UNIVERSITY OF MANNHEIM

University of Mannheim / Department of Economics

Working Paper Series

Ex-post Optimal Knapsack Procurement

Felix Jarman Vincent Meisner

Working Paper 15-02

May 2015

Ex-post Optimal Knapsack Procurement*

Felix Jarman† Vincent Meisner‡

May 28, 2015

Abstract

We consider a budget-constrained mechanism designer who selects an optimal set of projects to maximize her utility. A project's cost is private information and its value for the designer may vary. In this allocation problem, the selection of projects - both which and how many - is endogenously determined by the mechanism. The designer faces ex-post constraints: The participation and budget constraints must hold for each possible outcome while the mechanism must be implementable in dominant strategies. We derive the class of optimal mechanisms and show that it has a deferred acceptance auction representation. This feature guarantees an implementation with a descending clock auction. Only in the case of symmetric projects do price clocks descend synchronously such that the cheapest projects are executed. The case in which values or costs are asymmetrically distributed features a novel tradeoff between quantity and quality. Interestingly, this tradeoff mitigates the distortion due to the informational asymmetry compared to environments where quantity is exogenous.

JEL-Classification: D02, D44, D45, D82, H57.
Keywords: Mechanism Design, Knapsack, Budget, Procurement, Auction, Deferred Acceptance Auctions

*We thank Ludwig Ensthaler, Thomas Giebe, Hans Peter Grüner, Doh-Shin Jeon, Volker Nocke, Christoph Siemroth, André Stenzel, Thomas Tröger and participants of conferences and workshops in Berlin, Bielefeld, Lisbon, London, Mannheim, and Toulouse. Felix Jarman gratefully acknowledges financial support by the German Research Foundation (DFG) via SFB 884.
†CDSE, University of Mannheim, felix.jarman@gess.uni-mannheim.de
‡CDSE, University of Mannheim, vincent.meisner@gess.uni-mannheim.de

1 Introduction

We study the problem of a mechanism designer who can spend a fixed budget on a variable number of projects which differ in the value the designer derives from them. Projects (agents) have private information about their costs and want to get funding beyond the necessary minimum. The designer's goal is to select an affordable set of maximal aggregate quality. In other words, she faces a mechanism design variant of the knapsack problem with strategic behavior due to informational asymmetries.[1] Essentially, we approach this problem as an "up to possibly n-units" procurement problem with n agents with single-unit supply where demand quantity is determined after observing projects' reports under a budget constraint. The budget constraint, the individual rationality constraints, and the participation constraints are imposed ex-post, i.e., a project cannot be conducted when the assigned funds are insufficient, the sum of transfers must not exceed the budget for any cost realization, and truth telling must be a (weakly) dominant strategy.

This framework matches a large range of allocation problems, in which a designer needs to allocate a divisible but fixed capacity among agents. Allocation problems, in which a financial budget constraint represents the fixed capacity, include the procurement of bus lines, bridges, and streets, or the allocation of subsidies or research money. Alternatively, the capacity constraint can represent the payload limit on a freighter or on a space shuttle,[2] or a limited amount of time to be devoted to several tasks. Out of many suitable applications, we employ as our leading example a development fund, who desires to distribute money to nonprofit projects with non-monetary benefits.

Our paper not only helps to understand a class of economically relevant problems, the framework also presents a novel methodological challenge. The ex-post nature of both the participation and the budget constraint precludes the use of standard pointwise optimization techniques à la Myerson (1981). Nonetheless, rewriting the problem involves expressing expected transfers by the allocation function as an auxiliary step. As the designer maximizes expected payoff including residual money, we can employ the procurement analogue of Myerson's

[1]The knapsack problem is a classical combinatorial problem, dating as far back as 1897. A set of items is assigned values and weights. The knapsack should be filled with the maximal value, but can carry only up to a given weight. For an overview of the literature on knapsack problems see Kellerer, Pferschy, and Pisinger (2004).

[2]Clearly, the capacity of a space shuttle is limited. The problem of optimally allocating the capacity and incentivizing projects to reduce payload is economically relevant, see Ledyard, Porter, and Wessen (2000).

notion of "virtual surpluses". However, our results easily translate to a setting in which the designer does not value residual money.

By focusing on dominant strategy implementable deterministic mechanisms, we can reduce the problem to finding a set of optimal cutoff functions. These cutoff functions exhibit certain properties, and we call the corresponding mechanisms "z-mechanisms". A z-mechanism is characterized by a set of functions $\{z_i\}$ that depend only on the costs of non-executed projects and are weakly increasing in those costs. The function z_i is a cutoff such that project i is conducted whenever i's cost report falls below z_i. We show that any z-mechanism has an equivalent deferred acceptance (DA) auction representation as described in Milgrom and Segal (2014). A DA auction is an iterative algorithm that computes the allocation and transfers of an auction mechanism and possesses attractive features with respect to bidders' incentives that go beyond dominant strategy implementability.

First, any DA auction is weakly group-strategy proof. In other words, it is impossible for a coalition of projects to coordinate their bidding strategies such that it strictly increases the utility of all projects in the coalition. Second, the dominant strategy equilibrium outcome of any DA auction is the only outcome that survives iterated deletion of dominated strategies in the corresponding full information game with the same allocation rule but where players pay their own bid. Therefore predicting the dominant strategy equilibrium outcome in a DA auction can be considered robust. Milgrom and Segal (2014) argue that these properties make DA auctions suitable for many challenging environments such as radio spectrum reallocation. Among several potential applications, they also consider our budget constrained procurement setup (Example 5: "Adaptive Scoring for a Budget Constraint"). However, they do not show optimality of the DA auction. To the best of our knowledge, we are the first to do so in a non-trivial setting. Therefore we can strengthen the argument in favor of DA auctions.

We investigate symmetric and asymmetric environments separately and propose implementations. First, we focus on the case in which all projects are ex-ante symmetric and only the cost is private information. Having characterized the optimal mechanism as a z-mechanism, it follows that it is optimal to rank projects according to their cost and "greenlight" the cheapest ones. Because of the budget, the number of greenlighted projects is endogenously determined by the cost reports of all participating projects. Second, we examine the case of ex-ante asymmetric projects, i.e., costs are drawn from different distributions and/or project values differ. In applications, the designer may prefer some projects over others and might have different information over cost distributions. We restrict attention to the two-project case because it conveys the main insights while retaining tractability.

In standard procurement settings, the quantity of units to be procured is fixed. It is a well known result that in such settings projects are greenlighted in order of their virtual surpluses, e.g., Luton and McAfee (1986). When values are identical but costs are asymmetrically distributed, the ranking implied by costs and the ranking implied by virtual surpluses do not necessarily coincide. Broadly speaking, the designer discriminates against stochastically stronger projects.

Interestingly, the optimal allocation in the symmetric case does not easily generalize to the asymmetric case. Projects are not simply greenlighted in order of their virtual surpluses. We show that there are instances where out of two rival projects the project with lower virtual surplus is optimally chosen. The reasoning behind this result is that the number of procured units is endogenous. Always greenlighting in order of virtual surplus reduces the expected number of greenlighted projects compared to the optimal mechanism. Incentive compatibility constraints create a tradeoff between quantity and quality of the procured projects. Notably, the quantity-quality tradeoff mitigates the stochastic discrimination mentioned above.

Reducing the set of candidates for optimal mechanisms to z-mechanisms, which have a DA auction representation, enables us to implement any optimal allocation with an appropriately designed descending clock auction. Individual clock prices determine the transfer paid to each active project and continuously decrease. In the symmetric case, clocks run down synchronously. Therefore projects exit in order of their costs until all remaining projects can be financed.

In asymmetric settings with fixed quantities, a descending clock auction requires individual clock speeds as the order of virtual surpluses does not coincide with the order of costs. A designer can adjust the clocks' speed such that the virtual surplus of marginal projects is kept equal, see Caillaud and Robert (2005, Proposition 1). As the optimal mechanism in our asymmetric case does not always allocate in order of virtual surplus, we cannot adopt this approach. Instead, the descending clock implementation of the optimal allocation includes individual clocks stopping at certain times. Here, the quantity-quality tradeoff kicks in.

Clock auctions are generally easy to understand and hard to manipulate. Furthermore, they are less information hungry than, for example, sealed bid auctions. In descending clock auctions, the designer only learns the private information of those projects that are not greenlighted. These features of clock auctions make them attractive for applications in which there is limited trust between the involved parties. In practice, clock auctions are not only used in the eponymous Dutch flower auction or Japanese fish auction, they are also commonly used in the public sector, e.g., by the US Department of the Treasury.

To the best of our knowledge, this paper is the only one that considers purely ex-post constrained optimal procurement design. Such a restrictive setting can be seen as a "worst case scenario" for the designer, suiting many economic applications. In our leading example of the development fund, an ex-post budget constraint appears natural as budgets are usually fixed. The nonprofit nature of the projects might prohibit acquiring additional money on the financial market. Information rents are necessary, because a project might want to spend money on extra equipment that is convenient for the project's staff but has no value for the designer. In practice, such incentive problems are often resolved using dominant strategy implementable mechanisms, as they are easy to explain and not prone to manipulation or misspecification of beliefs. For similar reasons, we restrict attention to deterministic mechanisms. Deterministic mechanisms obviate the need for a credible randomization device and are therefore more easily applicable in practice. Finally, ex-post participation constraints are necessary because projects simply cannot be conducted with insufficient funds, and the designer wants to avoid costly renegotiations when the projects default.

1.1 Literature

Even though the knapsack problem has a wide range of economic applications, there are relatively few publications in economics on this issue. Most prominently, in his Nancy L. Schwartz memorial lecture Maskin (2002) addressed the related problem of the UK government that put aside a fixed fund to encourage firms to reduce their pollution. The government faces n firms that have private marginal cost of abatement θ_i and can commit to reduce x_i units of pollution. To reduce pollution as much as possible, the government pays expected compensation transfers t_i to the firms, who report costs and proposed abatement to maximize $t_i - \theta_i x_i$. For some distributions, Maskin (2002) proposes a mechanism that satisfies an ex-post participation constraint, an ex-post incentive compatibility constraint, and the condition that the budget is not exceeded in expectation. In response to Maskin (2002), Chung and Ely (2002b) look at a more general class of mechanism design problems with budget constraints and translate them into a setting à la Baron and Myerson (1982). Their approach nests Maskin (2002) and also Ensthaler and Giebe (2014a) as special cases. However, Ensthaler and Giebe (2014a) more explicitly derive a constructive solution. In contrast to us, they all consider a soft budget constraint that only requires the sum of expected transfers to be less than the budget.

Ensthaler and Giebe (2014a) circumvent this problem by using AGV-budget-balancing (such as Börgers and Norman, 2009) to obtain a mechanism which is ex-post budget-feasible. However, transformation into a mechanism with an ex-post balanced budget in such a way comes at the cost of sacrificing ex-post individual rationality. Many applications do not allow this constraint to be weakened. For instance, subsidy applicants usually cannot be forced to conduct their proposal when receiving only a small or possibly no subsidy. Alternatively, limited liability justifies insisting on ex-post individual rationality.

To the best of our knowledge, no paper exists that jointly considers optimal mechanism design under ex-post budget balance and ex-post individual rationality in a procurement setting. Ensthaler and Giebe (2014b) propose a clock mechanism that coincides with our optimal mechanism in the symmetric case for many parameterizations[3] but differs in the asymmetric case by holding the cost-benefit-ratio equal among projects. By simulating different settings, they conclude that this mechanism outperforms a mechanism used in practice. Although a comparison might be unfair as they consider a belief-free designer, this clock mechanism is outperformed by our mechanism.

Because of the appeal of dominant strategy incentive compatible (DIC) mechanisms compared to Bayesian incentive compatible (BIC) mechanisms, many researchers have produced valuable BIC-DIC equivalence results. These results characterize environments in which restricting attention to the more robust incentive criterion comes without loss. Our setup is not contained in these environments. For any BIC mechanism, Mookherjee and Reichelstein (1992) show that one can construct a DIC mechanism implementing the same ex-post allocation rule, whenever this allocation rule is monotone in each coordinate. However, the ex-post transfers of the constructed DIC mechanism are not guaranteed to satisfy ex-post budget balance. More recently, Gershkov, Goeree, Kushnir, Moldovanu, and Shi (2013) employ a definition of equivalence in terms of interim expected utilities introduced by Manelli and Vincent (2010). For any BIC mechanism, including the optimal one, they construct a DIC mechanism that yields the same interim expected utilities. Here, the ex-post allocation as well as the ex-post transfers might differ between the two. Therefore a DIC mechanism equivalent to a feasible BIC mechanism might violate the ex-post constraints in our setting.

Our budget constrained procurement setup with ex-post constraints has received

[3]In contrast to their setting, the mechanism designer in our model values residual money. Therefore the designer will not greenlight projects with negative virtual surplus. Ensthaler and Giebe (2014b) do not consider this case.

much attention in the computer science literature. Instead of specifying the optimal mechanism, the authors in this literature typically aim to construct allocation algorithms that give good approximation guarantees. In other words, they try to maximize the minimal payoff an algorithm can guarantee compared to the full information knapsack payoff. Apart from the seminal paper by Singer (2010), the works of Dobzinski, Papadimitriou, and Singer (2011) and Chen, Gravin, and Lu (2011) are notable examples of this approach. Anari, Goel, and Nikzad (2014) present a stochastic algorithm and show that it gives the best possible approximation guarantee in the many projects limit in which any individual project's costs are small compared to the budget. While the above papers examine the belief-free case, Bei, Chen, Gravin, and Lu (2012) propose an algorithm for setups in which the designer knows how the private information is distributed.

Dizdar, Gershkov, and Moldovanu (2011) investigate a dynamic knapsack problem where impatient projects with private capacity requirement w and private willingness to pay v arrive over time. The mechanism designer offers them a price p and a capacity w' where the sum of capacities offered is constrained. The projects' utility is given by $wv - p$ if the assigned capacity satisfies $w' > w$, and by $-p$ otherwise. However, the static version of their problem does not mirror our problem in the way procurement auctions mirror seller-buyer auctions. In their model, the mechanism designer is only interested in the sum of payments and the value only enters the individual projects' payoff. In our framework, the designer maximizes aggregate value of all greenlighted projects minus the sum of transfers. That is, the project's value is of first-order importance to the designer and the project's type only enter her objective indirectly via the transfers. A project may only benefit from a higher value indirectly because it cause a larger transfer.

There seems to be no reasonable analogy for our setting to another setting where the mechanism designer is a similarly constrained seller and the agents are buyers. Budget constrained buyers in auctions have been discussed in the literature, e.g., by Pai and Vohra (2014) or Che and Gale (1998). However, these authors study budget-constrained agents whereas in our setting the designer is budget-constrained.

In the following section, we introduce the model. In Section 3 we rewrite the problem as a problem of finding the optimal z-mechanism. Sections 3.1 and 3.2 cover symmetric and asymmetric environments, respectively. We discuss extensions and possible modifications to the model in Section 4. Finally, we conclude in Section 5.

2 Model

We consider a set of n projects $I = \{1, \dots, n\}$ and one mechanism designer. Each project can be conducted exactly once. The designer gains utility v_i if and only if project $i \in I$ is conducted. We consider projects to be utility maximizing agents. If project i is executed, it incurs cost $c_i \in [\underline{c}_i, \overline{c}_i]$. The costs are the projects' private information and are independently drawn from a distribution F_i. We assume F_i to be continuously differentiable with a strictly positive density f_i on the support. The value of the project v_i and the distribution F_i are common knowledge.

We restrict attention to deterministic mechanisms. This restriction implies that once all cost reports are collected, we know with certainty which project will be selected by the mechanism. In other words, the probability of implementation q_i is binary,

$$q_i = 1 \, \forall i \in G \text{ and } \; q_i = 0 \, \forall i \in R,$$

where the designer's allocation decision is represented by a partition of the set of projects into two disjoint sets $G \cup R = I$. We will say that projects in set G are "greenlighted" while projects in set R are "redlighted". We employ the revelation principle and without loss of generality only consider direct mechanisms.[4]

To compensate project i for its cost, the designer pays transfer t_i. Consequently, a direct mechanism is characterized by $\langle q_i, t_i \rangle$. It is a mapping from the vector of cost reports $\mathbf{c} \in \times_i^n [\underline{c}_i, \overline{c}_i]$ into provision decisions and transfers. When we talk about the allocation, we refer to the former. Project i's utility u_i is given by its transfer minus the cost it bears:

$$u_i(\mathbf{c}) = t_i(\mathbf{c}) - q_i(\mathbf{c}) c_i.$$

The designer derives value v_i from each greenlighted project $i \in G$ while having to pay the sum of transfers. Therefore she wants to maximize the aggregate value of greenlighted projects net of transfers paid. Her (ex-post) utility function u_D implies that, in our setting, the designer values residual money,

$$u_D(\mathbf{c}) = \sum_i \big(q_i(\mathbf{c}) v_i - t_i(\mathbf{c}) \big). \tag{1}$$

[4] In general, it is not always possible to employ the revelation principle while restricting attention to deterministic mechanisms. This caveat is a result of deterministic direct mechanisms being unable to replicate the players' equilibrium mixing strategies in deterministic indirect mechanisms, e.g., Strausz (2003). However, in our setting we do not lose generality, due to the ex-post nature of our constraints.

We impose an ex-post participation constraint. That is, if i is greenlighted the transfer must be at least as high as the cost,

$$t_i(c_i, \mathbf{c}_{-i}) - q_i(c_i, \mathbf{c}_{-i})c_i \geq 0 \quad \forall i, c_i, \mathbf{c}_{-i}. \tag{PC}$$

In addition, the designer has a budget constraint which is "hard" in the sense that she cannot spend more than her budget B for any realization of the cost vector. That is, the designer can never exceed her budget

$$\sum_i t_i(\mathbf{c}) \leq B \quad \forall \mathbf{c}. \tag{BC}$$

Finally, incentive compatibility has to hold ex-post. Alternatively, we can say that the mechanism has to be implementable in (weakly) dominant strategies.[5] Therefore, for every realization of the cost vector, project i's truthful report must yield at least as much utility as any possible deviation

$$t_i(c_i, \mathbf{c}_{-i}) - q_i(c_i, \mathbf{c}_{-i})c_i \geq t_i(\widetilde{c}_i, \mathbf{c}_{-i}) - q_i(\widetilde{c}_i, \mathbf{c}_{-i})c_i \quad \forall i, c_i, \mathbf{c}_{-i}, \widetilde{c}_i. \tag{IC}$$

One may think that a natural approach to this problem would be to express the ex-post transfer $t_i(c_i, \mathbf{c}_{-i})$ as a function of the ex-post allocation decision $q_i(c_i, \mathbf{c}_{-i})$, taking $\mathbf{c}_{-i}$ as given, and applying the envelope theorem. In that case, it would be possible to restrict attention to the allocation in order to solve for the optimal mechanism. However, this approach does not reduce the complexity of the problem. The reason is that the ex-post transfers and allocation for one cost vector restrict transfers and allocation for other cost vectors through the budget constraint in a manner much more involved than standard monotonicity. In particular, the budget constraint with the ex-post transfer expressed as a function of the ex-post allocation may be ill-behaved. Therefore we cannot straightforwardly arrive at sufficient conditions using convex optimization.[6]

3 Analysis

We search for the direct mechanism that maximizes the expected utility of the designer and refer to this mechanism as the optimal mechanism. Our first step

[5] In our private value environment, these two concepts are equivalent in a direct revelation mechanism. In general, however, ex-post incentive compatibility is essentially a generalization of dominant strategy implementability to interdependent value environments. See Chung and Ely (2002a).

[6] Requiring either the budget or the participation constraint to hold only in expectation would enable us to use the techniques employed by Ensthaler and Giebe (2014a).

is to show that the ex-post constraints imply that the optimal mechanism has to be in cutoffs.

Lemma 1. *The optimal mechanism can be represented by cutoff functions* $z_i : [\underline{c}_j, \overline{c}_j]^{j \in I \setminus \{i\}} \to [\underline{c}_i, \overline{c}_i]$, *where project* i *is greenlighted whenever it reports a cost weakly less than its cutoff:*

$$q_i(c_i, \mathbf{c}_{-i}) = \mathbb{I}(c_i \leq z_i(\mathbf{c}_{-i})).$$

The transfer to project i *will be its cutoff whenever it is greenlighted and zero otherwise:*

$$t_i(c_i, \mathbf{c}_{-i}) = q_i(c_i, \mathbf{c}_{-i}) z_i(\mathbf{c}_{-i}).$$

Proof. First note that for any two cost reports c_i, c_i' of project i and for some $\mathbf{c}_{-i}$, (IC) implies that if $q_i(c_i, \mathbf{c}_{-i}) = q_i(c_i', \mathbf{c}_{-i})$, then we must also have $t_i(c_i, \mathbf{c}_{-i}) = t_i(c_i', \mathbf{c}_{-i})$. Otherwise i could deviate to the report giving a higher transfer.

Suppose project i is greenlighted for some of its cost reports given $\mathbf{c}_{-i}$. Then there are only two possible values for t_i, depending on whether i is greenlighted or not: $t_i^{q_i=1}$ and $t_i^{q_i=0}$.

Define $z_i(\mathbf{c}_{-i}) = t_i^{q_i=1} - t_i^{q_i=0}$. Then (IC) yields

$$q_i(c_i, \mathbf{c}_{-i}) = \begin{cases} 1 & \text{if } c_i \leq z_i(\mathbf{c}_{-i}) \\ 0 & \text{if } c_i > z_i(\mathbf{c}_{-i}). \end{cases}$$

Suppose to the contrary that for some realization $\widehat{c}_i < z_i(\mathbf{c}_{-i})$ we had $q_i(\widehat{c}_i, \mathbf{c}_{-i}) = 0$. Deviating to a cost report that ensures the green light would imply a utility increase of $z_i - c_i$. An analogous argument applies for $\widehat{c}_i > z_i(\mathbf{c}_{-i}) > 0$.[7]

The last step is to show that $t_i^{q_i=0} = 0$. This result trivially follows from the mechanism being optimal, i.e., maximizing expected utility of the designer. □

As a direct consequence of dominant strategy implementability, Lemma 1 shows that allocation and transfers are characterized by cutoffs. Project i is greenlighted whenever it reports a cost that lies weakly below the cutoff. Crucially,

[7]When $c_i = z_i$, (IC) permits both $q_i = 0$ and $q_i = 1$. By convention, we will assume $q_i = 1$ in this case. However, writing a mechanism this way precludes the specification of tie-breakers, which might be necessary to conserve budget balance. For example, in a 2-project example we would write down the mechanism "greenlight the cheaper project" as $z_1(c_2) = c_2$ and $z_2(c_1) = c_1$. If $c_1 = c_2$ a tie-breaker is needed to select a project. As this is a zero-probability event, the choice of the tie-breaker does not impact the designer's payoff. Similarly, as projects are indifferent, their ex-post utility is unaffected. Therefore we refrain from specifying a tie-breaker and will write down our analysis as if both projects are greenlighted in these cases.

these cutoffs are functions of the other cost reports $\mathbf{c}_{-i}$. However, the optimal cutoffs remain to be determined. The maximization problem of the designer is given by

$$\begin{aligned} &\max_{z_i(\mathbf{c}_{-i})} \mathbb{E}\left[\sum_i q_i(\mathbf{c})v_i - t_i(\mathbf{c})\right] \\ &\text{s.t. (BC)}, \\ &q_i(\mathbf{c}) = \mathbb{I}(c_i \leq z_i(\mathbf{c}_{-i})), \\ &t_i = \mathbb{I}(c_i \leq z_i(\mathbf{c}_{-i}))z_i(\mathbf{c}_{-i}). \end{aligned} \tag{2}$$

Here q_i and t_i are determined by the cutoff z_i. Incentive compatibility and participation constraints, thus, hold by construction.

The next step towards solving this problem involves applying standard methods introduced by Myerson (1981). Let the conditional expected probability of being greenlighted and the conditional expected transfer be

$$Q_i(c_i) = \mathbb{E}[q_i(c_i, \mathbf{c}_{-i})|c_i]$$
$$\text{and } T_i(c_i) = \mathbb{E}[t_i(c_i, \mathbf{c}_{-i})|c_i].$$

The interim incentive compatibility required by Myerson (1981) is weaker than our condition (IC). Consequently, the expected transfer is determined by the allocation, $T_i(c_i) = Q_i(c_i)c_i + \int_{c_i}^{\overline{c}} Q_i(x)dx$. The usual monotonicity condition is trivially fulfilled as we are dealing with cutoff mechanisms. This reformulation in turn allows us to rewrite the objective function as a function of the allocation. Substituting into problem (2) and integrating by parts yields the following maximization problem,

$$\begin{aligned} &\max_{z_i(\mathbf{c}_{-i})} \mathbb{E}\left[\sum_i \mathbb{I}(c_i \leq z_i(\mathbf{c}_{-i}))\left(v_i - c_i - \frac{F_i(c_i)}{f_i(c_i)}\right)\right] \\ &\text{s.t.} \\ &\sum_i \mathbb{I}(c_i \leq z_i(\mathbf{c}_{-i}))z_i(\mathbf{c}_{-i}) \leq B \quad \forall \mathbf{c}. \end{aligned} \tag{3}$$

We call $\varphi_i(c_i) := c_i + \frac{F_i(c_i)}{f_i(c_i)}$ the *virtual cost* of project i and $\psi_i(c_i) := v_i - \varphi_i(c_i)$ the *virtual surplus*. Here, φ and ψ are the procurement analogues to standard auction terminology. We can directly see from problem (3) that the optimal mechanism maximizes the expected sum of greenlighted virtual surpluses.

Note that constrained optimization via Lagrangian is not straightforward here because of the non-differentiability of the indicator function. Instead, in the following we derive useful properties of the optimal cutoffs that can be exploited to characterize the optimal mechanism.

Assumption 1 (Log-concavity)**.** *For all i the cumulative distribution function F_i is log-concave.*

This assumption is standard in information economics. It is equivalent to the reverse hazard rate function f/F being a weakly decreasing function or the ratio F/f being weakly increasing. Hence, the standard regularity condition is implied: φ_i is strictly increasing and ψ_i is strictly decreasing. A decreasing reverse hazard rate is the procurement analogue to the assumption of increasing hazard rate functions in seller auction settings.

Lemma 2. *Disregarding (BC), the optimal cutoffs, here defined as z_i^{**}, are independent of the cost reports:*

$$z_i^{**} := \begin{cases} \psi_i^{-1}(0) & \text{if } \psi_i^{-1}(0) \in [\underline{c}_i, \overline{c}_i] \\ \overline{c} & \text{otherwise.} \end{cases}$$

In the symmetric case, $z_i^{**} = z^{**}$ for all $i \in I$.

Regularity ensures that a lower cost c_i translates to a higher virtual surplus $\psi_i(c_i)$. The designer wants to greenlight any project with cost weakly below z_i^{**}. Note that regularity implies the invertibility of ψ_i and thus allows for the above definition of z_i^{**}. Crucially, the arguments leading to Lemma 2 also imply that it is never optimal to greenlight a project with negative virtual surplus.

We have previously introduced G and R as the sets of greenlighted and redlighted agents. Consequently, we denote the cost vector of projects in G as $\mathbf{c}_G$ and similarly the cost vector of projects in R as $\mathbf{c}_R$. In general, $\mathbf{c}_X$ is a vector that contains all elements c_i from cost vector $\mathbf{c}$ for projects $i \in X \subset I$. We now define a class of mechanisms and then show that any mechanism outside of this class cannot be optimal.

Definition 1 (z-mechanism)**.** *A z-mechanism is characterized by a set of cutoff functions $\{z_i\}_{i\in I}$ that are almost everywhere equal to cutoff functions that are*

Property 1 *left-continuous for each of its arguments,*

Property 2 *always weakly less than z_i^{**},*

Property 3 *weakly increasing in each other project's cost report,*

Property 4 *independent of costs $\mathbf{c}_G$ conditional on $\{G, R\}$ being the partition of greenlighted and redlighted projects for projects in G.*

Note that z-mechanisms have some salient features. The cutoffs of those projects that are greenlighted are only determined by the cost reports of projects that are redlighted. This feature is due to the fact that project i's cost contains no information about the cost report of project j. What can be exploited however is the ordering of cost reports at the margin, allowing for an iterative allocation mechanism. Being able to restrict attention to z-mechanisms is highly useful, as the set of all feasible z-mechanisms is a much more tangible object than the substantially larger set of all permissible cutoff-mechanisms. In addition, we show at the end of this section that any z-mechanism has a DA auction representation.

For some of the following lemmata and propositions, we provide the proof for the two-project case in the main text and provide the proof of the general case in the appendix.

Proposition 1. *Among all mechanisms satisfying (PC), (BC) and (IC), any mechanism that maximizes the designer's payoff (1) is a z-mechanism.*

We divide the proof into several lemmata showing that any optimal mechanism can violate the properties of Definition 1 only on a set with Lebesgue-measure zero. There are infinitely many possible cutoff functions that differ on finitely many points that have Lebesgue-measure zero and therefore yield the same expected payoff. In order to make the functions we talk about unique, we will w.l.o.g. restrict attention to cutoff functions that satisfy the properties. Note that Property 1 does not require a proof because we can replace any function z_i with a left-continuous function that is identical up to a set of points with Lebesgue-measure zero. Property 2 follows directly from the rewritten objective function (3) and the argument behind Lemma 2. Let us now consider Property 3.

Lemma 3. *The optimal cutoff function z_i is weakly increasing in c_j for all i, j with $j \neq i$, i.e., $z_i(c_j, \mathbf{c}_{-i-j}) \geq z_i(c'_j, \mathbf{c}_{-i-j})$ for almost every $c_j > c'_j$ and $\mathbf{c}_{-i-j}$.*

Proof. (with $n = 2$, see appendix for the general proof)
For a graphical representation of the proof see Figure 1. By Property 2 of a z-mechanism, any optimal function z_i cannot exceed z_i^{**}. Greenlighting a project with negative virtual surplus decreases the designer's payoff and uses part of the budget.

Fix any function z_1. Suppose to the contrary that z_2 is decreasing on a set with positive Lebesgue-measure. Then, there exist sets H and L with positive Lebesgue-measure, such that

$$z_2(c^L) > z_2(c^H) \text{ for all } c^L \in L, c^H \in H,$$

where $c^L < c^H$ for all elements of the corresponding sets.

Now, consider the deviation $\widehat{z}_2(c^H) = \widehat{z}_2$ for all $c^H \in H$ with $\widehat{z}_2 = z_2(\widehat{c}^L)$ for an arbitrary $\widehat{c}^L \in L$. In other words, flatten z_2 over H and leave z_1 as it is. This deviation is depicted in Panel 1a.

The deviation increases the payoff by

$$\int_H \int_{z_2(c^H)}^{\widehat{z}_2} \psi_2(c_2) dF_2(c_2) dF_1(c^H) > 0.$$

The change in payoff is positive as $z_2(c^L) \leq z_2^{**}$. Graphically, the increase in payoff corresponds to the shaded areas in Panel 1b. In the lighter shaded area, both projects are implemented with the deviation whereas only project 1 is implemented with the initial candidate. In the darker shaded, area project 2 is implemented instead of no project.

It remains to show that the deviation is not only profitable but also feasible. Fix any c_1.

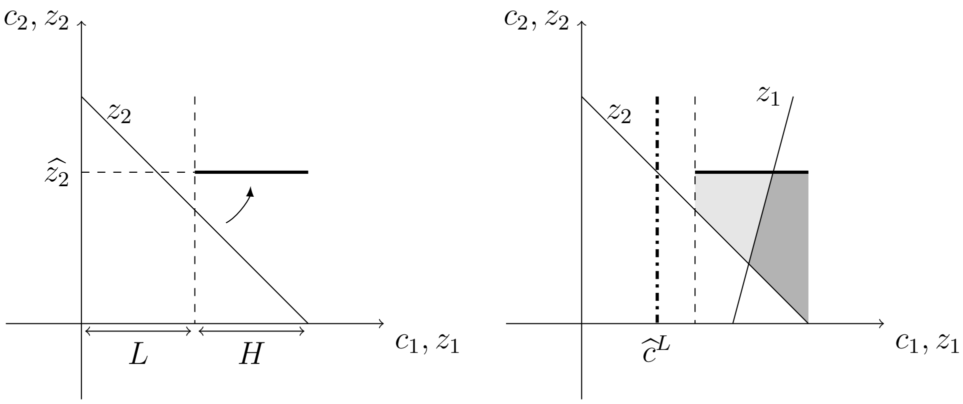
Case 1: For all $c_2 > \widehat{z}_2$. The deviation does not affect the budget constraint, because project 2 is redlighted regardless of the deviation.

Case 2: For all $c_2 \leq \widehat{z}_2$. The following establishes budget-feasibility:

- Suppose $z_1(c_2) \geq c_1$.
  Because the initial candidate mechanism was feasible and implemented both projects for cost realization $(\widehat{c}^L, c_2)$, the budget must satisfy $B \geq z_2(\widehat{c}^L) + z_1(c_2) = \widehat{z}_2 + z_1(c_2)$. Therefore the deviation is feasible.
- Suppose $z_1(c_2) < c_1$.
  Only project 2 is implemented. Again, the deviation is feasible.

Graphically, feasibility of the second case is represented by the dash-dotted line in Panel 1b. Any cost realization for which the deviation changes project 2's cutoff (that is, $c_1 \in H$) has a corresponding point on the dash-dotted line such that both points lead to the same cutoffs. In addition, points on the dash-dotted line can have a greenlighted project 1 while project 1 can be redlighted in corresponding points affected by the deviation (the darker shaded are in Panel 1b), as $c^H > \widehat{c}^L$. Finally, note that points on the dash-dotted line are not affected by the deviation. Therefore feasibility of the deviation follows from feasibility of the initial candidate. □

Lemma 3 establishes that cutoff functions must be weakly increasing in their arguments. The intuition is straightforward. The cost draws of all projects are independent. Therefore project i's cost report only matters for the payoff generated from project $j \neq i$ through the budget constraint. Project i's cost report only influences the budget through exceeding or lying below the cutoff. If project i exceeds its cutoff, this frees budget to be distributed among the other projects. Consequently, their cutoffs must remain constant or increase. While the intuition is the same for both $n = 2$ and $n > 2$, the proof is more involved in the general case. The reason is that the cost report of the project with the decreasing cutoff does not pin down all other cutoffs and hence the remaining budget - as it does when $n = 2$. We cannot trivially extend the proof above, if some cutoff of a third project z_3 increases in c_1 while z_2 decreases. The intuition of the general proof is that a decreasing cutoff cannot be optimal, because it essentially implies exchanging project 2 for project 1 while the virtual surplus of project 2 decreases relative to the virtual surplus of project 1.



(a) Candidate and deviation. (b) Feasibility and profitability.

Figure 1: A decreasing z_2 cannot be optimal (deviation in the proof of Lemma 3).

Remember that G represents the set of greenlighted projects and R represents the set of redlighted projects. In the following lemma, we establish that given that only the projects of some set G are greenlighted and given the remaining projects' costs $\mathbf{c}_R$, for all $g \in G$ all functions z_g intersect each other at some point $(a_1^G(\mathbf{c}_R), a_2^G(\mathbf{c}_R), ...)$. This point only depends on cost reports $\mathbf{c}_R$ of redlighted projects. Intuitively, optimal cutoffs cannot depend on greenlighted projects' cost, because for these projects the cutoff coincides with the transfer. For the two-project case, Figure 2 illustrates that (BC) must bind when both projects are greenlighted. But then project 1 influencing project 2's cutoff would change the remaining budget which is equal to project 1's transfer, given that (BC) binds. This contradicts the notion of a cutoff mechanism.

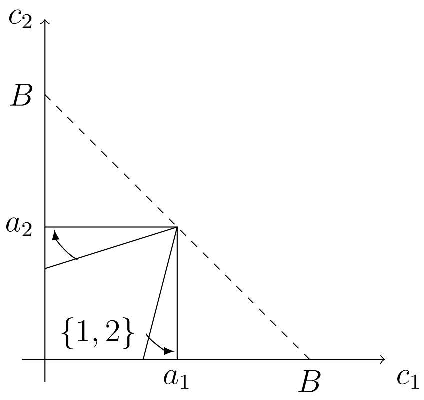


Figure 2: In Lemma 4, we show that in the non-trivial two-project case whenever $G = \{1, 2\}$, both projects get constant transfers summing up to the budget. For instance, the weakly increasing candidate mechanism depicted above is improved by the deviation indicated by the arrows.

Lemma 4. *Conditional on any arbitrary partition $\{G, R\}$, the optimal cutoff functions z_g for all $g \in G$ are independent of the costs of all greenlighted projects $\mathbf{c}_G$. That is,*

$$z_g(\mathbf{c}_{G-g}, \mathbf{c}_R) = z_g(\mathbf{c}'_{G-g}, \mathbf{c}_R),$$

for all $\mathbf{c}_{G-g}$ and $\mathbf{c}'_{G-g}$ such that G is the set of greenlighted agents.

Moreover, if cost vector $(\mathbf{c}_G, \mathbf{c}_R)$ induces allocation $\{G, R\}$, then cost vector $(\mathbf{c}'_G, \mathbf{c}_R)$ also induces $\{G, R\}$ if $c'_g \leq c_g$ for all $g \in G$.

Proof. (with $n = 2$, see appendix for the general proof and consult figure 2 for intuition)
By Lemma 1, the optimal mechanism has to be in cutoffs. What remains to be shown is that said cutoffs only depend on $\mathbf{c}_R$. For $G = \{1\}$ or $G = \{2\}$, i.e., when only one project is greenlighted, the statement follows from the nature of a cutoff function. Hence we need to show that the cutoffs must be constants whenever $G = \{1, 2\}$. Therefore suppose that $G = \{1, 2\}$ is induced with positive probability.

Take any feasible candidate mechanism with any increasing cutoff functions z_i and define

$$a_1 = \max\{c_1 | \exists c_2 : c_2 \leq z_2(c_1), c_1 \leq z_1(c_2)\}$$
$$a_2 = \max\{c_2 | \exists c_1 : c_1 \leq z_1(c_2), c_2 \leq z_2(c_1)\}.$$

The maximum exists by left-continuity of z_i, following Property 1 of a z-mechanism. Whenever greenlighting both projects, the sets over which we have defined a_1 and a_2 must be non-empty. Hence by definition of a_1, there exists $\widetilde{c}_2$ such that $a_1 = z_1(\widetilde{c}_2)$. Similarly, there exists $\widetilde{c}_1$ such that $a_2 = z_2(\widetilde{c}_1)$.

By definition $(\widetilde{c}_1, \widetilde{c}_2) \leq (a_1, a_2)$. Therefore $a_1 + a_2 \leq B$ is implied by the budget constraint.

Now we show that $z_1(c_2') = a_1$, for all $c_2' \leq a_2$, and $z_2(c_1') = a_2$, for all $c_1' \leq a_1$. Suppose not. Suppose (without loss of generality) there is some set $\Xi \subset [0, a_2]$ with positive Lebesgue-measure such that $z_1(c_2') < a_1$ for all $c_2' \in \Xi$. Denote $z_1^{\Xi} := \max_{c_2 \in \Xi} z_1(c_2)$. Since $a_1 + a_2 \leq B$, changing the mechanism to $z_1(c_2') = a_1, \quad \forall c_2' \leq a_2$ does not violate the budget constraint and increases the payoff by

$$\Delta > \Pr(c_2 \in \Xi) \int_{z_1^{\Xi}}^{a_1} \psi_1(c) dF(c) > 0.$$

□

By combining our earlier insights with the previous two lemmata, we have shown that the optimal mechanism must be a z-mechanism. The next step is to show that any z-mechanism can alternatively be described by a DA auction as proposed by Milgrom and Segal (2014). To this end, we first restate their definition.

Definition 2 (DA auction)**.** *A deferred acceptance (DA) auction is an iterative algorithm defined by a collection of scoring functions*

$$s_i^A : [\underline{c}_i, \overline{c}_i] \times [\underline{c}_j, \overline{c}_j]_{j \in I \setminus A} \to \mathbb{R}_+$$

that are weakly increasing in c_i for all $i \in A$ and for all $A \subset I$. Let $A_t \in I$ denote the set of active bidders in iteration t and initially $A_1 = I$. The algorithm stops in some period T when all active projects have a score of zero, $s_i^{A_T} = 0$ for all i in A_T. Then the set of greenlighted project is A_T. Otherwise, at each iteration t, the project with the highest score is removed. The payment p_i^t of project i at iteration t is either given by the highest possible cost that i could have had without being removed from the set of active bidders or by the last iteration's payment, depending on which payment is smaller,

$$p_i^t(c) = \begin{cases} \sup\{c_i' : s_i^{A_t}(c_i', c_{I \setminus A_t}) < s_j^{A_t}(c_j, c_{I \setminus A_t})\} & \textit{for } j \in A_t \setminus A_{t+1}, \\ \min\{\sup\{c_i' : s_i^{A_t}(c_i', c_{I \setminus A_t}) \leq 0\}, p_i^{t-1}\} & \textit{if } t = T. \end{cases}$$

The algorithm is initialized with $p_i^0 = \min\{\bar{c}_i, z_i^{**}, B\}$.[8]

The main appeal of DA auctions lies in their incentive guarantees on top of dominant strategy implementability. DA auction are weakly group-strategy proof. That is, no coalition of projects can manipulate their reports such that it strictly increases the utility of all projects in the coalition: At least one member of the coalition receives a weakly worse payoff whenever other coalition members benefit. Because collusion in auctions is generally illegal, compensating the worse off coalition member is not contractible. In addition, the dominant strategy equilibrium outcome in a DA auction can be interpreted as robust in the following sense: Consider the full information game where all cost reports are observed, projects can report any cost, the allocation is determined according to the DA auction's allocation rule, but projects receive their own report as payments. The dominant strategy equilibrium outcome of the DA auction is the only outcome that survives iterated deletion of dominated strategies in this game.

Proposition 2. *Any z-mechanism has an equivalent DA auction representation.*

The proof of Proposition 2 is relegated to a separate section in the appendix.

Corollary 1. *Any z-mechanism can be implemented with a descending clock auction.*

Corollary 1 follows directly from Proposition 2 and from Milgrom and Segal (2014, Proposition 3) who show that any DA auction can be implemented by a clock auction.[9]

3.1 The symmetric case

In this section, we focus on symmetric projects where $v_i = v$ and $F_i = F$ for every project i. An implication of this assumption is that the order of costs coincides with the order of virtual surpluses and that $z_i^{**} = z^{**}$ for all i. We show how to utilize the established results to characterize the optimal allocation and also how

[8] Compared to Milgrom and Segal (2014), we slightly tweak the updating function of payments without changing the deferred acceptance nature of the algorithm and any of its properties.

[9] For technical reasons, Milgrom and Segal (2014) only consider discrete type distributions. While we use continuous distributions for expositional reasons, we rely on the fact that any continuous type distribution can be approximated arbitrarily closely by a discrete distribution.

to implement it. As in previous proofs, the proof of Proposition 3 considers the two-project case while the general proof is relegated to the appendix. In the two-project case, the designer's optimization problem can be reduced to optimally solving for a single constant. Nevertheless, we discuss possible deviations from the optimal mechanism in greater detail to highlight the complications which arise in asymmetric environments.

Proposition 3. *Arrange the projects in ascending order of their reported costs, $c_1 \leq c_2 \leq \cdots \leq c_n$ and define $z^k := \min\left\{\frac{B}{k}, z^{**}, c_{k+1}\right\}$. In the symmetric case, the z-mechanism with $z_i(\mathbf{c}_{-i}) = z^{k^*}$ is the optimal budget-constrained mechanism. The optimal number of accepted projects k^* is given by $k^* := \max\{k | c_k \leq z^k\}$.*

Proof. (with $n = 2$, see appendix for the general proof)
In Proposition 1, we have shown that the optimal mechanism must be a z-mechanism. As a candidate for optimality, consider any z-mechanism M^z different from the mechanism proposed above. Suppose M^z greenlights both projects sometimes. For graphic intuition of the deviation consult Figure 3.

By Lemma 2, any optimal mechanism must never greenlight a project with negative virtual surplus. This property is depicted as the kink at (z^{**}, z^{**}).

In the area above the dashed budget line, $c_1 + c_2 > B$, the designer can, by (BC) and (PC), only execute one of the two projects. It can be directly seen from objective function (3) that the designer prefers the project with the higher virtual surplus, i.e., the one with lower cost. It does not, however, follow directly that $z_i(c_j) = c_j$ whenever $B - c_i < c_j < z^{**}$. The reason is that the designer may want to forgo executing the better project for some cost vectors (shaded triangle and crossed square in Figure 3) in order to execute both projects in an additional area (horizontally lined, Figure 3). In such a case, the designer is forced by incentive compatibility to execute the worse project (for cost vectors in the shaded triangle or the square that is both horizontally and vertically lined).

By Lemma 4, both cutoffs must be constant whenever both projects are executed. In optimum in that case, there can be no slack in the budget constraint and z_i is flat in that region. Otherwise increasing one of the cutoffs until the budget binds is both feasible and profitable.

Now, consider candidate mechanism M^z

$$z_i(c_j) = \begin{cases} z^{**} \text{ if } c_j \geq z^{**} \\ c_j \text{ if } z < c_j < z^{**} \\ B - z \text{ if } c_j < z \end{cases} \quad \text{and} \quad z_j(c_i) = \begin{cases} z^{**} \text{ if } c_j \geq z^{**} \\ c_i \text{ if } B - z < c_j < z^{**} \\ z \text{ if } c_j < B - z \end{cases} \tag{4}$$

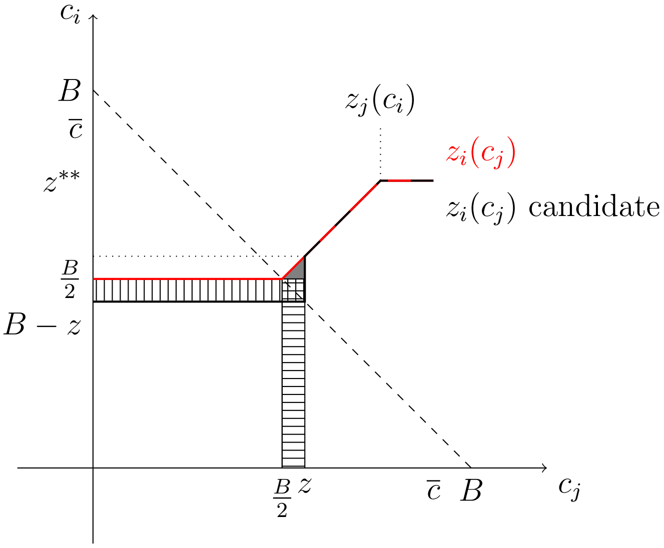


Figure 3: A candidate mechanism and the deviation to the proposed mechanism.

and see that a deviation to the mechanism in the proposition is always profitable.

For ease of exposition, let $A = \frac{B}{2}$. The proposed deviation to the z-mechanism M^A changes the designer's payoff in the following way

$$
\begin{aligned}
\Delta = F_j(z) \int_{B-z}^{A} \psi_i(x) dF_i(x) & \qquad \text{(vertical)} \\
- F_i(A) \int_{A}^{z} \psi_j(c) dF_j(c) & \qquad \text{(horizontal)} \\
+ \int_{A}^{z} \int_{A}^{c} \psi_i(x) dF_i(x) - (F_i(c) - F_i(A)) \psi_j(c) dF_j(c) & \qquad \text{(shaded)}
\end{aligned}
$$

where the patterns represent the area in Figure 3 where the allocation changes. Everywhere else the allocation and payoff remain the same.

To rewrite Δ define $\gamma(x) = F(x)(v - x)$ where $\gamma'(x) = \psi(x) f(x)$:

$$
\begin{aligned}
\Delta &= F(z)(\gamma(A) - \gamma(B - z)) - F(A)(\gamma(z) - \gamma(A)) \\
&\quad + F(A)(\gamma(z) - \gamma(A)) + \int_{A}^{z} \gamma(c) - \gamma(A) - F(c)\psi_j(c) dF(c) \\
&= F(z)(\gamma(A) - \gamma(B - z)) - F(A)(\gamma(z) - \gamma(A)) \\
&\quad + F(A)(\gamma(z) - \gamma(A)) - \gamma(A)(F(z) - F(A)) + \int_{A}^{z} F^2(c) dc
\end{aligned}
$$

because $(\psi(c)F(c) - F(c)(v-c))f(c) = F^2(c)$ and then since $\int_A^z F(c)^2 dc > F(A)^2 \int_A^z 1dc$,

$$\begin{aligned}\Delta &> F(z)(\gamma(A) - \gamma(B-z)) - \gamma(A)(F(z) - F(A)) + F(A)^2(z-A) \\ &= F(A)^2(v - A + z - A) - F(z)F(B-z)(v - B + z) \\ &= (v - B + z)(F(A)^2 - F(z)F(B-z)) \\ &> 0 \Leftrightarrow F(A)^2 > F(z)F(B-z).\end{aligned}$$

This statement is true under Assumption 1, log-concavity. Maximizing $F(z)F(B-z)$ with respect to z, the first order condition is given by

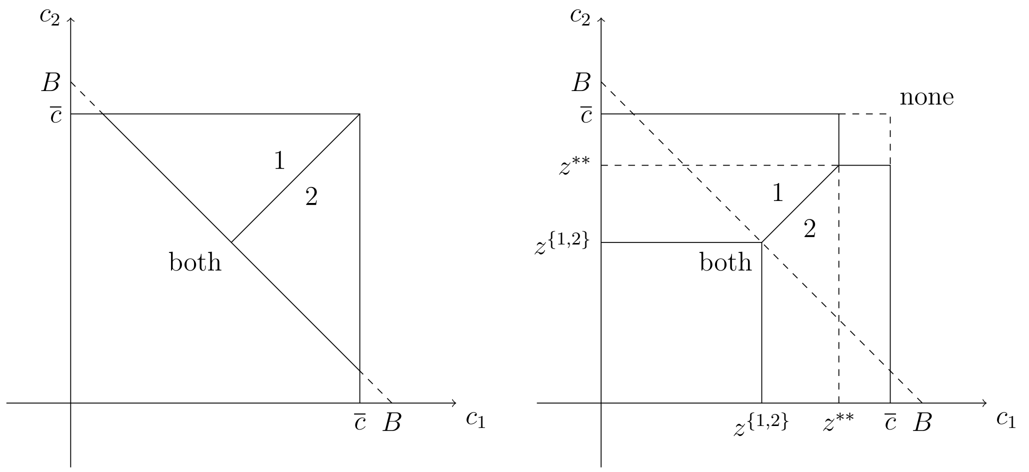
$$\frac{F(z)}{f(z)} = \frac{F(B-z)}{f(B-z)} \tag{5}$$

which is only true at $z = B/2$ since F/f is an increasing function. For the same reason, the left-hand side is greater (less) than the right-hand side for $z > B/2 (< B/2)$ making $z = B/2$ the maximum.

We have assumed that in the optimal mechanism both projects get greenlighted for some cost vectors. It remains to show that the optimal mechanism beats the best mechanism in which at most one project gets greenlighted. The best mechanism that selects at most one project would always select the project with higher virtual surplus. Clearly, the optimal mechanism of this proposition creates more payoff as it also always greenlights the project with higher virtual surplus. Additionally, it sometimes adds a second project with positive virtual surplus. □

In the symmetric case, all greenlighted projects get the same transfer. Those projects that are excluded do not prefer to instead get the green light with the associated transfer. There are two rationales for greenlighted projects to get the same transfer. First, as shown in the proof of Proposition 3, this way the probability of getting as many projects as possible is maximized. Ex-post incentive compatibility prevents the budget from being shifted away from projects with low cost reports to projects with high costs. Therefore offering equal cutoffs is the best the designer can do. Second, as shown in (3) - the rewritten maximization problem of the designer - the expected utility of the designer is given by the sum of virtual surpluses of greenlighted projects. Therefore she wants to greenlight those projects with the highest virtual surpluses. That goal is consistent with offering equal cutoffs to greenlighted projects and excluding those with higher cost. In the optimal allocation, greenlighted projects will have higher virtual surplus than those which are not greenlighted.

The compatibility of the two goals - get as many projects as possible and get those with the highest virtual surpluses - is a special feature of the symmetric case. It generically fails in the asymmetric case, as we demonstrate in the next section.



(a) Budget-constrained, full information. (b) Budget-constrained, private information.

Figure 4: An example of optimal allocations for the symmetric case with $n = 2$.

Figure 4 illustrates the optimal budget-constrained allocations in an example with two projects. Panel 4b shows the fully-constrained optimal allocation juxtaposed with the relaxed optimal allocation when (IC) is neglected, shown in Panel 4a. First, note that in this example $v \geq \overline{c}$ and $\overline{c} < B$. Therefore a completely unconstrained designer with full information would always greenlight both projects, and a budget-constrained designer with full information at least one. However, $z^{**} < \overline{c}$. Therefore for some realizations of $\mathbf{c}$ (the upper-right corner of Panel 4b), no project will get greenlighted in the optimal allocation, even though doing so would be profitable from an ex-post perspective. The negative virtual surpluses of the projects in these cases indicates that the cost of allocating to such a project - incentive compatibility will require higher transfers for other cost types - outweighs the benefit from an ex-ante perspective. The second major difference between the relaxed optimal allocation and the optimal allocation can be seen for those realizations of costs where allocating to both projects would be feasible only in the relaxed problem. This difference is a result of the designer's inability to shift budget from low-cost to relatively higher-cost projects.

Corollary 2. *In the symmetric case, the optimal direct mechanism can be implemented by a descending clock auction. The clock price, denoted by* τ*, starts* *at z^{**} and descends continuously down to $\frac{B}{n}$. Projects can drop out at any price but cannot re-enter. The auction stops once the clock price can be paid out to all projects remaining in the auction.*

In any iteration, a scoring function of the corresponding DA auction is

$$s_i^{A_t}(c_i, A_t) = \max\left\{c_i - \frac{B}{|A_t|}, 0\right\}.$$

We consider the descending clock auction of Corollary 2 to be a natural indirect mechanism that implements the outcome of the optimal z-mechanism. Project i's equilibrium strategy, which implements this outcome, has it staying in the auction as long as the price is weakly larger than its private cost, $\tau \geq c_i$. It is easily verifiable that this is a weakly dominant strategy for project i.

3.2 The asymmetric case

In this section, we demonstrate why the logic of the optimal mechanism in the symmetric case does not carry over to the asymmetric case. To preserve tractability, we restrict ourselves to the two-project case. However, we allow for differing values v_1 and v_2 as well as differing cost distributions F_1 and F_2.

First, note that we can draw on some of the observations from the symmetric case. We did not use symmetry in Lemma 1 and 2. Therefore just as in the symmetric case, we are faced with a problem of finding the right cutoff functions. The rewritten objective of the designer given by maximization problem (3) carries over to the asymmetric case. The designer still wants to maximize the expected virtual surplus of greenlighted projects and allocating to projects with negative virtual surplus is not profitable:

$$\begin{gathered}\max_{z_1(c_2), z_2(c_1)} \mathbb{E}\Big[\mathbb{I}(c_1 \leq z_1(c_2))\left(v_1 + c_1 + \tfrac{F_1(c_1)}{f_1(c_1)}\right) \\ +\mathbb{I}(c_2 \leq z_2(c_1))\left(v_2 + c_2 + \tfrac{F_2(c_2)}{f_2(c_2)}\right)\Big] \\ \text{s.t.} \\ \mathbb{I}(c_1 \leq z_1(c_2))z_1(c_2) + \mathbb{I}(c_2 \leq z_2(c_1))z_2(c_1) \leq B \quad \forall c_1, c_2.\end{gathered} \tag{6}$$

Given that we consider the non-trivial case, $z_1^{**} + z_2^{**} > B$, we know from Lemma 4 that the cutoffs must be constants whenever both projects are greenlighted. Furthermore, we know that these constants must add up to the budget. Otherwise, increasing one of the cutoffs until the budget binds is both feasible and profitable. Let project 1's cutoff be z and project 2's cutoff be $B - z$. These cutoffs pin down the allocation if at least one project has cost below its constant cutoff. Otherwise, we are free to choose one of the two projects. A glance at the objective function (6) reveals that in such a case it is desirable to greenlight the project with higher but positive virtual surplus, if feasible. This result allows us to rewrite the objective function (6) as a function of z:

$$\begin{aligned}
\max_z \pi(z) = & \int_0^z \psi_1(c_1)dF_1(c_1) + \int_0^{B-z} \psi_2(c_2)dF_2(c_2) \\
& + \int_{\max\{\psi_2^{-1}(\psi_1(z)),B-z\}}^{\bar{c}_2} \int_z^{\min\{\psi_1^{-1}(\psi_2(c_2)),z_1^{**},B\}} \psi_1(d)dF_1(d)dF_2(c_2) \\
& + \int_{\max\{\psi_1^{-1}(\psi_2(B-z)),z\}}^{\bar{c}_1} \int_{B-z}^{\min\{\psi_2^{-1}(\psi_1(c_1)),z_2^{**},B\}} \psi_2(d)dF_2(d)dF_1(c_1).
\end{aligned} \tag{7}$$

In the symmetric case, the order of virtual surpluses coincides with the reversed order of costs. A natural extension of the optimal allocation to the asymmetric case would involve adjusting the cutoffs so that they equalize virtual surplus. We will call this the "candidate" allocation.

The condition for optimality of the candidate allocation is stated in Proposition 4. By regularity, there is a unique z such that $\psi_1(z) = \psi_2(B - z)$. To implement the candidate allocation, the constant cutoffs at which both projects are greenlighted must be this z for project 1 and $B - z$ for project 2. But then, we only obtain optimality if $\frac{F_2(B-z)}{f_2(B-z)} = \frac{F_1(z)}{f_1(z)}$. The intuition behind this statement is straightforward. Selecting z in order to satisfy $\psi_1(z) = \psi_2(B - z)$ allows the designer to always get the project with the higher virtual surplus, if she cannot get both. However, if $\frac{F_2(B-z)}{f_2(B-z)} \neq \frac{F_1(z)}{f_1(z)}$ the cutoffs z and $B - z$ will not maximize the probability to get both projects.

Therefore the two aspects of the designer's payoff maximization - getting projects with high virtual surplus and getting as many projects as possible - are only aligned if the condition (8) is met. Note that the condition is met by construction in the symmetric case. However, in an asymmetric environment the condition is generically violated.

Proposition 4. *In the non-trivial asymmetric two-project case, i.e.,* $n = 2$ *and* $z_1^{**} + z_2^{**} > B$*, in which values or cost distributions differ across projects, it is generically not optimal to always allocate to the project with the higher virtual surplus.*

Proof. To obtain the derivative of (7) with respect to z we can use the rules for differentiation under the integral sign.[10] Given the max operators, the derivative will take a different form depending on whether $\psi_1(z) \gtrless \psi_2(B-z)$. However, as π is continuously differentiable, it suffices to look at one of the two forms.

$$\begin{aligned}\left.\frac{\partial \pi}{\partial z}\right|_{z:\psi_1(z)\geq\psi_2(B-z)} = &\int_z^{\psi_1^{-1}(\psi_2(B-z))} \psi_1(x)dF_1(x)f_2(B-z)+ \\ &+ \psi_1(z)f_1(z)F_2(B-z) \\ &- \psi_2(B-z)f_2(B-z)F_1(\psi_1^{-1}(\psi_2(B-z)))\end{aligned}$$

Now take the z corresponding to the candidate allocation with $\psi_1(z) = \psi_2(B-z)$. In this case we are left with

$$\frac{\partial \pi}{\partial z} = 0 \Leftrightarrow \frac{F_2(B-z)}{f_2(B-z)} = \frac{F_1(z)}{f_1(z)} \tag{8}$$

which is a non-generic case. Consequently, it is generically not optimal to always allocate to the project with the higher virtual surplus. □

Remark 1. *Proposition 4 is driven by a tradeoff between quantity and quality.*

Even though the designer always prefers the project with the higher virtual surplus, if she was to greenlight a single project, she sometimes greenlights the project with lower virtual surplus out of two rival projects, as quantity is endogenous here. The simplest way to lay out the intuition behind Proposition 4 is by an example.

Example 1. *There are two projects,* $(n = 2)$*, with* $v_1 = 5, v_2 = 4.5$*, and* c_1, c_2 *are uniformly distributed on support* $[0, 1]$*. The budget is given by* $B = 1$*. The optimal cutoff functions are given by:*

$$z_1(c_2) = \begin{cases} 0.53 & \text{if } c_2 \leq 0.47 \\ c_2 + 0.25 & \text{if } 0.47 < c_2 \leq 0.75 \\ 1 & \text{if } c_2 > 0.75 \end{cases}$$

$$z_2(c_1) = \begin{cases} 0.47 & \text{if } c_1 \leq 0.72 \\ c_1 - 0.25 & \text{if } c_1 > 0.72. \end{cases}$$

[10] Define $g(z, c_2) := \int_z^{\min\{\psi_1^{-1}(\psi_2(c_2)), z_1^{**}, B\}} \psi_1(x)dF_1(x)f_2(c_2)$ and then use $\frac{d}{dz}\left(\int_{a(z)}^{b(z)} g(z, c_2)dc_2\right) = g(z, b(z))b'(z) - g(z, a(z))a'(z) + \int_{a(z)}^{b(z)} g_z(z, c_2)dc_2$.

Possible scoring functions for a corresponding DA auction are given by:

$$s_1^{\{1,2\}}(c_1) = \begin{cases} c_1 + 0.47 & \text{if } 0.53 < c_1 < 0.72 \\ 2c_1 - 0.25 & \text{if } c_1 \geq 0.72 \\ 0 & \text{otherwise} \end{cases}$$

$$s_2^{\{1,2\}}(c_2) = \begin{cases} 2c_2 + 0.25 & \text{if } c_2 > 0.47 \\ 0 & \text{otherwise} \end{cases}$$

$$s_1^{\{1\}}(c_1) = 0$$

$$s_2^{\{2\}}(c_2) = 0.$$

The corresponding optimal allocation is:

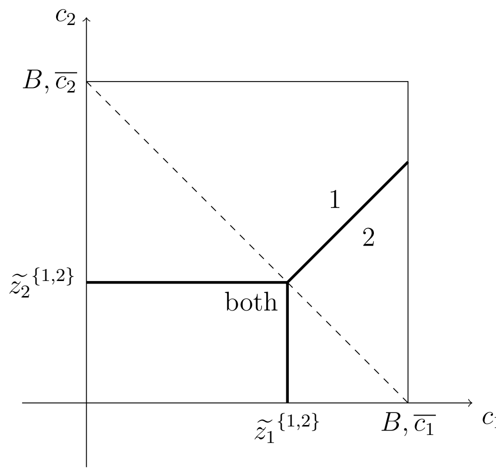
$$(q_1, q_2) = \begin{cases} (1,1) & \text{if } 0 \leq c_1 \leq 0.53 \text{ and } 0 \leq c_2 \leq 0.47 \\ (1,0) & \text{if } 0 \leq c_1 \leq 0.72 \text{ and } c_2 > 0.47 \\ (1,0) & \text{if } c_1 > 0.72 \text{ and } \psi_1 \geq \psi_2 \\ (0,1) & \text{if } 0.53 < c_1 \leq 0.72 \text{ and } c_2 \leq 0.47 \\ (0,1) & \text{if } c_1 > 0.72 \text{ and } \psi_1 < \psi_2. \end{cases}$$

The corresponding transfers are:

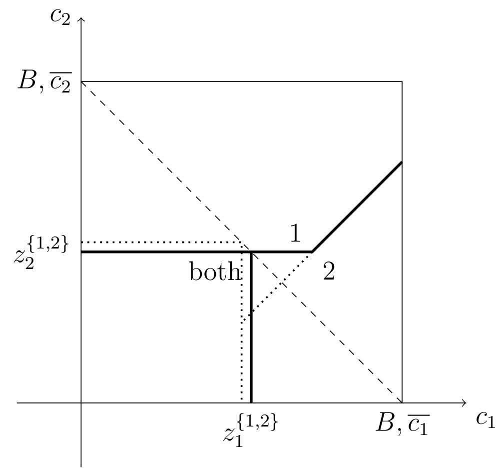
$$t_1(c_1, c_2) = \begin{cases} 0.53 & \text{if } c_2 \leq 0.47 \text{ and } c_1 \leq 0.53 \\ c_2 + 0.25 & \text{if } 0.47 < c_2 \leq 0.75 \text{ and } c_1 \leq c_2 + 0.25 \\ 1 & \text{if } c_2 > 0.75 \\ 0 & \text{otherwise} \end{cases}$$

$$t_2(c_1, c_2) = \begin{cases} 0.47 & \text{if } c_1 \leq 0.72 \text{ and } c_2 \leq 0.47 \\ c_1 - 0.25 & \text{if } c_1 > 0.72 \text{ and } c_2 < c_1 - 0.25 \\ 0 & \text{otherwise.} \end{cases}$$

Consider Example 1. The candidate allocation demands cutoffs $\widetilde{z}_1^{\{1,2\}} = 0.625$ and $\widetilde{z}_2^{\{1,2\}} = 0.375$ for allocating to both projects. At these cutoffs, the probability of allocating to both projects is $0.625 \cdot 0.375 \approx 0.234$. This allocation is depicted in Panel 5a. Now the maximal feasible probability to allocate to both projects is at equal cutoffs, $\widehat{z}_1^{\{1,2\}} = \widehat{z}_2^{\{1,2\}} = 0.5$. The corresponding area is the dotted square in the lower-left corner of Panel 5b. However, at these cutoffs it is not incentive compatible to always allocate to the project with higher virtual surplus, if at least one project exceeds $\widehat{z}_i^{\{1,2\}}$ - i.e., to allocate along the dotted diagonal line.[11] Hence, incentive compatibility introduces a tradeoff between maximizing the probability of greenlighting both projects and allocating to the preferred one if only one project is feasible. Consequently, the optimal cutoffs (z_1^*, z_2^*) for greenlighting both projects do not lie at $(0.625, 0.375)$ but rather at $(0.53, 0.47)$.



(a) Candidate allocation.



(b) Optimal allocation.

Figure 5: Candidate and optimal allocation for Example 1.

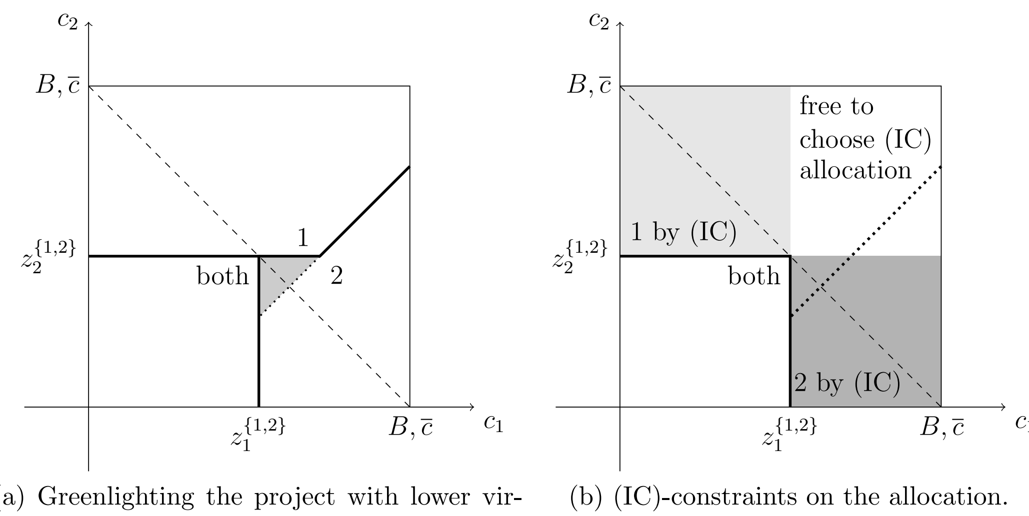
Given the optimal allocation in Example 1, there are some realizations of the cost vector in which the designer allocates to the project with lower virtual surplus. These realizations are represented by the shaded area in Panel 6a. Here, (IC), (PC), and the choice of $(z_1^{\{1,2\}}, z_2^{\{1,2\}})$ force the designer to allocate to project 2, even though project 1 has the higher virtual surplus.

The cost vectors for which the designer allocates to both projects is represented by the rectangular area in the lower-left corner of Panel 6a. The upper-right corner of this area lies on the dashed line representing the budget constraint. A point $(z_1^{\{1,2\}}, z_2^{\{1,2\}})$ on this line has $z_1^{\{1,2\}} + z_2^{\{1,2\}} = B$. Moving this corner point on the dashed line to the right has two effects: shrinking the shaded area and shrinking the area of the rectangle. While it is desirable to shrink the shaded area, in which the designer must allocate to project 2 despite its lower virtual surplus, shrinking the size of the rectangle lowers the probability of allocating

[11]Not to be confused with the dashed diagonal representing the budget constraint.

to both projects. Given that we have an interior solution in this example, at $(z_1^{\{1,2\}}, z_2^{\{1,2\}})$ these two effects balance each other out.

Graphically, the fact that there is no slack in the budget constraint, whenever both projects are greenlighted, implies that the area representing points at which both projects are executed touches the dashed line representing the (BC)-constraint at least once, as can be seen, for example, in Panel 6b. In fact, it can touch the (BC)-constraint exactly once, as it is not possible to greenlight both projects when $c_1 > z_1^{\{1,2\}}$ or $c_2 > z_2^{\{1,2\}}$ without violating (BC) sometimes. This result means that the area where both projects are greenlighted is the rectangle with corners $(0,0)$ and $(z_1^{\{1,2\}}, z_2^{\{1,2\}})$. Then, if $c_1 < z_1^{\{1,2\}}$ but $c_2 > z_2^{\{1,2\}}$, the nature of cutoffs prevents that the designer greenlights project 2. Therefore project 1 must be greenlighted, as represented by the lightly shaded area in Panel 6b. A similar argument applies to the darkly shaded area. Thus, looking at Panel 6b, the choice of $(z_1^{\{1,2\}}, z_2^{\{1,2\}})$ determines the allocation for all points except those in the upper-right corner. Here, the designer is free to choose the allocation, as long as the line delineating whether project 1 or 2 gets greenlighted is (weakly) increasing or vertical. Not surprisingly, it is optimal to greenlight the project with the higher virtual surplus.



(a) Greenlighting the project with lower virtual surplus.

(b) (IC)-constraints on the allocation.

Figure 6: Greenlighting the project with lower virtual surplus and (IC)-constraints on the allocation (Example 1).

Remark 2. *The tradeoff between quantity and quality mitigates the discrimi-*

nation against the stochastically stronger[12] *project compared to the case where quantity is exogenous.*

Example 2. *There are two projects, ($n = 2$), with $v_1 = v_2 = 5$ and c_1 is uniformly distributed on support $[0, 1]$ and $F_2(c_2) = \sqrt[3]{c_2}$ with support $[0, 1]$.*

The budget is given by $B = 1$. The optimal cutoff functions are given by:

$$z_1(c_2) = \begin{cases} 0.56 & \text{if } c_2 \leq 0.44 \\ 2c_2 & \text{if } 0.44 < c_2 \leq 0.5 \\ 1 & \text{if } c_2 > 0.5 \end{cases}$$

$$z_2(c_1) = \begin{cases} 0.44 & \text{if } c_1 \leq 0.88 \\ \frac{1}{2}c_1 & \text{if } c_1 > 0.88 \end{cases}$$

Scoring functions, allocation and transfers are omitted but can be easily computed from the cutoff functions as in Example 1.

To illustrate Remark 2, consider Example 2.[13] Here the designer chooses among two projects with identical value but different cost distributions. The notion of "weak" and "strong" is reversed to standard seller auction settings, e.g., as discussed in de Castro and de Frutos (2010). In the example, project 2 is stochastically stronger than project 1 in a sense that F_1 dominates F_2 in terms of the reverse hazard rate. That is, for all c

$$\frac{f_1(c)}{F_1(c)} \geq \frac{f_2(c)}{F_2(c)}.$$

Reverse hazard rate dominance implies first-order stochastic dominance, e.g., Krishna (2009, p. 47). Therefore project 2 tends to have lower cost. In Figure 7, the 45°-line represents the efficient allocation if only a single unit is procured. The dashed line below represents the allocation chosen by a designer maximizing her own payoff in the single unit case. Consequently, the horizontally striped wedge in between represents the cost vectors where the discrimination of project 2 creates an inefficiency. When quantity is endogenous however this inefficiency is mitigated. The size of this effect depends on the distributions and in Figure 7 corresponds to the shaded triangle. In contrast to the case where quantity is exogenously given, here the designer allocates efficiently.

[12] We say that project i is stochastically stronger than project j if F_j dominates F_i in terms of the reverse hazard rate.

[13] We choose to make this point by example. However, it should be clear that this point can easily be generalized.

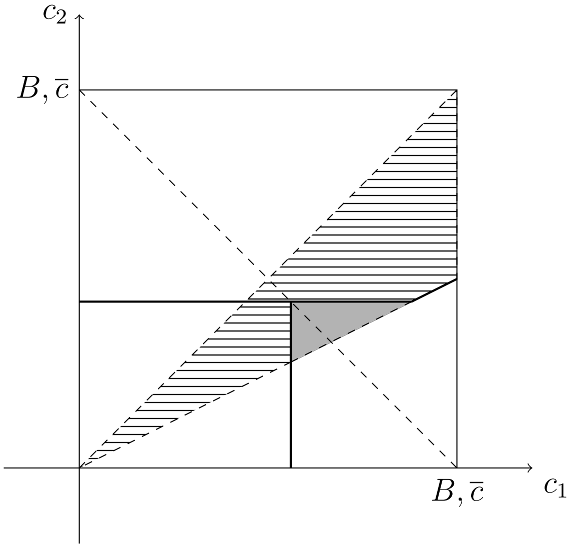


Figure 7: Quantity being endogenous mitigates the distortion against the stochastically stronger project (Example 2).

Remark 3. *In an optimal implementation with descending price clocks, the clocks not only run at individual speeds, occasionally some clocks also have to halt.*

By Corollary 1, we can implement the optimal mechanism with a descending clock auction. A crucial difference to the symmetric case is that each project has its own price clock, because individual virtual surplus functions require individual speeds. Interestingly, an implication of the quantity-quality tradeoff is that sometimes one clock has to halt. For Example 1, these price clocks are depicted in Figure 8 as a function of time. Note that the entire (maximal) duration of the auction can be divided into three segments. The auction starts with both clocks at $z_1^{**} = z_2^{**} = \overline{c}$. First, τ_2 decreases while τ_1 is held constant, which happens until both clocks imply the same virtual surplus, i.e., $\psi_2(\tau_2) = \psi_1(\overline{c}_2)$. Second, both τ_1 and τ_2 decrease simultaneously, keeping virtual surplus equal, $\psi_1(\tau_1) = \psi_2(\tau_2)$, until $\tau_2 = z_2^{\{1,2\}}$. Third, only τ_1 decreases until $\tau_1 = z_1^{\{1,2\}}$. If at this point both projects still remain in the auction, the auction stops and both are greenlighted. Otherwise, the inferior project 2 is greenlighted.

The cost vectors for which the designer greenlights project 2 despite its lower virtual surplus, represented by the shaded area in Panel 6a, are also represented graphically in Figure 8. If the auction ends in the third time segment (shaded area of Figure 8) before both projects can be greenlighted, project 1 must have exited

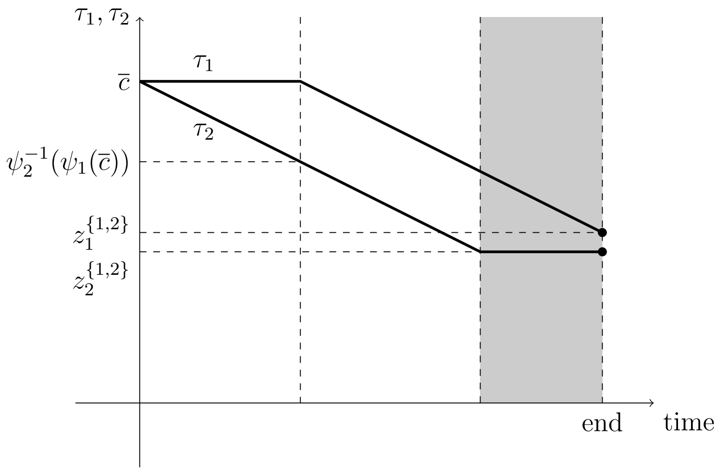


Figure 8: Optimal descending clock auction in Example 1.

because τ_1 dropped below c_1. Project 2 is greenlighted and receives transfer $z_2^{\{1,2\}}$ even though project 1 has the higher virtual surplus. Therefore if cost vectors in the shaded area of Panel 6a realize, the optimal descending clock auction will end in the third time segment.

We should emphasize again a novel feature of this descending clock auction. The clocks of both projects are paused asynchronously for some time of the auction. One project's clock runs down while the other project's clock stops. Since we have examined a very simple example, each project's clock is paused only once. In a more general setting, the projects' clocks may pause and resume several times.

Given the nature of our problem, we do not find a simple and general ($n > 2$) full characterization of the optimal mechanism in the asymmetric case. In our examples with two projects, the problem boils down to finding one point, $(z_1^{\{1,2\}}, z_2^{\{1,2\}})$, with respect to one crucial tradeoff. Naturally, the number of relevant tradeoffs increases with the number of projects. Therefore unfortunately, optimization with a larger set of projects quickly loses tractability.

4 Discussion

With our model as a starting point, there are several natural extensions. In this section, we will address the most natural alternative models or extensions.

Disregarding residual money - Whether it is reasonable to assume that the designer values residual money depends on the setting. To illustrate, this is not the case in Ensthaler and Giebe (2014a), where money does not enter the objective function, only the constraints. Note that in such a setting, the designer would want to allocate to projects with negative virtual surplus and would be indifferent between paying z^{k*} or c_{k*+1} in the optimal symmetric mechanism. Other than that, our results qualitatively carry over.

v_i **as private information, potentially correlated with** c_i - We can neglect asking for v_i directly since no meaningful non-babbling equilibria in the v_i-dimension exist. If the conditional density of $v_i|c_i$ has full support, project i cannot credibly announce being a "high" type, say $\overline{v}_i$. If we slightly change the regularity assumption such that $\mathbb{E}[v_i|c_i]-c_i-\frac{F(c_i)}{f(c_i}$ must be strictly increasing, our results generalize by exchanging the previously commonly known v_i with $\mathbb{E}[v_i|c_i]$. This regularity condition mildly restricts the degree of positive correlation.

Interdependent types - We can interpret the symmetric case as a setting in which identical projects are provided at individual costs. Hence, one may wonder about a setting where projects only draw an imperfect signal about the cost, which finally depends on other projects' signals as well. In a clock auction in such an environment, active projects update their belief about the cost whenever a project drops out. Moreover, the designer learns this information as well. Therefore the design of the optimal mechanism crucially depends on the information structure. This analysis is left for a follow-up paper.

5 Conclusion

Despite their importance, knapsack problems with private information have been somewhat overlooked by the economics literature. We examine a setting where a budget-constrained procurer faces privately informed sellers under ex-post constraints. Amongst many possible economic problems, this setting particularly applies to a development fund, who is typically endowed with a fixed budget and wants to finance both many projects and projects of high quality. Such problems often entail relationships in which sellers can renege on the terms of the agreement ex-post. In order to avoid non-delivery or costly renegotiation, it is appropriate to impose ex-post constraints on the seller's side. For such settings, we have have shown that z-mechanisms constitute the class of optimal deterministic dominant strategy implementable mechanisms.

A z-mechanism is described by a set of cutoff functions that are increasing in other projects' costs. Cutoffs only depend on the cost of other projects as they drop out of the allocation. In other words, if two different realizations of the cost vector lead to the same allocation, then the cutoffs of projects conducted only vary in the costs of projects not conducted.

We show that any z-mechanism can alternatively be characterized as a deferred acceptance (DA) auction, introduced by Milgrom and Segal (2014). The DA auction representation allows for a simple implementation via descending clock auctions, which are easy to understand and usable in practice. In addition, DA auctions have attractive properties regarding incentive compatibility which make the prediction of equilibrium play more robust.

We fully describe the optimal allocation and the corresponding descending clock auction in an environment where projects are ex-ante symmetric. The optimal mechanism is monotone in the sense that the k cheapest projects are greenlighted and all projects conducted receive the same transfer. This transfer either corresponds to the lowest cost among non-executed projects or the budget is distributed equally. The equivalent clock auction features a single price clock that continuously decreases until all active projects can be financed.

For asymmetric environments, where values and/or cost distributions differ, we demonstrate a novel tradeoff between quantity and quality of the greenlighted projects. The designer values both quantity and quality, expressed by the virtual surplus, of the projects. In settings where quantity is exogenous, the designer would always choose the projects with the highest virtual surpluses. If quantity is endogenously determined by the mechanism, as in our setup, it is not always desirable to conduct the best projects. When the best projects are always conducted, incentive compatibility would force the designer to reduce the expected number of greenlighted projects. This insight entails a consequence for the corresponding descending clock auction. Clocks not only run asynchronously, but also periodically have to stop for certain projects. In comparison to settings where quantity is exogenous, the allocation is less distorted away from efficiency, i.e., stochastically weaker projects are favored less.

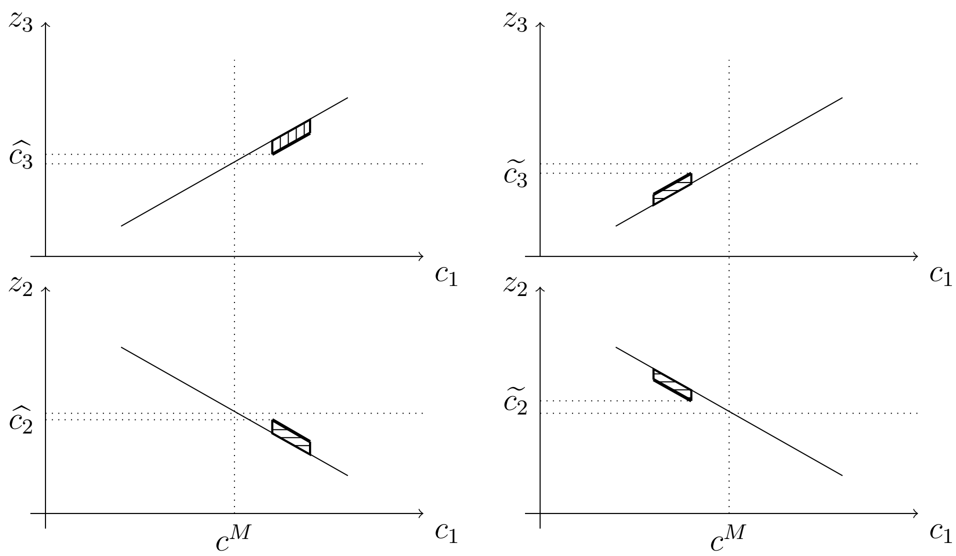
Other interesting extensions are left for future research, for example multiple projects per agent. For practitioners, a simple approximately optimal mechanism may be of great value. The characterization of the optimal mechanism as a z-mechanism sheds light on how to construct such an approximately optimal mechanism. Halting clocks should be a key feature for the corresponding clock auction in asymmetric environments.

In conclusion, our methodological approach contributes to a better understanding of of a class of relevant problems and opens the door for future research in this area. Furthermore, we provide an elegant indirect mechanism, that can be easily implemented in practice.

Appendix

A Properties of a z-mechanism: General proofs

Lemma 3. *The optimal cutoff function z_i is weakly increasing in c_j for all i, j with $j \neq i$, i.e., $z_i(c_j, \mathbf{c}_{-i-j}) \geq z_i(c'_j, \mathbf{c}_{-i-j})$ for almost every $c_j > c'_j$ and $\mathbf{c}_{-i-j}$.*



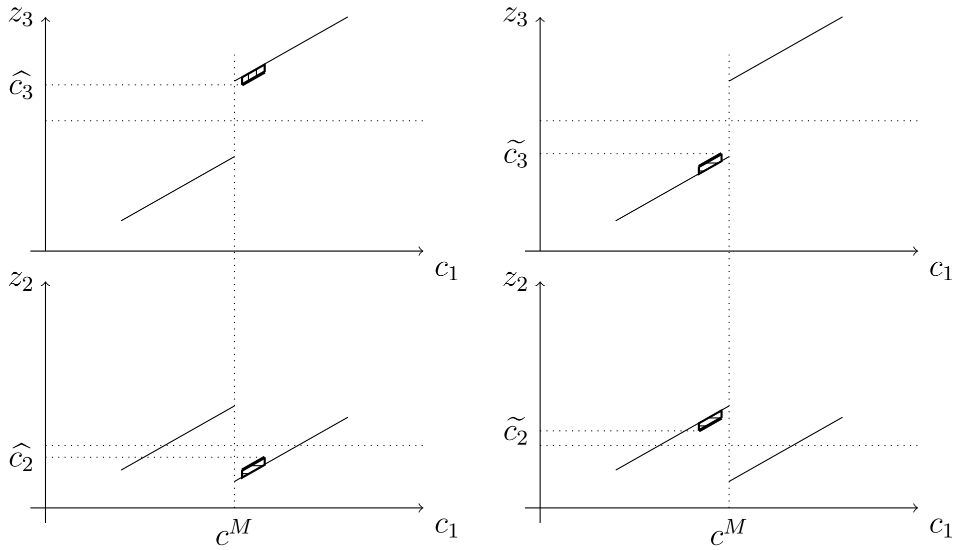
(a) Intuition for the hat deviation. (b) Intuition for the tilde deviation.

Figure 9: Continuous decrease / increase.

Proof. Suppose to the contrary that somewhere z_2 is decreasing in c_1. Then there exist some ${c_1}^M$ and $\eta > 0$ such that $z_2(\underline{\underline{c_1}}, \mathbf{c}_{-1-2}) > z_2(\overline{\overline{c_1}}, \mathbf{c}_{-1-2})$ for all $\underline{\underline{c_1}} \in ({c_1}^M - \eta, {c_1}^M)$, for all $\overline{\overline{c_1}} \in ({c_1}^M, {c_1}^M + \eta)$, and for all $\mathbf{c}_{-1-2} \in \chi_{-1-2} \subset \times_{j \in I \setminus \{1,2\}} [\underline{c}_j, \overline{c}_j]$, where χ_{-1-2} has positive Lebesgue-measure.

With more than two projects, the simple deviation of the two-project case - flattening the decreasing cutoff - is not necessarily feasible. It may be the case that other projects' cutoffs are strictly increasing and that for some cost vectors these cutoffs have to be paid along z_2. Then simply flattening z_2 could violate the budget constraint.

Suppose no other cutoff is increasing while z_2 is decreasing. Then the decrease of z_2 cannot be optimal and flattening z_2 will increase the designer's payoff much in the same way as in the two-project-case. Otherwise, pick a subset of



(a) Intuition for the hat deviation. (b) Intuition for the tilde deviation.

Figure 10: Jump decrease / increase.

$\widehat{\chi}_1 \subset ({c_1}^M, {c_1}^M + \eta)$ (with pos. Lebesgue-measure) such that w.l.o.g. project 3's cutoff increases in c_1 in the analogous sense to the decrease of z_2 defined above - for cost vectors where both project 2 and project 3 are eventually greenlighted, i.e., z_2 and z_3 both need to be paid.

The set

$$\widehat{\Xi}_{23}(c_1, \mathbf{c}_{-1-2-3}, \delta) = \{(c_2, c_3) | c_2 \in (z_2(c_1, c_3, \mathbf{c}_{-1-2-3}), z_2(c_1, c_3, \mathbf{c}_{-1-2-3}) + \delta]; \\ c_3 \in (z_3(c_1, c_2, \mathbf{c}_{-1-2-3}) - \delta, z_3(c_1, c_2, \mathbf{c}_{-1-2-3})]\}$$

must have positive measure on $\mathbb{R}^2$ for all $c_1 \in \widehat{\chi}_1$ and for any $\mathbf{c}_{-1-2-3} \in \chi_{-1-2-3}$, where χ_{-1-2-3} is a set with positive Lebesgue measure where the cutoff of project 2 is decreasing while the cutoff of project 3 is increasing. It is the set of (c_2, c_3) tuples, where c_2 just exceeds z_2 by no more than δ, while c_3 lies just below z_3 by no more than δ - given $\mathbf{c}_{-1-2-3}$ and c_1. By $\widehat{\Xi}^2_{23}(c_1, \mathbf{c}_{-1-2-3}, \delta)$ we denote the set of project 2 components of tuples in the set $\widehat{\Xi}_{23}(c_1, \mathbf{c}_{-1-2-3}, \delta)$, and similarly for project 3.

Now deviate from the candidate mechanism in setting

$$
\begin{aligned}
\widehat{z_2}(c_1, c_3, \mathbf{c}_{-1-2-3}) &:= z_2(c_1, c_3, \mathbf{c}_{-1-2-3}) + \delta \\
\widehat{z_3}(c_1, c_2, \mathbf{c}_{-1-2-3}) &:= z_3(c_1, c_2, \mathbf{c}_{-1-2-3}) - \delta \\
&\text{for all} \\
c_1 &\in (\widehat{c_1}, \widehat{c_1} + \varepsilon) \\
c_2 &\in \widehat{\Xi}^2_{23}(c_1, \mathbf{c}_{-1-2-3}) \\
c_3 &\in \widehat{\Xi}^3_{23}(c_1, \mathbf{c}_{-1-2-3}) \\
\mathbf{c}_{-1-2-3} &\in \widehat{\chi}_{-1-2-3} \subset \chi_{-1-2-3}.
\end{aligned}
$$

We call this deviation the *hat* deviation. The intuition for this deviation is the following. For an ε-environment of c_1 to the right of c_1^M (i.e., $\widehat{c_1} > c_1^M$), increase the decreasing cutoff $z_2(c_1, c_3, \mathbf{c}_{-1-2-3})$ by δ for all c_3 that drop out of the allocation if $z_3(c_1, c_2, \mathbf{c}_{-1-2-3})$ (at c_2) is decreased by δ. Likewise only increase $z_3(c_1, c_2, \mathbf{c}_{-1-2-3})$ by δ for those c_2 that are additionally greenlighted if $z_2(c_1, c_3, \mathbf{c}_{-1-2-3})$ is increased by δ. Therefore if the deviation changes the allocation, project 2 is now greenlighted whereas project 3 is not.

This deviation is feasible. Remember that there must be enough budget to pay both z_2 and z_3 - otherwise flattening z_2 would have been possible. But then there is enough budget for $z_2 + \delta$ and $z_3 - \delta$.

Now define

$$
\begin{aligned}
\widehat{c_2} &:= \sup_{c_1, \mathbf{c}_{-1-2-3}} \widehat{\Xi}^2_{23}(c_1, \mathbf{c}_{-1-2-3}) \\
\widehat{c_3} &:= \inf_{c_1, \mathbf{c}_{-1-2-3}} \widehat{\Xi}^3_{23}(c_1, \mathbf{c}_{-1-2-3}) \\
&\text{s.t.} \\
c_1 &\in (\widehat{c_1}, \widehat{c_1} + \varepsilon) \\
\mathbf{c}_{-1-2-3} &\in \widehat{\chi}_{-1-2-3}.
\end{aligned}
$$

In words, to bound the change in payoff we let $\widehat{c_2}$ be the highest cost type gained by the deviation and we let $\widehat{c_3}$ be the lowest cost type lost by the deviation. Then the change in payoff for the hat deviation is bounded in the following way:

$$
\widehat{\Delta} > (\psi_2(\widehat{c_2}) - \psi_3(\widehat{c_3})) *
\int_{\widehat{\chi}_{-1-2-3}} \int_{\widehat{c_1}}^{\widehat{c_1}+\varepsilon} \int_{\widehat{\Xi}^2_{23}(c_1, \mathbf{c}_{-1-2-3})} \int_{\widehat{\Xi}^3_{23}(c_1, \mathbf{c}_{-1-2-3})} 1 dF_3(\cdot) dF_2(\cdot) dF_1(\cdot) dF_{-1-2-3}(\cdot).
$$

If $\widehat{\Delta} > 0$, we have found a profitable deviation. If not, then consider the following *tilde* deviation.

Analogously to $\widehat{\Xi}_{23}$ we define the set

$$\widetilde{\Xi}_{23}(c_1, \mathbf{c}_{-1-2-3}, \delta) = \{(c_2, c_3) | c_2 \in (z_2(c_1, c_3, \mathbf{c}_{-1-2-3}) - \delta, z_2(c_1, c_3, \mathbf{c}_{-1-2-3})]; \\ c_3 \in (z_3(c_1, c_2, \mathbf{c}_{-1-2-3}), z_3(c_1, c_2, \mathbf{c}_{-1-2-3}) + \delta]\}$$

which again must have positive measure.

Now, we deviate for an ε-environment to the left of c_1^M (i.e., $\widetilde{c_1} < c_1^M$). But instead of increasing z_2 and decreasing z_3, we increase z_3 and decrease z_2:

$$\begin{aligned} \widetilde{z_2}(c_1, c_3, \mathbf{c}_{-1-2-3}) &:= z_2(c_1, c_3, \mathbf{c}_{-1-2-3}) - \delta \\ \widehat{z_3}(c_1, c_2, \mathbf{c}_{-1-2-3}) &:= z_3(c_1, c_2, \mathbf{c}_{-1-2-3}) + \delta \\ &\text{for all} \\ c_1 &\in (\widetilde{c_1} - \varepsilon, \widetilde{c_1}) \\ c_2 &\in \widetilde{\Xi}^2_{23}(c_1, \mathbf{c}_{-1-2-3}) \\ c_3 &\in \widetilde{\Xi}^3_{23}(c_1, \mathbf{c}_{-1-2-3}) \\ \mathbf{c}_{-1-2-3} &\in \widetilde{\chi}_{-1-2-3} \subset \chi_{-1-2-3}. \end{aligned}$$

The relevant bounds to bound the payoff are then given by

$$\begin{aligned} \widetilde{c_2} &:= \inf_{c_1, \mathbf{c}_{-1-2-3}} \widetilde{\Xi}^2_{23}(c_1, \mathbf{c}_{-1-2-3}) \\ \widetilde{c_3} &:= \sup_{c_1, \mathbf{c}_{-1-2-3}} \widetilde{\Xi}^3_{23}(c_1, \mathbf{c}_{-1-2-3}) \\ &\text{s.t.} \\ c_1 &\in (\widetilde{c_1} - \varepsilon, \widetilde{c_1}) \\ \mathbf{c}_{-1-2-3} &\in \widetilde{\chi}_{-1-2-3}. \end{aligned}$$

And this gives the following bound for the payoff

$$\widetilde{\Delta} > (\psi_2(\widetilde{c_3}) - \psi_3(\widetilde{c_2})) * \\ \int_{\chi_{-1-2-3}} \int_{\widetilde{c_1} - \varepsilon}^{\widetilde{c_1}} \int_{\widetilde{\Xi}^2_{23}(c_1, \mathbf{c}_{-1-2-3})} \int_{\widetilde{\Xi}^3_{23}(c_1, \mathbf{c}_{-1-2-3})} 1 dF_3(\cdot) dF_2(\cdot) dF_1(\cdot) dF_{-1-2-3}(\cdot).$$

By appropriately choosing δ, $\widehat{\Xi}_{-1-2-3}$, and $\widetilde{\Xi}_{-1-2-3}$, we can ensure that $\widehat{c_3} > \widetilde{c_3}$ and $\widehat{c_2} < \widetilde{c_2}$. This follows simply from the notion of increasing/decreasing cutoffs and is illustrated in Figures 9 and 10. Therefore $\widehat{\Delta} \leq 0$ implies $\widetilde{\Delta} > 0$. Consequently, there is always a profitable deviation and our candidate mechanism could not have been optimal. $\square$

Lemma 4. *Conditional on any arbitrary partition $\{G, R\}$, the optimal cutoff functions z_g for all $g \in G$ are independent of the costs of all greenlighted projects $\mathbf{c}_G$. That is,*

$$z_g(\mathbf{c}_{G-g}, \mathbf{c}_R) = z_g(\mathbf{c}'_{G-g}, \mathbf{c}_R),$$

for all $\mathbf{c}_{G-g}$ and $\mathbf{c}'_{G-g}$ such that G is the set of greenlighted agents.

Moreover, if cost vector $(\mathbf{c}_G, \mathbf{c}_R)$ induces allocation $\{G, R\}$, then cost vector $(\mathbf{c}'_G, \mathbf{c}_R)$ also induces $\{G, R\}$ if $c'_g \leq c_g$ for all $g \in G$.

Proof. Take any feasible candidate mechanism with any set of increasing cutoff functions $\{z_i\}_I$ for any individual project. Assume that for some cost vectors with positive Lebesgue-measure, only all projects in set $G \subseteq I$ are executed while all projects of set R are not conducted. Therefore there exists a set, C_R^G, with positive Lebesgue-measure containing the part of the cost vector for the projects in set R such that the partition $\{G, R\}$ is induced given some $\mathbf{c}$ where the redlighted projects have costs $\mathbf{c}_R \in C_R^G$. Then $a_i^G(\mathbf{c}_R)$ according to the following definition

$$\begin{aligned} a_i^G(\mathbf{c}_R) = \max\{c_i | \exists \mathbf{c}_{G-i} : c_i \leq z_i(\mathbf{c}_{G-i}, \mathbf{c}_R), \\ \text{and } c_g \leq z_g(\mathbf{c}_{G-j}, \mathbf{c}_{-G}) \forall g \in G, \\ \text{and } c_r > z_r(\mathbf{c}_G, \mathbf{c}_{-G-r}) \forall r \in R\} \end{aligned} \tag{9}$$

exists for all $i \in G$ given $\mathbf{c}_R \in C_R^G$. In words, $a_i^G(\mathbf{c}_R)$ is the highest cost of project i such that, given some cost vector $\mathbf{c}_R$ of projects that are not executed, there exists some vector $\mathbf{c}_{G-i}$ of costs of competing projects that induces a cutoff $z_i(\mathbf{c}_{G-i}, \mathbf{c}_{-G})$ above said cost while each element c_g of the vector $\mathbf{c}_{G-i}$ is lower than the cutoff induced by $a_i^G(\mathbf{c}_R)$ and the elements of the cost vectors $\mathbf{c}_R$ and $\mathbf{c}_{G-i-g}$,

$$\forall g \in G \setminus \{i\}, \;\; c_g \leq z_g(\mathbf{c}_R, \mathbf{c}_{G-i-g}, a_i^G(\mathbf{c}_R)).$$

Simultaneously, it must hold that these costs induce a cutoff such that no project $r \in R$ is conducted

$$\forall r \in R, \;\; c_r > z_r(\mathbf{c}_{R-r}, \mathbf{c}_{G-i}, a_i^G(\mathbf{c}_R)).$$

Moreover, we can replace any function z_i with a left-continuous function that is identical up to a set of points with Lebesgue-measure zero. Hence, the limit is reached from below and there exists at least one cost vector $(\widehat{\mathbf{c}}_{-i}, a_i^G(\mathbf{c}_R))$ where G is the set of executed projects and $a_i^G(\widehat{\mathbf{c}}_R) = z_i(\widehat{\mathbf{c}}_{-i})$ holds. Now, notice that

$$\widehat{c}_g \leq a_g^G(\widehat{\mathbf{c}}_R) \, \forall g \in G \setminus \{i\},$$

because, given $\widehat{\mathbf{c}}_R$, there cannot exist a cost vector where only all projects in G are executed and the cost of project g exceeds $a_g^G(\widehat{\mathbf{c}}_R)$ by its construction. Moreover, we have established that every cutoff function z_i is weakly increasing in each argument. Thus,

$$a_i^G(\widehat{\mathbf{c}}_R) = z_i(\widehat{\mathbf{c}}_{-i}) \leq z_i(a_{G-i}^G(\widehat{\mathbf{c}}_R), \widehat{\mathbf{c}}_R),$$

where a_{G-i}^G is the vector of all a_g^G defined according to (9) except a_i^G. This inequality tells us that, whenever some vector $(\mathbf{c}_R, \mathbf{c}_{G-i}) \geq (\widehat{\mathbf{c}}_R, a_{G-i}^G(\widehat{\mathbf{c}}_R))$[14] realizes, a sufficient condition for project $i \in G$ to be executed is $c_i \leq a_i^G(\widehat{\mathbf{c}}_R)$.

The same logic also applies to all projects in G other than i. Therefore at least all projects $g \in G$ are conducted whenever a cost vector realizes such that $c_g = a_g^G(\mathbf{c}_R)$.[15] Consequently, the budget constraint requires that

$$\sum_{g \in G} z_g(a_{-g}^G(\mathbf{c}_R), \mathbf{c}_R) \leq B. \tag{10}$$

Furthermore, given $\mathbf{c}_R$, for all projects $g \in G$, $z_g(\mathbf{c}_{-G}, \mathbf{c}_R) = a_g^G(\mathbf{c}_R)$ if $\mathbf{c}_{G-g} \leq a_{G-g}^G(\mathbf{c}_{-G})$. That is, the cutoffs are constant given the cost vector of non-executed projects.

Suppose to the contrary that $z_i(\mathbf{c}_{-i}) < a_i(\mathbf{c}_R)$ for some $i \in G$ and for all $\mathbf{c}_{-i} \in \Xi \subset C_{-i}^G$ with Ξ having positive Lebesgue measure.

Define $\Xi(\mathbf{c}_{G-i-j}, \mathbf{c}_R) \subset [0, \overline{c}_j]$ where $z_i(\mathbf{c}_{G-i-j}, c_j, \mathbf{c}_R) < a_i^G(\mathbf{c}_R)$ for all $c_j \in \Xi(\mathbf{c}_{G-i-j}, \mathbf{c}_R)$. For any $\mathbf{c}_{G-i-j} \leq a_{-i-j}^G(\mathbf{c}_R)$, let

$$z_i^{\Xi}(\mathbf{c}_{G-i-j}, \mathbf{c}_R) := \max_{c_j \in \Xi(\mathbf{c}_{G-i-j}, \mathbf{c}_R)} z_i(\mathbf{c}_{G-i-j}, c_j, \mathbf{c}_R)$$

By (10), changing the mechanism to

$$z_i(\mathbf{c}_{G-i,-j}, c_j, \mathbf{c}_R) = a_i^G(\mathbf{c}_R), \quad \forall c_j \leq a_j^G(\mathbf{c}_R)$$

[14] When $\mathbf{x}$ and $\mathbf{y}$ are vectors, $\mathbf{x} \geq \mathbf{y}$ means that every element x_i of $\mathbf{x}$ weakly exceeds the corresponding element y_i of $\mathbf{y}$.

[15] $a_i^G(\mathbf{c}_R)$ is only defined if $C^G \neq \emptyset$ and $\mathbf{c}_R \in C_R^G$, but this does not hinder the proof.

does not violate the budget constraint. This deviation increases the payoff conditional on $\mathbf{c}_R$ by

$$\Delta > \int_{\Xi_{-j}} \Pr(c_j \in \Xi(\mathbf{c}_{G-i-j}, \mathbf{c}_R)) \int_{z_i^{\Xi}(\mathbf{c}_{G-i-j}, \mathbf{c}_R)}^{a_i^G(\mathbf{c}_R)} \psi_i(c) dF_i(c) dF_{-i-j}(\mathbf{c}_{-i-j}) > 0.$$

Given that Ξ has positive Lebesgue-measure, this deviation will also strictly increase the unconditional payoff. □

B Constructing a scoring function for a z-mechanism: Proof of Proposition 2

To prove Proposition 2, it is helpful to consider the following lemmata. While Lemma 4 (Property 4 of a z-mechanism) is a statement that conditions on a fixed allocation, it also has implications on the cutoffs resulting from different cost vectors that induce different allocations.

Lemma 5. *Take any z-mechanism and any two cost vectors $\mathbf{c} \neq \widehat{\mathbf{c}}$ that induce partitions $\{G, R\}$ and $\{\widehat{G}, \widehat{R}\}$, respectively. Then*

$$\mathbf{c}_{R\cup\widehat{R}} = \widehat{\mathbf{c}}_{R\cup\widehat{R}}$$
$$\mathbf{c}_{G\cap\widehat{G}} \neq \widehat{\mathbf{c}}_{G\cap\widehat{G}}$$

implies

$$G = \widehat{G}$$
$$R = \widehat{R},$$

that is, $\mathbf{c}$ and $\widehat{\mathbf{c}}$ induce the same allocation.

Proof. Given cost vector $\mathbf{c}$, define a new cost vector $\mathbf{c}'$, where $c_i' = \min\{c_i, \widehat{c}_i\}$ for all $i \in G \cap \widehat{G}$ and $\mathbf{c}'_{R\cup\widehat{R}} = \mathbf{c}_{R\cup\widehat{R}}$. By Lemma 4, $\mathbf{c}'$ induces allocation $\{G, R\}$. Similarly, a perturbation of cost vector $\widehat{\mathbf{c}}$ in the same way with $\widehat{c}_i' = \min\{c_i, \widehat{c}_i\}$ for all $j \in G \cap \widehat{G}$ and $\widehat{\mathbf{c}}'_{R\cup\widehat{R}} = \widehat{\mathbf{c}}_{R\cup\widehat{R}}$ must induce allocation $\{\widehat{G}, \widehat{R}\}$. But $\mathbf{c}' = \widehat{\mathbf{c}}'$ by construction. Hence, $G = \widehat{G}$ and $R = \widehat{R}$. □

Lemma 6. *Take any z-mechanism and any two cost vectors $\mathbf{c} \neq \widetilde{\mathbf{c}}$ that induce partitions $\{G, R\}$ and $\{\widetilde{G}, \widetilde{R}\}$, respectively. Then*

$$z_i(\mathbf{c}_{G\cap\widetilde{G}}, \mathbf{c}_{R\cup\widetilde{R}}) = z_i(\widetilde{\mathbf{c}}_{G\cap\widetilde{G}}, \mathbf{c}_{R\cup\widetilde{R}})$$
$$z_j(\widetilde{\mathbf{c}}_{G\cap\widetilde{G}}, \widetilde{\mathbf{c}}_{R\cup\widetilde{R}}) = z_j(\mathbf{c}_{G\cap\widetilde{G}}, \widetilde{\mathbf{c}}_{R\cup\widetilde{R}})$$

for all $i \in G$ and for all $j \in \widetilde{G}$, respectively.

Proof. By Lemma 5, the vector $(\widetilde{\mathbf{c}}_{G\cap\widetilde{G}}, \mathbf{c}_{R\cup\widetilde{R}})$ leads to allocation $\{G, R\}$ and the vector $(\mathbf{c}_{G\cap\widetilde{G}}, \widetilde{\mathbf{c}}_{R\cup\widetilde{R}})$ leads to allocation $\{\widetilde{G}, \widetilde{R}\}$. The rest follows directly from Lemma 4 (Property 4 of a z-mechanism). □

Having established these properties we can prove Proposition 2 by induction. We construct a scoring function for each iteration of a DA auction that replicates the underlying z-mechanism. Conditional on all previous iterations having been constructed correctly, we can demonstrate how to construct an appropriate scoring function for any iteration.

Proposition 2. *Any z-mechanism has an equivalent DA representation.*

Proof. This proof is structured as follows. First, we construct scoring functions for each iteration of the DA auction. Then we explain how the zeros of the scoring functions are derived. Finally we show by induction that the constructed DA auction implements the same allocation as the underlying z-mechanism.

Scoring functions

First, we introduce some notation. Let A_t be the set of active projects in iteration t and let $O_t := I \setminus A_t$ be the set of inactive projects (O as in "out"). Let $O_tj := O_t \cup \{j\}$ be the union of dropped out projects and some individual project j.

Fix an optimal z-mechanism and consider the corresponding DA auction with scoring functions $\{s_i^A\}_{A\subset I, i\in A}$

$$s_i^A(c_i, \mathbf{c}_O) = \begin{cases} 0 & \text{if } c_i \leq a_i^A(\mathbf{c}_O), \\ c_i + \sum\limits_{i\neq j}^{j\in A} b_{Oi}^j(c_i, \mathbf{c}_O) & \text{otherwise,} \end{cases} \tag{11}$$

where $a_i^A(\mathbf{c}_O)$ is defined as in (9) and $b_{Oi}^j(c_i, \mathbf{c}_O)$ is defined as

$$\begin{aligned} b_{Oi}^j(c_i, \mathbf{c}_O) &:= \max\left\{c_j : \exists\tilde{\mathbf{c}}_{-Oi-j} : R = Oi | c_i, \mathbf{c}_O\right\} \\ &:= \max\Big\{c_j : \exists\tilde{\mathbf{c}}_{-Oi-j} : c_i > z_i(c_j, \tilde{\mathbf{c}}_{-Oi-j}, \mathbf{c}_O), \\ &\qquad \text{and } c_o > z_o(c_i, c_j, \tilde{\mathbf{c}}_{-Oi-j}, \mathbf{c}_{O-i}) \forall o \in O, \\ &\qquad \text{and } c_g \leq z_g(c_i, c_j, \tilde{\mathbf{c}}_{-Oi-j}, \mathbf{c}_O) \forall g \in A \setminus i\Big\}. \end{aligned}$$

In words, $b_{Oi}^j(c_i, \mathbf{c}_O)$ is the highest cost of project j such that given the vector $\mathbf{c}_{Oi}$ the corresponding z-mechanism implements the allocation partition $R = Oi$ and $G = A \setminus i$ for some realization of the cost vector $\tilde{\mathbf{c}}_{-Oi-j}$.

Zeros of the scoring functions

Suppose the DA auction ends in the t-th iteration. Then all projects $i \in A_t$ have score $s_i^{A_t} = 0$ and the cost vector must induce $G = A_t$ in the underlying z-mechanism. By Lemma 4 (Property 4 of a z-mechanism), cutoffs of projects in G are constant in the part of the cost vector $\mathbf{c}_{A_t}$ for all cost vectors inducing the same allocation.

Therefore we can characterize the zeros of the scoring function by a threshold and $s_i^{A_t} = 0$ whenever project i's cost is below this threshold. The threshold is given by $a_i^{A_t}(\mathbf{c}_O)$ as defined in (9). Notice that $c_i \leq a_i^{A_t}(\mathbf{c}_O)$ implies that project i will not be eliminated in the t-th iteration, even if other projects exceed their threshold. This implication does not rule out permissible z-mechanisms. Conditional on $\mathbf{c}_{O_t}$, some projects exceeding their threshold can at most lead to a higher cutoff for project i due to monotonicity.

Further notice that if $c_i > a_i^{A_t}(\mathbf{c}_O)$, there always exist cost vectors with $\mathbf{c}_{O_t}$ for previously eliminated projects that induce $G = A_t \setminus \{i\}$. For example, all cost vectors with $c_j \leq a_j^{A_t}(\mathbf{c}_{O_t})$ for all $j \in A_t \setminus \{i\}$ will induce that allocation. However, this condition is sufficient for $G = A_t \setminus \{i\}$ but not necessary. There can be other cost vectors inducing the same allocation.

Iteration 1

If multiple projects have a positive score, it also holds that

$$\text{If } \widehat{\mathbf{c}} \text{ induces } \widehat{R} = \{i\} \text{ then } s_i^I(c_i) > s_j^I(c_j) \text{ for all } j \neq i \tag{12}$$

The meaning of $\widehat{R} = \{i\}$ is that $\widehat{c}_i > z_i(\widehat{\mathbf{c}}_{-i})$ and $\widehat{c}_j \leq z_j(\widehat{\mathbf{c}}_{-j})$. Hence, by construction

$$\widehat{c}_j \leq z_j(\widehat{\mathbf{c}}_{-j}) \leq b_i^j(\widehat{c}_i) \tag{13}$$

as $b_i^j(\widehat{c}_i)$ is the highest cutoff z_j that allows allocation $\widehat{R} = \{i\}$ given $\widehat{c}_i$.

Next, we show

$$\widehat{c}_i > b_j^i(\widehat{c}_j). \tag{14}$$

Suppose that the contrary holds, then there exists a vector $\widetilde{\mathbf{c}}_{-i-j}$ such that

$$\widehat{c}_i \leq z_i(\widehat{c}_j, \widetilde{\mathbf{c}}_{-i-j})$$

and allocation $\widetilde{R} = \{j\}$ is implemented. By Lemma 6 we know that the cutoffs z are constant in costs of projects $\widehat{G} \cap \widetilde{G} = I \setminus \{i, j\}$. Consequently, we arrive at

$$\widehat{c}_i \leq z_i(\widehat{c}_j, \widetilde{\mathbf{c}}_{-i-j}) = z_i(\widehat{c}_j, \widehat{\mathbf{c}}_{-i-j})$$

which means that i is greenlighted for vector $\widehat{\mathbf{c}}$, a contradiction to our initial assumption that $\widehat{\mathbf{c}}$ implements $\widehat{R} = \{i\}$.

Next, we show

$$b_i^k(\widehat{c}_i) \geq b_j^k(\widehat{c}_j) \text{ for all } j \neq i \text{ and } k \neq i, j. \tag{15}$$

By definition

$$b_i^k(\widehat{c}_i) = z_k(\widehat{c}_i, \tilde{\mathbf{c}}_{-i-k}) \text{ for some } \tilde{\mathbf{c}}_{-i-k},$$
$$b_j^k(\widehat{c}_j) = z_k(\widehat{c}_j, \dot{\mathbf{c}}_{-j-k}) \text{ for some } \dot{\mathbf{c}}_{-j-k}.$$

Because projects $-i-j-k$ are greenlighted for both cost realizations $(\widehat{c}_k, \widehat{c}_i, \tilde{\mathbf{c}}_{-i-k})$ and $(\widehat{c}_k, \widehat{c}_i, \dot{\mathbf{c}}_{-i-k})$, it follows by Lemma 6 that

$$b_i^k(\widehat{c}_i) = z_k(\widehat{c}_i, \tilde{\mathbf{c}}_{-i-k}) = z_k(\widehat{c}_i, \widehat{\mathbf{c}}_{-i-k}),$$
$$b_j^k(\widehat{c}_j) = z_k(\widehat{c}_j, \dot{\mathbf{c}}_{-j-k}) = z_k(\dot{c}_i, \widehat{\mathbf{c}}_{-i-k}).$$

Furthermore, it must hold that $\widehat{c}_i > \dot{c}_i$, otherwise vector $\widehat{\mathbf{c}}$ would not optimally redlight project i while vector $(\widehat{\mathbf{c}}_{-\mathbf{i}}, \dot{c}_i)$ optimally greenlights project i. Then by property 2 of a z-mechanism (monotonicity),

$$b_i^k(\widehat{c}_i) = z_k(\widehat{c}_{-k}) \geq z_k(\dot{c}_i, \widehat{\mathbf{c}}_{-i-k}) = b_j^k(\widehat{c}_j).$$

Combining (13), (14) and (15) leads to (12). We have shown that the scoring function eliminates the correct project when $|R| = 1$, i.e., the redlighted,project.

Finally, we need to show that if $|R| > 1$, the project removed in the first iteration is redlighted in the allocation implemented by the underlying z-mechanism, i.e.,

$$A_1 \setminus A_2 = \{k\} \Rightarrow k \in R.$$

Now take cost vector $\widetilde{\mathbf{c}}$ with allocation $\{\widetilde{G}, \widetilde{R}\}$ and let $i \in \widetilde{G}$ be some greenlighted project and and let $j \in \widetilde{R}$ be some redlighted project, respectively. Since project j is redlighted, it must have cost $\widetilde{c}_j > a_j^I$. Hence there exists some cost vector $\widehat{\mathbf{c}}$ with $\widehat{c}_j = \widetilde{c}_j$ such that $\widehat{R} = \{j\}$. By Lemma 6, we can assume $\widehat{c}_i = \widetilde{c}_i$ since $i \in \widetilde{G} \cap \widehat{G}$. As our scoring function correctly matches all cases where $|R| = 1$, it must be that $s_j(\widetilde{c}_j) > s_i(\widetilde{c}_i)$. Given that we have chosen i and j arbitrarily, we have shown that any project removed in the first iteration must be in the redlighted set, which was to show.

Iteration 2

We can show with the same arguments as above, that the previously stated scoring function is correct for $t = 2$ as well. To this end, we inductively rely on the fact, that the project k removed in the first iteration is indeed redlighted by the z-mechanism - as we have shown above.

Iteration $t \geq 3$

With the appropriate scoring functions used in all previous iterations, we can then show that the t-th iteration removes the correct project for all cost vectors inducing $|R| = t$ given a z-mechanism and otherwise removes some project $i \in A_t$, where $i \in R$, for all cost vectors inducing $|R| > t$. □

C The asymmetric case

Proposition 4. *Arrange the projects by cost in ascending order,* $c_1 \leq c_2 \leq \cdots \leq c_n$ *and define* $z^k := \min\left\{\frac{B}{k}, z^{**}, c_{k+1}\right\}$*. In the symmetric case, the z-mechanism with* $z_i(\mathbf{c}_{-i}) = z^{k^*}$ *is the optimal budget-constrained mechanism. The optimal number of accepted projects* k^* *is given by* $k^* := \max\{k | c_k \leq z^k\}$.

Proof. The case $n = 2$ has been proven in Section 3.1.

Now, consider $n = 3$. Fix any c_3 and any mechanism as candidate for optimality. Either $c_3 > z_3(c_1, c_2)$ or $c_3 \leq z_3(c_1, c_2)$. In the first case, project 3 is not executed and the budget remaining for the other two is still B. In the second case, project 3 is executed and the budget remaining for the other two becomes $B - z_3(c_1, c_2)$.

Now, consider deviating to the proposed mechanism only for project 1 and 2. The change in profit looks like a probability weighted sum of terms similar to the two-project case, only that the distributions F are conditional on c_1 and c_2 being in some interval (that induces $z_3 >$ or $< c_3$) and the budget must be adjusted.

Because log-concavity of F implies log-concavity of $\frac{F(c)-F(a)}{F(b)-F(a)}$ this deviation is always positive like in the case $n = 2$. The same logic can be applied to any n, changing any mechanism by selecting two projects and then adjusting their cutoffs in the following way: The budget is shared equally if both projects are executed; if only one project is executed, it has to be the one with higher virtual surplus; never execute projects with negative virtual surplus. Iterating over these steps ultimately arrives at the proposed mechanism which has to be optimal. □

References

Anari, N., G. Goel, and A. Nikzad (2014): "Mechanism Design for Crowdsourcing: An Optimal 1-1/e Competitive Budget-Feasible Mechanism for

Large Markets," in *Foundations of Computer Science (FOCS), 2014 IEEE 55th Annual Symposium on*, pp. 266–275. IEEE.

Baron, D. P., and R. Myerson (1982): "Regulating a monopolist with unknown costs," *Econometrica*, pp. 911–930.

Bei, X., N. Chen, N. Gravin, and P. Lu (2012): "Budget feasible mechanism design: from prior-free to bayesian," in *Proceedings of the forty-fourth annual ACM symposium on Theory of computing*, pp. 449–458. ACM.

Börgers, T., and P. Norman (2009): "A note on budget balance under interim participation constraints: the case of independent types," *Economic Theory*, 39(3), 477–489.

Caillaud, B., and J. Robert (2005): "Implementation of the revenue-maximizing auction by an ignorant seller," *Review of Economic Design*, 9(2), 127–143.

Che, Y.-K., and I. Gale (1998): "Standard auctions with financially constrained bidders," *The Review of Economic Studies*, 65(1), 1–21.

Chen, N., N. Gravin, and P. Lu (2011): "On the approximability of budget feasible mechanisms," in *Proceedings of the twenty-second annual ACM-SIAM symposium on Discrete Algorithms*, pp. 685–699. SIAM.

Chung, K.-S., and J. C. Ely (2002a): "Ex-post incentive compatible mechanism design," *Mimeo, Northwestern University.*

——— (2002b): "Mechanism Design with Budget Constraint," *Mimeo, Northwestern University.*

de Castro, L. I., and M.-A. de Frutos (2010): "How to translate results from auctions to procurements," *Economics Letters*, 106(2), 115–118.

Dizdar, D., A. Gershkov, and B. Moldovanu (2011): "Revenue maximization in the dynamic knapsack problem," *Theoretical Economics*, 6(2), 157–184.

Dobzinski, S., C. H. Papadimitriou, and Y. Singer (2011): "Mechanisms for complement-free procurement," in *Proceedings of the 12th ACM conference on Electronic commerce*, pp. 273–282. ACM.

Ensthaler, L., and T. Giebe (2014a): "Bayesian optimal knapsack procurement," *European Journal of Operational Research*, 234(3), 774–779.

——— (2014b): "A dynamic auction for multi-object procurement under a hard budget constraint," *Research Policy*, 43(1), 179–189.

Gershkov, A., J. K. Goeree, A. Kushnir, B. Moldovanu, and X. Shi (2013): "On the equivalence of Bayesian and dominant strategy implementation," *Econometrica*, 81(1), 197–220.

Kellerer, H., U. Pferschy, and D. Pisinger (2004): *Knapsack problems.* Springer Science & Business Media.

Krishna, V. (2009): *Auction Theory.* Academic press.

Ledyard, J., D. Porter, and R. Wessen (2000): "A market-based mechanism for allocating space shuttle secondary payload priority," *Experimental Economics*, 2(3), 173–195.

Luton, R., and R. P. McAfee (1986): "Sequential procurement auctions," *Journal of Public Economics*, 31(2), 181–195.

Manelli, A. M., and D. R. Vincent (2010): "Bayesian and Dominant-Strategy Implementation in the Independent Private-Values Model," *Econometrica*, 78(6), 1905–1938.

Maskin, E. (2002): "How to Reduce Greenhouse Gas Emissions: An Application of Auction Theory," *Nancy L. Schwartz Memorial Lecture, Northwestern University.*

Milgrom, P., and I. Segal (2014): "Deferred-acceptance auctions and radio spectrum reallocation," in *Proceedings of the 15th ACM Conference on Economics and Computation.*

Mookherjee, D., and S. Reichelstein (1992): "Dominant strategy implementation of Bayesian incentive compatible allocation rules," *Journal of Economic Theory*, 56(2), 378–399.

Myerson, R. (1981): "Optimal Auction Design," *Mathematics of Operations Research*, pp. 58–73.

Pai, M. M., and R. Vohra (2014): "Optimal auctions with financially constrained buyers," *Journal of Economic Theory*, 150, 383–425.

Singer, Y. (2010): "Budget feasible mechanisms," in *Foundations of Computer Science (FOCS), 2010 51st Annual IEEE Symposium on*, pp. 765–774. IEEE.

Strausz, R. (2003): "Deterministic mechanisms and the revelation principle," *Economics Letters*, 79(3), 333–337.

US Department of the Treasury (2011): "Treasury Announces Intent to Sell Warrant Positions in Public Dutch Auctions," *Press Re-*

lease, http://www.treasury.gov/press-center/press-releases/Pages/tg1193.aspx, accessed: February 2, 2015.