1 ,$$

4) the coordinate expression of Y is of the type
$$Y = X^\lambda \, \partial_\lambda + (Y^1_1 + \mathrm{i}\, Y^2_1) \, (w^1 \, \partial_1 + w^2 \, \partial_2) , \qquad \text{with} \qquad X^\lambda, Y^1_1, Y^2_1 \in \mathrm{Map}(\boldsymbol{E}, I\!R) ,$$

5) the coordinate expression of Y is of the type
$$Y = X^\lambda \, \partial_\lambda + (Y^1_1 + \mathrm{i}\, Y^2_1) \, \mathbb{I} , \qquad \text{with} \qquad X^\lambda, Y^1_1, Y^2_1 \in \mathrm{Map}(\boldsymbol{E}, I\!R) .$$

PROOF. The equivalence of 1), 3), 4) and 5) follows from the above Lemma.
Moreover, the equivalence of 2) and 3) follows from the coordinate expression

$$L[Y]\,(\mathrm{i}\,\mathbb{I}) = (-Y^2_1\,w^1 - Y^2_2\,w^2 - Y^1_2\,w^1 + Y^1_1\,w^2)\,\partial_1 + (Y^1_1\,w^1 + Y^1_2\,w^2 - Y^2_2\,w^1 + Y^2_1\,w^2)\,\partial_2\,.\,\mathrm{QED}$$

Analogous results hold for the extended quantum bundle.

3.2.4. Proposition. Let us consider a vector field $Y^\uparrow \in \mathrm{Sec}(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow)$ projectable on $X^\uparrow \in \mathrm{Sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$. Then, $Y^\uparrow$ is (real) linear over $X^\uparrow$ if and only if its coordinate expression is of the type

$$Y = X^\lambda\,\partial_\lambda + X^i_0\,\partial^0_i + Y^{\mathrm{a}}_{\mathrm{b}}\,w^{\mathrm{b}}\,\partial_{\mathrm{a}}\,, \qquad \text{with} \qquad X^\lambda, X^i_0, Y^{\mathrm{a}}_{\mathrm{b}} \in \mathrm{Map}(J_1\boldsymbol{E}, I\!R)\,.\,\square$$

3.2.5. Lemma. Let $f \in \mathrm{Fib}(\boldsymbol{Q}^\uparrow, \boldsymbol{Q}^\uparrow)$ be (real) linear. Then, the following conditions are equivalent:

1) f is complex linear,

2)
$$f \circ \mathrm{i} = \mathrm{i} \circ f\,,$$

3) the coordinate expression of f is of the type

$$f = f^{\mathrm{a}}_{\mathrm{b}}\,w^{\mathrm{b}} \otimes \mathsf{b}_{\mathrm{a}}\,, \qquad \text{with} \qquad f^1_1 = f^2_2\,, \quad f^1_2 = -f^2_1\,,$$

4) the coordinate expression of f is of the type

$$f = (f^1_1 + \mathrm{i}\,f^2_1)\,(w^1 \otimes \mathsf{b}_1 + w^2 \otimes \mathsf{b}_2)\,, \qquad \text{with} \qquad f^1_1 = f^2_2\,, \quad f^1_2 = -f^2_1\,,$$

5)
$$f = (f^1_1 + \mathrm{i}\,f^2_1)\,\mathbb{I}\,.\,\square$$

3.2.6. Proposition. Let $Y^\uparrow \in \mathrm{Sec}(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow)$ projectable on $X^\uparrow \in \mathrm{Sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ and (real) linear over $X^\uparrow$. Then, the following conditions are equivalent:

1) $Y^\uparrow$ is complex linear,

2)
$$L[Y^\uparrow]\,(\mathrm{i}\,\mathbb{I}) = 0\,,$$

3) the coordinate expression of $Y^\uparrow$ is of the type

$$Y^\uparrow = X^\lambda\,\partial_\lambda + Y^i_0\,\partial^0_i + Y^{\mathrm{a}}_{\mathrm{b}}\,w^{\mathrm{b}}\,\partial_{\mathrm{a}}\,,$$

with

$$X^\lambda, Y^i_0, Y^{\mathrm{a}}_{\mathrm{b}} \in \mathrm{Map}(J_1\boldsymbol{E}, I\!R) \quad Y^1_1 = Y^2_2 \;\; Y^1_2 = -Y^2_1\,,$$

4) the coordinate expression of $Y^{\uparrow}$ is of the type

$$Y^{\uparrow} = X^{\lambda}\,\partial_{\lambda} + Y_0^i\,\partial_i^0 + (Y_1^1 + \mathrm{i}\,Y_1^2)\,(w^1\,\partial_1 + w^2\,\partial_2)\,,$$

with

$$X^{\lambda}, Y_0^i, Y_1^1, Y_1^2 \in \mathrm{Map}(J_1\boldsymbol{E}, I\!\!R)\,,$$

5) the coordinate expression of $Y^{\uparrow}$ is of the type

$$Y^{\uparrow} = X^{\lambda}\,\partial_{\lambda} + Y_0^i\,\partial_i^0 + (Y_1^1 + \mathrm{i}\,Y_1^2)\,\mathbb{I}\,,$$

with

$$X^{\lambda}, Y_0^i, Y_1^1, Y_1^2 \in \mathrm{Map}(\boldsymbol{E}, I\!\!R)\,.\,\square$$

3.2.2 Symmetries of the Hermitian metric of Q

First, we classify the vector fields of $\boldsymbol{Q}$, which preserve the Hermitian metric.

3.2.7. Remark. The Hermitian metric $\mathfrak{h}$ can be naturally regarded as a section

$$\mathfrak{h} : \boldsymbol{Q} \to V_{\boldsymbol{E}}^*\boldsymbol{Q} \underset{\boldsymbol{Q}}{\otimes} V_{\boldsymbol{E}}^*\boldsymbol{Q} \underset{\boldsymbol{Q}}{\otimes} \Lambda^3 V_{\boldsymbol{T}}^*\boldsymbol{Q}\,,$$

where $V_{\boldsymbol{E}}^*\boldsymbol{Q}$ denotes the vertical dual with respect to the fibring of $\boldsymbol{Q}$ over $\boldsymbol{E}$ and $V_{\boldsymbol{T}}^*\boldsymbol{Q}$ denotes the vertical dual with respect to the fibring of $\boldsymbol{Q}$ over $\boldsymbol{T}$. We have the coordinate expression

$$\mathfrak{h} = \Big((\check{d}^1 \otimes \check{d}^1 + \check{d}^2 \otimes \check{d}^2) + \mathrm{i}\,(\check{d}^1 \otimes \check{d}^2 - \check{d}^2 \otimes \check{d}^1)\Big) \otimes \eta.$$

Accordingly, if $Y \in \mathrm{Sec}(\boldsymbol{Q}, T\boldsymbol{Q})$, is projectable on $\boldsymbol{E}$ and $\boldsymbol{T}$, then the Lie derivative $L[Y]\mathfrak{h}$ is well defined (in spite of the fact that $\mathfrak{h}$ is vertical valued). $\square$

3.2.8. Definition. A vector field $Y \in \mathrm{Sec}(\boldsymbol{Q}, T\boldsymbol{Q})$ is said to be an *infinitesimal symmetry* (*i.s.*) of $\mathfrak{h}$ if it is projectable on $\boldsymbol{E}$ and $\boldsymbol{T}$ and fulfills the equality

$$L[Y]\,\mathfrak{h} = 0\,.$$

A vector field $Y \in \mathrm{Sec}(\boldsymbol{Q}, T\boldsymbol{Q})$ is said to be *Hermitian* if it is (real) linear projectable on $\boldsymbol{E}$ and projectable on $\boldsymbol{T}$ and fulfills the equality

$$L[Y]\,\mathfrak{h} = 0\,.\,\square$$

3.2.9. Remark. If $Y \in \mathrm{Sec}(\boldsymbol{Q}, T\boldsymbol{Q})$, is projectable on $\boldsymbol{E}$ and $\boldsymbol{T}$, then we obtain, for each $\Phi, \Psi \in \mathrm{Sec}(\boldsymbol{E}, \boldsymbol{Q})$,

$$L[Y]\Big(\mathfrak{h}(\Phi,\, \Psi)\Big) = (L[Y]\,\mathfrak{h})(\Phi,\, \Psi) + \mathfrak{h}\Big(L[Y]\,\Phi,\, \Psi\Big) + \mathfrak{h}\Big(\Phi,\, L[Y]\,\Psi\Big)\,,$$

and, if additionally Y is linear over $\boldsymbol{E}$,

$$L[Y]\Big(\mathfrak{h}(\Phi,\, \Psi)\Big) = (L[Y]\,\mathfrak{h})(\Phi,\, \Psi) + \mathfrak{h}\Big(Y.\Phi,\, \Psi\Big) + \mathfrak{h}\Big(\Phi,\, Y.\Psi\Big)\,.$$

PROOF. It follows immediately from the Leibnitz rule of the Lie derivative, by regarding Φ and Ψ as vertical valued vector fields of $\boldsymbol{Q}$, via pullback. QED

3.2.10. Lemma. Let $X \in \mathrm{Pro}(\boldsymbol{E}, T\boldsymbol{E})$ and $\Psi, \Phi \in \mathrm{Sec}(\boldsymbol{E}, \boldsymbol{Q})$. Then, we have

$$L[X]\Big(\mathfrak{h}(\Psi, \Phi)\Big) = \nabla[K']_X\Big(\mathfrak{h}(\Psi, \Phi)\Big) + \mathfrak{h}(\Psi, \Phi)\ \mathrm{div}\, X\,.$$

PROOF. We have

$$\nabla[K']\,\eta = 0\,, \qquad L[X]\,\eta = \mathrm{div}\, X\,\eta\,,$$

where K' is the linear connection induced on the vector bundle $V\boldsymbol{E} \to \boldsymbol{E}$ by the linear connection K.
Hence, we obtain

$$\begin{aligned} L[X]\big(\mathfrak{h}(\Psi,\, \Phi)\big) &= L[X](\bar\psi\,\phi\,\eta) = X.(\bar\psi\,\phi)\,\eta + (\bar\psi\,\phi)\ \mathrm{div}\, X\,\eta \\ \nabla[K']_X\big(\mathfrak{h}(\Psi,\, \Phi)\big) &= \nabla[K']_X(\bar\psi\,\phi\,\eta) = X.(\bar\psi\,\phi)\,\eta + (\bar\psi\,\phi)\,\nabla[K']_X\eta = X.(\bar\psi\,\phi)\,\eta\,. \text{ QED} \end{aligned}$$

Now, we classify the i.s.'s of $\mathfrak{h}$ and the Hermitian vector fields. For this purpose, we need several preliminary results.

3.2.11. Lemma. Let $Y \in \mathrm{Sec}(\boldsymbol{Q}, T\boldsymbol{Q})$ be projectable on $X \in \mathrm{Sec}(\boldsymbol{E}, T\boldsymbol{E})$ and

$$\partial_1 Y^1 = \partial_2 Y^2 := r \in \mathrm{Map}(\boldsymbol{E}, I\!R)\,, \qquad \partial_1 Y^2 = -\partial_2 Y^1 := s \in \mathrm{Map}(\boldsymbol{Q}, I\!R)\,.$$

Then, the coordinate expression of Y is of the type

$$Y = X^\lambda\,\partial_\lambda + (r + \mathrm{i}\,s)\,\mathbb{I} + c^{\mathrm{a}}\,\partial_{\mathrm{a}}\,, \qquad \text{with} \qquad r,\, s,\, c^{\mathrm{a}} \in \mathrm{Map}(\boldsymbol{E}, I\!R)\,.$$

PROOF. By integrating the equations

$$\partial_1 Y^1 = r \qquad \text{and} \qquad \partial_2 Y^2 = r$$

on the affine fibres of the quantum bundle, we obtain

$$Y^1 = r\,w^1 + a^1 \qquad \text{and} \qquad Y^2 = r\,w^2 + a^2\,,$$

where

*) $$a^1,\, a^2 \in \mathrm{Map}(\boldsymbol{Q}, I\!R)\,, \qquad \text{with} \qquad \partial_1 a^1 = 0 = \partial_2 a^2\,.$$

Then, the equation

$$\partial_1 Y^2 = -\partial_2 Y^1$$

reads

$$**) \qquad \partial_1 a^2 = -\partial_2 a^1 \,.$$

Moreover, (*) and (**) yield

$$\partial_1 a^2 = s \qquad \text{and} \qquad \partial_2 a^1 = -s \,, \qquad \text{with} \qquad s \in \mathrm{Map}(\boldsymbol{E}, I\!\!R) \,,$$

hence, by integrating the above equations on the affine fibres of the quantum bundle,

$$a^1 = -s\,w^2 + c^1 \qquad \text{and} \qquad a^2 = s\,w^1 + c^2 \,, \qquad \text{with} \qquad c^1,\, c^2 \in \mathrm{Map}(\boldsymbol{E}, I\!\!R) \,.$$

Therefore, we obtain

$$\begin{aligned} Y &= X^\lambda\,\partial_\lambda + r\,(w^1\,\partial_1 + w^2\,\partial_2) + s\,(w^1\,\partial_2 - w^2\,\partial_1) + c^1\,\partial_1 + c^2\,\partial_2 \\ &= X^\lambda\,\partial_\lambda + (r + \mathrm{i}\,s)\,(w^1\,\partial_1 + w^2\,\partial_2) + c^1\,\partial_1 + c^2\,\partial_2 \,.\ \text{QED} \end{aligned}$$

3.2.12. Lemma. Let $Y \in \mathrm{Sec}(\boldsymbol{Q}, V\boldsymbol{Q})$ and $r \in \mathrm{Map}(\boldsymbol{E}, \mathbb{C})$ and suppose that

$$\mathsf{h}\Big(L[Y]\,\Psi,\, \Phi\Big) + \mathsf{h}\Big(\Psi,\, L[Y]\,\Phi\Big) = r\,\mathsf{h}(\Psi,\, \Phi) \,, \qquad \forall \Psi, \Phi \in \mathrm{Sec}(\boldsymbol{E},\, \boldsymbol{Q}) \,.$$

Then, it turns out that $r \in \mathrm{Map}(\boldsymbol{E}, I\!\!R)$ and Y is of the type

$$Y = (\mathrm{i}\,s - \tfrac{1}{2}\,r)\,\mathbb{I} + c^{\mathrm{a}}\,\partial_{\mathrm{a}} \,, \qquad \text{with} \qquad s,\, c^{\mathrm{a}} \in \mathrm{Map}(\boldsymbol{E}, I\!\!R) \,.$$

Proof. We have

$$r\,\mathsf{h}(\Psi,\, \Phi) = r\,(\Psi^1\,\Phi^1 + \Psi^2\,\Phi^2) + r\,\mathrm{i}\,(\Psi^1\,\Phi^2 - \Psi^2\,\Phi^1) \,.$$

Moreover, we have

$$\begin{aligned} L[Y]\,\Psi &= -(\Psi^1\,\partial_1 Y^1 + \Psi^2\,\partial_2 Y^1)\,\partial_1 - (\Psi^1\,\partial_1 Y^2 + \Psi^2\,\partial_2 Y^2)\,\partial_2 \\ L[Y]\,\Phi &= -(\Phi^1\,\partial_1 Y^1 + \Phi^2\,\partial_2 Y^1)\,\partial_1 - (\Phi^1\,\partial_1 Y^2 + \Phi^2\,\partial_2 Y^2)\,\partial_2 \,, \end{aligned}$$

hence

$$\begin{aligned} \mathsf{h}\big(L[Y]\,\Psi,\, \Phi\big) + \mathsf{h}\big(\Psi,\, L[Y]\,\Phi\big) &= -(\Psi^1\,\partial_1 Y^1 + \Psi^2\,\partial_2 Y^1)\,\Phi^1 - (\Psi^1\,\partial_1 Y^2 + \Psi^2\,\partial_2 Y^2)\,\Phi^2 \\ &\quad - \mathrm{i}\,(\Psi^1\,\partial_1 Y^1 + \Psi^2\,\partial_2 Y^1)\,\Phi^2 + \mathrm{i}\,(\Psi^1\,\partial_1 Y^2 + \Psi^2\,\partial_2 Y^2)\,\Phi^1 \\ &\quad - \Psi^1\,(\Phi^1\,\partial_1 Y^1 + \Phi^2\,\partial_2 Y^1) - \Psi^2\,(\Phi^1\,\partial_1 Y^2 + \Phi^2\,\partial_2 Y^2) \\ &\quad - \mathrm{i}\,\Psi^1\,(\Phi^1\,\partial_1 Y^2 + \Phi^2\,\partial_2 Y^2) + \mathrm{i}\,\Psi^2\,(\Phi^1\,\partial_1 Y^1 + \Phi^2\,\partial_2 Y^1) \,. \end{aligned}$$

Therefore, we have

$$\mathsf{h}\big(L[Y]\,\Psi,\, \Phi\big) + \mathsf{h}\big(\Psi,\, L[Y]\,\Phi\big) = r\,\mathsf{h}(\Psi,\, \Phi)$$

if and only

$$\partial_1 Y^1 = \partial_2 Y^2 = -\tfrac{1}{2}\, r \qquad \text{and} \qquad \partial_1 Y^2 = -\partial_2 Y^1 \,.$$

Hence, the result follows from Lemma 3.2.11. QED

3.2.13. Lemma. Let us consider a function $f \in \mathrm{Quan}(J_1\boldsymbol{E},\, I\!\!R)$ and two observers o and $\bar{o} = o + v$.

Then, we have

$$\overset{\bar{o}}{\text{Ч}}(X[f]) + \mathrm{i}\, \overset{\bar{o}}{f}\, \mathbb{I} = \overset{o}{\text{Ч}}(X[f]) + \mathrm{i}\, \overset{o}{f}\, \mathbb{I}\,.$$

Proof. Let us refer to a quantum basis $\mathfrak{b}$ and to a quantum chart adapted to o and $\mathfrak{b}$. We have

$$\overset{\bar{o}}{\text{Ч}}(X[f]) + \mathrm{i}\, \overset{\bar{o}}{f}\, \mathbb{I} = \chi^{\shortparallel}[\mathfrak{b}](X[f]) + \mathrm{i}\,(\langle A[\mathfrak{b}, \bar{o}],\, X[f]\rangle + \overset{\bar{o}}{f})\, \mathbb{I}$$

Moreover, by recalling the formula

$$A[\mathfrak{b}, \bar{o}] = A[\mathfrak{b}, o] - \tfrac{1}{2} G(v, v) + \nu[o] \lrcorner G^{\flat}(v)\,,$$

we obtain

$$\begin{aligned}
&= \mathrm{i}\,(\langle A[\mathfrak{b}, \bar{o}],\, X[f]\rangle + \overset{\bar{o}}{f})\, \mathbb{I}\\
&= \mathrm{i}\,(\langle A[\mathfrak{b}, o],\, X[f]\rangle - \tfrac{1}{2}\,\langle f^{\shortparallel}, G(v,v)\rangle + G(v, \nu[o](X[f])) + \overset{\bar{o}}{f})\, \mathbb{I}\\
&= \mathrm{i}\,(\langle A[\mathfrak{b}, o],\, X[f]\rangle - (\tfrac{1}{2}\, f^0\, G^0_{ij}\, \bar{o}^i_0\, \bar{o}^j_0 + G^0_{ij}\, f^i\, \bar{o}^j_0) + \overset{\bar{o}}{f})\, \mathbb{I}\\
&= \mathrm{i}\,(\langle A[\mathfrak{b}, o],\, X[f]\rangle - (f - \overset{o}{f}) \circ \bar{o}) + \overset{\bar{o}}{f})\, \mathbb{I}\\
&= \mathrm{i}\,(\langle A[\mathfrak{b}, o],\, X[f]\rangle - (f \circ \bar{o} - \overset{o}{f}) + \overset{\bar{o}}{f})\, \mathbb{I}\\
&= \mathrm{i}\,(\langle A[\mathfrak{b}, o],\, X[f]\rangle + \overset{o}{f})\, \mathbb{I}
\end{aligned}$$

Hence, we obtain

$$\overset{\bar{o}}{\text{Ч}}(X[f]) + \mathrm{i}\, \overset{\bar{o}}{f}\, \mathbb{I} = \overset{o}{\text{Ч}}(X[f]) + \mathrm{i}\, \overset{o}{f}\, \mathbb{I}\,. \text{QED}$$

3.2.14. Theorem. *Let $Y \in \mathrm{Sec}(\boldsymbol{Q},\, T\boldsymbol{Q})$ be projectable on $\boldsymbol{E}$ and $\boldsymbol{T}$.*
The following conditions are equivalent:
1) Y is an i.s. of h;
2) with reference to an observer o, we have the expression

$$Y \,\equiv\, Y[f] + C = \overset{o}{\text{Ч}}(X[f]) + (\mathrm{i}\, \overset{o}{f} - \tfrac{1}{2}\, \mathrm{div}\, X[f])\, \mathbb{I} + C\,,$$

where $f \in \mathrm{Quan}(J_1\boldsymbol{E},\, I\!\!R)$ and $C \in \mathrm{Sec}(\boldsymbol{E},\, \boldsymbol{Q})$;

3) with reference to a quantum basis $\mathfrak{b}$ *and to an observer* o*, we have the expression*

$$Y = \chi^{\shortparallel}[\mathfrak{b}]\big(X[f]\big) + \Big(\mathrm{i}\,(A[\mathfrak{b}, o]\,(X[f]) + \overset{o}{f}) - \tfrac{1}{2}\,\mathrm{div}\,X[f]\Big)\,\mathbb{I} + C\,,$$

where $f \in \mathrm{Quan}(J_1\boldsymbol{E},\, I\!R)$ *and* $C \in \mathrm{Sec}(\boldsymbol{E},\, \boldsymbol{Q})$*;*

4) we have the coordinate expression

$$\begin{aligned} Y &= X^\lambda\,\partial_\lambda + g\,(w^1\,\partial_2 - w^2\,\partial_1) - \tfrac{1}{2}\,(X^0\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} + \frac{\partial_i(X^i\sqrt{|g|})}{\sqrt{|g|}})\,(w^1\,\partial_1 + w^2\,\partial_2) + C^{\mathrm{a}}\,\partial_{\mathrm{a}} \\ &= X^\lambda\,\partial_\lambda + \Big(\mathrm{i}\,g - \tfrac{1}{2}\,(X^0\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} + \frac{\partial_i(X^i\sqrt{|g|})}{\sqrt{|g|}})\Big)\,(w^1\,\partial_1 + w^2\,\partial_2) + C^{\mathrm{a}}\,\partial_{\mathrm{a}}\,, \end{aligned}$$

where $X^\lambda, C^{\mathrm{a}}, g \in \mathrm{Map}(\boldsymbol{E}, I\!R)$*.*

Moreover, if the quantum chart in 4) is adapted to the quantum basis $\mathfrak{b}$ *and to the observer* o*, then we obtain*

$$X = X[f] = f^0\,\partial_0 - f^i\,\partial_i\,, \qquad g = \overset{o}{f} + A_0\,f^0 - A_i\,f^i\,.$$

Indeed, the expression $Y[f]$ *in 2) does not depend on the choice of the observer* o*.*

Proof. Let us prove the equivalence 1) $\Leftrightarrow$ 2).
We can split Y into its horizontal and vertical components as

$$Y = \overset{o}{\mathrm{Ч}}(X) + \bar{Y} \qquad \text{where} \qquad \bar{Y} := \nu[\overset{o}{\mathrm{Ч}}](Y)\,.$$

Now, Y is an i.s. of $\mathfrak{h}$ if and only if, for each $\Psi, \Phi \in \mathrm{Sec}(\boldsymbol{E},\, \boldsymbol{Q})$,

$$L[X]\big(h(\Psi, \Phi)\big) = h(L[Y]\,\Psi,\, \Phi) + h(\Psi,\, L[Y]\,\Phi)$$

i.e., in virtue of Lemma 3.2.10, if and only if

$$\begin{aligned} &\nabla[K']_X\big(\mathfrak{h}(\Psi,\, \Phi)\big) + \mathrm{div}\,X\,\mathfrak{h}(\Psi,\, \Phi) = \\ &= \mathfrak{h}(\nabla[\overset{o}{\mathrm{Ч}}]_X\Psi,\, \Phi) + \mathfrak{h}(\Psi,\, \nabla[\overset{o}{\mathrm{Ч}}]_X\Phi) + \mathfrak{h}\big(L[\bar{Y}]\,\Psi,\, \Phi\big) + \mathfrak{h}\big(\Psi,\, L[\bar{Y}]\,\Phi\big)\,, \end{aligned}$$

i.e., in virtue of the Hermitianity of $\overset{o}{\mathrm{Ч}}$, if and only if

$$\mathrm{div}\,X\,\mathfrak{h}(\Psi,\, \Phi) = \mathfrak{h}\big(L[\bar{Y}]\,\Psi,\, \Phi\big) + \mathfrak{h}\big(\Psi,\, L[\bar{Y}]\,\Phi\big)\,,$$

i.e., in virtue of Lemma 3.2.12 (by setting $\overset{o}{f} \equiv s$), if and only if

$$\bar{Y} = (\mathrm{i}\,\overset{o}{f} - \tfrac{1}{2}\,\mathrm{div}\,X)\,\mathbb{I} + C\,, \qquad \text{with} \qquad \overset{o}{f} \in \mathrm{Map}(\boldsymbol{E},\, I\!R)\,, \quad C \in \mathrm{Sec}(\boldsymbol{E},\, \boldsymbol{Q})\,.$$

Hence, Y is an i.s. of $\mathfrak{h}$ if and only if it is of the type

$$Y = \overset{o}{\mathrm{Ч}}(X) + (\mathrm{i}\,\overset{o}{f} - \tfrac{1}{2}\,\mathrm{div}\,X)\,\mathbb{I} + C\,,$$

with

$$X \in \mathrm{Sec}(\boldsymbol{E}, T\boldsymbol{E})\,, \qquad \overset{o}{f} \in \mathrm{Map}(\boldsymbol{E}, \mathbb{R})\,, \qquad C \in \mathrm{Sec}(\boldsymbol{E}, \boldsymbol{Q})\,.$$

Moreover, we recall that the map

$$\mathrm{Quan}(J_1\boldsymbol{E}, \mathbb{R}) \to \big(\mathrm{Pro}(\boldsymbol{E}, T\boldsymbol{E}), \mathrm{Map}(\boldsymbol{E}, \mathbb{R})\big) : f \mapsto (X[f], \overset{o}{f})$$

is a bijection. Hence, Y is Hermitian if and only if it is of the type

$$Y = \overset{o}{\text{Ч}}(X[f]) + (\mathrm{i}\,\overset{o}{f} - \tfrac{1}{2}\,\mathrm{div}\,X)\,\mathbb{I} + C\,,$$

with

$$f \in \mathrm{Quan}(J_1\boldsymbol{E}, \mathbb{R})\,, \qquad C \in \mathrm{Sec}(\boldsymbol{E}, \boldsymbol{Q})\,.$$

Eventually, $Y[f]$ does not depend on the choice of the observer o in virtue of Lemma 3.2.13.

Then, the equivalences 2) $\Leftrightarrow$ 3) and 3) $\Leftrightarrow$ 4) follow easily from the expression of the observed quantum connection.

We can also prove the equivalence 1) $\Leftrightarrow$ 4) directly in coordinates in the following way.
We have

$$\begin{aligned}
L[Y]\,\mathfrak{h} = \big(&\partial_{\mathrm{a}} Y^1\, \check{d}^{\mathrm{a}} \otimes \check{d}^1 + \check{d}^1 \otimes \partial_{\mathrm{a}} Y^1\, \check{d}^{\mathrm{a}} + \partial_{\mathrm{a}} Y^2\, \check{d}^{\mathrm{a}} \otimes \check{d}^2 + \check{d}^2 \otimes \partial_{\mathrm{a}} Y^2\, \check{d}^{\mathrm{a}} \\
&+ \mathrm{div}\,X\, (\check{d}^1 \otimes \check{d}^1 + \check{d}^2 \otimes \check{d}^2) \\
&+ \mathrm{i}\,\partial_{\mathrm{a}} Y^1\, \check{d}^{\mathrm{a}} \otimes \check{d}^2 + \mathrm{i}\,\check{d}^1 \otimes \partial_{\mathrm{a}} Y^2\, \check{d}^{\mathrm{a}} - \mathrm{i}\,\partial_{\mathrm{a}} Y^2\, \check{d}^{\mathrm{a}} \otimes \check{d}^1 + \mathrm{i}\,\check{d}^2 \otimes \partial_{\mathrm{a}} Y^1\, \check{d}^{\mathrm{a}} \\
&+ \mathrm{i}\ \mathrm{div}\,X\, (\check{d}^1 \otimes \check{d}^2 - \check{d}^2 \otimes \check{d}^1)\big) \otimes \eta \\
\\
= \big(&\partial_2 Y^1\, (\check{d}^2 \otimes \check{d}^1 + \check{d}^1 \otimes \check{d}^2) + \partial_1 Y^2\, (\check{d}^1 \otimes \check{d}^2 + \check{d}^2 \otimes \check{d}^1) \\
&+ \partial_1 Y^1\, (\check{d}^1 \otimes \check{d}^1 + \check{d}^1 \otimes \check{d}^1) + \partial_2 Y^2\, (\check{d}^2 \otimes \check{d}^2 + \check{d}^2 \otimes \check{d}^2) \\
&+ \mathrm{div}\,X\, (\check{d}^1 \otimes \check{d}^1 + \check{d}^2 \otimes \check{d}^2) \\
&+ \mathrm{i}\,\partial_1 Y^1\, (\check{d}^1 \otimes \check{d}^2 - \check{d}^2 \otimes \check{d}^1) + \mathrm{i}\,\partial_2 Y^2\, (\check{d}^1 \otimes \check{d}^2 - \check{d}^2 \otimes \check{d}^1) \\
&+ \mathrm{i}\,\partial_2 Y^1\, (\check{d}^2 \otimes \check{d}^2 - \check{d}^2 \otimes \check{d}^2) + \mathrm{i}\,\partial_1 Y^2\, (\check{d}^1 \otimes \check{d}^2 - \check{d}^2 \otimes \check{d}^1) \\
&+ \mathrm{i}\ \mathrm{div}\,X\, (\check{d}^1 \otimes \check{d}^1 - \check{d}^1 \otimes \check{d}^1)\big) \otimes \eta
\end{aligned}$$

Hence, $L[Y]\,\mathfrak{h} = 0$ if and only if

$$\begin{aligned}
\partial_1 Y^1 + \tfrac{1}{2}\,\mathrm{div}\,X &= 0 \\
\partial_2 Y^2 + \tfrac{1}{2}\,\mathrm{div}\,X &= 0 \\
\partial_2 Y^1 + \partial_1 Y^2 &= 0
\end{aligned}$$

i.e. if and only if

$$\begin{aligned}
\partial_1 Y^1 = \partial_2 Y^2 &= -\tfrac{1}{2}\,\mathrm{div}\,X \\
\partial_2 Y^1 + \partial_1 Y^2 &= 0\,.
\end{aligned}$$

Therefore, Lemma 3.2.11 yields the equivalence 1) $\Leftrightarrow$ 4). QED

3.2.15. Remark. Condition 2) in the above theorem shows that each i.s. of $\mathfrak{h}$ is an affine vector field, whose fibre derivative is a complex linear vector field. □

3.2.16. Theorem. *Let $Y \in \mathrm{Sec}(\boldsymbol{Q}, T\boldsymbol{Q})$ be projectable on $\boldsymbol{E}$ and $\boldsymbol{T}$.*
The following conditions are equivalent:
1) Y is Hermitian;
2) with reference to an observer o, we have the expression

$$Y \equiv Y[f] = \overset{o}{Ч}(X[f]) + (\mathrm{i}\,\overset{o}{f} - \tfrac{1}{2}\,\mathrm{div}\,X[f])\,\mathbb{I}\,,$$

where $f \in \mathrm{Quan}(J_1\boldsymbol{E}, I\!R)$;
3) with reference to a quantum basis $\mathfrak{b}$ and to an observer o, we have the expression

$$Y = \chi^{\shortparallel}[\mathfrak{b}]\big(X[f]\big) + \big(\mathrm{i}\,(A[\mathfrak{b}, o]\,(X[f]) + \overset{o}{f}) - \tfrac{1}{2}\,\mathrm{div}\,X[f]\big)\,\mathbb{I}\,,$$

where $f \in \mathrm{Quan}(J_1\boldsymbol{E}, I\!R)$;
4) we have the coordinate expression

$$\begin{aligned} Y &= X^\lambda\,\partial_\lambda + g\,(w^1\,\partial_2 - w^2\,\partial_1) - \tfrac{1}{2}\,(X^0\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} + \frac{\partial_i(X^i\sqrt{|g|})}{\sqrt{|g|}})\,(w^1\,\partial_1 + w^2\,\partial_2) \\ &= X^\lambda\,\partial_\lambda + \Big(\mathrm{i}\,g - \tfrac{1}{2}\,(X^0\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} + \frac{\partial_i(X^i\sqrt{|g|})}{\sqrt{|g|}})\Big)\,(w^1\,\partial_1 + w^2\,\partial_2)\,, \end{aligned}$$

where $X^\lambda, g \in \mathrm{Map}(\boldsymbol{E}, I\!R)$.
Indeed, the expression $Y[f]$ in 2) does not depend on the choice of the observer o.
Moreover, let us suppose that the above conditions be fulfilled. If the quantum chart in 4) is adapted to the quantum basis $\mathfrak{b}$ and to the observer o, then we obtain

$$X = X[f] = f^0\,\partial_0 - f^i\,\partial_i\,, \qquad g = \overset{o}{f} + A_0\,f^0 - A_i\,f^i\,.$$

Proof. This Theorem is a particular case of the above Theorem, with the additional condition of the linearity of Y. QED

The sheaf of Hermitian vector fields is denoted by

$$\mathrm{Her}(\boldsymbol{Q}, T\boldsymbol{Q}) \subset \mathrm{Sec}(\boldsymbol{Q}, T\boldsymbol{Q})\,.$$

3.2.17. Definition. For each $f \in \mathrm{Quan}(J_1\boldsymbol{E}, I\!R)$, the vector field

$$Y[f] := \overset{o}{Ч}(X[f]) + (\mathrm{i}\,\overset{o}{f} - \tfrac{1}{2}\,\mathrm{div}\,X[f])\,\mathbb{I}$$

is called the *quantum lift* of f and the *quantum prolongation* of $X[f] \in \mathrm{Pro}(\boldsymbol{E}, T\boldsymbol{E})$. □

3.2.18. Corollary. Let $f \in \mathrm{Quan}(J_1\boldsymbol{E}, I\!\!R)$.
If $Y[f]$ is vertical, then

$$Y[f] = \mathrm{i}\, f\, \mathbb{I} \qquad \text{and} \qquad f \in \mathrm{Map}(\boldsymbol{E}, I\!\!R)\,.$$

Proof. If $\overset{o}{Ч}(X[f]) = 0$, then $X[f] = 0$. Hence, $\mathrm{div}\, X[f] = 0$ and $f \in \mathrm{Map}(\boldsymbol{E}, I\!\!R)$. QED

3.2.19. Corollary. Let $f \in \mathrm{Quan}(J_1\boldsymbol{E}, I\!\!R)$.
If $Y[f]$ is $\overset{o}{Ч}$–horizontal, with respect to a certain observer o, then

$$f \in \mathrm{Unim}(J_1\boldsymbol{E}, I\!\!R) \qquad \text{and} \qquad \overset{o}{f} = 0\,.$$

Proof. If $\mathrm{i}\,\overset{o}{f} - \frac{1}{2}\,\mathrm{div}\, X[f] = 0$, then we have separately $\mathrm{i}\,\overset{o}{f} = 0$ and $\frac{1}{2}\,\mathrm{div}\, X[f] = 0$. QED

3.2.20. Theorem. *For each $f, g \in \mathrm{Quan}(J_1\boldsymbol{E}, I\!\!R)$, we have*

$$\Big[Y[f],\, Y[g]\Big] = Y\Big[\,[\![\, f, g\,]\!]\,\Big]\,.$$

Hence, the sheaf $\mathrm{Her}(\boldsymbol{Q}, T\boldsymbol{Q})$ is closed with respect to the Lie bracket and the map

$$\mathrm{Quan}(J_1\boldsymbol{E}, I\!\!R) \to \mathrm{Her}(\boldsymbol{Q}, T\boldsymbol{Q}) : f \mapsto Y[f]$$

is a morphism of Lie algebras. Even more, this map is an isomorphism of Lie algebras.
Moreover, the map

$$\mathrm{Her}(\boldsymbol{Q}, T\boldsymbol{Q}) \to \mathrm{Pro}(\boldsymbol{E}, T\boldsymbol{E}) : Y[f] \mapsto X[f]$$

is a central extension of Lie algebras by $\mathrm{Map}(\boldsymbol{E}, I\!\!R)$.

Proof. Let us prove that the map $f \mapsto Y[f]$ is a morphism of Lie algebras.
We have

$$\begin{aligned}
\big[Y[f], Y[g]\big] &:= \big[\overset{o}{Ч}(X[f]) + (\mathrm{i}\,\overset{o}{f} - \tfrac{1}{2}\,\mathrm{div}\, X[f])\,\mathbb{I},\quad \overset{o}{Ч}(X[g]) + (\mathrm{i}\,\overset{o}{g} - \tfrac{1}{2}\,\mathrm{div}\, X[g])\,\mathbb{I}\big] \\
&= \big[\overset{o}{Ч}(X[f]),\ \overset{o}{Ч}(X[g])\big] \\
&\quad + \big[\overset{o}{Ч}(X[f]),\ \ (\mathrm{i}\,\overset{o}{g} - \tfrac{1}{2}\,\mathrm{div}\, X[g])\,\mathbb{I}\big] + \big[(\mathrm{i}\,\overset{o}{f} - \tfrac{1}{2}\,\mathrm{div}\, X[f])\,\mathbb{I},\ \ \overset{o}{Ч}(X[g])\big] \\
&\quad + \big[(\mathrm{i}\,\overset{o}{f} - \tfrac{1}{2}\,\mathrm{div}\, X[f])\,\mathbb{I},\quad (\mathrm{i}\,\overset{o}{g} - \tfrac{1}{2}\,\mathrm{div}\, X[g])\,\mathbb{I}\big]\,.
\end{aligned}$$

Moreover, in virtue of Lemma 1.2.1 and of our assumptions on the quantum connection, we obtain

$$\begin{aligned}
\big[\overset{o}{Ч}(X[f]),\ \overset{o}{Ч}(X[g])\big] &= \overset{o}{Ч}\big(X\,[\![\, f,\, g\,]\!]\,\big) + i(X[g])\, i(X[f])\, R[\overset{o}{Ч}] \\
&= \overset{o}{Ч}\big(X\,[\![\, f,\, g\,]\!]\,\big) + \frac{1}{2}\,\big(i(X[g])\, i(X[f])\, \Phi[o]\big)\,\mathrm{i}\,\mathbb{I}\,.
\end{aligned}$$

Furthermore, in virtue of Lemma 1.2.8, we obtain

$$\big[\overset{o}{\text{Ч}}(X[f]),\ \ (\mathrm{i}\,\overset{o}{g} - \tfrac{1}{2}\,\operatorname{div} X[g])\,\mathbb{I}\big] = \big(X[f].(\mathrm{i}\,\overset{o}{g} - \tfrac{1}{2}\,\operatorname{div} X[g])\big)\,\mathbb{I}$$
$$\big[\overset{o}{\text{Ч}}(X[g]),\ \ (\mathrm{i}\,\overset{o}{f} - \tfrac{1}{2}\,\operatorname{div} X[f])\,\mathbb{I}\big] = \big(X[g].(\mathrm{i}\,\overset{o}{f} - \tfrac{1}{2}\,\operatorname{div} X[f])\big)\,\mathbb{I}$$

Additionally, in virtue of Lemma 1.2.9, we obtain

$$\big[(\mathrm{i}\,\overset{o}{f} - \tfrac{1}{2}\,\operatorname{div} X[f])\,\mathbb{I},\quad (\mathrm{i}\,\overset{o}{g} - \tfrac{1}{2}\,\operatorname{div} X[g])\,\mathbb{I}\big] = 0\,.$$

Hence, in virtue of Lemma 2.3.29 and Lemma 2.1.15, we obtain

$$\begin{aligned}
\big[Y[f],\, Y[g]\big] &= \overset{o}{\text{Ч}}\big(X\,[\![\, f,\, g\,]\!]\,\big) + \frac{1}{2}\,\big(i(X[g])\,i(X[f])\,\Phi[o]\big)\,\mathrm{i}\,\mathbb{I} \\
&\quad + \big(\mathrm{i}\,(X[f].\overset{o}{g} - X[g].\overset{o}{f}) - \tfrac{1}{2}\,(X[f].\operatorname{div} X[g] - X[g].\operatorname{div} X[f])\big)\,\mathbb{I} \\
&= \overset{o}{\text{Ч}}\big(X\,[\![\, f,\, g\,]\!]\,\big) + \frac{1}{2}\,\big(i(X[g])\,i(X[f])\,\Phi[o]\big)\,\mathrm{i}\,\mathbb{I} \\
&\quad + \big(\mathrm{i}\,(X[f].\overset{o}{g} - X[g].\overset{o}{f}) - \tfrac{1}{2}\,(X[f].\operatorname{div} X[g] - X[g].\operatorname{div} X[f])\big)\,\mathbb{I} \\
&= \overset{o}{\text{Ч}}\big(X\,[\![\, f,\, g\,]\!]\,\big) + \big(\mathrm{i}\,\overset{o}{[\![\, f,\, g\,]\!]} - \tfrac{1}{2}\,\operatorname{div}(\,[\![\, f,\, g\,]\!]\,)\big)\,.
\end{aligned}$$

Next, let us prove that the map $f \mapsto Y[f]$ is a bijection. In fact, if $f, g \in \operatorname{Quan}(J_1\boldsymbol{E},\, I\!R)$, and

$$\overset{o}{\text{Ч}}(X[f]) + (\mathrm{i}\,\overset{o}{f} - \tfrac{1}{2}\,\operatorname{div} X[f])\,\mathbb{I} = \overset{o}{\text{Ч}}(X[g]) + (\mathrm{i}\,\overset{o}{g} - \tfrac{1}{2}\,\operatorname{div} X[g])\,\mathbb{I}\,,$$

then we obtain

$$X[f] = X[g] \qquad \text{and} \qquad \overset{o}{f} = \overset{o}{g}\,,$$

hence

$$f = g\,.$$

Eventually, let us prove that the map $Y[f] \mapsto X[f]$ is a central extension by $\operatorname{Map}(\boldsymbol{E},\, I\!R)$. In fact, this map is surjective and its kernel is constituted by the Hermitian vector fields of the type

$$Y[f] = \mathrm{i}\,\overset{o}{f}\,\mathbb{I}\,,$$

which are in bijection with the functions of $\boldsymbol{E}$. Moreover, these functions commute with all quantisable functions with respect to the special bracket. QED

3.2.3 Symmetries of the Hermitian metric of $Q^{\uparrow}$

Next, we classify the vector fields of $\boldsymbol{Q}^{\uparrow}$, which preserve the Hermitian metric.

3.2.21. Remark. The Hermitian metric $\mathfrak{h}^\uparrow$ can be naturally regarded as a section

$$\mathfrak{h}^\uparrow : \boldsymbol{Q}^\uparrow \to V^*\boldsymbol{Q}^\uparrow \underset{\boldsymbol{Q}}{\otimes} V^*\boldsymbol{Q}^\uparrow \,, \underset{\boldsymbol{Q}}{\otimes} \Lambda^3 V^*_{\boldsymbol{T}}\boldsymbol{Q}^\uparrow \,,$$

where $V^*_{\boldsymbol{E}}\boldsymbol{Q}^\uparrow$ denotes the vertical dual with respect to the fibring of $\boldsymbol{Q}^\uparrow$ over $\boldsymbol{E}$ and $V^*_{\boldsymbol{T}}\boldsymbol{Q}^\uparrow$ denotes the vertical dual with respect to the fibring of $\boldsymbol{Q}^\uparrow$ over $\boldsymbol{T}$. We have the coordinate expression

$$\mathfrak{h}^\uparrow = \Big((\check{d}^1 \otimes \check{d}^1 + \check{d}^2 \otimes \check{d}^2) + \mathrm{i}\,(\check{d}^1 \otimes \check{d}^2 - \check{d}^2 \otimes \check{d}^1)\Big) \otimes \eta$$

Accordingly, if $Y^\uparrow \in \mathrm{Sec}(\boldsymbol{Q}^\uparrow,\, T\boldsymbol{Q}^\uparrow)$ is projectable on $J_1\boldsymbol{E}$ and $\boldsymbol{T}$, then the Lie derivative $L[Y^\uparrow]\mathfrak{h}^\uparrow$ is well defined (in spite of the fact that $\mathfrak{h}^\uparrow$ is vertical valued). □

3.2.22. Definition. A vector field $Y^\uparrow \in \mathrm{Sec}(\boldsymbol{Q}^\uparrow,\, T\boldsymbol{Q}^\uparrow)$ is said to be an *infinitesimal symmetry* (*i.s.*) of $\mathfrak{h}^\uparrow$ if it is projectable on $J_1\boldsymbol{E}$ and $\boldsymbol{T}$ and fulfills the equality

$$L[Y^\uparrow]\, \mathfrak{h}^\uparrow = 0 \,.$$

A vector field $Y^\uparrow \in \mathrm{Sec}(\boldsymbol{Q}^\uparrow,\, T\boldsymbol{Q}^\uparrow)$ is said to be *Hermitian* if it is (real) linear projectable on $J_1\boldsymbol{E}$ and projectable on $\boldsymbol{T}$ and fulfills the equality

$$L[Y^\uparrow]\, \mathfrak{h}^\uparrow = 0 \,. \square$$

3.2.23. Remark. If $Y^\uparrow \in \mathrm{Sec}(\boldsymbol{Q}^\uparrow,\, T\boldsymbol{Q}^\uparrow)$ is projectable on $J_1\boldsymbol{E}$ and $\boldsymbol{T}$, then we obtain, for each $\Phi, \Psi \in \mathrm{Sec}(\boldsymbol{E},\, \boldsymbol{Q})$,

$$L[Y^\uparrow]\,\Big(\mathfrak{h}^\uparrow(\Phi,\, \Psi)\Big) = (L[Y^\uparrow]\, \mathfrak{h}^\uparrow)(\Phi,\, \Psi) + \mathfrak{h}^\uparrow\Big(L[Y^\uparrow]\, \Phi,\, \Psi\Big) + \mathfrak{h}^\uparrow\Big(\Phi,\, L[Y^\uparrow]\, \Psi\Big) \,,$$

and, if additionally $Y^\uparrow$ is linear over $J_1\boldsymbol{E}$,

$$L[Y^\uparrow]\,\Big(\mathfrak{h}^\uparrow(\Phi,\, \Psi)\Big) = (L[Y^\uparrow]\, \mathfrak{h}^\uparrow)(\Phi,\, \Psi) + \mathfrak{h}^\uparrow\Big(Y^\uparrow.\Phi,\, \Psi\Big) + \mathfrak{h}^\uparrow\Big(\Phi,\, Y^\uparrow.\Psi\Big) \,.$$

Proof. It follows immediately from the Leibnitz rule of the Lie derivative, by regarding Φ and Ψ as vertical valued vector fields of $\boldsymbol{Q}^\uparrow$, via pullback. QED

3.2.24. Lemma. Let $X^\uparrow \in \mathrm{Pro}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$ and $\Psi, \Phi \in \mathrm{Sec}(\boldsymbol{E},\, \boldsymbol{Q})$. Then, we have

$$L[X^\uparrow]\Big(\mathfrak{h}^\uparrow(\Psi, \Phi)\Big) = X^\uparrow \lrcorner \nabla[K']\Big(\mathfrak{h}^\uparrow(\Psi, \Phi)\Big) + \mathfrak{h}^\uparrow(\Psi, \Phi)\ \mathrm{div}\, X^\uparrow \,.$$

Proof. We have

$$\nabla[K']\, \eta = 0 \,, \qquad L[X^\uparrow]\, \eta = \mathrm{div}\, X^\uparrow\, \eta \,,$$

where K' is the linear connection induced on the vector bundle $V\boldsymbol{E} \to \boldsymbol{E}$ by the linear connection K.

Hence, we obtain

$$L[X^{\uparrow}]\big(\mathfrak{h}^{\uparrow}(\Psi,\Phi)\big) = L[X^{\uparrow}](\bar{\psi}\,\phi\,\eta) = X^{\uparrow}.(\bar{\psi}\,\phi)\,\eta + (\bar{\psi}\,\phi)\ \mathrm{div}\,X^{\uparrow}\,\eta$$
$$X^{\uparrow}\lrcorner\nabla[K']\big(\mathfrak{h}^{\uparrow}(\Psi,\Phi)\big) = X^{\uparrow}\lrcorner\nabla[K'](\bar{\psi}\,\phi\,\eta) = X^{\uparrow}.(\bar{\psi}\,\phi)\,\eta + (\bar{\psi}\,\phi)\,X^{\uparrow}\lrcorner\nabla[K']\eta = X^{\uparrow}.(\bar{\psi}\,\phi)\,\eta\,.\ \mathrm{QED}$$

Now, we classify the i.s.'s of $\mathfrak{h}^{\uparrow}$ and the Hermitian vector fields. For this purpose, we need several preliminary results.

3.2.25. Lemma. Let $Y^{\uparrow} \in \mathrm{Sec}(\boldsymbol{Q}^{\uparrow},\, T\boldsymbol{Q}^{\uparrow})$ be projectable on $X^{\uparrow} \in \mathrm{Sec}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$ and

$$\partial_1 Y^1 = \partial_2 Y^2 := r \in \mathrm{Map}(J_1\boldsymbol{E},\, I\!R)\,, \qquad \partial_1 Y^2 = -\partial_2 Y^1 := s \in \mathrm{Map}(\boldsymbol{Q}^{\uparrow},\, I\!R)\,.$$

Then, the coordinate expression of $Y^{\uparrow}$ is of the type

$$Y^{\uparrow} = X^{\lambda}\,\partial_{\lambda} + (r + \mathrm{i}\,s)\,\mathbb{I} + C^{\mathrm{a}}\,\partial_{\mathrm{a}}\,, \qquad \text{with} \qquad r,\ s,\ C^{\mathrm{a}} \in \mathrm{Map}(J_1\boldsymbol{E},\, I\!R)\,.$$

Proof. By integrating the equations

$$\partial_1 Y^1 = r \qquad \text{and} \qquad \partial_2 Y^2 = r$$

on the affine fibres of the extended quantum bundle, we obtain

$$Y^1 = r\,w^1 + a^1 \qquad \text{and} \qquad Y^2 = r\,w^2 + a^2\,,$$

where

*) $$a^1,\ a^2 \in \mathrm{Map}(\boldsymbol{Q}^{\uparrow},\, I\!R)\,, \qquad \text{with} \qquad \partial_1 a^1 = 0 = \partial_2 a^2\,.$$

Then, the equation

$$\partial_1 Y^2 = -\partial_2 Y^1$$

reads

**) $$\partial_1 a^2 = -\partial_2 a^1\,.$$

Moreover, (*) and (**) yield

$$\partial_1 a^2 = s \qquad \text{and} \qquad \partial_2 a^1 = -s\,, \qquad \text{with} \qquad s \in \mathrm{Map}(J_1\boldsymbol{E},\, I\!R)\,,$$

hence, by integrating the above equations on the affine fibres of the extended quantum bundle,

$$a^1 = -s\,w^2 + C^1 \qquad \text{and} \qquad a^2 = s\,w^1 + C^2\,, \qquad \text{with} \qquad C^1,\ C^2 \in \mathrm{Map}(J_1\boldsymbol{E},\, I\!R)\,.$$

Therefore, we obtain

$$\begin{aligned} Y &= X^{\lambda}\,\partial_{\lambda} + r\,(w^1\,\partial_1 + w^2\,\partial_2) + s\,(w^1\,\partial_2 - w^2\,\partial_1) + C^1\,\partial_1 + C^2\,\partial_2 \\ &= X^{\lambda}\,\partial_{\lambda} + (r + \mathrm{i}\,s)\,(w^1\,\partial_1 + w^2\,\partial_2) + C^1\,\partial_1 + C^2\,\partial_2\,.\ \mathrm{QED} \end{aligned}$$

3.2.26. Lemma. Let $Y^{\uparrow} \in \mathrm{Sec}(\boldsymbol{Q}^{\uparrow}, V\boldsymbol{Q}^{\uparrow})$ and $r \in \mathrm{Map}(J_1\boldsymbol{E}, \mathbb{C})$ and suppose that

$$\mathtt{h}^{\uparrow}\Big(L[Y^{\uparrow}]\,\Psi,\,\Phi\Big) + \mathtt{h}^{\uparrow}\Big(\Psi,\,L[Y^{\uparrow}]\,\Phi\Big) = r\,\mathtt{h}^{\uparrow}(\Psi,\,\Phi)\,, \qquad \forall \Psi, \Phi \in \mathrm{Sec}(J_1\boldsymbol{E},\,\boldsymbol{Q}^{\uparrow})\,.$$

Then, it turns out that $r \in \mathrm{Map}(J_1\boldsymbol{E},\,I\!\!R)$ and $Y^{\uparrow}$ is of the type

$$Y^{\uparrow} = (\mathrm{i}\,s - \tfrac{1}{2}\,r)\,\mathbb{I} + C^{\mathrm{a}}\,\partial_{\mathrm{a}}\,, \qquad \text{with} \qquad s,\,C^{\mathrm{a}} \in \mathrm{Map}(J_1\boldsymbol{E}, I\!\!R)\,.$$

Proof. We have

$$r\,\mathtt{h}^{\uparrow}(\Psi,\,\Phi) = r\,(\Psi^1\,\Phi^1 + \Psi^2\,\Phi^2) + r\,\mathrm{i}\,(\Psi^1\,\Phi^2 - \Psi^2\,\Phi^1)\,.$$

Moreover, we have

$$\begin{aligned} L[Y^{\uparrow}]\,\Psi &= -(\Psi^1\,\partial_1 Y^1 + \Psi^2\,\partial_2 Y^1)\,\partial_1 - (\Psi^1\,\partial_1 Y^2 + \Psi^2\,\partial_2 Y^2)\,\partial_2 \\ L[Y^{\uparrow}]\,\Phi &= -(\Phi^1\,\partial_1 Y^1 + \Phi^2\,\partial_2 Y^1)\,\partial_1 - (\Phi^1\,\partial_1 Y^2 + \Phi^2\,\partial_2 Y^2)\,\partial_2\,, \end{aligned}$$

hence

$$\begin{aligned} \mathtt{h}^{\uparrow}\big(L[Y^{\uparrow}]\,\Psi,\,\Phi\big) + \mathtt{h}^{\uparrow}\big(\Psi,\,L[Y^{\uparrow}]\,\Phi\big) = &-(\Psi^1\,\partial_1 Y^1 + \Psi^2\,\partial_2 Y^1)\,\Phi^1 - (\Psi^1\,\partial_1 Y^2 + \Psi^2\,\partial_2 Y^2)\,\Phi^2 \\ &- \mathrm{i}\,(\Psi^1\,\partial_1 Y^1 + \Psi^2\,\partial_2 Y^1)\,\Phi^2 + \mathrm{i}\,(\Psi^1\,\partial_1 Y^2 + \Psi^2\,\partial_2 Y^2)\,\Phi^1 \\ \\ &- \Psi^1\,(\Phi^1\,\partial_1 Y^1 + \Phi^2\,\partial_2 Y^1) - \Psi^2\,(\Phi^1\,\partial_1 Y^2 + \Phi^2\,\partial_2 Y^2) \\ &- \mathrm{i}\,\Psi^1\,(\Phi^1\,\partial_1 Y^2 + \Phi^2\,\partial_2 Y^2) + \mathrm{i}\,\Psi^2\,(\Phi^1\,\partial_1 Y^1 + \Phi^2\,\partial_2 Y^1)\,. \end{aligned}$$

Therefore, we have

$$\mathtt{h}^{\uparrow}\big(L[Y^{\uparrow}]\,\Psi,\,\Phi\big) + \mathtt{h}^{\uparrow}\big(\Psi,\,L[Y^{\uparrow}]\,\Phi\big) = r\,\mathtt{h}^{\uparrow}(\Psi,\,\Phi)$$

if and only

$$\partial_1 Y^1 = \partial_2 Y^2 = -\tfrac{1}{2}\,r \qquad \text{and} \qquad \partial_1 Y^2 = -\partial_2 Y^1\,.$$

Hence, the result follows from Lemma 3.2.25. QED

3.2.27. Theorem. *Let $Y^{\uparrow} \in \mathrm{Sec}(\boldsymbol{Q}^{\uparrow},\,T\boldsymbol{Q}^{\uparrow})$ be projectable on $J_1\boldsymbol{E}$ and $\boldsymbol{T}$.*
The following conditions are equivalent:
1) $Y^{\uparrow}$ is an i.s. of $\mathtt{h}^{\uparrow}$;
2) we have the expression

$$Y^{\uparrow} \;\equiv\; Y^{\uparrow}[X^{\uparrow}, f] + C^{\uparrow} = \text{Ч}(X^{\uparrow}) + (\mathrm{i}\,f - \tfrac{1}{2}\,\mathrm{div}\,X^{\uparrow})\,\mathbb{I} + C^{\uparrow}\,,$$

where $X^{\uparrow} \in \mathrm{Pro}(J_1\boldsymbol{E},\,TJ_1\boldsymbol{E})$, $f \in \mathrm{Map}(J_1\boldsymbol{E},\,I\!\!R)$ and $C^{\uparrow} \in \mathrm{Sec}(J_1\boldsymbol{E},\,\boldsymbol{Q}^{\uparrow})$;
3) we have the coordinate expression

$$Y^{\uparrow} = X^0\,\partial_0 + X^i\,\partial_i + X^i_0\,\partial^0_i + \Big(\mathrm{i}\,g - \tfrac{1}{2}\,(X^0\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} + \frac{\partial_i(X^i\sqrt{|g|})}{\sqrt{|g|}})\Big)\,\mathbb{I} + C^{\mathrm{a}}\,\partial_{\mathrm{a}}\,,$$

where $X^\uparrow \in \mathrm{Pro}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$, $\quad g \in \mathrm{Map}(J_1\boldsymbol{E},\, I\!R)$, $\quad C^{\mathrm{a}} \in \mathrm{Map}(J_1\boldsymbol{E},\, I\!R)$.

Moreover, let us suppose that the above conditions be fulfilled. Then we obtain

$$g = f - X^0\,(\tfrac{1}{2}\,G^0_{ij}\,x^i_0\,x^j_0 - A_0) + X^i\,(G^0_{ij}\,x^j_0 + A_i)\,.$$

PROOF. Let us prove the equivalence 1) $\Leftrightarrow$ 2).
We can split $Y^\uparrow$ into its horizontal and vertical components as

$$Y = \text{Ч}(X) + \bar{Y}^\uparrow \qquad \text{where} \qquad \bar{Y} := \nu[\text{Ч}](Y^\uparrow)\,.$$

Now, $Y^\uparrow$ is an i.s. of $\mathfrak{h}^\uparrow$ if and only if, for each $\Psi, \Phi \in \mathrm{Sec}(\boldsymbol{E},\, \boldsymbol{Q})$,

$$L[X^\uparrow]\big(h^\uparrow(\Psi, \Phi)\big) = h^\uparrow(L[Y^\uparrow]\,\Psi,\, \Phi) + h^\uparrow(\Psi,\, L[Y^\uparrow]\,\Phi)$$

i.e., in virtue of Lemma 3.2.24, if and only if

$$X^\uparrow \lrcorner \nabla[K']\big(\mathfrak{h}^\uparrow(\Psi,\,\Phi)\big) + \mathrm{div}\,X^\uparrow\,\mathfrak{h}^\uparrow(\Psi,\,\Phi) =$$
$$= \mathfrak{h}^\uparrow(X^\uparrow \lrcorner \nabla[\text{Ч}]\Psi,\,\Phi) + \mathfrak{h}^\uparrow(\Psi,\,X^\uparrow \lrcorner \nabla[\overset{o}{\text{Ч}}]\Phi) + \mathfrak{h}^\uparrow\big(L[\bar{Y}^\uparrow]\,\Psi,\,\Phi\big) + \mathfrak{h}^\uparrow\big(\Psi,\,L[\bar{Y}^\uparrow]\,\Phi\big)\,,$$

i.e., in virtue of the Hermitianity of Ч, if and only if

$$\mathrm{div}\,X^\uparrow\,\mathfrak{h}^\uparrow(\Psi,\,\Phi) = \mathfrak{h}^\uparrow\big(L[\bar{Y}^\uparrow]\,\Psi,\,\Phi\big) + \mathfrak{h}^\uparrow\big(\Psi,\,L[\bar{Y}^\uparrow]\,\Phi\big)\,,$$

i.e., in virtue of Lemma 3.2.26 (by setting $f \equiv s$), if and only if

$$\bar{Y}^\uparrow = (\mathrm{i}\,f - \tfrac{1}{2}\,\mathrm{div}\,X^\uparrow)\,\mathbb{I} + C\,, \qquad \text{with} \qquad f \in \mathrm{Map}(J_1\boldsymbol{E},\,I\!R)\,, \quad C \in \mathrm{Sec}(J_1\boldsymbol{E},\,\boldsymbol{Q}^\uparrow)\,.$$

Hence, $Y^\uparrow$ is an i.s. of $\mathfrak{h}^\uparrow$ if and only if it is of the type

$$Y^\uparrow = \text{Ч}(X^\uparrow) + (\mathrm{i}\,f - \tfrac{1}{2}\,\mathrm{div}\,X^\uparrow)\,\mathbb{I} + C^\uparrow\,,$$

with

$$X^\uparrow \in \mathrm{Pro}(J_1\boldsymbol{E},\,TJ_1\boldsymbol{E})\,, \qquad f \in \mathrm{Map}(J_1\boldsymbol{E},\,I\!R)\,, \qquad C^\uparrow \in \mathrm{Sec}(J_1\boldsymbol{E},\,\boldsymbol{Q}^\uparrow)\,.$$

Then, the equivalence 2) $\Leftrightarrow$ 3) follows easily from the expression of the quantum connection.

We can also prove the equivalence 1) $\Leftrightarrow$ 3) directly in coordinates in the following way.
We have

$$\begin{aligned}
L[Y^\uparrow]\,\mathfrak{h}^\uparrow = \big(&\partial_{\mathrm{a}}Y^1\,\check{d}^{\mathrm{a}} \otimes \check{d}^1 + \check{d}^1 \otimes \partial_{\mathrm{a}}Y^1\,\check{d}^{\mathrm{a}} + \partial_{\mathrm{a}}Y^2\,\check{d}^{\mathrm{a}} \otimes \check{d}^2 + \check{d}^2 \otimes \partial_{\mathrm{a}}Y^2\,\check{d}^{\mathrm{a}}\\
&+ \mathrm{div}\,X\,(\check{d}^1 \otimes \check{d}^1 + \check{d}^2 \otimes \check{d}^2)\\
&+ \mathrm{i}\,\partial_{\mathrm{a}}Y^1\,\check{d}^{\mathrm{a}} \otimes \check{d}^2 + \mathrm{i}\,\check{d}^1 \otimes \partial_{\mathrm{a}}Y^2\,\check{d}^{\mathrm{a}} - \mathrm{i}\,\partial_{\mathrm{a}}Y^2\,\check{d}^{\mathrm{a}} \otimes \check{d}^1 + \mathrm{i}\,\check{d}^2 \otimes \partial_{\mathrm{a}}Y^1\,\check{d}^{\mathrm{a}}\\
&+ \mathrm{i}\,\mathrm{div}\,X\,(\check{d}^1 \otimes \check{d}^2 - \check{d}^2 \otimes \check{d}^1)\big) \otimes \eta\\[2ex]
= \big(&\partial_2Y^1\,(\check{d}^2 \otimes \check{d}^1 + \check{d}^1 \otimes \check{d}^2) + \partial_1Y^2\,(\check{d}^1 \otimes \check{d}^2 + \check{d}^2 \otimes \check{d}^1)\\
&+ \partial_1Y^1\,(\check{d}^1 \otimes \check{d}^1 + \check{d}^1 \otimes \check{d}^1) + \partial_2Y^2\,(\check{d}^2 \otimes \check{d}^2 + \check{d}^2 \otimes \check{d}^2)\\
&+ \mathrm{div}\,X\,(\check{d}^1 \otimes \check{d}^1 + \check{d}^2 \otimes \check{d}^2)\\
&+ \mathrm{i}\,\partial_1Y^1\,(\check{d}^1 \otimes \check{d}^2 - \check{d}^2 \otimes \check{d}^1) + \mathrm{i}\,\partial_2Y^2\,(\check{d}^1 \otimes \check{d}^2 - \check{d}^2 \otimes \check{d}^1)\\
&+ \mathrm{i}\,\partial_2Y^1\,(\check{d}^2 \otimes \check{d}^2 - \check{d}^2 \otimes \check{d}^2) + \mathrm{i}\,\partial_1Y^2\,(\check{d}^1 \otimes \check{d}^2 - \check{d}^2 \otimes \check{d}^1)\\
&+ \mathrm{i}\,\mathrm{div}\,X\,(\check{d}^1 \otimes \check{d}^1 - \check{d}^1 \otimes \check{d}^1)\big) \otimes \eta
\end{aligned}$$

Hence, $L[Y^{\uparrow}]\,\mathfrak{h}^{\uparrow} = 0$ if and only if

$$\partial_1 Y^1 + \tfrac{1}{2}\,\operatorname{div} X^{\uparrow} = 0$$
$$\partial_2 Y^2 + \tfrac{1}{2}\,\operatorname{div} X^{\uparrow} = 0$$
$$\partial_2 Y^1 + \partial_1 Y^2 = 0$$

i.e. if and only if

$$\partial_1 Y^1 = \partial_2 Y^2 = -\tfrac{1}{2}\,\operatorname{div} X^{\uparrow}$$
$$\partial_2 Y^1 + \partial_1 Y^2 = 0\,.$$

Therefore, Lemma 3.2.25 yields the equivalence 1) $\Leftrightarrow$ 3). QED

3.2.28. Remark. Condition 2) in the above theorem shows that each i.s. of $\mathfrak{h}^{\uparrow}$ is an affine vector field, whose fibre derivative is a complex linear vector field. □

3.2.29. Theorem. *Let $Y^{\uparrow} \in \operatorname{Sec}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow})$ be projectable on $J_1\boldsymbol{E}$ and $\boldsymbol{T}$.*
The following conditions are equivalent:
1) $Y^{\uparrow}$ is Hermitian;
2) we have the expression

$$Y^{\uparrow} \equiv Y^{\uparrow}[X^{\uparrow}, f] = \text{Ч}(X^{\uparrow}) + (\mathrm{i}\, f - \tfrac{1}{2}\,\operatorname{div} X^{\uparrow})\,\mathbb{I}\,,$$

where $X^{\uparrow} \in \operatorname{Pro}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$, $\quad f \in \operatorname{Map}(J_1\boldsymbol{E}, I\!R)$;
3) we have the coordinate expression

$$Y^{\uparrow} = X^0\,\partial_0 + X^i\,\partial_i + X^i_0\,\partial^0_i + \Big(\mathrm{i}\, g - \tfrac{1}{2}\,(X^0\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} + \frac{\partial_i(X^i\sqrt{|g|})}{\sqrt{|g|}})\Big)\,\mathbb{I}\,,$$

where $X^{\uparrow} \in \operatorname{Pro}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$, $\quad g \in \operatorname{Map}(J_1\boldsymbol{E}, I\!R)$.
Moreover, let us suppose that the above conditions be fulfilled. Then we obtain

$$g = f - X^0\,(\tfrac{1}{2}\,G^0_{ij}\,x^i_0\,x^j_0 - A_0) + X^i\,(G^0_{ij}\,x^j_0 + A_i)\,.$$

Proof. This Theorem is a particular case of the above Theorem, with the additional condition of the linearity of $Y^{\uparrow}$. QED

The sheaf of Hermitian vector fields of $\boldsymbol{Q}^{\uparrow}$ is denoted by

$$\tilde{\operatorname{Her}}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow}) \subset \operatorname{Sec}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow})\,.$$

3.2.30. Definition. For each $X^{\uparrow} \in \operatorname{Pro}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ and $f \in \operatorname{Map}(J_1\boldsymbol{E}, I\!R)$, the vector field

$$Y^{\uparrow}[X^{\uparrow}, f] := \text{Ч}(X^{\uparrow}, f) + (\mathrm{i}\, f - \tfrac{1}{2}\,\operatorname{div} X^{\uparrow})\,\mathbb{I}$$

is called the *quantum lift* of the pair $(X^{\uparrow}, f)$. □

3.2.31. Corollary. Let $X^\uparrow \in \mathrm{Pro}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$ and $f \in \mathrm{Map}(J_1\boldsymbol{E},\, I\!R)$. If $Y^\uparrow[X^\uparrow, f]$ is vertical, then

$$Y^\uparrow[f] = \mathrm{i}\, f\, \mathbb{I} \qquad \text{and} \qquad X^\uparrow = 0\,.$$

Proof. If $\text{Ч}(X^\uparrow) = 0$, then $X^\uparrow = 0$. Hence, $\operatorname{div} X^\uparrow = 0$. QED

3.2.32. Corollary. Let $X^\uparrow \in \mathrm{Pro}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$ and $f \in \mathrm{Map}(J_1\boldsymbol{E},\, I\!R)$. If $Y[X^\uparrow, f]$ is Ч–horizontal, then

$$X^\uparrow \in \mathrm{Unim}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E}) \qquad \text{and} \qquad f = 0\,.$$

Proof. If $\mathrm{i}\, f - \frac{1}{2} \operatorname{div} X^\uparrow = 0$, then we have separately $\mathrm{i}\, f = 0$ and $\frac{1}{2} \operatorname{div} X^\uparrow = 0$. QED

3.2.33. Proposition. [126] If $X^\uparrow, \bar{X}^\uparrow \in \mathrm{Pro}(J_1\boldsymbol{E},\, TJ_1\boldsymbol{E})$ and $f, \bar{f} \in \mathrm{Map}(J_1\boldsymbol{E},\, I\!R)$, then we have

$$\Big[Y^\uparrow[X^\uparrow, f],\; Y^\uparrow[\bar{X}^\uparrow, \bar{f}]\Big] = Y^\uparrow\Big[[X^\uparrow, \bar{X}^\uparrow],\;\; (X^\uparrow.\bar{f} - \bar{X}^\uparrow.f + i(\bar{X}^\uparrow)\, i(X^\uparrow)\, \Omega)\Big]\,.$$

Hence, the sheaf $\widetilde{\mathrm{Her}}(\boldsymbol{Q}^\uparrow,\, T\boldsymbol{Q}^\uparrow)$ is closed with respect to the Lie bracket.

Proof. By recalling Lemmas 1.2.1, 1.2.8, 1.2.9, 2.1.21, we obtain

$$\begin{aligned}
\big[Y^\uparrow[X^\uparrow, f],\; Y^\uparrow[\bar{X}^\uparrow, \bar{f}]\big] &= \big[\text{Ч}(X^\uparrow),\, \text{Ч}(\bar{X}^\uparrow)\big] \\
&\quad + \big[\text{Ч}(X^\uparrow),\, (\mathrm{i}\, \bar{f} - \tfrac{1}{2} \operatorname{div} \bar{X}^\uparrow)\, \mathbb{I}\big] - \big[\text{Ч}(\bar{X}^\uparrow),\, (\mathrm{i}\, f - \tfrac{1}{2} \operatorname{div} X^\uparrow)\, \mathbb{I}\big] \\
&\quad + \big[(\mathrm{i}\, f - \tfrac{1}{2} \operatorname{div} X^\uparrow)\, \mathbb{I},\; (\mathrm{i}\, \bar{f} - \tfrac{1}{2} \operatorname{div} \bar{X}^\uparrow)\, \mathbb{I}\big] \\
&= \text{Ч}\big([X^\uparrow,\, \bar{X}^\uparrow]\big) + 2\,\mathrm{i}\; i(\bar{X}^\uparrow)\, i(X^\uparrow)\, \Omega\, \mathbb{I} \\
&\quad + \big(\mathrm{i}\, (X^\uparrow.\bar{f} - \bar{X}^\uparrow.f) - \tfrac{1}{2}\, (X^\uparrow.\operatorname{div} \bar{X}^\uparrow - \bar{X}^\uparrow.\operatorname{div} X^\uparrow)\big)\, \mathbb{I} \\
&= \text{Ч}\big([X^\uparrow,\, \bar{X}^\uparrow]\big) + 2\,\mathrm{i}\; i(\bar{X}^\uparrow)\, i(X^\uparrow)\, \Omega\, \mathbb{I} \\
&\quad + \big(\mathrm{i}\, (X^\uparrow.\bar{f} - \bar{X}^\uparrow.f) - \tfrac{1}{2} \operatorname{div}[X^\uparrow,\, \bar{X}^\uparrow]\big)\, \mathbb{I} \\
&= Y^\uparrow\big[[X^\uparrow,\, \bar{X}^\uparrow],\;\; (X^\uparrow.\bar{f} - \bar{X}^\uparrow.f + 2\, i(\bar{X}^\uparrow)\, i(X^\uparrow)\, \Omega)\big]\,.
\end{aligned}$$
QED

3.2.4 Projectable Hermitian vector fields of $Q^\uparrow$

In particular, we are interested in Hermitian vector fields of $\boldsymbol{Q}^\uparrow$, which are projectable on $\boldsymbol{Q}$.

3.2.34. Theorem. *A Hermitian vector field* $Y^{\uparrow}[X^{\uparrow}, f] \in \tilde{\mathrm{Her}}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow})$ *is projectable on* $\boldsymbol{Q}$ *if and only if*

(i) $f \in \mathrm{Quan}(J_1\boldsymbol{E}, I\!R)$,

(ii) $X^{\uparrow}$ *is projectable on the tangent lift* $X[f]$ *of* f.

Thus, the Hermitian vector fields of $\boldsymbol{Q}^{\uparrow}$ *projectable on* $\boldsymbol{Q}$ *are of the type*

$$Y^{\uparrow}[X^{\uparrow}, f] = \text{Ч}(X^{\uparrow}) + (\mathrm{i}\, f - \tfrac{1}{2}\, \mathrm{div}\, X9f])\, \mathbb{I}\,,$$

i.e., in coordinates, of the type

$$Y^{\uparrow}[X^{\uparrow}, f] = f^0\, \partial_0 - f^i\, \partial_i + X^i_0\, \partial^0_i + \Big(\mathrm{i}\,(f^0\, A_0 - f^i\, A_i + \overset{o}{f}) - \tfrac{1}{2}\,(f^0\, \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(f^i\sqrt{|g|})}{\sqrt{|g|}})\Big)\, \mathbb{I}\,,$$

where $f \in \mathrm{Quan}(J_1\boldsymbol{E}, I\!R)$ *and* $X^{\uparrow} \in \mathrm{Pro}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ *is any prolongation of* $X[f]$.

Moreover, if a vector field $Y^{\uparrow}[X^{\uparrow}, f]$ *as above is projectable on* $\boldsymbol{Q}$, *then its projection is the quantum prolongation of* $f \in \mathrm{Quan}(J_1\boldsymbol{E}, I\!R)$

$$Y[f] \in \mathrm{Sec}(\boldsymbol{Q}, T\boldsymbol{Q})\,,$$

with coordinate expression

$$Y[f] = f^0\, \partial_0 - f^i\, \partial_i + \Big(\mathrm{i}\,(\overset{o}{f} + f^0\, A_0 - f^i\, A_i) - \tfrac{1}{2}\,(f^0\, \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(f^i\sqrt{|g|})}{\sqrt{|g|}})\Big)\, \mathbb{I}\,.$$

Proof. According to Theorem 3.2.29, the coordinate expression of Hermitian vector fields of $\boldsymbol{Q}^{\uparrow}$ is of the type

$$Y^{\uparrow}[X^{\uparrow}, f] = X^0\, \partial_0 + X^i\, \partial_i + X^i_0\, \partial^0_i + \big(\mathrm{i}\,(f - X^0\,(\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 - A_0) + X^i\,(G^0_{ij}\, x^j_0 + A_i)) - \tfrac{1}{2}\,(X^0\, \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} + \frac{\partial_i(X^i\sqrt{|g|})}{\sqrt{|g|}})\big)\, \mathbb{I}\,,$$

where $X^{\uparrow} \in \mathrm{Pro}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$, $f \in \mathrm{Map}(J_1\boldsymbol{E}, I\!R)$.

Now, if we set

$$f^0 := X^0\,, \qquad f^i := -X^i\,, \qquad \overset{o}{f} := f - f^0\, \tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 - f^i\, G^0_{ij}\, x^j_0\,,$$

the above expression becomes

$$Y^{\uparrow}[X^{\uparrow}, f] = f^0\, \partial_0 - f^i\, \partial_i + X^i_0\, \partial^0_i + \big(\mathrm{i}\,(\overset{o}{f} + f^0\, A_0 - f^i\, A_i) - \tfrac{1}{2}\,(f^0\, \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(f^i\sqrt{|g|})}{\sqrt{|g|}})\big)\, \mathbb{I}\,,$$

where f^0, f^i, $\overset{o}{f} \in \mathrm{Map}(J_1\boldsymbol{E}, I\!R)$.

Then, $Y^{\uparrow}[X^{\uparrow}, f]$ is projectable on $\boldsymbol{Q}$ if and only if

$$f^0,\ f^i,\ \overset{o}{f} \in \operatorname{Map}(\boldsymbol{E},\ I\!R)\,.$$

Moreover, if this condition is fulfilled, then

$$f = f^0\,\tfrac{1}{2}\,G^0_{ij}\,x^i_0\,x^j_0 + f^0_i\,x^i_0 + \overset{o}{f}$$

turns out to be a quantisable function,

$$X = f^0\,\partial_0 - f^i\,\partial_i$$

turns out to be the tangent prolongation $X[f]$ of f and the projection Y of $Y[X^{\uparrow}, f]$ turns out to be the vector field with coordinate expression

$$Y = f^0\,\partial_0 - f^i\,\partial_i + \big(\mathrm{i}\,(\overset{o}{f} + f^0\,A_0 - f^i\,A_i) - \tfrac{1}{2}\,(f^0\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(f^i\sqrt{|g|})}{\sqrt{|g|}})\big)\,\mathbb{I}\,.\ \text{QED}$$

We denote the sheaf of Hermitian vector fields of $\boldsymbol{Q}^{\uparrow}$ projectable on $\boldsymbol{Q}$ by

$$\operatorname{Her}(\boldsymbol{Q}^{\uparrow},\ T\boldsymbol{Q}^{\uparrow}) \subset \tilde{\operatorname{Her}}(\boldsymbol{Q}^{\uparrow},\ T\boldsymbol{Q}^{\uparrow})\,.$$

3.2.35. Theorem. *If $f, \bar{f} \in \operatorname{Quan}(J_1\boldsymbol{E},\ I\!R)$ and $X^{\uparrow}, \bar{X}^{\uparrow} \in \operatorname{Pro}(J_1\boldsymbol{E},\ TJ_1\boldsymbol{E})$ are prolongations of $X[f]$ and $X[\bar{f}]$, then we obtain*

$$(*)\qquad \Big[Y^{\uparrow}[X^{\uparrow}, f],\ Y^{\uparrow}[\bar{X}^{\uparrow}, \bar{f}]\Big] = Y^{\uparrow}\Big[[X^{\uparrow}, \bar{X}^{\uparrow}],\ [\![\, f,\ \bar{f}\,]\!]\Big]\,,$$

where $[\![\, f, \bar{f}\,]\!] \in \operatorname{Quan}(J_1\boldsymbol{E},\ I\!R)$ and $[X^{\uparrow}, \bar{X}^{\uparrow}] \in \operatorname{Pro}(J_1\boldsymbol{E},\ TJ_1\boldsymbol{E})$ is projectable on the tangent prolongation $[X[f],\ X\bar{f}]]$.

Hence, the sheaf $\operatorname{Her}(\boldsymbol{Q}^{\uparrow},\ T\boldsymbol{Q}^{\uparrow})$ is closed with respect to the Lie bracket.

Moreover, the map

$$\operatorname{Her}(\boldsymbol{Q}^{\uparrow},\ T\boldsymbol{Q}^{\uparrow}) \to \operatorname{Her}(\boldsymbol{Q},\ T\boldsymbol{Q}) : Y^{\uparrow}[X^{\uparrow}, f] \mapsto Y[f]$$

is a morphism of Lie algebras. Its kernel is the horizontal prolongation of the sheaf $\operatorname{Sec}(J_1\boldsymbol{E},\ V_{\boldsymbol{E}}J_1\boldsymbol{E})$.

Proof. Formula (*) follows immediately from Proposition 3.2.33 and Proposition 2.3.34. Moreover, the Lie bracket of two projectable vector fields is projectable on the Lie bracket of their projections. Hence $\operatorname{Her}(\boldsymbol{Q}^{\uparrow},\ T\boldsymbol{Q}^{\uparrow})$ is closed with respect to the Lie bracket.

Formula *) shows that the map $\operatorname{Her}(\boldsymbol{Q}^{\uparrow},\ T\boldsymbol{Q}^{\uparrow}) \to \operatorname{Her}(\boldsymbol{Q},\ T\boldsymbol{Q})$ is a morphism of Lie algebras. Moreover, the coordinate expressions of $Y^{\uparrow}[X^{\uparrow}, f]$ and $Y[f]$ show that the kernel of the above map is the natural prolongation of the sheaf (see Remark 3.1.7) $\operatorname{Sec}(J_1\boldsymbol{E},\ V_{\boldsymbol{E}}J_1\boldsymbol{E})$. QED

Actually, we are interested in the two distinguished prolongations $X^{\uparrow}$ of $X[f]$, determined, respectively, by the fibred structure of spacetime and by the cosymplectic structure of the phase space.

3.2.36. Definition. Let us consider a quantisable function f, the tangent lift $X[f]$ of f, the Hamiltonian prolongation $X^{\uparrow}_{\mathrm{Ham}}[f]$ of $X[f]$, the holonomic prolongation $X^{\uparrow}_{\mathrm{hol}}[f]$ of $X[f]$ and the quantum lift $Y[f]$ of f.

The Hermitian vector field of $\boldsymbol{Q}^{\uparrow}$ projectable on $\boldsymbol{Q}$

$$Y^{\uparrow}_{\mathrm{Ham}}[f] := Y^{\uparrow}[X^{\uparrow}_{\mathrm{Ham}}[f], f] = Ч[X^{\uparrow}_{\mathrm{Ham}}[f]] + (\mathrm{i}\, f - \tfrac{1}{2}\operatorname{div} X[f])\, \mathbb{I}\,,$$

is said to be the *Hamiltonian quantum lift* of f and the *Hamiltonian quantum prolongation* of $Y[f]$.

The Hermitian vector field of $\boldsymbol{Q}^{\uparrow}$ projectable on $\boldsymbol{Q}$

$$Y^{\uparrow}_{\mathrm{hol}}[f] := Y^{\uparrow}[X^{\uparrow}_{\mathrm{hol}}[f], f] = Ч[X^{\uparrow}_{\mathrm{hol}}[f]] + (\mathrm{i}\, f - \tfrac{1}{2}\operatorname{div} X[f])\, \mathbb{I}\,,$$

is said to be the *holonomic quantum lift* of f and the *holonomic quantum prolongation* of $Y[f]$. □

We denote the sheaves of Hamiltonian and holonomic vector field of $\boldsymbol{Q}^{\uparrow}$ projectable on $\boldsymbol{Q}$, respectively, by

$$\mathrm{Her}_{\mathrm{Ham}}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow}) \subset \mathrm{Her}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow}) \qquad \text{and} \qquad \mathrm{Her}_{\mathrm{Hol}}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow}) \subset \mathrm{Her}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow})\,.$$

3.2.37. Theorem. *If* $f, \bar f \in \mathrm{Quan}(J_1\boldsymbol{E}, I\!\!R)$*, then we obtain*

$$\Big[Y^{\uparrow}_{\mathrm{Ham}}[f],\, Y^{\uparrow}_{\mathrm{Ham}}[\bar f]\Big] = Y^{\uparrow}_{\mathrm{Ham}}\Big[\,[\![f, \bar f]\!]\,\Big]$$
$$\Big[Y^{\uparrow}_{\mathrm{hol}}[f],\, Y^{\uparrow}_{\mathrm{hol}}[\bar f]\Big] = Y^{\uparrow}_{\mathrm{hol}}\Big[\,[\![f, \bar f]\!]\,\Big]\,.$$

Hence, the sheaves $\mathrm{Her}_{\mathrm{Ham}}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow})$ *and* $\mathrm{Her}_{\mathrm{Hol}}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow})$ *are closed with respect to the Lie bracket and the maps*

$$\mathrm{Quan}(J_1\boldsymbol{E}, I\!\!R) \to \mathrm{Her}_{\mathrm{Ham}}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow}) : f \mapsto Y^{\uparrow}_{\mathrm{Ham}}[f]$$
$$\mathrm{Quan}(J_1\boldsymbol{E}, I\!\!R) \to \mathrm{Her}_{\mathrm{hol}}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow}) : f \mapsto Y^{\uparrow}_{\mathrm{hol}}[f]$$

are morphisms of Lie algebras. Even more, these maps are isomorphisms of Lie algebras.

Moreover, the maps

$$\mathrm{Her}_{\mathrm{Ham}}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow}) \to \mathrm{Her}(\boldsymbol{Q}, T\boldsymbol{Q}) : Y^{\uparrow}_{\mathrm{Ham}}[f] \mapsto Y[f]$$
$$\mathrm{Her}_{\mathrm{Hol}}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow}) \to \mathrm{Her}(\boldsymbol{Q}, T\boldsymbol{Q}) : Y^{\uparrow}_{\mathrm{hol}}[f] \mapsto Y[f]$$

are Lie algebra isomorphisms and the maps

$$\mathrm{Her}_{\mathrm{Ham}}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow}) \to \mathrm{Pro}(\boldsymbol{E}, T\boldsymbol{E}) : Y^{\uparrow}_{\mathrm{Ham}}[f] \mapsto X[f]$$
$$\mathrm{Her}_{\mathrm{Hol}}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow}) \to \mathrm{Pro}(\boldsymbol{E}, T\boldsymbol{E}) : Y^{\uparrow}_{\mathrm{hol}}[f] \mapsto X[f]$$

are central extensions of the Lie algebra $\mathrm{Pro}(\boldsymbol{E}, T\boldsymbol{E})$*, by* $\mathrm{Map}(\boldsymbol{E}, I\!\!R) \otimes \mathrm{i}\,\mathbb{I}$.

PROOF. The closure of the sheaves follows from Theorem 3.2.35 Theorem 2.3.49, Lemma 1.6.3 and Proposition 3.2.33.

The above isomorphisms and central extensions follow from Theorem 3.2.35 and from the expression of Hermitian vector fields. QED

3.2.5 Symmetries of the quantum connection

Next, we analyze the infinitesimal symmetries of the quantum connection.

3.2.38. Definition. An *infinitesimal symmetry* (*i.s.*) of the quantum connection Ч is defined to be a vector field $Y^{\uparrow} \in \mathrm{Sec}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow})$, such that

$$L[Y^{\uparrow}]\,\text{Ч} = 0\,.\square$$

3.2.39. Lemma. Let $Y^{\uparrow} \in \mathrm{Sec}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow})$.
Then, we have the following coordinate expression:

$$\begin{aligned} L[Y^{\uparrow}]\,\text{Ч} = &- \text{Ч}^{\mathrm{a}}_{\lambda}\,\partial_{\mathrm{a}}Y^{\mu}\,d^{\lambda}\otimes\partial_{\mu} \\ &- \text{Ч}^{\mathrm{a}}_{\lambda}\,\partial_{\mathrm{a}}Y^{i}_{0}\,d^{\lambda}\otimes\partial^{0}_{i} \\ &+ (-\partial_{\lambda}Y^{\mathrm{a}} + Y^{\mu}\,\partial_{\mu}\text{Ч}^{\mathrm{a}}_{\lambda} + Y^{i}_{0}\,\partial^{0}_{i}\text{Ч}^{\mathrm{a}}_{\lambda} + Y^{\mathrm{b}}\,\partial_{\mathrm{b}}\text{Ч}^{\mathrm{a}}_{\lambda} + \text{Ч}^{\mathrm{a}}_{\lambda}\,\partial_{\mu}Y^{\lambda} - \text{Ч}^{\mathrm{b}}_{\lambda}\,\partial_{\mathrm{b}}Y^{\mathrm{a}})\,d^{\lambda}\otimes\partial_{\mathrm{a}} \\ &+ (\text{Ч}^{\mathrm{a}}_{\lambda}\,\partial^{0}_{i}Y^{\lambda} - \partial^{0}_{i}Y^{\mathrm{a}})\,d^{i}_{0}\otimes\partial_{\mathrm{a}} \\ &+ \partial_{\mathrm{a}}Y^{\lambda}\,d^{\mathrm{a}}\otimes\partial_{\lambda} \\ &+ \partial_{\mathrm{a}}Y^{i}_{0}\,d^{\mathrm{a}}\otimes\partial^{0}_{i} \\ &+ \text{Ч}^{\mathrm{a}}_{\lambda}\,\partial_{\mathrm{b}}Y^{\lambda}\,d^{\mathrm{b}}\otimes\partial_{\mathrm{a}}\,.\square \end{aligned}$$

3.2.40. Proposition. Let $Y^{\uparrow} \in \mathrm{Sec}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow})$.
Then, the following conditions are equivalent:
1) $Y^{\uparrow}$ is an i.s. of Ч;
2) the following equalities hold in a chart

$$\begin{aligned} \partial_{\mathrm{a}}Y^{\lambda} &= 0 \\ \partial_{\mathrm{a}}Y^{i}_{0} &= 0 \\ \partial_{\lambda}Y^{\mathrm{a}} - Y^{\mu}\,\partial_{\mu}\text{Ч}^{\mathrm{a}}_{\lambda} - Y^{i}_{0}\,\partial^{0}_{i}\text{Ч}^{\mathrm{a}}_{\lambda} - \text{Ч}^{\mathrm{a}}_{\lambda}\,\partial_{\mu}Y^{\lambda} + \text{Ч}^{\mathrm{b}}_{\lambda}\,\partial_{\mathrm{b}}Y^{\mathrm{a}} - Y^{\mathrm{b}}\,\partial_{\mathrm{b}}\text{Ч}^{\mathrm{a}}_{\lambda} &= 0 \\ \partial^{0}_{i}Y^{\mathrm{a}} - \text{Ч}^{\mathrm{a}}_{\lambda}\,\partial^{0}_{i}Y^{\lambda} &= 0\,. \end{aligned}$$

PROOF. It follows immediately from Lemma 3.2.39. QED

3.2.41. Corollary. Let $Y^{\uparrow} \in \mathrm{Sec}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow})$. be an is an i.s. of Ч.
Then, $Y^{\uparrow}$ is projectable on a vector field $X^{\uparrow} \in \mathrm{Sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$.

PROOF. It follows from the first two equalities of the above system. QED

3.2.42. Lemma. Let $X^{\uparrow} \in \mathrm{Sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$.
Then, we have the following coordinate expression

$$\begin{aligned} i(X^{\uparrow})\,\Omega = &-\Big(X_j\,x_0^j - \gamma_{0j}X^j + (\Gamma_{jh} - \Gamma_{hj})\,(X^h - x_0^h\,X^0)\,x_0^j\Big)\,d^0 \\ &+ \Big(X_j - \gamma_{0j}\,X^0 + (\Gamma_{jh} - \Gamma_{hj})\,(X^h - x_0^h\,X^0)\Big)\,d^j \\ &- G^0_{ij}\,(X^j - x_0^j\,X^0)\,d_0^i\,, \end{aligned}$$

where $X_j := G^0_{ij}\,X_0^i$.

Proof. We have

$$\begin{aligned} i(X^{\uparrow})\,\Omega &= i(X^\lambda\,\partial_\lambda + X_0^i\,\partial_i^0)\,\big(G^0_{ij}\,(d_0^i - \gamma_0{}^i{}_0\,d^0 - \Gamma_h{}^i{}_0\,\theta^h) \wedge \theta^j\big) \\ &= G^0_{ij}\,(X_0^i - \gamma_0{}^i{}_0\,X^0 - \Gamma_h{}^i{}_0\,(X^h - x_0^h\,X^0))\,\theta^j \\ &\quad - G^0_{ij}\,(d_0^i - \gamma_0{}^i{}_0\,d^0 - \Gamma_h{}^i{}_0\,\theta^h)\,(X^j - x_0^j\,X^0)\,. \text{QED} \end{aligned}$$

3.2.43. Proposition. Let $Y^{\uparrow} \in \mathrm{Sec}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow})$.
Then, the following conditions are equivalent:
1) $Y^{\uparrow}$ is an i.s. of Ч;
2) $Y^{\uparrow}$ is of the type

$$Y^{\uparrow} = Ч(X^{\uparrow}) + \bar{Y}^{\uparrow}\,,$$

where

$$X^{\uparrow} \in \mathrm{Sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})\,, \qquad \bar{Y}^{\uparrow} \in \mathrm{Sec}(\boldsymbol{Q}^{\uparrow}, V_{J_1\boldsymbol{E}}\boldsymbol{Q}^{\uparrow})$$

with

$$L[\bar{Y}^{\uparrow}]\,Ч = -i(X^{\uparrow})\,\Omega \otimes \mathbb{I}\,.$$

Proof. If $Y^{\uparrow}$ is projectable on $X^{\uparrow} \in \mathrm{Sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$, then we can write

$$Y^{\uparrow} = Ч(X^{\uparrow}) + \bar{Y}^{\uparrow}\,,$$

with $\bar{Y}^{\uparrow} \in \mathrm{Sec}(\boldsymbol{Q}, V_{J_1\boldsymbol{E}}\boldsymbol{Q}^{\uparrow})$.
By assumption, $R[Ч] = \mathrm{i}\,\Omega \otimes \mathbb{I}$. Hence, by recalling Lemma 1.2.2, we obtain

$$\begin{aligned} L[Y^{\uparrow}]\,Ч &= L[Ч(X^{\uparrow})]\,Ч + L[\bar{Y}^{\uparrow}]\,Ч \\ &= i(X^{\uparrow})\,\Omega \otimes (\mathrm{i}\,\mathbb{I}) + L[\bar{Y}^{\uparrow}]\,Ч\,. \text{QED} \end{aligned}$$

Next, we restrict our attention to i.s. of Ч, which are Hermitian vector fields of $\boldsymbol{Q}^{\uparrow}$.

3.2.44. Definition. A *Hermitian infinitesimal symmetry* of Ч is defined to be a Hermitian vector field of $\boldsymbol{Q}^{\uparrow}$, which is an i.s. of Ч. □

The sheaf of Hermitian i.s.'s of Ч is denoted by

$$\tilde{\mathrm{Cov}}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow}) \subset \tilde{\mathrm{Her}}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow}) .$$

3.2.45. Proposition. Let $Y^{\uparrow}[X^{\uparrow}, f] \in \tilde{\mathrm{Her}}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow})$, with $X^{\uparrow} \in \mathrm{Pro}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ and $f \in \mathrm{Map}(J_1\boldsymbol{E}, I\!\!R)$.

Then, the following conditions are equivalent:

1) $$L\Big[Y^{\uparrow}[X^{\uparrow}, f]\Big]\, \text{Ч} = 0 ,$$

2) $$i(X^{\uparrow})\,\Omega = df, \qquad d(\mathrm{div}\, X^{\uparrow}) = 0 .$$

PROOF. By assumption, $R[\text{Ч}] = \mathrm{i}\,\Omega \otimes \mathbb{I}$. Hence, in virtue of Theorem 3.2.29, Lemma 1.2.2 and Lemma 1.2.7, we obtain

$$\begin{aligned} L\big[Y^{\uparrow}[X^{\uparrow}, f]\big]\, \text{Ч} &= L\big[\text{Ч}[X^{\uparrow}]\big]\, \text{Ч} + L\big[(\mathrm{i}\, f - \tfrac{1}{2}\, \mathrm{div}\, X^{\uparrow})\,\mathbb{I}\big]\, \text{Ч} \\ &= \big(\mathrm{i}\, i[X^{\uparrow}]\,\Omega - \mathrm{i}\, df + \tfrac{1}{2}\, d(\mathrm{div}\, X^{\uparrow})\big)\,\mathbb{I} . \end{aligned}$$

Then, the result follows by splitting the above equality into the real and imaginary components. QED

3.2.46. Theorem. *Let* $Y^{\uparrow} \in \mathrm{Sec}(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow})$. *Then, the following conditions are equivalent:*

1) $Y^{\uparrow}$ *is a Hermitian i.s. of* Ч*;*

2) we have the expression

$$\begin{aligned} Y^{\uparrow} = Y^{\uparrow}[\tau, f] &:= Y^{\uparrow}\Big[X^{\uparrow}_{\mathrm{Ham}}[\tau, f],\, f\Big] \\ &:= \text{Ч}(X^{\uparrow}_{\mathrm{Ham}}[\tau, f]) + (\mathrm{i}\, f - \tfrac{1}{2}\, \mathrm{div}\, X^{\uparrow}_{\mathrm{Ham}}[\tau, f])\,\mathbb{I} , \end{aligned}$$

where

$$\tau \in \mathrm{Map}(\boldsymbol{T}, \bar{\mathbb{T}}) , \qquad f \in \mathrm{Con}(J_1\boldsymbol{E}, I\!\!R) \qquad \textit{and} \qquad d(\mathrm{div}\, X^{\uparrow}_{\mathrm{Ham}}[\tau, f]) = 0 .$$

PROOF. Let $X^{\uparrow} \in \mathrm{Pro}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$.

Then, in virtue of Theorem 2.2.4, $X^{\uparrow}$ can be uniquely written as

$$X^{\uparrow} = \gamma(\tau) + (\Lambda^{\sharp} \circ \Omega^{\flat})\,(X^{\uparrow}) , \qquad \text{with} \qquad \tau := dt(X^{\uparrow})\,\mathrm{Map}(\boldsymbol{T}, , \bar{\mathbb{T}}) .$$

Moreover, if $i[X^{\uparrow}]\,\Omega = df$, then we obtain

$$X^{\uparrow} = \gamma(\tau) + \Lambda^{\sharp}(df) := X^{\uparrow}_{\mathrm{Ham}}[\tau, f]$$

and

$$f \in \mathrm{Con}(J_1\boldsymbol{E}, I\!\!R) .$$

Conversely, if $\tau \in \operatorname{Map}(\boldsymbol{T}, \,, \bar{\mathbb{T}})$, $f \in \operatorname{Map}(J_1\boldsymbol{E}, I\!R)$ and $X^\uparrow = \gamma(\tau) + \Lambda^\sharp(df)$, then, in virtue of Lemma 2.2.3, we obtain

$$i(X^\uparrow)\,\Omega = df - \gamma(df)\,.$$

Moreover, if $f \in \operatorname{Con}(J_1\boldsymbol{E}, I\!R)$, then we obtain

$$i(X^\uparrow)\,\Omega = df\,.$$

Hence, the Theorem follows from Proposition 3.2.45. QED

3.2.6 Symmetries of the quantum structure

Here we study the infinitesimal symmetries of the full quantum structure, i.e. the vector fields of the extended quantum bundle, which are projectable on the quantum bundle and preserve the linear structure, the complex structure, the Hermitian metric and the quantum connection. These infinitesimal symmetries turn out to preserve the classical structure, as well.

3.2.47. Definition. An *infinitesimal symmetry of the quantum structure* is defined to be a Hermitian infinitesimal symmetry $Y^\uparrow \in \operatorname{Sec}(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow)$ of Ч, which is projectable on $\boldsymbol{Q}$. □

We denote the sheaf of i.s.'s of the quantum structure by

$$\operatorname{Cov}(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow) \subset \tilde{\operatorname{Cov}}(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow)\,.$$

We observe that an i.s. of the quantum structure turns out to be projectable with respect to all fibrings of $\boldsymbol{Q}^\uparrow$, i.e. it is projectable on $\boldsymbol{Q}$, $J_1\boldsymbol{E}$, $\boldsymbol{E}$, $\boldsymbol{T}$.

3.2.48. Theorem. *Let $Y^\uparrow \in \operatorname{Sec}(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow)$. Then, the following conditions are equivalent:*

1) $Y^\uparrow$ is an i.s. of the quantum structure;

2) we have the expression

$$\begin{aligned} Y^\uparrow = Y^\uparrow[f] &:= Y^\uparrow_{\mathrm{Ham}}[f] \\ &= Ч[X^\uparrow_{\mathrm{Ham}}[f]] + (\mathrm{i}\, f - \tfrac{1}{2}\operatorname{div} X[f])\,\mathbb{I}\,, \end{aligned}$$

with $f \in \operatorname{Self}(J_1\boldsymbol{E}, I\!R) \cap \tilde{\operatorname{Unim}}(J_1\boldsymbol{E}, I\!R)$.

Proof. In virtue of Theorem 3.2.34 $Y^\uparrow$ is a projectable Hermitian vector field if and only if

$$Y^\uparrow = Ч(X^\uparrow) + (\mathrm{i}\, f - \tfrac{1}{2}\operatorname{div} X[f])\,\mathbb{I}\,,$$

where $f \in \operatorname{Quan}(J_1\boldsymbol{E}, I\!R)$ and $X^\uparrow \in \operatorname{Pro}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ is any prolongation of $X[f]$.

In virtue of Theorem 3.2.46, $Y^\uparrow$ is a Hermitian i.s. of the quantum connection if and only

$$Y^\uparrow = \text{Ч}(X^\uparrow_{\text{Ham}}[\tau, f]) + (\mathrm{i}\, f - \tfrac{1}{2}\, \operatorname{div} X[f])\, \mathbb{I}\,,$$

where

$$\tau \in \operatorname{Map}(\boldsymbol{T}, \bar{\mathbb{T}})\,, \qquad f \in \operatorname{Con}(J_1\boldsymbol{E}, I\!R) \qquad \text{and} \qquad d(\operatorname{div} X^\uparrow_{\text{Ham}}[\tau, f]) = 0\,.$$

But, by Theorem 2.4.33, $Self(J_1\boldsymbol{E}, I\!R) = \operatorname{Con}(J_1\boldsymbol{E}, I\!R) \cap \operatorname{Quan}(J_1\boldsymbol{E}, I\!R)$.

On the other hand, in virtue of Remark 3.1.8, if a vector field $Y^\uparrow \in \operatorname{Sec}(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow)$ projectable on $J_1\boldsymbol{E}$ is also projectable on $\boldsymbol{Q}$, then $X^\uparrow$ is projectable on $\boldsymbol{E}$.

Moreover, in virtue of Theorem 2.3.5, if $X^\uparrow_{\text{Ham}}[\tau, f]$ is projectable on $\boldsymbol{E}$, then $\tau = f$, hence $X^\uparrow_{\text{Ham}}[\tau, f] = X^\uparrow_{\text{Ham}}[f]$. QED

We recall (see Theorem 2.4.20 and Theorem 2.4.33) that

$$\operatorname{Self}(J_1\boldsymbol{E}, I\!R) = \operatorname{Con}(J_1\boldsymbol{E}, I\!R) \cap \operatorname{Quan}(J_1\boldsymbol{E}, I\!R) \subset \operatorname{Hol}(J_1\boldsymbol{E}, I\!R)$$

3.2.49. Definition. For each $f \in \operatorname{Self}(J_1\boldsymbol{E}, I\!R)$, we call the vector field

$$Y^\uparrow[f] := Y^\uparrow_{\text{Ham}}[f] = Y^\uparrow_{\text{hol}}[f] = \text{Ч}[X^\uparrow_{\text{Ham}}[f]] + (\mathrm{i}\, f - \tfrac{1}{2}\, \operatorname{div} X[f])\, \mathbb{I} \in \operatorname{Sec}(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow)$$

the *quantum lift* of f. □

3.2.50. Theorem. *The sheaf* $\operatorname{Cov}(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow)$ *is closed with respect to the Lie bracket. Moreover, the map*

$$\operatorname{Self}(J_1\boldsymbol{E}, I\!R) \cap \widetilde{\operatorname{Unim}}(J_1\boldsymbol{E}, I\!R) \to \operatorname{Cov}(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow) : f \mapsto Y^\uparrow[f]$$

is an isomorphism of Lie algebras.

Furthermore, the map

$$\operatorname{Cov}(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow) \to \operatorname{Pro}(\boldsymbol{E}, T\boldsymbol{E}) : Y^\uparrow[f] \mapsto X[f]$$

is a central extension of Lie algebras, by $\mathrm{i}\, I\!R \otimes \mathbb{I}$.

Proof. The closure of $\operatorname{Cov}(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow)$ follows from Theorem 3.2.37 and from the closure of the sheaf $\operatorname{Self}(J_1\boldsymbol{E}, I\!R) \cap \widetilde{\operatorname{Unim}}(J_1\boldsymbol{E}, I\!R)$ with respect to the special bracket.

The isomorphism $\operatorname{Self}(J_1\boldsymbol{E}, I\!R) \cap \widetilde{\operatorname{Unim}}(J_1\boldsymbol{E}, I\!R) \to \operatorname{Cov}(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow)$ follows from the isomorphism $\operatorname{Quan}(J_1\boldsymbol{E}, I\!R) \to \operatorname{Her}_{\text{Ham}}(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow)$.

In virtue of Theorem 3.2.37,

$$\operatorname{Her}_{\text{Ham}}(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow) \to \operatorname{Pro}(\boldsymbol{E}, T\boldsymbol{E}) : Y^\uparrow_{\text{Ham}}[f] \mapsto X[f]$$

is a central extension by $\operatorname{Map}(\boldsymbol{E}, I\!R) \otimes \mathrm{i}\, \mathbb{I}$.

Moreover, in virtue of Corollary 2.4.6,

$$\operatorname{Con}(J_1\boldsymbol{E}, I\!R) \cap \operatorname{Map}(\boldsymbol{E}, I\!R) = I\!R\,.\ \text{QED}$$

Eventually, we can restrict the above constructions to the classical generators, by recalling

$$\mathrm{Clas}(J_1\boldsymbol{E},\, I\!\!R) := \mathrm{Self}(J_1\boldsymbol{E},\, I\!\!R) \cap \mathrm{Time}(J_1\boldsymbol{E},\, I\!\!R) \subset \mathrm{Unim}(J_1\boldsymbol{E},\, I\!\!R)\,.$$

In fact, the vector fields generated by quantisable functions of the above type preserve the affine structure of time.

3.2.51. Definition. An *infinitesimal quantum symmetry* is defined to be an i.s. of the quantum structure, which is also an i.s. of dt. □

We denote the sheaf of infinitesimal quantum symmetries by

$$\mathrm{Quan}(\boldsymbol{Q}^{\uparrow},\, T\boldsymbol{Q}^{\uparrow}) \subset \mathrm{Cov}(\boldsymbol{Q}^{\uparrow},\, T\boldsymbol{Q}^{\uparrow})\,.$$

3.2.52. Corollary. Let $Y^{\uparrow} \in \mathrm{Sec}(\boldsymbol{Q}^{\uparrow},\, T\boldsymbol{Q}^{\uparrow})$. Then, the following conditions are equivalent:

1) $Y^{\uparrow}$ is an infinitesimal quantum symmetry;

2) we have the expression

$$\begin{aligned} Y^{\uparrow} &= Y^{\uparrow}[f] := Y^{\uparrow}_{\mathrm{Ham}}[f] \\ &= \text{Ч}[X^{\uparrow}_{\mathrm{Ham}}[f]] + (\mathrm{i}\, f - \tfrac{1}{2}\, \mathrm{div}\, X[f])\, \mathbb{I}\,, \end{aligned}$$

with $f \in \mathrm{Clas}(J_1\boldsymbol{E}, I\!\!R)$. □

3.2.53. Corollary. The sheaf $\mathrm{Quan}(\boldsymbol{Q}^{\uparrow},\, T\boldsymbol{Q}^{\uparrow})$ is closed with respect to the Lie bracket.

Moreover, the map

$$\mathrm{Clas}(J_1\boldsymbol{E},\, I\!\!R) \to \mathrm{Quan}(\boldsymbol{Q}^{\uparrow},\, T\boldsymbol{Q}^{\uparrow}) : f \mapsto Y^{\uparrow}[f]$$

is an isomorphism of Lie algebras.

Furthermore, the map

$$\mathrm{Quan}(\boldsymbol{Q}^{\uparrow},\, T\boldsymbol{Q}^{\uparrow}) \to \mathrm{Time}(\boldsymbol{E},\, T\boldsymbol{E}) : Y^{\uparrow}[f] \mapsto X[f]$$

is a central extension of Lie algebras, by $\mathrm{i}\, I\!\!R \otimes \mathbb{I}$. □

3.3 Quantum Nöther symmetries

3.3.1 Holonomic symmetries of the quantum Lagrangian

I apply the above results to the quantum Lagrangian formalism. It turns out that the (holonomic) vector fields which preserve the quantum Lagrangian are also generated by the function $\mathrm{Clas}(J_1\boldsymbol{E}, I\!\!R)$. Moreover, the map which associates with a classical generator the corresponding symmetry of the quantum Lagrangian is an isomorphism of Lie algebras. The standard Lagrangian formalism suggests me to associate with any quantisable function a quantum current. It turns out that the quantum currents which are associated with a function of $\mathrm{Clas}(J_1\boldsymbol{E}, I\!R)$ are conserved along the solutions of the covariant Schrödinger equation.

3.3.1. Proposition. Let us consider a function $f \in \mathrm{Time}(J_1\boldsymbol{E},\, I\!R)$ and the associated Hermitian vector field $Y[f] \in \mathrm{Her}(\boldsymbol{Q},\, T\boldsymbol{Q})$. Then, we obtain the holonomic prolongation

$$Y_{\mathrm{hol}}[f] := (Y[f])_{(1)} \in \mathrm{Pro}(J_1\boldsymbol{Q},\, TJ_1\boldsymbol{Q})\,,$$

with real coordinate expression

$$\begin{aligned} Y_{\mathrm{hol}}[f] =& X[f]^\mu \partial_\mu + (\overset{o}{f} + X[f]^\lambda A_\lambda)(w^1 b_2 - w^2 b_1) - \frac{1}{2}\operatorname{div} X[f](w^1 b_1 + w^2 b_2) \\ &+ \partial_\mu(\overset{o}{f} + X[f]^\lambda A_\lambda)(w^1 b_2^\mu - w^2 b_1^\mu) - \frac{1}{2}\partial_\lambda(\operatorname{div} X[f])(w^1 b_1^\lambda + w^2 b_2^\lambda) \\ &+ (\overset{o}{f} + X[f]^\lambda A_\lambda)(w^1_\mu b_2^\mu - w^2_\mu b_1^\mu) - \frac{1}{2}\operatorname{div} X[f](w^1_\mu b_1^\mu + w^2_\mu b_2^\mu) \\ &- \partial_\lambda X[f]^\mu (w^1_\mu b_1^\lambda + w^2_\mu b_2^\lambda)\,.\square \end{aligned}$$

3.3.2. Lemma. Let $f \in \mathrm{Time}(J_1\boldsymbol{E}, I\!R)$. Then,

$$L\Big[Y_{\mathrm{hol}}[f]\Big]\,\mathsf{L} = 0$$

if and only if

$$\begin{aligned} 0 =& -\partial_0 \overset{o}{f} + X[f]^j(\partial_j A_0 - \partial_0 A_j) + G^{ij}_0 A_i(\partial_j \overset{o}{f} + X[f]^\lambda(\partial_j A_\lambda - \partial_\lambda A_j) \\ &- G^{jk}_0(\partial_0 X[f]^k)) + \frac{1}{2}A_i A_j(G^{ik}_0(\partial_k X[f]^j) + G^{jk}_0(\partial_k X[f]^i) - X[f]^\lambda(\partial_\lambda G^{ij}_0)) \\ &- \frac{1}{2}kX[f]^\lambda(\partial_\lambda r_0)\,, \\ 0 =& G^{ij}_0(\partial_j \overset{o}{f} + X[f]^\lambda(\partial_j A_\lambda - \partial_\lambda A_j) - G^{jm}_0(\partial_0 X[f]^m)) \\ &+ A_i(G^{jk}_0(\partial_k X[f]^i) + G^{ik}_0(\partial_k X[f]^j) - X[f]^\lambda(\partial_\lambda G^{ij}_0))\,, \\ 0 =& \frac{1}{2}\sqrt{|g|}G^{ij}_0 \partial_i(\operatorname{div} X[f])(w^1_j w^1 + w^2_j w^2)\,, \\ 0 =& \Big(G^{ik}_0\sqrt{|g|}(\partial_k X[f]^j) - \frac{1}{2}X[f]^\lambda\sqrt{|g|}(\partial_\lambda G^{ij}_0)\Big)(w^1_i w^1_j + w^2_i w^2_j)\,. \end{aligned}$$

Proof. We have

$$L\big[Y_{\mathrm{hol}[f]}\big]\,\mathsf{L} =$$

$$
\begin{aligned}
0 =& \big(-\frac{1}{2} \operatorname{div} X[f] \sqrt{|g|}\, (2A_0 - G_0^{ij} A_i A_j - kr_0) + G_0^{ij} A_i \sqrt{|g|}\, \partial_j\, (\overset{o}{f} + X[f]^\lambda A_\lambda) \\
& - \sqrt{|g|} \partial_0 (\overset{o}{f} + X[f]^\lambda A_\lambda) + \frac{1}{2} \sqrt{|g|} \partial_\lambda X[f]^\lambda (2A_0 - G_0^{ij} A_i A_j - kr_0) \big) (w^1 w^1 + w^2 w^2) \,, \\
0 =& \big(\frac{1}{2} G_0^{ij} A_i \operatorname{div} X[f] \sqrt{|g|} + G_0^{ij} \sqrt{|g|} \partial_i (\overset{o}{f} + X[f]^\lambda A_\lambda) + \frac{1}{2} G_0^{ij} A_i \operatorname{div} X[f] \sqrt{|g|} \\
& + G_0^{ik} \sqrt{|g|} A_i (\partial_k X[f]^j) - \sqrt{|g|} \partial_0 X[f]^j - X[f]^\lambda \partial_\lambda (G_0^{ij} \sqrt{|g|} A_i) \\
& - G_0^{ij} \sqrt{|g|} A_i \partial_\lambda X[f]^\lambda \big) (w^1_j w^2 - w^2_j w^1) \,, \\
0 =& \big(-\frac{1}{2} G_0^{ij} \sqrt{|g|} \partial_i \operatorname{div} X[f] \big) (w^1_j w^1 + w^2_j w^2) \,, \\
0 =& \big(-\frac{1}{2} \sqrt{|g|} \operatorname{div} X[f] - \frac{1}{2} \sqrt{|g|} \operatorname{div} X[f] - \sqrt{|g|} \partial_0 X[f]^0 + G_0^{ik} \sqrt{|g|} A_i \partial_k X[f]^0 \\
& + X[f]^\lambda (\partial_\lambda \sqrt{|g|}) + \sqrt{|g|} (\partial_\lambda X[f]^\lambda) \big) (w^1_0 w^2 - w^2_0 w^1) \,, \\
0 =& \big(\frac{1}{2} G_0^{ij} \sqrt{|g|} \operatorname{div} X[f] + G_0^{ik} \sqrt{|g|} \partial_k X[f]^j + \frac{1}{2} X[f]^\lambda \partial_\lambda\, (G_0^{ij} \sqrt{|g|}) \\
& - \frac{1}{2} G_0^{ij} \sqrt{|g|} (\partial_\lambda X[f]^\lambda) \big) (w^1_i w^1_j + w^2_i w^2_j) \,.
\end{aligned}
$$

Performing the partial derivatives, we find the following equivalent system

$$
\begin{aligned}
0 =& \big(G_0^{ij} A_i \sqrt{|g|} (\partial_j \overset{o}{f} + A_\lambda (\partial_j X[f]^\lambda) + X[f]^\lambda (\partial_j A_\lambda)) - \sqrt{|g|} (\partial_0 \overset{o}{f} + A_\lambda (\partial_0 X[f]^\lambda) \\
& + X[f]^\lambda (\partial_0 A_\lambda)) + \frac{1}{2} \sqrt{|g|} X[f]^\lambda (2 \partial_\lambda A_0 - (\partial_\lambda G_0^{ij}) A_i A_j \\
& - 2 G_0^{ij} A_i (\partial_\lambda A_j) - k (\partial_\lambda r_0)) \big) (w^1 w^1 + w^2 w^2) \,, \\
0 =& \big(G_0^{ij} \sqrt{|g|} (\partial_j \overset{o}{f} + A_\lambda (\partial_j X[f]^\lambda) + X[f]^\lambda (\partial_j A_\lambda)) + G_0^{ik} \sqrt{|g|} A_i (\partial_k X[f]^j) \\
& - \sqrt{|g|} (\partial_0 X[f]^j) - \sqrt{|g|} X[f]^\lambda ((\partial_\lambda G_0^{ij}) A_i \\
& + G_0^{ij} (\partial_\lambda A_i)) \big) (w^1_j w^2 - w^2_j w^1) \,, \\
0 =& \big(\frac{1}{2} \sqrt{|g|} G_0^{ij} \partial_i (\operatorname{div} X[f]) \big) (w^1_j w^1 + w^2_j w^2) \,, \\
0 =& \, 0 \\
0 =& \big(G_0^{ik} \sqrt{|g|} (\partial_k X[f]^j) - \frac{1}{2} X[f]^\lambda \sqrt{|g|} (\partial_\lambda G_0^{ij}) \big) (w^1_i w^1_j + w^2_i w^2_j) \,.
\end{aligned}
$$

Ordering the equations with respect to different algebraic orders in the potentials A_λ and using the equality $A_i A_j (G_0^{ik} (\partial_k X[f]^j) - \frac{1}{2} X[f]^\lambda (\partial_\mu G_0^{ij})) = \frac{1}{2} A_i A_j (G_0^{ik} (\partial_k X[f]^j) + G_0^{jk} (\partial_k X[f]^i) - X[f]^\lambda (\partial_\mu G_0^{ij}))$

the system is equivalent to

$$\begin{aligned}
0 =& -\partial_0 \overset{o}{f} + X[f]^j(\partial_j A_0 - \partial_0 A_j) + G_0^{ij} A_i(\partial_j \overset{o}{f} + X[f]^\lambda(\partial_j A_\lambda - \partial_\lambda A_j) \\
& - G_0^{jk}(\partial_0 X[f]^k)) + \frac{1}{2} A_i A_j (G_0^{ik}(\partial_k X[f]^j) + G_0^{jk}(\partial_k X[f]^i) - X[f]^\lambda(\partial_\lambda G_0^{ij})) \\
& - \frac{1}{2} k X[f]^\lambda(\partial_\lambda r_0)\,, \\
0 =& G_0^{ij}(\partial_j \overset{o}{f} + X[f]^\lambda(\partial_j A_\lambda - \partial_\lambda A_j) - G_0^{jm}(\partial_0 X[f]^m)) \\
& + A_i(G_0^{jk}(\partial_k X[f]^i) + G_0^{ik}(\partial_k X[f]^j) - X[f]^\lambda(\partial_\lambda G_0^{ij}))\,, \\
0 =& \frac{1}{2}\sqrt{|g|} G_0^{ij} \partial_i(\operatorname{div} X[f])(w_j^1 w^1 + w_j^2 w^2)\,, \\
0 =& \big(G_0^{ik}\sqrt{|g|}(\partial_k X[f]^j) - \frac{1}{2} X[f]^\lambda \sqrt{|g|}(\partial_\lambda G_0^{ij})\big)(w_i^1 w_j^1 + w_i^2 w_j^2)\,. \text{QED}
\end{aligned}$$

3.3.3. Theorem. *Let* $f \in \mathrm{Time}(J_1\boldsymbol{E}, I\!R)$ *Then, the following equivalence holds*

$$L\Big[Y^{\uparrow}_{\mathrm{hol}}[f]\Big]\, Ч = 0 \qquad \Leftrightarrow \qquad L\Big[Y_{\mathrm{hol}}[f]\Big]\, \mathsf{L} = 0$$

Proof. The last equation in Lemma 3.3.2, by means of the polynomial identities for coordinates w_i^1 and w_i^2 (symmetrisation!!), turns out to be equivalent to the equation

$$0 \;=\; G_0^{ik}(\partial_k X[f]^j) + G_0^{jk}(\partial_k X[f]^i) - X[f]^\lambda(\partial_\lambda G_0^{ij})\,,$$

which is nothing else than the intrinsic condition $L[X[f]]\,\bar{G} = 0$.

On the other hand, by means of Lemma 2.3.16, $L[X[f]]\,\bar{G} = 0$ implies that $\operatorname{div} X[f] = 0$ and $X[f].r_0 = 0$.

Hence, the third equation in Lemma 3.3.2 becomes a consequence of the last equation.

Moreover, it makes the first and the second equation to become

$$0 \;=\; \big(\partial_j \overset{o}{f} - G^0_{ij}(\partial_0 X[f]^i) - X[f]^0 \Phi_{0j} - X[f]^k \Phi_{kj}\big) x_0^j$$

and

$$0 \;=\; \partial_0 \overset{o}{f} + X[f]^j \Phi_{0j}\,.$$

Hence, the system becomes equivalent to ... QED

3.3.4. Corollary. The map $f \to Y[f]$ is an isomorphism of Lie algebras between functions $f \in (\mathrm{Con}(J_1\boldsymbol{E}) \cap \mathrm{Quan}(J_1\boldsymbol{E}))$ with respect to the Poisson bracket and holonomic infinitesimal symmetries $Y[f]_{\mathrm{Hol}}$ of the quantum Lagrangian with respect to the Lie bracket.

We can summarize the above symmetry results by the following theorem

3.3.5. Theorem. *Let $f \in \mathrm{Time}(J_1\boldsymbol{E}, I\!R)$. Then the following conditions are equivalent*

$$
\begin{array}{lll}
1. & L[Y^{\uparrow}_{\mathrm{hol}}[f]]\, \mathsf{Ч} = 0 & (3.3.1)\\
2. & L[Y_{\mathrm{hol}}[f]]\, \mathsf{L} = 0 & (3.3.2)\\
3. & i(X^{\uparrow}_{\mathrm{hol}}[f])\, \Omega = df & (3.3.3)\\
4. & \gamma.f = 0 & (3.3.4)\\
5. & f \in \mathrm{Clas}(J_1\boldsymbol{E}, \mathbb{R}). & (3.3.5)
\end{array}
$$

3.3.2 Quantum currents

The quantum Poincaré-Cartan form and each quantisable function yields, in a covariant way, an horizontal n–form on the first jet prolongation of the quantum bundle. In the case when the quantisable function generates a symmetry of the quantum Lagrangian, the above form is conserved along the solutions of the Schrödinger equation.

3.3.6. Definition. Let $f \in \mathrm{Quan}(J_1\boldsymbol{E}, \mathbb{R})$. We say the 3-form form

$$\mathsf{j}[f] := -i[Y[f]]\,\Pi : J_1\boldsymbol{Q} \to \Lambda^3 T^*\boldsymbol{Q}$$

to be the *quantum current* associated with f.

By direct calculation in coordinates we find the following result

3.3.7. Proposition. Let $f \in \mathrm{Quan}(J_1\boldsymbol{E}, \mathbb{R})$. Then, the coordinate expression of $\mathsf{j}[f]$ is

$$
\begin{aligned}
\mathsf{j}[f] \;=\; & -\tfrac{1}{2}\, G^{ij}_0 (w^1_i w^1_j + w^2_i w^2_j)(f^0 v^0_0 - f^k v^0_k)\\
& -(A_0 - \tfrac{1}{2}\, G^{ij}_0 A_i A_j + \frac{1}{2} k \rho_0)(w^1 w^1 + w^2 w^2)(f^0 v^0_0 - f^k v^0_k)\\
& -(\overset{o}{f} + f^0 A_0 - f^k A_k)(w^1 w^1 + w^2 w^2)\,(G^{ij}_0 A_i\, v^0_j - v^0_0)\\
& -\tfrac{1}{2}\; G^{ij}_0 \ \mathrm{div}\ X[f](w^1 w^1_i + w^2 w^2_i) v^0_j\\
& +G^{ij}_0\,(\overset{o}{f} + f^0 A_0 - f^k A_k)(w^1 w^2_i - w^2 w^1_i) v^0_j\\
& -G^{ij}_0 (w^1_i dw^1 + w^2_i dw^2) \wedge ((f^0 \partial_0 - f^k \partial_k) \lrcorner\, v^0_j)\\
& +(w^1 dw^2 - w^2 dw^1) \wedge (f^k \partial_k \lrcorner\, v^0_0)\\
& +G^{ij}_0 A_i (w^1 dw^2 - w^2 dw^1) \wedge ((f^0 \partial_0 - f^k \partial_k) \lrcorner\, v^0_j),
\end{aligned}
$$

where we have set $v^0_\alpha = \partial_\alpha \lrcorner (d^0 \wedge \eta)$.

3.3.8. Lemma. For any quantum section Ψ the pullback $j_1\Psi^*(\mathsf{j}[f])$ of a quantum current $\mathsf{j}[f]$ is a 3–form of the type

$$j_1\Psi^*(\mathsf{j}[f]) : \boldsymbol{E} \to \Lambda^3 T^*\boldsymbol{E}.$$

We have the coordinate expression,

$$\begin{aligned}
j_1\Psi^*(\mathfrak{j}[f]) \;=\;& \Big((\tfrac{1}{2}\, f^0 G_0^{ij} A_i A_j - \tfrac{1}{2}\, f^0 k\rho_0 + \overset{o}{f} - f^h A_h)(\psi^1\psi^1 + \psi^2\psi^2) \\
& + \tfrac{1}{2}\, f^0 G_0^{ij}(\partial_i\psi^1\partial_j\psi^1 + \partial_i\psi^2\partial_j\psi^2) - (f^i - f^0 G_0^{ij} A_j)(\psi^2\partial_i\psi^1 - \psi^1\partial_i\psi^2)\Big) \quad \upsilon_0^0 \\
+\;& \Big((f^k A_0 - \tfrac{1}{2}\, f^k G_0^{ij} A_i A_j + \tfrac{1}{2}\, f^k k\rho_0) - (\overset{o}{f} + f^0 A_0 - f^h A_h) G_0^{ik} A_i)(\psi^1\psi^1 + \psi^2\psi^2) \\
& + \tfrac{1}{2}\, f^k G_0^{ij}(\partial_i\psi^1\partial_j\psi^1 + \partial_i\psi^2\partial_j\psi^2) - (\overset{o}{f} + f^0 A_0 - f^h A_h) G_0^{ik}(\psi^1\partial_i\psi^2 - \psi^2\partial_i\psi^1) \\
& - \tfrac{1}{2}\, \operatorname{div} X[f] G_0^{ik}(\psi^1\partial_i\psi^1 + \psi^2\partial_i\psi^2) - (f^k - f^0 G_0^{ik} A_i)(\psi^2\partial_0\psi^1 - \psi^1\partial_0\psi^2) \\
& + f^0 G_0^{ik}(\partial_i\psi^1\partial_0\psi^1 + \partial_i\psi^2\partial_0\psi^2)\Big) \quad \upsilon_k^0 \\
-\;& \Big(f^k G_0^{ij} A_i(\psi^2\partial_l\psi^1 - \psi^1\partial_l\psi^2) - f^k G_0^{ij}(\partial_i\psi^1\partial_l\psi^1 + \partial_i\psi^2\partial_l\psi^2)\Big) \quad d^l \wedge (\partial_k \lrcorner\, \upsilon_j^0)\, \square
\end{aligned}$$

Let us set $\upsilon_{\alpha\beta}^0 := \partial_\alpha \lrcorner\, \upsilon_\beta^0$.

3.3.9. Example. If $f \in \operatorname{Aff}(J_1\boldsymbol{E}, I\!R)$, then we obtain

$$\begin{aligned}
& \mathfrak{j}[f] = \\
&= \Big(\big(- f^i\, A_i + \overset{o}{f}\big)\, (w^1\, w^1 + w^2\, w^2)\Big)\, \upsilon_0^0 \\
&+ \Big(\tfrac{1}{2}\, f^h\, G_0^{ij}\, (w_i^1\, w_j^1 + w_i^2\, w_j^2) + (-f^i\, A_i + \overset{o}{f})\, G_0^{hj}\, (w^1\, w_j^2 - w^2\, w_j^1)\Big)\, \upsilon_h^0 \\
&- \Big((-f^i\, A_i + \overset{o}{f})\, A_0^h - f^h\, (A_0 - \tfrac{1}{2}\, A_0^i\, A_i + \tfrac{1}{2}\, k\, \rho_0)\Big)\, (w^1\, w^1 + w^2\, w^2)\, \upsilon_h^0 \\
&- \tfrac{1}{2}\, \operatorname{div} X[f]\, G_0^{hj}\, (w^1\, w_j^1 + w^2\, w_j^2)\, \upsilon_h^0 \\
&+ f^h\, (w^2\, dw^1 - w^1\, dw^2) \wedge \upsilon_{0h}^0 \\
&- f^k \Big(G_0^{hj}\, (w_j^1\, dw^1 + w_j^2\, dw^2) + A_0^h\, (w^2\, dw^1 - w^1\, dw^2)\Big) \wedge \upsilon_{hk}^0\,,
\end{aligned}$$

and

$$\begin{aligned}
& \Psi^*(\mathfrak{j}[f]) = \\
&= -f^i\, (\Psi^2\, \partial_i\Psi^1 - \Psi^1\, \partial_i\Psi^2)\, \upsilon_0^0 \\
&+ \big(- f^i\, A_i + \overset{o}{f}\big)\, (\Psi^1\, \Psi^1 + \Psi^2\, \Psi^2)\, \upsilon_0^0 \\
&+ (f^i\, G_0^{hj} - \tfrac{1}{2}\, f^h G_0^{ij})\, (\partial_i\Psi^1\, \partial_j\Psi^1 + \partial_i\Psi^2\, \partial_j\Psi^2)\, \upsilon_h^0 \\
&+ f^h\, (\Psi^2\, \partial_0\Psi^1 - \Psi^1\, \partial_0\Psi^2)\, \upsilon_h^0 \\
&+ (-f^i\, A_i + \overset{o}{f})\, G_0^{hj}\, (\Psi^1\, \partial_j\Psi^2 - \Psi^2\, \partial_j\Psi^1)\, \upsilon_h^0 \\
&- \Big(\tfrac{1}{2}\, \operatorname{div} X[f]\, G_0^{hj}\, (\Psi^1\, \partial_j\Psi^1 + \Psi^2\, \partial_j\Psi^2) + (f^h\, A_0^i - f^i\, A_0^h)\, (\Psi^2\, \partial_i\Psi^1 - \Psi^1\, \partial_i\Psi^2)\Big)\, \upsilon_h^0 \\
&+ \Big((f^i\, A_i - \overset{o}{f})\, A_0^h + f^h\, (A_0 - \tfrac{1}{2}\, A_0^i\, A_i + \tfrac{1}{2}\, k\, \rho_0)\Big)\, (\Psi^1\, \Psi^1 + \Psi^2\, \Psi^2)\, \upsilon_h^0\,. \square
\end{aligned}$$

3.3.10. Example. If $f \in \mathrm{Map}(\boldsymbol{E}, I\!\!R)$, then we obtain

$$\mathfrak{j}[f] = f\,\Big((w^1\,w^1 + w^2\,w^2)\,\upsilon^0_0 + \Big(G^{hj}_0\,(w^1\,w^2_j - w^2\,w^1_j) - A^h_0\,(w^1\,w^1 + w^2\,w^2)\Big)\,\upsilon^0_h\Big)$$

and

$$\begin{aligned}\Psi^*(\mathfrak{j}[f]) = f\,\Big(&(\Psi^1\,\Psi^1 + \Psi^2\,\Psi^2)\,\upsilon^0_0 \\ &+ \Big(G^{hj}_0\,(\Psi^1\,\partial_j\Psi^2 - \Psi^2\,\partial_j\Psi^1) - A^h_0\,(\Psi^1\,\Psi^1 + \Psi^2\,\Psi^2)\Big)\,\upsilon^0_h\Big)\,.\,\square\end{aligned}$$

3.3.11. Example. We obtain

$$\begin{aligned}&\mathfrak{j}[\mathcal{H}_0] = \\ &= \Big(-\tfrac{1}{2}\,G^{ij}_0\,(w^1_i\,w^1_j + w^2_i\,w^2_j) + \Big(\tfrac{1}{2}\,(A^i_0\,A_i - k\,\rho_0) - A_0\Big)\,(w^1\,w^1 + w^2\,w^2)\Big)\,\upsilon^0_0 \\ &- \tfrac{1}{2}\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\,G^{hj}_0\,(w^1\,w^1_j + w^2\,w^2_j)\,\upsilon^0_h \\ &- \Big(G^{hj}_0\,(w^1_j\,dw^1 + w^2_j\,dw^2) + A^h_0\,(w^2\,dw^1 - w^1\,dw^2)\Big)\Big) \wedge \upsilon^0_{0h}\end{aligned}$$

and

$$\begin{aligned}&\Psi^*(\mathfrak{j}[\mathcal{H}_0]) = \\ &= \Big(\tfrac{1}{2}\,G^{ij}_0\,(\partial_i\Psi^1\,\partial_j\Psi^1 + \partial_i\Psi^2\,\partial_j\Psi^2) + A^i_0\,(\Psi^2\,\partial_i\Psi^1 - \Psi^1\,\partial_i\Psi^2)\Big)\,\upsilon^0_0 \\ &+ \Big(\tfrac{1}{2}\,(A^i_0\,A_i - k\,\rho_0) - A_0\Big)\,(\Psi^1\,\Psi^1 + \Psi^2\,\Psi^2)\,\upsilon^0_0 \\ &- \Big(G^{hj}_0\,(\partial_j\Psi^1\,\partial_0\Psi^1 + \partial_j\Psi^2\,\partial_0\Psi^2) + A^h_0\,(\Psi^2\,\partial_0\Psi^1 - \Psi^1\,\partial_0\Psi^2)\Big)\,\upsilon^0_h \\ &+ \Big(-\tfrac{1}{2}\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\,G^{hj}_0\,(\Psi^1\,\partial_j\Psi^1 + \Psi^2\,\partial_j\Psi^2)\Big)\,\upsilon^0_h\,.\,\square\end{aligned}$$

3.3.12. Example. We obtain

$$\begin{aligned}&\mathfrak{j}[\mathcal{P}_j] = \\ &+ \tfrac{1}{2}\,G^{hk}_0\,(w^1_h\,w^1_k + w^2_h\,w^2_k)\,\upsilon^0_j + (A_0 - \tfrac{1}{2}\,A^i_0\,A_i + \tfrac{1}{2}\,k\,\rho_0)\,(w^1\,w^1 + w^2\,w^2)\,\upsilon^0_j \\ &+ \tfrac{1}{2}\,\frac{\partial_j\sqrt{|g|}}{\sqrt{|g|}}\,G^{hk}_0\,(w^1\,w^1_k + w^2\,w^2_k)\,\upsilon^0_h \\ &+ (w^2\,dw^1 - w^1\,dw^2) \wedge \upsilon^0_{0j} - \Big(G^{hk}_0\,(w^1_k\,dw^1 + w^2_k\,dw^2) + A^h_0\,(w^2\,dw^1 - w^1\,dw^2)\Big) \wedge \upsilon^0_{hj}\,,\end{aligned}$$

and

$$
\begin{aligned}
&\Psi^*(\mathfrak{j}[\mathcal{P}_j]) = \\
&= -(\Psi^2\,\partial_j\Psi^1 - \Psi^1\,\partial_j\Psi^2)\,\upsilon^0_0 \\
&+ G^{hk}_0\,(\partial_j\Psi^1\,\partial_k\Psi^1 + \partial_j\Psi^2\,\partial_k\Psi^2)\,\upsilon^0_h - \tfrac{1}{2}\,G^{hk}_0\,(\partial_h\Psi^1\,\partial_k\Psi^1 + \partial_h\Psi^2\,\partial_k\Psi^2)\,\upsilon^0_j \\
&+ (\Psi^2\,\partial_0\Psi^1 - \Psi^1\,\partial_0\Psi^2)\,\upsilon^0_j \\
&+ \Big(\tfrac{1}{2}\,\frac{\partial_j\sqrt{|g|}}{\sqrt{|g|}}\,G^{hj}_0\,(\Psi^1\,\partial_j\Psi^1 + \Psi^2\,\partial_j\Psi^2) + A^h_0\,(\Psi^2\,\partial_j\Psi^1 - \Psi^1\,\partial_j\Psi^2)\Big)\,\upsilon^0_h \\
&- A^i_0\,(\Psi^2\,\partial_i\Psi^1 - \Psi^1\,\partial_i\Psi^2)\,\upsilon^0_j \\
&+ (A_0 - \tfrac{1}{2}\,A^i_0\,A_i + \tfrac{1}{2}\,k\,\rho_0)\,(\Psi^1\,\Psi^1 + \Psi^2\,\Psi^2)\,\upsilon^0_j\,.\,\square
\end{aligned}
$$

3.3.13. Corollary. Vertical restriction with respect to the fibres of spacetime yields the vertical 3–form $\Big(j_1\Psi^*(j[f])\Big)^V : \boldsymbol{E} \to \Lambda^3 T^*\boldsymbol{E}$ with coordinate expression

$$
\begin{aligned}
\Big(j_1\Psi^*(j[f])\Big)^V \;=\;& \Big((\tfrac{1}{2}\,f^0 G^{ij}_0 A_i A_j - \tfrac{1}{2}\,f^0 k\rho_0 + \overset{o}{f} - f^h A_h)\,(\psi^1\psi^1 + \psi^2\psi^2) \\
&+\tfrac{1}{2}\,f^0 G^{ij}_0(\partial_i\psi^1\partial_j\psi^1 + \partial_i\psi^2\partial_j\psi^2) - (f^i - f^0 G^{ij}_0 A_j)(\psi^2\partial_i\psi^1 - \psi^1\partial_i\psi^2)\Big)\quad \eta \qquad \square
\end{aligned}
$$

On the other hand, by means of the Hermitian product, a function $f \in \mathrm{Quan}(J_1\boldsymbol{E}, I\!R)$ and a quantum section Ψ yield in a covariant way the real valued vertical 3–form

$$
\frac{1}{2i}\Big(\mathfrak{h}(\Psi, Y[f].\Psi) - h(Y[f].\Psi, \Psi)\Big) : \boldsymbol{E} \to \Lambda^3 V^*\boldsymbol{E}. \tag{3.3.6}
$$

3.3.14. Remark. Recalling that $Z[f] := \mathrm{i}\,Y[f]$, we can write the expression 3.3.6, equivalently, as

$$
\frac{1}{2}\Big(\mathfrak{h}(\Psi, Z[f].\Psi) + h(Z[f].\Psi, \Psi)\Big) : \boldsymbol{E} \to \Lambda^3 V^*\boldsymbol{E}.
$$

3.3.15. Lemma. We have the coordinate expression

$$
\begin{aligned}
\frac{1}{2i}\Big(\mathfrak{h}(\Psi, Y[f].\Psi) - h(Y[f].\Psi, \Psi)\Big) \;=\;& \Big(f^0(\psi^1\partial_0\psi^2 - \psi^2\partial_0\psi^1) + f^i(\psi^2\partial_i\psi^1 - \psi^1\partial_i\psi^2) \\
&-(\overset{o}{f} + f^0 A_0 - f^h A_h)(\psi^1\psi^1 + \psi^2\psi^2)\Big)\,\eta \qquad \square
\end{aligned}
$$

3.3.16. Corollary. Let $f \in \mathrm{Aff}(J_1\boldsymbol{E}, I\!R)$ and $\Psi \in \mathrm{Sec}(\boldsymbol{Q}, T\boldsymbol{Q})$. Then, the following equality holds

$$
(j[f] \circ j_1\Psi)^V = \frac{1}{2i}\Big(\mathfrak{h}(\Psi, Y[f].\Psi) - h(Y[f].\Psi, \Psi)\Big) \tag{3.3.7}
$$

We have the coordinate expression

$$
(\mathfrak{j}[f] \circ j_1\Psi)^V = f^i(\psi^2\partial_i\psi^1 - \psi^1\partial_i\psi^2) - (\overset{o}{f} - f^h A_h)(\psi^1\psi^1 + \psi^2\psi^2)\eta.\,\square \tag{3.3.8}
$$

3.3.17. Remark. The results in this section can be generalized. In [69], we introduce, for any quantisable function in analogy with expression (3.3.6) the real vertical form

$$\tfrac{1}{2}\Big(\mathsf{h}\big(\mathsf{S}[f].\Psi,\,\Psi\big) + \mathsf{h}\big(\Psi,\,\mathsf{S}[f].\Psi\big)\Big),$$

where $\mathsf{S}[f]$ is the Schrödinger operator associated with the quantisable function. The results of the sections 3.1.5 and 3.3.2 suggest to introduce the notion of *quantum expectation forms*, where $\mathsf{e}_1[f] := \frac{1}{2}\big(\mathsf{h}(\Psi, Z[f].\Psi) + h(Z[f].\Psi, \Psi)\big)$ is called *first order quantum expectation form* and where $\mathsf{e}_2[f] := \frac{1}{2}\Big(\mathsf{h}\big(\mathsf{S}[f].\Psi,\,\Psi\big) + \mathsf{h}\big(\Psi,\,\mathsf{S}[f].\Psi\big)\Big)$ is called *second order quantum expectation form*. There is a unique combination (see introduction) $\widehat{Y[f]} - \widehat{\mathsf{S}[f]}$ of the naturally induced operators $\widehat{Y[f]}$ and $\widehat{\mathsf{S}[f]}$ which acts on the fibres of the Hilbert bundle. Thus, we call

$$\mathsf{e} := \mathsf{e}_1 - \mathsf{e}_2$$

the *quantum expectation form* associated with f.

I think that this could be the pre-quantum source of the probabilistic interpretation in covariant quantum mechanics. □

Now, let us recall the well known result of Nöther

3.3.18. Proposition. Let Y be a vector field of $\boldsymbol{Q}$. If $L[Y_{(1)}]\,\mathsf{L} = 0$, then, the current $\mathsf{j} = -i_Y\,\Pi$ is conserved along critical sections Ψ.

3.3.19. Definition. We say the current j associated with a holonomic symmetry of L to be a *quantum conserved current.*

Now, we are going to apply the Nöther theorem to our results above

3.3.20. Proposition. Let $f \in \mathrm{Clas}(J_1\boldsymbol{E}, I\!R)$. Then, the quantum current $\mathsf{j}[f]$ is a conserved current.

If, additionally, $f \in \mathrm{Aff}(J_1\boldsymbol{E}, I\!R)$ and $\Psi \in \mathrm{Sec}(\boldsymbol{E}, \boldsymbol{Q})$, we have the equality

$$\Big(j_1\Psi^*(\mathsf{j}[f])\Big)^V = \frac{1}{2i}\Big(\mathsf{h}(\Psi, Y[f].\Psi) - h(Y[f].\Psi, \Psi)\Big).$$

Finally, I apply the machinery above to the classical backgrounds, which we have modeled at the end of section 2.6.

3.3.21. Example. Let $(\boldsymbol{Q}, Ч)$ a quantum structure and $(J_1\boldsymbol{E}, \Omega, dt)$ the associated classical background. We consider the natural infinitesimal action of the $I\!R$–Lie algebra $\mathrm{i} \otimes I\!R$ (associated to the group action of $U(1)$) on the fibres of $\boldsymbol{Q}$) given by $Y[f] = i \otimes \mathbf{1}$.

We observe that $f = 1 \in I\!R$ has no classical meaning since it corresponds to the classical gauge freedom in the choice of a potential function. Clearly, $f \in \mathrm{Clas}(J_1\boldsymbol{E}, I\!R)$. The associated (globally) conserved quantum current has the coordinate expression

$$\begin{aligned}\mathsf{j}[1] \;\; = \;\; & -(w^1w^1 + w^2w^2)\,(G_0^{ij} A_i\, v_j^0 - v_0^0) \\ & + G_0^{ij}\,(w^1w_i^2 - w^2w_i^1)v_j^0\end{aligned}$$

We say $\mathfrak{j}[1]$ to be the *conserved probability current.* In particular, for any quantum section Ψ, we have

$$\Big(j_1\Psi^*(\mathfrak{j}[1])\Big)^V = (\psi^1\psi^1 + \psi^2\psi^2)\eta$$

.

3.3.22. Example. Let $t : \boldsymbol{E} \to \boldsymbol{T}$ an affine space with affine projection t. Let G be the euclidian metric, $K^\natural$ the natural flat connection and $F = 0$. We consider the natural vertical action of the associated vector space $\boldsymbol{S}$ of $\boldsymbol{E}$ given by $\mathbb{S} \times \boldsymbol{E} \to \boldsymbol{E} : (u, e) \mapsto (e + u)$. Then, any Poincaré-Cartan form Θ is globally defined and $f = \mathcal{P}(u) := -i_u\Theta \in \mathrm{Clas}(J_1\boldsymbol{E}, I\!R)$ is the conserved kinetic momentum (defined up to an additive constant $c \in I\!R$). In a cartesian chart, we have the expression $\mathcal{P}(u) = G^0_{ij}u^ix^j_0$.

Let $(\boldsymbol{Q}, Ч)$ be a quantum structure for this background. Then, in a cartesian chart, the associated conserved quantum current has the coordinate expression

$$\begin{aligned}
\mathfrak{j}[\mathcal{P}(u)] \;\; = \;\; & -\tfrac{1}{2}\, G^{ij}_0 (w^1_i w^1_j + w^2_i w^2_j)(-u^k v^0_k) \\
& -G^{ij}_0 (w^1_i dw^1 + w^2_i dw^2) \wedge (-u^k\partial_k \lrcorner\, v^0_j) \\
& +(w^1 dw^2 - w^2 dw^1) \wedge (u^k\partial_k \lrcorner\, v^0_0)
\end{aligned}$$

We say $\mathfrak{j}[\mathcal{P}(u)]$ to be the *conserved quantum momentum current.* In particular, we have $\Big(j_1\Psi^*(\mathfrak{j}[\mathcal{P}])\Big)^V = +u^i(\psi^1\partial_i\psi^2 - \psi^2\partial_i\psi^1)\eta.$

3.3.23. Example. Let $\boldsymbol{E} := \boldsymbol{T} \times SO(3)$ be the configuration space for a rigid body with a fixed point. We assume the *inertia tensor* I as the scaled vertical metric. Let $K^\natural$ be the inertial connection and $F = 0$. We have a global (classical) Θ, the rotational energy. Θ is invariant w.r.t. the action of $SO(3)$ on itself. The conserved quantity associated with every infinitesimal generator $\sigma so(3) \simeq \mathbb{L}^{-1} \otimes \boldsymbol{S} \ni X(\sigma)$ is the angular momentum $\mathcal{J}(\sigma c$.

The induced infinitesimal quantum symmetry is $Y[\mathcal{J}(\sigma)] = X(\sigma)$. Clearly,

$$i_{X[\mathcal{J}(\sigma)]_1}\,\Omega = d(\mathcal{J}(\sigma))\,.$$

This induces a *conserved quantum angular momentum current* $j[\mathcal{J}(\sigma)]$. We get its expression from equation **??** by substituting $G^{ij}_0 \to I^{ij}_0$ and $u^i \to \sigma^i$ and adding a mass term due to a non vanishing scalar curvature for this configuration space.

CONCLUSIONS AND OUTLOOK

At this point, I spend some words on my results and I give an outlook on further developments within this field.

I have studied the symmetries of the (full) classical structure, i.e. the vector field of phase space which preserve what we call the "classical structure", and the symmetries of the (full) quantum structure, i.e. vector field which preserve the "quantum structure". Both symmetries can be classified by means of a subalgebra of the algebra of special functions of phase space, namely the classical generators $\mathrm{Clas}(J_1\boldsymbol{E}, I\!R)$ using a covariant lift of functions of phase space to vector fields of phase space. In particular, I have found a morphism of Lie algebras between $\mathrm{Clas}(J_1\boldsymbol{E}, I\!R)$ and the symmetries of the classical structure and an isomorphism of Lie algebras between $\mathrm{Clas}(J_1\boldsymbol{E}, I\!R)$ and the symmetries of the quantum structure. In our formalism, we can directly express our results in terms of central extensions. In particular, we recover well known results found by other approaches in other contexts.

On the other hand, I have "systematically" studied the symmetries of classical and quantum structure. That is, I have classified step by step the vector fields which preserve only a part of the structure. It turned out that many of the above (well known) results are due to more general geometric structures. Let us recall, at this point, the isomorphism of Lie algebras between quantisable functions and Hermitian vector fields of the quantum bundle. Here, the main contribution is due to the algebra of special functions, more precisely, the special subalgebra of quantisable functions.

Moreover, the covariance in the sense of naturality and observer independence yield a new insight into the notion of holonomicity within a concrete physical model of fundamental type. That is, the symmetries which preserve the classical or quantum structure are automatically of holonomic type.

Furthermore, the model provides a canonical quantum Lagrangian, hence, a canonical quantum Lagrangian formalism. This leads to a new insight into a quantum theory and the quantum symmetries by means of standard (classic field theoretic) methods. In particular, with any quantisable function, we can associate a quantum current. For functions in $\mathrm{Clas}(J_1\boldsymbol{E}, I\!R)$ the associated quantum current is conserved along the solutions of the (covariant) Schrödinger equation. We have explicit coordinate expressions for these currents. They look like well known expressions of field theory. More generally, these currents are completely determined by the classical structure, since they are associated with a quantisable function. Moreover, for each affine function and quantum section, we obtain, in a covariant way, a spacelike 3–form (which can be integrated on the fibres of spacetime). These results can be generalised and seem to be the pre-quantum source of the (standard) probabilistic interpretation within covariant quantum mechanics.

The results are promising for further investigation. In the following I give an outlook on the next main steps.

We study the quantum Lagrangian formalism. In particular, we proceed with our analysis of the quantum currents and quantum expectation forms. Additionally, we look for a quantum momentum map.

Moreover, we apply our pre-quantum results to the (natural) Hilbert bundle in order to get the standard probabilistic interpretation for these currents.

On the other hand, we plan to go back, from the Hilbert bundle to the pre-quantum structure, in order to compare standard techniques in quantum mechanics with (covariant) geometric techniques on the pre-quantum structure.

There are other possible directions of future research. Some of them are still too far away, hence, they are not worth to be mentioned right now.

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I hope I can give a few things back.

ERKLÄRUNG

Ich erkläre, dass ich die Arbeit *Symmetries in covariant quantum mechanics* selbständig angefertigt und keine anderen als die angegebenen Hilfsmittel verwendet habe.

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