University of Mannheim / Department of Economics

Working Paper Series

Ex-post Optimal Knapsack Procurement

Felix Jarman Vincent Meisner

Working Paper 15-02

January 2015

Ex-post Optimal Knapsack Procurement*

Felix Jarman† Vincent Meisner‡

January 12, 2015

Abstract

We consider a budget-constrained mechanism designer who wants to select an optimal subset of projects to maximize her utility. Project costs are private information and the value the designer derives from their provision may vary. In this allocation problem the choice of projects - both which and how many - is endogenously determined by the mechanism. The designer faces hard ex-post constraints: The participation and budget constraint must hold for each possible outcome while the mechanism must be implementable in dominant strategies. We derive the class of optimal mechanisms that are characterized by cutoff functions. These cutoff functions exhibit properties that allow an implementation through a descending clock auction. Only in the case of symmetric projects price clocks descend synchronously such that always the cheapest projects are executed. However, the asymmetric case, where values or costs are asymmetrically distributed, features a novel tradeoff between quantity and quality. Interestingly, this tradeoff mitigates the distortion due to the informational asymmetry compared to environments where quantity is exogenous.

JEL-Classification: D02, D44, D45, D82, H57.
Keywords: Mechanism Design, Knapsack, Budget, Procurement, Auction

*We thank Ludwig Ensthaler, Thomas Giebe, Hans Peter Grüner, Doh-Shin Jeon, Volker Nocke, Thomas Tröger and participants of the 10th SFB Young Researchers Workshop, the ENTER MaCCI-TSE Workshop, the UECE Lisbon Meetings 2014, and the 9th BiGSEM workshop for useful comments and suggestions.
†CDSE, University of Mannheim, felix.jarman@gess.uni-mannheim.de
‡CDSE, University of Mannheim, vincent.meisner@gess.uni-mannheim.de

1 Introduction

We study the problem of an institution that can spend a fixed budget on a variable number of projects. For instance, a development fund desires to distribute money to nonprofit projects with non-monetary benefits. Even if the benefit outweighs the cost for all available projects, the budget prohibits that all projects' costs can be covered. Therefore, the fund's problem is to select an affordable subset of maximal aggregate quality. Under complete information this is a variant knapsack problem, well known in computer science.[1] In economic applications the assumption of complete information is generally problematic. Here, a project might desire to get funding beyond the necessary minimum. For example, a project could spend money on extra equipment that is convenient for the project's staff but has no value for the designer. When costs are private information a strategic game is induced. The mechanism designer, i.e. the fund, can set the rules for this game and the projects act as economic agents in this environment. Essentially, we view this problem as an "up to possibly n-units" procurement problem with n agents with single-unit supply where the demand is determined after observing projects' reports and under a budget constraint.

Importantly, the budget and participation constraints are imposed ex-post, i.e. an applicant cannot be forced to conduct a project when the assigned funds are insufficient and the sum of transfers must not exceed the budget for any cost realization. In the example of a development fund, the nonprofit nature of the projects prohibits acquiring money on the market. Therefore underfunded projects are not executed, making their participation constraint ex-post.[2] Imposing the budget constraint to hold ex-post comes natural, as budgets are usually fixed. Our results are applicable to settings in which the designer may or may not value residual money. We focus on the first case. In addition, we examine mechanisms that are implementable in dominant strategies. Dominant strategy implementable mechanisms are widely used in practice as they are easy to explain and not prone to manipulation or misspecification of beliefs. For similar reasons we restrict attention to deterministic mechanisms. Deterministic mechanisms obviate the need for

[1] A classical combinatorial problem: A set of items is assigned values and weights. The knapsack should be filled with the maximal value, but can carry only up to a given weight.

[2] Alternatively, think of a scientist applying for a research grant at some science foundation in order to buy a data set.

a credible randomization device and are therefore more easily applicable in practice.

This paper not only helps in understanding a class of economically relevant problems, its framework also presents itself with a novel methodological challenge. The ex-post nature of both the participation and the budget constraint precludes the standard pointwise optimization techniques à la Myerson (1981). By focusing on dominant strategy implementable deterministic mechanisms, we can reduce the problem to finding a set of optimal cutoff functions. We call the corresponding mechanisms "z-mechanism". A z-mechanism is characterized by a set of functions $\{z_i\}$ that only depend on the costs of non-executed projects and weakly increase in those costs. The function z_i is a cutoff such that project i is conducted whenever i's cost report falls below z_i.

Furthermore we investigate symmetric and asymmetric environments separately and propose implementations. First, we focus on the case where all projects are ex-ante symmetric and only cost is independent private information. Having characterized the optimal mechanism as a z-mechanism, it follows that it is optimal to rank projects according to their cost and greenlight the cheapest ones. The number of greenlighted projects is then endogenously determined by the budget and the cost reports of all participating projects. Second, we examine the case of ex-ante asymmetric projects, i.e. costs are drawn from different distributions and/or project values differ. In applications, the designer may prefer some projects over others and might have different information over cost distributions. We restrict attention to the two project case because it conveys the main insights while retaining tractability. Interestingly, the optimal allocation of the symmetric case does not easily generalize to the asymmetric case. That is, we do not simply greenlight projects in order of their virtual values.[3] We show that there are instances where out of two rival projects optimally the project with lower virtual value is chosen. The cause underlying this result is that the amount of units procured is endogenous. As a consequence, incentive compatibility constraints create a tradeoff between quantity and quality of the procured projects.

[3] As a fist step, rewriting the problem involves expressing expected transfers by the allocation function. As the designer maximizes expected payoff we can employ the procurement analogue of Myerson's notion of virtual values.

In auction theory it is a known result (e.g. Luton and McAfee, 1986) that when only one unit is procured out of identical projects among agents with asymmetrically distributed cost, it is not necessarily the lower cost project that is greenlighted - but the one with lowest virtual cost. Broadly speaking, the designer discriminates against stochastically stronger projects. Notably, the quantity-quality tradeoff introduced above mitigates this discrimination.

Reducing the set of candidates for optimal mechanisms to z-mechanisms enables us to implement any optimal allocation with an appropriately designed descending clock auction. Individual clock prices determine the transfer paid to each active project and continuously decrease. In the symmetric case, all clocks run down synchronously. Therefore, projects drop out in order of their costs until all active projects can be financed.

As the optimal mechanism in the asymmetric case does not always allocate in order of virtual value, we cannot adjust the descending clock auction in the usual way where clocks are asynchronous such that the virtual value is kept equal. Instead, the descending-clock implementation of the optimal allocation includes individual clocks stopping completely at certain times. This effect emerges so that the probability of executing a larger number of projects is not decreased too much, which comes at the cost that in some cases the provision is allocated to an "inferior project" in terms of the designer's payoff. Put differently, the incentive constraints prohibit the designer from always allocating to the better project while holding the probability to execute a larger number of projects constant. This is where the quantity-quality tradeoff kicks in.

Other than distribution of monetary transfers, this framework matches a whole range of allocation problems. In general, our mechanism is applicable to all problems, in which a designer needs to allocate a divisible but fixed quantity among agents. For instance, time on a production facility or on a supercomputer can be allocated with our mechanism. Similarly, consider the allocation of payload on a freighter or a space shuttle. Clearly, the capacity of a space shuttle is limited. The problem of optimally allocating the capacity and incentivizing projects to reduce payload is economically relevant, see Ledyard, Porter, and Wessen (2000). Of course, we can also consider classical applications such as the procurement of bus lines, bridges, or streets as well as the allocation of R&D money.

1.1 Literature

The knapsack problem is a classical problem in combinatorial optimization with a wide range of economic applications. Even though it has been studied since at least the late 19th century, publications in economics have remained relatively silent on this issue. Most prominently, in his Nancy L. Schwartz memorial lecture Maskin (2002) addressed the related problem of the UK government that put aside a fixed fund to encourage firms to reduce their pollution. The government faces n firms that have private marginal cost of reduction θ_i and can commit to reduce x_i units of pollution. To reduce pollution as much as possible, the government pays expected compensation transfers t_i to the firms, who report costs and reduction to maximize $t_i - \theta_i x_i$. For some distributions, Maskin (2002) proposes a mechanism that satisfies an ex-post participation constraint, an ex-post incentive compatibility constraint, and the condition that the budget is not exceeded in expectation. In their response to Maskin (2002), Chung and Ely (2002) look at a more general class of mechanism design problems with budget constraints and translate them into a setting à la Baron and Myerson (1982). Their approach nests Maskin (2002) and also Ensthaler and Giebe (2014a) as special cases. However, the latter more explicitly derive a constructive solution. In contrast to us, they consider a soft budget constraint that only requires the sum of expected transfers to be less than the budget. This is a relevant setting as well, but we think that in many of the examples provided so far a hard budget constraint must be imposed because funds exceeding the budget might simply not be available.

Ensthaler and Giebe (2014a) circumvent this problem by using AGV-budget-balancing (such as Börgers and Norman, 2009) to get a mechanism that is ex-post budget-feasible. However, getting an ex-post balanced budget in such a way comes at the cost of sacrificing ex-post individual rationality. Many applications do not allow this constraint to be weakened. For instance, subsidy applicants usually cannot be forced to conduct their proposal when receiving only a small or possibly no subsidy. Alternatively, limited liability justifies insisting on ex-post individual rationality. To the best of our knowledge, no paper exists that jointly considers optimal mechanism design under ex-post budget balance and ex-post individual rationality in a procurement setting. Ensthaler and Giebe (2014b) propose a clock mechanism that coincides with our optimal mechanism in the symmetric case for many parameterizations[4] but differs in the asymmetric case by holding the cost-benefit-ratio equal among projects. By simulating different settings, they conclude that this mechanism outperforms a mechanism used in practice. This clock mechanism is outperformed by our mechanism.

Our problem is also investigated by computer scientists such as Dobzinski, Papadimitriou, and Singer (2011). However, instead of specifying the private information optimum they search for an algorithmic mechanism that approximates the full information optimum within specific bounds. In contrast to the algorithmic approach that is used by them and to solve the canonical knapsack problem, we directly characterize the optimal allocation under private information.

Dizdar, Gershkov, and Moldovanu (2011) investigate a dynamic knapsack problem where impatient projects with (possibly) private capacity requirement w and willingness to pay v arrive over time and a mechanism designer offers them a (constrained) capacity w' and a price p. The projects' utility is given by $wv - p$ if the assigned capacity suffices $w' > w$ and by $-p$ otherwise. However, the static version of their problem does not mirror ours in the way procurement auctions mirror seller-buyer auctions. In their model, the mechanism designer is only interested in the sum of payments. In our framework, the designer not only wants to minimize payments but also maximize aggregate value of all greenlighted projects.

There seems to be no reasonable analogy to a setting where the mechanism designer is a similarly constrained seller and the agents are buyers. Budget constrained buyers in auctions have been discussed in the literature, e.g. by Pai and Vohra (2014) or Che and Gale (1998). However, note that these authors study constrained agents whereas in our setting the designer is constrained.

In the following section, we introduce the model. In section 2.1 we rewrite the problem as a problem of finding the optimal z-mechanism. Sections 2.2 and 2.3 cover symmetric and asymmetric environments separately. Next, we discuss extensions and possible modifications to the model in section 3 and

[4]In contrast to their setting, the mechanism designer in our model values residual money. Therefore the designer will not greenlight projects with negative virtual value. Ensthaler and Giebe (2014b) do not consider this case.

finally we conclude in section 4.

2 Model

This section is organized as follows. First we introduce the notation and elaborate technical details that are helpful towards finding properties of the optimal mechanism. Subsequently, we argue that the optimal mechanism belongs to a class of mechanism that we call z-mechanisms. A z-mechanism is characterized by monotone cutoff functions z that only depend on the cost of projects that are not conducted. We lay out how to obtain the optimal z-mechanism. In subsection 2.2 we fully characterize the optimal z-mechanism for the symmetric case and propose a clock-auction implementation. Finally we illustrate the difference to a non-symmetric environment and characterize properties of the optimal z-mechanism in the non-symmetric environment. The optimal clock-auction for the general case not only features asynchronous clocks but also requires that subsets of clocks periodically hold.

There is a set $I = \{1, \ldots, n\}$ of n projects and one mechanism designer. Each project can be conducted exactly once. The designer gains utility v for each project that is conducted. Each project acts as an economic agent. If project i is executed, it incurs cost $c_i \in [\underline{c}_i, \overline{c}_i]$. The costs are the projects' private information and are drawn independently and identically from a distribution $F_i(c)$. We assume $F_i(c)$ to be continuous and continuously differentiable with a positive continuous density $f_i(c)$. The value of the project, v, and the distribution, $F_i(c)$, are common knowledge.

We employ the revelation principle and without loss of generality limit attention to direct mechanisms. In addition, we restrict ourselves to deterministic mechanisms. This restriction implies that once all cost reports are collected we know with certainty which project will be selected by the mechanism. Formally, the designer partitions the set of projects into two disjoint sets $G \cup R = I$. We will say that projects in set G are "greenlighted" while the other are "redlighted". This decision is represented by

$$q_i = 1 \, \forall i \in G \text{ and } \, q_i = 0 \, \forall i \in R.$$

To compensate project i for its cost, the designer pays transfer t_i. Therefore a direct mechanism is characterized by $\langle q_i, t_i \rangle$. It is a mapping from the vector of cost reports $\mathbf{c} \in \times_i^n[\underline{c}_i, \overline{c}_i]$ both into provision decisions and transfers. When we talk about the allocation, we refer to the former. Project i's utility u_i is given by its transfer minus the cost it bears.

$$u_i = t_i - q_i c_i$$

The designer derives value v_i from each greenlighted project $i \in G$ while having to pay the sum of transfers. Therefore she wants to maximize the aggregate value of greenlighted projects while minimizing cost. Her (ex-post) utility function u_P implies that, in our setting, the designer values residual money.

$$u_D = \sum_i (q_i v_i - t_i) \tag{1}$$

We impose an ex-post participation constraint. Thus, if i is greenlighted the transfer must be at least as high as the cost.

$$t_i(c_i, c_{-i}) - q_i(c_i, c_{-i})c_i \geq 0 \quad \forall i, c_i, c_{-i} \tag{PC}$$

In addition, the designer has a budget constraint that is hard in the sense that she cannot spend more than her budget B for any realization of the cost vector. That is, the designer can never exceed her budget

$$\sum_i t_i(\mathbf{c}) \leq B \quad \forall \mathbf{c}. \tag{BC}$$

Finally, incentive compatibility has to hold in (weakly) dominant strategies. Therefore, for every realization of the cost vector, project i's truthful report must give it at least as much utility as any possible deviation.

$$t_i(c_i, c_{-i}) - q_i(c_i, c_{-i})c_i \geq t_i(\widetilde{c}_i, c_{-i}) - q_i(\widetilde{c}_i, c_{-i})c_i \quad \forall i, c_i, c_{-i}, \widetilde{c}_i \tag{IC}$$

One may think that a natural approach to this problem would be to express the ex-post transfer $t_i(c_i, c_{-i})$ as a function of the ex-post allocation decision $q_i(c_i, c_{-i})$, both taking c_{-i} as given, applying the envelope theorem. Then it is possible to restrict attention to the allocation in order to solve for the optimal mechanism. However, this approach does not reduce the complexity of the problem. The reason is that the ex-post transfers and allocation for one cost vector restrict transfers and allocation for other cost vectors in a manner much more involved than standard monotonicity. In particular, the ex-post transfer expressed as a function of the ex-post allocation might be ill-behaved. Therefore, we cannot arrive at sufficient conditions using convex optimization. [5]

2.1 Rewriting the problem

We search for the direct mechanism that maximizes the expected utility of the designer. We call this mechanism the optimal mechanism. Our first step is to show that the ex-post constraints imply that the optimal mechanism has to be in cutoffs.

Lemma 1. *The optimal mechanism can be represented by cutoff functions $z_i(c_{-i})$, where project i is greenlighted whenever it reports cost weakly below its cutoff.*

$$q_i(c_i, c_{-i}) = \mathbb{I}(c_i \leq z_i(c_{-i}))$$

The transfer to project i will be its cutoff whenever it is greenlighted and zero otherwise.

$$t_i(c_i, c_{-i}) = q_i(c_i, c_{-i}) z_i(c_{-i})$$

Proof. First note that for any two cost reports c_i, c_i' of project i and for some c_{-i} (IC) implies that if $q_i(c_i, c_{-i}) = q_i(c_i', c_{-i})$, then we must also have $t_i(c_i, c_{-i}) = t_i(c_i', c_{-i})$. Otherwise i could deviate to the report giving a higher transfer.

Suppose project i is greenlighted for some cost reports given c_{-i}. Then there are only two possible values for t_i, depending on whether i is greenlighted or not: $t_i^{q_i=1}$ and $t_i^{q_i=0}$.

Define $z_i(c_{-i}) = t_i^{q_i=1} - t_i^{q_i=0}$. Then (IC) again gives the following.

[5] Note however that relaxing either budget or participation constraint to hold only in expectation would enable us to use the techniques employed by Ensthaler and Giebe (2014a).

$$q_i(c_i, c_{-i}) = \begin{cases} 1 & \text{if } c_i \leq z_i(c_{-i}) \\ 0 & \text{if } c_i > z_i(c_{-i}) \end{cases}$$

Suppose to the contrary that for some realization $\widehat{c}_i < z_i(c_{-i})$ we would have $q_i(\widehat{c}_i, c_{-i}) = 0$. Then deviating to a cost report that ensures the green light would imply a utility increase of $z_i - c_i$. An analogous argument applies for $\widehat{c}_i > z_i(c_{-i}) > 0$.[6]

The last step is to show that $t_i^{q_i=0} = 0$. This result trivially follows from the mechanism being optimal, i.e. maximizing expected utility of the designer. □

Lemma 1 shows that the allocation and the transfers are characterized by cutoffs, with which project i is greenlighted whenever it reports cost that lies weakly below the cutoff. These cutoffs are functions of the other cost reports c_{-i}. Therefore what remains to be determined is the nature of the optimal cutoffs. The maximization problem of the designer is given by the following.

$$\begin{aligned} &\max_{z_i(c_{-i})} \mathbb{E}\left[\textstyle\sum_i q_i(\mathbf{c})v_i - t_i(\mathbf{c})\right] \\ &\qquad \text{s.t. (BC)}, \\ \\ &q_i(\mathbf{c}) = \mathbb{I}(c_i \leq z_i(c_{-i})) \\ &\quad t_i = \mathbb{I}(c_i \leq z_i(c_{-i}))z_i(c_{-i}) \end{aligned} \tag{2}$$

Here q_i and t_i are determined by cutoff z_i. Therefore incentive compatibility and participation constraints come for free. The next step towards solving this problem involves applying standard methods introduced by Myerson (1981). Now we define the conditional expected probability of getting greenlighted and the conditional expected transfer.

$$Q_i(c_i) = \mathbb{E}[q_i(c_i, c_{-i})|c_i]$$
$$T_i(c_i) = \mathbb{E}[t_i(c_i, c_{-i})|c_i]$$

The interim incentive compatibility required by Myerson (1981) is weaker than our condition (IC). Consequently, the expected transfer is determined

[6]When $c_i = z_i$, (IC) permits both $q_i = 0$ and $q_i = 1$. For ease of exposition, we stick to $q_i = 1$.

by the allocation, $T_i(c_i) = Q_i(c_i)c_i + \int_{c_i}^{\overline{c}} Q_i(x)dx$. The usual monotonicity condition is trivially fulfilled as we are dealing with cutoff mechanisms. This reformulation in turn gives us a way to rewrite the objective function as a function of the allocation. Substituting into problem (2) and integrating by parts gives the following.

$$\begin{gathered}\max_{z_i(c_{-i})} \mathbb{E}\left[\sum_i \mathbb{I}(c_i \leq z_i(c_{-i}))\left(v_i - c_i - \frac{F_i(c_i)}{f_i(c_i)}\right)\right] \\ \text{s.t.} \\ \sum_i \mathbb{I}(c_i \leq z_i(c_{-i}))z_i(c_{-i}) \leq B \quad \forall \mathbf{c}\end{gathered} \tag{3}$$

We will call $\varphi_i(c_i) := c_i + \frac{F_i(c_i)}{f_i(c_i)}$ the *virtual cost* of project i and $\psi_i(c_i) := v_i - \varphi_i(c_i)$ the *virtual value*. Here, φ and ψ are the procurement analogues to standard auction terminology. Now we can directly see from problem (3) that the optimal mechanism will maximize the expected sum of greenlighted virtual values.

Note that constrained optimization via Lagrangian is not straightforward here because of the non-differentiability of the indicator function. Instead, in the following we derive useful properties of the optimal cutoffs that can be exploited to characterize the optimal mechanism.

Assumption 1 (Log-concavity)**.** *For all i, the cumulative distribution function of c_i, $F_i(\cdot)$, is log-concave and has a continuous density function $f_i(\cdot)$.*

This assumption is a very common in information economics. This property is equivalent to the ratio f/F being a monotone decreasing function or the ratio F/f being monotonically increasing. Hence, the common regularity assumption is implied: $\varphi_i(c_i)$ is strictly increasing and $\psi_i(c_i)$ is strictly decreasing.

Lemma 2. *Disregarding (BC) the optimal cutoffs are given by z_i^{**} and are independent of the cost reports.*

$$z_i^{**} := \begin{cases} \psi_i^{-1}(0) & \text{if } \psi_i^{-1}(0) \in [\underline{c}_i, \overline{c}_i] \\ \overline{c} & \text{otherwise} \end{cases}$$

In the symmetric case, $z_i^{**} = z^{**}$.

Regularity ensures that a lower cost c_i translates to higher virtual value $\psi_i(c_i)$. Therefore the designer wants to greenlight any project with cost weakly below z_i^{**}. Note that regularity implies the invertibility of $\psi_i(c_i)$ and thus allows for the above definition of z_i^{**}. Crucially, Lemma 2 implies that it is never optimal to allocate the provision right to a project with negative virtual value.

We have previously introduced G and R as the sets of greenlighted and redlighted agents. Consequently, we denote the cost vector of projects in G as c_G and similarly the cost vector of projects in R as c_R. We now define a class of mechanisms and then show that any mechanism outside of this class cannot be optimal.

Definition 1 (z-mechanism)**.** *A z-mechanism is characterized by a set of cutoff functions $\{z_i(\cdot)\}_{i\in I}$ that are almost everywhere equal to cutoff functions that are*

1. *left-continuous for each of its arguments,*
2. *always weakly below z_i^{**},*
3. *weakly increasing in the other projects' costs,*
4. *independent of costs c_G conditional on $\{G, R\}$ being the partition of greenlighted and redlighted projects.*

The corresponding direct mechanism $\langle q_i, t_i\rangle$ is given by:

$$q_i(c_i, c_{-i}) = \mathbb{I}(c_i \leq z_i(c_R))$$
$$t_i(c_i, c_{-i}) = q_i(c_i, c^R) z_i(c_R).$$

Note that z-mechanisms have some salient features. The cutoffs of those projects that get greenlighted are only determined by the cost report of projects that get redlighted. This feature is due to the fact that project i's cost contains no information about the cost report of project j. What can be exploited however is the ordering of cost reports at the margin. Here, clearly, there is no incentive to misreport. Being able to restrict attention to z-mechanisms is highly useful, as the set of all feasible z-mechanisms is a much more tangible object than the substantially larger set of all permissible cutoff-mechanisms. In particular, the nature of z-mechanisms allows the designer to implement any optimal allocation with an appropriately designed descending clock auction.

For some of the following lemmata and propositions, we provide the proof for the two-project case while giving the general proof in the appendix. We do this because the logic of the proof will be the same for both $n = 2$ and $n > 2$ but a proof for the former is much easier to read. In addition, it provides comparability between the symmetric case and the asymmetric case, as we restrict ourselves to $n = 2$ in the asymmetric case.

Proposition 1. *Among all mechanisms satisfying (PC), (BC) and (IC), any mechanism that maximizes the designer's payoff (1) is a z-mechanism.*

We divide the proof into several lemmata showing that any optimal mechanism can only violate the properties of definition 1 on a set with Lebesgue-measure zero. In order to make the functions we talk about unique, we will w.l.o.g. restrict attention to cutoff functions that satisfy the properties. Note that property 1 does not require a proof because we can replace any function $z_i(\cdot)$ with a left-continuous function that is identical up to a set of points with Lebesgue-measure zero. Property 2 follows directly from the rewritten objective function (3). Let us now continue with property 3.

Lemma 3. *The optimal cutoff function $z_i(c_{-i-j}, c_j)$ is almost everywhere equal to a left-continuous function that is weakly increasing in c_j for all i, j with $j \neq i$, i.e. $z_i(c_j, c_{-i-j}) \geq z_i(c'_j, c_{-i-j})$ for almost every $(c_j, c'_j) : c_j > c'_j$ and c_{-i-j}.*

Proof. (with $n = 2$, see appendix for the general proof)
It follows form Lemma 2 that any optimal function $z_i(\cdot)$ cannot exceed z_i^{**}.

Given any function $z_1(\cdot)$, suppose to the contrary that there exists a set $H \subset (c^L, \overline{c}_2]$ of the other projects' cost with positive Lebesgue-measure such that $z_2(c^L) = z^L$ and $z_2(c_1) \leq z^H$ for all $c_1 \in H$ with $z_2^{**} \geq z^L > z^H$. That is, $z_2(\cdot)$ is decreasing somewhere.

Now, consider the deviation $\widehat{z}_2(c_1) = z^L$ for all $c_1 \in H$. In words, flatten the decreasing part in $z_2(\cdot)$ and leave $z_1(\cdot)$ as it is.

Case 1: For all $c_2 > z^L > z^H$. The deviation affects neither the budget constraint nor the profit, because project i is not executed for both cutoffs z^L and z^H.

Case 2: For all $c_2 \leq z^H < z^L$. The deviation again has no impact on profit, because project 2 is executed for both cutoffs. Moreover, c_2 pins down $z_1(c_2)$ and at cost vector (c^L, c_2) both projects are executed. Therefore, $B \geq z_2(c^L) + z_1(c_2)$, i.e. the deviation is budget-feasible.

Case 3: For all $c_2, z^L \geq c_1 > z^H$, the deviation is feasible and profitable.

1. $z_1(c_2) \geq c_1 > c^L$: Then it must be that $B \geq z_1(c_2) + z^L$ as the initial (feasible) mechanism executes both projects when costs (c^L, c_2) realize.
2. $c_1 > z_1(c_2) \geq c^L$: Similarly, it must be that $B \geq z_1(c_2) + z^L > z^L$.
3. $c_1 > c^L > z_1(c_2)$: Then it must be that $B \geq z^L$.

Hence, it cannot be optimal that $z_1(\cdot)$ is decreasing anywhere. □

Lemma 3 establishes that cutoff functions must be weakly increasing in their arguments. The intuition is straightforward. The cost draws of all projects are independent. Therefore project i's cost report only matters for the payoff generated from project $j \neq i$ through the budget constraint. Project i's cost report only influences the budget through exceeding or lying below the cutoff. If project i exceeds its cutoff, this frees budget to be distributed among the other projects. Consequently, their cutoffs must remain constant or increase. There are infinitely many possible cutoff functions that differ on finitely many points that have Lebesgue-measure zero. In the following we restrict attention to the left-continuous version of any such function.

Remember that G represents the set of greenlighted projects and R represents the set of redlighted projects. We establish that given that only the projects of some set G are greenlighted and given the remaining projects costs' c_R, for all $g \in G$ all functions $z_g(\cdot)$ intersect each other at the point $(a_1^G(c_R), a_2^G(c_R), ...)$.

Lemma 4. *Conditional on any arbitrary partition $\{G, R\}$, the optimal cutoff functions z_g for all $g \in G$ are independent of costs of all greenlighted projects c_G. That is,*

$$z_g(c_{G-g}, c_R) = z_g(c'_{G-g}, c_R),$$

for all c_{G-g} and c'_{G-g} such that G is the set of greenlighted agents.

Proof. (with $n = 2$, see appendix for the general proof and consult figure 1 for intuition)
By lemma 1 the optimal mechanism has to be in cutoffs. What remains to be shown is that said cutoffs only depend on c^R. For $G = \{1\}$ or $G = \{2\}$,

i.e. when only one project is greenlighted, the statement follows from the nature of a cutoff function. Hence we need to show that the cutoffs must be constants whenever $G = \{1, 2\}$. Therefore suppose $C^{\{1,2\}}$ has positive Lebesgue-measure.

Take any feasible candidate mechanism with any increasing cutoff functions $z_i(\cdot)$ and define

$$a_1 = \max\{c_1 | \exists c_2 : c_2 \le z_2(c_1), c_1 \le z_1(c_2)\}$$
$$a_2 = \max\{c_2 | \exists c_1 : c_1 \le z_1(c_2), c_2 \le z_2(c_1)\}.$$

The maximum exists by left-continuity of $z_i(\cdot)$ following lemma 3. Since we sometimes greenlight both projects, the sets over which we have defined a_1 and a_2 must be non-empty. Hence by definition of a_1, there exists $\widetilde{c}_2 : a_1 = z_1(\widetilde{c}_2)$. Similarly, there exists $\widetilde{c}_1 : a_2 = z_2(\widetilde{c}_1)$.

By definition $(\widetilde{c}_1, \widetilde{c}_2) \le (a_1, a_2)$. Therefore, $a_1 + a_2 \le B$ is implied by the budget constraint.

Now we show that $z_1(c_2') = a_1$, for all $c_2' \le a_2$ and $z_2(c_1') = a_2$ for all $c_1' \le a_1$. Suppose not. Suppose (without loss of generality) there is some $\Xi \subset [0, a_2]$ with positive Lebesgue-measure such that $z_1(c_2') < a_1$ for all $c_2' \in \Xi$. Call $z_1^{\Xi} := \max_{c_2 \in \Xi} z_1(c_2)$. Since $a_1 + a_2 \le B$, changing the mechanism to $z_1(c_2') = a_1, \quad \forall c_2' \le a_2$ does not violate the budget-constraint and increases the payoff by:

$$\Delta > \Pr(c_2 \in \Xi) \int_{z_1^{\Xi}}^{a_1} \psi_1(c) dF(c) > 0.$$

□

By combining our earlier insights with the previous two lemmata, we have shown that the optimal mechanism must be a z-mechanism.

2.2 The symmetric case

In this section, we focus on symmetric projects where $v_i = v$ and $F_i = F$ for all projects i. An implication of this assumption is that the order of

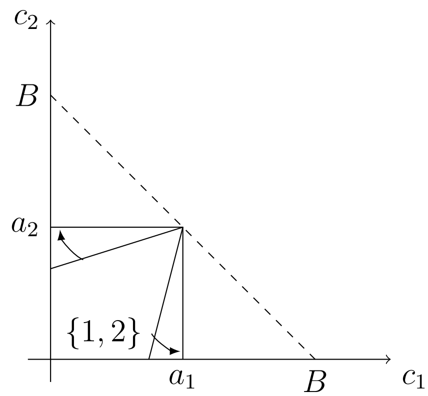


Figure 1: In Lemma 4, we show that in the non-trivial 2-project case whenever $G = \{1,2\}$, both projects get constant transfers summing up to the budget. For instance, the weakly increasing candidate mechanism depicted above is improved by the deviation indicated by the arrows.

costs coincides with the order of virtual values and that $z_i^{**} = z^{**}$ for all i. Now we show how to utilize the established results to characterize the optimal allocation and also how to implement it. Although we know that we only have to solve an optimization problem for optimal constants, we will discuss possible deviations in greater detail to highlight where the intuition goes wrong when considering asymmetric environments.

Proposition 2. *Arrange the projects by cost in ascending order,* $c_1 \leq c_2 \leq \cdots \leq c_n$ *and define* $z^k := \min\left\{\frac{B}{k}, z^{**}, c_{k+1}\right\}$. *In the symmetric case, the* z*-mechanism with* $z_i(c_{-i}) = z^{k^*}$ *is the optimal budget-constrained mechanism. The optimal number of accepted projects* k^* *is given by* $k^* := \max\{k | c_k \leq z^k\}$.

Proof. (with $n = 2$, see appendix for the general proof)
In proposition 1 we have shown that the optimal mechanism must be a z-mechanism. Consider any z-mechanism M^z different from the mechanism proposed above as a candidate for optimality, in which both projects get greenlighted sometimes. For graphic intuition of the deviation consult figure 2.

By Lemma 2 this candidate mechanism must never greenlight a project with negative virtual value. This is depicted as the kink at (z^{**}, z^{**}).

In the area above the dashed budget line, $c_1 + c_2 > B$, the designer can, by (BC) and (PC), only execute one of the two projects. It can be directly seen from objective function (3) that the designer prefers the project with the higher virtual value, i.e. the one with lower cost. It does not, however, follow directly that $z_i(c_j) = c_j$ whenever $B - c_i < c_j < z^{**}$. The reason is that the designer may want to forgo executing the better project for some cost vectors (green triangle and violet square in figure 2) in order to execute both projects in an additional area (cyan bar, figure 2). In such a case, the designer is forced by incentive compatibility to execute the worse project (for cost vectors in the green triangle or violet square).

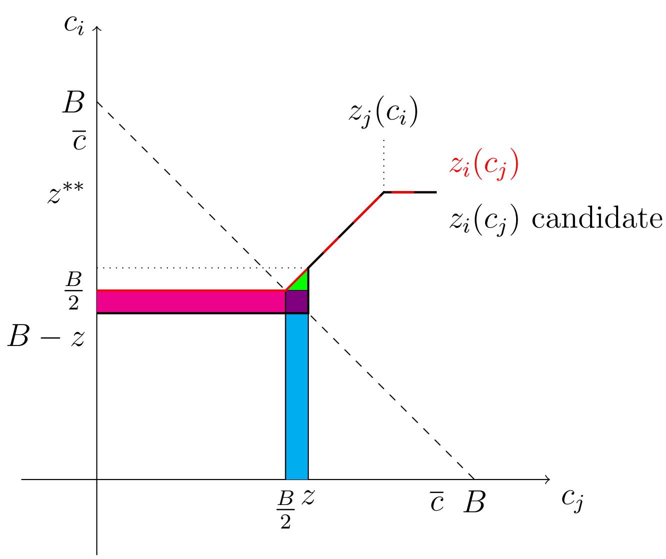


Figure 2: A candidate mechanism and the deviation to the proposed mechanism.

By lemma 4 both cutoffs must be constant whenever both projects are executed. In optimum in that case, there can be no slack in the budget constraint and z_i is flat in that region.

Now, consider candidate mechanism M^z

$$z_i(c_j) = \begin{cases} z^{**} \text{ if } c_j \geq z^{**} \\ c_j \text{ if } z < c_j < z^{**} \\ B - z \text{ if } c_j < z \end{cases} \quad \text{and} \quad z_j(c_i) = \begin{cases} z^{**} \text{ if } c_j \geq z^{**} \\ c_i \text{ if } B - z < c_j < z^{**} \\ z \text{ if } c_j < B - z \end{cases} \tag{4}$$

and see that a deviation to the mechanism in the proposition is always profitable.

For ease of exposition, let $A = \frac{B}{2}$. This proposed deviation to the z-mechanism M^A changes the designer's payoff in the following way

$$\begin{aligned}\Delta = {} & F_j(z)\int_{B-z}^{A}\psi_i(x)dF_i(x) && \text{(magenta+violet)}\\ & - F_i(A)\int_{A}^{z}\psi_j(c)dF_j(c) && \text{(cyan+violet)}\\ & + \int_{A}^{z}\int_{A}^{c}\psi_i(x)dF_i(x) - (F_i(c)-F_i(A))\psi_j(c)dF_j(c) && \text{(green)}\end{aligned}$$

where the colors represent the area in figure 2 where the allocation changes. Everywhere else the allocation and payoff remain the same.

To rewrite Δ define $\gamma(x) = F(x)(v-x)$ where $\gamma'(x) = \psi(x)f(x)$.

$$\begin{aligned}\Delta = {} & F(z)(\gamma(A)-\gamma(B-z)) - F(A)(\gamma(z)-\gamma(A))\\ & + F(A)(\gamma(z)-\gamma(A)) + \int_{A}^{z}\gamma(c)-\gamma(A)-F(c)\psi_j(c)dF(c)\\ = {} & F(z)(\gamma(A)-\gamma(B-z)) - F(A)(\gamma(z)-\gamma(A))\\ & + F(A)(\gamma(z)-\gamma(A)) - \gamma(A)(F(z)-F(A)) + \int_{A}^{z}F^2(c)dc\end{aligned}$$

because $(\psi(c)F(c) - F(c)(v-c))f(c) = F^2(c)$ and then since $\int_A^z F(c)^2dc > F(A)^2\int_A^z 1dc$,

$$\begin{aligned}\Delta & > F(z)(\gamma(A)-\gamma(B-z)) - \gamma(A)(F(z)-F(A)) + F(A)^2(z-A)\\ & = F(A)^2(v-A+z-A) - F(z)F(B-z)(v-B+z)\\ & = (v-B+z)(F(A)^2 - F(z)F(B-z))\\ & > 0 \Leftrightarrow F(A)^2 > F(z)F(B-z)\end{aligned}$$

This is true under assumption 1, log-concavity. If you maximize $F(z)F(B-z)$ with respect to z, the first order condition is given by

$$\frac{F(z)}{f(z)} = \frac{F(B-z)}{f(B-z)} \tag{5}$$

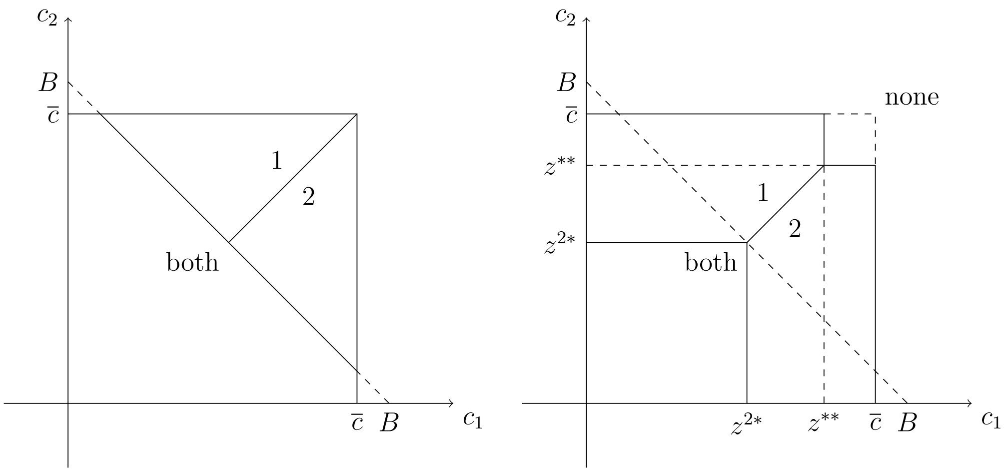
which is only true at $z = B/2$ since $F(x)/f(x)$ is an increasing function. For the same reason, the left-hand side is greater (less) than the right-hand side for $z > B/2(< B/2)$ making $z = B/2$ the maximum.

We have assumed that in the optimal mechanism both projects get greenlighted for some cost vectors. Therefore it remains to show that the optimal mechanism beats the best mechanism in which at most one project gets greenlighted. The best mechanism that selects at most one project would always select the project with higher virtual value. Clearly the optimal mechanism of this proposition creates more payoff as it also always greenlights the project with higher virtual value. Additionally, it sometimes adds a second project with positive virtual value. □

All greenlighted projects get the same transfer, and those projects that are excluded do not prefer to instead get the green light with the associated transfer. There are two rationales for greenlighted projects to get the same transfer. First, as shown in the proof of proposition 2, this way the probability of getting as many projects as possible is maximized. Ex-post incentive compatibility prevents budget to be shifted away from projects with low cost reports to projects with high costs. Therefore offering equal cutoffs is the best the designer can do. Second, as shown in (3) - the rewritten maximization problem of the designer - the expected utility of the designer is given by the sum of virtual values of greenlighted projects. Therefore she wants to greenlight those projects with the highest virtual values. That goal is exactly consistent with offering equal cutoffs to greenlighted projects and excluding those with higher cost. In the optimal allocation greenlighted projects will have higher virtual value than those who are not greenlighted.

The compatibility of the two objectives - get as many projects as possible and get those with the highest virtual values - is a special feature of the symmetric case. It generally fails in the asymmetric case, as we demonstrate in the next section.

Figure 3 illustrates the optimal budget-constrained allocations in an example with two projects. Panel 3b shows the fully-constrained optimal allocation juxtaposed with the relaxed optimal allocation when (IC) is neglected, shown in panel 3a. First, note that in this example $v \geq \overline{c}$ and $\overline{c} < B$. Therefore a completely unconstrained designer with full information would always greenlight both projects, and a budget-constrained designer with full information



(a) Budget-constrained, full information (b) Budget-constrained, private information

Figure 3: An example of optimal allocations for the symmetric case with $n = 2$

at least one. However, $z^{**} < \overline{c}$. Therefore for some realizations of $\mathbf{c}$ (the upper-right corner of panel 3b), no project will get greenlighted in the optimal allocation, even though doing so would be profitable from an ex-post perspective. The negative virtual value of the projects in these cases indicates that the cost of allocating to such a project - incentive compatibility will require higher transfers for other cost types - outweighs the benefit from an ex-ante perspective. The second major difference between the relaxed optimal allocation and the optimal allocation can be seen for those realizations of costs where allocating to both projects would be feasible only in the relaxed problem. This difference comes from the designer's inability to shift budget from low-cost to relatively higher-cost projects.

Corollary 1. *In the symmetric case, the optimal direct mechanism can be implemented by a descending-clock auction. The clock price, denoted by τ, will start at z^{**} and descend continuously down to $\frac{B}{n}$. Projects can drop out at any price but cannot re-enter. The auction stops once the clock price can be paid out to all projects remaining in the auction.*

We consider the descending-clock auction of corollary 1 to be a natural indirect mechanism that implements the outcome of the optimal z-mechanism.

Project i's equilibrium strategy, which implements this outcome, has it staying in the auction as long as the price is weakly larger than its private cost, $\tau \geq c_i$. It is easily verifiable that this is a weakly dominant strategy for project i.

The main advantages of clock auctions are twofold. Clock auctions are generally easy to understand and hard to manipulate. Furthermore they are less information hungry than, for example, sealed bid auctions. In this case, the designer only learns the private information of those projects that are not greenlighted. These features of clock auctions make them attractive for applications in which there is limited trust between the involved parties.

2.3 The asymmetric case

In the previous section, we examined the very special symmetric case. In this section, we demonstrate why the logic of the optimal mechanism in the symmetric case does not carry through to the asymmetric case. To preserve tractability, we restrict ourselves to the two-project case. However, now we allow for differing values as well as differing distributions for c_1 and c_2.

First note that we can draw on some of the observations from the symmetric case. We did not use symmetry in Lemma 1 and 2. Therefore, just as in the symmetric case, we are faced with a problem of finding the right cutoff functions. Similarly, the rewritten objective of the designer given by maximization problem (3) is also valid for the asymmetric case. The designer still wants to maximize the expected virtual value of greenlighted projects and allocating to projects with negative virtual value is not profitable.

$$\begin{gathered}\max_{z_1(c_2), z_2(c_1)} \mathbb{E}\Big[\mathbb{I}(c_1 \leq z_1(c_2)) \left(v_1 + c_1 + \tfrac{F_1(c_1)}{f_1(c_1)}\right) \\ + \mathbb{I}(c_2 \leq z_2(c_1)) \left(v_2 + c_2 + \tfrac{F_2(c_2)}{f_2(c_2)}\right)\Big] \\ \text{s.t.} \\ \mathbb{I}(c_1 \leq z_1(c_2)) z_1(c_2) + \mathbb{I}(c_2 \leq z_2(c_1)) z_2(c_1) \leq B \quad \forall c_1, c_2\end{gathered} \tag{6}$$

In the symmetric case, the order of virtual values coincides with the reversed order of costs. A natural way to extend the optimal allocation to the asymmetric case is to adjust the cutoffs so that they equalize virtual value. We will call this the *candidate* allocation. The corresponding descending-clock auction would have project i's clock start at the now individual z_i^{**}. Then clocks descend while keeping the virtual value implied by the clock price equal for all projects. Again, the auction stops when the entire sum of active clock prices can be covered. However, the designer can generally gain a higher expected payoff with another mechanism.

The condition for optimality of the candidate allocation is stated in proposition 3. By regularity, there can only be one z such that $\psi_1(z) = \psi_2(B - z)$. To implement the candidate allocation, the constant cutoffs at which both projects are greenlighted must be this z for project 1 and $B - z$ for project 2. But then, we only obtain optimality if $\frac{F_2(B-z)}{f_2(B-z)} = \frac{F_1(z)}{f_1(z)}$. The intuition behind this statement is straightforward. Selecting z in order to satisfy $\psi_1(z) = \psi_2(B - z)$ allows the designer to always get the project with the higher virtual value, if she cannot get both. However, if $\frac{F_2(B-z)}{f_2(B-z)} \neq \frac{F_1(z)}{f_1(z)}$ the cutoffs z and $B - z$ will not maximize the probability to get both projects.

Therefore, the two goals of the designer - getting the projects with the highest virtual value and getting as many projects as possible - are only aligned if the condition of proposition 3 is met. Note that the condition is met by construction in the symmetric case. However, in an asymmetric environment the condition will be violated in general.

Proposition 3. *In the non-trivial asymmetric two-project case, i.e.* $n = 2$, $z_1^{**} + z_2^{**} > B$, *where values or cost distributions differ across projects, it is not generally optimal to always allocate to the project with the higher virtual value. In fact, it is only optimal if there exists a cutoff* z *such that*

$$\begin{aligned} \psi_1(z) &= \psi_2(B - z) \\ \frac{F_1(z)}{f_1(z)} &= \frac{F_2(B - z)}{f_2(B - z)}, \end{aligned}$$

which is not generally the case.

Proof. Given that we are in the non-trivial case, $z_1^{**} + z_2^{**} > B$, we know from lemma 4 that the cutoffs must be constants whenever both projects are greenlighted. Furthermore, we know that these constants must add up to the budget. We will call project 1's cutoff z and project 2's cutoff $B - z$. These cutoffs pin down the allocation if at least one project has cost below its constant cutoff. Otherwise, we are free to choose the allocation. A glance at the objective function (6) reveals that in such a case it is optimal to greenlight the project with higher but still positive virtual value, if feasible.

This allows us to rewrite the objective function (6) as a function of z.

$$\begin{aligned}\pi(z) = &\int_0^z \psi_1(c_1)dF_1(c_1) + \int_0^{B-z} \psi_2(c_2)dF_2(c_2)\\ &+ \int_{\max\{\psi_2^{-1}(\psi_1(z)),B-z\}}^{\bar{c}_2} \int_z^{\min\{\psi_1^{-1}(\psi_2(c_2)),z_1^{**},B\}} \psi_1(d)dF_1(d)dF_2(c_2)\\ &+ \int_{\max\{\psi_1^{-1}(\psi_2(B-z)),z\}}^{\bar{c}_1} \int_{B-z}^{\min\{\psi_2^{-1}(\psi_1(c_1)),z_2^{**},B\}} \psi_2(d)dF_2(d)dF_1(c_1)\end{aligned}$$

To obtain the derivative with respect to z we can use the rules for differentiation under the integral sign.[7] Given the max operators, the derivative will take a different form depending on whether $\psi_1(z) \gtrless \psi_2(B-z)$. However, as $\pi(z)$ is continuously differentiable, it suffices to look at one of the two forms.

$$\begin{aligned}\left.\frac{\partial \pi}{\partial z}\right|_{z:\psi_1(z)\geq\psi_2(B-z)} = &\int_z^{\psi_1^{-1}(\psi_2(B-z))} \psi_1(x)dF_1(x)f_2(B-z)+\\ &+ \psi_1(z)f_1(z)F_2(B-z)\\ &- \psi_2(B-z)f_2(B-z)F_1(\psi_1^{-1}(\psi_2(B-z)))\end{aligned}$$

Now take the z corresponding to the candidate allocation with $\psi_1(z) = \psi_2(B-z)$. In this case we are left with

$$\frac{\partial \pi}{\partial z} = 0 \Leftrightarrow \frac{F_2(B-z)}{f_2(B-z)} = \frac{F_1(z)}{f_1(z)}$$

□

[7] Define $g(z, c_2) := \int_z^{\min\{\psi_1^{-1}(\psi_2(c_2)),z_1^{**},B\}} \psi_1(x)dF_1(x)f_2(c_2)$ and then use $\frac{d}{dz}\left(\int_{a(z)}^{b(z)} g(z,c_2)dc_2\right) = g(z,b(z))b'(z) - g(z,a(z))a'(z) + \int_{a(z)}^{b(z)} g_z(z,c_2)dc_2$.

The simplest way to lay out the intuition behind proposition 3 is by an example. Consider example 1. The candidate allocation demands cutoffs $\widetilde{z_1}^2 = 0.625$ and $\widetilde{z_2}^2 = 0.375$ for allocating to both projects. At these cutoffs, the probability of allocating to both projects is $0.625 \cdot 0.375 \approx 0.234$. This allocation is depicted in panel 4a. Now the maximal feasible probability to allocate to both projects is at equal cutoffs, $\widehat{z_1}^2 = \widehat{z_2}^2 = 0.5$. The corresponding area is the dotted square in the lower-left corner of panel 4b. However, at these cutoffs it is not incentive compatible to always allocate to the project with higher virtual value if at least one project exceeds $\widehat{z_i}^2$ - i.e. to allocate along the dotted diagonal line.[8] Hence, incentive compatibility introduces a trade-off between maximizing the probability of allocation to both projects and allocating to the preferred one if only one project is feasible. Consequently, the optimal cutoffs (z_1^*, z_2^*) for allocating to both projects do not lie at $(0.625, 0.375)$ - given that the condition of proposition 3 fails - but rather at $(0.53, 0.47)$.

Example 1. *There are two projects, $(n = 2)$, with $v_1 = 5, v_2 = 4.5$, and $c_1, c_2 \sim U[0,1]$. The budget is given by $B = 1$. The optimal cutoff functions are given by:*

$$z_1(c_2) = \begin{cases} 0.53 & \text{if } c_2 \leq 0.47 \\ c_2 + 0.25 & \text{if } 0.47 < c_2 \leq 0.75 \\ 1 & \text{if } c_2 > 0.75 \end{cases}$$

$$z_2(c_1) = \begin{cases} 0.47 & \text{if } c_1 \leq 0.72 \\ c_1 - 0.25 & \text{if } c_1 > 0.72 \end{cases}$$

The corresponding optimal allocation is:

$$(q_1, q_2) = \begin{cases} (1,1) & \text{if } 0 \leq c_1 \leq 0.53 \text{ and } 0 \leq c_2 \leq 0.47 \\ (1,0) & \text{if } 0 \leq c_1 \leq 0.72 \text{ and } c_2 > 0.47 \\ (1,0) & \text{if } c_1 > 0.72 \text{ and } \psi_1 \geq \psi_2 \\ (0,1) & \text{if } 0.53 < c_1 \leq 0.72 \text{ and } c_2 \leq 0.47 \\ (0,1) & \text{if } c_1 > 0.72 \text{ and } \psi_1 < \psi_2 \end{cases}$$

[8] Not to be confused with the dashed diagonal representing the budget constraint.

The corresponding transfers are:

$$t_1(c_1, c_2) = \begin{cases} 0.53 & \text{if } c_2 \leq 0.47 \text{ and } c_1 \leq 0.53 \\ c_2 + 0.25 & \text{if } 0.47 < c_2 \leq 0.75 \text{ and } c_1 \leq c_2 + 0.25 \\ 1 & \text{if } c_2 > 0.75 \\ 0 & \text{otherwise} \end{cases}$$

$$t_2(c_1, c_2) = \begin{cases} 0.47 & \text{if } c_1 \leq 0.72 \text{ and } c_2 \leq 0.47 \\ c_1 - 0.25 & \text{if } c_1 > 0.72 \text{ and } c_2 < c_1 - 0.25 \\ 0 & \text{otherwise} \end{cases}$$

Example 2. *There are two projects, ($n = 2$), with $v_1 = v_2 = 5$,$c_1 \sim U[0, 1]$, and*

$$F_2(c_2) = \begin{cases} 1 & \text{if } c_2 > 1 \\ c_2^{\frac{1}{3}} & \text{if } 0 \leq c_2 \leq 1 \\ 0 & \text{if } 0 \leq c_2 < 0 \end{cases}$$

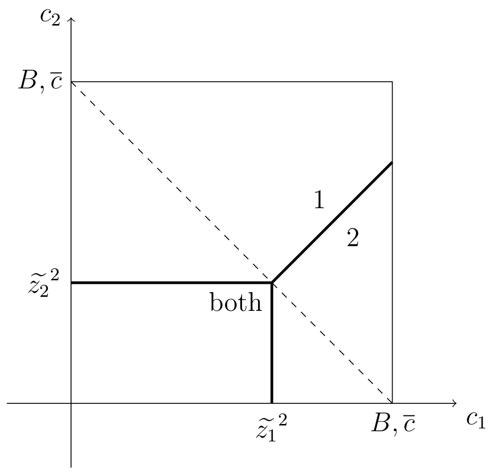
The budget is given by $B = 1$. The optimal cutoff functions are given by:

$$z_1(c_2) = \begin{cases} 0.56 & \text{if } c_2 \leq 0.44 \\ 2c_2 & \text{if } 0.44 < c_2 \leq 0.5 \\ 1 & \text{if } c_2 > 0.5 \end{cases}$$

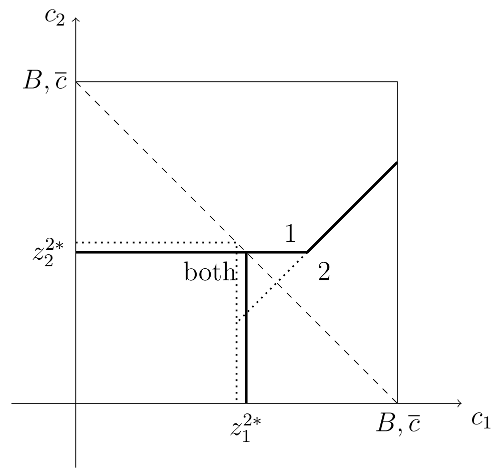
$$z_2(c_1) = \begin{cases} 0.44 & \text{if } c_1 \leq 0.88 \\ \frac{1}{2}c_1 & \text{if } c_1 > 0.88 \end{cases}$$

Allocation and transfers are omitted but can be easily computed from the cutoff functions as in example 1.

Given the optimal allocation in example 1, there are some realizations of the cost vector in which the designer allocates to the project with lower virtual value. These realizations are represented by the shaded area in panel 5a. Here, (IC), (PC), and the choice of (z_1^{2*}, z_2^{2*}) force the designer to allocate to project 2, even though project 1 has the higher virtual value.



(a) Candidate allocation

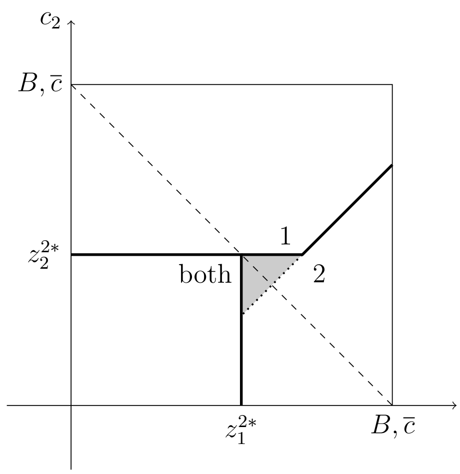


(b) Optimal allocation

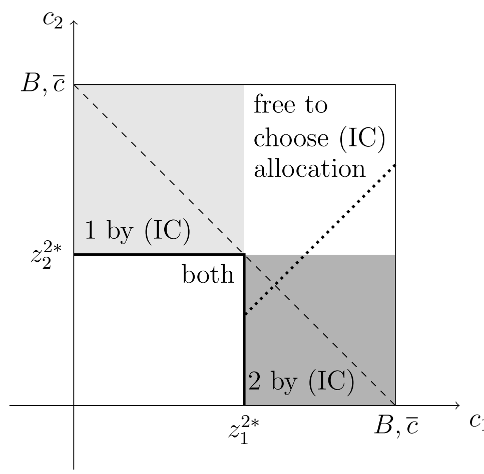
Figure 4: Candidate and optimal allocation for example 1

The cost vectors for which the designer allocates to both projects is represented by the rectangular area in the lower-left corner of panel 5a. The upper-right corner of this area lies on the dashed line representing the budget constraint. A point (z_1^2, z_2^2) on this line has $z_1^2 + z_2^2 = B$. Moving this corner point on the dashed line to the right has two effects: shrinking the shaded area and shrinking the area of the rectangle. While it is desirable to shrink the shaded area, in which the designer must allocate to project 2 despite its lower virtual value, shrinking the size of the rectangle lowers the probability of allocating to both projects. Given that we have an interior solution in this example, at (z_1^{2*}, z_2^{2*}) these two effects balance each other out.

Graphically, the fact that there is no slack in the budget constraint whenever both projects are greenlighted implies that the area representing points at which both projects are executed touches the dashed line representing the (BC)-constraint at least once, as can be seen for example in panel 5b. In fact, it can touch the (BC)-constraint exactly once, as it is not possible to allocate to both projects when $c_1 > z_1^{2*}$ or $c_2 > z_2^{2*}$ without violating (BC) sometimes. As it is optimal to greenlight both projects whenever possible, this result means that the area where both projects are greenlighted is the rectangle with corners $(0, 0)$ and (z_1^{2*}, z_2^{2*}). Then, if $c_1 < z_1^{2*}$ but $c_2 > z_2^{2*}$, the nature of cutoffs prevents that the designer greenlights project 2. Therefore, project 1 must be greenlighted, as represented by the lightly shaded area in panel 5b. A symmetric argument applies to the darkly shaded area. Thus, looking at panel 5b, the choice of (z_1^{2*}, z_2^{2*}) determines the allocation for all points except those in the upper-right corner. Here, the designer is free to choose the allocation, as long as the line delineating whether project 1 or 2 gets greenlighted is (weakly) increasing or vertical. Not surprisingly, it is optimal to greenlight the project with the higher virtual value.



(a) Greenlighting the project with lower virtual value



(b) (IC)-constraints on the allocation

Figure 5: Greenlighting the project with lower virtual value and (IC)-constraints on the allocation (example 1)

In proposition 3 and example 1 we have addressed the existence of the tradeoff between quantity and quality, for generic asymmetric environments. Even though the designer always prefers the project with the higher virtual value, if she was to greenlight a single project, she sometimes greenlights the project with lower virtual value out of two rival projects, as quantity is endogenous here. Graphically, this is represented by the shaded triangle in panel 5a. An interesting consequence of this tradeoff is that it mitigates the discrimination against the stochastically stronger project compared to the case where quantity is exogenous.

To illustrate this property, consider example 2.[9] Here the designer chooses

[9]We choose to make this point by example. However, it should be clear that this point

among two projects with identical value but different cost distributions. Project 2 is stochastically stronger than project 1 in the sense that F_1 first-order stochastically dominates F_2 and therefore project 2 tends to have lower cost. In figure 6 the 45°-line represents the efficient allocation if only a single unit is procured. The dashed line below represents the allocation chosen by a designer maximizing her own payoff in the single unit case. Consequently, the horizontally dashed wedge in between represents the cost vectors where the discrimination of project 2 creates an inefficiency. When quantity is endogenous however the inefficiency is mitigated. The size of this effect depends on the distributions and in figure 6 corresponds to the shaded triangle. In contrast to the case where quantity is exogenously given, here the designer allocates efficiently.

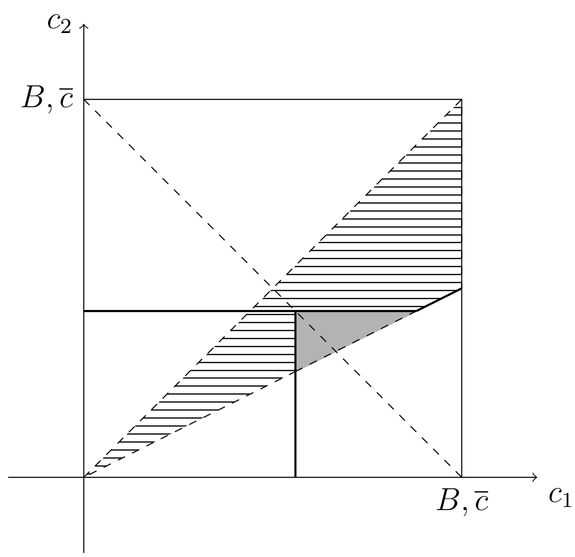


Figure 6: The endogenous quantity mitigates the distortion against the stochastically stronger project (example 2)

We can implement the optimal mechanism for example 1 with individual and asynchronous descending price clocks. The price clocks not only run at individual speeds, occasionally one clock also has to halt. The auction ends either when one of the projects drops out or when the clocks reach $\tau_1 = z_1^{2*}$ and $\tau_2 = z_2^{2*}$ - whichever happens first. These price clocks are depicted in figure 7 as a function of time. Note that the entire (maximal) duration of the

can easily be generalized.

auction can be divided into three segments. The auction starts with both clocks at $z_1^{**} = z_2^{**} = \bar{c}$. First, τ_2 decreases while τ_1 is held constant, which happens until both clocks imply the same virtual value, i.e. $\psi_2(\tau_2) = \psi_1(\bar{c}_2)$. Second, both τ_1 and τ_2 decrease synchronously, keeping virtual value equal, $\psi_1(\tau_1) = \psi_2(\tau_2)$, until $\tau_2 = z_2^{2*}$. Third, now only τ_1 decreases until $\tau_1 = z_1^{2*}$. If at this point both projects still remain in the auction, the auction stops and both are greenlighted.

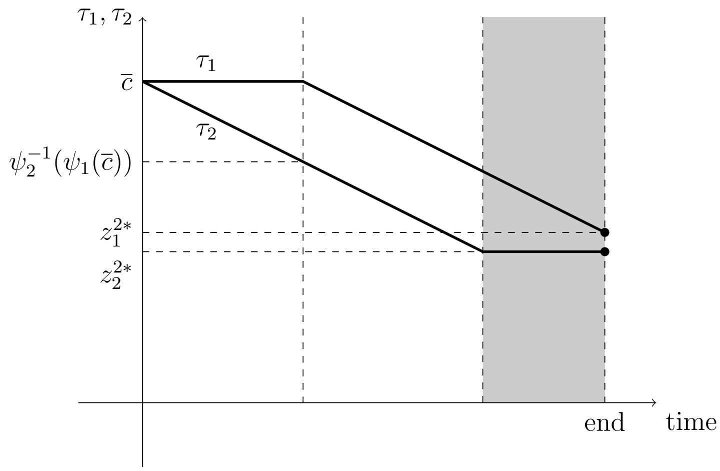


Figure 7: Optimal descending-clock auction in example 1

The cost vectors for which the designer greenlights project 2 despite its lower virtual value, represented by the shaded area in panel 5a, are also represented graphically in figure 7. If the auction ends in the third time segment (shaded area of figure 7) before both projects can be greenlighted, project 1 must have dropped out because τ_1 dropped below c_1. Project 2 is greenlighted and receives transfer z_2^{2*} even though project 1 has the higher virtual value. Therefore, if cost vectors in the shaded area of panel 5a realize, the optimal descending clock auction will end in the third time segment.

We should note again a novel feature of this descending-clock auction. The clocks of both projects are paused asynchronously for some time of the auction. One project's clock runs down while the other project's clock is paused. Since we have examined a very simple example, each project's clock is paused only once. In a more general setting, the projects' clocks could possibly pause and resume several times.

Given the nature of our problem, we do not find a simple and general characterization of the optimal mechanism in the asymmetric case. In our example 1 with two projects, the problem boils down to finding one point, (z_1^{2*}, z_2^{2*}), with respect to one crucial tradeoff. Naturally, the number of relevant tradeoffs increases with the number of projects. Therefore, unfortunately, characterizing the optimal allocation with a larger set of projects quickly becomes analytically intractable.

3 Discussion

Considering our model as a starting point, several extensions come to mind. In this section, we will address the most natural alternative models or extensions.

Disregarding residual money - It depends on the setting whether it is reasonable to assume that the designer values residual money. To illustrate, this is not the case in Ensthaler and Giebe (2014a) where money does not enter the objective function, but only the constraints. Such an assumption especially suits applications where quality-ranking the projects is possible, but it is hard to determine a monetary value of said quality. Then, an ordinal ranking would suffice. Note that in such a setting, the designer would want to allocate to projects with negative virtual value and would be indifferent between paying z^{k*} or c_{k*+1} in the optimal symmetric mechanism. Other than that, qualitatively our results carry over.

v_i **as private information, maybe correlated with** c_i - We can neglect asking for v_i directly since no meaningful non-babbling equilibria in the v_i-dimension exist. If the conditional density of $v_i|c_i$ has full support, project i can not credibly announce to be a "high" type, say $\overline{v}_i$. If we slightly change the regularity assumption such that $\mathbb{E}[v_i|c_i]-c_i-\frac{F(c_i)}{f(c_i}$ must be strictly increasing, our results generalize by exchanging common knowledge v_i with $\mathbb{E}[v_i|c_i]$. Note that this regularity condition mildly restricts the degree of positive correlation.

Interdependent types - We can interpret the symmetric case as identical projects that can be provided at individual costs. Hence, one may wonder about a setting where projects only draw an imperfect signal about the cost which finally depends on other projects' signals as well. In such an environment, it is clear that our price clock implementation can only be optimal when it is not publicly revealed when high signal projects drop out. In such an environment, the designer can adjust the cutoffs with information from the reports of projects that dropped out. This analysis is left for a follow-up paper.

Other interesting extensions are left for future research, for example: Multiple projects per agent. For practitioners, a simple approximately optimal mechanism such as Ensthaler and Giebe (2014b) may be of great value.

4 Conclusion

Despite their importance, knapsack problems with private information have been somewhat overlooked by the literature on economics. We examine a setting where a budget-constrained procurer faces privately informed sellers under ex-post constraints. Amongst many possible economic problems, this applies to subsidy allocation as well as scientific research funding, where funding institutions are typically endowed with a fixed budget and want to finance both many projects and projects of high quality. Such problems often entail relationships in which sellers can renege on the terms of the agreement ex-post. In order to avoid non-delivery or costly renegotiation it is then appropriate to impose ex-post constraints on the seller's side. For such settings, we have have shown that z-mechanisms constitute the class of optimal deterministic dominant strategy implementable mechanisms. Moreover, we propose an implementation with a descending clock auction that is easy to understand and could be used in practice.

A z-mechanism is described by a set of cutoff functions that are increasing in other projects' costs. Cutoffs only depend on the cost of other projects as they drop out of the allocation. In other words, if two different realizations of the cost vector lead to the same allocation, then the cutoffs of projects conducted only vary in the costs of projects not conducted. This feature allows for a simple implementation via descending clocks.

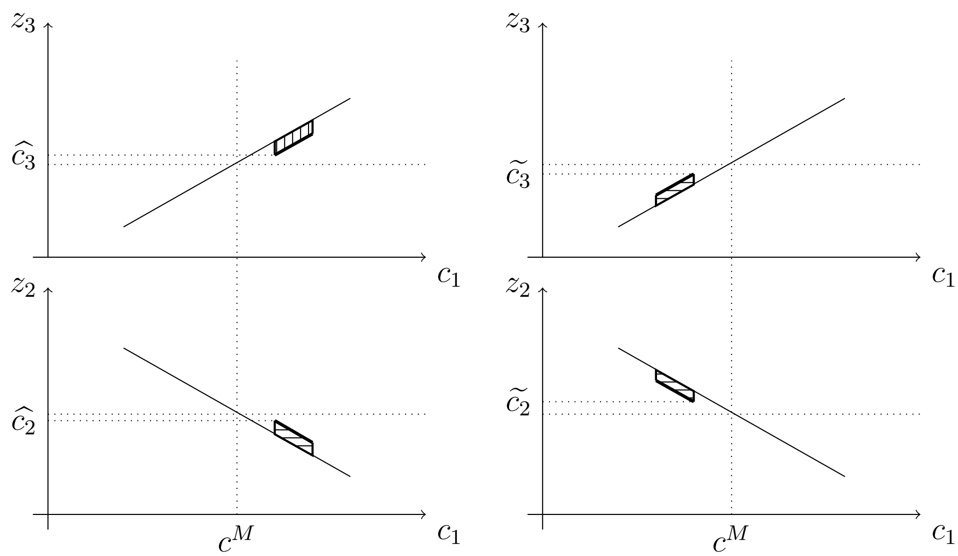
We fully describe the optimal allocation and the corresponding descending clock auction in an environment where projects are ex-ante symmetric. The optimal mechanism is monotonic in the sense that the k cheapest projects are greenlighted and all projects conducted receive the same transfer. This transfer corresponds to either the lowest cost among non-executed projects or the budget is distributed equally. The equivalent clock auction features a single price clock that continuously decreases until all active projects can be financed.

For asymmetric environments, where values and/or cost distributions differ, we derive a novel tradeoff between quantity and quality of the greenlighted projects. The designer values both quantity and quality, expressed by the virtual value, of the projects. In settings where quantity is exogenous, the designer would always choose the projects with the highest virtual values. Here quantity is endogenously determined by the mechanism and therefore it is not always desirable to conduct the best projects. In doing so incentive compatibility would force the designer to reduce the expected number of greenlighted projects. This insight entails a consequence for the corresponding descending clock auction. Clocks not only run asynchronously but also periodically have to stop for certain projects. In comparison to settings where quantity is exogenous, here the allocation is less distorted away from efficiency, i.e. stochastically weaker projects are favored less.

Finally, we think the class of problems discussed is relevant. We hope that our methodological approach will contribute to a better understanding of such problems and open the door for future research in this area.

Appendix

Lemma 3. *The optimal cutoff function* $z_i(c_{-i-j}, c_j)$ *is almost everywhere equal to a left-continuous function that is weakly increasing in* c_j *for all* i, j *with* $j \neq i$, *i.e.* $z_i(c_j, c_{-i-j}) \geq z_i(c_j', c_{-i-j})$ *for almost every* $(c_j, c_j') : c_j > c_j'$ *and* c_{-i-j}.

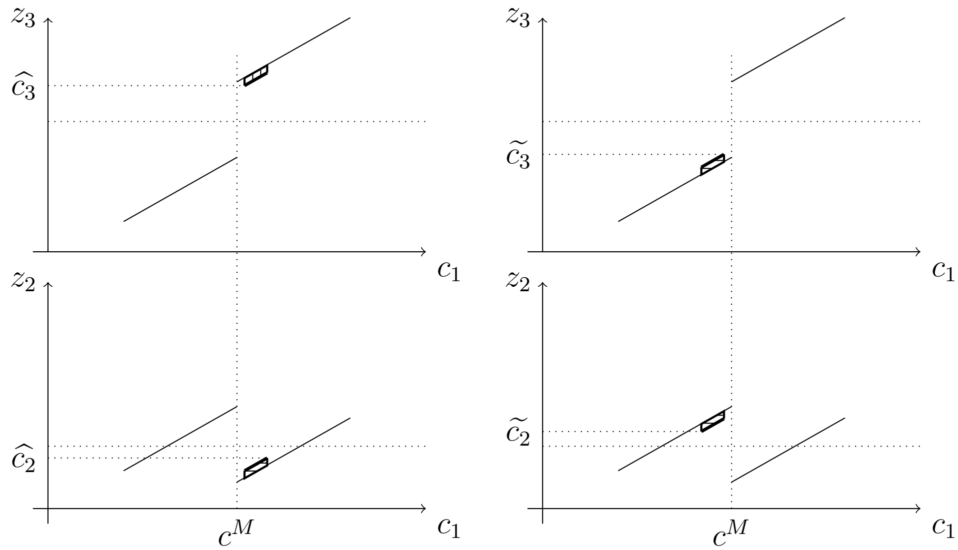


(a) Intuition for the hat deviation. (b) Intuition for the tilde deviation.

Figure 8: Continuous decrease / increase

Proof. Suppose to the contrary, that somewhere $z_2(\cdot)$ is decreasing in c_1. Then there exists a ${c_1}^M$ and $\eta > 0$ such that $z_2(\underline{\underline{c_1}}, c_{-1-2}) > z_2(\overline{\overline{c_1}}, c_{-1-2})$ for all $\underline{\underline{c_1}} \in ({c_1}^M - \eta, {c_1}^M)$, for all $\overline{\overline{c_1}} \in ({c_1}^M, {c_1}^M + \eta)$, and for all $c_{-1-2} \in \chi_{-1-2} \subset \times_{j \in I \setminus \{1,2\}} [\underline{c}_j, \overline{c}_j]$, where χ_{-1-2} has positive Lebesgue-measure.

With more than two projects the simple deviation of the two project case - flattening the decreasing cutoff - must not necessarily be feasible. It could be the case that other projects' cutoffs are strictly increasing and that for some cost vectors these cutoffs have to be paid along z_2. Then simply flattening z_2 could violate the budget constraint.



(a) Intuition for the hat deviation. (b) Intuition for the tilde deviation.

Figure 9: Jump decrease / increase

Suppose there are no such cutoffs. Then a decreasing z_2 cannot be optimal and flattening z_2 will increase the designer's payoff much in the same way as in the two-project-case. Otherwise, pick a subset of $\widehat{\chi}_1 \subset ({c_1}^M, {c_1}^M + \eta)$ (with pos. Lebesgue-measure) such that w.l.o.g. project 3's cutoff increases in c_1 in the analogous sense to the decrease of z_2 defined above - for cost vectors where both project 2 and project 3 are eventually greenlighted, i.e. z_2 and z_3 need to be paid both.

The set

$$\begin{aligned}\widehat{\Xi}_{23}(c_1, c_{-1-2-3}, \delta) = \{(c_2, c_3) | c_2 \in (z_2(c_1, c_3, c_{-1-2-3}), z_2(c_1, c_3, c_{-1-2-3}) + \delta]; \\ c_3 \in (z_3(c_1, c_2, c_{-1-2-3}) - \delta, z_3(c_1, c_2, c_{-1-2-3})]\}\end{aligned}$$

must have positive measure on $\mathbb{R}^2$ for all $c_1 \in \widehat{\chi}_1$ and for any $c_{-1-2-3} \in \chi_{-1-2-3}$, where χ_{-1-2-3} is a set with positive Lebesgue measure where the cutoff of project 2 is decreasing while the cutoff of project 3 is increasing. It is the set of (c_2, c_3) tuples, where c_2 just exceeds z_2 by no more than δ, while c_3 lies just below z_3 by no more than δ - given c_{-1-2-3} and c_1. By

$\widehat{\Xi}^2_{23}(c_1, c_{-1-2-3}, \delta)$ we denote the set of project 2 components of tuples in the set $\widehat{\Xi}_{23}(c_1, c_{-1-2-3}, \delta)$, and similarly for project 3.

Now deviate from the candidate mechanism in setting

$$\begin{aligned}
\widehat{z_2}(c_1, c_3, c_{-1-2-3}) &:= z_2(c_1, c_3, c_{-1-2-3}) + \delta \\
\widehat{z_3}(c_1, c_2, c_{-1-2-3}) &:= z_3(c_1, c_2, c_{-1-2-3}) - \delta \\
&\text{for all} \\
c_1 &\in (\widehat{c_1}, \widehat{c_1} + \varepsilon) \\
c_2 &\in \widehat{\Xi}^2_{23}(c_1, c_{-1-2-3}) \\
c_3 &\in \widehat{\Xi}^3_{23}(c_1, c_{-1-2-3}) \\
c_{-1-2-3} &\in \widehat{\chi}_{-1-2-3} \subset \chi_{-1-2-3}.
\end{aligned}$$

We call this deviation the *hat* deviation. The intuition for this deviation is the following. For an ε-environment of c_1 to the right of c_1^M (i.e. $\widehat{c_1} > c_1^M$), increase the decreasing cutoff $z_2(c_1, c_3, c_{-1-2-3})$ by δ for all c_3 that drop out of the allocation if $z_3(c_1, c_2, c_{-1-2-3})$ (at c_2) is decreased by δ. Likewise only increase $z_3(c_1, c_2, c_{-1-2-3})$ by δ for those c_2 that are additionally greenlighted if $z_2(c_1, c_3, c_{-1-2-3})$ is increased by δ. Therefore, if the deviation changes the allocation, project 2 is now greenlighted whereas project 3 is not.

This deviation is feasible. Remember that there must be enough budget to pay both z_2 and z_3 - otherwise flattening z_2 would have been possible. But then there is enough budget for $z_2 + \delta$ and $z_3 - \delta$.

Now define

$$\begin{aligned}
\widehat{c_2} &:= \sup_{c_1, c_{-1-2-3}} \widehat{\Xi}^2_{23}(c_1, c_{-1-2-3}) \\
\widehat{c_3} &:= \inf_{c_1, c_{-1-2-3}} \widehat{\Xi}^3_{23}(c_1, c_{-1-2-3}) \\
&\text{s.t.} \\
c_1 &\in (\widehat{c_1}, \widehat{c_1} + \varepsilon) \\
c_{-1-2-3} &\in \widehat{\chi}_{-1-2-3}.
\end{aligned}$$

In words, to get bounds on the payoff change we let $\widehat{c_2}$ be the highest cost type gained by the deviation and we let $\widehat{c_3}$ be the lowest cost type lost by the deviation. Then the change in payoff for the hat deviation is bounded in the following way:

$$\widehat{\Delta} > (\psi_2(\widehat{c_2}) - \psi_3(\widehat{c_3}))* \\ \int_{\widehat{\chi}_{-1-2-3}} \int_{\widehat{c_1}}^{\widehat{c_1}+\varepsilon} \int_{\widehat{\Xi}^2_{23}(c_1, c_{-1-2-3})} \int_{\widehat{\Xi}^3_{23}(c_1, c_{-1-2-3})} 1 dF_3(\cdot) dF_2(\cdot) dF_1(\cdot) dF_{-1-2-3}(\cdot).$$

If $\widehat{\Delta} > 0$, we are set. If not, then consider the following *tilde* deviation.

Analogously to $\widehat{\Xi}_{23}$ we define the set

$$\widetilde{\Xi}_{23}(c_1, c_{-1-2-3}, \delta) = \{(c_2, c_3) | c_2 \in (z_2(c_1, c_3, c_{-1-2-3}) - \delta, z_2(c_1, c_3, c_{-1-2-3})]; \\ c_3 \in (z_3(c_1, c_2, c_{-1-2-3}), z_3(c_1, c_2, c_{-1-2-3}) + \delta]\}$$

which again must have positive measure.

Now we deviate for an ε-environment to the left of c_1^M (i.e. $\widetilde{c_1} < c_1^M$). But instead of increasing z_2 and decreasing z_3, we increase z_3 and decrease z_2.

$$\begin{aligned} \widetilde{z_2}(c_1, c_3, c_{-1-2-3}) &:= z_2(c_1, c_3, c_{-1-2-3}) - \delta \\ \widehat{z_3}(c_1, c_2, c_{-1-2-3}) &:= z_3(c_1, c_2, c_{-1-2-3}) + \delta \\ &\text{for all} \\ c_1 &\in (\widetilde{c_1} - \varepsilon, \widetilde{c_1}) \\ c_2 &\in \widetilde{\Xi}^2_{23}(c_1, c_{-1-2-3}) \\ c_3 &\in \widetilde{\Xi}^3_{23}(c_1, c_{-1-2-3}) \\ c_{-1-2-3} &\in \widetilde{\chi}_{-1-2-3} \subset \chi_{-1-2-3}. \end{aligned}$$

The relevant bounds to bound the payoff are then given by

$$\widetilde{c_2} := \inf_{c_1, c_{-1-2-3}} \widetilde{\Xi}^2_{23}(c_1, c_{-1-2-3})$$
$$\widetilde{c_3} := \sup_{c_1, c_{-1-2-3}} \widetilde{\Xi}^3_{23}(c_1, c_{-1-2-3})$$
$$\text{s.t.}$$
$$c_1 \in (\widetilde{c_1} - \varepsilon, \widetilde{c_1})$$
$$c_{-1-2-3} \in \widetilde{\chi}_{-1-2-3}.$$

And this gives the following bound for the payoff

$$\widetilde{\Delta} > (\psi_2(\widetilde{c_3}) - \psi_3(\widetilde{c_2}))*$$
$$\int_{\chi_{-1-2-3}} \int_{\widetilde{c_1}-\varepsilon}^{\widetilde{c_1}} \int_{\widetilde{\Xi}^2_{23}(c_1, c_{-1-2-3})} \int_{\widetilde{\Xi}^3_{23}(c_1, c_{-1-2-3})} 1 dF_3(\cdot) dF_2(\cdot) dF_1(\cdot) dF_{-1-2-3}(\cdot).$$

By appropriately choosing δ, $\widehat{\Xi}_{-1-2-3}$, and $\widetilde{\Xi}_{-1-2-3}$, we can ensure that $\widehat{c_3} > \widetilde{c_3}$ and $\widehat{c_2} < \widetilde{c_2}$. This follows simply from the notion of increasing/decreasing cutoffs and is illustrated in figures 8 and 9. Therefore $\widehat{\Delta} \leq 0$ implies $\widetilde{\Delta} > 0$. Consequently, there is always a profitable deviation and our candidate mechanism could not have been optimal. □

Lemma 4. *Conditional on any arbitrary partition $\{G, R\}$, the optimal cutoff functions z_g for all $g \in G$ are independent of costs of all greenlighted projects c_G. That is,*
$$z_g(c_{-g}) = z_g(c_R).$$

Proof. Take any feasible candidate mechanism with any increasing cutoff functions $z_i(\cdot)$ for any individual project. Assume that for some cost vectors with positive Lebesgue-measure, only all projects in set $G \subseteq I$ are executed while all projects of set R are not conducted. Therefore, there exists a set, C^G_R, of cost vectors of projects of set R with positive Lebesgue-measure where $a^G_i(\mathbf{c}_R)$ according to the following definition

$$\begin{aligned} a^G_i(\mathbf{c}_R) = \max\{c_i | \exists \mathbf{c}_{G-i} : c_i \leq z_i(\mathbf{c}_{G-i}, \mathbf{c}_R), \\ \text{and } c_g \leq z_g(\mathbf{c}_{G-j}, \mathbf{c}_{-G}) \forall g \in G, \\ \text{and } c_r > z_r(\mathbf{c}_G, \mathbf{c}_{-G-r}) \forall r \in R\} \end{aligned} \quad (7)$$

exists for all $i \in G$ given some cost vector $\mathbf{c}_R \in C_R^G$. In words, $a_i^G(\mathbf{c}_R)$ is the highest cost of project i such that, given some cost vector $\mathbf{c}_R$ of projects that are not executed, there exists some vector $\mathbf{c}_{G-i}$ of costs of competing projects that induces a cutoff $z_i(\mathbf{c}_{G-i}, \mathbf{c}_{-G})$ above said cost while each element c_g of the vector $\mathbf{c}_{G-i}$ is lower than the cutoff induced by $a_i^G(\mathbf{c}_R)$ and the elements of the cost vectors $\mathbf{c}_R$ and $\mathbf{c}_{G-i-g}$,

$$\forall c_g \in \mathbf{c}_{G-i},\ \ c_g \leq z_g(\mathbf{c}_R, \mathbf{c}_{G-i-g}, a_i^G(\mathbf{c}_R)).$$

Simultaneously, it must hold that these costs induce a cutoff such that no project $r \in R$ is conducted

$$\forall c_r \in \mathbf{c}_R,\ \ c_r > z_r(\mathbf{c}_{R-r}, \mathbf{c}_{G-i}, a_i^G(\mathbf{c}_R)).$$

Moreover, we can replace any function $z_i(\cdot)$ with a left-continuous function that is identical up to a set of points with Lebesgue-measure zero. Hence, the limit is reached from below and there exists at least one cost vector $(\widehat{\mathbf{c}}_{-i}, a_i^G(\mathbf{c}_R))$ where G is the set of executed projects and $a_i^G(\widehat{\mathbf{c}}_R) = z_i(\widehat{\mathbf{c}}_{-i})$ holds. Now, notice that

$$\widehat{c}_g \leq a_g^G(\widehat{\mathbf{c}}_R)\ \forall \widehat{c}_g \in \widehat{\mathbf{c}}_{G-i},$$

because, given $\widehat{\mathbf{c}}_R$, there cannot exist a cost vector where only all projects in G are executed and the cost of project g exceeds $a_g^G(\widehat{\mathbf{c}}_R)$ by its construction. Moreover, we have established that every cutoff function $z_i(\cdot)$ is weakly increasing in each argument. Thus,

$$a_i^G(\widehat{\mathbf{c}}_R) = z_i(\widehat{\mathbf{c}}_{-i}) \leq z_i(a_{G-i}^G(\widehat{\mathbf{c}}_R), \widehat{\mathbf{c}}_R),$$

where a_{G-i}^G is the vector of all a_g^G defined according to (7) except a_i^G. This inequality tells us that, whenever some vector $(\mathbf{c}_R, \mathbf{c}_{G-i}) \geq (\widehat{\mathbf{c}}_R, a_{G-i}^G(\widehat{\mathbf{c}}_R))$[10] realizes, a sufficient condition for project $i \in G$ to be executed is $c_i \leq a_i^G(\widehat{\mathbf{c}}_R)$.

The same logic also applies to all projects in G other than i. Therefore, only all projects $g \in G$ are conducted whenever a cost vector realizes such that $a_i^G(\mathbf{c}_R)$ is defined for all $i \in G$[11] and where for each element $g \in G, c_g \leq$

[10]When $\mathbf{x}$ and $\mathbf{y}$ are vectors, $\mathbf{x} \geq \mathbf{y}$ means that every element x_i of $\mathbf{x}$ weakly exceeds the corresponding element y_i of $\mathbf{y}$.

[11]$a_i^G(\mathbf{c}_R)$ is only defined if $C^G \neq \emptyset$ and $\mathbf{c}_R \in C_R^G$, but this does not hinder the proof.

$a_g^G(\mathbf{c}_R)$ and for all $r \in R, c_r > z_r(\mathbf{c}_G, \mathbf{c}_{R-r})$. Consequently, the budget constraint requires that

$$\sum_{g \in G} z_g(a_{-g}^G(\mathbf{c}_R)) \leq B. \tag{8}$$

Furthermore, given $\mathbf{c}_R$, for all projects $g \in G$ $z_g(a_{-g}^G(\mathbf{c}_R), \mathbf{c}_R) = a_g^G(\mathbf{c}_R)$ if $\mathbf{c}_{G-g} \leq a_{G-g}^G(\mathbf{c}_{-G})$. That is, the cutoffs are constant given the cost vector of non-executed projects.

Suppose to the contrary that $z_i(\mathbf{c}_{-i}) < a_i(\mathbf{c}_R)$ for some $i \in G$ and for all $\mathbf{c}_{-i} \in \Xi \subset C_{-i}^G$ with Ξ having positive Lebesgue measure.

Define $\Xi(\mathbf{c}_{G-i-j}, \mathbf{c}_R) \subset [0, \bar{c}_j]$ where $z_i(\mathbf{c}_{G-i-j}, c_j, \mathbf{c}_R) < a_i^G(\mathbf{c}_R)$ for all $c_j \in \Xi(\mathbf{c}_{G-i-j}, \mathbf{c}_R)$. For any $\mathbf{c}_{G-i-j} \leq a_{-i-j}^G(\mathbf{c}_R)$, let

$$z_i^{\Xi}(\mathbf{c}_{G-i-j}, \mathbf{c}_R) := \max_{c_j \in \Xi(\mathbf{c}_{G-i-j}, \mathbf{c}_R)} z_i(\mathbf{c}_{G-i-j}, c_j, \mathbf{c}_R)$$

By (8), changing the mechanism to

$$z_i(\mathbf{c}_{G-i,-j}, c_j, \mathbf{c}_R) = a_i^G(\mathbf{c}_R), \quad \forall c_j \leq a_j^G(\mathbf{c}_R)$$

does not violate the budget-constraint. This deviation increases the payoff conditional on $\mathbf{c}_R$ by

$$\Delta > \int_{\Xi_{-j}} \Pr(c_j \in \Xi(\mathbf{c}_{G-i-j}, \mathbf{c}_R)) \int_{z_i^{\Xi}(\mathbf{c}_{G-i-j}, \mathbf{c}_R)}^{a_i^G(\mathbf{c}_R)} \psi_i(c) dF_i(c) dF_{-i-j}(c_{-i-j}) > 0.$$

Given that Ξ has positive Lebesgue-measure, this deviation will also strictly increase the unconditional payoff. □

Proposition 2. *Arrange the projects by cost in ascending order, $c_1 \leq c_2 \leq \cdots \leq c_n$ and define $z^k := \min\left\{\frac{B}{k}, z^{**}, c_{k+1}\right\}$. In the symmetric case, the z-mechanism with $z_i(c_{-i}) = z^{k^*}$ is the optimal budget-constrained mechanism. The optimal number of accepted projects k^* is given by $k^* := \max\{k | c_k \leq z^k\}$.*

Proof. The case $n = 2$ has been proven in section 2.2.

Now, consider $n = 3$. Fix any c_3 and any mechanism as candidate for optimality. Either $c_3 > z_3(c_1, c_2)$ or $c_3 \leq z_3(c_1, c_2)$. In the first case, project

3 is not executed and the budget remaining for the other two is still B. In the second case, project 3 is executed and the budget remaining for the other two becomes $B - z_3(c_1, c_2)$.

Now, consider deviating to the proposed mechanism only for project 1 and 2. The change in profit looks like a probability weighted sum of terms similar to the two project case, only that the distributions F are conditional on c_1 and c_2 being in some interval (that induces $z_3 >$ or $< c_3$) and the budget must be adjusted.

Because log-concavity of F implies log-concavity of $\frac{F(c)-F(a)}{F(b)-F(a)}$ this deviation is always positive like in the case $n = 2$. The same logic can be applied to any n, changing any mechanism by selecting two projects and then adjusting their cutoffs in the following way: The budget is shared equally if both projects are executed; if only one project is executed, it has to be the one with higher virtual value; never execute projects with negative virtual value. Iterating over these steps ultimately arrives at the proposed mechanism which has to be optimal. □

References

Baron, D. P., and R. Myerson (1982): "Regulating a monopolist with unknown costs," *Econometrica*, pp. 911–930.

Börgers, T., and P. Norman (2009): "A note on budget balance under interim participation constraints: the case of independent types," *Economic Theory*, 39(3), 477–489.

Che, Y.-K., and I. Gale (1998): "Standard auctions with financially constrained bidders," *The Review of Economic Studies*, 65(1), 1–21.

Chung, K.-S., and J. C. Ely (2002): "Mechanism Design with Budget Constraint," *Mimeo, Northwestern University.*

Dizdar, D., A. Gershkov, and B. Moldovanu (2011): "Revenue maximization in the dynamic knapsack problem," *Theoretical Economics*, 6(2), 157–184.

Dobzinski, S., C. H. Papadimitriou, and Y. Singer (2011): "Mechanisms for complement-free procurement," in *Proceedings of the 12th ACM conference on Electronic commerce*, pp. 273–282. ACM.

Ensthaler, L., and T. Giebe (2014a): "Bayesian optimal knapsack procurement," *European Journal of Operational Research*, 234(3), 774–779.

——— (2014b): "A dynamic auction for multi-object procurement under a hard budget constraint," *Research Policy*, 43(1), 179–189.

Ledyard, J., D. Porter, and R. Wessen (2000): "A market-based mechanism for allocating space shuttle secondary payload priority," *Experimental Economics*, 2(3), 173–195.

Luton, R., and R. P. McAfee (1986): "Sequential procurement auctions," *Journal of Public Economics*, 31(2), 181–195.

Maskin, E. (2002): "How to Reduce Greenhouse Gas Emissions: An Application of Auction Theory," *Nancy L. Schwartz Memorial Lecture, Northwestern University.*

Myerson, R. (1981): "Optimal Auction Design," *Mathematics of operations research*, pp. 58–73.

Pai, M. M., and R. Vohra (2014): "Optimal auctions with financially constrained buyers," *Journal of Economic Theory*, 150, 383–425.