# Substitution Effects in Supply Chains with Asymmetric Information Distribution and Upstream Competition

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Inventory management in markets with substituting customers is extremely challenging, not only for a downstream wholesaler, but also for upstream manufacturers. Motivated by the structures in the agrochemical market, we analyze the optimal production and stocking quantities of a manufacturer and a wholesaler, respectively, in a two-stage supply chain with upstream competition and vertical information asymmetries. We characterize a monopolistic wholesaler's optimal stocking quantities and show that these quantities are not necessarily monotonic, neither in the available production quantities nor in the customers' substitution rates. We further derive the optimal production quantities of a monopolistic and a competitive manufacturer when they are incompletely informed about the wholesaler's stocking quantities. We find that the introduction of competition may lead to decreasing production quantities for some products. Furthermore, a product's end-of-season inventories at the manufacturer which arise due to information asymmetries may decrease even when initial production levels increase.

*Key words:* customer substitution; supply chain; asymmetric information; competition; inventory management

## 1 Introduction

In recent years, a diverse body of research has focused on how firms should react to customers' substitution behavior. For firms that directly serve customers, investigations range from strategic assortment planning (Kök and Fisher, 2007; Honhon et al., 2010) over promotion strategies (Walters, 1991) to optimal stocking decisions (Netessine and Rudi, 2003; Jiang et al., 2011). In a supply chain setting, only the downstream stage experiences the immediate effects of customer substitution; however, substitution effects also diffuse across the entire supply chain. This paper therefore investigates how different stages of a supply chain are affected by customer substitution. In particular, we examine the optimal production and stocking decisions of different supply chain members under upstream competition and vertical information asymmetries.

We are interested in situations when competition and substitution effects arise simultaneously within the supply chain. While competition occurs due to the non-cooperative behavior of independent firms, substitution emerges from the competitive structures within the set of available products. Note that competition and substitution are neither inclusive nor exclusive concepts: Competition without substitution arises if multiple independent firms offer an identical product (in a supply chain setting, e.g., Cachon, 2001; Adida and DeMiguel, 2011), while substitution without competition occurs if a monopolistic firm offers non-identical, yet similar products that serve a common customer base.

In this paper, we concentrate on markets where substitution and competition effects exist simultaneously. Initially, our work is motivated by the agrochemical market. Agrochemical manufacturers sell their products through locally monopolistic wholesalers to their customers, mostly farmers or farmer unions. Substitution in this market arises from customers' focus on active ingredients, resulting in low brand loyalty. In consequence, stock-outs at the wholesaler lead to high substitution rates among products. This effect is even further enhanced by the inherent finiteness of the selling season for agrochemicals and the non-durability of some chemical components.

Information asymmetries in this market stem from the wholesaler's bargaining power and substantial production lead-times at the manufacturers which can amount to two years (Shah, 2004). While production needs to be initiated well in advance of the desired selling season, the wholesaler cannot be forced to commit to order quantities at this early stage. Final orders are typically released close to the selling season when (weather-dependent) demand can be predicted sufficiently well. In essence, production and ordering decisions are based on potentially different information sets, and thus, vertical information asymmetries arise.

To analyze the manufacturer's (wholesaler's) optimal production (stocking) quantities, we consider a supply chain in which potentially multiple manufacturers sell partially substitutable products for a single season through a monopolistic wholesaler. We focus on a single period setting because (i) it yields a very good approximation of the agrochemical market where the selling season is finite and some chemical components cannot be stored until the next season; and (ii) it is a necessary first step in the analysis of substitution effects within supply chains which is in line with the existing literature. To capture the effects of upstream competition, we compare two distinct supply chain scenarios: a horizontally integrated (hereafter 'non-competitive') supply chain with a single manufacturer producing all available products; and a horizontally competitive (hereafter 'competitive') supply chain with multiple manufacturers, each producing only one product. While inspired by the agrochemical market, our framework generally suits industries in which (1) products are partial substitutes, (2) products and market structures exhibit typical newsvendor characteristics, and (3) customers are served by a monopolistic wholesaler.

Our work contributes to the literature on (i) vertical information asymmetries in supply chains; and, most importantly, (ii) optimal stocking levels under customer substitution. Information sharing within supply chains has been a prevalent research area in the last decades (Li, 2002; Özer and Wei, 2006). Apart from the issue of truthful information sharing, literature also investigates the effects that asymmetric information exert on operational problems. In the presence of short capacity at the manufacturer, Cachon and Lariviere (1999) show that wholesalers exploit their informational advantage by manipulating the manufacturer's allocation mechanism. Under asymmetric information, Corbett (2001) depicts that the introduction of consignment stocks at the wholesaler leads to reduced cycle stocks at the expense of increased safety stocks. If wholesalers are allowed to share inventories, Yan and Zhao (2011) conclude that wholesalers share demand information with each other, but not with the manufacturer. We depart from this stream by incorporating an asymmetric information structure into a supply chain prone to customer substitution.

There has been an extensive literature on the repercussions of customer substitution on the wholesaler's optimal stocking quantities. As common building block, the single-stage newsvendor inventory (competition) model with stock-out-based substitution as pioneered by McGillivray and Silver (1978), Parlar (1988), Lippman and McCardle (1997), Bassok et al. (1999), Smith and Agrawal (2000), and Netessine and Rudi (2003) has evolved. In a seminal paper, Netessine and Rudi (2003) extend the preceding work by characterizing the structure of the optimal stocking levels for an arbitrary number of products under both centralization and competition. Based on these results, recent work has investigated various competitive environments under customer substitution. Mishra and Raghunathan (2004), Kraiselburd et al. (2004), and Kim (2008) explore the consequences of introducing Vendor Managed Inventory for the wholesaler's stocking levels and advertisement efforts. Nagarajan and Rajagopalan (2008) embed the substitution framework into a multi-period setting and Jiang et al. (2011) provide a robust optimization approach that determines stocking levels by minimizing absolute regret. Recently, Vulcano et al. (2012) develop an efficient procedure to empirically estimate required substitution parameters.

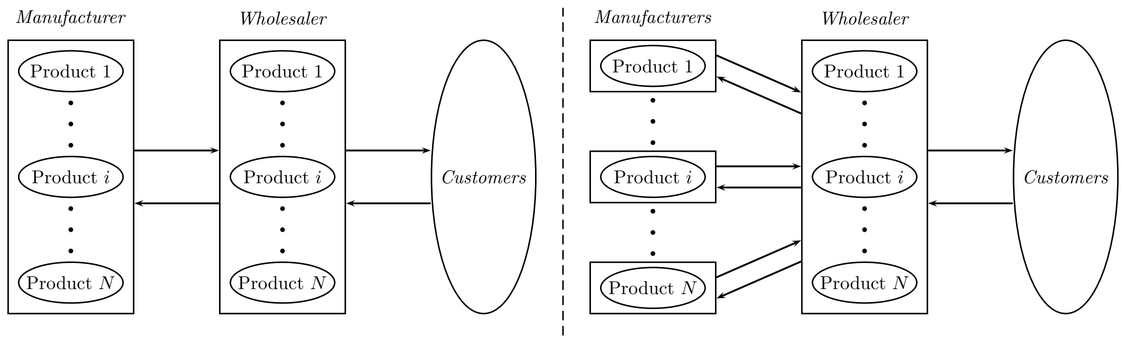
As common in the newsvendor framework, existing models assume that the wholesaler is unconstrained in his stocking decision, i.e., any arbitrary amount of products can be ordered. Being true in a single-stage setting, this assumption fails to hold in a supply chain setting. Here, a manufacturer's production or capacity decision constitutes a natural upper bound on the wholesaler's decision space (compare this to the literature on capacity choice, e.g., Cachon and Lariviere, 1999; Montez, 2007). By explicitly integrating these dependencies into our model we make a two-fold contribution: first, we investigate a constrained wholesaler's behavior; second, to the best of our knowledge, we are the first to examine how customer substitution affects upstream stages. To be specific, the contributions of this paper are as follows: (1) We derive the optimal stocking quantities of a constrained wholesaler and characterize the non-monotonic effects that a change in a manufacturer's production quantity exerts on optimal stocking levels. (2) We formally analyze the influence of changing substitution rates on the wholesaler's stocking quantities. In contrast to an intuitive conjecture of Netessine and Rudi (2003), we show that stocking levels for certain products may increase even if customer substitution away from these products increases. (3) We characterize the optimal production quantities of an incompletely informed manufacturer both under centralization and competition by applying a Bayesian (Nash-)Stackelberg game. (4) We explicitly compare optimal production levels under competition and centralization and find that competition may lead to reduced production. (5) We show that for some products, end-of-season inventories at the manufacturer may decrease even when initial production levels are increased under competition.

The remainder of this paper is organized as follows. The structure of the supply chain under consideration and the distribution of information are described in §2. Furthermore, we elaborate on the properties of the resulting supply chain game. In §3, we present our model of a constrained wholesaler and derive the optimal stocking quantities. We proceed by analyzing the effects of changing substitution rates on these optimal stocking levels. The manufacturer's production quantities are the focus of §4. We first characterize the equilibrium production quantities of a manufacturer under competition, before investigating the structure of a monopolistic manufacturer's optimal production quantities. We then compare production levels under centralization and competition, and examine the manufacturer's end-of-season inventories under both scenarios. Section 5 provides a discussion of our results and concluding remarks.

## 2 Supply Chain Structure and Information Distribution

We consider a two-stage supply chain with possibly multiple manufacturers (she) and a single wholesaler (he) selling $N \geq 2$ partially substitutable products for one period. While competition among manufacturers at the upstream stage may arise, we restrict attention to a monopolistic downstream wholesaler. In the non-competitive situation, a single manufacturer provides all $N$ products (bilateral monopoly), whereas in the competitive scenario, $N$ independent manufacturers each produce a different product (unilateral monopoly with upstream competition). Figure 1 illustrates both supply chain structures. In the agrochemical market, a centralized manufacturer occurs whenever a family of patents that allows for the provision of different, yet substitutable products is exclusively held by a single firm. In contrast, upstream competition is introduced if different manufacturers hold different patents for similar, but not identical

Figure 1: Bilateral monopoly (left) and unilateral monopoly with upstream competition (right).



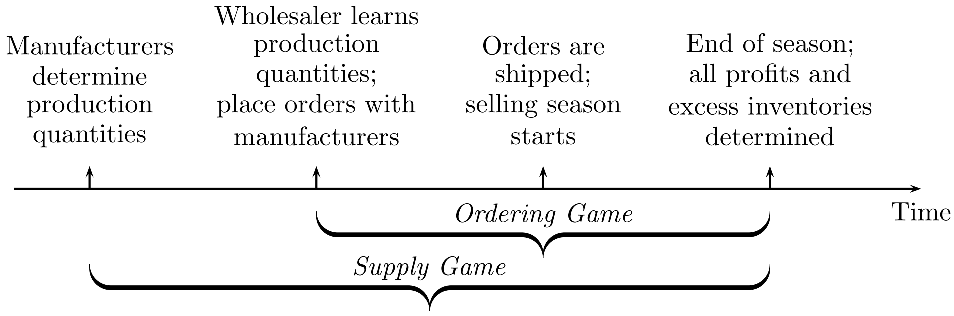

products or if patents run out.

We assume that information is asymmetrically distributed between manufacturers and the wholesaler. As mentioned earlier, this vertical information asymmetry between supply chain stages arises naturally in the agrochemical market due to the wholesaler's bargaining power and manufacturers' lead-times. Besides such natural causes for differing information sets, literature has also identified many other reasons including technological issues (Lee and Whang, 2000) and the fear of information leakage (Anand and Goyal, 2009). Our modeling approach allows for the inclusion of any such cause for information asymmetries within the supply chain.

To be precise, in line with the literature on vertical information asymmetries, e.g., Li (2002), Özer and Wei (2006) and Yan and Zhao (2011), we assume that manufacturers are incompletely informed about the wholesaler's optimal stocking quantities. In contrast, upstream information are common knowledge across manufacturers, i.e., no horizontal information asymmetry arises, and production quantities are commonly verifiable. This assumption is reasonable in the agrochemical market since manufacturers produce substitutable, hence comparable products and thus, they are able to credibly estimate their competitors' cost structures. Furthermore, to analyze the change in production quantities under competitive effects, we need to ensure that decisions are based on identical information sets under both supply chain structures. Following the argument of Harsanyi (1968) and Myerson (2004), we assume that manufacturers hold a common prior belief about the wholesaler's optimal stocking levels. Hence, manufacturers' beliefs are consistent. This prior belief represents the manufacturers' perception about the collection of information that are not common knowledge. In summary, supply chain structure and information distribution imply a Bayesian (Nash-)Stackelberg Game as first introduced by Gal-Or (1987). The for our work relevant case of multiple-leader Stackelberg games has first been studied by Sherali (1984) and recently by DeMiguel and Xu (2009), but only for complete, non-Bayesian information structures.

The sequence of events is as follows: In the first stage, manufacturers maximize expected profits and determine their optimal production quantities based on their beliefs about the wholesaler's subsequent stocking quantities. In the second stage, before the start of the selling

Figure 2: Sequence of Events.




season, the wholesaler learns these production quantities and, given his private information, derives his optimal stocking levels by maximizing expected profits. Afterwards, orders are submitted and shipped before the selling season starts. Throughout the selling season the wholesaler experiences customer demand and realizes profits. We refer to the subgame with given production quantities as the Ordering Game, while the entire game is denoted as the Supply Game. Hence, production quantities are exogenously given in the Ordering Game, whereas they are decision variables in the Supply Game. Figure 2 summarizes the chronology.

We assume that stochastic customer demand appears exclusively at the wholesaler and no manufacturer can pursue a direct selling strategy. Prices are exogenously given by the market and neither player can negotiate on the price to pay. Furthermore, we restrict attention to pure-strategy equilibria.

# 3 The Ordering Game

Focusing on the Stackelberg follower in this section, we derive the wholesaler's optimal stocking levels given the manufacturers' production quantities and characterize its sensitivity with respect to (i) changes in a manufacturer's production quantity, and (ii) substitution effects.

## 3.1 Optimal Stocking Quantities

For each product $i \in \{1, \dots, N\}$, the wholesaler pays a unit wholesale price $w_i$ to the manufacturer and sells the product at a unit retail price $r_i$, satisfying $r_i > w_i > 0$. Additionally, the wholesaler incurs a unit holding or disposal cost of $h_i \geq 0$ for each unsold item. Total demand occurrence follows the standard model of stock-out-based substitution processes as defined by Netessine and Rudi (2003), Kök et al. (2009), and Jiang et al. (2011). Customers arrive at the wholesaler with an initial product preference. Thus, the wholesaler faces random initial demand for product $i$ given by $D_i$, which is assumed to have a continuous demand distribution with positive support. Second choice (substitution) demand stems from customers whose initially preferred product is out of stock. If a stock-out of product $i$ occurs, an exogenously given fraction $\alpha_{ij}$ of unserved customers is willing to substitute from product $i$ to $j$; naturally $\sum_{j\neq i}\alpha_{ij} \leq 1$ for all $i$. Each initially unserved customer makes at most one substitution attempt, which, if again unserved, results in a lost sale. Total demand for product $i$ after substitution is denoted by $D_i^s = D_i + \sum_{j\neq i}\alpha_{ji}\max\{0, D_j - x_j\}$, where $x_j$ is the wholesaler's stocking level for product $j$. For future reference, denote by $x_{-j}$ the $(N-1)$-dimensional vector of stocking levels for all products $i \neq j$.

Let $x$ be the vector of stocking levels and $\Pi_W(x)$ be the wholesaler's expected profit when choosing $x$. Since the vector of production quantities $y$ is common knowledge and verifiable, the wholesaler faces an optimization problem under complete information. Thus, he determines his optimal stocking quantities by solving the following maximization problem $P_y$:

$$\begin{aligned}\max_{0\leq x\leq y} \Pi_W(x) &= \mathbb{E}\left[\sum_i r_i \min\{x_i, D_i^s\} - w_i x_i - h_i \max\{x_i - D_i^s, 0\}\right] \\ &= \mathbb{E}\left[\sum_i u_i x_i - (u_i + o_i)\max\{x_i - D_i^s, 0\}\right], \qquad (1)\end{aligned}$$

where $u_i = r_i - w_i$ and $o_i = h_i + w_i$ are the wholesaler's underage and overage costs, respectively. The wholesaler's objective is to maximize his expected profit under the quantity restrictions imposed by the manufacturers' production quantities $y$. If there are no such restrictions, we let $y = \infty$ and refer to this case as the unconstrained problem $P_\infty$. We start our analysis of the optimal stocking quantities with a brief discussion on the properties of $\Pi_W(x)$. All proofs are in the appendix.

**Lemma 1.** *For arbitrary $i$, $\Pi_W(x)$ is not concave in $x_i$, in general, for given $x_{-i}$.*

Lemma 1 formalizes the numerical results in Netessine and Rudi (2003) that $\Pi_W(x)$ is not always concave in each individual stocking level $x_i$. This also implies that $\Pi_W(x)$ is not necessarily jointly concave in $x$, either. Thus, there may exist multiple local optima.

For the unconstrained problem $P_\infty$, we know from Proposition 1 in Netessine and Rudi (2003) that the optimal stocking quantities $\hat{x}$ must simultaneously satisfy the following first-order necessary optimality conditions for all $i \in \{1, \ldots, N\}$:

$$\mathbb{P}(D_i < \hat{x}_i) - \mathbb{P}(D_i < \hat{x}_i < D_i^s) + \sum_{j\neq i}\alpha_{ij}\frac{u_j + o_j}{u_i + o_i}\mathbb{P}(D_j^s < \hat{x}_j, D_i > \hat{x}_i) = \frac{u_i}{u_i + o_i}. \qquad (2)$$

In the remainder, denote by $\hat{x}_i(x_{-i})$ the solution to product $i$'s optimality condition (2) for given fixed values of $x_{-i}$. Analogously, let $\hat{x}_{-i}(x_i)$ be the solution vector of the remaining $(N-1)$ optimality conditions in (2) for products $j \neq i$ if $x_i$ is fixed. We further refer to product $j$'s entry in $\hat{x}_{-i}(x_i)$ as $\hat{x}_j(x_i)$. By Lemma 1, it is not ensured that $\hat{x}_i(x_{-i})$ is unique. Therefore, for a given problem instance $P_y$, we define $\hat{x}_i(x_{-i})$ to be the largest solution that is feasible in $P_y$ and for simplicity, we let $\hat{x}_i(x_{-i}) \equiv \infty$ if there exists no feasible solution. The introduction of this tie-breaking rule ensures uniqueness of $\hat{x}_i(x_{-i})$ and helps us to avoid ambiguities when comparing two scenarios with multiple optima.

The interpretation of (2) is appealing. It is a standard newsvendor fractile solution, adjusted by substitution effects. The second term on the left hand side increases the optimal stocking level to account for additional second choice demand, whereas the third term reduces the optimal stocking level by considering that a stock-out need not result in a lost sale.

The optimal solution of the constrained problem $P_y$ follows a similar pattern. Whenever feasible, the wholesaler tries to stock the quantity that solves (2), given the other products' optimal stocking levels. If this is not possible, he procures the entire available production quantity $y_i$. Proposition 1 formalizes this intuition.

**Proposition 1.** *Denote the vector of the wholesaler's optimal stocking quantities for the constrained problem $P_y$ by $x^\star(y)$. Further, refer to $x^\star(y)$ as a partially largest optimal solution if there exists no other optimal solution $x'^\star(y)$ with $x^\star_{-i}(y) = x'^\star_{-i}(y)$ and $x^\star_i(y) < x'^\star_i(y)$ for any $i$. Then, any partially largest optimal solution simultaneously satisfies*

$$x^\star_i(y) = \min\{\hat{x}_i(x^\star_{-i}(y)), y_i\}, \tag{3}$$

*for all $i = 1, \ldots, N$.*

In the remainder, we explicitly restrict our analysis to partially largest optimal solutions. Thus, from now on, $x^\star(y)$ refers only to partially largest optimal solutions. Analogously to our tie-breaking rule for $\hat{x}_i(x_{-i})$, we employ this selection criterion to avoid ambiguities and to enhance the expositional clarity of our analysis. Obviously, each optimization problem $P_y$ has at least one partially largest optimal solution. In contrast, our numerical experiments indicate that optimal solutions that are not partially largest occur very rarely. Moreover, we emphasize that most of our subsequent results also hold for optimal solutions that are not partially largest.

Note that $x^\star(\infty) = \hat{x}$. Therefore, the optimal stocking quantities given in (3) are consistent with the solution to the unconstrained problem $P_\infty$ given in Netessine and Rudi (2003). Furthermore, in any Bayesian (Nash-)Stackelberg equilibrium, the wholesaler plays his best-response against the manufacturers' initial decision $y$, which is given by $x^\star(y)$.

We now investigate the sensitivity of the wholesaler's optimal stocking quantities with respect to changes in a manufacturer's production quantity. In particular, we are interested in the question if the wholesaler's optimal reaction to changes in $y$ is monotonic. From a manufacturer's perspective, when altering $y_i$, monotonicity of the wholesaler's best-response function at least guarantees predictability of the direction of change of $x^\star(y)$, even in the asymmetric information case. In contrast, under information asymmetries, a non-monotonic best-response function is much harder to predict. We start our analysis by exogenously forcing one stocking level to increase in the unconstrained problem $P_\infty$.

**Lemma 2.** *Let $\varepsilon > 0$ and denote by $e_i$ the unit vector for product $i$.*

*(i) For given $x_{-i}$ and $x'_{-i} = x_{-i} + \varepsilon e_j$ with $j \neq i$, $\hat{x}_i(x_{-i}) \geq \hat{x}_i(x'_{-i})$.*

*(ii) For given $x_j$ and $x'_j = x_j + \varepsilon$, there are instances of $P_\infty$ for which $\hat{x}_i(x_j) < \hat{x}_i(x'_j)$ for some $i \neq j$.*

Using the results of Lemma 2, we can now endogenize the increasing stocking level by explicitly considering changes in a manufacturer's production quantity $y_j$. This is done in the first part of Proposition 2. Building on this result, the second and third part transfer the findings of Lemma 2 to the solution of the constrained problem $P_y$.

**Proposition 2.** *Let $y' = y + \varepsilon e_j$, $\varepsilon > 0$, for arbitrary $j$. Then:*

*(i) $x^\star_j(y') \geq x^\star_j(y)$.*

*(ii) For arbitrary $i$ and $j$, fix $x^\star_k$ for all $k \neq i, j$ and solve (3) for $i$ and $j$. Then, there always exists one optimal solution for which $x^\star_i(y') \leq x^\star_i(y)$.*

*(iii) Solve (3) for $k = 1, \ldots, N$. There are instances of $P_y$ for which $x^\star_i(y') > x^\star_i(y)$ for some $i \neq j$.*

In essence, Proposition 2 highlights that the wholesaler's best-response is not necessarily monotonic in a manufacturer's production decision. The reason for this is the multidimensionality of substitution which comprises of direct and indirect effects. If the available production quantity for one product $j$ is increased, (i) and (ii) indicate that the wholesaler increases his stocks for product $j$ and, considered in isolation, reduces any other stock $i \neq j$. This is the direct effect which is in line with our common understanding of economic substitutes. However, each increase or decrease in any one product's stocking quantity has immediate effects on all other products' optimal stocking levels. Hence, if the wholesaler optimizes his stocking quantities across all products, a cascade of indirect effects arises due to all products' mutual interdependency. We find that in some situations these indirect effects dominate the direct effects so that, in optimum, the wholesaler may increase stocking levels for more than one product (iii). Indirect effects are dominant if, e.g., the market's substitution structure is heterogeneous in the sense that there is few direct substitution between products $j$ and $i$, but frequent substitution between products $j$ and $k$, and $k$ and $i$.

## 3.2 Substitution Effects

We now investigate the sensitivity of the wholesaler's optimal stocking quantities and expected profit with respect to changing substitution rates. A change in the customers' reaction to product stock-outs implies changing substitution rates. Naturally, this also affects the total demand for the wholesaler's products. To be specific, increasing substitution rates imply a stochastically larger total demand at the wholesaler, or mathematically, $D^s_i$ is stochastically increasing in $\alpha_{ji}$ for all $j \neq i$. Intuition suggests that this increased demand is always beneficial for the wholesaler since the probability of incurring lost sales decreases. Moreover, Netessine and Rudi (2003) conjecture intuitively that optimal stocking levels for a product increase (decrease) if substitution rates to (from) this product increase. We now test this intuition.

We start our analysis with the sensitivity of the wholesaler's expected profit. As already argued, demand is stochastically increasing in any substitution rate. Furthermore, it is well known that, on expectation, a wholesaler benefits from increased demand if trade is profitable (Li, 1992). Accordingly, the wholesaler's expected profit increases in any substitution rate. The following proposition formally states this argument.

**Proposition 3.** *Suppose $\sum_{i \neq j} \alpha_{ji} < 1$. The wholesaler's expected profit $\Pi_W(x)$ is increasing in any substitution rate $\alpha_{ji}$ if stocking quantities $x$ are adjusted optimally to changes in substitution rates.*

Note that Proposition 3 is true for the constrained and unconstrained problems $P_y$ and $P_\infty$, respectively. If $\sum_{i \neq j} \alpha_{ji} = 1$, then $\alpha_{ji}$ can only increase if at least one other substitution rate $\alpha_{jk}$, $k \neq i$, simultaneously decreases. In this case, $\Pi_W$ may actually decrease in $\alpha_{ji}$.

While the sensitivity of the wholesaler's expected profit has a monotonic behavior, we now show that, in contrast to common intuition, his optimal stocking quantities might be non-monotonic in substitution rates. As a starting point we analyze how $\hat{x}$ changes in $\alpha_{ji}$.

**Lemma 3.** *(i) For arbitrary $i$, $\partial \hat{x}_i / \partial \alpha_{ji} \geq 0$ for all $j \neq i$.*
*(ii) There are instances of $P_y$ for which $\partial \hat{x}_j / \partial \alpha_{ji} > 0$ for some $i$ and $j$.*

Ceteris paribus, $\hat{x}_i$ is monotone increasing in the substitution rates to product $i$, $\alpha_{ji}$, while $\hat{x}_j$ may change non-monotonically in the substitution rates from product $j$. Thus, Lemma 3 partially contradicts common intuition. In particular, $\hat{x}_j$ need not decrease in $\alpha_{ji}$. Thus, in a situation where (ii) holds, it is optimal for the wholesaler to limit substitution behavior by increasing initial stocking levels. Now, the results of Lemma 3 allow us to examine the total effects that $\alpha_{ji}$ exerts on $x^\star$.

**Proposition 4.** *There are instances of $P_y$ for which $dx^\star_j / d\alpha_{ji} > 0$ for some $i$ and $j$.*

For the constrained optimization problem $P_y$, Lemma 3(ii) remains valid even when including all indirect substitution dynamics and not only direct effects. A trade-off argument between sales volumes and product margins explains these non-intuitive results. With increased substitution the wholesaler achieves a higher total sales volume, but potentially at the cost of reduced sales for certain high margin products. (Note that the overall sales volume increases, but not necessarily each single product's volume.) Consider a high margin product $j$ and a low margin product $i$. To restrict substitution from product $j$ to $i$, the wholesaler may raise $x^\star_j$ even when $\alpha_{ji}$ increases. In such a situation, the wholesaler deliberately reduces his sales volume, because this negative effect is dominated by the positive effect of more expected sales of the high margin product. To conclude, Lemma 3 together with Proposition 4 indicate that the wholesaler's optimal stocking quantities are in general non-monotonic in changing substitution rates.

## 4 The Supply Game

In this section, we analyze the manufacturer's optimal production quantities under incomplete information about the wholesaler's stocking levels. We first focus on the competitive scenario with multiple Stackelberg leaders and then investigate the situation with a single Stackelberg leader. Afterwards, we compare the optimal production quantities for both scenarios and illustrate our findings with a numerical example.

The Ordering Game which takes production quantities $y$ as given is the second stage of the Supply Game. In the first stage, manufacturers choose $y$ to maximize their expected profits given their beliefs about the wholesaler's subsequent behavior. The manufacturers' unit production cost and selling price for product $i$ are $c_i$ and $w_i$, respectively, with $w_i > c_i > 0$, $i \in \{1, \ldots, N\}$. We assume that manufacturers credibly and simultaneously announce their production quantities $y_i$. Further, $y_i \in [0, K]$, with $K$ sufficiently large so that it never constrains any manufacturer. Since the wholesaler has private information on his optimal stocking quantities, manufacturers can only hold a belief about the wholesaler's equilibrium stocking levels. We explicitly model this uncertainty about the wholesaler's orders as a random variable which depends on the chosen production quantities $y$. To be specific, let $\chi_i \in X_i(y)$ with cumulative distribution $\Phi_i(\chi_i, y)$ and density $\phi_i(\chi_i, y) > 0$. We assume $\Phi_i(\chi_i, y)$ to be twice continuously differentiable in all arguments $y$ and define $\mu_i(y) \equiv \int_{X_i(y)} \chi_i d\Phi_i(\chi_i, y)$. We restrict attention to rational beliefs.

**Definition 1.** *We say that a manufacturer's belief about the wholesaler's stocking quantities is rational if it satisfies the following conditions for all products $i$:*

1. $X_i(y) = [0, y_i]$;
2. $\partial^2 \Phi_i(\chi_i, y)/\partial y_i \partial y_j \geq 0$, $j \neq i$;
3. $\partial \Phi_i(\chi_i, y)/\partial y_i \leq 0$ *and* $\partial^2 \Phi_i(\chi_i, y)/\partial y_i^2 \geq 0$.

Definition 1 ensures three structural properties of a manufacturer's belief. First, manufacturers assign a positive probability mass only to non-negative stocking levels which are naturally bounded from above by the chosen production quantity $y_i$. Second, ceteris paribus, manufacturers consider all products to be economic substitutes. Third, production quantities exert a stimulating effect on the wholesaler's stocking decision, i.e. stocking levels stochastically increase with the available production quantities, but at a decreasing rate (for a thorough discussion on stimulating effects of inventories, see Balakrishnan et al., 2008).

We emphasize that Definition 1 imposes very mild restrictions on a manufacturer's belief. The wholesaler, by construction, never orders more than $y$. Therefore, the first property is in line with the results of Proposition 1. The second property ensures that manufacturers correctly believe that they compete in a substitution market. Finally, the third property follows immediately from Proposition 2(i) which states that $x_i^\star(y)$ increases in $y_i$. Irrespective of the kind of information asymmetries, any rational manufacturer can always predict these properties, only the magnitude of these effects may be unknown to her. Note that we neither require beliefs to be correct on expectation, nor do we make any assumption on how the belief for product $i$ changes with $y_j$, since Propositions 2(ii) and (iii) indicate that $x_i^\star(y)$ can increase or decrease in $y_j$.

The manufacturer's decision problem structurally differs in two ways from the wholesaler's optimization problem. First, the wholesaler's reaction to limited production quantities is fundamentally different from the customers' reaction to stock-outs. While customers only try to substitute once with a given probability, the wholesaler's reaction to short production capacities is based on a non-monotonic optimization strategy across all products. Second, the manufacturer can influence the wholesaler's stocking quantity for product $i$ by changing $y_i$, whereas the wholesaler cannot influence customer demand for product $i$ by varying $x_i$.

### 4.1 Competing Manufacturers

We now establish the equilibrium of the first stage of the Supply Game when there are $N$ competing manufacturers, each selling a different, yet partially substitutable product through a monopolistic wholesaler. Before the wholesaler communicates his stocking quantities, manufacturers simultaneously choose their production levels. Accordingly, manufacturers act as Bayesian Stackelberg leaders with respect to the wholesaler, but as Nash competitors with respect to other manufacturers. Thus, each manufacturer maximizes her expected profit, given the other manufacturers' production quantities and given her rational beliefs about the wholesaler's subsequent reaction. Her decision problem for given $y_{-i}$ is

$$\max_{y_i \geq 0} \Pi_{M_i}(y_i|y_{-i}) = w_i \mu_i(y) - c_i y_i, \tag{4}$$

where $\Pi_{M_i}(y_i|y_{-i})$ is the $i$th manufacturer's expected profit. For brevity, let $\Pi_{M_i} = \Pi_{M_i}(y_i|y_{-i})$ and denote by $y_i^c = \arg\max_{y_i \geq 0} \Pi_{M_i}$ the $i$th manufacturer's best-response to her competitors' production quantities $y_{-i}$.

We start our equilibrium analysis by noting that rational beliefs are sufficient to guarantee concavity of each manufacturer's expected profit.

**Lemma 4.** *Assume rational beliefs. Given $y_{-i}$, $\Pi_{M_i}$ is a concave function of the production quantity $y_i$ for all $i$.*

Due to the concavity of $\Pi_{M_i}$ we can derive each manufacturer's best-response $y_i^c$ by examining the first-order conditions which provide necessary and sufficient optimality conditions.

**Proposition 5.** *Assume rational beliefs. The following system of necessary first-order optimality conditions characterizes any manufacturer Nash equilibrium:*

$$\left.\frac{\partial \mu_i(y)}{\partial y_i}\right|_{y=y^c} = \frac{c_i}{w_i}, \tag{5}$$

$i = 1, \ldots, N$.

A simple trade-off argument explains the optimality conditions (5). On expectation, increasing the production level raises the wholesaler's stocking level. This generates a marginal increase in revenue given by $w_i \partial\mu_i(y)/\partial y_i$, while simultaneously inducing a marginal cost of $c_i$. Equating marginal revenue and marginal costs provides the desired result. Note that $y_i^c$ constitutes an upper bound on the wholesaler's decision space. Hence, in any case, the wholesaler's stocking level is smaller than $y_i^c$.

Naturally, (5) not only determines each manufacturer's best-response in the manufacturer Nash game, i.e. in the competition among leaders, but also persists in the entire Bayesian Nash-Stackelberg game. Here, any Bayesian Nash-Stackelberg equilibrium is given by the wholesaler's optimal stocking levels $x^\star(y^c)$ and the manufacturers' production quantities $y^c$ which form a Nash equilibrium in the manufacturer Nash game. In a next step, we establish existence and uniqueness of the manufacturer Nash equilibrium.

**Proposition 6.** *Assume rational beliefs. For the competitive scenario, a pure-strategy manufacturer Nash equilibrium exists and is found by solving (5). If $\Pi_{M_i}$ is strictly concave in $y_i$ and*

$$2 + \sum_{j \neq i} \frac{\partial y_i^c}{\partial y_j} - \sum_{j \neq i} \frac{\partial^2 \mu_j(y)/\partial y_i \partial y_j}{\partial^2 \mu_i(y)/\partial y_i^2} > 0, \tag{6}$$

*$i = 1, \ldots, N$, for all $y$, then the manufacturer Nash equilibrium is unique.*

Proposition 6 states two sufficient conditions for uniqueness of the manufacturer Nash equilibrium. Each manufacturer's expected profit $\Pi_{M_i}$ is strictly concave in $y_i$ if and only if each manufacturer's belief satisfies $\partial^2\Phi_i(\chi_i, y)/\partial y_i^2 > 0$. Further note that a necessary condition for (6) to hold is given by $\sum_{j \neq i} |\partial y_i^c/\partial y_j| < 2$. Intuitively, the sensitivity of each manufacturer's best-response with respect to the other manufacturers' production decisions should be bounded.

A special case where (6) is automatically satisfied occurs if the effects of $y_i$ and $y_{-i}$ on $\mu_i(y)$ are additive separable, i.e. $\mu_i(y) = g_i(y_i) + h_i(y_{-i})$ for arbitrary differentiable functions $g_i$ and $h_i$. If $g_i$ is furthermore strictly concave, then the manufacturer Nash equilibrium is unique.

While Proposition 6 ensures uniqueness of the manufacturer Nash equilibrium, the stated conditions are not sufficient to generally guarantee uniqueness of the Bayesian Nash-Stackelberg equilibrium. As discussed in §3, the wholesaler's optimal stocking levels given the manufacturers' production quantities are not necessarily unique. Consequently, the wholesaler might have multiple best-responses. Accordingly, to gain a unique equilibrium in the Supply Game, the wholesaler's optimal stocking quantities must be unique. Corollary 1 states a simple condition that guarantees uniqueness.

**Corollary 1.** *Let the conditions of Proposition 6 hold. Suppose $\Pi_W(x)$ is jointly concave in $x$. Then, the Supply Game has a unique Bayesian Nash-Stackelberg equilibrium in the competitive scenario.*

### 4.2 Monopolistic Manufacturer

As a benchmark, we now derive the Bayesian Stackelberg equilibrium of the Supply Game without manufacturer competition. To be specific, a monopolistic manufacturer simultaneously produces all $N$ substitutable products and sells them through a monopolistic wholesaler. Therefore, the manufacturer serves as Bayesian Stackelberg leader with respect to the wholesaler. Thus, she maximizes her expected profit $\Pi_M$ across all products given her belief about the wholesaler's subsequent stocking levels. Her decision problem is

$$\max_{y \geq 0} \Pi_M(y) = \sum_i w_i \mu_i(y) - c_i y_i. \tag{7}$$

For given rational beliefs, denote by $y^{nc} = \arg\max_{y \geq 0} \Pi_M$ a vector of optimal production quantities. In contrast to the competitive scenario, the manufacturer's expected profit $\Pi_M$ is not generally concave in $y$. Thus, first-order optimality conditions provide only necessary, but not sufficient conditions for the manufacturer's optimal production quantities.

**Proposition 7.** *Assume rational beliefs. In any Bayesian Stackelberg equilibrium of the non-competitive scenario, the manufacturer's production quantities satisfy the system of first-order necessary optimality conditions*

$$\left. \frac{\partial \mu_i(y)}{\partial y_i} + \sum_{j \neq i} \frac{w_j}{w_i} \frac{\partial \mu_j(y)}{\partial y_i} \right|_{y = y^{nc}} = \frac{c_i}{w_i}, \tag{8}$$

$i = 1, \ldots, N$.

Analogously to the optimality conditions of the competitive scenario, the monopolistic manufacturer's optimal decision also follows a trade-off argument. Again, the manufacturer equates marginal costs and marginal revenues. This time, the shift in revenue accounts not only for the increased revenue for product $i$, but also for the decreased revenue for all other products $j \neq i$. Intuitively, the monopolistic manufacturer considers the influence of her production quantities on the revenue for all products, whereas each competitive manufacturer only cares about her own product.

Neither the manufacturer's optimal production quantities $y^{nc}$ nor the wholesaler's optimal stocking levels $x^\star(y^{nc})$ are necessarily unique. In consequence, the Bayesian Stackelberg equilibrium of the Supply Game is not guaranteed to be unique. A sufficient condition for uniqueness is given in Corollary 2.

**Corollary 2.** *Suppose $\Pi_W(x)$ and $\Pi_M(y)$ are jointly concave in $x$ and $y$, respectively. Then, the Supply Game has a unique Bayesian Stackelberg equilibrium in the non-competitive scenario.*

## 4.3 The Consequences of Manufacturer Competition

Intuitively, competing manufacturers adopt production quantities $y^c$ that differ substantially from a monopolistic manufacturer's production quantities $y^{nc}$ even though they may hold identical beliefs about the wholesaler's subsequent stocking levels. In this context, the natural question arises whether competition causes manufacturers to increase production quantities, i.e., $y^c > y^{nc}$? Furthermore, vertical information asymmetries induce supply chain inefficiencies that manifest in end-of-season inventories at the manufacturer. However, are these effects smaller or larger under upstream competition? We now explore these issues.

Intuition suggests that the wholesaler prefers competing manufacturers to a monopolistic manufacturer because we expect production quantities to increase under competition. Hence, the wholesaler's decision space is less restricted under manufacturer competition and so, he can provide a more profitable service level to his customers. Proposition 8 shows that this intuition is not always true.

**Proposition 8.** *For given rational beliefs, the relationship between $y^c$ and $y^{nc}$ is as follows:*

*(i) If*

$$\sum_{j \neq i} w_j \frac{\partial \mu_j(y)}{\partial y_i} \leq 0 \tag{9}$$

*for all products $i$, then $y_i^c \geq y_i^{nc}$ for at least one product $i$.*

*(ii) There are rational beliefs such that $y_i^c < y_i^{nc}$ for some product $i$.*

It can never happen that all production quantities decrease under competition, if (9) holds. This condition ensures that each product has in total a negative effect on the other products, which is the nature of substitute products. A sufficient condition for (9) are rational beliefs that additionally satisfy $\partial \Phi_i(\chi_i, y)/\partial y_j \geq 0$ for all $j \neq i$, or intuitively, each product $j$ should exert a negative influence on every other product $i$. Note that Proposition 2(iii) indicates that this need not be true for all products. There exist situations where two products have a positive effect on each other, i.e. $\partial \Phi_i(\chi_i, y)/\partial y_j < 0$ for some $i$ and $j$. Condition (9) also captures these contingencies because we only require the weighted sum over all effects to be negative, not each single effect. We propose that any product that violates (9) is no longer an economic substitute, but rather an economic complement for the other products.

An availability trade-off explains why a monopolistic manufacturer sometimes stocks more than a competitive manufacturer (ii). A monopolistic manufacturer can coordinate the availability of all products, i.e. she can optimally build large stocks of a product $i$, while simultaneously decreasing the availability for products $j \neq i$. Under competition, a manufacturer cannot accomplish this availability trade-off since she cannot force her competitors to reduce production quantities. This contingency occurs for a product $i$ if, e.g., manufacturers believe that $y_i$ exerts only a limited influence on the wholesaler's stocking decision for the other products $x^\star_{-i}$. Markets with such a heterogeneous substitution structure typically include no-name and brand products (Ailawadi and Keller, 2004) or heterogeneous products.

If the effects of $y_i$ and $y_{-i}$ on $\mu_i(y)$ are additive separable for all $i$, or if (9) holds and all products are homogeneous and symmetric, then production increases under competition for all products, i.e. $y^c \geq y^{nc}$.

Note that the results of Proposition 8 are similar to the findings of Netessine and Rudi (2003) for competition among wholesalers. However, these two results are based on different problem characteristics because the wholesaler's and manufacturer's problem differ structurally in numerous ways. In particular, substitution dynamics and demand characteristics are completely different. Therefore, Proposition 8 establishes the transferability of the previous results to the manufacturer stage.

Naturally, as manufacturers' production quantities change under competition, the wholesaler also adjusts his stocking quantities. This implies that end-of-season inventories at the manufacturer, i.e., excess inventories after trading, change if competition is introduced. Note that these residual inventories are a direct consequence of the vertical information asymmetry within the supply chain. If manufacturers could perfectly determine the wholesaler's best-response stocking quantities, they would never produce more than this quantity. Accordingly, manufacturers would never incur end-of-season inventories. Therefore, we now examine the change in manufacturers' end-of-season inventories under competition in the case of information asymmetries. We denote the end-of-season inventory level of product $i$ at the manufacturer by $I_i(y) = y_i - x_i^\star(y)$.

**Proposition 9.** *Let $y' \geq y$. Then, the following relations between $I(y')$ and $I(y)$ hold:*

*(i) $I_i(y') \geq I_i(y)$ for at least one product $i$.*

*(ii) There are instances of the Supply Game where $I_i(y') < I_i(y)$ for some product $i$.*

The wholesaler is always less restricted in his decision under $y'$ than under $y$. This reflects, e.g., a situation where all production quantities increase under upstream competition. Even though all production quantities (weakly) increase, end-of-season inventories for some (ii), but not all (i) products may decrease. In such a case, the wholesaler increases his stocking quantity for product $i$ more than the manufacturer increases $y_i$. This behavior is closely related to the findings of Proposition 2. Thus, indirect substitution dynamics at the wholesaler can lead to such a disproportionate adjustment of stocking levels.

### 4.4 Numerical Illustration

We now provide a small numerical example to illustrate our theoretical findings. Consider a market with three substitutable products. For the sake of analytical tractability, suppose that each manufacturer believes that the wholesaler's stocking quantities follow a truncated exponential distribution with support on $[0, y_i]$ and rate parameter $\lambda_i(y)$, i.e. $\Phi_i(\chi_i, y) = [1 - exp(-\lambda_i(y)\chi_i)]/[1 - exp(-\lambda_i(y)y_i)]$. Note that our framework also works for any other common distribution such as truncated Normal, Gamma, or Weibull distributions, but at the cost of analytical tractability.

| Scenario | Parameters | | | | | | | | | Optimal decision | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $w_1$ | $w_2$ | $w_3$ | $k_{21}$ | $k_{31}$ | $k_{12}$ | $k_{32}$ | $k_{13}$ | $k_{23}$ | $y_1^c$ | $y_2^c$ | $y_3^c$ | $y_1^{nc}$ | $y_2^{nc}$ | $y_3^{nc}$ |
| A | 8 | 8 | 8 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.70 | 0.70 | 0.70 | 0.61 | 0.61 | 0.61 |
| B | 8 | 8 | 8 | 0 | 0 | 0.5 | 0.5 | 0.5 | 0.5 | 1.20 | 0.63 | 0.63 | 1.00 | 0.60 | 0.60 |
| C | 8 | 8 | 8 | 0.5 | 0.5 | 0 | 0.5 | 0 | 0.5 | 0.65 | 0.84 | 0.84 | 0.71 | 0.69 | 0.69 |
| D | 8 | 8 | 8 | 0 | 0 | 0.5 | 0 | 0.5 | 0 | 1.20 | 0.75 | 0.75 | 0.81 | 0.85 | 0.85 |
| E | 10 | 8 | 8 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 1.00 | 0.66 | 0.66 | 1.07 | 0.45 | 0.45 |
| F | 11.9 | 8 | 8 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 1.24 | 0.62 | 0.62 | 1.89 | 0.06 | 0.06 |
| G | 11.9 | 10 | 7 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 1.20 | 0.91 | 0.44 | 1.44 | 0.59 | 0.05 |

Table 1: Optimal production decisions.

Following Definition 1, beliefs about the wholesaler's stocking level for product $i$ should be stochastically increasing in $y_i$. Thus, each rate parameter $\lambda_i(y)$ is a function of the manufacturers' production quantities which decreases in $y_i$. To be specific, we employ the following simple structural form: $\lambda_i(y) = y_i^{-1} + \sum_{j \neq i} k_{ji} y_j + 1$. By setting $k_{ji} \geq 0$ we ensure that the other requirements of Definition 1 are met. We work with the inverse of $y_i$ and not with $-y_i$ to ensure non-negativity of $\lambda_i(y)$. Intuitively, each scale parameter $k_{ji}$ reflects the magnitude of influence that $y_j$ exerts on the wholesaler's stocking decision for product $i$.

The truncated exponential distribution together with the specification of $\lambda_i(y)$ ensures that each manufacturer holds rational beliefs as described in Definition 1. It is readily shown that $\mu_i(y) = [1/\lambda_i(y)] - [y_i exp(-\lambda_i(y) y_i)/(1 - exp(-\lambda_i(y) y_i))]$. Thus, the influence of $y_i$ and $y_{-i}$ on $\mu_i(y)$ is not additive separable. For all investigated scenarios, we assume $c_i = 2$ for all $i$. All other parameter values $w_i$ and $k_{ji}$ are given in Table 1. Parameters include high and low margin cases, and high and low substitution rates. Note that for all displayed parameter values, a unique Bayesian (Nash-)Stackelberg equilibrium exists. For each scenario, we display the optimal production decisions for both supply chain configurations.

Obviously, in a market with symmetric price and substitution structure, production quantities increase if manufacturer competition is introduced (A). In our example, this result remains valid if there is no substitution *to* one product in the assortment (B). If instead one product does not influence the other products, i.e., there is no substitution *away* from the product, then production levels decrease for this product under competition (C,D). In such a scenario, a monopolistic manufacturer optimally increases the availability of the product at the cost of decreasing the other products' availability. In a competitive environment, a manufacturer cannot coordinate product availability across multiple products because her competitors are reluctant to lose market shares. In the agrochemical market, these heterogeneous substitution structures arise due to the coexistence of single- and multi-purpose products. While single-purpose products are specialized to fight a single plant disease such as mildew, multi-purpose products are effective against a wider class of diseases. Naturally, substitution from the specialized to the more general product is likely to occur, because the specialized product lies within the application range of the general product. In contrast, the specialized product need not be useful for a customer initially desiring the general product.

In our example, production quantities for high margin products decrease under competition, while production increases for low and medium margin products (E,F,G). We observe this behavior because a monopolistic manufacturer shifts as much demand as possible to the high margin products, thereby reducing the other products' availability to a minimum. In contrast, a similar demand shift cannot be accomplished under competition. Note that under a monopolistic manufacturer, low margin products almost disappear from the market, while competition ensures product diversity (F,G). Concurrent with intuition, overall production increases with the introduction of manufacturer competition.

# 5 Discussion and Conclusions

In this paper, we analyzed the optimal production and stocking decisions of a manufacturer and a wholesaler in a two-stage supply chain with upstream competition and vertical information asymmetries. We characterize the wholesaler's equilibrium stocking levels and show that these quantities are non-monotonic in both, available production quantities and customer substitution rates. For the upstream stage of the supply chain, we derive the equilibrium production levels of a monopolistic and a competitive manufacturer, respectively. We find that production levels for some products decrease if upstream competition is introduced. Furthermore, we highlight the counterintuitive situation that some end-of-season inventories at the manufacturer decrease although initial production levels increase.

## 5.1 Robustness

We now discuss the robustness of our results with respect to changes in the information and supply chain structure. Additionally, we delineate opportunities for future research.

Concerning the *information structure*, we assume that (i) manufacturers' production quantities $y$ are verifiable, and (ii) $\Phi_i(\chi_i, y)$ is differentiable in $y$. Verifiability of $y$ ensures that the wholesaler determines his stocking quantities under complete information about the manufacturer's strategy. Consequently, we can ignore communication issues between manufacturer and wholesaler. This is not true if $y$ is unverifiable and thus privately observed by the manufacturer. In this case, the manufacturer's equilibrium behavior consists of her production and communication strategy, which introduces an additional inference problem for the wholesaler. Under strategic communication, the manufacturer need not pursue a truth-telling strategy or she may not communicate any information at all, which inherently changes the timing of the game to simultaneous moves. Whether the structure of our results remains valid under such a scenario, or not, is an interesting question for future research.

We further assume that a manufacturer's belief $\Phi_i(\chi_i, y)$ about the wholesaler's optimal stocking quantities $x_i^\star(y)$ is differentiable with respect to $y$. This is a common assumption (Cachon and Lariviere, 1999; Özer and Wei, 2006), but clearly, it is not ensured that, in equilibrium, $x_i^\star(y)$ is actually differentiable. Nevertheless, it is guaranteed that $x_i^\star(y)$ is continuous in

$y$. For such a situation, Cachon and Lariviere (1999) show numerically that the differentiability assumption provides an excellent approximation. We therefore expect our results to be robust with respect to differentiability of beliefs.

Concerning the *supply chain structure*, we assume that competition occurs only among manufacturers. This assumption is inspired by our observations in the agrochemical market, but obviously, a general extension of our framework is to allow for downstream competition as well. Such an extension introduces two new issues that need to be incorporated into the model. First, manufacturers need to decide on allocation mechanisms for their production quantities in case that total orders exceed the available production quantities. Second, these allocation schemes induce strategic ordering behavior of the wholesalers. The influence of these allocation problems on supply chains in substitution markets should be a focal point of future work.

Additionally, under downstream competition, the assumption that $\Phi_i(\chi_i, y)$ is differentiable in $y$ becomes much more problematic. At some point, competition among heterogeneous wholesalers can induce some competitors to leave the market. Generally, such a market exit results in discontinuities in the stocking quantities of the remaining competitors. Therefore, the differentiability assumption provides a less reliable approximation. Nevertheless, we expect that such an approximation yields structurally valid results, even under downstream competition.

To deepen our understanding of the repercussions that substitution exerts on the individual supply chain members, more fundamental extensions should also be examined. In particular, we believe that future models should also incorporate pricing decisions, but this might come at the expense of analytical tractability. Another aspect that deserves future research is the introduction of multiple time periods. In such a setting, initial product demand changes dynamically over time because there is a probability of a substituting customer changing his product preferences due to product unavailability.

## 5.2 Concluding Remarks

Our analysis demonstrates that substituting customers affect the production and stocking decisions within a supply chain in non-monotonic and partially counterintuitive ways. Thus, intuition may fail to capture all relevant substitution dynamics and this effect becomes stronger the more heterogeneous the competing products are. While in completely homogeneous (symmetric) markets intuition correctly predicts each supply chain member's behavior, intuitive reasoning is prone to crucial misinterpretations as soon as the market becomes heterogeneous. Reasons for such heterogeneities are widely spread in reality and can be found in terms of profit margins, brands, and product and demand characteristics.

The agrochemical market, e.g., is shaped by these heterogeneities. Brand manufacturers and (former) patent holders compete with generic products, which oftentimes differ in price and profit margins. Furthermore, the market's substitution structure is skewed due to the coexistence of single- and multi-purpose products. Hence, in such a heterogeneous market, it is very important to understand the substitution structures among products to take the right decisions.

## Appendix

*Proof of Lemma 1.* For given $x_{-i}$, the first-order and second-order derivatives of $\Pi_W(x)$ with respect to $x_i$ are

$$\begin{aligned}\frac{\partial \Pi_W(x)}{\partial x_i} =& u_i - (u_i+o_i)\mathbb{P}(D_i^s < x_i) - \sum_{j\neq i}(u_j+o_j)\alpha_{ij}\mathbb{P}(D_j^s < x_j, D_i > x_i)\\ =& u_i - (u_i+o_i)\mathbb{P}(D_i^s < x_i) - \sum_{j\neq i}(u_j+o_j)\alpha_{ij}\mathbb{P}(D_j^s < x_j | D_i > x_i)\mathbb{P}(D_i > x_i)\\ \frac{\partial^2 \Pi_W(x)}{\partial x_i^2} =& -(u_i+o_i)f_{D_i^s}(x_i)\\ &+\sum_{j\neq i}(u_j+o_j)\alpha_{ij}\left[f_{D_i}(x_i)\mathbb{P}(D_j^s < x_j|D_i > x_i) - \alpha_{ij}f_{D_j^s|D_i>x_i}(x_j)\mathbb{P}(D_i > x_i)\right],\end{aligned}$$

$i = 1, \ldots, N$, with $f_Y$ being the density function of random variable $Y$. By rearranging terms, $\Pi_W(x)$ is concave in $x_i$ if and only if

$$(u_i+o_i)f_{D_i^s}(x_i) + \sum_{j\neq i}(u_j+o_j)\alpha_{ij}^2 f_{D_j^s|D_i>x_i}(x_j)\mathbb{P}(D_i > x_i) \geq \sum_{j\neq i}(u_j+o_j)\alpha_{ij}f_{D_i}(x_i)\mathbb{P}(D_j^s < x_j|D_i > x_i) \tag{10}$$

for all $x$. To prove the lemma, we construct a scenario for which (10) is violated for some $x$.

Let $\eta > 0$, and for given $x_i$, let $X_\eta(x_i)$ be the set of stocking quantities $x_{-i}$ such that $\mathbb{P}(D_j^s < x_j|D_i > x_i) \geq 1/(N-1)$ and $f_{D_j^s|D_i>x_i}(x_j) < \eta$. Note that for any $x_i$, $X_\eta(x_i)$ is non-empty because $\mathbb{P}(D_j^s < x_j|D_i > x_i) \to 1$ and $f_{D_j^s|D_i>x_i}(x_j) \to 0$ for $x_j \to \infty$. For all $j \neq i$, let (i) $\alpha_{ji} = 0$, i.e., $D_i^s =_{st} D_i$; (ii) $\alpha_{ij} = 1/(N-1)$; and (iii) $(u_j+o_j) = (1+\nu)(u_i+o_i)(N-1)$, $\nu > 0$. Further assume that $D_i \sim Normal(\mu_i, \sigma_i)$ with $\sigma_i < \nu/\left[(1+\nu)\eta\sqrt{2\pi}\right]$.

Given these assumptions,

$$(u_i+o_i)\left[f_{D_i}(x_i) + (1+\nu)\eta\right] > (u_i+o_i)f_{D_i^s}(x_i) + \sum_{j\neq i}(u_j+o_j)\alpha_{ij}^2 f_{D_j^s|D_i>x_i}(x_j)\mathbb{P}(D_i > x_i) \tag{11}$$

and

$$\sum_{j\neq i}(u_j+o_j)\alpha_{ij}f_{D_i}(x_i)\mathbb{P}(D_j^s < x_j|D_i > x_i) \geq (u_i+o_i)(1+\nu)f_{D_i}(x_i). \tag{12}$$

By (10)-(12), it follows that $\Pi_W(x)$ is not concave in $x_i$, if for some $x_i$,

$$(1+\nu)f_{D_i}(x_i) > \left[f_{D_i}(x_i) + (1+\nu)\eta\right], \tag{13}$$

or equivalently,

$$f_{D_i}(x_i) > \frac{1+\nu}{\nu}\eta. \tag{14}$$

Since $D_i$ is normally distributed, we can choose $x_i$ such that $f_{D_i}(x_i) = 1/(\sigma_i\sqrt{2\pi})$ and hence, (14) holds for any $\sigma_i < \nu/\left[(1+\nu)\eta\sqrt{2\pi}\right]$. □

*Proof of Proposition 1.* Consider the maximization problem $P_y$. Since $\Pi_W(x)$ and all constraints are continuously differentiable in $x$ and all constraints are linear in $x$, there exists a unique vector $\lambda$ such that $(x^\star, \lambda)$ satisfies the Karush-Kuhn-Tucker (KKT) conditions:

$$\frac{\partial \Pi_W(x^\star)}{\partial x_i} - \lambda_i = 0 \tag{15}$$

$$\lambda_i(x_i^\star - y_i) = 0 \tag{16}$$

$$x_i^\star - y_i \leq 0 \tag{17}$$

$$x^\star, \lambda \geq 0, \tag{18}$$

$i = 1, \ldots, N$. Now, suppose $x^\star$ is a partially largest optimal solution.

*Case 1:* $x_i^\star < y_i$. For (16) to hold, we need $\lambda_i = 0$, which implies by (15) and (2) that $x_i^\star = \hat{x}_i(x_{-i}^\star)$.

*Case 2:* $x_i^\star = y_i$. We need to show that $y_i \leq \hat{x}_i(x_{-i}^\star)$. Suppose to the contrary that there exist situations where $x_i^\star = y_i > \hat{x}_i(x_{-i}^\star)$. By (2), $\hat{x}_i(x_{-i}^\star)$ is the wholesaler's optimal stocking quantity if he is unrestricted in his stocking decision for product $i$. Now, if this stocking quantity is also feasible for the bounded problem $P_y$, then it must also be optimal in $P_y$. Thus, $x_i^\star = \hat{x}_i(x_{-i}^\star) < y_i = x_i^\star$ which is a contradiction.

Combining Case 1 and 2 for all $i$ yields $x_i^\star(y) = \min\{\hat{x}_i(x_{-i}^\star(y)), y_i\}$. □

*Proof of Lemma 2.* Given $x_{-i}$, the wholesaler's optimization problem is now one-dimensional in $x_i$. Thus, to analyze how $\hat{x}_i(x_{-i})$ changes in $x_j$, $j \neq i$, we apply the Implicit Function Theorem to gain the required differential

$$\frac{\partial \hat{x}_i(x_{-i})}{\partial x_j} = -\frac{\partial^2 \Pi_W(\hat{x}_i, x_{-i})/\partial x_i \partial x_j}{\partial^2 \Pi_W(\hat{x}_i, x_{-i})/\partial x_i^2}.$$

Due to the optimality of $\hat{x}_i(x_{-i})$, we know that $\partial^2 \Pi_W(\hat{x}_i, x_{-i})/\partial x_i^2 \leq 0$. Furthermore, analysis of the cross-partial yields

$$\frac{\partial^2 \Pi_W(\hat{x}_i, x_{-i})}{\partial x_i \partial x_j} = -(u_i + o_i)\frac{\partial}{\partial x_j}\mathbb{P}(D_i^s < \hat{x}_i) - \sum_{k \neq i}(u_k + o_k)\alpha_{ik}\frac{\partial}{\partial x_j}\mathbb{P}(D_k^s < x_k | D_i > \hat{x}_i)\mathbb{P}(D_i > x_i).$$

By construction, $D_k^s$, $k \neq j$, is stochastically decreasing in $x_j$ and so, $\partial \mathbb{P}(D_i^s < \hat{x}_i)/\partial x_j \geq 0$ and $\partial \mathbb{P}(D_k^s < x_k | D_i > \hat{x}_i)/\partial x_j \geq 0$ for all $k \neq i, j$. Additionally, $D_j^s$ does not dependent on $x_j$ and therefore $\partial \mathbb{P}(D_j^s < x_j | D_i > \hat{x}_i)/\partial x_j \geq 0$. Combining these arguments gives

$\partial^2 \Pi_W(\hat{x}_i, x_{-i})/\partial x_i \partial x_j \leq 0$ and finally

$$\frac{\partial \hat{x}_i(x_{-i})}{\partial x_j} \leq 0.$$

Thus, it follows that $\hat{x}_i(x_{-i}) \geq \hat{x}_i(x'_{-i})$.

(ii) Consider a three-product scenario with products denoted by $i$, $j$, and $k$, respectively, and suppose that the density functions of $D_i$, $D_j$, and $D_k$ are strictly positive on $\mathbb{R}^+$. This implies that the inequality in Part (i) is strict because $\partial^2 \Pi_W(\hat{x}_i, x_{-i})/\partial x_i \partial x_j < 0$. Assume $\alpha_{jk} > 0$, $\alpha_{ki} > 0$, and any other substitution rate to be zero. Note that $\hat{x}_i(x_j)$ depends on $x_j$ only indirectly through $\hat{x}_k(x_j)$. We now prove the lemma by a sequential argument.

First, we analyze the direct effects between the three products. By Part (i), $x'_j > x_j$ implies $\hat{x}_k(x'_j) < \hat{x}_k(x_j)$, and thus $\hat{x}_i(x'_j) > \hat{x}_i(x_j)$. Second, to complete the proof, we need to show that an increased stocking quantity for product $i$ also leads to a decreased stocking quantity for $k$, but this is again just an application of Part (i).

Accordingly, since direct and indirect substitution effects point in the same direction, we can conclude that $\hat{x}_i(x_j) < \hat{x}_i(x'_j)$. □

*Proof of Proposition 2.* (i) Suppose $x^\star_j(y') < x^\star_j(y)$. This can never happen because $x^\star_j(y')$ is feasible in $P_y$, but by assumption, it is dominated in $P_y$ by $x^\star_j(y)$. This must also be true in $P_{y'}$ because any feasible solution of $P_y$ is feasible in $P_{y'}$. Thus, $x^\star_j(y')$ cannot be optimal in $P_{y'}$. This is a contradiction and therefore $x^\star_j(y') \geq x^\star_j(y)$.

(ii) By Part (i) and Lemma 2(i), it is always true that $\hat{x}_i(x^\star_j(y), x^\star_{-j}) \geq \hat{x}_i(x^\star_j(y'), x^\star_{-j})$. It follows immediately that $x^\star_i(y) = \min\{\hat{x}_i(x^\star_j(y), x^\star_{-j}), y_i\} \geq \min\{\hat{x}_i(x^\star_j(y'), x^\star_{-j}), y_i\} = x^\star_i(y')$.

(iii) Assume $y_i$ large enough so that it never constrains the wholesaler. This assumption ensures the applicability of Lemma 2 because we are guaranteed to find an interior solution to the wholesaler's optimization problem. Hence, by Part (i) and Lemma 2(ii), there exist situations where $\hat{x}_i(x^\star_{-i}(y)) < \hat{x}_i(x^\star_{-i}(y'))$ for some $i \neq j$. Thus,

$$x^\star_i(y) = \min\{\hat{x}_i(x^\star_{-i}(y)), y_i\} = \hat{x}_i(x^\star_{-i}(y)) < \hat{x}_i(x^\star_{-i}(y')) = \min\{\hat{x}_i(x^\star_{-i}(y')), y_i\} = x^\star_i(y')$$

for some $i \neq j$. □

*Proof of Proposition 3.* The total differential of $\Pi_W(x)$ with respect to substitution rates is

$$\frac{d\Pi_W(x^\star(\alpha_{ji}), \alpha_{ji})}{d\alpha_{ji}} = \frac{\partial \Pi_W}{\partial \alpha_{ji}} + \sum_k \frac{\partial \Pi_W}{\partial x^\star_k} \frac{\partial x^\star_k}{\partial \alpha_{ji}}.$$

In a first step, we show that $\partial \Pi_W / \partial \alpha_{ji} \geq 0$ for all $i$ and $j$, $i \neq j$, i.e.

$$\frac{\partial \Pi_W}{\partial \alpha_{ji}} = (u_i + o_i)\mathbb{E}\left[(D_j - x_j)1_{\{D^s_i < x_i, D_j > x_j\}}\right] \geq 0. \tag{19}$$

This holds true, since the term under the expectation in (19) is non-negative.

In a second step, we investigate the indirect effects of $\alpha_{ji}$ on $\Pi_W$. If $x$ is optimally adjusted, then, for all $k$, $\partial\Pi_W/\partial x_k = 0$ if $x_k^\star < y_k$ and $\partial x_k^\star/\partial\alpha_{ji} = 0$ if $x_k^\star = y_k$. Thus, $d\Pi_W/d\alpha_{ji} = \partial\Pi_W/\partial\alpha_{ji} \geq 0$ for all $i$ and $j$, if $x$ is adjusted optimally. $\square$

*Proof of Lemma 3.* (i) Choose an arbitrary product $i$. Application of the Implicit Function Theorem yields

$$\frac{\partial\hat{x}_i(\alpha)}{\partial\alpha_{ji}} = -\frac{\partial^2\Pi_W(\hat{x}(\alpha),\alpha)/\partial x_i\partial\alpha_{ji}}{\partial^2\Pi_W(\hat{x}(\alpha),\alpha)/\partial x_i^2}. \tag{20}$$

Due to the optimality of $\hat{x}(\alpha)$, we know that $\partial^2\Pi_W(\hat{x}(\alpha),\alpha)/\partial x_i^2 \leq 0$. In addition, the cross-partial $\partial^2\Pi_W/\partial x_i\partial\alpha_{ji}$ is explicitly given by

$$\frac{\partial^2\Pi_W}{\partial x_i\partial\alpha_{ji}} = -(u_i + o_i)\frac{\partial}{\partial\alpha_{ji}}\mathbb{P}(D_i^s < \hat{x}_i), \tag{21}$$

for all $j \neq i$. By construction, $D_i^s = D_i + \sum_{k\neq i}\alpha_{ki}(D_k - x_k)^+$. Thus, $D_i^s$ is stochastically increasing in $\alpha_{ji}$. It follows that $\partial\mathbb{P}(D_i^s < x_i)/\partial\alpha_{ji} \leq 0$, and hence, $\partial^2\Pi_W/\partial x_i\partial\alpha_{ji} \geq 0$. Now, by (20) and (21), $\partial\hat{x}_i/\partial\alpha_{ji} \geq 0$ for all $j \neq i$ due to the optimality of $\hat{x}(\alpha)$.

(ii) Similar to Part (i), the proof proceeds by evaluating

$$\frac{\partial\hat{x}_j(\alpha)}{\partial\alpha_{ji}} = -\frac{\partial^2\Pi_W(\hat{x}(\alpha),\alpha)/\partial x_j\partial\alpha_{ji}}{\partial^2\Pi_W(\hat{x}(\alpha),\alpha)/\partial x_j^2}. \tag{22}$$

In contrast to the proof of Part (i), the cross-partial can now be positive or negative, since

$$\frac{\partial^2\Pi_W}{\partial x_j\partial\alpha_{ji}} = -(u_i + o_i)\left[\mathbb{P}(D_i^s < \hat{x}_i, D_j > \hat{x}_j) + \alpha_{ji}\frac{\partial}{\partial\alpha_{ji}}\mathbb{P}(D_i^s < \hat{x}_i, D_j > \hat{x}_j)\right], \tag{23}$$

where $\partial\mathbb{P}(D_i^s < \hat{x}_i, D_j > \hat{x}_j)/\partial\alpha_{ji} \leq 0$.

We therefore prove the lemma by providing an example. Consider a two-product portfolio with heterogeneous initial demands $D_i \sim Uniform(0,1)$ and $D_j \sim Beta(2,1)$, i.e. $F_j(x_j) = x_j^2$. Assume all other parameters to be symmetric across products. To be concrete: $u_i = u_j = 2$, $o_i = o_j = 8$, and $\alpha_{ij} = \alpha_{ji} = 0.8$. In this setting, we obtain $\partial^2\Pi_W/\partial x_j^2 = -10\left[(x_i + x_j)^2 + x_j^2/4\right] \leq 0$, and $\partial^2\Pi_W/\partial x_j\partial\alpha_{ji} = 125x_i^3/24 \geq 0$. Consequently, $\partial\hat{x}_j/\partial\alpha_{ji} = 25/48\cdot\hat{x}_i^3/\left[(\hat{x}_i + \hat{x}_j)^2 + \hat{x}_j^2/4\right] > 0$ for $\hat{x} > 0$, which is satisfied because $\hat{x} = 0$ is not an optimum since there exist stocking quantities that yield a strictly positive profit. $\square$

*Proof of Proposition 4.* The total differential of the optimal stocking level for product $j$ with respect to substitution rates is

$$\frac{dx_j^\star(x_{-j}^\star(\alpha_{ji}),\alpha_{ji})}{d\alpha_{ji}} = \frac{\partial x_j^\star}{\partial\alpha_{ji}} + \sum_{k\neq j}\frac{\partial x_j^\star}{\partial x_k^\star}\frac{\partial x_k^\star}{\partial\alpha_{ji}}.$$

To prove the claim, we make use of the following two properties: For all $k$, (a) if $x_k^\star = y_k$, then $\partial x_k^\star/\partial \alpha_{ji} = 0$; and (b) if $x_k^\star < y_k$, then $\partial x_k^\star/\partial \alpha_{ji} = \partial \hat{x}_k/\partial \alpha_{ji}$. From Lemma 3(ii), for some $i$ and $j$, $i \neq j$, there are instances of $P_y$ where $\partial \hat{x}_j/\partial \alpha_{ji} > 0$. Combining this result with property (b), we find that there are instances of $P_y$ with $\partial x_j^\star/\partial \alpha_{ji} > 0$. Now assume that $x_k^\star = \hat{x}_k = y_k$ for all $k \neq i$, yielding $dx_j^\star/d\alpha_{ji} = \partial x_j^\star/\partial \alpha_{ji} > 0$ and the proposition follows. □

*Proof of Lemma 4.* To prove the desired result, we make use of the inverse distribution function $\Phi_i^{-1}(\rho_i, y)$, $\rho_i \in [0,1]$. In particular, $\Phi_i(\chi_i, y) = \rho_i$ and $\Phi_i^{-1}(\rho_i, y) = \chi_i$. Note that the assumptions on rational beliefs imply $\partial^2 \Phi_i^{-1}(\rho_i, y)/\partial y_i^2 \leq 0$. Further, $\Phi_i(0, y) = 0$ and $\Phi_i(y_i, y) = 1$.

Assuming rational beliefs and given $y_{-i}$, each manufacturer's expected profit can be written as

$$\Pi_{M_i}(y_i|y_{-i}) = w_i \int_0^{y_i} \chi_i d\Phi_i(\chi_i, y) - c_i y_i = w_i \int_0^{y_i} (1 - \Phi_i(\chi_i, y)) d\chi_i - c_i y_i. \tag{24}$$

Using the inverse distribution function, we can rewrite (24) as

$$\Pi_{M_i}(y_i|y_{-i}) = w_i \int_0^1 (1 - \rho_i) d\Phi_i^{-1}(\rho_i, y) - c_i y_i = w_i \int_0^1 \Phi_i^{-1}(\rho_i, y) d\rho_i - c_i y_i.$$

Therefore,

$$\frac{\partial^2 \Pi_{M_i}(y_i|y_{-i})}{\partial y_i^2} = w_i \int_0^1 \frac{\partial^2 \Phi_i^{-1}(\rho_i, y)}{\partial y_i^2} d\rho_i \leq 0.$$

□

*Proof of Proposition 5.* Assuming rational beliefs, each manufacturer's expected profit given her competitors' production levels is

$$\Pi_{M_i}(y_i|y_{-i}) = w_i \mu_i(y) - c_i y_i.$$

Taking the first-order derivative and satisfying the optimality condition yields

$$\frac{\partial \Pi_{M_i}(y_i|y_{-i})}{\partial y_i} = w_i \frac{\partial \mu_i(y)}{\partial y_i} - c_i = 0,$$

and the result follows immediately. □

*Proof of Proposition 6.* A pure-strategy manufacturer Nash equilibrium exists if (i) each manufacturer's strategy space is a non-empty, compact and convex set, and (ii) each manufacturer's profit function $\Pi_{M_i}$ is continuous in $y$ and quasi-concave in $y_i$ (Debreu, 1952). Lemma 4 together with our assumptions ensures that these conditions are satisfied. Thus, there exists at least one pure-strategy manufacturer Nash equilibrium.

To derive our uniqueness conditions, we rely on the fundamental results of Rosen (1965). In particular, Theorem 2 in Rosen (1965) asserts that the manufacturer Nash equilibrium defined by (5) is unique if (i) $\Pi_{M_i}$ is twice continuously differentiable in $y$ for all $i$, and (ii) $\sigma(y, \delta) = \sum_{i=1}^{N} \delta_i \Pi_{M_i}(y_i|y_{-i})$ is diagonally strictly concave for some fixed $\delta > 0$. While condition (i) is guaranteed by our assumptions, we need some more definitions to verify condition (ii).

Let $g(y, \delta)$ be the pseudogradient of $\sigma(y, \delta)$ for fixed $\delta$, i.e.,

$$g(y, \delta) = \begin{pmatrix} \delta_1 \partial \Pi_{M_1} / \partial y_1 \\ \vdots \\ \delta_N \partial \Pi_{M_N} / \partial y_N \end{pmatrix},$$

and denote by $G(y, \delta)$ the Jacobian of $g(y, \delta)$ with respect to $y$, i.e.,

$$G(y, \delta) = \nabla_y g(y, \delta) = \left( \delta_i \partial^2 \Pi_{M_i} / \partial y_i \partial y_j \right)_{ij}.$$

Now, Theorem 6 in Rosen (1965) states that $\sigma(y, \delta)$ is diagonally strictly concave if $G(y, \delta)$ is negative definite for all $y \in \times_i [0, y_i] \subseteq [0, K]^N$ and some fixed $\delta > 0$. Thus, the manufacturer Nash equilibrium is unique if, for some $\delta > 0$, $G(y, \delta)$ is negative definite for all $y$.

*Negative definiteness of $G(y, \delta)$:* Denote by $G^T(y, \delta)$ the transposed of $G(y, \delta)$. A basic result in fundamental algebra states that $G(y, \delta)$ is negative definite if its symmetric part $G_{sym}(y, \delta) = \left[ G(y, \delta) + G^T(y, \delta) \right] / 2$ is negative definite. This is true if all eigenvalues of $G_{sym}(y, \delta)$ are negative. Note that, due to Definition 1, all elements of $G_{sym}(y, \delta)$ are non-positive. Hence, by the Gershgorin Circle Theorem (see Varga, 2004), an upper bound for the $i$th eigenvalue of $G_{sym}(y, \delta)$ is given by

$$ub_i = \delta_i \frac{\partial^2 \Pi_{M_i}}{\partial y_i^2} - \frac{1}{2} \sum_{j \neq i} \left[ \delta_i \frac{\partial^2 \Pi_{M_i}}{\partial y_i \partial y_j} + \delta_j \frac{\partial^2 \Pi_{M_j}}{\partial y_i \partial y_j} \right],$$

$i = 1, \ldots, N$. Therefore, $G_{sym}(y, \delta)$ is negative definite if, for all $i$, $ub_i < 0$. This is true if $\Pi_{M_i}$ is strictly concave in $y_i$, and

$$2 + \sum_{j \neq i} \frac{\partial y_i^C}{\partial y_j} - \sum_{j \neq i} \frac{\delta_j}{\delta_i} \frac{\partial^2 \Pi_{M_j} / \partial y_i \partial y_j}{\partial^2 \Pi_{M_i} / \partial y_i^2} > 0 \tag{25}$$

for all $y$, where we make use of the Implicit Function Theorem

$$\frac{\partial y_i^C}{\partial y_j} = - \frac{\partial^2 \Pi_{M_i} / \partial y_i \partial y_j}{\partial^2 \Pi_{M_i} / \partial y_i^2}.$$

By choosing $\delta_i = 1/w_i > 0$ for all $i$, (25) reduces to (6), which proves the proposition. $\square$

*Proof of Corollary 1.* If $\Pi_W(x)$ is jointly concave in $x$, then the wholesaler's optimal stocking quantity $x^\star(y)$ is unique for any given $y$. In addition, under the conditions of Proposition 6, the manufacturer Nash equilibrium $y^c$ is unique. It follows that $(x^\star(y^c), y^c)$ defines the unique Bayesian Nash-Stackelberg equilibrium in the competitive scenario of the Supply Game. $\square$

*Proof of Proposition 7.* Assuming rational beliefs, the manufacturer's expected profit is

$$\Pi_M(y) = \sum_i w_i \mu_i(y) - c_i y_i.$$

Taking first-order derivatives yields

$$\frac{\partial \Pi_M(y)}{\partial y_i} = w_i \frac{\partial \mu_i(y)}{\partial y_i} + \sum_{j \neq i} w_j \frac{\partial \mu_j(y)}{\partial y_i} - c_i,$$

$i = 1, \ldots, N$. Rearranging terms and satisfying the optimality conditions gives (8). □

*Proof of Corollary 2.* If $\Pi_W(x)$ and $\Pi_M(y)$ are jointly concave in $x$ and $y$, respectively, then the wholesaler's optimal stocking quantity given $y$, $x^\star(y)$, and the manufacturer's optimal production quantity $y^{nc}$ are both unique. Thus, in the non-competitive scenario of the Supply Game, $(x^\star(y^{nc}), y^{nc})$ defines the unique Bayesian Stackelberg equilibrium. □

*Proof of Proposition 8.* We start this proof with a preliminary result that is useful in the remainder. Let $y'_{-i} \geq y_{-i}$ and note that

$$\frac{\partial^2 \mu_i(y)}{\partial y_i \partial y_j} = -\int_0^{y_i} \frac{\partial^2 \Phi_i(\chi_i, y)}{\partial y_i \partial y_j} d\chi_i \leq 0$$

by the definition of rational beliefs. It follows that for arbitrarily fixed $\tilde{y}_i$

$$\left.\frac{\partial \mu_i(y_i, y'_{-i})}{\partial y_i}\right|_{y_i = \tilde{y}_i} \leq \left.\frac{\partial \mu_i(y_i, y_{-i})}{\partial y_i}\right|_{y_i = \tilde{y}_i}. \tag{26}$$

(i) The proof proceeds by contradiction. Assume $y^c < y^{nc}$. Now, by comparing and equating the optimality conditions (5) and (8), we require

$$\left.\frac{\partial \mu_i(y_i, y^c_{-i})}{\partial y_i}\right|_{y_i = y_i^c} = \frac{c_i}{w_i} = \left.\frac{\partial \mu_i(y_i, y^{nc}_{-i})}{\partial y_i} + \sum_{j \neq i} \frac{w_j}{w_i} \frac{\partial \mu_j(y_i, y^{nc}_{-i})}{\partial y_i}\right|_{y_i = y_i^{nc}} \tag{27}$$

to be true. By assumption (9), the second term on the right-hand side of (27) is always non-positive. So, for (27) to hold, we need

$$\left.\frac{\partial \mu_i(y_i, y^c_{-i})}{\partial y_i}\right|_{y_i = y_i^c} \leq \left.\frac{\partial \mu_i(y_i, y^{nc}_{-i})}{\partial y_i}\right|_{y_i = y_i^{nc}}.$$

By (26) and concavity of $\mu_i$ with respect to $y_i$, this can only be true if $y_i^c \geq y_i^{nc}$, a contradiction to our initial assumption.

(ii) An example provides the proof. Assume manufacturers' beliefs about the wholesaler's stocking levels for products $j \neq i$ are independent of the production quantity of product $i$, i.e.

$\mu_j(y_i, y_{-i}) = \mu_j(y_{-i})$ for all $j \neq i$. Hence, $\partial\mu_j/\partial y_i = 0$ for all $j \neq i$. Assume further that $\partial^2\Phi_i(\chi_i, y)/\partial y_i \partial y_j > 0$ for all $j \neq i$. Then, the inequality in (26) becomes strict.

Comparing the optimality conditions (5) and (8) for product $i$ gives

$$\left.\frac{\partial\mu_i(y_i, y_{-i}^c)}{\partial y_i}\right|_{y_i=y_i^c} = \frac{c_i}{w_i} = \left.\frac{\partial\mu_i(y_i, y_{-i}^{nc})}{\partial y_i}\right|_{y_i=y_i^{nc}}. \tag{28}$$

Now, assume $y_{-i}^c \geq y_{-i}^{nc}$; otherwise the proof would already be complete. By (26) and concavity of $\mu_i$ with respect to $y_i$, (28) can only be true if $y_i^c < y_i^{nc}$. □

*Proof of Proposition 9.* (i) The proof proceeds by contradiction. Let $y' \geq y$ and suppose $I(y') < I(y)$. Then, for arbitrary $i$,

$$y'_{-i} - x^\star_{-i}(y') < y_{-i} - x^\star_{-i}(y). \tag{29}$$

As an immediate consequence of (29), we know that $x^\star_{-i}(y') > x^\star_{-i}(y)$. Now, by repeatedly applying Lemma 2(i),

$$\hat{x}_i(x^\star_{-i}(y')) \leq \hat{x}_i(x^\star_{-i}(y)), \tag{30}$$

and recall that $x_i^\star(y) = \min\{\hat{x}_i(x^\star_{-i}(y)), y_i\}$.

If $\hat{x}_i(x^\star_{-i}(y)) \geq y_i$, then $I_i(y) = y_i - y_i = 0$, and thus $I_i(y') \geq I_i(y)$. If, to the contrary, $\hat{x}_i(x^\star_{-i}(y)) < y_i$, then applying (30) yields

$$I_i(y) = y_i - \hat{x}_i(x^\star_{-i}(y)) \leq y'_i - \hat{x}_i(x^\star_{-i}(y')) = I_i(y').$$

Accordingly, $I_i(y') \geq I_i(y)$; a contradiction to our initial assumption that $I(y') < I(y)$.

(ii) The proof is an application of Proposition 2. Suppose $y' = y + \varepsilon e_j$, $\varepsilon > 0$, for arbitrary $j$. Then, by Proposition 2(iii), there exist situations where $x_i^\star(y') > x_i^\star(y)$ for some $i \neq j$. Thus,

$$I_i(y') = y'_i - x_i^\star(y') < y'_i - x_i^\star(y) = y_i - x_i^\star(y) = I_i(y).$$

□

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