# Essays in Microeconomic Theory

vorgelegt von

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Inauguraldissertation zur Erlangung des akademischen Grades eines
Doktors der Wirtschaftswissenschaften
der Universität Mannheim

an der
Fakultät für Rechtswissenschaft und Volkswirtschaftslehre
Abteilung Volkswirtschaftslehre

Juni 2016

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Verteidigung: 29.06.2016

# Declaration of Authorship

I, VINCENT MEISNER, declare that this thesis titled, 'Essays in Microeconomic Theory' and the work presented in it are my own. I confirm that:

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- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
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"[...] we abandoned reality altogether and entered the world of mathematical make-believe. The practical-minded reader may rightfully ask whether any contribution has been made toward an actual solution of the original problem."

Gale and Shapley (1962, p. 14)

# *Acknowledgements*

First of all, I would like to thank my supervisors Thomas Tröger and Volker Nocke for their incredible support and encouragement throughout my doctoral studies in Mannheim. I am truly grateful for the time they took in advising me, and the detailed feedback I received. I am greatly indebted to Thomas and his admirable patience when listening to my half-baked ideas, sometimes for several hours. From him, I have learned the importance of rigor although even in this respect I still have a lot to learn from him. Similarly, I am greatly indebted to Volker who often read my papers over the weekend and who always provided incredibly helpful feedback and suggestions. I will miss the interaction during seminars and his quick mind. Both of them are outstanding scholars and I hope that someday I can be an academic role model like them.

I owe a special thanks to my co-author and close friend Felix Jarman. Many people compare the relationship between authors and papers to the one between parents and kids: You spend a lot of time on the paper, it makes you dream, you love it unconditionally, and you will always defend it. However, if it had not been for Felix' continuous effort, this particular paper would have been aborted at an early stage. Once we gave birth to it and after we saw its first steps, we were determined to see this paper grow up. It did not want to go to school, hung out with the wrong friends, and all in all was a disrespectful brat. Together we raised this paper to what it is now: Considering the humble upbringing, I am quite proud of it.

I am grateful to all aunts and uncles of my papers, especially those who believed in my second paper, a true love-child. I thank all fellow graduate students and researchers who attended my seminars and provided helpful feedback. In particular, I want to mention (in alphabetical order): Benni Balzer, Andras Niedermeyer, Martin Peitz, Philipp Schmidt-Dengler, Johannes Schneider, Nicolas Schutz, André Stenzel, Emanuele Tarantino, and Chris Wolf. In addition, I want to thank the GESS team for their smooth administration.

Finally, I thank the people outside of the research community for being there, always. First of all, my parents, Sylvia and Hans-Ulrich Meisner: Without you, I obviously would not be where I am now. My girlfriend Andrea: I am looking forward to our new life in Berlin. The Quadratekids: It is a short-train ride between Mannheim and Berlin.

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# 1. General Introduction

One of the central questions defining the field of Economics is how scarce goods should be allocated among a pool of economic agents. Since at least the seminal work of Akerlof (1970), asymmetric information became a focus of study as a major source of inefficiencies in many markets. This dissertation consists of three self-contained papers on this topic: Two mechanism design papers aiming at the strategyproof elicitation of private information, and one dynamic pricing paper where sellers can screen private information with continuous price paths. The chapters are linked through the presence of capacity constraints as a common theme.

Chapter 2 is joint work with Felix Jarman. We derive mechanisms that maximize a budget-constrained procurer's payoff under ex-post constraints. Chapter 3, also joint with Felix Jarman, is a note that formulates a revelation principle in terms of payoff for deterministic mechanisms under ex-post constraints. In Chapter 4, I investigate the interaction between forward-looking buyers and multiple sellers in a continuous-time revenue management setting.

## I Chapter 2 and 3

Public procurement affects a substantial share of world trade flows, amounting to 10 - 25 % of GDP. According to the European Commission,[1] 16 % of the EU GDP stems from public procurement. As a consequence, even small increases in efficiency amount to billions of dollars saved. Many procurement commissions are endowed with a fixed budget, and they are allowed to finance as many projects as the budget allows. In such settings, problems often arise as the allocated funding may be insufficient and hence bidders default. Either the provider goes bankrupt

[1] Source: http://ec.europa.eu/trade/policy/accessing-markets/public-procurement/

which means the project is not provided and the money is lost, or he requests additional funding. The second and the third chapter address mechanism design under such ex-post constraints.

For example, a development fund with a fixed budget wants to install wells for heterogeneous developing communities. The fund knows how much it values water supply for those communities, but does not know the wells' building costs. The communities cannot pay for the well themselves. Hence, costs have to be covered by a compensation payment out of the procurer's budget. We ask the question, how should the budget be allocated, and which projects should be implemented? Crucially, how can this be done when the budget and the participation constraint hold ex-post. That is, for every possible state of the world, the sum of transfers does not exceed the budget and costs are always fully compensated. We find that the optimal allocation can be implemented with a descending-clock auction with deferred-acceptance rule.

Without informational asymmetries, this problem is the classical knapsack problem: The procurer can carry up to $B$ kg/lbs in a knapsack, and faces a set of items, each characterized by a value and a weight. He wants to maximize the aggregate value packed without exceeding the weight limit. We add asymmetric information to this setting: We want the items to tell us their weight, and use transfers to provide incentives. The combinatorial problems arising due to the ex-post budget constraint make Vickrey-Clarke-Groves outcomes almost impossible to compute. Because we impose individual rationality, incentive compatibility and the budget constraint ex-post, we cannot solve the problem using standard pointwise optimization techniques. In settings as described above, insights from mechanism design under ex-ante or interim constraints are not directly applicable. Instead, we derive a set of properties that every optimal mechanism must have.

First, we show that strategyproofness (ex-post incentive compatibility) implies that the optimal mechanism is a cutoff mechanism: Every project obtains an individual cutoff cost level determined by others' cost reports, and gets implemented if and only if the own cost is below the cutoff. Second, we show that the optimal allocation rule has substitutes. That is, if a project gets implemented, it also gets implemented when, all else equal, another project's cost increases. Third, the optimal allocation rule has non-bossy winners. That is, an individual provider who is implemented cannot affect the allocation without losing his own allocation status. These three properties imply that the optimal mechanism belongs to a special class of deferred-acceptance auctions, recently introduced by Milgrom and Segal (2014).

They show that any DA auction has a corresponding implementation with a descending-clock auction with a deferred-acceptance rule: Each agent faces a clock with a continuously decreasing price, and indicates whether he is willing to provide the project for the price currently shown on his clock. The optimal allocation takes a simple form in the symmetric case, when all projects have the same value and costs are drawn from the same distribution: All projects get the same transfer and the most expensive projects are rejected iteratively until the budget suffices. A single price clock can implement this allocation by having projects drop at their reservation prices out over time. However, when projects are asymmetric, every project is assigned an individual clock. Clocks not only descend asynchronously, sometimes individual clocks have to stop. This is due to a quantity-quality trade-off: The procurer not only prefers high-value projects over low-value projects, but also prefers more over fewer projects. If the procurer did always implement the best projects, the properties of the allocation rule enforce that the number of conducted projects is reduced.

In Chapter 3, we address that the classical revelation principle does not hold when attention is restricted to deterministic mechanisms. However, we show that deterministic direct truthful mechanisms are optimal when constraints have to hold ex-post.

## II Chapter 4

The final chapter deals with revenue management. Revenue management is the technical term for dynamic pricing under capacity constraints with heterogeneous consumers and a deadline before which the good has to be sold. It is practiced in multimillion dollar businesses such as airlines, hotels, cruise ships, rental cars, seasonal clothing, sporting events and many more. The lead example for this literature is the sale of airline tickets: There is a fixed number of seats on a plane and having a ticket loses its value after departure. Almost all such papers consider a monopolistic seller, although the markets of application are almost never monopolistic. I address this gap in the literature by considering oligopolistic competition on the seller side.

I find that for all model parameters for which a monopolist would want to sell all his goods with probability one, the same price path, the same allocation, the same consumer surplus and the same industry profit is obtained under monopoly and under oligopoly. The intuition is that, because the good is scarce, a seller does not want to undercut every price, because he can just let his competitors sell out and thereby become a monopolist. Because forward-looking buyers arbitrage away any differences between current and future prices, prices are a martingale. Hence, in equilibrium a seller is at each time indifferent between selling the current and selling the next item, exactly as a monopolist.

In contrast to static models, I can elaborate on differences between sellers with and without the ability to commit to future prices. While a seller with commitment power might be able to commit to withhold some capacity of the good profitably, this may not be the case when she lacks the commitment power. In equilibrium, sellers replicate sequential Dutch auctions by continuously decreasing the price. As a purely technical contribution, I generalize the continuous-time inertia approach by Bergin and MacLeod (1993) to stochastic games with private information.

# 2. Ex-post Optimal Knapsack Procurement

*with Felix Jarman*

## I Introduction

We study the problem of a procurer who can spend a fixed budget on any of $n$ available projects which differ in the value the designer derives from them. Projects (agents) have private information about their costs and want to get funding beyond the necessary minimum. The designer's goal is to select an affordable set of maximal aggregate quality. In other words, she faces a mechanism design variant of the knapsack problem with strategic behavior due to informational asymmetries.[1] Essentially, we approach this problem as an "up to possibly $n$-units" procurement problem with $n$ agents with single-unit supply where demand quantity is determined after observing projects' reports under a budget constraint. The budget constraint, the individual rationality constraints, and the incentive compatibility constraints are imposed ex-post, i.e., for any cost realization, implemented projects are always at least fully compensated, the sum of transfers must not exceed the budget, and truth-telling must be a (weakly) dominant strategy. We find that the optimal mechanism can be implemented with a descending-clock auction with a deferred acceptance rule. Because of a tradeoff between quantity and quality, an optimal price clock may have to stop for a period of time leading to instances in which an inferior project is implemented instead of a superior one.

This framework matches a large range of allocation problems, in which a designer needs to allocate a divisible but fixed capacity among agents. Allocation problems,

[1]The knapsack problem is a classical combinatorial problem, dating as far back as 1897. A set of items is assigned values and weights. The knapsack should be filled with the maximal value, but can carry only up to a given weight. For an overview of the literature on knapsack problems, see Kellerer, Pferschy, and Pisinger (2004).

in which a financial budget constraint represents the fixed capacity, include the procurement of bus lines, bridges, and streets, or the allocation of subsidies or research money. Alternatively, the capacity constraint can represent the payload limit on a freighter or on a space shuttle,[2] or a limited amount of time to be devoted to several tasks. Out of many suitable applications, we employ as our leading example a development fund that desires to distribute money to nonprofit projects with nonmonetary benefits.

Our paper not only helps to understand a class of economically relevant problems, the framework also presents a novel methodological challenge. The ex-post nature of both the participation and the budget constraint precludes the use of standard pointwise optimization techniques à la Myerson (1981). Nonetheless, rewriting the problem involves expressing expected transfers in terms of the allocation function as an auxiliary step. As the designer maximizes expected payoff including residual money, we can employ the procurement analogue of Myerson's notion of "virtual values". However, our results qualitatively translate to a setting in which the designer does not value residual money.

By focusing on strategyproof deterministic mechanisms, we can reduce the problem to finding a set of optimal cutoff functions $z_i$ that, for each project $i$, map the cost vector of other projects $\mathbf{c}_{-i}$ into a cutoff cost level. Project $i$ is conducted if and only if $i$'s cost report falls weakly below cutoff $z_i(\mathbf{c}_{-i})$ and the corresponding compensation payment for that case equals the cutoff $z_i(\mathbf{c}_{-i})$. In optimum, these cutoff functions implement an allocation rule that exhibits certain properties. First, the optimal allocation rule has substitutes: Given a project is implemented for some cost vector, it is also implemented when, all else being equal, the cost of a rival project is increased. Second, the optimal allocation rule has non-bossy winners: A single project that is implemented cannot affect the allocation without changing its own allocation status. Third, the optimal allocation rule excludes all projects with negative "virtual surplus" from the allocation.

By virtue of these properties, any optimal mechanism has an equivalent deferred acceptance (DA) auction representation as described in Milgrom and Segal (2014). A DA auction is an iterative algorithm that computes the allocation and transfers of an auction mechanism and possesses attractive features with respect to bidders'

[2]Clearly, the capacity of a space shuttle is limited. The problem of optimally allocating the capacity and incentivizing projects to reduce payload is economically relevant, see Ledyard, Porter, and Wessen (2000).

incentives that go beyond dominant-strategy implementability. First, in any DA auction, revealing the type truthfully is an "obviously dominant strategy" as defined by Li (2015).[3] Second, any DA auction is weakly group-strategyproof. In other words, it is impossible for a coalition of projects to coordinate their bidding strategies such that it strictly increases the utility of all projects in the coalition. Third, the dominant strategy equilibrium outcome of any DA auction is the only outcome that survives iterated deletion of dominated strategies in the corresponding full information game with the same allocation rule but where players pay their own bid. Therefore predicting the dominant-strategy equilibrium outcome in a DA auction can be considered robust.

Milgrom and Segal (2014) argue that these properties make DA auctions suitable for many challenging environments such as radio spectrum reallocations. Most importantly, they show that every DA auction can be represented by a descending-clock auction. Among several potential applications, they also consider our budget-constrained procurement setup (Example 5: "Adaptive Scoring for a Budget Constraint"). However, they do not show optimality of the DA auction. To the best of our knowledge, we are the first to do so in a nontrivial setting. Therefore we can strengthen the argument in favor of DA auctions. The techniques established in our paper may be helpful to prove optimality of DA auctions in the other settings mentioned in their paper.

Reducing the set of candidates for optimality to a special kind of DA auction implies that any optimal allocation can be implemented with an appropriately designed descending-clock auction: Any project faces a clock with a continuously decreasing price on it, and indicates whether it is willing to conduct its project at this price. In this auction it is a weakly dominant strategy for any project to exit the auction once the clock price hits the project's cost level. At first, we focus on the case in which all projects are ex-ante symmetric: They have the same value and costs are drawn from the same distribution. Here, we show that it is optimal to rank projects according to their cost and "greenlight" the cheapest ones. In optimum, price clocks run down synchronously and hence projects exit in order of their costs until the budget suffices to pay the current clock price to all remaining active projects.

[3] There does not exist any deviation such that, in any information set in which a deviating action is played, the best-case deviation payoff (against even the most favorable profile of strategies of the other players that is consistent with this information set) is strictly larger than the worst-case payoff from truthful bidding (achieved against the least favorable such strategy profile).

Next, we examine the case of ex-ante asymmetric projects, i.e., costs are drawn from different distributions and/or project values differ. Here, we restrict attention to the two-project case because it conveys the main insights while retaining tractability. In applications, the designer may prefer some projects over others and might have different information over cost distributions. In standard procurement settings, the quantity of units to be procured is not endogenously determined as in our model, but it is exogenously fixed to be some quantity $k$. It is well known that in $k$-unit procurement auctions the $k$ projects with the greatest non-negative virtual surpluses are implemented, e.g., Luton and McAfee (1986). In the asymmetric case, the ranking implied by costs and the ranking implied by virtual surpluses do not necessarily coincide. Broadly speaking, the designer discriminates against stochastically stronger projects, and favors projects with higher values. The asymmetry requires that each project faces an individual clock and prices decrease asynchronously. In optimum when quantities are exogenous, the clocks' speed is adjusted such that the virtual surplus of marginal projects is kept equal at all times, see Caillaud and Robert (2005, Proposition 1).

Interestingly, the optimal allocation of this environment does not simply translate into the asymmetric case of our environment. In contrast, projects are not always greenlighted in order of their virtual surpluses. Therefore we cannot adopt the approach of Caillaud and Robert (2005). Instead, the descending-clock implementation of the optimal allocation includes individual clocks stopping at certain times. Here, the quantity-quality tradeoff kicks in: We show that the optimal allocation generically features instances in which out of two rival projects the project with lower virtual surplus is chosen. The reasoning behind this result is that the number of procured units is endogenous. In the asymmetric case, always greenlighting in order of virtual surplus reduces the expected number of greenlighted projects compared to the optimal mechanism. Strategyproofness creates a tradeoff between quantity and quality of the procured projects. This discrimination of the stronger project is employed on top of the discrimination due to stochastic domination through the virtual costs.

Clock auctions are generally easy to understand and hard to manipulate. Furthermore, they are less information hungry than, for example, sealed bid auctions. In descending-clock auctions, the designer only learns the private information of those projects that are not greenlighted. In fact, Milgrom and Segal (2014) show that clock auctions are the only strategyproof mechanisms that preserve winners' unconditional privacy: Winners only need to reveal the minimum of their private information that is necessary to prove that they should be winning. These features of clock auctions make them attractive for applications in which there is limited trust between the involved parties. In practice, clock auctions are commonly used to sell fish in Japan and they are often found in the public sector, e.g., when the US Department of the Treasury sells warrant positions.

To the best of our knowledge, this paper is the only one that considers purely ex-post constrained optimal procurement design. Such a restrictive setting can be seen as a "worst-case scenario" for the designer, suiting many economic applications. In our leading example of the development fund, an ex-post budget constraint appears natural as budgets are usually fixed. The nonprofit nature of the projects might prohibit acquiring additional money on the financial market. Information rents are necessary, because a project might want to spend money on extra equipment that is convenient for the project's staff but has no value for the designer. In practice, such incentive problems are often resolved using dominant-strategy implementable mechanisms. In strategyproof mechanisms, agents have no incentive to invest in espionage activities or to hire consultants to avoid misspecification of beliefs. Mainly, dominant strategies are desirable as they are easy to explain and not prone to manipulation. For similar reasons, we restrict attention to deterministic mechanisms. Deterministic mechanisms obviate the need for a credible randomization device and are therefore more easily applicable in practice. Finally, ex-post participation constraints are necessary because projects simply cannot be conducted with insufficient funds, and the designer wants to avoid costly renegotiations when the projects default.

### I.i Literature

Even though the knapsack problem has a wide range of economic applications, there are relatively few publications in economics on this issue. Most prominently, Maskin (2002), in his Nancy L. Schwartz memorial lecture, addressed the related problem of the UK government that put aside a fixed fund to encourage firms to reduce their pollution. The government faces $n$ firms that have private marginal cost of abatement $\theta_i$ and can commit to reduce $x_i$ units of pollution. To reduce pollution as much as possible, the government pays expected compensation transfers $t_i$ to the firms, who report costs and proposed abatement to maximize $t_i - \theta_i x_i$. For some distributions, Maskin (2002) proposes a mechanism that satisfies an ex-post participation constraint, an ex-post incentive compatibility constraint, and the condition that the budget is not exceeded in expectation. In response to Maskin (2002), Chung and Ely (2002b) look at a more general class of mechanism design problems with budget constraints and translate them into a setting à la Baron and Myerson (1982). Their approach nests Maskin (2002) and also Ensthaler and Giebe (2014a) as special cases. However, Ensthaler and Giebe (2014a) more explicitly derive a constructive solution. In contrast to us, they all consider a soft budget constraint that only requires the sum of expected transfers to be less than the budget. By incorporating the budget constraint into a Lagrangian function and ignoring the monotonicity (incentive) constraint, they find a mechanism that, under the standard regularity condition, indeed is incentive compatible.

In addition, Ensthaler and Giebe (2014a) use AGV-budget-balancing (such as Börgers and Norman, 2009) to obtain a mechanism which is ex-post budget-feasible. However, transformation into a mechanism with an ex-post balanced budget in such a way comes at the cost of sacrificing ex-post individual rationality. Many applications do not allow this constraint to be weakened. For instance, subsidy applicants usually cannot be forced to conduct their proposal when receiving only a small or possibly no subsidy. Alternatively, limited liability justifies insisting on ex-post individual rationality. Because we want both constraints to hold ex-post, we cannot build on their techniques and, thus, we approach the problem by characterizing the optimal allocation rule.

To the best of our knowledge, no paper exists that jointly considers optimal mechanism design under ex-post budget balance and ex-post individual rationality in a procurement setting. Ensthaler and Giebe (2014b) propose a belief-free clock mechanism that coincides with our optimal mechanism in the symmetric case for many parameterizations[4] but differs in the asymmetric case by holding the cost-benefit-ratio equal among projects. By simulating different settings, they conclude that this mechanism outperforms a mechanism used in practice. In contrast to their setting, the mechanism designer in our model values residual money. In Section V, we discuss the meaning of residual money and find that our main results qualitatively carry to the case where residual money is neglected.

[4]For all parameter constellations such that virtual surplus is always nonnegative.

Because of the appeal of dominant-strategy incentive-compatible (DIC) mechanisms compared to Bayesian incentive-compatible (BIC) mechanisms, many researchers have produced valuable BIC-DIC equivalence results. These results characterize environments in which restricting attention to the more robust incentive criterion comes without loss. Our setup is not contained in these environments. For any BIC mechanism, Mookherjee and Reichelstein (1992) show that one can construct a DIC mechanism implementing the same ex-post allocation rule, whenever this allocation rule is monotone in each coordinate. However, the ex-post transfers of the constructed DIC mechanism are not guaranteed to satisfy ex-post budget balance. More recently, Gershkov, Goeree, Kushnir, Moldovanu, and Shi (2013) employ a definition of equivalence in terms of interim expected utilities introduced by Manelli and Vincent (2010). For any BIC mechanism, including the optimal one, they construct a DIC mechanism that yields the same interim expected utilities. Here, the ex-post allocation as well as the ex-post transfers might differ between the two. Therefore a DIC mechanism equivalent to a feasible BIC mechanism might violate the ex-post constraints in our setting.

Our budget-constrained procurement setup with ex-post constraints has received much attention in the computer science literature. Instead of specifying the optimal mechanism, the authors in this literature typically aim to construct allocation algorithms that give good approximation guarantees. In other words, they try to maximize the minimal payoff an algorithm can guarantee compared to the full information knapsack payoff. Apart from the seminal paper by Singer (2010), the works of Dobzinski, Papadimitriou, and Singer (2011) and Chen, Gravin, and Lu (2011) are notable examples of this approach. Anari, Goel, and Nikzad (2014) present a stochastic algorithm and show that it gives the best possible approximation guarantee in the many projects limit in which any individual project's costs are small compared to the budget. While the above papers examine the belief-free case, Bei, Chen, Gravin, and Lu (2012) propose an algorithm for setups in which the designer knows how the private information is distributed.

Other auction theoretic papers featuring "knapsack auctions" deal with a slightly different problem compared to us. Aggarwal and Hartline (2006) consider a setting in which each agent is characterized by his object of commonly known size and a privately known valuation for having his object placed in the auctioneer's knapsack with commonly known capacity. They are looking for the truthful auction that best approximates the optimal full-information monotone pricing rule which maximizes the auctioneer's profit. Mu'Alem and Nisan (2008) cover the case of an auctioneer maximizing social welfare instead. Dütting, Gkatzelis, and Roughgarden (2014) study the performance of DA auctions for knapsack auctions, i.e., they show DA auctions fail to achieve a constant factor approximation of the optimal social welfare in knapsack auctions Dizdar, Gershkov, and Moldovanu (2011) investigate a similar knapsack problem of a profit maximizing auctioneer in a dynamic setting: Agents sequentially arrive over time and are either included in the knapsack immediately or lost forever. Thereby they avoid combinatorial issues, which gives rise to a threshold property of the optimal mechanism. In such knapsack auctions, the mechanism designer maximizes the sum of transfers, and the value only enters the individual projects' payoff while the capacity constraint is imposed on the weight assigned to agents. In our framework, the value is collected by the auctioneer and the capacity constraint is imposed on the sum of transfers. Because of the latter, knapsack auctions and our knapsack procurement auctions are not dual problems

There seems to be no reasonable analogy for our setting to another setting in which the mechanism designer is a similarly constrained seller and the agents are buyers. The literature on group-strategyproof cost-sharing mechanisms, initiated by Moulin (1999), considers the dual of a "surplus-sharing" problem. The crucial difference between this problem and our "budget-sharing" problem is that the agents themselves produce the output to be distributed, while in our case the budget to be distributed is fixed and unrelated to the surplus created by the agents, which is collected by the mechanism designer. Budget-constrained buyers in auctions have been discussed in the literature, e.g., by Che and Gale (1998) or Pai and Vohra (2014). However, these authors study budget-constrained agents whereas in our setting the designer is budget-constrained.

In the following section, we introduce the model. In Section III, we rewrite the problem as a problem of finding the optimal cutoff functions and derive a set of properties that any optimal mechanism must have. Sections III.i and III.ii cover symmetric and asymmetric environments, respectively. We discuss extensions and possible modifications to the model in Section V. Finally, we conclude in Section VI.

## II Model

We consider a set of $n$ projects $I = \{1, \ldots, n\}$ and one mechanism designer. Each project can be conducted exactly once. The designer gains utility $v_i$ if and only if project $i \in I$ is conducted. We consider projects to be utility maximizing agents. If project $i$ is executed, it incurs cost $c_i \in C_i := [\underline{c}_i, \overline{c}_i]$, where in the following we restrict $\underline{c} = 0$.[5] Let $C := \times_{i \in I} C_i$ and $C_{-i} := \times_{j \in I \setminus \{i\}} C_j$. Let the realization of a cost vector be denoted by $\mathbf{c} \in C$. The costs are the projects' private information and are independently drawn from a distribution $F_i$. We assume $F_i$ to be continuously differentiable with a strictly positive density $f_i$ on the support. The value of the project $v_i$ and the distribution $F_i$ are common knowledge.

To compensate project $i$ for its cost, the designer pays transfer $t_i$. A direct mechanism is characterized by $\langle q_i, t_i \rangle$. It is a mapping from the vector of cost reports $\mathbf{c} \in C$ into provision decisions and transfers. We denote the allocation function by $\gamma : C \to \mathcal{P}(I)$, and it maps a cost vector into the set of "greenlighted" projects, an element of the power set of $I$. Correspondingly, we call $I \setminus \gamma(\mathbf{c})$ the set of "redlighted" projects.

We restrict attention to deterministic mechanisms. This restriction implies that once all cost reports are collected, we know with certainty which project is selected by the mechanism. In other words, the decision of implementation $q_i$ is binary,

$$q_i(\mathbf{c}) = \mathbb{I}(i \in \gamma(\mathbf{c})),$$

where $\mathbb{I}$ denotes an indicator function that is one if the corresponding condition is true and zero otherwise. We employ a revelation-principle argument and without loss of generality only consider direct mechanisms.[6]

[5] The impact of this assumption it discussed in Appendix VI.D.

[6] In general, the revelation principle does not hold when restricting attention to deterministic mechanisms: Deterministic direct mechanisms are unable to replicate mixed strategy equilibria in deterministic indirect mechanisms, as noted by, e.g., Strausz (2003). However, in our setting we do not lose generality. A mixed strategy equilibrium consists of a distribution over pure strategy profiles. Because the mechanism is implementable in dominant strategies any of these pure strategy profiles also constitutes a pure strategy equilibrium, in particular the pure strategy equilibrium associated with the designer's most preferred outcome. Similarly, because the mechanism is ex-post constrained, this outcome is feasible. Therefore, while there are allocations that (in the class of deterministic mechanisms) can only be implemented by indirect mechanisms, the designer's most preferred feasible allocation can truthfully be implemented in a direct mechanism.

Project $i$'s utility $u_i$ is given by its transfer minus the cost it bears,

$$u_i(\mathbf{c}) = t_i(\mathbf{c}) - q_i(\mathbf{c})c_i.$$

The designer derives value $v_i$ from each greenlighted project $i$ while having to pay the sum of transfers. Therefore she wants to maximize the aggregate value of greenlighted projects net of transfers paid. Her (ex-post) utility function $u_D$ implies that, in our setting, the designer values residual money,

$$u_D(\mathbf{c}) = \sum_i \big(q_i(\mathbf{c})v_i - t_i(\mathbf{c})\big). \tag{2.1}$$

We impose an ex-post participation constraint. That is, if $i$ is greenlighted the transfer must be at least as high as the cost,

$$t_i(c_i, \mathbf{c}_{-i}) - q_i(c_i, \mathbf{c}_{-i})c_i \geq 0 \quad \forall i \in I, (c_i, \mathbf{c}_{-i}) \in C. \tag{PC}$$

In addition, the designer has a budget constraint which is "hard" in the sense that she cannot spend more than her budget $B$ for any realization of the cost vector. That is, the designer can never exceed her budget,

$$\sum_i t_i(\mathbf{c}) \leq B \quad \forall \mathbf{c} \in C. \tag{BC}$$

Finally, incentive compatibility has to hold ex-post. Alternatively, we can say that the mechanism has to be implementable in (weakly) dominant strategies[7] or that the mechanism must be strategyproof. Therefore for every realization of the cost vector, project $i$'s truthful report must yield at least as much utility as any possible deviation,

$$t_i(c_i, \mathbf{c}_{-i}) - q_i(c_i, \mathbf{c}_{-i})c_i \geq t_i(\widetilde{c}_i, \mathbf{c}_{-i}) - q_i(\widetilde{c}_i, \mathbf{c}_{-i})c_i$$
$$\forall i \in I, \mathbf{c}_{-i} \in C_{-i} \text{ and } c_i, \widetilde{c}_i \in C_i. \tag{IC}$$

[7]In our private value environment, these two concepts are equivalent in a direct revelation mechanism. In general, however, ex-post incentive compatibility is essentially a generalization of dominant-strategy implementability to interdependent value environments. See Chung and Ely (2002a).

## III Analysis

We search for the direct mechanism that maximizes the expected utility of the designer and refer to this mechanism as the optimal mechanism. One may think that a natural approach to this problem would be to express the ex-post transfer $t_i(c_i, \mathbf{c}_{-i})$ as a function of the ex-post allocation decision $q_i(c_i, \mathbf{c}_{-i})$, taking $\mathbf{c}_{-i}$ as given, and applying the envelope theorem. In that case, it would be possible to restrict attention to the allocation in order to solve for the optimal mechanism. However, this approach does not reduce the complexity of the problem. The reason is that the ex-post transfers and allocation for one cost vector restrict transfers and allocation for other cost vectors through the budget constraint in a manner much more involved than standard monotonicity. In particular, the budget constraint with the ex-post transfer expressed as a function of the ex-post allocation may be ill-behaved. Therefore we cannot straightforwardly arrive at sufficient conditions using convex optimization.[8]

Instead, we derive a set of properties that every mechanism must inherit to be optimal. In general, we establish these properties by showing that the expected payoff yielded by any feasible mechanism not having one of the properties can be increased by adopting the properties. For some of the following lemmata, we provide the proof for the two-project case in the main text and provide the proof of the general case in the appendix. Our first step is to show that strategyproofness implies that the optimal mechanism has to be a cutoff mechanism.

**Lemma 2.1.** *The optimal mechanism can be represented by cutoff functions $z_i : C_{-i} \to C_i$, such that project $i$ is greenlighted whenever it reports a cost weakly less than its cutoff,*

$$q_i(c_i, \mathbf{c}_{-i}) = \mathbb{I}(c_i \leq z_i(\mathbf{c}_{-i})).$$

*The transfer to project $i$ equals its cutoff whenever it is greenlighted and zero otherwise,*

$$t_i(c_i, \mathbf{c}_{-i}) = q_i(c_i, \mathbf{c}_{-i}) z_i(\mathbf{c}_{-i}).$$

*Proof.* For any two cost reports $c_i, c_i' \in C_i$ of project $i$ and for some $\mathbf{c}_{-i} \in C_{-i}$, (IC) implies that if the allocation of $i$ is the same, $q_i(c_i, \mathbf{c}_{-i}) = q_i(c_i', \mathbf{c}_{-i})$, also the transfer has to be the same, $t_i(c_i, \mathbf{c}_{-i}) = t_i(c_i', \mathbf{c}_{-i})$. Otherwise, project $i$ could, as one of the cost types, deviate to the report yielding the higher transfer.

[8] Requiring either the budget or the participation constraint to hold only in expectation would enable us to use the techniques employed by Ensthaler and Giebe (2014a).

Conditional on $i$'s allocation and given any cost reports $\mathbf{c}_{-i}$, the transfer is fixed and does not vary with $i$'s cost report. Hence, given $\mathbf{c}_{-i}$, there can only be two different transfers $t_i$ for project $i$, one for each allocation status, $t_i^{q_i=1}(\mathbf{c}_{-i})$ and $t_i^{q_i=0}(\mathbf{c}_{-i})$.

Define $z_i(\mathbf{c}_{-i}) := t_i^{q_i=1}(\mathbf{c}_{-i}) - t_i^{q_i=0}(\mathbf{c}_{-i})$. Then, (IC) implies

$$q_i(c_i, \mathbf{c}_{-i}) = \begin{cases} 1 & \text{if } c_i \leq z_i(\mathbf{c}_{-i}) \\ 0 & \text{if } c_i > z_i(\mathbf{c}_{-i}) \end{cases}.$$

Suppose to the contrary that for some realization $\widehat{c}_i < z_i(\mathbf{c}_{-i})$ and some other $\widetilde{c}_i < z_i(\mathbf{c}_{-i})$, $q_i(\widehat{c}_i, \mathbf{c}_{-i}) = 0$ and $q_i(\widetilde{c}_i, \mathbf{c}_{-i}) = 1$. Then, type $\widehat{c}_i$ can profitably deviate to reporting $\widetilde{c}_i$ to ensure the green light which yields a utility increase of $z_i(\mathbf{c}_{-i}) - \widehat{c}_i$. An analogous argument applies for $\widehat{c}_i > z_i(\mathbf{c}_{-i}) > 0$.[9]

The last step is to show that $t_i^{q_i=0}(\mathbf{c}_{-i}) = 0$. This result follows from the mechanism being optimal, i.e., maximizing expected utility of the designer. □

As a direct consequence of dominant-strategy implementability, Lemma 2.1 shows that allocation and transfers are characterized by cutoffs. Project $i$ is greenlighted whenever it reports a cost that lies weakly below the cutoff. Crucially, these cutoffs are functions of the other cost reports $\mathbf{c}_{-i}$. However, the optimal cutoffs remain to be determined. The maximization problem of the designer is given by

$$\max_{\{z_i\}_{i \in I}} \mathbb{E}_{\mathbf{c}} \left[ \sum_i q_i(\mathbf{c}) v_i - t_i(\mathbf{c}) \right]$$

$$\begin{aligned} &\text{s.t. (BC)}, \\ q_i(\mathbf{c}) &= \mathbb{I}(c_i \leq z_i(\mathbf{c}_{-i})) \quad \forall \mathbf{c} \in C, \\ t_i(\mathbf{c}) &= \mathbb{I}(c_i \leq z_i(\mathbf{c}_{-i})) z_i(\mathbf{c}_{-i}) \quad \forall \mathbf{c} \in C. \end{aligned} \tag{2.2}$$

[9]When $c_i = z_i(\mathbf{c}_{-i})$, (IC) permits both $q_i(c_i, \mathbf{c}_{-i}) = 0$ and $q_i(c_i, \mathbf{c}_{-i}) = 1$. By convention, we assume $q_i(c_i, \mathbf{c}_{-i}) = 1$ in this case. However, writing a mechanism this way precludes the specification of tie-breakers, which might be necessary to conserve budget balance. For example, in a two-project example we would write down the mechanism "greenlight the cheaper project" as $z_1(c_2) = c_2$ and $z_2(c_1) = c_1$. If $c_1 = c_2$ a tie-breaker is needed to select a project. As this is a zero-probability event, the choice of the tie-breaker does not impact the designer's payoff. Similarly, as projects are indifferent, their ex-post utility is unaffected. Therefore we refrain from specifying a tie-breaker and proceed with our analysis as if both projects are greenlighted in these cases.

Here, $q_i$ and $t_i$ are determined by the cutoff function $z_i$. Incentive compatibility and participation constraints, thus, hold by construction.

The next step towards solving this problem involves applying standard methods introduced by Myerson (1981). Let the conditional expected probability of being greenlighted and the conditional expected transfer be

$$\begin{aligned} Q_i(c_i) &= \mathbb{E}_{\mathbf{c}}[q_i(c_i, \mathbf{c}_{-i})|c_i] \\ \text{and } T_i(c_i) &= \mathbb{E}_{\mathbf{c}}[t_i(c_i, \mathbf{c}_{-i})|c_i]. \end{aligned}$$

The interim incentive compatibility required by Myerson (1981) is weaker than our condition (IC). Consequently, the expected transfer is determined by the allocation, $T_i(c_i) = Q_i(c_i)c_i + \int_{c_i}^{\bar{c}_i} Q_i(x)dx$. The usual monotonicity condition is trivially fulfilled as we are dealing with cutoff mechanisms. This reformulation in turn allows us to rewrite the objective function as a function of the allocation. Substituting into problem (2.2) and integrating by parts yields the following maximization problem,

$$\begin{gathered} \max_{\{z_i\}_{i\in I}} \mathbb{E}_{\mathbf{c}} \left[\sum_i \mathbb{I}(c_i \leq z_i(\mathbf{c}_{-i})) \left(v_i - c_i - \tfrac{F_i(c_i)}{f_i(c_i)}\right)\right] \\ \text{s.t.} \\ \sum_{i\in I} \mathbb{I}(c_i \leq z_i(\mathbf{c}_{-i}))z_i(\mathbf{c}_{-i}) \leq B \quad \forall \mathbf{c} \in C. \end{gathered} \tag{2.3}$$

We call $\varphi_i(c_i) := c_i + \frac{F_i(c_i)}{f_i(c_i)}$ the *virtual cost* of project $i$ and $\psi_i(c_i) := v_i - \varphi_i(c_i)$ the *virtual surplus*. Here, $\varphi$ and $\psi$ are the procurement analogues to standard auction terminology. We can directly see from problem (2.3) that the optimal mechanism maximizes the expected sum of greenlighted virtual surpluses.

Note that constrained optimization by Lagrangian is not straightforward here because of the nondifferentiability of the indicator function. Instead, in the following we derive useful properties of the optimal cutoffs that can be exploited to characterize the optimal mechanism. A cutoff mechanism is by construction monotonic in the following sense:

**Definition 2.2.** An allocation rule $\gamma$ is monotonic in costs if $i \in \gamma(c_i, \mathbf{c}_{-i})$ and $c_i' < c_i$ imply $i \in \gamma(c_i', \mathbf{c}_{-i})$ for all $\mathbf{c}_{-i} \in C_{-i}$.

In words, if a project gets greenlighted for some cost vector, it also gets greenlighted when, all else equal, its cost is lower. To proceed, we restrict the class of distributions from which costs can be drawn.

*Assumption* 1 (Log-concavity). For all $i$, the cumulative distribution function $F_i$ is log-concave.

This assumption is standard in information economics. It is equivalent to the reverse hazard rate function $f/F$ being a weakly decreasing function or the ratio $F/f$ being weakly increasing. Hence, the standard regularity condition is implied: $\varphi_i$ is strictly increasing and $\psi_i$ is strictly decreasing. A decreasing reverse hazard rate is the procurement analogue to the assumption of increasing hazard rate functions in seller auction settings.

Regularity ensures that a lower cost $c_i$ translates to a higher virtual surplus $\psi_i(c_i)$. Hence, we can define the following cutoff cost type

$$z_i^{**} := \begin{cases} \psi_i^{-1}(0) & \text{if } \psi_i^{-1}(0) \in C_i \\ \overline{c}_i & \text{otherwise} \end{cases}, \tag{2.4}$$

where regularity implies the invertibility of $\psi_i$ and thus allows for the above definition of $z_i^{**}$. In the symmetric case, $z_i^{**} = z^{**}$ for all $i \in I$. Let $\boldsymbol{\zeta}^{**}$ be the $n$-dimensional vector with $z_i^{**}$ as $i$-th element for all $i \in I$.

**Definition 2.3.** An allocation rule $\gamma$ is $\boldsymbol{\zeta}^{**}$-exclusive if, for all $i \in I$, $c_i > z_i^{**}$ implies that $i \notin \gamma(c_i, \mathbf{c}_{-i})$ for all $\mathbf{c}_{-i} \in C_{-i}$.

A cutoff mechanism is $\boldsymbol{\zeta}^{**}$-exclusive if and only if $z_i(\mathbf{c}_{-i}) \leq z_i^{**}$ for all $\mathbf{c}_{-i} \in C_{-i}$ and for all $i \in I$. If the budget sufficed, a designer would want to greenlight all projects with nonnegative virtual surplus. Crucially, the arguments leading to this statement also imply that it is never optimal to greenlight a project with negative virtual surplus.

**Lemma 2.4.** *The optimal mechanism is $\boldsymbol{\zeta}^{**}$-exclusive. In the trivial case, $\sum z_i^{**} \leq B$, the optimal cutoffs are independent of the cost reports,*

$$z_i(\mathbf{c}_{-i}) = z_i^{**} \quad \forall \mathbf{c}_{-i} \in C_{-i} \text{ and } \forall i \in I.$$

The proof of this lemma is standard and hence omitted. It immediately follows from the rewritten objective function (2.3): Greenlighting a project with negative virtual surplus decreases the designer's payoff and uses part of the budget. Guaranteeing the green light for high-cost types comes at the cost of having to pay higher information rents to all cost types. For the same reason, also a budget-unconstrained designer would implement a $\boldsymbol{\zeta}^{**}$-exclusive mechanism, even when the surplus $v_i - \overline{c}_i$ is positive for all projects. Next, we show that an optimal mechanism possesses the following property:

**Definition 2.5.** An allocation rule $\gamma$ has substitutes if $i \in \gamma(\mathbf{c})$ and $c'_j > c_j$ for some $j \neq i$ implies $i \in \gamma(c'_j, \mathbf{c}_{-j})$.

That is, if a project gets greenlighted for some cost vector $\mathbf{c}$, it is also greenlighted when, all else equal, another project's cost is increased. This property relates to the cross-monotonicity defined in the cost sharing problem of Moulin and Shenker (2001): an agent's cost share cannot increase when the allocation set expands.

Having in mind a setting with an exogenously determined amount of projects to be procured and without a budget constraint, this property is clearly optimal, because if $i$ is among the projects with the highest virtual surpluses for some cost vector, it is also among them when the cost of some other project $j$ is increased, i.e., when $j$'s virtual surplus is decreased. However, with the budget constraint, this property does not hold in a full-information setting.[10] A cutoff mechanism has substitutes if all functions $z_i$ are weakly increasing in each argument.

**Lemma 2.6.** *The optimal mechanism has substitutes,*

$$z_i(\widetilde{c}_j, \mathbf{c}_{-i-j}) \geq z_i(\widehat{c}_j, \mathbf{c}_{-i-j}) \text{ for almost every } \widetilde{c}_j > \widehat{c}_j \text{ and } \mathbf{c}_{-i-j} \in C_{-i-j}. \quad (2.5)$$

*Proof.* (with $n = 2$, see appendix for the general proof)

For a graphical representation of the proof, consult Figure 2.1. We show that for any feasible cutoff mechanism that does not have substitutes, there exists a feasible alternative mechanism with substitutes that outperforms the initial candidate in terms of the designer's payoff. In fact, the alternative mechanism outperforms the initial candidate state-by-state and not only in expected terms.

As a first step, we can, without loss of generality, restrict the range of any optimal function $z_i$: By $\boldsymbol{\zeta}^{**}$-exclusivity, any optimal functional value $z_i(\mathbf{c}_{-i})$ cannot exceed

[10] For example, there are two projects, $v_1 > v_2$. Under full information, both projects get implemented for a cost vector $(c_1, c_2) = (B - z, z)$. Then, increasing $c_1$ would kick project 2 out of the allocation. In contrast, in our asymmetric-information setting where $c_2$ pins down a cutoff $z_1(c_2)$ for project 1, project 1 instead loses the green light status, when its cost increases while $c_2$ remains constant.

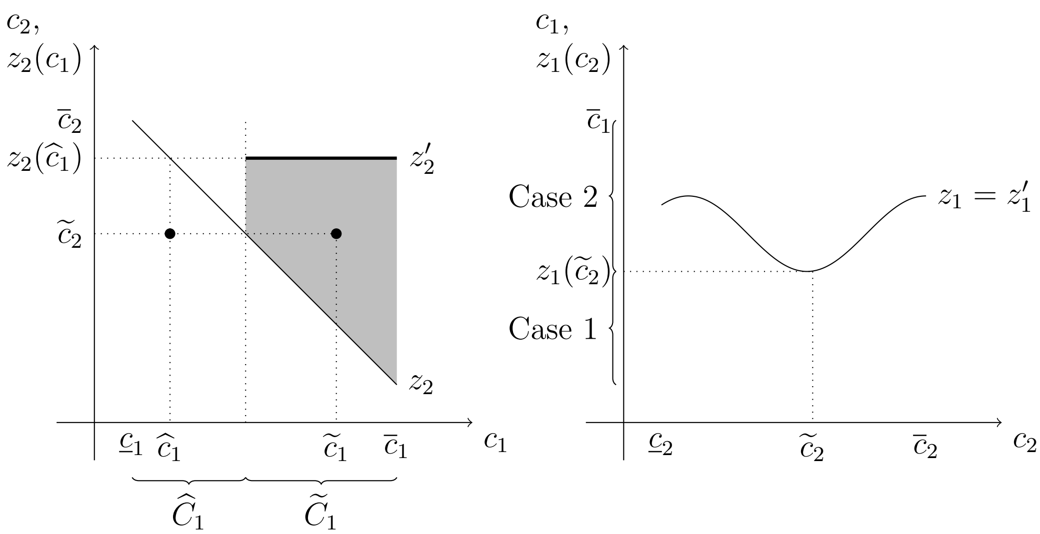
$z_i^{**}$.

Next, fix an arbitrary feasible pair of cutoff functions $\{z_1, z_2\}$ as a candidate for optimality. Contrary to (2.5), suppose that $z_2$ is decreasing on a set with positive Lebesgue-measure. Then, there exist sets $\widetilde{C}_1$ and $\widehat{C}_1$ with positive Lebesgue-measure, such that

$$z_2(\widehat{c}_1) > z_2(\widetilde{c}_1) \text{ for all } \widehat{c}_1 \in \widehat{C}_1, \widetilde{c}_1 \in \widetilde{C}_1,$$

and $\widehat{c}_1 < \widetilde{c}_1$ for all elements of the corresponding sets.

FIGURE 2.1: The alternative cutoff mechanism $\{z_1, z_2'\}$ outperforms the initial candidate $\{z_1, z_2\}$ for all cost vectors in the light gray area and otherwise yields the same allocation.




Now, consider an alternative cutoff mechanism $\{z_1, z_2'\}$ that leaves cutoff function $z_1$ unchanged, but modifies the cutoff function of project 2 in the following way

$$z_2'(c_1) = \begin{cases} z_2(\widehat{c}_1) & \text{if } c_1 \in \widetilde{C}_1 \\ z_2(c_1) & \text{otherwise} \end{cases},$$

with an arbitrary $\widehat{c}_1 \in \widehat{C}_1$. In words, the alternative flattens $z_2$ over region $\widetilde{C}_1$ and otherwise leaves the initial mechanism as it is. This alternative cutoff function is depicted in Figure 2.1 as the thick flat line.

The alternative mechanism implements the same allocation, except in the gray area depicted in Figure 2.1 where it additionally greenlights project 2. Because

$z_2(\widehat{c}_1) \leq z_2^{**}$ by $\boldsymbol{\zeta}^{**}$-exclusivity, the alternative mechanism clearly yields a higher payoff.

It remains to be shown that the alternative mechanism is not only more profitable but also feasible. First of all, the initial mechanism is, by assumption, budget-feasible everywhere. In particular, it is feasible at any point $(\widehat{c}_1, \widetilde{c}_2)$ with $\widehat{c}_2 \leq z_2(\widehat{c}_1)$ and $\widehat{c}_1 \in \widehat{C}_1$. Formally, for any such points, the budget constraint holds,

$$q_1(\widehat{c}_1, \widetilde{c}_2) z_1(\widetilde{c}_2) + q_2(\widehat{c}_1, \widetilde{c}_2) z_2(\widehat{c}_1) \leq B. \tag{*}$$

To any point $(\widehat{c}_1, \widetilde{c}_2)$, there is a range of corresponding points $(\widetilde{c}_1, \widetilde{c}_2)$ with $\widetilde{c}_1 \in \widetilde{C}_1$. We now check feasibility for any such point $(\widetilde{c}_1, \widetilde{c}_2)$. Referring to Figure 2.1, we are addressing all points that live in the rectangle below the thick flat line of $z_2'$.

Under the alternative mechanism, for all $\widetilde{c}_1 \in \widetilde{C}_1$, $q_2'(\widetilde{c}_1, \widetilde{c}_2) = q_2(\widehat{c}_1, \widetilde{c}_2) = 1$. Regarding $\widehat{c}_1$, there can be two cases:

Case 1: If $\widehat{c}_1 \leq z_1(\widehat{c}_2)$, then $q_1(\widehat{c}_1, \widetilde{c}_2) = 1$, i.e., both projects are implemented and have to be compensated. The alternative is feasible in any point $(\widetilde{c}_1, \widetilde{c}_2)$ as

$$\begin{aligned} q_1'(\widetilde{c}_1, \widetilde{c}_2) z_1'(\widetilde{c}_2) + q_2'(\widetilde{c}_1, \widetilde{c}_2) z_2'(\widetilde{c}_1) &= q_1'(\widetilde{c}_1, \widetilde{c}_2) z_1(\widetilde{c}_2) + z_2(\widehat{c}_1) \\ &\leq z_1(\widetilde{c}_2) + z_2(\widehat{c}_1) \qquad \leq B, \end{aligned}$$

where the final inequality follows from (*).

Case 2: If $\widehat{c}_1 > z_1(\widehat{c}_2)$, then $q_1(\widehat{c}_1, \widetilde{c}_2) = 0$, i.e., only project 2 is financed. The alternative is feasible in any point $(\widetilde{c}_1, \widetilde{c}_2)$ as

$$\begin{aligned} q_1'(\widetilde{c}_1, \widetilde{c}_2) z_1'(\widetilde{c}_2) + q_2'(\widetilde{c}_1, \widetilde{c}_2) z_2'(\widetilde{c}_1) &= 0 + z_2'(\widetilde{c}_1) \\ &\leq z_2(\widehat{c}_1) \qquad \leq B, \end{aligned}$$

where the first equality follows from $\widetilde{c}_1 \geq \widehat{c}_1 > z_1(\widetilde{c}_2)$ and the final inequality again follows from (*).

Since, for any feasible cutoff mechanism with a cutoff function that is somewhere decreasing, we can find an alternative more profitable cutoff mechanism with cutoff functions that are weakly increasing, the optimal mechanism's allocation rule must have substitutes. □

Lemma 2.6 establishes that optimal cutoff functions are weakly increasing in each of their arguments. The intuition is straightforward. The cost realizations of all projects are independent. Therefore project $i$'s cost report only influences the allocation of project $j \neq i$ via the budget constraint. Project $i$'s cost report only influences the budget through exceeding or lying below the cutoff. If project $i$ exceeds its cutoff, this frees budget to be distributed among the other projects. Consequently, their cutoffs should remain constant or increase. While the intuition is the same for both $n = 2$ and $n > 2$, the proof is more involved in the general case. The reason is that the cost report of the project with the decreasing cutoff does not simultaneously pin down all other cutoffs and the remaining budget - as it does when $n = 2$. We cannot trivially extend the proof above, if some cutoff of a third project $z_3$ increases in $c_1$ while $z_2$ decreases. The intuition of the general proof is that a decreasing cutoff cannot be optimal, because it essentially implies exchanging project 2 for project 1 while the virtual surplus of project 2 decreases relative to the virtual surplus of project 1.

We continue by establishing the next property of the optimal mechanism:

**Definition 2.7.** An allocation rule $\gamma$ has non-bossy winners if for any $i \in I$, $\mathbf{c} \in C$, and $c_i' \in C_i$, $i \in \gamma(c_i', \mathbf{c}_{-i}) \cap \gamma(\mathbf{c})$ implies $\gamma(c_i', \mathbf{c}_{-i}) = \gamma(\mathbf{c})$.

In words, a non-bossy winner cannot affect the allocation without changing its own green-light status. In restricted environments, it can be shown that the optimal allocation rule is non-bossy: $\gamma(c_i', \mathbf{c}_{-i}) \cap \{i\} = \gamma(\mathbf{c}) \cap \{i\}$ implies $\gamma(c_i', \mathbf{c}_{-i}) = \gamma(\mathbf{c})$. However, we only need the winners to be non-bossy and examples of environments with bossy losers in the optimal mechanism can be constructed, see Appendix VI.D.

Given some cost vector, let $G$ represent the set of greenlighted projects and $R$ represent the set of redlighted projects. In the following lemma, we show that given that only the projects in some set $G$ are greenlighted and given the remaining projects' costs $\mathbf{c}_R$, for all $g \in G$ all functions $z_g$ intersect at some point $(a_1^G(\mathbf{c}_R), a_2^G(\mathbf{c}_R), ...)$. This point only depends on cost reports $\mathbf{c}_R$ of redlighted projects. Intuitively, optimal cutoffs cannot depend on greenlighted projects' cost, because for these projects the cutoff coincides with the transfer. For the two-project case, Figure 2.2 illustrates that $(BC)$ must bind when both projects are greenlighted. However, then project 1 influencing project 2's cutoff would change the remaining budget which is equal to project 1's transfer, given that $(BC)$ binds. This contradicts the notion of a cutoff mechanism.

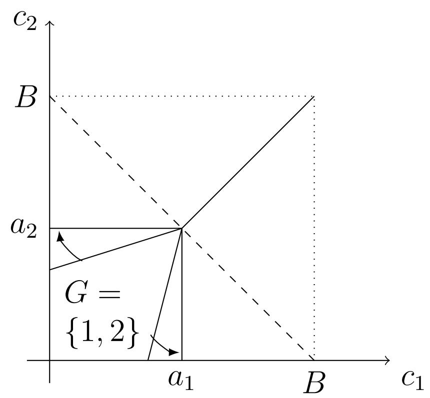
**Lemma 2.8.** *For any cost vectors* $(\mathbf{c}_G, \mathbf{c}_R) \in C$ *and* $(\mathbf{c}'_G, \mathbf{c}_R) \in C$ *such that* $G = \gamma(\mathbf{c}_G, \mathbf{c}_R) = \gamma(\mathbf{c}'_G, \mathbf{c}_R)$ *and* $R = I \setminus \gamma(\mathbf{c}_G, \mathbf{c}_R)$*, the optimal cutoff function* $z_g$ *for all* $g \in G$ *is (almost everywhere) independent of the costs of all greenlighted projects* $\mathbf{c}_G$*. That is,*

$$z_g(\mathbf{c}_{G-g}, \mathbf{c}_R) = z_g(\mathbf{c}'_{G-g}, \mathbf{c}_R),$$

*for all* $\mathbf{c}_{G-g}$ *and* $\mathbf{c}'_{G-g}$ *such that* $G$ *is the set of greenlighted agents.*

*Proof.* (with $n = 2$, see appendix for the general proof and consult Figure 2.2 for intuition)
By Lemma 2.1, the optimal mechanism has to be a cutoff mechanism. What remains to be shown is that the cutoff functions $\{z_i\}_{i \in I}$ only depend on $\mathbf{c}_R$. When $\gamma(\mathbf{c})$ is a singleton, i.e., when only one project is greenlighted, the statement follows from the nature of a cutoff function. Hence, we need to show that the cutoffs must be constants whenever $\gamma(\mathbf{c}) = \{1, 2\}$. Therefore suppose that $\gamma(\mathbf{c}) = \{1, 2\}$ is induced with positive probability.

FIGURE 2.2: In Lemma 2.8, we show that in the nontrivial two-project case whenever $G = \{1, 2\}$ both projects get constant transfers summing up to the budget. For instance, the candidate mechanism (with substitutes) depicted above is outperformed by an alternative mechanism indicated by the arrows.




Take any feasible candidate mechanism with any increasing cutoff functions $z_i$ and define

$$\begin{aligned} a_1 &= \max\{c_1 | \exists c_2 : c_2 \leq z_2(c_1), c_1 \leq z_1(c_2)\} \\ a_2 &= \max\{c_2 | \exists c_1 : c_1 \leq z_1(c_2), c_2 \leq z_2(c_1)\}, \end{aligned} \tag{2.6}$$

i.e., $a_i$ is the highest cost of project $i$ such that both projects are implemented.

Whenever greenlighting both projects, the sets over which we have defined $a_1$ and $a_2$ must be non-empty. The maximum exists by left-continuity of any optimal function $z_i$.[11] Hence by definition of $a_1$, there exists $\widetilde{c}_2$ such that $a_1 = z_1(\widetilde{c}_2)$. Similarly, there exists $\widetilde{c}_1$ such that $a_2 = z_2(\widetilde{c}_1)$.

By definition, $(\widetilde{c}_1, \widetilde{c}_2) \leq (a_1, a_2)$ and at cost realization $(\widetilde{c}_1, \widetilde{c}_2)$ both projects are implemented. The budget feasibility of the candidate mechanism implies $a_1 + a_2 \leq B$.

Now we show that, in optimum, $z_1(c_2') = a_1$, for all $c_2' \leq a_2$, and $z_2(c_1') = a_2$, for all $c_1' \leq a_1$. Suppose not. Suppose (without loss of generality) there is some set $\Xi \subset [0, a_2]$ with positive Lebesgue-measure such that $z_1(c_2') < a_1$ for all $c_2' \in \Xi$. Denote $z_1^\Xi := \max_{c_2 \in \Xi} z_1(c_2)$. Since $a_1 + a_2 \leq B$, changing the mechanism to $z_1(c_2') = a_1, \quad \forall c_2' \leq a_2$ does not violate the budget constraint and increases the payoff by

$$\Delta > \Pr(c_2 \in \Xi) \int_{z_1^\Xi}^{a_1} \psi_1(c) dF(c) > 0.$$

In fact, this alternative mechanism outperforms the initial candidate state-by-state and not only in expectation. □

The following corollary is an immediate consequence of Lemma 2.8 combined with monotonicity and bidder substitutability. It establishes that any optimal mechanism satisfies non-bossiness of greenlighted projects.

**Corollary 2.9.** *For any optimal mechanism with $G = \gamma(\mathbf{c}_G, \mathbf{c}_R)$ for some $(\mathbf{c}_G, \mathbf{c}_R) \in C$, also $\gamma(\mathbf{c}'_G, \mathbf{c}_R) = G$ for any cost vector $(\mathbf{c}'_G, \mathbf{c}_R) \in C$ with $c_g' \leq c_g$ for all $g \in G$. Hence, for all $i \in I$, for all $\mathbf{c}_{-i} \in C_{-i}$, and for all $\widehat{c}_i$, $\widetilde{c}_i \in C_i$ with $\widehat{c}_i < \widetilde{c}_i$, in any optimal mechanism,*

$$\widehat{c}_i < \widetilde{c}_i \leq z_i(\mathbf{c}_{-i}) \qquad \textit{implies} \qquad \gamma(\widehat{c}_i, \mathbf{c}_{-i}) = \gamma(\widetilde{c}_i, \mathbf{c}_{-i}).$$

Taking stock, among all mechanisms satisfying (PC), (BC) and (IC), any mechanism that maximizes the designer's expected payoff (2.1) belongs to a certain class

[11] We can replace any function $z_i$ with a left-continuous function that is identical up to a set of points with Lebesgue-measure zero. Hence, if there exists an optimal function $z_i$ that is not left-continuous, then there also exists a left-continuous version of the same function that yields the same payoff and hence is also optimal.

of mechanisms: We have shown that the optimal mechanism is characterized by a set of cutoff functions $\{z_i\}_{i\in I}$ and the corresponding allocation rule is

**Property 1** monotonic in costs,

**Property 2** $\zeta^{**}$-exclusive,

**Property 3** has substitutes, and

**Property 4** has non-bossy winners.

Being able to restrict attention to mechanisms with these properties is highly useful, as these mechanisms are a much more tangible class than the substantially larger set of all permissible cutoff mechanisms. In addition, all mechanisms with these properties can be implemented with a DA auction as proposed by Milgrom and Segal (2014). To this end, we first restate their definition adapted to our setting.

**Definition 2.10** (DA auction)**.** A deferred acceptance (DA) auction is an iterative algorithm defined by a collection of scoring functions

$$s_i^A : C_i \times C_{I\setminus A} \to \mathbb{R}_+$$

that are weakly increasing in $c_i$ for all $i \in A$ and for all $A \subset I$. Let $A_t \subset I$ denote the set of active bidders in iteration $t$ and initially $A_1 = I$. The algorithm stops in some period $T$ when all active projects have a score of zero, $s_i^{A_T} = 0$ for all $i \in A_T$. Then the set of greenlighted project is $A_T$. Otherwise, at each iteration $t$, the project with the highest score is removed. The payment $p_i^t$ of project $i$ at iteration $t$ is either given by the highest possible cost that $i$ could have had without being removed from the set of active bidders or by the last iteration's payment, depending on which payment is smaller,

$$p_i^t(\mathbf{c}) = \begin{cases} \sup\{c_i' : s_i^{A_t}(c_i', \mathbf{c}_{I\setminus A_t}) < s_j^{A_t}(c_j, \mathbf{c}_{I\setminus A_t})\} & \text{for } j \in A_t \setminus A_{t+1}, \\ \min\{\sup\{c_i' : s_i^{A_t}(c_i', \mathbf{c}_{I\setminus A_t}) \leq 0\}, p_i^{t-1}\} & \text{if } t = T. \end{cases}$$

The algorithm is initialized with $p_i^0 = \min\{\overline{c}_i, z_i^{**}, B\}$.[12]

The main appeal of DA auctions lies in their incentive guarantees. They are not only strategyproof, they are obviously strategyproof, as defined by Li (2015).

[12] Compared to Milgrom and Segal (2014), we slightly tweak the updating function of payments without changing the deferred acceptance nature of the algorithm and any of its properties.

Moreover, DA auctions are weakly group-strategyproof. That is, no coalition of projects can manipulate their reports such that it strictly increases the utility of all projects in the coalition: At least one member of the coalition receives a weakly worse payoff whenever other coalition members benefit. Because collusion in auctions is generally illegal, compensating the worse off coalition member is not contractible. In addition, the dominant-strategy equilibrium outcome in a DA auction can be interpreted as robust in the following sense: Consider the full-information game in which all cost reports are observed, projects can report any cost, the allocation is determined according to the DA auction's allocation rule, but projects receive their own report as payments. The dominant-strategy equilibrium outcome of the DA auction is the only outcome that survives iterated deletion of dominated strategies in this game.

**Proposition 2.11.** *Any optimal mechanism has a DA auction representation and can be implemented with a descending-clock auction.*

The proof of Proposition 2.11 is relegated to a separate section in the appendix. Milgrom and Segal (2014) show that with a finite type space, any mechanism satisfying monotonicity, bidder substitutability, and non-bossiness of winners can be implemented by a myopic clock auction.

### III.i The symmetric case

In this section, we focus on symmetric projects, i.e., environments with $v_i = v$ and $F_i = F$ for every project $i \in I$. An implication of this assumption is that the order of costs coincides with the order of virtual surpluses and that $z_i^{**} = z^{**}$ for all $i \in I$. We show how to utilize the established results to characterize the optimal allocation and also how to implement it. As in previous proofs, the proof of Proposition 2.12 considers the two-project case while the general proof is relegated to the appendix. In the two-project case, the designer's optimization problem can be reduced to optimally solving for a single constant. Nevertheless, we discuss possible alternatives to the optimal mechanism in greater detail to foreshadow the complications which arise in asymmetric environments.

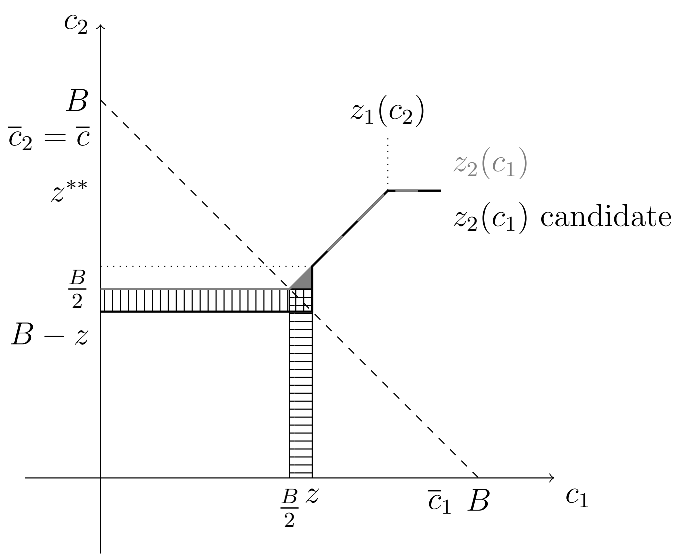
**Proposition 2.12.** *Arrange the projects in ascending order of their reported costs,* $c_1 \leq c_2 \leq \cdots \leq c_n \leq c_{n+1} := \overline{c}$*, and define* $z^k := \min\left\{\frac{B}{k}, z^{**}, c_{k+1}\right\}$*. In the symmetric case, the cutoff mechanism with* $z_i(\mathbf{c}_{-i}) = z^{k^*}$ *is the optimal mechanism. The optimal number of accepted projects* $k^*$ *is given by* $k^* := \max\{k | c_k \leq z^k\}$*.*

*Proof.* (with $n = 2$, see appendix for the general proof)
In Proposition 2.11, we have shown that the optimal mechanism must be a special kind of DA auction. We call the mechanism in the proposition the proposed mechanism and, as a candidate for optimality, consider $\{z_1, z_2\}$, some different cutoff mechanism with the properties we derived. Suppose $\{z_1, z_2\}$ greenlights both projects with nonzero probability and that it differs from the proposed mechanism in a way such that $a_1 = z > B/2$ and $a_2 = B - z < B/2$ with $a_i$ defined as in (2.6). For graphic intuition of the deviation consult Figure 2.3.

By Lemma 2.4, any optimal mechanism must never greenlight a project with negative virtual surplus. This property is depicted as the kink at $(z^{**}, z^{**})$.

In the area northwest of the dashed budget line, $c_1 + c_2 > B$, the designer can, by (BC) and (PC), only execute one of the two projects. It can be directly seen from objective function (2.3) that the designer prefers the project with the higher virtual surplus, i.e., the one with lower cost. It does not, however, follow directly that $z_i(c_j) = c_j$ whenever $B - c_i < c_j < z^{**}$. It could be optimal for the designer to forgo executing the lower-cost project for some cost vectors (shaded triangle and crossed square in Figure 2.3) in order to execute both projects in an additional area (horizontally lined, Figure 2.3). In such a case, the designer is forced by incentive compatibility to execute the higher-cost project (for cost vectors in the shaded triangle or the square that is both horizontally and vertically lined).

FIGURE 2.3: A candidate mechanism compared to the proposed mechanism.



By Lemma 2.8, both cutoffs must be constant whenever both projects are executed. In optimum in that case, there can be no slack in the budget constraint and $z_i$ is flat in that region. Otherwise increasing one of the cutoffs until the budget binds is both feasible and profitable.

Formally, candidate mechanism $\{z_1, z_2\}$ is given by

$$z_2(c_1) = \begin{cases} z^{**} \text{ if } c_1 \geq z^{**} \\ c_1 \text{ if } z < c_1 < z^{**} \\ B - z \text{ if } c_1 < z \end{cases} \quad \text{and} \quad z_1(c_2) = \begin{cases} z^{**} \text{ if } c_2 \geq z^{**} \\ c_2 \text{ if } B - z < c_2 < z^{**} \\ z \text{ if } c_2 < B - z \end{cases} \quad . \tag{2.7}$$

For ease of exposition, let $A = \frac{B}{2}$. Let $\Delta$ be the increase in the designer's expected payoff from implementing the proposed mechanism instead of candidate $\{z_1, z_2\}$.

$$\begin{aligned} \Delta = {} & F(z) \int_{B-z}^{A} \psi(x_2) dF(x_2) && \text{(vertical)} \\ & - F(A) \int_{A}^{z} \psi(x_1) dF(x_1) && \text{(horizontal)} \\ & + \int_{A}^{z} \int_{A}^{c} \psi(x_2) dF(x_2) - (F(c) - F(A)) \psi(x_1) dF(x_1) && \text{(shaded)} \end{aligned}$$

where the patterns represent the area in Figure 2.3 where the allocation changes. Everywhere else the allocation and payoff remain the same.

To rewrite $\Delta$, define $\xi(x) = F(x)(v - x)$ with $\xi'(x) = \psi(x) f(x)$:

$$\begin{aligned} \Delta = {} & F(z)(\xi(A) - \xi(B - z)) - F(A)(\xi(z) - \xi(A)) \\ & + F(A)(\xi(z) - \xi(A)) + \int_{A}^{z} \xi(x_1) - \xi(A) - F(x_1)\psi_j(x_1) dF(x_1) \\ = {} & F(z)(\xi(A) - \xi(B - z)) - F(A)(\xi(z) - \xi(A)) \\ & + F(A)(\xi(z) - \xi(A)) - \xi(A)(F(z) - F(A)) + \int_{A}^{z} F^2(x_1) dx_1 \end{aligned}$$

because $(\psi(c)F(c) - F(c)(v-c))f(c) = F^2(c)$ and then since $\int_A^z F(x_1)^2 dx_1 > F(A)^2 \int_A^z 1 dx_1$,

$$\begin{aligned}\Delta &> F(z)(\xi(A) - \xi(B-z)) - \xi(A)(F(z) - F(A)) + F(A)^2(z-A)\\ &= F(A)^2(v - A + z - A) - F(z)F(B-z)(v - B + z)\\ &= (v - B + z)(F(A)^2 - F(z)F(B-z))\\ &> 0 \quad \Leftrightarrow \quad F(A)^2 > F(z)F(B-z).\end{aligned}$$

This statement is true under Assumption 1, log-concavity. Maximizing $F(z)F(B-z)$ with respect to $z$, the first order condition is given by

$$\frac{F(z)}{f(z)} = \frac{F(B-z)}{f(B-z)} \tag{2.8}$$

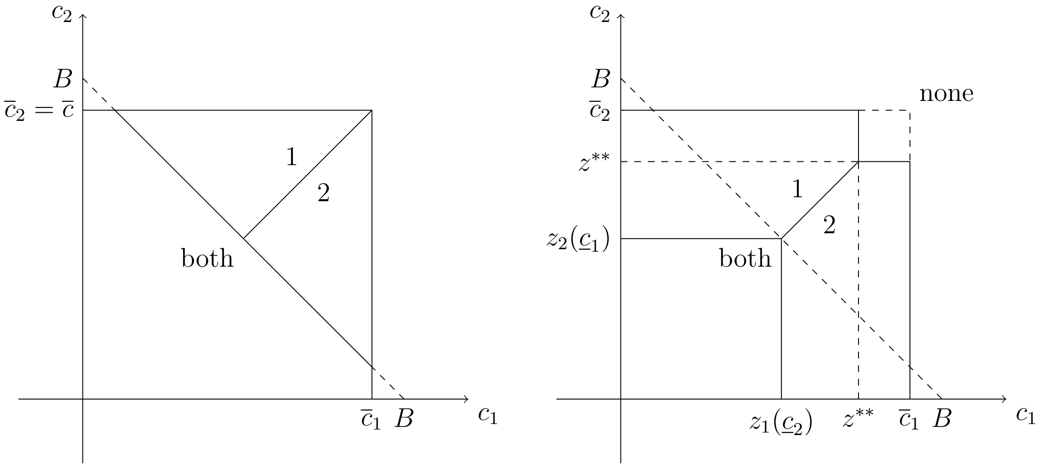
which is only true at $z = B/2$ since $F/f$ is an increasing function. For the same reason, the left-hand side is greater (less) than the right-hand side for $z > B/2 (< B/2)$ making $z = B/2$ the maximum.

We have assumed that in the optimal mechanism both projects get greenlighted for some cost vectors. It remains to show that the optimal mechanism beats the best mechanism in which at most one project gets greenlighted. The best mechanism that selects at most one project always greenlights the project with higher virtual surplus. Clearly, the proposed mechanism outperforms this mechanism as it also always greenlights the project with higher virtual surplus, and it, additionally, sometimes greenlights a second project with positive virtual surplus. □

To sum up, in the symmetric case, the optimal allocation rule takes a simple form: The cheapest projects are greenlighted and the mechanism greenlights as many projects as the budget allows, while each procured project receives the same compensation. Any project that is redlighted prefers this allocation status over having to conduct the project with the associated compensation.

There are two rationales for greenlighted projects to get the same transfer. First, as shown in the proof of Proposition 2.12, this cutoff rule maximizes the probability of getting as many projects as possible. Dominant-strategy incentive compatibility prevents the budget from being shifted away from projects with low cost reports to projects with high costs. Therefore offering equal cutoffs is the best the designer can do. Second, as seen in (2.3), the rewritten maximization problem of the designer, the expected utility of the designer is given by the sum of virtual surpluses of greenlighted projects. Therefore she wants to greenlight those projects with the highest virtual surpluses. That goal is consistent with offering equal cutoffs to greenlighted projects and excluding those with higher cost. In the optimal allocation, greenlighted projects have higher virtual surplus than those which are not greenlighted. The compatibility of the two goals - get as many projects as possible and get those with the highest virtual surpluses - is a special feature of the symmetric case. It generically fails in the asymmetric case, as we demonstrate in the next section.

FIGURE 2.4: An example of optimal allocations for the symmetric case with $n = 2$.




(A) Budget-constrained, full information.

(B) Budget-constrained, private information.

Figure 2.4 illustrates the optimal budget-constrained allocations in an example with two projects. Panel 2.4b shows the fully-constrained optimal allocation juxtaposed with the relaxed optimal allocation when (IC) is neglected, shown in Panel 2.4a. First, note that in this example $v \geq \bar{c}$ and $\bar{c} < B$. Therefore a fully-unconstrained designer with full information would always greenlight both projects, and a budget-constrained designer with full information would always greenlight at least one project. However, since $z^{**} < \bar{c}$, there exist realizations of $\mathbf{c}$ (the upper-right corner of Panel 2.4b) such that no project gets greenlighted in the (IC)-constrained optimal allocation, even though doing so would be profitable from an ex-post perspective. The negative virtual surpluses of the projects in these cases indicates that the cost of allocating to such a project - incentive compatibility requires higher transfers for other cost types - outweighs the benefit from an ex-ante perspective. The second major difference between the relaxed optimal allocation and the optimal allocation can be seen for those realizations of costs such that allocating to both projects would be feasible only in the relaxed problem. This difference is a result of the designer's inability to shift budget from low-cost to relatively higher-cost projects with a strategyproof mechanism.

**Corollary 2.13.** *In the symmetric case, the optimal direct mechanism can be implemented by a descending-clock auction. The clock price, denoted by $\tau$, starts at $z^{**}$ and descends continuously and synchronously down to $\frac{B}{n}$. Projects can drop out at any price but cannot re-enter. The auction stops once the clock price can be paid out to all projects remaining in the auction.*

*In any iteration, a scoring function of the corresponding DA auction is*

$$s_i^{A_t}(c_i, A_t) = \max\left\{c_i - \frac{B}{|A_t|}, 0\right\}.$$

We consider the descending-clock auction of Corollary 2.13 to be a natural indirect mechanism that implements the outcome of the optimal allocation. Project $i$'s equilibrium strategy, which implements this outcome, has it staying active as long as the price is weakly larger than its private cost, $\tau \geq c_i$. It is easily verifiable that this is a weakly dominant strategy for project $i$.

### III.ii The asymmetric case

In this section, we demonstrate why the logic of the optimal mechanism in the symmetric case does not carry over to the asymmetric case. To preserve tractability, we restrict the analysis to the two-project case which conveys the intuition behind the forces at work in the general case. However, we allow for general values $v_1$ and $v_2$ as well as differing cost distributions $F_1$ and $F_2$. We consider the non-trivial case, $z_1^{**} + z_2^{**} > B$

Since we did not impose symmetry to prove Proposition 2.11, we can without loss of generality restrict attention to mechanisms inheriting the optimal properties to find an optimal mechanism for the asymmetric case as well. The rewritten maximization problem of the designer (2.3) for the asymmetric two-project case is given by

$$\begin{gathered}\max_{z_1(c_2),z_2(c_1)} \mathbb{E}\Big[\mathbb{I}(c_1 \leq z_1(c_2))\left(v_1 - c_1 - \tfrac{F_1(c_1)}{f_1(c_1)}\right) \\ +\mathbb{I}(c_2 \leq z_2(c_1))\left(v_2 - c_2 - \tfrac{F_2(c_2)}{f_2(c_2)}\right)\Big] \\ \text{s.t.} \\ \mathbb{I}(c_1 \leq z_1(c_2))z_1(c_2) + \mathbb{I}(c_2 \leq z_2(c_1))z_2(c_1) \leq B \quad \forall (c_1, c_2) \in C.\end{gathered} \tag{2.9}$$

By Lemma 2.8, the cutoffs must be constants whenever both projects are greenlighted. Since we consider the non-trivial case, these constants must sum up to the budget. Otherwise, increasing one of the cutoffs until the budget binds is both feasible and profitable. Let project 1's cutoff for this case be $z_1(\underline{c}_2) = z$ and project 2's cutoff be $z_2(\underline{c}_1) = B - z$. By virtue of the optimal properties, the designer must greenlight a project once its cost is below the constant cutoff $z_i(\underline{c}_{-i})$. If both projects report greater costs, the designer is free to choose one of them. A glance at the objective function (2.9) reveals that in such a case it is desirable to greenlight the project with greater positive virtual surplus, if feasible. This result allows us to rewrite the objective function (2.9) as a function of $z$,

$$\begin{aligned}\max_z \pi(z) = &\int_0^z \psi_1(c_1)dF_1(c_1) + \int_0^{B-z} \psi_2(c_2)dF_2(c_2) \\ &+ \int_{\max\{\psi_2^{-1}(\psi_1(z)),B-z\}}^{\bar{c}_2} \int_z^{\min\{\psi_1^{-1}(\psi_2(c_2)),z_1^{**},B\}} \psi_1(x)dF_1(x)dF_2(c_2) \\ &+ \int_{\max\{\psi_1^{-1}(\psi_2(B-z)),z\}}^{\bar{c}_1} \int_{B-z}^{\min\{\psi_2^{-1}(\psi_1(c_1)),z_2^{**},B\}} \psi_2(x)dF_2(x)dF_1(c_1).\end{aligned} \tag{2.10}$$

In the symmetric case, the ranking of virtual surpluses coincides with the reversed order of costs. Hence, the optimal DA auction in the symmetric case rejects in each round the least attractive project in terms of virtual surplus. A natural extension of this mechanism to the asymmetric case would involve adjusting the cutoffs so that they equalize virtual surplus. This modification ensures that again in each round the least attractive project in terms of virtual surplus is rejected. We call this the candidate allocation.

The condition for optimality of the candidate allocation is stated in (2.11). To implement the candidate allocation, the constant cutoffs at which both projects are greenlighted must be a pair $(a_1, a_2) = (z, B - z)$ such that $\psi_1(z) = \psi_2(B - z)$. Then, however, optimality is only obtained if $\frac{F_2(B-z)}{f_2(B-z)} = \frac{F_1(z)}{f_1(z)}$. The intuition behind this statement is straightforward. Selecting $z$ in order to satisfy $\psi_1(z) = \psi_2(B-z)$ allows the designer to always program the price clocks such that they greenlight the project with the higher virtual surplus, whenever it is not feasible to greenlight both projects. However, if $\frac{F_2(B-z)}{f_2(B-z)} \neq \frac{F_1(z)}{f_1(z)}$ the cutoffs $z$ and $B-z$ do not maximize the probability to greenlight both projects. Consequently, the designer can adjust the cutoffs $\{z, B-z\}$ to trade off a higher probability of implementing the most favorable allocation ($\gamma(c_1, c_2) = \{1, 2\}$) against a positive probability of having to implement the less preferred of two possible singleton allocations ($\gamma(\mathbf{c}) = j$, when project $j$ has lower virtual surplus).

Therefore the two aspects of the designer's payoff maximization - getting projects with high virtual surplus and getting as many projects as possible - are only aligned if condition (2.11) is met. In the symmetric case, the condition holds by construction. However, in an asymmetric environment it is generically violated.

**Proposition 2.14.** *In the nontrivial asymmetric two-project case, i.e.,* $n = 2$ *and* $z_1^{**} + z_2^{**} > B$*, in which values or cost distributions differ across projects, it is generically not optimal to always greenlight the project with the higher virtual surplus. That is, under the optimal allocation rule* $\gamma$*, there may exist cost vectors* $(c_i, c_j, \mathbf{c}_{-i-j}) \in C$ *such that*

$$i \notin \gamma(c_i, c_j, \mathbf{c}_{-i-j}), \text{ and } j \in \gamma(c_i, c_j, \mathbf{c}_{-i-j})$$

*although*

$$\psi_i(c_i) > \psi_j(c_j).$$

*Proof.* To obtain the derivative of $\pi(z)$ given in (2.10) with respect to $z$ we can use the rules for differentiation under the integral sign.[13] Given the max operators, the derivative takes a different form depending on whether $\psi_1(z) \gtrless \psi_2(B-z)$. However, as $\pi$ is continuously differentiable, it suffices to look at one of the two forms,

$$\begin{aligned} \left.\frac{\partial \pi}{\partial z}\right|_{z:\psi_1(z)\geq\psi_2(B-z)} &= \int_z^{\psi_1^{-1}(\psi_2(B-z))} \psi_1(x) dF_1(x) f_2(B-z) + \\ &+ \psi_1(z) f_1(z) F_2(B-z) \\ &- \psi_2(B-z) f_2(B-z) F_1(\psi_1^{-1}(\psi_2(B-z))). \end{aligned}$$

[13] Define $g(z, c_2) := \int_z^{\min\{\psi_1^{-1}(\psi_2(c_2)), z_1^{**}, B\}} \psi_1(x) dF_1(x) f_2(c_2)$ and then use $\frac{d}{dz}\left(\int_{a(z)}^{b(z)} g(z, c_2) dc_2\right) = g(z, b(z)) b'(z) - g(z, a(z)) a'(z) + \int_{a(z)}^{b(z)} g_z(z, c_2) dc_2$.

Now, consider $z$ corresponding to the candidate allocation with $\psi_1(z) = \psi_2(B-z)$, which yields

$$\frac{\partial \pi}{\partial z} = 0 \Leftrightarrow \frac{F_2(B-z)}{f_2(B-z)} = \frac{F_1(z)}{f_1(z)}, \tag{2.11}$$

a nongeneric case. Consequently, it is generically not optimal to always allocate to the project with the higher virtual surplus. □

Proposition 2.14 is driven by a tradeoff between quantity and quality: Even though the designer always prefers the project with the higher virtual surplus, if she greenlights a single project she sometimes greenlights the project with lower virtual surplus out of two rival projects, as quantity is endogenous here. The simplest way to lay out the intuition behind Proposition 2.14 is by an example.

**Example 2.1.** *There are two projects,* $(n = 2)$ *with* $v_1 = 5, v_2 = 4.5$ *and* $c_1$ *and* $c_2$ *are uniformly distributed on support* $[0, 1]$. *The budget is given by* $B = 1$. *The optimal cutoff functions are given by:*

$$z_1(c_2) = \begin{cases} 0.53 & \text{if } c_2 \leq 0.47 \\ c_2 + 0.25 & \text{if } 0.47 < c_2 \leq 0.75 \\ 1 & \text{if } c_2 > 0.75 \end{cases}$$

$$z_2(c_1) = \begin{cases} 0.47 & \text{if } c_1 \leq 0.72 \\ c_1 - 0.25 & \text{if } c_1 > 0.72. \end{cases}$$

*Possible scoring functions for a corresponding DA auction are given by:*

$$s_1^{\{1,2\}}(c_1) = \begin{cases} c_1 + 0.47 & \text{if } 0.53 < c_1 < 0.72 \\ 2c_1 - 0.25 & \text{if } c_1 \geq 0.72 \\ 0 & \text{otherwise} \end{cases}$$

$$s_2^{\{1,2\}}(c_2) = \begin{cases} 2c_2 + 0.25 & \text{if } c_2 > 0.47 \\ 0 & \text{otherwise} \end{cases}$$

$$s_1^{\{1\}}(c_1) = 0$$

$$s_2^{\{2\}}(c_2) = 0.$$

*The corresponding optimal allocation is:*

$$(q_1, q_2) = \begin{cases} (1,1) & \textit{if } 0 \leq c_1 \leq 0.53 \textit{ and } 0 \leq c_2 \leq 0.47 \\ (1,0) & \textit{if } 0 \leq c_1 \leq 0.72 \textit{ and } c_2 > 0.47 \\ (1,0) & \textit{if } c_1 > 0.72 \textit{ and } \psi_1 \geq \psi_2 \\ (0,1) & \textit{if } 0.53 < c_1 \leq 0.72 \textit{ and } c_2 \leq 0.47 \\ (0,1) & \textit{if } c_1 > 0.72 \textit{ and } \psi_1 < \psi_2. \end{cases}$$

*The corresponding transfers are:*

$$t_1(c_1, c_2) = \begin{cases} 0.53 & \textit{if } c_2 \leq 0.47 \textit{ and } c_1 \leq 0.53 \\ c_2 + 0.25 & \textit{if } 0.47 < c_2 \leq 0.75 \textit{ and } c_1 \leq c_2 + 0.25 \\ 1 & \textit{if } c_2 > 0.75 \\ 0 & \textit{otherwise} \end{cases}$$

$$t_2(c_1, c_2) = \begin{cases} 0.47 & \textit{if } c_1 \leq 0.72 \textit{ and } c_2 \leq 0.47 \\ c_1 - 0.25 & \textit{if } c_1 > 0.72 \textit{ and } c_2 < c_1 - 0.25 \\ 0 & \textit{otherwise.} \end{cases}$$

Consider Example 2.1. The candidate allocation demands cutoffs such that $\widetilde{z}_1(\underline{c}_2) = 0.625$ and $\widetilde{z}_2(\underline{c}_1) = 0.375$ for allocating to both projects. At these cutoffs, the probability of greenlighting both projects is $0.625 \cdot 0.375 \approx 0.234$. This allocation is depicted in Panel 2.5a. In contrast, the maximal feasible probability to greenlight both projects is at equal cutoffs, $\widehat{z}_1(\underline{c}_2) = \widehat{z}_2(\underline{c}_1) = 0.5$. The corresponding area is the dotted square in the lower-left corner of Panel 2.5b. However, at these cutoffs it is not incentive compatible to guarantee the green light for the project with higher virtual surplus in every case. More specifically, it is not incentive compatible to allocate along the dotted[14] diagonal line, if at least one project exceeds $\widehat{z}_i(\underline{c}_{-i})$. Hence, strategyproofness introduces a tradeoff between maximizing the probability of greenlighting both projects and allocating to the preferred one if only one project is feasible. Consequently, the optimal cutoffs $(z_1^*, z_2^*)$ for greenlighting

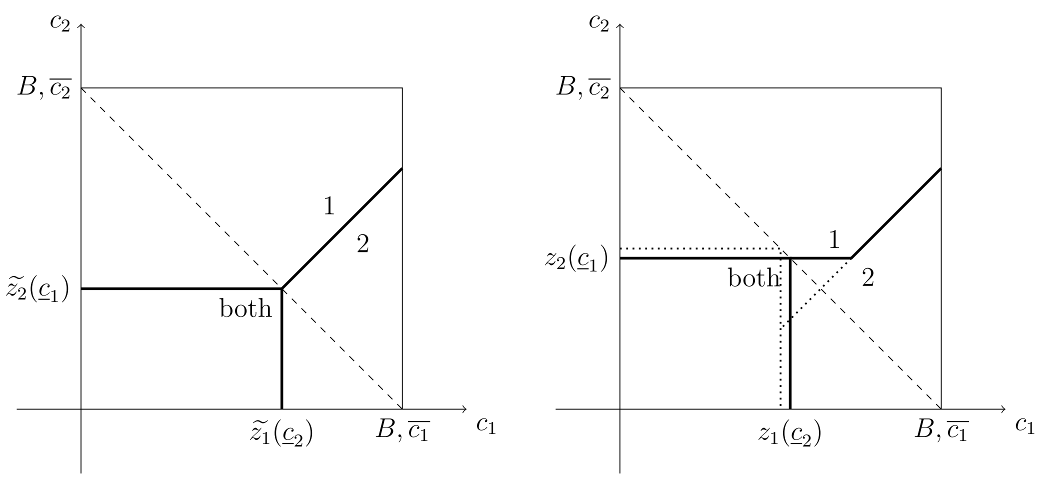
[14]Not to be confused with the dashed diagonal representing the budget constraint.

both projects do not lie at $(0.625, 0.375)$ but rather at $(0.53, 0.47)$. Importantly, this optimal discrimination of the stronger project is pursued independently of the discrimination due to the stochastic dominance reflected in the virtual costs.

Given the optimal allocation in Example 2.1, there are some realizations of the cost vector for which the designer greenlights the project with lower virtual surplus. These realizations are represented by the shaded area in Panel 2.6a. Here, (IC), (PC), and the choice of $(z_1(\underline{c}_2), z_2(\underline{c}_1))$ force the designer to greenlight project 2, even though project 1 has the higher virtual surplus.

The cost vectors for which the designer implements both projects are represented by the rectangular area in the lower-left corner of Panel 2.6a. Any point $(z_1(\underline{c}_2), z_2(\underline{c}_1))$ on the dashed line representing the budget constraint satisfies $z_1(\underline{c}_2) + z_2(\underline{c}_1) = B$. Moving this corner point southwest along the dashed budget line has two effects: shrinking the shaded area and shrinking the area of the rectangle, which in this example represents the probability that both projects are conducted. While it is desirable to shrink the shaded area, in which the designer must allocate to project 2 despite its lower virtual surplus, shrinking the size of the rectangle lowers the probability of allocating to both projects. Given that we have an interior solution in this example, at $(z_1(\underline{c}_2), z_2(\underline{c}_1))$ these two effects balance each other out.

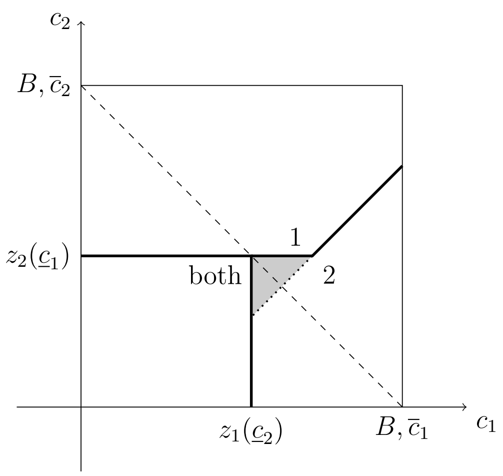
FIGURE 2.5: Candidate and optimal allocation for Example 2.1.



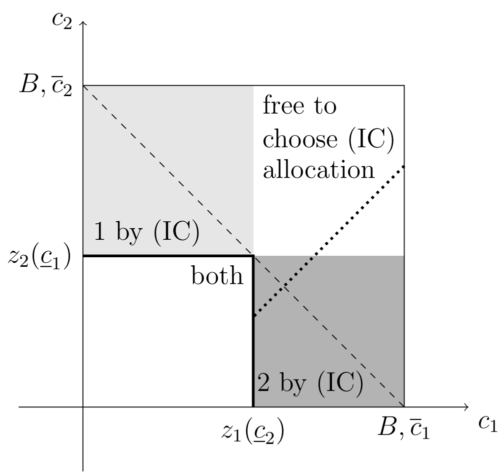

(A) Candidate allocation. (B) Optimal allocation.

Graphically, the fact that there is no slack in the budget constraint whenever both projects are greenlighted implies that the area representing points at which both projects are executed touches the dashed line at least once, as can be seen, for example, in Panel 2.6b. In fact, it can touch the (BC)-constraint exactly once, as it is not possible to greenlight both projects when $c_1 > z_1(\underline{c}_2)$ or $c_2 > z_2(\underline{c}_1)$ without violating (BC) sometimes. This result means that the area where both projects are greenlighted is the rectangle with corners $(0, 0)$ and $(z_1(\underline{c}_2), z_2(\underline{c}_1))$. Then, if $c_1 < z_1(\underline{c}_2)$ but $c_2 > z_2(\underline{c}_1)$, the nature of cutoffs prevents the designer from greenlighting project 2. Therefore project 1 must be greenlighted, as represented by the lightly shaded area in Panel 2.6b. A similar argument applies to the darkly shaded area. Thus, looking at Panel 2.6b, the choice of $(z_1(\underline{c}_2), z_2(\underline{c}_1))$ determines the allocation for all cost realizations except those in the upper-right corner. Here, the designer is free to choose the allocation, as long as the line delineating whether project 1 or 2 gets greenlighted is (weakly) increasing or vertical. Not surprisingly, it is optimal to greenlight the project with the higher virtual surplus.

FIGURE 2.6: Greenlighting the project with lower virtual surplus and (IC)-constraints on the allocation (Example 2.1).




(A) Greenlighting the project with lower virtual surplus.



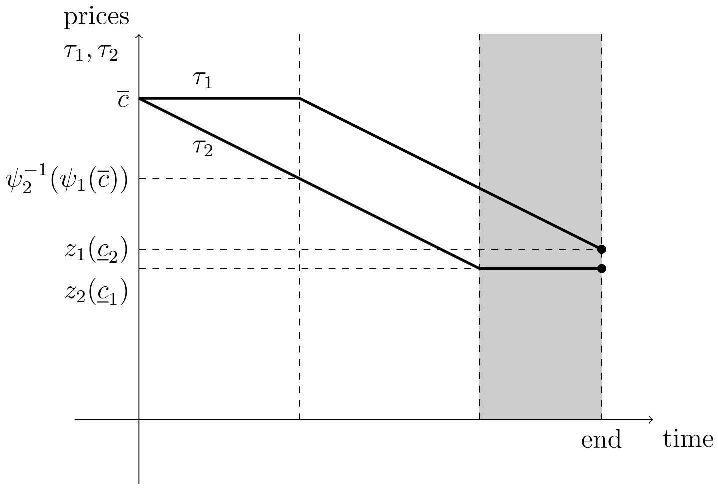

(B) (IC)-constraints on the allocation.

By Proposition 2.11, the optimal allocation can be implemented with a descending-clock auction. In the following, we show how to accommodate the tradeoff between quantity and quality in a modified clock auction.

**Corollary 2.15.** *In an optimal implementation with descending price clocks, the clocks not only run at individual speeds, occasionally some clocks also have to halt.*

A crucial difference to the symmetric case is that each project must have an individual price clock, because heterogeneous virtual surplus functions require individual speeds. Interestingly, an implication of the quantity-quality tradeoff is that sometimes one clock has to halt. For Example 2.1, the clock prices, denoted by $\tau_i$, are depicted in Figure 2.7 as a function of time. The entire (maximal) duration of the auction can be divided into three segments. The auction starts with both clocks at $z_1^{**} = z_2^{**} = \bar{c}$. First, $\tau_2$ decreases while $\tau_1$ is held constant, which happens until both clock prices lead to the same virtual surplus, i.e., $\psi_2(\tau_2) = \psi_1(\bar{c}_2)$. Second, both $\tau_1$ and $\tau_2$ decrease simultaneously, but asynchronously keeping virtual surplus equal, $\psi_1(\tau_1) = \psi_2(\tau_2)$, until $\tau_2 = z_2(\underline{c}_1)$. Third, only $\tau_1$ decreases until $\tau_1 = z_1(\underline{c}_2)$. If at this point both projects still remain in the auction, the auction stops and both are greenlighted. Otherwise, the inferior project 2 is greenlighted.

FIGURE 2.7: Optimal descending-clock auction in Example 2.1.




The cost vectors for which the designer greenlights project 2 despite its lower virtual surplus, represented by the shaded area in Panel 2.6a, are also represented graphically in Figure 2.7: If the auction ends in the third time segment (shaded area of Figure 2.7) before both projects can be greenlighted, project 1 must have exited because $\tau_1$ dropped below $c_1$. Project 2 is greenlighted and receives transfer $z_2(\underline{c}_1)$ even though project 1 has the higher virtual surplus. Therefore if cost vectors in the shaded area of Panel 2.6a realize, the optimal descending-clock auction ends in the third time segment.

We should emphasize again a novel feature of this descending-clock auction. The clocks of both projects are paused asynchronously over some time of the auction. One project's clock runs down while the other project's clock stops. Since we have examined a very simple example, each project's clock is paused only once. In a more general setting, the projects' clocks may pause and resume several times.

Given the complexity of our problem, we do not find a simple and general ($n > 2$) full characterization of the optimal mechanism in the asymmetric case. In our examples with two projects, the problem boils down to finding one point, $(z_1(\underline{c}_2), z_2(\underline{c}_1))$, with respect to one crucial tradeoff. Naturally, the number of relevant tradeoffs increases with the number of projects. Therefore unfortunately, optimization with a larger set of projects quickly loses tractability.

## IV Discussion

With our model as a starting point, there are several interesting modifications. In this section, we address the most natural alternative models or extensions.

**$\mathbf{v}_i$ as private information, potentially correlated with $\mathbf{c}_i$** - The designer can neglect asking for $v_i$ directly since no meaningful non-babbling equilibria in the $v_i$-dimension exist. If the conditional density of $v_i|c_i$ has full support, project $i$ cannot credibly announce being a "high" type, say $\overline{v}_i$. If we slightly change the regularity assumption such that $\mathbb{E}[v_i|c_i] - c_i - \frac{F(c_i)}{f(c_i}$ must be strictly increasing, our results generalize by exchanging the previously commonly known $v_i$ with $\mathbb{E}[v_i|c_i]$. This regularity condition mildly restricts the degree of positive correlation.

**Interdependent types** - We can interpret the symmetric case as a setting in which identical projects are provided at individual costs. Hence, one may wonder about a setting in which projects only draw an imperfect signal about the cost, which finally depends on other projects' signals as well. In a clock auction in such an environment, active projects update their belief about the cost whenever a project drops out. Moreover, the designer learns this information as well. Therefore the design of the optimal mechanism crucially depends on the information structure. This analysis is left for a follow-up paper.

### IV.i Residual money

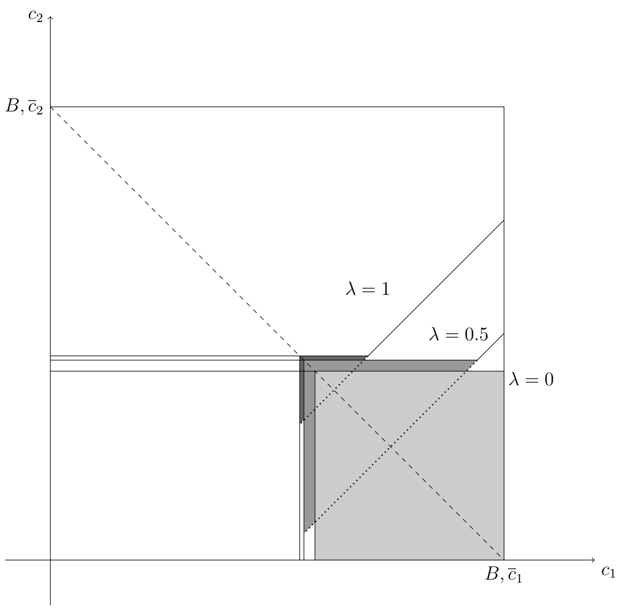
Whether it is reasonable to assume that the designer values residual money depends on the setting. In Ensthaler and Giebe (2014a), money does not enter the objective function, only the constraints. To clarify the relation to their paper, we introduce a linear weighting $\lambda \in [0, 1]$ of residual money, and provide comparative statics on parameter $\lambda$. The objective function can be rewritten as in (2.3),

$$\max_{\{z_i\}_{i\in I}} \mathbb{E}_{\mathbf{c}} \left[\sum_i \mathbb{I}(c_i \leq z_i(\mathbf{c}_{-i})) \left(v_i - \lambda \left(c_i + \frac{F_i(c_i)}{f_i(c_i)}\right)\right)\right]$$
$$\text{s.t.}$$
$$\sum_{i\in I} \mathbb{I}(c_i \leq z_i(\mathbf{c}_{-i})) z_i(\mathbf{c}_{-i}) \leq B \quad \forall \mathbf{c} \in C.$$

This objective function highlights one difference to the original setting. Instead of $\boldsymbol{\zeta}^{**}$-exclusive the optimal mechanism is $\boldsymbol{\zeta}^{**}_\lambda$-exclusive: Define $\psi_{i,\lambda}(c) = v_i - \lambda(c + \frac{F_i(c)}{f_i(c)})$ as the $\lambda$-adjusted virtual surplus and define the vector $\boldsymbol{\zeta}^{**}_\lambda$ with $i$-the element $z^{**}_{i,\lambda} = \min\{\bar{c}_i, \psi^{-1}_{i,\lambda}(0)\}$.

It can be shown that the other properties that are sufficient to allow a DA-auction implementation continue to hold. In fact, the optimal allocation in the symmetric case remains unchanged if $\boldsymbol{\zeta}^{**}_\lambda = (\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_n)$ for all $\lambda \in [0, 1]$, i.e., when the original optimal mechanism did not exclude any cost types. For any combination of cost supports and values, there exists a sufficiently small $\lambda' > 0$ such that the designer's ranking over projects is lexicographic. In other words, $\lambda'$ must be sufficiently small such that no $\lambda'$-weighted difference in cost can offset any difference in values.

FIGURE 2.8: Decreasing $\lambda$ augments the quantity-quality tradeoff: The gray areas, where the project with lower $\lambda$-adjusted virtual surplus is implemented, increases.




In the asymmetric case, however, the quantity-quality tradeoff is affected as well. To illustrate how the optimal allocation varies when $\lambda$ is perturbed, we consider the example again, see Figure 2.8. A lower $\lambda$ means that the designer prefers the high-value project 1 for higher cost reports relative to the low-value project 2 for a given cost report. This difference is illustrated by a right-shift in the diagonal that represents the loci such that both projects have equal ($\lambda$-adjusted) virtual surplus.

Reducing the weight of residual money increases the measure of cost reports for which the optimal mechanism implements project 2 despite project 1 having the larger $\lambda$-adjusted virtual surplus. Thus changing $\lambda$ directly affects the quantity-quality tradeoff. As illustrated in Figure 2.8, reducing $\lambda$ means that in the optimal mechanism the cutoffs at which both projects are greenlighted moves southeast, thus reducing the probability to greenlight both projects. The reason is that for lower $\lambda$ a higher weight is placed on the high-value project 1.

## V Conclusion

Despite their importance, knapsack problems with private information have been somewhat overlooked by the economics literature. We examine a setting in which a budget-constrained procurer faces privately-informed sellers under ex-post constraints. Amongst many possible economic problems, this setting particularly applies to development funds, which are typically endowed with a fixed budget and want to finance both many projects and projects of high quality. Such problems often entail relationships in which sellers can renege on the terms of the agreement ex-post. To avoid nondelivery, shelving the project or costly renegotiation, it is appropriate to impose ex-post constraints on the agents' participation. For such settings, we have shown that a subset of DA auctions constitutes the class of optimal deterministic strategyproof mechanisms.

An optimal mechanism is described by a set of cutoff functions: All projects that report costs below their cutoff are greenlighted and receive a transfer equal to the cutoff. These cutoff functions are weakly increasing in other projects' costs, which means that the optimal allocation rule has substitutes: Given a project is implemented for some cost vector, it is also implemented when, all else being equal, the cost of a rival project is increased. Moreover, we show that the optimal allocation rule has non-bossy winners: A project that is implemented cannot affect the allocation without changing its own allocation status. In particular, if two different realizations of the cost vector lead to the same allocation, then the cutoffs of conducted projects only vary in the costs of projects not conducted. Finally, the optimal allocation rule excludes all projects with negative "virtual surplus" from the allocation.

These properties allow for a characterization as a deferred acceptance (DA) auction, introduced by Milgrom and Segal (2014). The DA auction representation provides a simple implementation via descending-clock auctions, which are easy to understand and usable in practice. In addition, DA auctions have attractive properties regarding incentive compatibility which make the prediction of equilibrium play more robust.

We fully describe the optimal allocation and the corresponding descending-clock auction in an environment in which projects are ex-ante symmetric. The optimal mechanism is monotone in the sense that the cheapest projects are greenlighted and all projects conducted receive the same transfer. This transfer either corresponds to the lowest cost among redlighted projects or the budget is distributed equally. The equivalent clock auction features a single price clock that continuously descends until all active projects can be financed.

For asymmetric environments, in which values and/or cost distributions differ, we demonstrate a novel tradeoff between quantity and quality of the greenlighted projects. The designer values both quantity and quality of the projects: She prefers projects with high virtual surplus over projects with low virtual surplus and she prefers more projects over fewer projects. In models in which the designer wants to procure a fixed number of projects, she would always choose the projects with the highest virtual surpluses. If quantity is endogenously determined by the mechanism, as in our setup, it is ex-ante not always desirable to conduct the best projects. When the best projects are always conducted, incentive compatibility would force the designer to reduce the expected number of greenlighted projects. This insight entails a consequence for the corresponding descending-clock auction. Clocks not only run asynchronously, but also periodically have to stop for certain projects.

Other interesting extensions are left for future research, for example, multiple projects per agent or projects that are complements instead of perfect substitutes. For practitioners, a simple approximately optimal mechanism may be of great value. The characterization of the optimal mechanism as a DA auction sheds light on how to construct such an approximately optimal mechanism. Halting clocks should be a key feature for the corresponding clock auction in asymmetric environments. However, we showed that the optimal strategyproof mechanism is not detail-free.

In conclusion, our methodological approach contributes to a better understanding of a class of relevant problems and opens the door for future research in this area. Furthermore, we provide an elegant indirect mechanism that can be easily implemented in practice.

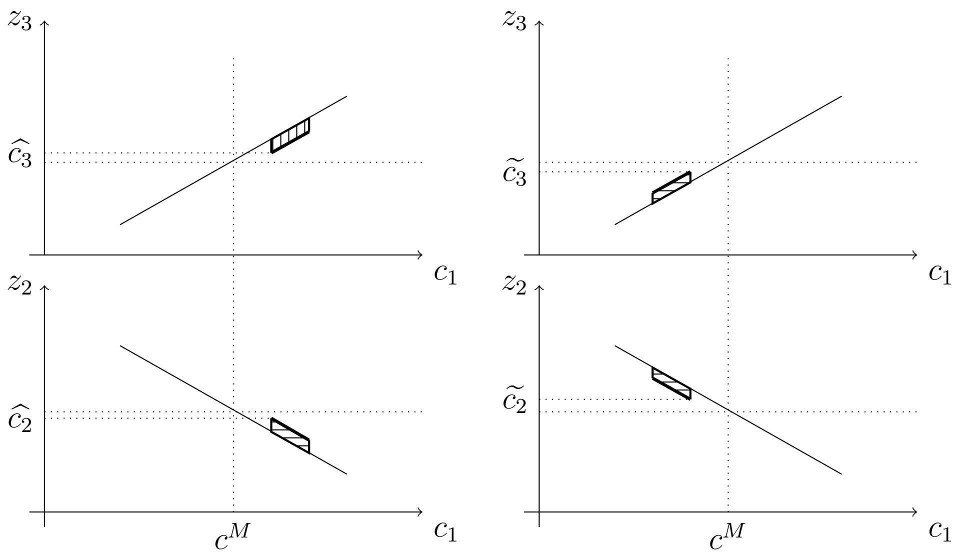
# VI Appendix

## VI.A Properties of optimal mechanisms: General proofs

**Lemma 2.6.** *The optimal mechanism has substitutes,*

$$z_i(\widetilde{c}_j, \mathbf{c}_{-i-j}) \geq z_i(\widehat{c}_j, \mathbf{c}_{-i-j}) \textit{ for almost every } \widetilde{c}_j > \widehat{c}_j \textit{ and } \mathbf{c}_{-i-j} \in C_{-i-j}. \quad (2.5)$$

Figure 2.9: Continuous decrease / increase.




(A) Intuition for the hat deviation.

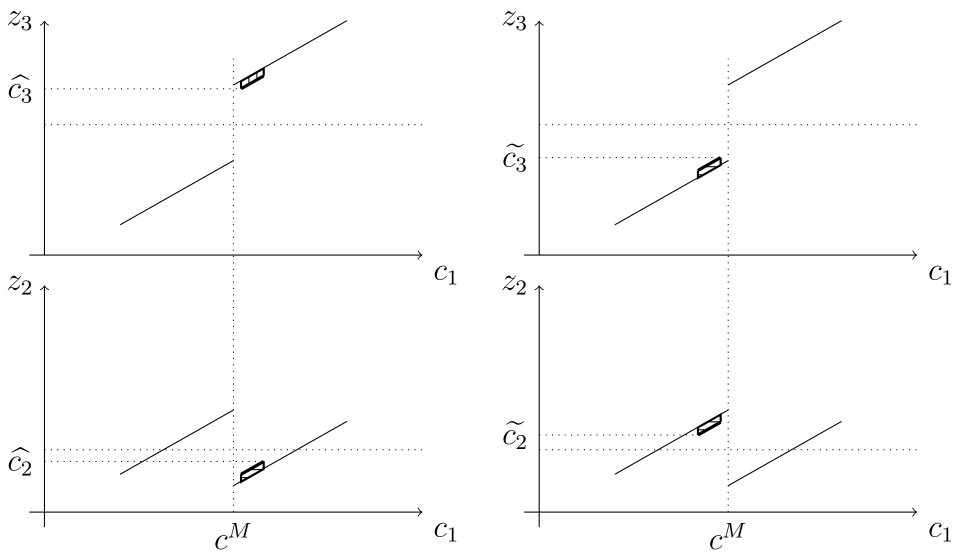
(B) Intuition for the tilde deviation.

*Proof.* Suppose to the contrary that somewhere $z_2$ is decreasing in $c_1$. Then there exist some ${c_1}^M$ and $\eta > 0$ such that $z_2(\underline{\underline{c_1}}, \mathbf{c}_{-1-2}) > z_2(\overline{\overline{c_1}}, \mathbf{c}_{-1-2})$ for all $\underline{\underline{c_1}} \in ({c_1}^M - \eta, {c_1}^M)$, for all $\overline{\overline{c_1}} \in ({c_1}^M, {c_1}^M + \eta)$, and for all $\mathbf{c}_{-1-2} \in \chi_{-1-2} \subset C_{-1-2}$, and $\chi_{-1-2}$ has positive Lebesgue-measure.

With more than two projects, the simple deviation of the two-project case - flattening the decreasing cutoff - is not necessarily feasible. It may be the case that other projects' cutoff functions are strictly increasing in $c_1$ over the same region and that for some cost vectors these cutoffs have to be paid along $z_2$. Then simply flattening $z_2$ could violate the budget constraint.

Suppose no other cutoff function is increasing while $z_2$ is decreasing. Then the decrease of $z_2$ cannot be optimal and flattening $z_2$ increases the designer's payoff much in the same way as in the two-project-case. Otherwise, pick a subset of

Figure 2.10: Jump decrease / increase.




(A) Intuition for the hat deviation.

(B) Intuition for the tilde deviation.

$\widehat{\chi}_1 \subset ({c_1}^M, {c_1}^M + \eta)$ (with pos. Lebesgue-measure) such that w.l.o.g. project 3's cutoff increases in $c_1$ in the analogous sense to the decrease of $z_2$ defined above - for cost vectors where both project 2 and project 3 are eventually greenlighted, i.e., $z_2$ and $z_3$ both need to be paid.

The set

$$\widehat{\Xi}_{23}(c_1, \mathbf{c}_{-1-2-3}, \delta) = \{(c_2, c_3) | c_2 \in (z_2(c_1, c_3, \mathbf{c}_{-1-2-3}), z_2(c_1, c_3, \mathbf{c}_{-1-2-3}) + \delta];$$
$$c_3 \in (z_3(c_1, c_2, \mathbf{c}_{-1-2-3}) - \delta, z_3(c_1, c_2, \mathbf{c}_{-1-2-3})]\}$$

must have positive measure on $\mathbb{R}^2$ for all $c_1 \in \widehat{\chi}_1$ and for any $\mathbf{c}_{-1-2-3} \in \chi_{-1-2-3}$, where $\chi_{-1-2-3}$ is a set with positive Lebesgue measure where the cutoff of project 2 is decreasing while the cutoff of project 3 is increasing. It is the set of $(c_2, c_3)$ tuples, where $c_2$ just exceeds $z_2$ by no more than $\delta$, while $c_3$ lies just below $z_3$ by no more than $\delta$ - given $\mathbf{c}_{-1-2-3}$ and $c_1$. By $\widehat{\Xi}^2_{23}(c_1, \mathbf{c}_{-1-2-3}, \delta)$ we denote the set of project 2 components of tuples in the set $\widehat{\Xi}_{23}(c_1, \mathbf{c}_{-1-2-3}, \delta)$, and similarly for project 3.

Now deviate from the candidate mechanism in setting

$$\begin{aligned}
\widehat{z_2}(c_1, c_3, \mathbf{c}_{-1-2-3}) &:= z_2(c_1, c_3, \mathbf{c}_{-1-2-3}) + \delta \\
\widehat{z_3}(c_1, c_2, \mathbf{c}_{-1-2-3}) &:= z_3(c_1, c_2, \mathbf{c}_{-1-2-3}) - \delta \\
&\text{for all} \\
c_1 &\in (\widehat{c_1}, \widehat{c_1} + \varepsilon) \\
c_2 &\in \widehat{\Xi}^2_{23}(c_1, \mathbf{c}_{-1-2-3}) \\
c_3 &\in \widehat{\Xi}^3_{23}(c_1, \mathbf{c}_{-1-2-3}) \\
\mathbf{c}_{-1-2-3} &\in \widehat{\chi}_{-1-2-3} \subset \chi_{-1-2-3}.
\end{aligned}$$

We call this deviation the *hat* deviation. The intuition for this deviation is the following. For an $\varepsilon$-environment of $c_1$ to the right of $c_1^M$ (i.e., $\widehat{c_1} > c_1^M$), increase the decreasing cutoff $z_2(c_1, c_3, \mathbf{c}_{-1-2-3})$ by $\delta$ for all $c_3$ that drop out of the allocation if $z_3(c_1, c_2, \mathbf{c}_{-1-2-3})$ (at $c_2$) is decreased by $\delta$. Likewise only increase $z_3(c_1, c_2, \mathbf{c}_{-1-2-3})$ by $\delta$ for those $c_2$ that are additionally greenlighted if $z_2(c_1, c_3, \mathbf{c}_{-1-2-3})$ is increased by $\delta$. Therefore if the deviation changes the allocation, project 2 is now greenlighted whereas project 3 is not.

This deviation is feasible. Remember that there must be enough budget to pay both $z_2$ and $z_3$ - otherwise flattening $z_2$ would have been possible. But then there is enough budget for $z_2 + \delta$ and $z_3 - \delta$.

Now define

$$\begin{aligned}
\widehat{c_2} &:= \sup_{c_1, \mathbf{c}_{-1-2-3}} \widehat{\Xi}^2_{23}(c_1, \mathbf{c}_{-1-2-3}) \\
\widehat{c_3} &:= \inf_{c_1, \mathbf{c}_{-1-2-3}} \widehat{\Xi}^3_{23}(c_1, \mathbf{c}_{-1-2-3}) \\
&\text{s.t.} \\
c_1 &\in (\widehat{c_1}, \widehat{c_1} + \varepsilon) \\
\mathbf{c}_{-1-2-3} &\in \widehat{\chi}_{-1-2-3}.
\end{aligned}$$

In words, to bound the change in payoff we let $\widehat{c_2}$ be the highest cost type gained by the deviation and we let $\widehat{c_3}$ be the lowest cost type lost by the deviation. Then the change in payoff for the hat deviation is bounded in the following way:

$$\widehat{\Delta} > (\psi_2(\widehat{c_2}) - \psi_3(\widehat{c_3}))* \int_{\widehat{\chi}_{-1-2-3}} \int_{\widehat{c_1}}^{\widehat{c_1}+\varepsilon} \int_{\widehat{\Xi}^2_{23}(c_1,\mathbf{c}_{-1-2-3})} \int_{\widehat{\Xi}^3_{23}(c_1,\mathbf{c}_{-1-2-3})} 1dF_3(\cdot)dF_2(\cdot)dF_1(\cdot)dF_{-1-2-3}(\cdot).$$

If $\widehat{\Delta} > 0$, we have found a profitable deviation. If not, then consider the following *tilde* deviation.

Analogously to $\widehat{\Xi}_{23}$ we define the set

$$\widetilde{\Xi}_{23}(c_1, \mathbf{c}_{-1-2-3}, \delta) = \{(c_2, c_3)|c_2 \in (z_2(c_1, c_3, \mathbf{c}_{-1-2-3}) - \delta, z_2(c_1, c_3, \mathbf{c}_{-1-2-3})]; \\ c_3 \in (z_3(c_1, c_2, \mathbf{c}_{-1-2-3}), z_3(c_1, c_2, \mathbf{c}_{-1-2-3}) + \delta]\}$$

which again must have positive measure.

Now, we deviate for an $\varepsilon$-environment to the left of $c_1^M$ (i.e., $\widetilde{c}_1 < c_1^M$). But instead of increasing $z_2$ and decreasing $z_3$, we increase $z_3$ and decrease $z_2$:

$$\begin{aligned} \widetilde{z_2}(c_1, c_3, \mathbf{c}_{-1-2-3}) &:= z_2(c_1, c_3, \mathbf{c}_{-1-2-3}) - \delta \\ \widehat{z_3}(c_1, c_2, \mathbf{c}_{-1-2-3}) &:= z_3(c_1, c_2, \mathbf{c}_{-1-2-3}) + \delta \\ &\text{for all} \\ c_1 &\in (\widetilde{c_1} - \varepsilon, \widetilde{c_1}) \\ c_2 &\in \widetilde{\Xi}^2_{23}(c_1, \mathbf{c}_{-1-2-3}) \\ c_3 &\in \widetilde{\Xi}^3_{23}(c_1, \mathbf{c}_{-1-2-3}) \\ \mathbf{c}_{-1-2-3} &\in \widetilde{\chi}_{-1-2-3} \subset \chi_{-1-2-3}. \end{aligned}$$

The relevant bounds to bound the payoff are then given by

$$\begin{aligned} \widetilde{c_2} &:= \inf_{c_1, \mathbf{c}_{-1-2-3}} \widetilde{\Xi}^2_{23}(c_1, \mathbf{c}_{-1-2-3}) \\ \widetilde{c_3} &:= \sup_{c_1, \mathbf{c}_{-1-2-3}} \widetilde{\Xi}^3_{23}(c_1, \mathbf{c}_{-1-2-3}) \\ &\text{s.t.} \\ c_1 &\in (\widetilde{c_1} - \varepsilon, \widetilde{c_1}) \\ \mathbf{c}_{-1-2-3} &\in \widetilde{\chi}_{-1-2-3}. \end{aligned}$$

And this gives the following bound for the payoff

$$\widetilde{\Delta} > (\psi_2(\widetilde{c_3}) - \psi_3(\widetilde{c_2}))*$$
$$\int_{\chi_{-1-2-3}} \int_{\widetilde{c_1}-\varepsilon}^{\widetilde{c_1}} \int_{\widetilde{\Xi}^2_{23}(c_1,\mathbf{c}_{-1-2-3})} \int_{\widetilde{\Xi}^3_{23}(c_1,\mathbf{c}_{-1-2-3})} 1 dF_3(\cdot)dF_2(\cdot)dF_1(\cdot)dF_{-1-2-3}(\cdot).$$

By appropriately choosing $\delta$, $\widehat{\Xi}_{-1-2-3}$, and $\widetilde{\Xi}_{-1-2-3}$, we can ensure that $\widehat{c_3} > \widetilde{c_3}$ and $\widehat{c_2} < \widetilde{c_2}$. This follows simply from the notion of increasing/decreasing cutoffs and is illustrated in Figures 2.9 and 2.10. Therefore $\widehat{\Delta} \leq 0$ implies $\widetilde{\Delta} > 0$. Consequently, there is always a profitable deviation and our candidate mechanism could not have been optimal. □

**Lemma 2.8.** *For any cost vectors $(\mathbf{c}_G, \mathbf{c}_R) \in C$ and $(\mathbf{c}'_G, \mathbf{c}_R) \in C$ such that $G = \gamma(\mathbf{c}_G, \mathbf{c}_R) = \gamma(\mathbf{c}'_G, \mathbf{c}_R)$ and $R = I \setminus \gamma(\mathbf{c}_G, \mathbf{c}_R)$, the optimal cutoff function $z_g$ for all $g \in G$ is (almost everywhere) independent of the costs of all greenlighted projects $\mathbf{c}_G$. That is,*

$$z_g(\mathbf{c}_{G-g}, \mathbf{c}_R) = z_g(\mathbf{c}'_{G-g}, \mathbf{c}_R),$$

*for all $\mathbf{c}_{G-g}$ and $\mathbf{c}'_{G-g}$ such that $G$ is the set of greenlighted agents.*

*Proof.* Take any feasible candidate mechanism with any set of increasing cutoff functions $\{z_i\}_{i\in I}$ for any individual project. Assume that for some cost vectors with positive Lebesgue-measure, only all projects in set $G \subseteq I$ are executed while all projects of set $R$ are not conducted. Therefore there exists a set, $C^G_R$, with positive Lebesgue-measure containing the part of the cost vector for the projects in set $R$ such that the partition $\{G, R\}$ is induced given some $\mathbf{c}$ where the redlighted projects have costs $\mathbf{c}_R \in C^G_R$. Then $a^G_i(\mathbf{c}_R)$ according to the following definition

$$\begin{aligned} a^G_i(\mathbf{c}_R) = \max\{c_i | \exists \mathbf{c}_{G-i} : c_i \leq z_i(\mathbf{c}_{G-i}, \mathbf{c}_R), \\ \text{and } c_g \leq z_g(\mathbf{c}_{G-j}, \mathbf{c}_{-G}) \forall g \in G, \\ \text{and } c_r > z_r(\mathbf{c}_G, \mathbf{c}_{-G-r}) \forall r \in R\} \end{aligned} \tag{2.12}$$

exists for all $i \in G$ given $\mathbf{c}_R \in C^G_R$. In words, $a^G_i(\mathbf{c}_R)$ is the highest cost of project $i$ such that, given some cost vector $\mathbf{c}_R$ of projects that are not executed, there exists some vector $\mathbf{c}_{G-i}$ of costs of competing projects that induces a cutoff $z_i(\mathbf{c}_{G-i}, \mathbf{c}_{-G})$ above said cost while each element $c_g$ of the vector $\mathbf{c}_{G-i}$ is lower than the cutoff induced by $a_i^G(\mathbf{c}_R)$ and the elements of the cost vectors $\mathbf{c}_R$ and $\mathbf{c}_{G-i-g}$,

$$\forall g \in G \setminus \{i\},\;\; c_g \leq z_g(\mathbf{c}_R, \mathbf{c}_{G-i-g}, a_i^G(\mathbf{c}_R)).$$

Simultaneously, it must hold that these costs induce a cutoff such that no project $r \in R$ is conducted

$$\forall r \in R,\;\; c_r > z_r(\mathbf{c}_{R-r}, \mathbf{c}_{G-i}, a_i^G(\mathbf{c}_R)).$$

Moreover, we can replace any function $z_i$ with a left-continuous function that is identical up to a set of points with Lebesgue-measure zero. Hence, the limit is reached from below and there exists at least one cost vector $(\widehat{\mathbf{c}}_{-i}, a_i^G(\mathbf{c}_R))$ where $G$ is the set of executed projects and $a_i^G(\widehat{\mathbf{c}}_R) = z_i(\widehat{\mathbf{c}}_{-i})$ holds. Now, notice that

$$\widehat{c}_g \leq a_g^G(\widehat{\mathbf{c}}_R)\; \forall g \in G \setminus \{i\},$$

because, given $\widehat{\mathbf{c}}_R$, there cannot exist a cost vector where only all projects in $G$ are executed and the cost of project $g$ exceeds $a_g^G(\widehat{\mathbf{c}}_R)$ by its construction. Moreover, we have established that every cutoff function $z_i$ is weakly increasing in each argument. Thus,

$$a_i^G(\widehat{\mathbf{c}}_R) = z_i(\widehat{\mathbf{c}}_{-i}) \leq z_i(a_{G-i}^G(\widehat{\mathbf{c}}_R), \widehat{\mathbf{c}}_R), \tag{2.13}$$

where $a_{G-i}^G$ is the vector of all $a_g^G$ defined according to (2.12) except $a_i^G$. This inequality tells us that, whenever some vector $(\mathbf{c}_R, \mathbf{c}_{G-i}) \geq (\widehat{\mathbf{c}}_R, a_{G-i}^G(\widehat{\mathbf{c}}_R))$[15] realizes, a sufficient condition for project $i \in G$ to be executed is $c_i \leq a_i^G(\widehat{\mathbf{c}}_R)$.

The same logic also applies to all projects in $G$ other than $i$. Therefore at least all projects $g \in G$ are conducted whenever a cost vector realizes such that $c_g = a_g^G(\mathbf{c}_R)$.[16] Consequently, the budget constraint requires that

$$\sum_{g \in G} z_g(a_{-g}^G(\mathbf{c}_R), \mathbf{c}_R) \leq B. \tag{2.14}$$

Furthermore, given $\mathbf{c}_R$, for all projects $g \in G$, $z_g(\mathbf{c}_{-G}, \mathbf{c}_R) = a_g^G(\mathbf{c}_R)$ if $\mathbf{c}_{G-g} \leq a_{G-g}^G(\mathbf{c}_{-G})$. That is, the cutoffs are constant given the cost vector of redlighted projects.

[15] When $\mathbf{x}$ and $\mathbf{y}$ are vectors, $\mathbf{x} \geq \mathbf{y}$ means that every element $x_i$ of $\mathbf{x}$ weakly exceeds the corresponding element $y_i$ of $\mathbf{y}$.

[16] $a_i^G(\mathbf{c}_R)$ is only defined if $C^G \neq \emptyset$ and $\mathbf{c}_R \in C_R^G$, but this does not hinder the proof.

Suppose to the contrary that $z_i(\mathbf{c}_{-i}) < a_i(\mathbf{c}_R)$ for some $i \in G$ and for all $\mathbf{c}_{-i} \in \Xi \subset C_{-i}^G$ with $\Xi$ having positive Lebesgue measure.

Define $\Xi(\mathbf{c}_{G-i-j}, \mathbf{c}_R) \subset [0, \bar{c}_j]$ where $z_i(\mathbf{c}_{G-i-j}, c_j, \mathbf{c}_R) < a_i^G(\mathbf{c}_R)$ for all $c_j \in \Xi(\mathbf{c}_{G-i-j}, \mathbf{c}_R)$. For any $\mathbf{c}_{G-i-j} \leq a_{-i-j}^G(\mathbf{c}_R)$, let

$$z_i^{\Xi}(\mathbf{c}_{G-i-j}, \mathbf{c}_R) := \max_{c_j \in \Xi(\mathbf{c}_{G-i-j}, \mathbf{c}_R)} z_i(\mathbf{c}_{G-i-j}, c_j, \mathbf{c}_R)$$

By (2.14), changing the mechanism to

$$z_i(\mathbf{c}_{G-i,-j}, c_j, \mathbf{c}_R) = a_i^G(\mathbf{c}_R), \quad \forall c_j \leq a_j^G(\mathbf{c}_R)$$

does not violate the budget constraint. This deviation increases the payoff conditional on $\mathbf{c}_R$ by

$$\Delta > \int_{\Xi_{-j}} \Pr(c_j \in \Xi(\mathbf{c}_{G-i-j}, \mathbf{c}_R)) \int_{z_i^{\Xi}(\mathbf{c}_{G-i-j}, \mathbf{c}_R)}^{a_i^G(\mathbf{c}_R)} \psi_i(c) dF_i(c) dF_{-i-j}(\mathbf{c}_{-i-j}) > 0.$$

Given that $\Xi$ has positive Lebesgue-measure, this deviation also strictly increases the unconditional payoff. □

## VI.B Constructing a scoring function: Proof of Proposition 2.11

To prove Proposition 2.11, it is helpful to consider the following lemmata. While Lemma 2.8 (non-bossy winners) is a statement that conditions on a fixed allocation, it also has implications on the cutoffs resulting from different cost vectors that induce different allocations.

**Lemma 2.16.** *Take any mechanism and any two cost vectors $\mathbf{c} \neq \widehat{\mathbf{c}}$ that induce partitions $\{G, R\}$ and $\{\widehat{G}, \widehat{R}\}$, respectively. Then*

$$\mathbf{c}_{R \cup \widehat{R}} = \widehat{\mathbf{c}}_{R \cup \widehat{R}}$$
$$\mathbf{c}_{G \cap \widehat{G}} \neq \widehat{\mathbf{c}}_{G \cap \widehat{G}}$$

*implies*

$$G = \widehat{G}$$
$$R = \widehat{R},$$

*that is,* $\mathbf{c}$ *and* $\widehat{\mathbf{c}}$ *induce the same allocation.*

*Proof.* Given cost vector $\mathbf{c}$, define a new cost vector $\mathbf{c}'$, where $c_i' = \min\{c_i, \widehat{c}_i\}$ for all $i \in G \cap \widehat{G}$ and $\mathbf{c}'_{R\cup\widehat{R}} = \mathbf{c}_{R\cup\widehat{R}}$. By Lemma 2.8, $\mathbf{c}'$ induces allocation $\{G, R\}$. Similarly, a perturbation of cost vector $\widehat{\mathbf{c}}$ in the same way with $\widehat{c}_i' = \min\{c_i, \widehat{c}_i\}$ for all $j \in G \cap \widehat{G}$ and $\widehat{\mathbf{c}}'_{R\cup\widehat{R}} = \widehat{\mathbf{c}}_{R\cup\widehat{R}}$ must induce allocation $\{\widehat{G}, \widehat{R}\}$. But $\mathbf{c}' = \widehat{\mathbf{c}}'$ by construction. Hence, $G = \widehat{G}$ and $R = \widehat{R}$. □

**Lemma 2.17.** *Take any mechanism and any two cost vectors* $\mathbf{c} \neq \widetilde{\mathbf{c}}$ *that induce partitions* $\{G, R\}$ *and* $\{\widetilde{G}, \widetilde{R}\}$*, respectively. Then*

$$z_i(\mathbf{c}_{G\cap\widetilde{G}}, \mathbf{c}_{R\cup\widetilde{R}}) = z_i(\widetilde{\mathbf{c}}_{G\cap\widetilde{G}}, \mathbf{c}_{R\cup\widetilde{R}})$$
$$z_j(\widetilde{\mathbf{c}}_{G\cap\widetilde{G}}, \widetilde{\mathbf{c}}_{R\cup\widetilde{R}}) = z_j(\mathbf{c}_{G\cap\widetilde{G}}, \widetilde{\mathbf{c}}_{R\cup\widetilde{R}})$$

*for all* $i \in G$ *and for all* $j \in \widetilde{G}$*, respectively.*

*Proof.* By Lemma 2.16, the vector $(\widetilde{\mathbf{c}}_{G\cap\widetilde{G}}, \mathbf{c}_{R\cup\widetilde{R}})$ leads to allocation $\{G, R\}$ and the vector $(\mathbf{c}_{G\cap\widetilde{G}}, \widetilde{\mathbf{c}}_{R\cup\widetilde{R}})$ leads to allocation $\{\widetilde{G}, \widetilde{R}\}$. The rest follows directly from Lemma 2.8 (non-bossy winners). □

Having established these properties we can prove Proposition 2.11 by induction. We construct a DA scoring function for each iteration. Conditional on all previous iterations having been constructed correctly, we can demonstrate how to construct an appropriate scoring function for any iteration.

**Proposition 2.11.** *Any optimal mechanism has a DA auction representation and can be implemented with a descending-clock auction.*

*Proof.* This proof is structured as follows. First, we construct scoring functions for each iteration of the DA auction. Then we explain how the zeros of the scoring functions are derived. Finally we show by induction that the constructed DA auction implements the same allocation as the underlying $z$-mechanism.

### Scoring functions

First, we introduce some notation. Let $A_t$ be the set of active projects in iteration $t$ and let $O_t := I \setminus A_t$ be the set of inactive projects ($O$ as in "out"). Let $O_tj := O_t \cup \{j\}$ be the union of dropped out projects and some individual project $j$.

Fix an optimal $z$-mechanism and consider the corresponding DA auction with scoring functions $\{s_i^A\}_{A\subset I, i\in A}$

$$s_i^A(c_i, \mathbf{c}_O) = \begin{cases} 0 & \text{if } c_i \leq a_i^A(\mathbf{c}_O), \\ c_i + \sum\limits_{i\neq j}^{j\in A} b_{Oi}^j(c_i, \mathbf{c}_O) & \text{otherwise,} \end{cases} \tag{2.15}$$

where $a_i^A(\mathbf{c}_O)$ is defined as in (2.12) and $b_{Oi}^j(c_i, \mathbf{c}_O)$ is defined as

$$\begin{aligned} b_{Oi}^j(c_i, \mathbf{c}_O) :=& \max\left\{c_j : \exists \tilde{\mathbf{c}}_{-Oi-j} : R = Oi | c_i, \mathbf{c}_O\right\} \\ :=& \max\Big\{c_j : \exists \tilde{\mathbf{c}}_{-Oi-j} : c_i > z_i(c_j, \tilde{\mathbf{c}}_{-Oi-j}, \mathbf{c}_O), \\ & \quad \text{and } c_o > z_o(c_i, c_j, \tilde{\mathbf{c}}_{-Oi-j}, \mathbf{c}_{O-i}) \forall o \in O, \\ & \quad \text{and } c_g \leq z_g(c_i, c_j, \tilde{\mathbf{c}}_{-Oi-j}, \mathbf{c}_O) \forall g \in A \setminus i\Big\}. \end{aligned}$$

In words, $b_{Oi}^j(c_i, \mathbf{c}_O)$ is the highest cost of project $j$ such that given the vector $\mathbf{c}_{Oi}$ the corresponding $z$-mechanism implements the allocation partition $R = Oi$ and $G = A \setminus i$ for some realization of the cost vector $\tilde{\mathbf{c}}_{-Oi-j}$.

**Zeros of the scoring functions**

Suppose the DA auction ends in the $t$-th iteration. Then all projects $i \in A_t$ have score $s_i^{A_t} = 0$ and the cost vector must induce $G = A_t$ in the underlying $z$-mechanism. By non-bossiness of winners, cutoffs of projects in $G$ are constant in the part of the cost vector $\mathbf{c}_{A_t}$ for all cost vectors inducing the same allocation.

Therefore we can characterize the zeros of the scoring function by a threshold and $s_i^{A_t} = 0$ whenever project $i$'s cost is below this threshold. The threshold is given by $a_i^{A_t}(\mathbf{c}_O)$ as defined in (2.12). Notice that $c_i \leq a_i^{A_t}(\mathbf{c}_O)$ implies that project $i$ is not eliminated in the $t$-th iteration, even if other projects exceed their threshold. This implication does not rule out permissible $z$-mechanisms. Conditional on $\mathbf{c}_{O_t}$, some projects exceeding their threshold can at most lead to a higher cutoff for project $i$ due to monotonicity.

Further notice that if $c_i > a_i^{A_t}(\mathbf{c}_O)$, there always exist cost vectors with $\mathbf{c}_{O_t}$ for previously eliminated projects that induce $G = A_t \setminus \{i\}$. For example, all cost vectors with $c_j \leq a_j^{A_t}(\mathbf{c}_{O_t})$ for all $j \in A_t \setminus \{i\}$ induce that allocation. However, this condition is sufficient for $G = A_t \setminus \{i\}$ but not necessary. There can be other cost vectors inducing the same allocation.

**Iteration 1**

If multiple projects have a positive score, it also holds that

$$\text{If } \widehat{\mathbf{c}} \text{ induces } \widehat{R} = \{i\} \text{ then } s_i^I(c_i) > s_j^I(c_j) \text{ for all } j \neq i \tag{2.16}$$

The meaning of $\widehat{R} = \{i\}$ is that $\widehat{c}_i > z_i(\widehat{\mathbf{c}}_{-i})$ and $\widehat{c}_j \leq z_j(\widehat{\mathbf{c}}_{-j})$. Hence, by construction

$$\widehat{c}_j \leq z_j(\widehat{\mathbf{c}}_{-j}) \leq b_i^j(\widehat{c}_i) \tag{2.17}$$

as $b_i^j(\widehat{c}_i)$ is the highest cutoff $z_j$ that allows allocation $\widehat{R} = \{i\}$ given $\widehat{c}_i$.

Next, we show

$$\widehat{c}_i > b_j^i(\widehat{c}_j). \tag{2.18}$$

Suppose that the contrary holds, then there exists a vector $\widetilde{\mathbf{c}}_{-i-j}$ such that

$$\widehat{c}_i \leq z_i(\widehat{c}_j, \widetilde{\mathbf{c}}_{-i-j})$$

and allocation $\widetilde{R} = \{j\}$ is implemented. By Lemma 2.17 we know that the cutoffs $z$ are constant in costs of projects $\widehat{G} \cap \widetilde{G} = I \setminus \{i, j\}$. Consequently, we arrive at

$$\widehat{c}_i \leq z_i(\widehat{c}_j, \tilde{\mathbf{c}}_{-i-j}) = z_i(\widehat{c}_j, \widehat{\mathbf{c}}_{-i-j})$$

which means that $i$ is greenlighted for vector $\widehat{\mathbf{c}}$, a contradiction to our initial assumption that $\widehat{\mathbf{c}}$ implements $\widehat{R} = \{i\}$.

Next, we show

$$b_i^k(\widehat{c}_i) \geq b_j^k(\widehat{c}_j) \text{ for all } j \neq i \text{ and } k \neq i, j. \tag{2.19}$$

By definition

$$b_i^k(\widehat{c}_i) = z_k(\widehat{c}_i, \tilde{\mathbf{c}}_{-i-k}) \text{ for some } \tilde{\mathbf{c}}_{-i-k},$$
$$b_j^k(\widehat{c}_j) = z_k(\widehat{c}_j, \dot{\mathbf{c}}_{-j-k}) \text{ for some } \dot{\mathbf{c}}_{-j-k}.$$

Because projects $-i-j-k$ are greenlighted for both cost realizations $(\widehat{c}_k, \widehat{c}_i, \tilde{\mathbf{c}}_{-i-k})$ and $(\widehat{c}_k, \widehat{c}_i, \dot{\mathbf{c}}_{-i-k})$, it follows by Lemma 2.17 that

$$b_i^k(\widehat{c}_i) = z_k(\widehat{c}_i, \tilde{\mathbf{c}}_{-i-k}) = z_k(\widehat{c}_i, \widehat{\mathbf{c}}_{-i-k}),$$
$$b_j^k(\widehat{c}_j) = z_k(\widehat{c}_j, \dot{\mathbf{c}}_{-j-k}) = z_k(\dot{c}_i, \widehat{\mathbf{c}}_{-i-k}).$$

Furthermore, it must hold that $\widehat{c}_i > \dot{c}_i$, otherwise vector $\widehat{\mathbf{c}}$ would not optimally redlight project $i$ while vector $(\widehat{\mathbf{c}}_{-\mathbf{i}}, \dot{c}_i)$ optimally greenlights project $i$. Then by bidder substitutability,

$$b_i^k(\widehat{c}_i) = z_k(\widehat{c}_{-k}) \geq z_k(\dot{c}_i, \widehat{\mathbf{c}}_{-i-k}) = b_j^k(\widehat{c}_j).$$

Combining (2.17), (2.18) and (2.19) leads to (2.16). We have shown that the scoring function eliminates the correct project when $|R| = 1$, i.e., the redlighted project.

Finally, we need to show that if $|R| > 1$, the project removed in the first iteration is redlighted in the allocation implemented by the underlying $z$-mechanism, i.e.,

$$A_1 \setminus A_2 = \{k\} \Rightarrow k \in R.$$

Now take cost vector $\widetilde{\mathbf{c}}$ with allocation $\{\widetilde{G}, \widetilde{R}\}$ and let $i \in \widetilde{G}$ be some greenlighted project and and let $j \in \widetilde{R}$ be some redlighted project, respectively. Since project $j$ is redlighted, it must have cost $\widetilde{c}_j > a_j^I$. Hence there exists some cost vector $\widehat{\mathbf{c}}$ with $\widehat{c}_j = \widetilde{c}_j$ such that $\widehat{R} = \{j\}$. By Lemma 2.17, we can assume $\widehat{c}_i = \widetilde{c}_i$ since $i \in \widetilde{G} \cap \widehat{G}$. As our scoring function correctly matches all cases in which $|R| = 1$, it must be that $s_j(\widetilde{c}_j) > s_i(\widetilde{c}_i)$. Given that we have chosen $i$ and $j$ arbitrarily, we have shown that any project removed in the first iteration must be in the redlighted set, which was to show.

**Iteration 2**

We can show with the same arguments as above, that the previously stated scoring function is correct for $t = 2$ as well. To this end, we inductively rely on the fact that the project $k$ removed in the first iteration is indeed redlighted by the $z$-mechanism - as we have shown above.

**Iteration $t \geq 3$**

With the appropriate scoring functions used in all previous iterations, we can then show that the $t$-th iteration removes the correct project for all cost vectors inducing $|R| = t$ given a $z$-mechanism and otherwise removes some project $i \in A_t$, where $i \in R$, for all cost vectors inducing $|R| > t$. □

## VI.C The symmetric case

**Proposition 2.12.** *Arrange the projects in ascending order of their reported costs, $c_1 \leq c_2 \leq \cdots \leq c_n \leq c_{n+1} := \bar{c}$, and define $z^k := \min\left\{\frac{B}{k}, z^{**}, c_{k+1}\right\}$. In the symmetric case, the cutoff mechanism with $z_i(\mathbf{c}_{-i}) = z^{k^*}$ is the optimal mechanism. The optimal number of accepted projects $k^*$ is given by $k^* := \max\{k | c_k \leq z^k\}$.*

*Proof.* The case $n = 2$ has been proven in Section III.i.

Now, consider $n = 3$. Fix any $c_3$ and any mechanism as candidate for optimality. Either $c_3 > z_3(c_1, c_2)$ or $c_3 \leq z_3(c_1, c_2)$. In the first case, project 3 is not executed and the budget remaining for the other two is still $B$. In the second case, project 3 is executed and the budget remaining for the other two becomes $B - z_3(c_1, c_2)$.

Now, consider deviating to the proposed mechanism only for project 1 and 2. The change in profit looks like a probability weighted sum of terms similar to the two-project case, only that the distributions $F$ are conditional on $c_1$ and $c_2$ being in some interval (that induces $z_3 >$ or $< c_3$) and the budget must be adjusted.

Because log-concavity of $F$ implies log-concavity of $\frac{F(c)-F(a)}{F(b)-F(a)}$ this deviation is always positive like in the case $n = 2$. The same logic can be applied to any $n$, changing any mechanism by selecting two projects and then adjusting their cutoffs in the following way: The budget is shared equally if both projects are executed; if only one project is executed, it has to be the one with higher virtual surplus; never execute projects with negative virtual surplus. Iterating over these steps ultimately arrives at the proposed mechanism which has to be optimal. □

## VI.D Bidder Substitutability and Complementarity

In the main text, we made the crucial assumption that $\underline{c} = 0$. As a consequence, complementaries as in the following example are excluded. The example shows that an optimal mechanism may not have substitutes. When the lower bound of all projects' costs is zero, it is always possible to improve a mechanism that does not have substitutes. The idea of the proof of optimality of bidder substitutability in Appendix A is that it cannot be optimal to decrease $i$'s cutoff to the benefit of increasing $j$'s cutoff when some third project's cost increases from $c_k$ to $c_k' > c_k$, because then it would either be better to raise project $z_j(\cdot, c_k)$ at the cost of lowering $z_i(\cdot, c_k)$ as well or it would be better to raise $z_i(\cdot, c_k')$ at the cost of lowering $z_j(\cdot, c_k')$.

In the following example, this approach is not feasible. Through the lower cost bounds and the values, projects 1 and 2 inherit endogenous complementarities. The designer prefers implementing 1 and 2 together over implementing 3 alone, but once either 1 or 2 becomes too expensive the other project is dropped as well in favor of implementing only project 3.

**Example 2.2.** *Suppose $I = \{1, 2, 3\}$ and $B = 300$. Let the costs be arbitrarily distributed on the following supports:*

$$c_1 \sim [200, 400], c_2 \sim [20, 200], c_3 \sim [290, 300],$$

*and let the values be*

$$v_1 = 700, v_2 = 500, v_3 = 1000.$$

*Let the corresponding optimal mechanism be given by*

$$z_1(c_2, c_3) = \begin{cases} 250 \text{ if } c_2 \leq 50 \\ 0 \text{ otherwise} \end{cases}, z_2(c_1, c_3) = \begin{cases} 50 \text{ if } c_1 \leq 250 \\ 0 \text{ otherwise} \end{cases},$$

$$z_3(c_1, c_2) = \begin{cases} 300 \text{ if } c_2 > 50 \text{ or } c_1 > 250 \\ 0 \text{ otherwise} \end{cases}.$$

Bidder substitutability fails because, e.g., as $c_1$ increases from 249 to 251, project 2 with, say, cost 40 gets dropped from the allocation set. The designer cannot, as in the main text with $\underline{c} = 0$, lower $z_3(40, 249)$ as it is already zero or profitably raise $z_2(251, \cdot)$ at the cost of project 3 as the lower cost bounds prohibit that projects 2 and 3 are ever conducted together and implementing $G = \{3\}$ is preferred to $G' = \{2\}$.

However, it is still possible to construct an implementation with price clocks: All clocks start at the upper bounds. Then (at arbitrary speed) the prices of 1 and 2 decrease to $(250, 50)$. If both projects are still active, the price for project 3 decreases to zero while clocks 1 and 2 halt: 1 and 2 are implemented. If any project $i \in \{1,2\}$ drops out earlier, then the price for $j \neq i$, $j \in \{1,2\}$ drops to zero, while price 3 remains at 300. 3 is implemented.

The next example features another kind of complementatity. In this example project 3 can be a bossy loser. Again, there exists a DA-auction implementation. The lower cost bounds of the (stochastically) identical projects 1 and 2 are too high for both projects to ever be conducted together. The cheaper of the two is greenlighted. Project 3 is then only implemented if enough money remains.

**Example 2.3.** *Suppose $I = \{1,2,3\}$ and $B = 300$. Let the costs be arbitrarily distributed on the following supports:*

$$c_1, c_2 \sim [151, 200], c_3 \sim [50, 300],$$

*and let the values be*

$$v_1 = v_2 = 1000, v_3 = 500.$$

*Let the corresponding optimal mechanism be given by*

$$z_1(c_2, c_3) = c_2, \quad z_2(c_1, c_3) = c_1, \quad z_3(c_1, c_2) = B - \max\{c_1, c_2\}.$$

Suppose $c_2 > c_1$, then project 2 can be a bossy loser: It can increase its cost report without changing its status to the green light and thereby kick project 3 out of the allocation.

While substitutes and non-bossiness are sufficient for an implementation with a DA auction, they are clearly not necessary. From the matching literature, it is apparent that some kind of substitutes condition is needed and non-bossy winners seem to be important for DA implementations. We have constructed a scoring function that implements the exemplary allocations above. However, in the proof of non-bossiness of winners, we need the strong substitutes condition for inequality (2.13).

A weaker substitutes condition, such as our groupwise substitutes, does not suffice for the optimality of non-bossy winners. This condition is satisfied by the examples above and is helpful for the construction of a scoring function.

**Definition 2.18.** An allocation rule $\gamma$ has groupwise substitutes, if $\sum_{g \in G} z_g(c_{-g})$ is increasing in any cost report $c_r$ with $r \notin G$ for all allocation sets $G$ that are admitted by $\gamma$.

# 3. Deterministic mechanisms, the revelation principle, and ex-post constraints

*with Felix Jarman*

## I Introduction

In the analysis of mechanism design problems, economists often restrict attention to deterministic mechanisms. In applications, stochastic mechanisms are often deemed unfair as they require that the mechanism designer has access to a credible randomization device which can be implausible in some environments or, alternatively, may be prone to manipulation. However, restricting attention to deterministic mechanisms is not innocuous.

As shown by Strausz (2003), the classical revelation principle (e.g., Myerson, 1979) does not hold if the environment contains more than one agent. More precisely, there are social choice functions that can be implemented by deterministic indirect mechanisms but that cannot be implemented by a deterministic direct mechanism in which agents truthfully reveal their type. We generalize his formulation of the revelation principle in terms of payoff from the one-agent case to the multiple-agents case under ex-post constraints: Any optimal deterministic mechanism corresponds to a payoff-equivalent feasible truthful direct deterministic mechanism. Hence, when constraints have to hold regardless of the strategy of other players (including "nature"), there is no loss of generality when restricting attention to direct truthful mechanisms in optimal mechanism design.

The failure of the revelation principle is due to the possibility that agents play a mixed-strategy equilibrium in a discrete indirect mechanism. In this equilibrium, a stochastic social choice function can be implemented, even though the outcome function of an indirect mechanism is restricted to be deterministic. While a stochastic deterministic direct mechanism can replicate mixing using a randomization device, a deterministic direct mechanism cannot. In this note, we show that despite the failure of the revelation principle, it is still without loss of generality to neglect indirect mechanisms if the objective is to identify a social choice function that

(a) maximizes the expectation of some objective function over outcomes,

(b) is implementable in dominant strategies, and

(c) satisfies additional constraints (if there are any) ex-post.

We use this result in Jarman and Meisner (2015).

## II Model

A mechanism designer faces is a set of agents $I = \{1, 2, ..., N\}$. Each agent $i \in I$ is privately informed about type $\theta_i$, drawn from type space $\Theta_i$. The type profile $\theta = (\theta_1, \ldots, \theta_N)$ is drawn from $\Theta = \Theta_1 \times \cdots \times \Theta_N$ according to some distribution. The mechanism designer's problem is to select an outcome $x \in X$ to maximize her expected payoff $w(\theta, x)$, while constraints imposed ex-post have to be satisfied.

A deterministic social choice function $f_d$ is a mapping from the set of type profiles into the set of outcomes $X$,

$$f_d : \Theta \to X,$$

while a stochastic social choice function $f_s$ maps type profiles into distributions over outcomes,

$$f_s : \Theta \to \Delta X.$$

We call the set of deterministic social choice functions $F_d$ and the set of stochastic social choice functions $F_s$, where $F_d \subset F_s$.

A deterministic mechanism $M = (S, g)$ consists of a collection of strategy spaces $S = S_1 \times \ldots S_N$ and an outcome function $g$ that maps the strategy profile $s = (s_1, \ldots, s_N) \in S$ into outcomes, $g : S \to X$. We say that the mechanism $M$ implements a potentially stochastic social choice function $f$ if $f(\theta)$ is an equilibrium outcome of the game induced by $M$ and $\theta$. Similarly, $M$ implements

$f$ in dominant strategies if the equilibrium strategies that lead to $f$ are weakly dominant. Let agent $i$'s payoff from playing strategy $s_i$ against strategies $s_{-i}$ in mechanism $M$ be denoted by $u_i^M(s_i, s_{-i})$. Strategy $s_i$ is a dominant strategy if

$$u_i^M(s_i, s_{-i}) \geq u_i^M(s_i', s_{-i}) \quad \forall s_i', s_{-i}. \tag{3.1}$$

The designer generally cannot implement any social choice function but might face some feasibility constraints. We say that the designer faces ex-post constraints if these constraints must be satisfied at the ex-post stage, i.e., regardless of which strategy other players (including nature) play. The resulting set of implementable social choice functions is given by $F \subset F_s$.

The mechanism designer searches for an implementable social choice function that maximizes her objective function. The value of the objective function is $w(\theta, x)$ for type profile $\theta$ if outcome $x$ is realized. For a stochastic social choice function $f_s$ several outcomes can potentially realize for type profile $\theta$. The expected value of the designer's objective conditional on $\theta$ is given by

$$\omega(\theta, f_s) := E[w(\theta, x)|\theta] = \int_{x \in X} w(\theta, x) f_s(\theta, x) dx.$$

Consequently, the designer's optimization problem is:

$$\max_{f_s \in F} E[\omega(\theta, f_s)].$$

## III A revelation principle in terms of payoff

The following proposition gives the main result of this note.

**Proposition 3.1.** *For any stochastic social choice function $f$ that is implemented with an indirect deterministic mechanism in dominant strategies and under ex-post constraints (or no constraints) there exists a deterministic social choice function $\widehat{f}$ that*

1. *is implementable under the same set of ex-post constraints in a deterministic direct revelation mechanism,*

2. *weakly dominates $f$ in the sense of a general objective function $w(\theta, x)$:*

$$w(\theta, \widehat{f}(\theta)) \geq \omega(\theta, f).$$

*Proof.* Suppose $f \in F_s$ and $f \notin F_d$. The deterministic indirect mechanism $M = (S, g)$ implements $f$ in dominant strategies and for some type profiles some players mix. Let $\sigma(\theta) = (\sigma_1(\theta_1), \ldots, \sigma_N(\theta_N))$ be the corresponding mixed strategy profile for type profile $\theta$, mixing over pure strategy profiles $\widehat{s} \in \widehat{S} \subset S$ with density $\gamma$. Because $\sigma_i(\theta_i)$ is a dominant strategy for agent $i$, every pure strategy $\widehat{s}_i \in \widehat{s} \in \widehat{S}$ over which $\sigma_i$ randomizes must be a pure strategy, too, and $u_i^M(\sigma_i(\theta_i), s_{-i}) = u_i^M(\widehat{s}_i, s_{-i})$ for all $\widehat{s}_i \in \widehat{s} \in \widehat{S}$. Otherwise (3.1) would be violated for $\sigma_i(\theta_i)$, as it must be a best-response for agent $i$ regardless of the other agents' strategies.

The designer's payoff is given by

$$\omega(\theta, f) = \int_{\widehat{s} \in \widehat{S}} w(\theta, g(\widehat{s}))\gamma(\widehat{s})d\widehat{s}.$$

Define strategy profile $\overline{s}(\theta) = (\overline{s}_1(\theta), ..., \overline{s}_n(\theta))$ such that

$$\overline{s}(\theta) \in \arg\max_{\widehat{s} \in \widehat{S}} w(\theta, g(\widehat{s})).$$

By the argument above, $\overline{s}(\theta)$ is a pure strategy equilibrium profile for type profile $\theta$ in mechanism $M$. Similarly, any outcome that can result from $f(\theta)$ for type profile $\theta$ must be ex-post feasible. Therefore $g(\overline{s}(\theta))$ is feasible as well.

Set $\widehat{f}(\theta) = g(\overline{s}(\theta))$ for any type profile $\theta$ for which agents mix according to $f$. By construction, $\widehat{f}$ generates a weakly higher payoff for any type profile,

$$\omega(\theta, \widehat{f}) = w(\theta, g(\overline{s}(\theta))) \geq \omega(\theta, f),$$

and consequently $\widehat{f}$ also yields a weakly larger payoff in expectation,

$$E[\omega(\theta, \widehat{f})] \geq E[\omega(\theta, f)].$$

Because $\widehat{f}$ is feasible, $\widehat{f} \in F$, and deterministic, $\widehat{f} \in F_d$, $\widehat{f}$ can be implemented in a direct revelation mechanism. $\square$

Proposition 3.1 states that for any stochastic social choice function that is implementable by a deterministic indirect mechanisms there exists a deterministic social choice function that is also implementable and weakly dominates the stochastic social choice function in terms of the designer's payoff. This result holds under some conditions on the initial social choice function. It must be implementable in dominant strategies and, if there are any additional feasibility constraints, it must satisfy them ex-post. Such a deterministic social choice function can also be implemented by a deterministic direct revelation mechanism. Therefore the result can be interpreted as a variation of the revelation principle, formulated in terms of payoff. While not any social choice function that can be implemented by an indirect deterministic mechanism can also be implemented by a direct deterministic mechanism, the optimal social choice function can be implemented by a direct mechanism under the above conditions.

This result is an extension of an argument made by Strausz (2003) who obtains a similar result for mechanisms with one agent. In such mechanisms, the agent's best response is necessary a dominant strategy. Similarly, with only one agent in a deterministic mechanism a participation constraint that holds interim also holds ex-post.

Strausz (2003) provides an example with more than one agent such that his revelation principle in terms of payoff fails. In his example, he imposes an interim participation constraint (individual rationality), and mixing in the indirect mechanism guarantees the agents their reservation utility. In contrast, in our setting the participation constraint would have to hold ex-post, i.e., agents must obtain at least their reservation utility regardless of the other agents' strategies. Therefore agents cannot play a mixed strategy that attaches positive weight to a pure strategy that could, against any possible strategies of the other agents, yield a payoff less than the reservation utility.

## IV Conclusion

It is known that the classical revelation principle fails when attention is restricted to deterministic mechanisms. In this note, we establish that deterministic direct truthful mechanisms are optimal when dominant-strategy implementability is considered and all constraints are imposed ex-post.

# 4. Competeing for Strategic Buyers

## I Introduction

In this paper, I investigate the interaction between forward-looking buyers and multiple sellers in a continuous-time revenue management setting. Perhaps surprisingly, allocations, prices, joint industry profits and buyer payoffs are equivalent under monopoly and oligopoly if a monopolist prefers to sell efficiently all her goods with probability one. For example, for uniformly distributed valuations, such a pricing strategy is optimal when sellers are unable to commit to future prices and goods are sufficiently scarce. In contrast, if a monopolist can commit on future prices, she only wants to sell her full capacity if she values the good sufficiently less than the lowest buyer type. The irrelevance of the distribution of goods over sellers is driven by the insight that a seller can let her competitors sell their entire stock, and then gain a monopoly continuation payoff. Hence, she is not willing to undercut every positive price. The results follow because intertemporal arbitrage of the forward-looking buyers entails martingale equilibrium prices. In equilibrium, a seller is, at each point in time, indifferent between selling at the current price and letting a competitor sell at that price and instead having the next trade at the same price in expectation. In contrast, if a monopolist in expectation profits from withholding some capacity with positive probability, the profit of a monopolist is higher than the industry profit of oligopolists. For example, this condition holds when sellers can commit to future prices in "no-gap cases".

Since the Airline Deregulation Act of 1978, revenue management (RM) has been a standard business practice to price airline tickets and subsequently became a tool to price goods in a wide range of industries with similar characteristics, for example, cruise ships, hotels, rental cars, seasonal clothing, freight, electricity or sporting and entertainment events. Key business conditions conducive to RM are that (i) customers have heterogeneous valuations, (ii) (short-term) capacity is fixed and (iii) the goods lose their value after a deadline. Although none of the industries mentioned above is monopolistic, the literature on RM in oligopolistic settings is scant. In this paper, I ask how the interaction between forward-looking buyers and competing sellers shapes market outcomes.

I consider a RM environment in which $M \geq 2$ price-posting sellers desire to sell in total $K$ homogeneous goods, which are exogenously distributed among the sellers. Sellers can post prices at any point in a time continuum before the deadline. All $n$ buyers enter the market at the same time, privately draw a persistent valuation for the good, and strategically time their purchase decision. Importantly, the good is scarce, $n > K$. Sellers exit the market once they are stocked out and buyers exit the market once their single-unit demand is satisfied.

I find that for all model parameters such that a monopolist would optimally sell her goods with certainty, it is irrelevant for consumer rents and industry profits how the $K$ units of the good are distributed among sellers. Hence, a thorough understanding of the monopoly benchmark is essential for the analysis of the oligopoly setting. The monopoly benchmark for the case without price commitment is provided by Hörner and Samuelson (2011). Their most important result for my setting is that a monopolist with $K$ goods replicates an efficient Dutch auction when facing $n > K + 5$ buyers with uniformly distributed value. Unfortunately, the analysis is quite involved, making it hard to expand this result qualitatively to other distributions.

To grasp the intuition behind the oligopoly prices, suppose that two sellers, each offering one good, jointly replicate sequential Dutch auctions without reserve prices: At first, they simultaneously post the choke price and then synchronously and continuously decrease the price until a sale occurs and the corresponding seller exits the market. Immediately after the sale, the remaining seller discontinuously raises the price to a choke level and continuously decreases it until the next sale occurs. The price must jump to avoid frenzies as in Bulow and Klemperer (1994), because supply decreased relative to demand. Because in sequential auctions forward-looking buyers arbitrage away any differences between current and expected future prices, both sellers are at each point in time indifferent between selling at the current price and letting the competitor sell and then replicating a Dutch auction in the monopoly continuation game. Consequently, the same price path arises, leading to the same allocation and the same expected payoffs per trade for all players as in a setting in which a monopolist sells two goods in an auction without reserve price.

However, a monopolist may profit from setting an exclusive reserve price implying that she may not sell her entire capacity. In my setting, only a monopolist with the ability to commit to future prices can replicate an exclusive optimal mechanism: Prices decrease continuously, jump to a choke price immediately after each sale and finally remain at an optimal reserve price. However, in the presence of a competitor, a single seller has an incentive to decrease the price further than the optimal reserve price of a monopolist. In equilibrium under oligopoly, sellers sell out over time and the terminal price is determined by the last active seller once all competitors are stocked out. Although equilibrium prices decrease continuously as well, any buyer type who would get a good under both market conditions pays a lower price. Moreover, the price decreases below the level optimal under monopoly such that possibly more goods are sold in comparison. At the time a seller becomes a monopolist, she commits to (replicating) a Dutch auction that is optimal with respect to her updated prior about the remaining buyers' valuations. The payoffs of players are bounded from above by the monopoly payoffs (the mechanism design optimal profit) and bounded from below by the payoffs from sequential Dutch auctions without reserve price. Consequently, prices under oligopoly are lower and buyers are better off, while competing sellers are worse off compared to a situation in which they share a jointly maximized profit.

In traditional RM models, a monopolist faces sequentially arriving and perfectly impatient buyers, but there is survey evidence[1] that buyers strategically time their purchase decision. For a review of dynamic pricing with forward-looking consumers, consult Gönsch, Klein, Neugebauer, and Steinhardt (2013) who report losses between 7% and 50% in the surveyed articles when sellers treat forward-looking consumers as myopic. In my model, the buyers' strategic purchase timing drives an important ingredient for the equivalence result, the martingale property of prices: In equilibrium, the expected sale price of the next unit of the good is at each time equal to the current price. As a consequence, it is important for antitrust authorities to know whether buyers are forward-looking or myopic.

[1] According to the consumer report "America's Bargain-Hunting Habits", Apr. 30th 2014, around 60% of consumers "wait for a sale to buy what they want." See also the survey of American Research Group, Inc on "2014 Christmas Gift Spending Plans Stall", Nov. 21st 2014.

In terms of policy advice, my findings have to be interpreted with caution as they suggest that, under conditions, an industry with RM characteristics and forward-looking buyers does not require any merger control. There is no need to protect forward-looking-consumers from a monopolistic price discrimination by breaking the monopoly into several smaller firms. This benchmark result, however, raises the question of what kind of additional features have to be included into the model to yield the more intuitive result that seller competition increases buyers' rents. The insight that the irrelevance result does not hold when sellers prefer to commit to excluding low buyer types sheds light on the role of commitment in RM markets which is valuable for evaluating antitrust issues. Because the martingale property of prices is key for the results, I discuss extensions for which it is known that prices do not follow a martingale process.

My oligopoly setting emphasizes results from the sequential auctions literature from a novel angle, and thereby links two seemingly unconnected insights: First, a price posting monopolist without price commitment replicates Dutch auctions by posting continuous price paths in equilibrium and, second, prices in sequential auctions are a martingale. The martingale property of prices in sequential auctions was derived by Milgrom and Weber (2000) and sparked the academic debate around the "declining price anomaly" discussed in Section V. Settings with interdependent valuations, unknown size of inventory or background risk would be interesting to study as such models feature upward or downward trends in prices. In light of major applications such as airline tickets or hotel rooms, the role of sequentially arriving buyers is of great interest as well.

This RM model of multiple sellers facing buyers with private information fills an important gap in the literature. One reason why current research is paradoxically silent on competing sellers in a private value environment might be that it is not clear how the buyers' selection strategies might look like if sellers do not post identical prices. One may think about correlated equilibria or alternatively introduce a coordination device or a search game. The approach taken here is to allow at most one good to be traded at each instant and this single good is traded at the lowest current price. Either all buyers reject the posted prices or a single buyer trades and the remaining buyers face new prices in the future. This procedure has convenient implications: First, sellers' profits feature a discontinuity reminiscent of Bertrand (1883). Second, the buyers' optimal dynamic strategy is easy to characterize. Third, matching frictions such as those described in Burdett, Shi, and Wright (2001) are circumvented in a game-theoretically consistent way.

Only allowing a single transaction at each trading instant sounds more restrictive than it actually is. Primarily, it is a succinct way to capture the idea that, following a sale, sellers can adapt prices faster than buyers can react. Alternatively, I could put all buyers into a queue in random order. Neither sellers nor buyers have any knowledge about the positions in the queue except that they are drawn uniformly at random before each purchase decision. Then, at each time buyers are released sequentially from the queue and observe the prices and how many items were sold. Because buyers are released one-by-one and have single-unit demand, a buyer's optimal strategy is to randomize among the cheapest sellers if he wants to buy. Sellers set a menu of prices contingent on how many sales have already occurred at that time. Consequently, with each price, sellers only compete for the first purchasing buyer in the queue and then the queue is redrawn. In equilibrium, each trade occurs between a randomly chosen interested buyer (the first accepting buyer in the random queue) and a randomly chosen cheapest seller (the one randomly selected by that buyer). This approach is similar to the model by Deneckere and Peck (2012) in which, however, the queue is not reformed in each period.

Another reason why the RM literature with competing seller is relatively sparse might be that it appears to be complicated to keep track of intertemporal arbitrage conditions of buyers and sellers simultaneously. In my setting, tractability can be sustained when prices are well-behaved. Importantly, prices are driven by continuation payoffs which makes the game easy to solve when the continuation payoffs are easy to solve for. In particular, I can incorporate the tractable solution to the problem of Hörner and Samuelson (2011) as the payoff of a monopoly continuation game of my richer oligopoly setting. To construct a well-defined game in continuous time, I have to consider a restricted "inertia" strategy space that permits the use of discrete-time game theory, and then I complete the strategy space with respect to an appropriately defined metric.

The following subsection relates my paper to the existing literature. Section II presents the model. After I introduce the full-commitment monopoly optimum, I analyze of the model without price commitment in Section III. In Section IV, I solve the model with full price commitment under oligopoly. The discussion in Section V serves the purpose to identify which assumptions are important for the main result, Proposition 4.15, and touches on a few interesting modifications of the model. Finally, I conclude in Section VI.

**Literature**

Initiating the literature on RM, Gallego and Van Ryzin (1994) consider a single seller facing demand by short-lived buyers, whose arrival is modeled as a Poisson process with intensity $\lambda(p)$. The take-away result of such models is that average prices fall over time as the approaching deadline diminishes the option value of selling. The main focus of this literature has been to improve the modeling of buyer behavior (such as strategic buyers) or making the monopolist's problem more complex by introducing additional resources to manage (network RM). Talluri and Van Ryzin (2005) provide an excellent overview of RM in their book that became the main reference of the field. There have been only few studies on RM with oligopoly. One reason might be that capacity constraints are a definitive characteristic of RM models, and equilibriua in a simple static benchmark model such as Bertrand-Edgeworth competition (Edgeworth (1897)) is widely unexplored beyond special cases.[2] In such models, it is known that assumptions about how buyers are rationed are not innocuous. In my model, efficient rationing arises endogenously. Moreover, a static model obviously cannot quantify the value of commitment to future prices like my model is able to do.

Martínez-de Albéniz and Talluri (2011) generalize the model of Dudey (1992), who shows that a dynamic version of Bertrand-Edgeworth duopoly competition has a unique subgame-perfect equilibrium. They model sequentially and randomly arriving short-lived buyers with commonly known valuations. In contrast, the buyers in my setting are long-lived and forward-looking, and have private information. Similar to my results, Martínez-de Albéniz and Talluri (2011) find that continuation payoffs determine prices. Contrary to my results, the seller with the fewest goods sells her entire stock first, always priced at the reservation value of the next smallest seller, and the largest seller sells her goods at last and at a constant monopoly price. Gallego and Hu (2014) consider a similar framework with differentiated products.

[2] See Levitan and Shubik (1972), Kreps and Scheinkman (1983), Osborne and Pitchik (1986) for a full characterization for the duopoly case, Hirata (2009) and De Francesco and Salvadori (2009) for the triopoly case and Vives (1986) for the case of equal capacities among all sellers.

Deneckere and Peck (2012) model a perfectly competitive dynamic market with a continuum of sellers, who have to produce output in advance, and a continuum of buyers who can costly delay their purchase. Moreover, demand uncertainty is innovatively modeled through a demand state. The unobserved demand state then determines the value distribution of a new batch of buyers that joins the remaining active buyers of the previous period. Sellers price under partial knowledge of the demand state: Prices within a period rise as sellers become more optimistic about the demand realization and then prices have to be corrected when demand dries up. Prices are dispersed as some sellers only want to sell when demand is sufficiently strong. However, as a consequence of intertemporal arbitrage conditions, lowest prices available are a martingale. My model differs in multiple respects: I model oligopolistic competition for (exogenously) scarce goods, there is no buyer entry and the possibility of being rationed is the only cost from delaying purchase.

The literature on the Coase conjecture (1972) was the first to investigate the role of a seller's (lack of) commitment power. Surprisingly, Gul (1987) and Ausubel and Deneckere (1987) show that the competitive allocation result of durable-goods monopoly (e.g. Stokey (1981), Bulow (1982), Gul, Sonnenschein, and Wilson (1986), Ausubel and Deneckere (1989)) is reversed when additional sellers populate the model: While the monopolist prices at marginal cost, oligopolists can attain (total industry) profits arbitrarily close to the static monopoly profit. The reason why monopoly is more competitive than oligopoly is that a competitor can help to sustain higher prices through credible punishments, which is not possible when a monopolist only competes with the future self. In comparison, my equivalence result does not stem from punishment strategies. In fact, strategies only depend on a market state. The equilibrium in this paper rather reflects that the market cannot become more competitive when the good is scarce and buyers are forward-looking, because there is no incentive to exert competitive pressure. Therefore, despite the similarities, the durable-goods monopolist, who can offer as many goods as buyers are present, is not the relevant monopoly benchmark of my RM setting. In contrast, my buyers want to buy early to avoid being rationed and the good is paid and consumed at a fixed date in the undiscounted future. Hence, the monopoly benchmark in a setting with price commitment is given by an optimal Dutch auction that screens types perfectly and can maintain an exclusive reserve price and the benchmark in a setting without price commitment is explored by Hörner and Samuelson (2011),

## II The Model

**Players:** I consider a dynamic game with $M$ sellers (she) and $n$ buyers (he) over the normalized time interval $T := [0,1]$. Each seller $m \in \mathcal{M} := \{m_1, ..., m_M\}$ is endowed with $K_m \in \mathbb{N}$ homogeneous goods, respectively. All buyers simultaneously enter the market at time 0. Each buyer $i \in \mathcal{I} := \{i_1, ..., i_n\}$ demands a single good and exits the market after purchasing. Similarly, a seller exits the market after selling all her goods. All players who have not exited the market are called active. The good is scarce, $n > K := \sum K_m$, and the endowment of all sellers is common knowledge. In addition, the game has three nature players: Nature 1 (N1), a "trade selector", Nature 2 (N2), a "trade processor", and Nature 3 (N3) who draws private types. Nature 1 determines who trades at time $t \in T$, while Nature 2 determines if a trade can occur. A generic player is denoted by $\iota \in I := \mathcal{I} \cup \mathcal{M} \cup \{N1, N2, N3\}$.

**Actions:** Each player $\iota$ has a corresponding action space $A_\iota$ from which an action is selected at any time $t \in T$. Let $A := \times_{\iota \in I} A_\iota$. Each seller $m$ posts a price $p_t^m \in A_m := \mathbb{R}_+$ at each time $t$. Each buyer $i$ either decides to buy at a current price or to delay purchase to the next purchasing opportunity, $d_t^i \in A_i := \{0,1\}$. At each time $t$, N1 randomly draws a buyer and a seller, $(i^t, m^t) \in A_{N1} := \mathcal{I} \cup \{0\} \times \mathcal{M} \cup \{0\}$. At each time $t$, at most one good is traded, and this trade is selected by N1: The seller is randomly selected among the sellers posting the lowest price at the time, some $m \in \{m : p_t^m \leq p_t^{m'} \quad \forall m' \in \mathcal{M}\}$, and the buyer is randomly selected among the accepting buyers, some $i \in \{i : d_t^i = 1\}$.[3] That is, a necessary condition for buyer $i$ and seller $m$ to trade at time $t$ is that they are selected. If no buyer wants to purchase, $(i^t, m^t) = (0,0)$. For each time $t$, N2 sets a time $\tau_t \in A_{N2} := T$. A necessary condition for a trade to occur at time $t$ is that $\tau_t = t$. In other words, by setting some $\tau_t \neq t$, N2 can prohibit all trade activity at time $t$. N3 draws for each buyer $i$ a persistent valuation (or type) $v_i$. Each $v_i$ is an iid draw from commonly known continuous distribution $F$ with support $[\underline{v}, \overline{v}]$ and positive density $f$, $A_{N3} := [\underline{v}, \overline{v}]^n$.

**Outcomes:** An outcome for player $\iota$ is a function $o_\iota : T \to A_\iota$ and $O_\iota$ is the set of outcomes for player $\iota$. Let $o = (o_\iota)_{\iota \in I}$ denote an outcome vector, while $O = \times_{\iota \in I} O_\iota$ is the set of possible outcomes of the game.

[3] This assumption is merely for simplicity of notation. Alternatively, I could let the buyers decide which seller to select. Since only one trade can occur at each time, in equilibrium, a buyer would randomize over the cheapest sellers.

**Timing and Information:** The goods are traded within the normalized time interval $T = [0, 1]$. At $t = 0$, N3 draws the types, and each $v_i = o_{N3}(t)_i$ is privately observed by the corresponding buyer $i$. The type is constant over time. At $t = 0$, N2 publicly sets trading times $\tau_t \geq t$ for all $t \in T$ contingent on the game's history. I am interested in the game where $\tau_t = t$ for all $t \in T$ except those times at which inertia (see below) prohibits this action of N2. I set up a continuous-time game in which actions are taken sequentially at each time $t \in T$ in the following order:

1. Sellers and buyers update their belief about the buyers' valuation corresponding to the history. Go to step 2.

2. All active sellers publicly post individual prices. Go to step 3.

3. All active buyers privately and simultaneously decide whether they want to purchase. Go to step 4a or 4b.

4a. If $\tau_t = t$, N1 selects the trade $o_{N1}(t) = (i^t, m^t)$. Trade $(i^t, m^t)$ occurs and the corresponding seller is publicly observed. End of time $t$.

4b. If $\tau_t \neq t$, N1 sets $o_{N1}(t) = (i^{t'}, m^{t'})$ with $t' = \max\{\widehat{t} : \tau_{\widehat{t}} = \widehat{t}, \widehat{t} < t\}$ and no trade occurs. End of time $t$.

Importantly, after a sale, sellers do not observe which or how many other buyers tried to purchase. Let some history be denoted by $h$ and let a posterior following history $h$ be denoted by $F_h$. More specific, let a seller history be denoted by $h^m \in \mathcal{H}^m$ and a buyer history be denoted by $h^i \in \mathcal{H}^i$, where $\mathcal{H}^\iota$ is the set of all possible histories of player $\iota$. A seller history at time $t$ given by

$$h_t^m := \left((o_\iota(t'))_{\iota \in \mathcal{M} \cup \{N1\}}\right)_{t' < t : \tau_{t'} = t'}, \tag{4.1}$$

that is a seller remembers all past prices and all past trades for all times at which a trade could have occurred, i.e., times $t'$ at which N2 $\tau_{t'} = t'$. A buyer history at time $t$ given by

$$h_t^i := (o_{N3}(t)_i, \left((o_\iota(t'))_{\iota \in \mathcal{M} \cup \{N1\} \cup \{i\}}\right)_{t' < t : \tau_{t'} = t'}). \tag{4.2}$$

Thus, compared to a seller, a buyer additionally recalls all of his own actions and knows his own valuation, $v_i = o_{N3}(t)_i$, the $i$-th element of constant vector $o_{N3}(t)$.

Let a market state for player $\iota$ following history $h$ be denoted by

$$\omega_\iota(h) := (F_h, (k_m(h))_{m\in\mathcal{M}}), \tag{4.3}$$

where $k_m(h)$ is the number of goods seller $m$ is offering following history $h$. From this information, the number of active sellers, $M_h = \sum_m \mathbb{I}_{k_m(h)>0}$, as well as the number of active buyers, $n_h = n - \sum_m (K_m - k_m(h))$, can be inferred.

**Payoffs:** Obtaining a good is only valued at a time $t \in T$, after time 1 the good loses its value. From outcome $o$, buyer $i$ of type $v_i = o_{N3}(t)_i$ gains payoff

$$U_i(o) = \begin{cases} v_i - p_t \text{ if for some } t \in T\text{: } o_i(t) = 1; o_{N1}(t) = (i, \cdot); o_{N2}(t) = t \\ 0 \text{ otherwise} \end{cases}, \tag{4.4}$$

that is, a buyer only gains a positive payoff if he, at some point in time, accepted a price $p_t = \min_{m\in\mathcal{M}}\{o_m(t)\}$, he was selected for this trade by N1 and the trade was possible according to N2.

It is commonly known that sellers do not value the good and $\underline{v} \geq 0$. A seller $m$ has a time-$t$ payoff from outcome $o$ given by

$$U_m(o) = \sum_{t\in T_m} p_t^m, \tag{4.5}$$

where $T_m = \{t' : o_{N1}(t') = (\cdot, m); o_{N2}(t') = t\}$ is the set of all times when seller $m$ traded and $p_t^m = o_m(t)$ is the corresponding sale price. Nature players don't have a payoff function. As seen from $U_\iota$, there is no discounting.

**Inertia:** When setting up a continuous-time model, unavoidable pathologies arise. Namely, well-defined strategies may be consistent with multiple outcomes.[4] Here, I circumvent these issues by generalizing the approach that Bergin and MacLeod (1993) introduced for full-information repeated games to asymmetric-information stochastic games. I look at the restricted space of inertia strategies $S$ and take the completion $S^*$ of the strategy space with respect to an appropriately defined metric. Thereby I include strategy profiles $\zeta^* \in S^*$ that arise as limits of inertia strategies $\zeta \in S$. A more detailed description and the preliminary analysis that guarantees that such limits and its outcomes are well-defined is executed in Appendix VII.A.

[4] "Next" instants are not well-defined, see Simon and Stinchcombe (1989) and Bergin and MacLeod (1993) for a detailed discussion.

Roughly speaking, I require that each player's action (including the nature players) at any time $t$ is held constant during some time interval $[t, t+\epsilon)$. That is, players can only adjust their action after some time lag $\epsilon$ has passed, implying that players can adjust actions only a countably number of times. The limits $\epsilon \to 0$ capture the idea that players can react "instantaneously". Importantly, this $\epsilon$ may depend on the time $t$ and all outcomes up to that time. Note that no player receives any payoff or any valuable new information during the inertia lags: N2's action is held constant as well, prohibiting all trading activity, and the prices and the corresponding trade outcome set by N1 was already observed at time $t$. The inertia formulation of the game allows me to employ discrete-time game theory and then translate the analysis to continuous time.

**Equilibrium:** I restrict attention to a tractable class of equilibria: In an $\varepsilon$-**Strongly Symmetric Markov Perfect Bayesian Equilibrium (SSMPBE)**,

1. sellers post identical prices given the market state,
2. buyers of the same type take the same purchase decision given the market state,
3. all actions are sequentially rational, given the history of previous play and anticipations of optimal continuation play,
4. beliefs are $\varepsilon$-consistent (Definition 4.1) with beliefs derived according to Bayes' rule.

I call this equilibrium strongly symmetric, because sellers set the same price even if they do not have the same stock of goods $k_m(h)$.

I analyze the model under different assumptions regarding sellers' ability to commit to future prices. In Section IV, I analyze the model with full price commitment, i.e., I look for semi-perfect equilibria. They are defined as above, but bullet point 3 is replaced with

3a. buyers' purchase decisions are sequentially rational, given the history of previous play and anticipations of optimal continuation play,

3b. each seller $m$ commits to a price plan contingent on each possible market state in the beginning of the game.

It remains to be shown that any sequence of $\varepsilon$-SSMPBE strategy profiles with $\varepsilon \to 0$ converges to a 0-SSMPBE strategy profile, the equilibrium I am eventually interested in.

**Definition 4.1.** A distribution $G$ is $\varepsilon$-consistent with a true Bayes' update $F$, if and only if

$$1 \geq \frac{F(v)}{G(v)} \geq 1-\varepsilon \quad \forall v > \underline{v} \quad \text{and} \quad F(v) > F(v') \Rightarrow G(v) > G(v'). \tag{4.6}$$

## III Analysis

### The full-commitment monopoly benchmark

This subsection serves the purpose to provide an upper bound of industry profits that turns out to be helpful over the course of the analysis. The reader familiar with basic auction design with single-unit demand following Myerson (1981), Riley and Samuelson (1981) and Maskin and Riley (1989) may want to skip to the definition of Condition 4.4 immediately.

Let the $i$-th highest order statistic of $n$ draws from distribution $F$ be denoted by $Y_i^{(n)}$ such that $Y_1^{(n)} \geq ... \geq Y_i^{(n)} \geq ... \geq Y_n^{(n)}$ is a rearrangement of $V_1, \dots, V_n$. Moreover, let the virtual valuation be denoted by

$$\psi(v) = v - \frac{1-F(v)}{f(v)}. \tag{4.7}$$

The literature on auctions speaks of a regular environment when $\psi$ is a strictly increasing function of the valuation.

Lemma 4.2 establishes the first important benchmark: If sellers with the ability to commit to future prices collude and jointly maximize profits, each seller $m$ can get her fraction $K_m/K$ of the (mechanism design) optimal industry profit.

**Lemma 4.2** (Monopoly, full commitment)**.** *A monopolist with the ability to commit to future prices can replicate sequential Dutch auctions with any reserve price. In regular environments, this mechanism is optimal when the reserve price $r^*$ is such that $\psi(r^*) = \max\{0, \psi(\underline{v})\}$. With $K$ goods, this price path yields a profit of*

$$\mathbb{E}\left[\sum_{l=1}^{K} \max\{\psi(Y_l^{(n)}), 0\}\right]. \tag{4.8}$$

I omit a formal proof and give the intuition. Since there is no competition and the price path is not restricted to be sequentially rational, a monopolist can implement the optimal allocation (Maskin and Riley (1989)) by replicating an optimal Dutch auction: In continuous time, a seller has an infinite amount of pricing possibilities and, thus, she can optimally screen and exclude buyer types by setting a continuously decreasing price path that becomes flat at an optimal reserve price $r^*$ defined by $\psi(r^*) = \max\{0, \psi(\underline{v})\}$. It is irrelevant for the monopolist's payoff whether the price decreases rapidly or slowly because there is no discounting. By the revenue equivalence theorem the implemented allocation yields the optimal profit. Under the condition below, the optimal auction is efficient as it allocates the goods to the $K$ highest types, and the seller never keeps a good.

**Corollary 4.3** (Monopoly, no exclusion)**.** *In regular environments, the optimal allocation is efficient if and only if*

$$\psi(\underline{v}) \geq 0 \quad \Longleftrightarrow \underline{v} f(\underline{v}) \geq 1. \tag{4.9}$$

*Replicating a sequential Dutch auctions without or with non-exclusive reserve price* $r \leq \underline{v}$ *is optimal for the seller.*

**Condition 4.4** (No exclusion)**.** *A monopolist wants to sell all of her goods efficiently with probability one.*

**The buyers' cutoff valuation x**

For every potential sale, every active buyer faces a stopping problem, i.e., he chooses a point along a path of minimum prices $(p_t)_{t \in T}$ at which he wants to apply for a good, taking as given the stopping strategy of other buyers. If another buyer got to buy the good at some price, a similar stopping problem arises for the next sale and so on. In the following, consider an arbitrary buyer $i$ and take as given an inertia strategy profile of all other players, $\zeta_{-i} \in S_{-i}$.

Let $h$ be a history with no sale so far, so that all players are in the same market state $\omega = \omega_m(h^m) = \omega_i(h^i) = (F_h, (K_m)_{m \in \mathcal{M}})$ for all $i \in \mathcal{I}$ and $m \in \mathcal{M}$, all players have the same beliefs. Consider a type-$v$ buyer $i$ with corresponding buyer history $h^i$. Let $W(v, h^i)$ be buyer $i$'s continuation payoff at buyer history $h^i$,

$$W(v, h^i) := \max_{\zeta_i \in S_i} \mathbb{E}_o \left[ U_i(o) | \, \zeta_{-i}, h^i, v \right],$$

i.e., his expected payoff from optimal continuation play under uncertainty of the true outcome conditional on being type $v$ and having observed buyer history $h^i$ so far when playing against inertia strategy profile $\zeta_{-i} \in S_{-i}$. A buyer knows that once he sets his purchase decision at time $t$, all players are inertial for some known time $\epsilon$ and the next trade can occur at time $t+\epsilon$, while all outcomes in $(t, t+\epsilon)$ are meaningless.

If buyer $i$ decides to buy at time $t$, then either he gets the good and exits the market or another buyer got the good. Similar, if $i$ decides to delay, then either someone obtained a good or all buyers declined, and he faces a new price with an updated belief about the other buyers. Define

$$W_+(v, h^i, p_t, d_t) := \mathbb{E}_{h^i_{t+\epsilon}} \left[ W(v, h^i_{t+\epsilon}) \middle| \, d_t, p_t, o_{N1}(t) \neq (0,0) \right]$$

as the expected continuation payoff of a type-$v$ buyer observing a sale at $t$, after setting action $d_t$ at time $t$ when $p_t$ was the minimum price. The expectation is taken with respect to history $h^i_{t+\epsilon}$ as it is uncertain which seller is selected and, thus, how the market state evolves. Let $\widetilde{h}^i_{t+\epsilon}$ be the continuation of history $h^i_t$ such that, at $t$, sellers posted a price consistent with $\zeta$ and all buyers rejected prices, and players were inertial during $(t, t+\epsilon)$. Let

$$W_-(v, h^i, p_t) := W(v, \widetilde{h}^i_{t+\epsilon})$$

be the continuation payoff of a type-$v$ buyer when no sale occurred at price $p_t$ with history $h^i$. I can dispense with the expectation with respect to history $\widetilde{h}^i_{t+\epsilon}$, as necessarily all buyers must have delayed purchase.

Buyer $i$'s payoff when deciding to purchase, $d^i_t = 1$, at price $p_t$ following buyer history $h^i$ is

$$\phi_\omega(v_i - p_t) + (1 - \phi_\omega) W_+(v, h^i, p_t, 1) \tag{4.10}$$

where $\phi_\omega$ denotes the probability that any given accepting buyer is selected for purchase at time $t$ when accepting the minimum price in market state $\omega = \omega_i(h^i)$. Obviously, this probability depends on how many other buyers accept the price which in turn depends on the given strategy profile $\zeta_{-i}$. With probability $(1 - \phi_\omega)$ another buyer gets the good and buyer $i$ obtains the expected continuation payoff of the corresponding history.

The expected payoff from delaying is given by

$$\sigma_\omega W_-(v_i, h^i, p_t, 0) + (1-\sigma_\omega) W_+(v_i, h^i, p_t, 0),$$

where $\sigma_\omega$ is the probability that no sale occurred during time $[t, t+\epsilon)$. I can specify the probabilities $\phi$ and $\sigma$ after the statement of Lemma 4.5. To find a critical type, who is indifferent between taking the current and the next price, the following expected payoff is crucial: From delaying purchase and accepting the minimum price at the next opportunity, time $t+\epsilon$, buyer $i$ garners

$$\sigma_\omega \big[\phi_{\widetilde{\omega}}(v_i - p_{t+\epsilon}) + (1-\phi_{\widetilde{\omega}}) W_+(v_i, \widetilde{h}^i_{t+\epsilon}), p_{t+\epsilon}, 1)\big] \\ +(1-\sigma_\omega) W_+(v_i, h^i, p_t, 0), \tag{4.11}$$

where $\widetilde{\omega}$ is a market state following from $\omega$ after no sale occurred during time $[t, t+\epsilon)$.

Since the good is scarce and each sale reduces the supply further (and finally the good may be sold out), a form of discounting arises endogenously through the probability that the good becomes more expensive or sells out. Consequently, higher types are more eager to buy. Remember that up to this point my analysis solely covers the case of histories without a sale, but I will extend it to the case of histories with nice price paths after the statement of the following lemma and its implications.

**Lemma 4.5.** *Consider some time $t \in T$ with a market state $\omega$ without a sale so far. In equilibrium, there exists an $\omega$-dependent cutoff type $x_t \in [\underline{v}, \overline{v}]$ such that all types $v \geq x_t$ decide to accept price $p_t$ and all types $v < x_t$ delay purchase.*

*Proof.* Fix some equilibrium and consider a market state $\omega$ with corresponding history $h_t$. Suppose that some buyer $i$ with valuation $v_i$ prefers to buy at price $p$ over delaying purchase to the next opportunity at price $p'$. Then it must be that (4.10) is larger than (4.11). It remains to be shown that all types $v > v_i$ decide to buy as well. I do this by showing that the derivative of (4.10) with respect to $v$ is larger than the derivative of (4.11), i.e.,

$$\phi_\omega + (1-\phi_\omega) W'_+(v_i, h_t, p, 1) \geq \tag{4.12}$$
$$\sigma_\omega \left[\phi_{\widetilde{\omega}} + (1-\phi_{\widetilde{\omega}}) W'_+(v_i, \widetilde{h}^i_{t+\epsilon}, p', 1)\right] + (1-\sigma_\omega) W'_+(v_i, h_t, p, 0),$$

where $W'$ denotes the derivative of $W$ with respect to the first argument, the valuation $v$.

By the envelope theorem, any derivative $W'(v_i, h_t, p, d)$ is a type-independent probability (the probability that buyer $i$ gets selected for purchase before the good is sold out). Hence, $W'_+(v_i, h_t, p, d)$ is bounded from above by one and it follows that $\left[\phi_{t'} + (1-\phi_{t'})W'_+(v_i, {}_0h_t^0(p), p', 1)\right] \leq 1$. Therefore a sufficient condition for the inequality above is

$$\phi_\omega(1 - W'_+(v_i, h_t, p, 1)) \geq \sigma_\omega(1 - W'_+(v_i, h_t, p, 0)). \tag{4.13}$$

If

$$W'_+(v_i, h, p, 0) \geq W'_+(v_i, h, p, 1) \quad \text{for a.e. } v_i \in [\underline{v}, \overline{v}] \tag{4.14}$$

holds, (4.13) is clearly true, because the probability that no other buyer at all accepts the price cannot be larger than the probability that no other buyer is selected for purchase by definition, $\phi_\omega \geq \sigma_\omega$: If no other buyer $i' \neq i$ accepts the price, buyer $i$ is selected with certainty, and even when other buyers $i' \neq i$ want to purchase as well, buyer $i$ is still selected with positive probability.

The sufficient condition (4.14) follows as a corollary from Lemma 4.8 which holds for arbitrary symmetric strategy profiles. The reason is that a declining buyer attaches a higher probability to obtaining a good than an accepting buyer. Hence, any symmetric equilibrium strategy for the first purchase is necessarily a cutoff strategy. □

Define

$$x_t := \min\{v_i : (4.10) \geq (4.11)\}. \tag{4.15}$$

This cutoff varies with previous and future prices. Moreover, the cutoff is not necessarily unique as the $\phi_\omega$ and $\sigma_\omega$ depend on the corresponding cutoff type $x$ as well.

Having established this lemma, I can express the probabilities $\phi_\omega$ and $\sigma_\omega$ as functions of cutoff valuations $x_t$. The event that no sale occurred reveals that every buyer rejected the current prices and, hence, there is no asymmetric information about the buyer decisions in the market. As a consequence, a common prior is maintained: All buyers and sellers learn that all buyer types are below the cutoff level. Straightforwardly, $\sigma_\omega$ then is the probability that all $(n-1)$ other buyers' types are below the cutoff. Similarly for the first sale, $\phi_\omega$ also is a simple function of the cutoff valuation. Given that $j$ other buyer types are above the cutoff, the probability that a buyer who is willing to purchase at price $p_t$ gets to buy is given by $1/(j+1)$.[5]

**Corollary 4.6.** *Suppose market state $\omega$ with prior $F_h$ does not feature a sale so far. The probabilities $\phi_\omega$ (probability of getting the good when accepting price $p_t$) and $\sigma_\omega$ (probability that no buyer accepted price $p_t$) can be expressed as*

$$\phi_\omega = \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{1}{j+1} F_h(x_t)^{n-1-j}(1-F_h(x_t))^j = \frac{1-F_h(x_t)^n}{n\,(1-F_h(x_t))} \tag{4.16}$$

*and*

$$\sigma_\omega = F_h(x_t)^{n-1}. \tag{4.17}$$

It remains to be formalized how $F_h$ is formed for general histories. Lemma 4.5 establishes that given that no sale occurred following history $h$, all players know, that no active buyer's type is greater than cutoff $x_t$, and update the prior appropriately. However, if a sale occurred at $t$, it is possible that types greater than cutoff $x_t$ remain in the market because they were not (randomly) selected for purchase. Then the analysis is complicated by the fact that different players can have different beliefs on how likely it is that such types remained in the market.

*Remark* 4.7. After a sale occurred, buyers and sellers update their prior differently. Moreover, accepting buyers update their prior in a different way from declining buyers. The reason is that the individual histories differ.

For some seller history $h$, let $F_{s,h}$ denote a seller's update of $F_h$ upon observing a sale. Similarly, for some buyer history $h$, let $F_{a,h}$ and $F_{d,h}$ denote an accepting and a declining buyer's update of $F_h$ upon observing a sale, respectively. Moreover, let $F_{no,h}$ denote a buyer or seller's update of $F_h$ upon observing no sale at a time at which N3 would have permitted a trade. The calculation of the posteriors involves

[5] Mathematically, probability $\phi_\omega$ is identical to an allocation according to a queue as motivated in the introduction. Since the queue is unobserved and redrawn uniformly at random after each sale, the probability that exactly $j$ buyers are in front in the queue is given by $1/n = (n-1)!/n!$ for all integer $j \in [0, n-1]$. The probability $\phi_\omega$ is thus a finite geometric series weighted by $1/n$, leading to an equivalent formulation of (4.16),

$$\phi_\omega = \frac{1}{n}\sum_{j=0}^{n-1} F_h(x_t)^j = \frac{1-F_h(x_t)^n}{n\,(1-F_h(x_t))}.$$

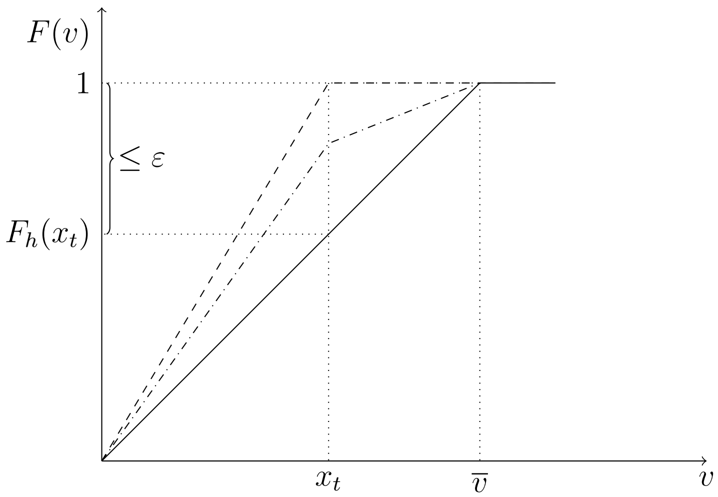



FIGURE 4.1: An illustration of a prior $F_h$ and two different updates: $F_{s,h}$(dashed) following a sale, $F_{no,h}$ (dash-dotted) following no sale. Moving $x_t$ to the right decreases $\varepsilon$ and makes the posteriors approach the prior.

a straightforward application of Bayes' Rule that is explained in greater detail in Appendix VII.B, where the posteriors are formally sated in Corollary 4.20.

**Lemma 4.8.** *The following first-order stochastic dominance results hold for any equilibrium in the buyers' game*

$$F_{no,h}(v) > F_{d,h}(v) > F_{a,h}(v) > F_h(v),$$

*for all* $v, x : F_h(v) \in (0,1), F_v(x) \in (0,1)$*. For* $v, x : F_h(v), F_h(x) \in \{0,1\}$*, all four distributions are equal.*

*Proof.* See Appendix VII.B. □

When a sale occurred, sellers and buyers update their priors differently because a buyer has one piece of information more compared to the seller, i.e., he knows his own decision. To grasp the intuition, suppose there are three buyers and a sale occurs. A seller conducts the following thought experiment: The valuation of a remaining buyer can only be below the cutoff, if at least one of the other two buyers accepted the price, i.e., has a value above the cutoff. Otherwise, all buyers would have rejected the price and no sale would have occurred. Similarly, the valuation of a remaining buyer can only be above the cutoff, if at least one other buyer accepted the trade (has a valuation above the cutof)f, and one of the two was selected for trade. In comparison, one of the two remaining buyers, $i$, updates his prior about the valuation of the other buyer $i'$ in a similar manner, but with the additional knowledge of his own decision. Whenever buyer $i$ delayed purchase, he updates his prior like a seller, but he considers only $(n-2)$ instead of $(n-1)$ other buyers, as $i$ himself declined. However, if a sale occurred and buyer $i$ had accepted the price, type $v_{i'}$ can only be above the cutoff, if the third buyer accepted the price as well and was selected for purchase. Put differently, if $i$ accepted and was not selected for trade, it is more likely that $i'$ accepted as well, reducing the probability that $i$ is selected. Therefore an accepting buyer $i$ who was not selected attaches a higher weight to $i'$ having a value above the cutoff.

It is straightforward to show that any (buyer or seller) prior first-order stochastically dominates any (buyer or seller) posterior following any event for any cutoff valuation strictly within the support. If $x_t$ is either the lower or the upper bound of the support, posterior and prior coincide, because either all buyers take the price or a sale happens with probability zero. Following the notion of the queue elaborated on in the introduction, the distribution of valuations of the buyer remaining in the market is the same as before, because either all buyers reject the price or the first buyer in the queue takes the price with probability one and the remaining buyers had no chance to buy. *Nice* cutoff paths give rise to this equivalence. For price paths that are not nice, the analysis leaves the realm of common priors. The individual posteriors give rise to individual cutoff valuations $x$. However, for $\varepsilon$-*nice* cutoff sequences, the posteriors are consistent in the sense of Definition 4.1.

**Definition 4.9.** A cutoff sequence is $\varepsilon$-*nice* over history $h$ if, along history $h$, at each point in time trade occurs with probability less than $\varepsilon$, i.e., for any truncation $h_t$ of $h$

$$F_{h_t}(x_t)) \geq 1 - \varepsilon. \tag{4.18}$$

A cutoff sequence is *nice* when $\varepsilon = 0$.

I later provide conditions such that nice sequences arise on equilibrium path. It turns out that the price path in any strongly symmetric equilibrium produces a nice cutoff sequence. When the prices imply cutoffs within the support of the updated priors, only continuous price paths with discontinuous jumps upwards after each sale are consistent with nice cutoff sequences. The price jump has to be high enough such that a sale occurs with probability zero. Obviously, continuous price paths can only exist in continuous time, i.e., with strategy profiles $\zeta \in S^*$.

Nice cutoff sequences are analytically convenient: Because at each time a sale occurs with probability zero, all buyers' and sellers' posteriors are identical regardless of the outcome, and the each buyer's strategy is characterized by the same time-specific cutoff type $x_t$. Therefore the insights from Lemma 4.5 carry over to histories that only feature sales within nice cutoff sequences. In particular, any player's posterior is equal to the prior of the previous instant.

For inertia strategies, I capture the notion that prior and posterior are "roughly the same" when the cutoff sequence is $\varepsilon$-nice with the following definition and the corresponding lemma.

**Lemma 4.10.** *If the cutoff sequence is $\varepsilon$-nice over history $h$, then for all truncations $h_t$ of $h$, the prior $F_{h_t}$ is $\varepsilon$-consistent with posteriors $F_{s,h_t}$, $F_{a,h_t}$ and $F_{d,h_t}$ following a sale at $h$, and $\widehat{\varepsilon}$-consistent with posterior $F_{no,h_t}$ following no sale at the same history for some $\widehat{\varepsilon} \geq \varepsilon$.*

*Proof.* See Appendix VII.B. □

Sloppily speaking, Lemma 4.10 says that, if the cutoff sequence is $\varepsilon$-nice, players only make a bounded mistake when they always update their prior as if no sale occurred or don't update at all, and this mistake vanishes as $\varepsilon \to 0$. This approach does not ($\varepsilon$-)solve all problems because even if all buyers had the same updated prior, there still might be multiple cutoff values $x_t$ that solve the corresponding equation.

In a strongly symmetric equilibrium, given any history $h$, the prices $p_t$ and $p_{t+\epsilon}$ together with the continuation payoffs $W$ determine critical types $x_t$ in the equation (4.10) = (4.11) evaluated at valuation $x_t$.

Solving for a sequence $(x_t)$ corresponding to buyers' inerita strategies, $\zeta_{\mathcal{I}} \in S_{\mathcal{I}}$, involves a higher-order difference equation with boundary conditions that the first cutoff is equal to $\overline{v}$ (the highest valuation, as players have not yet learned anything) and the last cutoff is equal to the final minimum price. If multiple sales have occurred, the critical types of all rounds in which a sale occurred along the history path enter the posterior and hence the difference equation is of higher order.

Like Hörner and Samuelson (2011), I face the issue of multiple solutions as well. For general distributions, such problems often feature multiple or even no solutions at all (see, for example, Agarwal (2000)). Following your economic intuition, you may have expected multiple solutions because buyers are strategic complements: If a buyer believes all other buyers are more likely to buy, he has more incentive to buy himself. Vice versa, a buyer's incentive to delay increases if he believes that other buyers are more likely to delay. For this reason, it is unclear whether a general characterization of perfect inertia $\epsilon$-equilibria for general distributions with $\epsilon = 0$ exists. Hence, Hörner and Samuelson (2011) later restrict attention to the uniformly distributed valuations for which they can show a unique equilibrium.

After elaborating on the sellers' game, I specify the buyers' stopping strategy, i.e., at which price a type-$v$ buyer wants to purchase given a pricing strategy suggested by the analysis of the sellers' game.

### The oligopolists' tradeoff: selling vs. not selling

As an implication of the assumption that, after a sale, sellers can adjust their price before buyers can buy again, sellers face a simple tradeoff. In each round, at most one trade occurs and buyers only patronize the cheapest sellers. Therefore seller $m$'s time $t$ expected profit has a discontinuity in the price at

$$\underline{p}_t^m = \min\{p_t^{m'}\}_{m' \in M \setminus \{m\}},$$

i.e., at the minimum among the prices of $m$'s active competitors.

Consider some seller $m$ playing against an inertia strategy profile $\zeta_{-m} \in S_{-m}$. I am looking for a strongly symmetric perfect equilibrium in which all sellers post the same price at each round. In such an equilibrium, no seller can have a profitable one-shot deviation $p^m \neq p_t$ when price $p_t$ is set by all active competitors.

Consider some seller $m$ with seller history $h$ at time $t$. Let ${}^m h(p)$ be the time-$(t+\epsilon)$ continuation of history $h$ in which a sale with seller $m$ occurred at at price $p$ at $t$ and time $(t, t+\epsilon)$ was inertial. Consequently, ${}^0 h(p)$ is a continuation seller history in which no sale occurred at time $t$.

Seller $m$'s expected revenue, when offering $k$ units and setting price $p^m$ while all other sellers set price $p$, at time $t$ with seller history $h$ can expressed piecewise as

$$R_t^m(p^m, p, h, k) = \begin{cases} (4.19) & \text{if } p^m > p \\ (4.20) & \text{if } p^m = p \\ (4.21) & \text{if } p^m < p \end{cases},$$

where each line is explained below. Let $C^m_{t+\epsilon}(k', h')$ be seller $m$'s continuation payoff under her strategy when she owns $k$ units at time $t + \epsilon$ with history $h'$.

$$C^m_{t+\epsilon}(k', h') := R^m_{t+\epsilon}(p', p'', h', k'),$$

where $p'$ is the price that strategy $\zeta_m$ intends for market state $\omega_m(h')$ and $p''$ is the symmetric price that consistent with $\zeta_{-m}$ and $h'$.

The first line of the revenue is given by

$$(1 - F_h(x_t)^{n_h})\mathbb{E}_{m' \neq m}\left[C^m_{t+\epsilon}(k, {}^{m'}h(p))\right] + F_h(x_t)^{n_h} \cdot C^m_{t+\epsilon}(k, {}^0h(p)), \tag{4.19}$$

representing the expected revenue from raising the price. Here, the cutoff type $x_t$ is unaffected by the deviator's price, because only the minimum price is relevant for trade. Consequently, $(1 - F_h(x_t)^{n_h})$ is the probability that one of $m$'s competitors sells a good at $p$. The expectation for the continuation payoff is needed, because the identity of the selected seller is uncertain, which affects the market state. With probability $F_h(x_t)^{n_h}$, no good is sold, leading to a continuation payoff of the the corresponding history, $C^m_{t+\epsilon}(k, {}^0h(p))$.

Analogously, seller $m$'s profit from complying, $p^m = p$, is given by

$$\begin{aligned}(1 - F_h(x_t)^{n_h}) \left(\tfrac{1}{M_h}(p + C^m_{t+\epsilon}(k-1, {}^mh(p)) + \tfrac{M_h - 1}{M_h}\mathbb{E}_{m' \neq m}\left[C^m_{t+\epsilon}(k, {}^{m'}h(p))\right]\right) \\ + F_h(x_t)^{n_h} \cdot C^m_{t+\epsilon}(k, {}^0h(p)),\end{aligned} \tag{4.20}$$

because, if a sale occurs at $t$, seller $m$ gets selected with probability $1/M_h$. In that case, she gets the price and the continuation payoff of having one good fewer following the corresponding history, ${}^mh(p)$. With the complementary probability, the good is bought from another seller. Similarly, if no good is traded, the game continues as described in the previous case (4.19).

Finally, the payoff from undercutting the competitors' price, $p^m < p$, is given by

$$(1 - F_h(y_t)^{n_h})\left(p^m + C^m_{t+\epsilon}(k-1, {}^mh(p^m))\right) + F_h(y_t)^{n_h} \cdot C^m_{t+\epsilon}(k, {}^0h(p^m)), \tag{4.21}$$

where $y_t > x_t$ is the cutoff type when $p^m$ is the minimum price at $t$. It must be larger than $x_t$ because buying at $t$ becomes more attractive. The continuation games differs from the two previous case as the minimum price as well. If a good is traded, it is sold by the deviator with certainty. If no good is traded, the continuation game is also different because a lower price got rejected and hence the belief about the remaining buyers' valuation is updated more pessimistically from the sellers' point of view.

Verbalizing this profit function, a seller playing against strongly symmetric prices has three choices:

1. She can raise the price, which means abstaining from selling;
2. She can match the price, which means all sellers get selected with equal probability if a buyer accepts;
3. She can undercut the price, which means she gets selected with certainty if a buyer accepts.

The setup of the model has two implications on the continuation payoffs: First, because a seller exits the market when all her goods are sold, $C^m_{t'}(0, h') = 0$ for all $h \in \mathcal{H}$ and $t' \in T$. Second, if $t'' > 1$, $C^m_{t''}(k', h'') = 0$ for any $k' \in \mathbb{N}$ and any $h'' \in \mathcal{H}$. That is, the continuation payoff is 0 at the (in terms of inertia) last trading opportunity.

Let $m$'s *marginal continuation payoff* be denoted by

$$MCP^m_{t+\epsilon}(k, h, p) = \mathbb{E}_{m' \neq m} \left[ C^m_{t+\epsilon}(k, {}^{m'}h(p)) \right] - C^m_{t+\epsilon}(k, {}^{0}h(p)). \tag{4.22}$$

**Lemma 4.11.** *In any strongly symmetric equilibrium, following any history $h$ at $t$, each seller posts a price*

$$p_t = MCP^m_{t+\epsilon}(k_m(h), h, p_t), \tag{4.23}$$

*when competing with at least one other seller.*

*Proof.* In a strongly symmetric equilibrium, no seller has an incentive to deviate from symmetric price $p_t$. A seller $m$ does not want to raise the price if

$$\begin{aligned} & (4.19) \leq (4.20) \\ \iff & p_t \geq MCP^m_{t+\epsilon}(k_m(h), h, p_t). \end{aligned}$$

A seller $m$ does not want to undercut price $p_t$ with some price $p^m < p_t$ if

$$(4.21) \leq (4.20).$$

In particular, the inequality above must hold for any arbitrarily small cut $p^m \approx p_t$, such that the inequality can be rewritten as

$$p_t \leq MCP^m_{t+\epsilon}(k_m(h), h, p_t),$$

because for any sequence of prices $({}^l p^m)_{l=1,...}$ with $({}^l p^m) \to p_t$, $F({}^l y_t) \to F(x_t)$ and $C^m_{t+\epsilon}(k, {}^m h({}^l p^m)) \to C^m_{t+\epsilon}(k, {}^m h(p_t))$.

Combining the two conditions on $p_t$ leads to a necessary condition for strongly symmetric equilibrium prices, (4.23). □

This lemma pins down a condition for strongly symmetric equilibria, namely, that continuation payoffs are symmetric over sellers even when goods are asymmetrically distributed, i.e.,

$$MCP^m_{t+\epsilon}(k, h, p) = MCP^{m'}_{t+\epsilon}(k', h, p) \quad \forall m, m' \in \mathcal{M} \text{ and } k, k' \in \mathbb{N}. \tag{4.24}$$

In an environment with $k_m(h) \in \{0, 1\}$ for all $m \in \mathcal{M}$, the continuation payoffs are clearly symmetric, because each seller either offers a single good or has exited the market.

**Pricing under monopoly: No commitment**

Now, I delineate the pricing strategy of a monopolist who lacks the ability to commit to future prices. In particular, I investigate in which aspects it differs from the monopolist's full-commitment strategy discussed in the beginning of the analysis. The monopolist's game is relevant for two reasons: First, it establishes the benchmark outcome under collusion when sellers maximize joint profits and, second, it is a continuation game of the dynamic oligopoly game. Since continuation payoffs determine equilibrium prices, the expected single-unit monopoly profit with an updated prior determines the price under duopoly when each seller offers a single good. Finally, I solve the game backwards sale-by-sale. Importantly, the assumption of no commitment is taken seriously here in a sense that the monopolist cannot (commit to) destroy any units of the good. For example, for uniformly distributed valuations, disposal would be profitable as the seller's revenue is only increasing in the amount of goods when there are at least twice as many buyers as goods.

I say that a history $h$ *generates monopoly* if and only if $M_h = 1$, i.e., if and only if following history $h$ only a single active seller remains in the market. By Lemma 4.11, the continuation payoffs pin down strongly symmetric equilibrium oligopoly prices. Since the good is scarce, every oligopoly can at some point become a monopoly when all but one sellers are stocked out. Under a duopoly in which both sellers offer a single good, i.e., for some $h$ with $M_h = \sum k_m(h) = 2$ , the marginal continuation payoff is a single-unit monopoly payoff. If I can determine the expected sequentially rational single-good monopoly payoff $C^m_{t'}(1, {}^{m'}h)$ for any history ${}^{m'}h \in \mathcal{H}$ and any $t \in T$. that generates monopoly along equilibrium path, (4.23) pins down the duopoly price for any corresponding history $h$ with $M_h = \sum k_m(h) = 2$. With this insight, I can proceed to solve the game backwards sale-by-sale and continue in a similar fashion starting from any other expected sequentially rational $k$-goods monopoly payoff $C^m_{t'}(k, {}^{m'}h(p))$ for any history ${}^{m'}h(p)$ that generates monopoly.

In this analysis, the history dependence of the inertia lag turns out to be helpful. By allowing the number of remaining pricing opportunities to approach infinity for the remaining time as soon as the market endogenously becomes monopolistic, I can exploit existing monopoly results. Because there is no discounting, the length of the inertia lags is irrelevant, only the number of lags remaining before the deadline matters. Hence, I can create a sequence of inertia strategy profiles $\zeta \in S$ that converges to a continuous-time strategy profile $\zeta^* \in S^*$, and I can do this without affecting the discrete-time grid of the oligopoly game before. That is, the only impact of the lags getting finer for the monopoly continuation game is that the continuation payoff converges to the continuous-time continuation payoff. This trick allows me to incorporate the convergence results of Hörner and Samuelson (2011) seamlessly.

In contrast to the full-commitment benchmark case outlined in the beginning, here, prices have to be sequentially rational. Intuitively, a monopolist without the ability to commit to future prices faces a tradeoff between perfect separation of buyer types and a positive terminal price that excludes low types. Hörner and Samuelson (2011) analyze the monopolist's game and provide two lower bounds on her profit: The static monopoly profit (achieved by posting some price above the choke price at all histories that are not terminal and posting the static monopoly price at the final opportunity) and the profit of sequential Dutch auctions without reserve price. In contrast to the full-commitment case, the seller cannot sustain a positive terminal price while screening types perfectly. I am especially interested in the latter bound.

Lemma 4.10 implies that an initially regular environment remains regular following any update at any history over which the cutoff path is nice. Moreover, for any initially regular environment, there is some $\widehat{\varepsilon}$-consistent belief update that conserves regularity if the cutoff sequence is $\varepsilon$-nice. If a positive measure of types was to accept a price at some non-terminal history $h$, the correctly updated CDF corresponding to (4.33) has a kink at the cutoff type. This kink introduces a downward jump discontinuity in the updated virtual value function and thus the updated prior generically fails regularity - even when the initial prior was regular. By Lemma 4.10, the updated virtual valuation at some history $h$ that corresponds to a nice cutoff path is identical whether a sale occurred or not. Let this updated virtual value be

$$\widetilde{\psi}(v, x) := v - \frac{F(x) - F(v)}{f(v)} = v - \frac{1 - F_h(v)}{f_h(v)}, \tag{4.25}$$

where $x$ is the cutoff type at that history. Hence, if the cutoff sequence along a monopoly generating history ${}^{m'}h$ is $\varepsilon$-nice, then the $k$-unit monopoly continuation payoff is given by

$$C^m_{t+\epsilon}(k, {}^{m'}h) = \mathbb{E}\left[\sum_{l=1}^{k} \widetilde{\psi}(Y_l^{(n')}, x_t) \middle| Y_1^{(n')} < x_t\right], \tag{4.26}$$

where $Y_i^{(n)}$ is defined as the $i$-th highest order statistic of $n$ draws from distribution $F$ and $n' = n - (K_m - k) - \sum_{m' \neq m} K_{m'}$ is the number of remaining buyers.

Unfortunately, Hörner and Samuelson (2011) restrict attention to the workhorse uniform distribution for their results on multi-unit monopoly: For sufficiently many buyers, $n > K + 5$, a replication of sequential Dutch auctions is sequentially rational.

**Lemma 4.12** (Monopoly, no commitment). *If values are uniformly distributed and $n > K + 5$, Condition 4.4 holds.*

*Proof.* See Hörner and Samuelson (2011). □

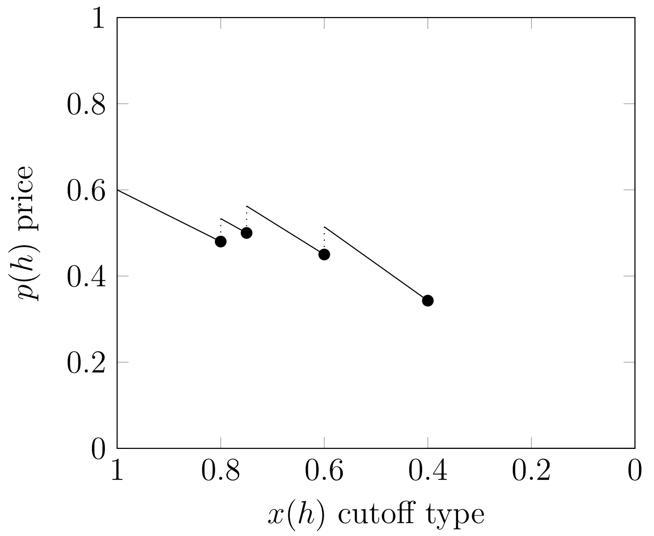



FIGURE 4.2: An exemplary price path of a four-goods monopolist who cannot commit to future prices. Sales are denoted by the black dots. The price jumps after each sale.

An exemplary monopoly price path is depicted in Figure 4.2. In words, for sufficiently scarce goods, as soon as all competitors are sold out, the last remaining seller replicates a series of Dutch auctions by continuously decreasing the price. Once a sale occurs, the monopolist immediately raises the price to a choke level and again continuously lowers the price. The price has to jump after any sale, because buyers follow a more aggressive stopping strategy. The reason is that buyers observe the sale and hence they are aware that the relative supply of the scarce good has decreased. In Bulow and Klemperer (1994), sellers cannot immediately raise the price, but buyers are repeatedly allowed to buy at the same sale price. Whenever excess demand occurs, the price in their model jumps. As a consequence either a frenzy occurs (several buyer buy at the same price) or the price "crashes", i.e., drops discontinuously.

## The buyers' stopping strategy

A sequence of decreasing prices corresponds to a sequence of decreasing cutoff types, and a buyer accepts a price if and only if his valuations is above the corresponding cutoff type. That is, taking a symmetric $\zeta \in S$ as given, a type-$v$ buyers accepts a price

$$\beta_k(v) := \min\{p_t : x_t \leq v\} \tag{4.27}$$

for the $k$-th good that is sold.

When the cutoff sequence is nice, a buyer faces a strategic tradeoff exactly as in sequential Dutch auctions with any reserve price. To compute an optimal stopping strategy at the start of one of the sequential auctions, a buyer only needs to know his valuation, a prior about the other buyers' valuations and how many auctions take place after this particular auction. Let $\beta_k$ be the symmetric stopping strategy in the $k$-th auction, let $\chi_k$ be the buyer type who was awarded the good in auction $k$ and let $r$ be a reserve price. If the cutoff sequence is only $\varepsilon$-nice, there exists an $\epsilon$-consistent posterior such that the strategic tradeoffs are "close" to sequential Dutch auctions.

**Claim 4.13.** *If cutoffs sequences are nice, the stopping strategies $(\beta_k)_{k\leq K}$ of all auctions are strictly increasing and differentiable in the valuation.*

This claim is a standard method in auction theory. It simplifies the analysis, and is easy to verify ex-post. By inverting $\beta_k$, buyers and sellers can infer a purchasing buyer's valuation $\chi$ from the trading price. By Claim 4.13, buyers buy the goods in order of their valuations. Moreover, the updated prior at the beginning of any $k$-th auction is given by $F(v)/F(\chi_{k-1})$ and any types of earlier buyers, $\chi_s$ with $s < k-1$, are irrelevant for the nice belief update. Imposing a seller strategy profile that replicates sequential Dutch auctions with reserve price $r$, let $\beta_{k,r}(v)$ be the price at which a type-$v$ buyer wants to purchase in the $k$-th auction. It turns out that the type of the buyer who purchased in the previous auction cancels out and hence the stopping strategy is also independent of the previous buyer's valuation, $\chi_{k-1}$.

**Lemma 4.14.** *Suppose $K$ units are offered in $K$ sequential Dutch auctions with reserve price $r$. The desired purchase price of a type-$v$ buyer in the $k$-th auction is given by*

$$\begin{aligned}\beta_{k,r}(v) &= \mathbb{E}\left[\beta_{k+1,r}(Y_1^{(n-k)})\middle| Y_1^{(n-k)} < v\right] \\ &= \mathbb{E}\left[\max\{Y_K^{(n-1)}, r\}\middle| Y_k^{(n-1)} < v < Y_{k-1}^{(n-1)}\right]. \end{aligned} \tag{4.28}$$

Now, Claim 4.13 now can be verified easily. The proof of this statement proceeds along the lines of, e.g., Krishna (2009, Proposition 15.2). I provide some details in Appendix VII.C. Following the lines of the proof of Lemma 4.22, also in Appendix VII.C, with $r(x) = r$ for all $x$ establishes the result as well.

In a $K$-unit sequential Dutch auction, forward-looking buyers, in equilibrium, arbitrage away the gains from preponing or postponing purchase. Consequently, sale prices are a martingale. If sale prices had, say, an upward trend, a buyer would benefit from employing a more aggressive stopping strategy in the current auction as the next item will be more expensive in expectation. If sale prices had a downward trend, a buyer would want to shade his current bid more, because the option value of the following auction is higher. On the one hand side, a buyer is willing to pay more for the good when a sale occurred, because supply decreased. On the other hand side, fewer buyers are active and they have a lower value than the buyer who purchased the previous item. In equilibrium in a symmetric independent private value environment, both effects exactly offset each other. The martingale property in sequential auctions was derived by Milgrom and Weber (2000). In the following subsection, I show a perhaps surprising implication of the martingale property: Proposition 4.15.

**Pricing under oligopoly yields monopoly profits**

The main result below appears to be counterintuitive at first glance: Why does a competing seller not have an incentive to undercut a monopoly price path at any time? The underlying reason is the martingale property derived in the previous section. Because the monopolist sells with probability one, the sellers' expected payoff from selling to the currently highest type is at each point in time equal to the expected payoff from selling to the next highest type after the highest type purchased. Under Condition, 4.4 buyers and sellers know that as soon as all but one sellers are stocked out, the remaining single seller replicates a non-exclusive Dutch auction. Because all players anticipate this sequentially rational continuation play, the proposed price path offers no opportunity for intertemporal arbitrage on the seller side as well. The following proposition establishes that, under Condition 4.4, sellers post identical prices that decrease synchronously and continuously and jump immediately after each sale.

**Proposition 4.15.** *Under Condition 4.4, there exists a SSMPBE in which the outcome of efficient sequential Dutch auctions is replicated, which is independent of the distribution of the $K$ goods.*

*Proof.* I begin with the analysis on equilibrium path: Lemma 4.14 describes the buyers' best response given the sellers use the pricing strategy proposed. I now show that the sellers have no incentive to deviate from replicating sequential Dutch auctions.

The cutoff sequence is nice over any history along equilibrium path. By Condition 4.4, the last remaining seller replicates a non-exclusive sequential Dutch auction, $r \leq \underline{v}$. The reason is that, when already starting as a monopolist, doing so when arriving at exactly the same market state is sequentially rational. Hence, the $K$-th good, which is provided under monopoly by construction, is allocated according to the proposition. Similarly, the same is true for the penultimate, the $(K-1)$-th, sale. In fact, if, for any $k \in \mathbb{N}$, the $k$-th good is offered by a monopolist, it is again allocated as proposed by assumption of Condition 4.4.

Suppose the market for the $(K-1)$-th good is duopolistic, each seller offers one good. Let $h$ be a history with $M_h = 2$ and $k_m(h) = 1$ for both active sellers. By Lemma 4.11, a duopolist wants to deviate from any price that differs from the monopoly profit she can make when her competitor sells at this price, $p_t = C^m_{t+\epsilon}(1, {}^m h(p_t))$.

Let $G_i^{(n)}(v,x)$ be defined as

$$G_i^{(n)}(v,x) := \sum_{l=0}^{i-1} \binom{n}{l} \left(\frac{F(v)}{F(x)}\right)^{n-l} \left(1 - \frac{F(v)}{F(x)}\right)^{l},$$

the cdf of $Y_i^{(n)} < x$, and let $g_i^{(n)}(v,x)$ be the corresponding density.

Define $\widetilde{h}_t$ as a continuation of $h$, in which the price decreased continuously and the $(K-1)$-th trade has not occurred yet, i.e., all prices have been rejected so far. The following term is rearranged in Appendix VII.D. For any such history $\widetilde{h}_t$, there exists a corresponding buyer type $\chi = x_t$ who wants to accept price $p_t$. The price paid for the $(K-1)$-th good by this buyer is given by (4.28) of Lemma 4.14,

$$\begin{aligned} p_t = \beta_{K-1,\underline{v}}(\chi) &= \underline{v} + \int_{\underline{v}}^{\chi} g_1^{(n-K+1)}(v,\chi)\beta_{K,\underline{v}}(v)dv \\ &= \mathbb{E}\big[\widetilde{\psi}(Y_1^{(n-K+1)},\chi)\big|Y_1^{(n-K+1)} < \chi\big] = C^m_{t+\epsilon}(1, {}^{m'}\widetilde{h}_t), \end{aligned} \quad (4.29)$$

which is exactly the expected profit of a monopolist selling the last good when the penultimate good got sold to type $\chi$: The expected value of the virtual valuation of the highest buyer type left in the market. Therefore both sellers $m$ and $m'$ have no incentive to deviate from any price $\beta_{K-1,\underline{v}}(\chi)$ for any $\chi = x(\widetilde{h})$ along the price sequence for the $(K-1)$-th sale, i.e., at no time after the $(K-2)$-th sale.

Next, consider the $k$-th sale for any $k < K - 1$. From Lemma 4.14,

$$\beta_{k,\underline{v}}(\chi_k) \quad = \quad \mathbb{E}\left[\beta_{k+1,\underline{v}}(Y_1^{(n-k)}) \middle| Y_1^{(n-k)} < \chi_k\right].$$

Hence, a seller is indifferent between selling the $k$-th traded good in the market to some type $\chi_k$ or trading the $(k+1)$-th good in the market with the highest type at the time, given that the $k$-th good was sold to type $\chi_k$.

This statements also holds true when the market is asymmetric. Because of the martingale property, the expected prices of future sales are equal. Then the symmetry of continuation payoffs pin down equilibrium prices at each time, that is, for any history $h$ and the corresponding cutoff type $x_t = \chi_k$. Hence,

$$p_t = \beta_{k,\underline{v}}(\chi_k) = k\beta_{k,\underline{v}}(\chi_k) - (k-1)\beta_{k,\underline{v}}(\chi_k) = MCP^m_{t+\epsilon}(k, h_t, p_t)$$

and the necessary condition of Lemma 4.11 holds. This condition is also sufficient, because a deviator can at no history gain more than $p_t$: While undercutting is dominated by letting the competitor sell, raising the price essentially means letting the competitor sell which yields payoff also equal to $p_t$. Raising the price does not even influence the market sate.

Suppose that a seller at some time $t$ with history $h$ sets a lower price $p' < p_t$ than she is supposed to set. As a consequence, the cutoff sequence off equilibrium path is only $\varepsilon$-nice, where $\varepsilon$ depends on the size of the price cut. Then, there exists a continuum of $\varepsilon$-consistent beliefs approximating correct off-path beliefs $F_{no,h}$ and $F_{s,h}$ that are first-order stochastically dominated by the on-path belief. Since the correct off-path belief is first-order stochastically dominated by the on-path belief as well, one can always find such an $\varepsilon$-consistent belief. Therefore not only the price, also the off-path continuation payoff is weakly below the on-path continuation payoff. In particular, by Lemma 4.10, the on-path belief is $\varepsilon$-consistent with the correct off-path belief. With this belief, the deviation leads to equal continuation payoffs, but is strictly non-profitable as $p' < p_t$. □

In words, when sequential Dutch auctions are replicated, any seller has, at any price along the continuous price path, no incentive to deviate. The reason is that, in equilibrium, the marginal continuation payoff is at each point in time equal to the current price as prices are a martingale. Under Condition 4.4, all goods gets traded with certainty, and hence the price is equal to the marignal continuation payoff.

Proposition 4.15 only holds when the monopoly continuation game induces an efficient allocation. From earlier analysis, it is known that a seller with price commitment wants to exclude low buyer types, and thus Condition 4.4 fails. I address sellers with price commitment in the following section. However, even for the case of no price commitment, Hörner and Samuelson (2011) show that a monopolist only prefers to implement an efficient allocation when the good is sufficiently scarce. For example, a monopolist facing a single buyer maximizes her profit by always posting unacceptable prices up to the deadline, at which she posts the static monopoly price. Similarly, for few buyers and few goods, the monopolist also posts unacceptable prices until *few* pricing opportunities before the deadline.

## IV Full price commitment

In this section, I shed light on the role of commitment to future prices. Although the no-commitment case appears to be more suitable for applications, the full-commitment solution is a relevant benchmark case to quantify the value of commitment. In this section, I consider the same model as in the previous one, but I relax the no-commitment restriction on the sellers' behavior: In the beginning of the game, each seller $m$ commits to a price plan contingent on each possible market state (full price commitment).

By Corollary 4.3, Condition 4.4 holds if and only if (4.9) is true. Thus, under this condition, Proposition 4.15 continues to hold. If $\underline{v}f(\underline{v}) < 1$, the optimal allocation excludes low buyer types and hence the good is not sold with probability one. In a no-gap case, $\underline{v} = 0$, a monopolist prefers to exclude low-type buyers for any updated prior $f_h$. Hence, if sellers' strategies are not restricted by sequential rationality, the remaining seller at any history that generates monopoly would not want to sell with probability one. However, the measure of excluded types with respect to the initial type distribution is smaller.

Suppose, all sellers replicate the monopoly price path of an optimal Dutch auction. Then, a single seller can profitably deviate by decreasing the price further than the optimal reserve price $r^*$. On the one hand, the deviator gains in case she becomes the only seller posting the minimum price because she exploits the revealed information that active buyers' valuations are lower. On the other hand, the deviator loses from the fact that buyers employ a less aggressive stopping strategy as they anticipate that with some probability (in the case in which only the deviator remains) the terminal price will be lower. For a monopolist, the loss from the second effect exceeds the gain from the first effect. Under oligopoly, however, the second effect is shared with all other sellers while the gains of the first effect are solely pocketed by the deviator. In other words, the buyers shade their bid less compared to the monopoly case because the reserve price is lower once the deviator becomes the monopolist which occurs with a positive probability. By a Bertrand argument, posting a positive price at the final trading possibility cannot form an equilibrium when the market is oligopolistic at that time.

The following proposition pins down an equilibrium price path: All sellers post identical synchronously and continuously decreasing prices that jump to the choke price after each sale if at least two sellers are active. When only a single seller remains active, she sells her goods by replicating sequential Dutch auctions with an exclusive reserve price that is optimal with respect to the updated prior of the history that generated the monopoly. By Claim 4.13, the type of the buyer who purchased the last good traded in oligopoly is learned from the corresponding price paid. Since the proposed price path is nice, the updated virtual valuation is given by (4.25). Figure 4.3 shows (a) an exemplary oligopoly price path, when sellers can commit, juxtaposed with (b) the corresponding monopoly price path.

Let $\widetilde{\psi}^{-1}(0, x_t)$ be the inverse of $\widetilde{\psi}$ with respect to the first argument evaluated at 0 and $x_t$.

**Proposition 4.16.** *Suppose* $\underline{v}f(\underline{v}) < 1$. *When sellers can commit to a price path contingent on market states, the price continuously decreases with upward jumps whenever a sale occurs. For any history* $h$ *that generates monopoly at time* $t + \epsilon$*, the monopolist commits to a Dutch auction with reserve price* $r^*(x_t) = \max\{\widetilde{\psi}^{-1}(0, x_t), \underline{v}\}$ *Prices and expected industry profits are higher under monopoly than under oligopoly.*

*Proof.* The reserve price $r(\chi)$ set by a monopolist who emerged endogenously at some time $t$ is determined by valuation $\chi = x_t$, the type of the last buyer who purchased under oligopoly. Function $r$ maps a buyer type $x$ into a reserve price such that $\widetilde{\psi}(r(x), x) = 0$ with $\widetilde{\psi}$ given in (4.25). From then on, the buyers' best response to the monopolist's strategy is given by Lemma 4.14 with $r = r^*(\chi)$, $\beta_{k,r^*(\chi)}(v)$.

Next consider the buyers' best response to the sellers' proposed pricing strategy under oligopoly. Let $\beta_k^{oli}(v,\omega)$ be stopping price of type $v$ when the $k$-th sale takes place under an oligopolistic market state $\omega$, as defined in (4.3). The strategy depends on $\omega$ because the distribution of goods determines how likely it is that the market becomes monopolistic for the next sale(s).

The intertemporal arbitrage condition of a buyer implies that he must be indifferent between getting the $k$-th good at some price $\widehat{p}$ and entering an auction for the remaining $(K-k)$ goods when the $k$-th good was sold at the same price $\widehat{p}$. In particular, this is true for any sale that could potentially be the last sale under oligopoly. In Lemma 4.22 in Appendix VII.C, I show that

$$\begin{aligned} \beta_k^{oli}(v,\omega) &= \sum_{w\in\Omega_\omega} \Pr(\omega' = w) \\ &\cdot \mathbb{E}\left[\beta_{k+1}^{oli}(Y_k^{(n-1)}, w)\,\middle|\, Y_k^{(n-1)} < v < Y_{k-1}^{(n-1)}\right], \end{aligned} \tag{4.30}$$

where $\Omega_\omega$ is the set of all market states that can possibly arise from $\omega$ when a single sale occurred. Similar to the procedure before, this formulation allows me to solve the game backwards from the $K$-th sale on, which by construction occurs in a monopolistic market. For details, see Lemma 4.22 in Appendix VII.C.

I now show that the proposed seller behavior is indeed the best reply to the buyers' strategy.

Consider the penultimate sale $(K-1)$ with a duopolistic market state $\omega = (F_h,(1,1))$ at some history $h$ such that the next market state after a sale is monopolistic with certainty. Suppose all players have behaved as proposed so far. Let $\widetilde{h}_t$ be some continuation of $h$ in which all prices along the continuous price path until time $t$ were rejected.

The stopping strategy $\beta_{K-1}^{oli}(\chi,\omega)$ (see (4.45) in Appendix VII.C) of a buyer type $\chi$ is given by

$$\begin{aligned} \beta_{K-1}^{oli}(\chi,\omega) &= \underline{v} + (n-K-1)\int_{\underline{v}}^{\chi}\left(\frac{F_h(z)}{F_h(\chi)}\right)^{n-K}\frac{f_h(z)}{F_h(\chi)} \\ &\cdot \left[z - \int_{r(z)}^{z}\left(\frac{F_h(y)}{F_h(z)}\right)^{n-K} dy\right] dz. \end{aligned}$$

I can rewrite this term (see the details in Appendix VII.E) as

$$\begin{aligned}\beta^{oli}_{K-1}(\chi,\omega) &= \int_{r(\chi)}^{\chi}(n-K+1)\frac{f(y)}{F(\chi)}\left(\frac{F(y)}{F(\chi)}\right)^{n-K}\left(y-\frac{F(\chi)-F(y)}{f(y)}\right)dy \\ &= \mathbb{E}\left[\max\{\widetilde{\psi}(Y_1^{(n-K+1)},\chi),0\}\right] = C^m_{t+\epsilon}(1,{}^m\widetilde{h})\end{aligned}$$

where the second line holds for exactly one function $r$: $r^*(x)$ such that $\widetilde{\psi}(r^*(x),x) = 0$, the reserve price function proposed in this statement. Each $\chi$ corresponds to a history $\widetilde{h}_t$ with $x_t = \chi$ and

$$p_t = \beta^{oli}_{K-1}(\chi,\omega) = C^m_{t+\epsilon}(1,{}^m\widetilde{h})$$

That is, both sellers are indifferent between selling to type $\chi$ at price $\beta^{oli}_{K-1}(\chi,\omega)$ and obtaining the expected monopoly profit of the final sale when type $\chi$ purchased the penultimate good. The equality of the price and the corresponding monopoly continuation payoff holds at every point in time when players follow the proposed strategy profile. The resulting price path is nice everywhere.

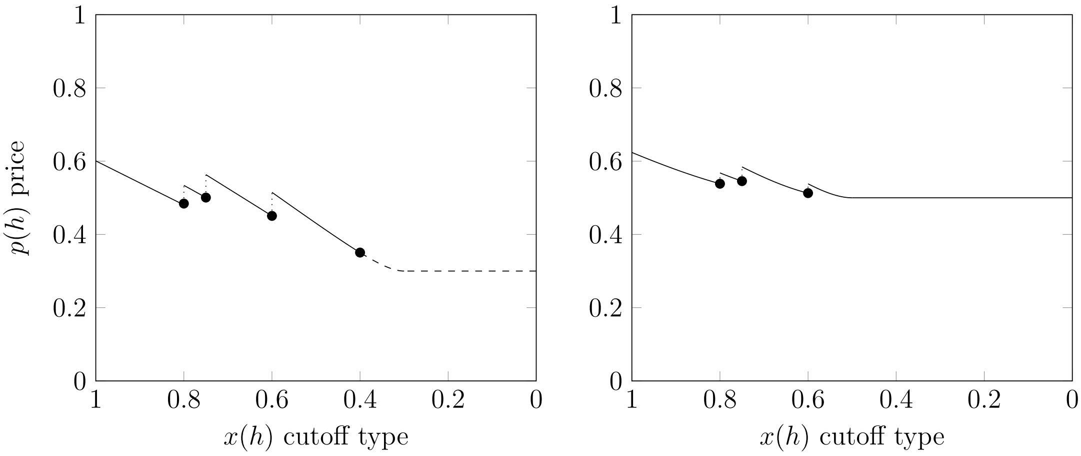
The off equilibrium path analysis is more involved compared to Proposition 4.15 since after deviations continuation play does not have to be sequentially rational. Discontinuous price cuts are not profitable deviations following the same argument as in the proof of Proposition 4.15: It strictly reduces the payoff from the current price and it weakly decreases the continuation payoff which is maximized under the proposed rule.

Next suppose that some seller commits to some price path other than a sequential Dutch auction with reserve price rule $r^*(x)$ in a monopolistic market state. Still the same types as on equilibrium path buy at the oligopolistic market states and hence the same monopoly posterior is induced. By definition the monopoly continuation payoff decreases, as it is maximized under the proposed rule. The intertemporal arbitrage condition of the buyers requires that the marginal type $\chi$ that accept the last price posted under oligopoly is indifferent between buying at this price and entering the monopoly continuation game. Suppose the deviating monopoly continuation game increases the payoff of type $\chi$ in compared to the proposed equilibrium. By incentive compatibility, it also increases the payoff of all types larger than $\chi$. As a consequence the stopping strategy of all types that buy in equilibrium becomes less aggressive such that the accepted prices are lower, making this deviation non-profitable for the deviating seller.

Suppose the deviating monopoly continuation game decreases the utility of type $\chi$. The idea of this deviation is to increase the price to be gained under oligopoly at the cost of decreasing the monopoly continuation payoff. An upper bound of this deviation is the profit of the same deviation while also decreasing the oligopoly price slightly faster than the other sellers. That way it is guaranteed that the deviator sells all her goods under oligopoly without reaching the monopoly continuation game which yields less payoff than the equilibrium payoff. However, in this case the continuation game after the deviator is sold out is exactly the same as in equilibrium. Hence, the bidding strategies are the same and the deviator does not gain from this deviation. □

The proposed monopoly continuation play is the only strategy that ensures that the buyers' and the sellers' intertemporal arbitrage conditions hold simultaneously. That is, only when sellers, who endogenously become monopolists, commit to conducting sequential Dutch auctions with the given reserve price function, the sellers' marginal continuation payoffs have the martingale property as well. In comparison with the no-commitment case, buyers purchase at higher prices, but in expectation fewer goods are sold because the prices do not decrease as much. The opposite is true in the comparison with the full-commitment monopoly case.

Figure 4.3 illustrates the difference between monopoly and oligopoly prices with price commitment. Although the first three units are sold to the same buyer types, monopoly prices are higher. The reason is that an ab-initio $K$-unit monopolist commits to a higher terminal price than an endogenously emerging monopolist. The latter's reserve price is ex-ante unknown, but lower than the ab-initio monopolist's reserve price with probability one.




(A) Oligopoly price path, full commitment

(B) Monopoly price path, full commitment

FIGURE 4.3: Comparison of monopoly and oligopoly (each seller one good) prices. Price paths under full commitment with $K = 4$ goods and $n = 10$ buyers with uniformly distributed values. The black dots indicate the sales to types 0.8, 0.75, 0.6(, 0.4 - not served with $M = 1$).

## V Discussion

In this model, the strategic interaction between forward-looking buyers and sellers without price commitment in continuous time suggests that it is irrelevant for profits and buyer surplus how goods are distributed among sellers, because monopolistic market power can be sustained anyway. Proposition 4.15 is a counterintuitive result because economists instinctively promote competition in standard settings (without innovation, synergies, natural monopoly cost structures etc). From any real world angle, it appears to be a questionable policy advice to ignore market conditions in the industries mentioned in the introduction. To provide a better understanding of Proposition 4.15, I suggest some modifications which may overturn the result.

I interpret Proposition 4.15 as a benchmark result that sheds light on which characteristics of an industry are important when, e.g., evaluating the welfare effect of a merger. In Section IV, I analyzed the role of commitment to future prices in destroying the irrelevance result. In the following, I investigate the role of other important aspects and connect my results to the literature on sequential auctions.

In standard oligopoly models, sellers' incentives to undercut prices provide benefits to consumers. In the equilibrium of Proposition 4.15, these incentives are not present because of the simultaneous intertemporal arbitrage conditions of buyers and sellers which result in martingale prices. Since the benchmark result is driven by the martingale property of equilibrium prices, it opens the door for research investigating a similar setting in which prices do not follow a martingale.

**Interdependent values:** The martingale property of prices in sequential auctions in was derived by Milgrom and Weber (2000, written in 1982). In addition, they show that prices tend to drift upward in a model with interdependent values with affiliated signals. For example, a reasonable application of my model with affiliated signals are fashion fads: There is a sales season and the previously produced goods are only fashionable for a given time after which the market dries up. Because the sale price of items bought earlier reveals information about the value of the good to other buyers, remaining buyers are willing to pay higher prices for the next items. Then, however, a seller prefers to trade later rather than earlier. Extending my model in this extension would be interesting.

**Risk:** Empirically, prices in real world sequential auctions appear to show a downward trend, a stylized fact known as the "declining price anomaly". This term was popularized by Ashenfelter (1989) who notes such a trend in prices of sequential art and wine auctions. Since then many empirical papers ( e.g. Van den Berg, Van Ours, and Pradhan (2001)) reported declining prices and many theoretical papers provided possible explanations for the finding. A natural explanation for declining prices is risk aversion. McAfee (1993) can explain the discrepancy between theory and empirics with nondecreasing absolute risk aversion which appears to be unconventional. More recently, Hu and Zou (2015) set up a model with "background risk", i.e., bidders participate in auctions not only to seek profits, but also to avoid losses. Bidders exhibit non-quasilinear utilities and the risk exposure is type dependent. In such a setting, they show that a pure strategy equilibrium in sequential first- or second price auctions exists when marginal utilities of income are log-supermodular in payment and type. Equilibria feature a declining price path when bidders are risk-averse and an increasing price path when bidders are risk-loving. Buyers' background risk can easily be incorporated into my model and the implications of background risk in a setting with competition on the seller side remain to be investigated.

**Unobserved inventory:** I assume that buyers are always aware of how many goods are left to allocate. Internet platforms often reveal the inventory (e.g. number of seats, rooms or tickets left), but in many settings, especially in bigger markets, this assumption might be implausible. Because airlines sometimes reserve seats for special passengers, the number of remaining seats observed online is only an informative proxy for real inventory. In Jeitschko (1999) prices in sequential auctions can decline because the number of units is unknown. Due to the uncertainty whether a next auction takes place, the option value of participating in the next auction declines which drives up the price in the current auction. Similarly, an increasing expected price path can be found when information arrives that fewer units than anticipated will be sold.

**Arriving buyers:** Another intriguing extension would be to allow additional buyer entry over time. A dynamic buyer population could be incorporated into my setting by dividing the continuous time interval into several continuous time intervals which start with the arrival of additional buyers. This extension is particularly interesting in the context of airline tickets as, say, business travelers find out about their need to travel much later than leisure travelers. The vast majority of theory papers predict falling prices as the deadline approaches, contradicting the date (see McAfee and Te Velde (2006) for stylized facts about pricing in the airline industry). Board and Skrzypacz (2015) consider such a model with a single seller. Remarkably, they show that the optimal allocation in the continuous-time limit of their setting can be implemented with an optimal path of posted prices. However, this result heavily hinges on their assumption of discounting. For several applications, discounting is of second order importance. To illustrate, a hotel room is consumed and paid at the day of arrival and hence the time of purchase is only indirectly relevant through the price and the probability that there still is a hotel room available. This indirect form of discounting is endogenously part of my model and an explicit discount factor may only reflect a reduced form approach to model an urge to buy early. Without a discount factor, their monopolist would simply wait until the deadline when all buyers have arrived and conduct an optimal auction following Myerson (1981). This strategy is clearly not an equilibrium if additional sellers were present. Competition gives rise to an interesting dynamic of preponing sales to attract already present high-value buyers versus postponing sales to include buyers entering in the future.

**Heterogeneous goods:** Although many typical applications, such as low-cost bus and plane travel or small-sized rental cars, do not display significant brand or product differentiation, my assumption of homogeneous goods limits the scope for reasonable applications. Here, I want to stress that good homogeneity is not driving the results qualitatively per se. Suppose there is a quality difference between two goods offered by two different sellers. For example, two flights with the same destination departing on the same day, one leaving at 4 am and the other at 11 am. If the quality difference is modeled as a shift of the distribution, qualitatively the analysis remains the same. That is, if for any buyer whose willingness to pay for the good of seller A amounts to $v$, then this seller values the good of seller B $v + \Delta$. Because the quality difference is assessed unanimously, incentives are not distorted and prices continuously decrease at the same speed, but at different levels.

As discussed in the introduction, strategic buyers are prevalent in many markets. However, the purchase of some goods is rather the result of impulsive decision making instead of fully strategic considerations. The proofs of Propositions 4.15 and 4.16 hinge on the buyers' objective to optimally time their purchase. Strategic buyer arbitrage away any expected intertemporal price differences. When, in contrast, buyers are fully myopic, i.e., when they have a discount factor of zero, they buy as soon the price is below their valuation, $x_t = p_t$ for any history $h_t$.

**Myopic buyers:** I consider the extreme case of fully-forward-looking buyers. For a better understanding of this assumption, it is helpful to study the opposite extreme assumption, fully myopic buyers as in Lazear (1986). Assume the environment of Section II, but suppose buyers have a discount factor of zero. As a result, a $K$-good monopolist maximizes profit by continuously decreasing prices to make a profit of

$$\mathbb{E}\left[\sum_{k=1}^{K} Y_k^{(n)}\right]$$

and all buyers obtain zero utility. Clearly, prices are decreasing over time.

Under oligopoly, a competitor has incentive to undercut prices to attract the highest-type buyer. Consequently, a positive measure of buyer types accepts the first price and, hence, the price path is not nice in equilibrium under oligopoly. Because Lemma 4.11 continues to hold and all goods are sold in equilibrium, oligopoly prices are a martingale. To illustrate, consider two sellers, each offering one good. The first price is equal to the continuation payoff of not selling and becoming a monopolist, which is equal to the expectation of the highest order statistic of $(n-1)$ draws from the updated prior. If no sale occurs, the same procedure is repeated with an updated prior. Prices jump discontinuously in duopoly and when a sale occurs, the remaining seller continuously decreases prices. The expected revenue of the monopolist is equal to the expected price which is equal to the duopoly price. For asymmetrically distributed capacities the martingale property of sale prices is lost.

## VI Conclusion

In this paper, I contribute to filling a gap in the RM literature by analyzing oligopolistic competition. Virtually none of the industries characterized by RM business conditions is monopolistic and hence this paper adds to a better understanding of real world RM industries. Surprisingly, my setting features equal allocations, prices, joint industry profits and buyer payoffs under monopoly and oligopoly if under monopoly, an efficient allocation arises with probability one. With uniformly distributed values, the latter is true when sellers are unable to commit to future prices and goods are sufficiently scarce. With commitment, it holds when sellers value the good sufficiently less than the lowest buyer type. This result is driven by the forward-looking buyers and the scarcity of the good, because the buyers' intertemporal optimization entails martingale equilibrium prices and hence, in equilibrium, sellers have no incentive to deviate from the monopoly price path. If, however, a single seller optimally want to commit to excluding low buyer types from trade, competition on the seller side leads to lower prices accompanied by higher consumer surplus and lower industry profits.

The main result of this paper is puzzling: Why do price paths observed in reality differ when the competition is introduced? In the previous section, I offer modifications of the model that produce the more intuitive result that competition on the seller side benefits the consumers and harms the sellers. Nevertheless, this model is a relevant benchmark that contributes to a better understanding of oligopolistic RM markets. It highlights the role of commitment and the role of forward-looking buyers. Moreover, I derive sharp predictions about the behavior of prices and thereby open the door for intriguing empirical research. For example, the observation that competition beats down prices in the airline industry suggests that buyers are myopic instead of forward-looking. In combination with a more elaborate form of the proposed modifications of the model, it could be interesting to see how myopic they are.

# VII Appendix

## VII.A Defining the Continuous-Time Game

This section establishes that the continuous-time game I set up is well-defined. I expand the approach of Bergin and MacLeod (1993) for complete-information repeated games to imperfect-information stochastic games. The following proofs mimic the corresponding ones in their paper.

Let $B_T$ and $B_A$ be the Borel sets of $T$ and $A = \times_{\iota \in I} A_\iota$, respectively. Let $\mu_\iota$ be a metric on $A_\iota$ and $\mu = \sum \mu_\iota$ be a metric on $A$. Let

$$D_\iota(o_\iota, \widetilde{o}_\iota, T') = \int_{T'} \mu_\iota(o_\iota(t), \widetilde{o}_\iota(t))dt \quad \forall o_i, \widetilde{o}_i \in O_i$$

be a metric on outcome paths relative to time interval $T'$. Let $D(o, \widetilde{o}, T') := \sum_\iota D_\iota(o_\iota, \widetilde{o}_\iota, T')$ for all $o, \widetilde{o} \in O$ be a metric on $O$ relative to $T' \in B_T$. Let $B_O$ be the Borel $\sigma$-algebra determined by $D$. This metric on outcome paths is explicitly used to generate the appropriate metric $\rho$ on the space of inertia strategies. With respect to $\rho$, the completion $S^*$ of the space of inertia strategies is then taken.

Let player $\iota$'s strategy given by the mapping

$$\zeta_\iota : O \times T \to A_\iota$$

where $\zeta_\iota(o, t)$ is the action chosen by player $\iota$ at time $t$, given outcome $o$, while $\zeta_\iota(o)$ represents $\zeta_\iota(o, \cdot) \in O_i$. Note that the private information Bergin and MacLeod (1993) use this formulation because then the domain of a strategy conveniently is time invariant.

Let me, for now, ignore the restriction that strategies have to be sequentially rational, are only set contingent on a market state and a strategy of a player cannot depend on the private information of another player. The requirement that a player's strategy at time $t$ can only depend on the past is reflected by A2 below. First of all, $\zeta_\iota$ has to satisfy the following conditions:

A1. $\zeta_\iota$ is a $B_O \times B_T$ measurable function on $O \times T$.

A2. For all $t \in T$, and $o, o' \in O$ such that $D(o, o', [0, t)) = 0$, $\zeta_\iota(o, t) = \zeta_\iota(o', t)$.

These conditions are necessary, but not sufficient for well-defined strategies. See Bergin and MacLeod (1993). I also need the following condition:

**Definition 4.17.** A strategy $\zeta_\iota$ satisfies inertia if given $t \in T$, and $o \in O$, there exists an $\epsilon > 0$ and $a_\iota \in A_\iota$ such that

$$D_\iota(\zeta_\iota(o'), a_\iota, [t, t+\epsilon)) = 0,$$

for every $o' \in O$, such that $D(o, o', [0, t)) = 0$.

In other words, at every point in time, a strategy has to be constant for a small period of time. Denote by $S_i$ the set of strategies satisfying A1, A2 and inertia. The next lemma shows that any $|I|$-tuple of functions $\zeta = (\zeta_{i_1}, \ldots, \zeta_{i_n}, \zeta_{m_1}, \ldots, \zeta_{m_M}, \zeta_{N1}, \zeta_{N2}, \zeta_{N3}) \in S = \times_\iota S_\iota$ determines a unique outcome on every continuation game.

**Lemma 4.18.** *Let $\zeta \in S$, then for every $o \in O$, and $t \in T$, there exists a unique outcome $\widetilde{o} \in O$ such that $D(o, \widetilde{o}, [0, t)) = 0$, and $D(\zeta(\widetilde{o}), \widetilde{o}, [t, 1]) = 0$.*

*Proof.* See Bergin and MacLeod (1993) Theorem 1, different notation: $o = h$, $\zeta = x$. □

Given $(\bar{o}, \bar{t}) \in O \times T$, and $\zeta \in S$, let $\sigma(\zeta, \bar{o}, \bar{t})$ be the outcome that is identical to $\bar{o}$ on $[0, \bar{t})$ and is determined by $\zeta$ on $[t, 1]$. Now, define a metric on $S_\iota$,

$$\begin{aligned}\rho_\iota(\zeta_\iota, \zeta'_\iota) &= \sup\{D(\sigma((\zeta_\iota, \zeta_{-\iota}), o, t)), \sigma((\zeta'_\iota, \zeta_{-\iota}), h, t)\big| \\ &\qquad (o, t) \in O \times T, \zeta_{-\iota} \in S_{-\iota}\} \quad \forall \zeta_\iota, \zeta'_\iota \in S_\iota. \end{aligned} \tag{4.31}$$

I take the completion with respect to this metric $\rho$ to guarantee a well-defined outcome. This metric considers two strategies to be equal if they give the same outcome starting at an arbitrary history and time, when played against the same strategies of other players. Extend this metric to $\rho(\zeta, \zeta') = \sum_\iota \rho_\iota(\zeta, \zeta')$, for all $\zeta, \zeta' \in S$. Let $S^*_\iota$ be the completion relative to $\rho_\iota$, and let $S^* = \times_\iota S^*_\iota$: Two Cauchy sequences $(x_l)_{l=1,\ldots}$ and $(y_l)_{l=1,\ldots}$ in $S$ converge to the same strategy vector $\zeta \in S^*$ if and only if $\rho(x_l, y_l) \to 0$. Extend the metric to $\rho^*$ on $S^*$ by letting $\rho^*(x, y) = \lim \rho(x_l, y_l)$ for $x, y \in S^*$ and $(x_l) \to x$ and $(y_l) \to y$.

The next result establishes that every strategy in $S^*$ is identified with a unique outcome $o^* \in O$.

**Lemma 4.19.** *For every $\zeta \in S^*$, and every $(o,t) \in O \times T$, there exists a unique $o^* \in H$, such that $\sigma((\zeta_l), o, t) \to o^*$ for any Cauchy sequence $(\zeta_l)_{l=1,\ldots}$ in $S$ converging to $\zeta \in S^*$.*

*Proof.* See Bergin and MacLeod (1993) Theorem 2, different notation: $o = h$, $\zeta = x$. □

The limit is taken only within the strategy space and the outcome function is not changed. We have constructed a game such that for any $\zeta \in S^*$, a well-defined payoff of outcome $\sigma(\zeta, o, t)$ for any $(o,t) \in O \times T$ exists.

## VII.B The Posterior

**Corollary 4.20.** *Suppose history $h$ of time $t$ features no sale so far. Following a sale or no sale, sellers update their prior according to Bayes' Rule as follows:*

$$f_{no,h}(v) = \frac{f_h(v)}{F_h(x)} \text{ and} \tag{4.32}$$

$$f_{s,h}(v) = \begin{cases} \frac{f_h(v)(1-\sigma_\omega)}{(1-F_h(x)^n)\frac{n-1}{n}} & \text{if } v < x \\ \frac{f_h(v)(1-\phi_\omega)}{(1-F_h(x)^n)\frac{n-1}{n}} & \text{if } v \geq x \end{cases} \text{ for any } m \in \mathcal{M}, \tag{4.33}$$

*where $x$ is given by (4.15), $\sigma_\omega$ is given by (4.17) and $\phi_\omega$ is given by (4.16).*

*Similarly, buyers update their prior after a sale as follows:*

$$f_{d,h}(v) = \begin{cases} \frac{f_h(v)(1-\sigma'_\omega)}{(1-F_h(x)^{n-1})\frac{n-2}{n-1}} & \text{if } v < x \\ \frac{f_h(v)(1-\phi'_\omega)}{(1-F_h(x)^{n-1})\frac{n-2}{n-1}} & \text{if } v \geq x \end{cases} \text{ for any } m \in \mathcal{M}, \tag{4.34}$$

$$f_{a,h}(v) = \begin{cases} \frac{f_h(v)(1-\phi'_t)}{(1-\phi_\omega)\frac{n-2}{n-1}} & \text{if } v < x \\ \frac{f_{h^i}(v)(1-\phi''_\omega)}{(1-\phi_\omega)\frac{n-2}{n-1}} & \text{if } v \geq x \end{cases} \text{ for any } m \in \mathcal{M}, \tag{4.35}$$

*with $\sigma'_\omega = F_h(x)^{n-2}$ and $\phi'_\omega = \frac{1-F_h(x)^{n-1}}{(n-1)(1-F_h(x))}$ and $\phi''_\omega = \frac{2(n-1-nF(x)+F(x)^n)}{(n-1)n(1-F_h(x))^2}$*

Let $h'$ be a continuation of history $h$ with one additional trading opportunity Bayes' Rule states

$$f_h(v_i|h') = \frac{f_h(v)\Pr(h'|h, v_i)}{\Pr(h'|h)}.$$

The posterior when no sale occurred is straightforward to derive. Suppose a sale occurred. Let a seller consider buyer 1 wlog.

If $v_1 < x_t$, a sale to some buyer $i \neq i_1$ could only occur when at least one of the other $(n-1)$ buyers has a value greater than $x_t$. Hence,

$$\Pr(h'|v_1, h) = 1 - F_h(x_t)^{(n-1)}.$$

If $v_1 \geq x_t$, buyer 1 only remains in the market following a sale when 1 was not selected to trade. There must have been another buyer type larger than the cutoff and this buyer was selected instead. Otherwise, buyer 1 would have bought the good and would have exited the market. Hence, $\Pr(h'|h, v_i) = 1 - \phi_\omega$.

The denominator is the probability that a good gets traded, but not with buyer $i$:

$$\int_0^{x_t} f(v)(1 - F(x_t)^{n-1})dv + \int_{x_t}^1 f(v) \left( 1 - \overbrace{\frac{1 - F(x_t)^n}{(1 - F(x_t))n}}^{\phi} \right) dv.$$

Next, suppose a sale occurred following a buyer history $h$, and wlog consider buyer 2 forming a posterior about buyer 1. Suppose 2 declined purchase. As 2 declined himself, 1 only could have declined as well ($v_1 < x_t$) if one of the other $(n-2)$ buyers accepted the price. Similarly for valuations $v_1 \geq x$, buyer 2 updates his belief exactly as a seller, but considers only $(n-2)$ other buyers.

Now, suppose 2 accepted the price. In case 1 declined ($v_1 < x$), it must have been that some buyer other than 2 was selected for purchase. If 1 tried to purchase ($v_1 \geq x$) as well, it must have been that one buyer other than 1 or 2 was selected,

$$1 - \sum_{j=0}^{n-2} \binom{n-2}{j} \frac{2}{j+2} F(x_t)^{n-2-j}(1 - F(x_t))^j = 1 - \phi''.$$

The following lemma holds for all symmetric equilibria, even when no cutoff strategies are played:

**Lemma 4.8.** *The following first-order stochastic dominance results hold for any equilibrium in the buyers' game*

$$F_{no,h}(v) > F_{d,h}(v) > F_{a,h}(v) > F_h(v),$$

*for all $v, x : F_h(v) \in (0,1), F_v(x) \in (0,1)$. For $v, x : F_h(v), F_h(x) \in \{0,1\}$, all four distributions are equal.*

*Proof.* For ease of exposition, I suppress the subscripts. I proof this inequality for symmetric cutoff strategies, so that the posteriors look like the ones in Corollary 4.20. In any equilibrium not in cutoff strategies, it must be that higher types are more likely to accept a price. Otherwise, the equilibrium candidate would violate the principle of incentive compatibility. The following extends to mixed strategies that are weakly increasing in the type.

The last inequality is obviously true. I start with the first in inequality. It holds for $v < x$ if and only if

$$\frac{1 \cdot F(v)}{F(x)} > \frac{(1-\sigma')F(v)}{(1-\sigma')F(x) + (1-F(x))(1-\phi')},$$

which is obviously true. For $v > x, F_{no,h}(v) = 1$ such that the inequality holds, too.

The second inequality holds for $v < x$ if and only if

$$\frac{(1-\sigma')F(v)}{(1-\sigma')F(x) + (1-F(x))(1-\phi')} > \frac{(1-\phi')F(v)}{(1-\phi')F(x) + (1-F(x))(1-\phi'')}$$
$$\frac{1-\sigma'}{1-\phi'} > \frac{1-\sigma'}{1-\phi'} \cdot \frac{F(x) + (1-F(x))\frac{1-\phi'}{1-\sigma'}}{F(x) + (1-F(x))\frac{1-\phi''}{1-\phi'}}$$

which holds when the second term on the RHS is $\leq 1$. Hence, it is to show that

$$(1-\phi')^2 < (1-\phi'')(1-\sigma') \tag{4.36}$$
$$\left(\sum_{j=1}^{n-2} a_j\right)\left(\sum_{k=1}^{n-2} a_k\right) < \left(\sum_{j=1}^{n-2} b_j\right)\left(\sum_{k=1}^{n-2} c_k\right)$$

with

$$2a_j a_k = 2\left(\frac{j}{j+1}\right)\left(\frac{k}{k+1}\right)\binom{n-2}{j}\binom{n-2}{k} F(x)^{2(n-1)-j-k}(1-F(x))^{j+k} <$$
$$b_j c_k + b_k c_j = \left(\frac{j}{j+2} + \frac{k}{k+2}\right)\binom{n-2}{j}\binom{n-2}{k} F(x)^{2(n-1)-j-k}(1-F(x))^{j+k}$$

for all $j \neq k \in (1, \ldots, n-2)$ and

$$a_j a_k = \left(\frac{j}{j+1}\right)^2 \binom{n-2}{j}^2 F(x)^{2(n-1)-2j}(1-F(x))^{2j} <$$
$$b_j c_k = \left(\frac{j}{j+2}\right) \binom{n-2}{j}^2 F(x)^{2(n-1)-2j}(1-F(x))^{2j}$$

for all $j = k \in (1, \ldots, n-2)$.

The second inequality holds for $v \geq x$ if and only if

$$\frac{(1-\sigma')F(x) + (F(v)-F(x))(1-\phi')}{(1-\sigma')F(x) + (1-F(x))(1-\phi')} > \frac{(1-\phi')F(x) + (F(v)-F(x))(1-\phi'')}{(1-\phi')F(x) + (1-F(x))(1-\phi'')}$$
$$\frac{(1-\sigma')}{(1-\phi')} \cdot \frac{F(x) + (F(v)-F(x))\frac{(1-\phi')}{(1-\sigma')}}{\frac{(1-\sigma')}{(1-\phi')}F(x) + (1-F(x))} > \frac{(1-\phi')}{(1-\phi'')} \cdot \frac{F(x) + (F(v)-F(x))\frac{(1-\phi'')}{(1-\phi')}}{\frac{(1-\phi')}{(1-\phi'')}F(x) + (1-F(x))}$$

which is true because of (4.36). □

**Corollary 4.21.**

$$W'_+(v_i, h, p, 0) \geq W'_+(v_i, h, p, 1) \quad \textit{for a.e. } v_i \in [\underline{v}, \overline{v}]$$

We have shown that a declining buyer is more optimistic about getting a good than an accepting buyer the Lemma 4.8 .

**Lemma 4.10.** *If the cutoff sequence is $\varepsilon$-nice over history $h$, then for all truncations $h_t$ of $h$, the prior $F_{h_t}$ is $\varepsilon$-consistent with posteriors $F_{s,h_t}$, $F_{a,h_t}$ and $F_{d,h_t}$ following a sale at $h$, and $\widehat{\varepsilon}$-consistent with posterior $F_{no,h_t}$ following no sale at the same history for some $\widehat{\varepsilon} \geq \varepsilon$.*

*Proof.* Since the (buyer or seller) posterior after sale is first-order stochastically dominated by the prior, and it first-order stochastically dominates the posterior following no sale, for all $G \in \{F_{s,h_t}, F_{a,h_t}, F_{d,h_t}\}$,

$$\frac{F_{h_t}(v)}{F_{h_t}(x_t)} \geq G(v) \geq F_{h_t}(v) \tag{4.37}$$
$$1 \geq \tfrac{F_{h_t}(v)}{G(v)} \geq F_{h_t}(x_t) \geq 1-\varepsilon, \tag{4.38}$$

which shows the first statement.

From inequality (4.37), it also follows that any such $G$ is $\varepsilon$-consistent with $\frac{F_{h_t}}{F_{h_t}(x_t)}$,

$$1 \quad \geq \frac{G(v)}{F_{h_t}(v)/F_{h_t}(x_t)} \geq \quad F_{h_t}(x_t) \geq 1-\varepsilon$$

As a consequence $F_{h_t}$ is $\widehat{\varepsilon}$-consistent with $\frac{F_{h_t}}{F_{h_t}(x_t)}$, because

$$\frac{F_{h_t}(v)}{F_{h_t}(x_t)} \geq G(v) \geq (1-\varepsilon)\frac{F_h(v)}{F_h(x(h))}$$

together with (4.38) implies

$$\begin{aligned} 1 \geq \frac{F_{h_t}(v)}{G(v)} &\geq \frac{F_{h_t}(v)}{F_{h_t}(v)/F_{h_t}(x_t)} \qquad \text{and} \\ 1-\varepsilon \leq \frac{F_{h_t}(v)}{G(v)} &\leq \frac{F_{h_t}(v)}{(1-\varepsilon)F_{h_t}(v)/F_{h_t}(x_t)}. \end{aligned}$$

Hence,

$$(1-\widehat{\varepsilon}) = (1-\varepsilon)^2 \leq \frac{F_{h_t}(v)}{F_{h_t}(v)/F_{h_t}(x_t)} \leq 1.$$

□

### VII.C Stopping Strategies $\beta$

Proof of Lemma 4.14

*Proof.* Suppose that Claim 4.13 is true. Then buyers purchase in order of their values and players can infer the valuation of a buyer from the price he paid. Let $h$ be a history in which the penultimate good got sold to type $\chi$. Then, the (unique, see, e.g., Maskin and Riley (2000)) equilibrium stopping strategy in the final Dutch auction with $n_h$ buyers and reserve price $r \geq \underline{v}$ is given by

$$\begin{aligned} \beta_{K,r}(v) &= r\frac{G_1^{(n_h-1)}(r)}{G_1^{(n_h-1)}(v,\chi)} + \frac{1}{G_1^{(n_h-1)}(v,\chi)}\int_r^v y g_1^{(n_h-1)}(y,\chi)dy \qquad (4.39) \\ &= \mathbb{E}\left[\max\{r, Y_1^{(n-K)}\}\middle| Y_1^{(n-K)} < v\right] \\ &= \mathbb{E}\left[\max\{r, Y_K^{(n-1)}\}\middle| Y_K^{(n-1)} < v < Y_{K-1}^{(n-1)}\right] \ \forall v \in [r,\chi]. \qquad (4.40) \end{aligned}$$

Types $v < r$ abstain from buying and types above $v > \chi$ purchase at $\beta_1(\chi)$ which only happens off path. The strategy $\beta_{K,r}$ is independent of $\chi$.

The iterative arguments behind Krishna (2009, Proposition 15.2) or Lemma 4.22 (see the proof below) straightforwardly extend to a non-zero and fixed reserve price,

$$\begin{aligned} \beta_{k,r}(v) &= \mathbb{E}\left[\beta_{k+1,r}(Y_1^{(n-k)})\middle| Y_1^{(n-k)} < v\right] \qquad (4.41) \\ &= \mathbb{E}\left[\beta_{k+1,r}(Y_k^{(n-1)})\middle| Y_k^{(n-1)} < v < Y_{k-1}^{(n-1)}\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\max\{r, Y_K^{(n-1)}\}\middle| Y_k < Y_{k-1}\right]\middle| Y_k^{(n-1)} < v < Y_{k-1}^{(n-1)}\right] \\ &= \mathbb{E}\left[\max\{r, Y_K^{(n-1)}\}\middle| Y_k^{(n-1)} < v < Y_{k-1}^{(n-1)}\right] \ \forall v \in [r,\chi], \qquad (4.42) \end{aligned}$$

which is (4.28). □

**Lemma 4.22.** *Suppose $K$ units are offered and prices behave as suggested in Proposition 4.16. When the market has an oligopolisitc market state $\omega$ at history $h$, the desired purchase price of a type-$v$ buyer when the $k$-th good is offered is given by*

$$\begin{aligned} \beta_k^{oli}(v,\omega) = \sum_{w\in\Omega_\omega} & Pr(\omega_{h^m} = w) \\ & \cdot \mathbb{E}\left[\beta_{k+1}^{oli}(Y_k^{(n-1)}, w)\middle| Y_k^{(n-1)} < v < Y_{k-1}^{(n-1)}\right], \qquad (4.30) \end{aligned}$$

*where $\Omega_\omega$ is the history-dependent set of all possible market states that can follow from $\omega$ when a single sale occurred.*

*Proof.* Consider the sale of the $K$-th good. Suppose that sellers have acted according to Proposition 4.16 so far. At history $h$, the market was duopolistic, $\omega = (F_h, (1,1))$ and then the $(K-1)$-th good was the last good traded in oligopoly, leading to history ${}^m\widetilde{h}_t(p)$. Let the corresponding buyer type be denoted by $\chi = x_t$ and he bought at price $p = \beta^{oli}_{K-1}(\chi, \omega)$. Then, by the Revenue Equivalence Theorem, a buyer's expected utility from entering the last auction at history ${}^m\widetilde{h}(\beta^{oli}_{K-1}(\chi, \omega))$ is given by

$$u_K(v, r(\chi), {}^m\widetilde{h}) = \begin{cases} \int_{r(\chi)}^{v} G_1^{(n-K)}(y, \chi)dy & \text{for } v \geq r(\chi) \\ 0 & \text{for } v < r(\chi) \end{cases}, \tag{4.43}$$

with $r(x)$ such that $\widetilde{\psi}(r(x), x) = 0$, because $G_1^(n-K)$ is the probability of winning when $n - K + 1$ buyers are active.

Next, consider the penultimate sale at history $h$ (with $\chi_2$ as upper bound of the support of $F_h$) and suppose two sellers are active, each offers one good, $\omega = (F_h, (1,1))$. The expected utility of a type-$v$ buyer from disguising as another type $z$ in the penultimate auction is given by

$$\begin{gathered} G_1^{(n-K+1)}(z, \chi_2)(v - \beta^{oli}_{K-1}(z, \omega)) + \\ \int_z^{\chi_2} g_1^{(n-K+1)}(x, \chi_2) u_K(v, r(x), {}^m\widetilde{h}(\beta^{oli}_{K-1}(x, \omega)))dx \end{gathered} \tag{4.44}$$

where $G_1^{(n-K+1)}(z, \chi_2)$ is the probability of winning the penultimate good and $\beta^{oli}_{K-1}(z, \omega)$ is the price to be paid when disguising as type $z$. Expression (4.43) is the expected utility when a type-$\chi$ buyer snatched the penultimate good ($\chi > z$).

Dropping some super- and subscripts for convenience, the FOC wrt $z$ is given by

$$g(z)(v - \beta(z)) - \beta'(z)G(z) - g(z)u_K(v, r(z), {}^m\widetilde{h}(\beta(z))) = 0.$$

Imposing $z = v$, rearranging and then integrating yields

$$g(z)(z - u_K(z, r(z), {}^m\widetilde{h}(\beta(z)))) = g(z)\beta(z) + G(z)\beta'(z)$$

which can be rewritten as

$$\begin{aligned}\beta^{oli}_{K-1}(v,\omega) &= \underline{v} + \frac{1}{G(v)}\int_{\underline{v}}^{v} g(z)\left[z - u_K(z, r(z), \widetilde{h}^m(\beta(z)))\right] dz \quad (4.45)\\ &= \underline{v} + \frac{1}{G(v)}\int_{\underline{v}}^{v} g(z)\cdot \beta_{K,r(z)}(z)dz\\ &= \mathbb{E}\left[\beta_{K,r(Y_1^{(n-K+1)})}(Y_1^{(n-K+1)})\middle| Y_1^{(n-K+1)} < v\right]\\ &= \mathbb{E}\left[\beta_{K,r(Y_{K-1}^{(n-1)})}(Y_{K-1}^{(n-1)})\middle| Y_{K-1}^{(n-1)} < v < Y_{K-2}^{(n-1)}\right], \quad (4.46)\end{aligned}$$

where the second line follows from the fact that type $z$, as the highest type of the support of $F_{^m\widetilde{h}(\beta(z))}$, wins the last auction with certainty and pays $\beta_{K,r(z)}(z)$.

The penultimate trade is the last trade that could possibly occur in an oligopolistic market state. Thus, the next sale is monopolistic and $\chi$, the type of the penultimate buyer, determines the reserve price of the final auction, $r(\chi)$. For earlier sales, however, the probability mass function that assigns a probability with which any of the following auctions is monopolistic depends on how the goods are distributed, which depends on the history.

Consider another history $h'$ and suppose there are three active sellers offering the last good but two, the $(K-2)$-th sale, i.e., $\omega' = (F_{h'}, (1,1,1))$. Then, with probability one, the corresponding continuation game is the duopoly analyzed above. If type $\chi_2$ buys the $(K-2)$-th good, type $v$ either gets the $(K-1)$-th good, when all other types are lower, or he gets the $K$-th good when there exists only one active type $\chi \geq v$ and all other active types less than $v$, conditional on $v \geq r(\chi)$.

Consider another history $h''$ and suppose there are two sellers offering the last good but two, and suppose that seller 1 has one good and seller 2 has two goods. Because I look for symmetric equilibria, it is irrelevant whether $\omega'' = (F_{h''}, (1,2))$ or $\omega'' = (F_{h''}, (2,1))$. Then, the continuation game is either duopolistic or monopolistic. Both continuation games are equally likely because both sellers sell with equal probability,

$$\mathbb{E}_H[u_{K-1}(v, r_h, H)] = \frac{1}{2}u^{mon}_{K-1}(v, r(\chi_2), h^{mon}) + \frac{1}{2}u^{oli}_{K-1}(v, 0, h),$$

where $h^{mon}$ generates monopoly and $h$ generates duopoly (as analyzed above). If type $\chi_2$ buys from seller 1, seller 2 becomes a monopolist and replicates a sequential Dutch auction with a reserve price $r(\chi_2)$. A type-$v$ buyer gets to buy the good if and only if $v$ is among the highest two valuations and $v \geq r(\chi_2)$. If type $\chi_2$ buys from seller 2, type $v$ gets the next good if and only if all other types are lower or he gets to buy the last good when $v$ has the highest valuation among the then active buyers, conditional on $v \geq r(\chi)$.

From (4.44), it follows that

$$\mathbb{E}_h[u_{K-1}(z, r(z), {}^m h(\beta(z))] = \frac{1}{2}(z - \beta_{K-1}(z, r(z))) + \frac{1}{2}(z - \beta^{oli}_{K-1}(z, \omega_h)),$$

because $z$ is the highest type and, on equilibrium path, wins the good with certainty.

To find $\beta^{oli}_{K-2}$, I maximize the expected utility of a buyer of type $v$ masking as a type $z$ when stopping along a price path for the $(K-2)$-th good. The objective looks just like (4.44) and the FOC corresponds to (4.45). It can be rearranged to

$$\begin{aligned}
\beta^{oli}_{K-2}(v, \omega'') &= \underline{v} + \frac{1}{G(v)} \int_{\underline{v}}^{v} g(z) \left[ z - \mathbb{E}_h[u_{K-1}(z, r(z), {}^m\widetilde{h}(\beta(z)))] \right] dz \qquad (4.47) \\
&= \underline{v} + \frac{1}{G(v)} \int_{\underline{v}}^{v} g(z) \frac{1}{2} \left[ \beta_{K-1, r(z)}(z) + \beta^{oli}_{K-1}(z, \omega) \right] dz \\
&= \frac{1}{2} \mathbb{E} \left[ \beta_{K-1, r(Y^{(n-1)}_{K-2})}(Y^{(n-1)}_{K-2}) \middle| Y^{(n-1)}_{K-2} < v < Y^{(n-1)}_{K-3} \right] \\
&+ \frac{1}{2} \mathbb{E} \left[ \beta^{oli}_{K-1}(Y^{(n-1)}_{K-2}, \omega) \middle| Y^{(n-1)}_{K-2} < v < Y^{(n-1)}_{K-3} \right] \\
&= \frac{1}{2} \mathbb{E} \left[ \beta_{K, r(Y^{(n-1)}_{K-2})}(Y^{(n-1)}_{K-1}) \middle| Y^{(n-1)}_{K-2} < v < Y^{(n-1)}_{K-3} \right] \\
&+ \frac{1}{2} \mathbb{E} \left[ \beta_{K, r(Y^{(n-1)}_{K-1})}(Y^{(n-1)}_{K-1}) \middle| Y^{(n-1)}_{K-2} < v < Y^{(n-1)}_{K-3} \right],
\end{aligned}$$

where I plugged in (4.46).

Iteratively, I arrive at (4.30).

□

## VII.D Details of Proof of Proposition 4.15

$$\begin{aligned}
\beta_{K-1, \underline{v}}(\chi) &= \underline{v} + \int_{\underline{v}}^{\chi} g_1^{(n-K+1)}(v, \chi) \beta_{K, \underline{v}}(v) dv \\
&= \underline{v} + \int_{\underline{v}}^{\chi} (n-K+1) \left( \frac{F(v)}{F(\chi)} \right)^{n-K} \frac{f(v)}{F(\chi)} \left[ v - \int_{\underline{v}}^{v} \left( \frac{F(y)}{F(v)} \right)^{n-K} dy \right] dv.
\end{aligned}$$

Then change the order of integration in the second term and cancel $F(v)$ once

$$\begin{aligned}
& \int_0^{\chi}\int_0^{v}(n-K+1)\left(\frac{F(v)}{F(\chi)}\right)^{n-K}\frac{f(v)}{F(\chi)}\left(\frac{F(y)}{F(v)}\right)^{n-K}dydv \\
= & \int_0^{\chi}\int_y^{\chi}(n-K+1)\left(\frac{F(y)}{F(\chi)}\right)^{n-K}\frac{f(v)}{F(\chi)}dvdy \\
= & \int_0^{\chi}(n-K+1)\left(\frac{F(y)}{F(\chi)}\right)^{n-K}\int_y^{\chi}\frac{f(v)}{F(\chi)}dvdy.
\end{aligned}$$

Then (after swapping $v$ and $y$ wlog) plug this term back into the original term to yield

$$\begin{aligned}
\beta_{K-1,\underline{v}}(\chi) &= \underline{v}+\int_{\underline{v}}^{\chi}(n-K+1)\left(\frac{F(v)}{F(\chi)}\right)^{n-K}\frac{f(v)}{F(\chi)}\left[v-\frac{F(\chi)-F(v)}{f(v)}\right]dv \\
&= \underline{v}+\int_{\underline{v}}^{\chi}(n-K+1)\left(\frac{F(v)}{F(\chi)}\right)^{n-K}\frac{f(v)}{F(\chi)}\left[v-\frac{F(\chi)-F(v)}{f(v)}\right]dv \\
&= \mathbb{E}\left[\widetilde{\psi}(Y_1^{(n-K+1)},\chi)\middle|\, Y_1^{(n-K+1)}\leq\chi\right].
\end{aligned}$$

## VII.E Details of Proof of Proposition 4.16

Dropping some subscripts, I can rewrite the second part of term (4.45) as

$$\begin{aligned}
& \int_{\underline{v}}^{\chi}\int_{r(z)}^{z}(n-K+1)\frac{f(z)}{F(\chi)}\left(\frac{F(y)}{F(\chi)}\right)^{(n-K)}dydz \\
= & \int_{\underline{v}}^{\chi}\int_y^{\min\{r^{-1}(y),\chi\}}(n-K+1)\frac{f(z)}{F(\chi)}\left(\frac{F(y)}{F(\chi)}\right)^{(n-K)}dzdy \\
= & \int_{\underline{v}}^{r(\chi)}(n-K+1)\left(\frac{F(y)}{F(\chi)}\right)^{(n-K)}\int_y^{r^{-1}(y)}\frac{f(z)}{F(\chi)}dzdy \\
+ & \int_{r(\chi)}^{\chi}(n-K+1))\left(\frac{F(y)}{F(\chi)}\right)^{(n-K)}\int_y^{\chi}\frac{f(z)}{F(\chi)}dzdy \\
= & \int_{\underline{v}}^{r(\chi)}(n-K+1)\left(\frac{F(y)}{F(\chi)}\right)^{(n-K)}\frac{F(r^{-1}(y))-F(y)}{F(\chi)}dy \\
+ & \int_{r(\chi)}^{\chi}(n-K+1)\left(\frac{F(y)}{F(\chi)}\right)^{(n-K)}\frac{F(\chi)-F(y)}{F(\chi)}dy.
\end{aligned}$$

The integral of the first term of (4.45) can be split into $\int_0^{r(\chi)}...+\int_{r(\chi)}^{\chi}$ and the integration variable can be renamed as $y$. Adding the two parts of (4.45) again yields

$$\begin{aligned}
\beta^{oli}_{K-1}(\chi,\omega_h) &= \int_{r(\chi)}^{\chi}(n-K+1)\frac{f(y)}{F(\chi)}\left(\frac{F(y)}{F(\chi)}\right)^{(n-K)}\left(y-\frac{F(\chi)-F(y)}{f(y)}\right)dy \\
&+ \int_{\underline{v}}^{r(\chi)}(n-K+1)\left(\frac{F(y)}{F(\chi)}\right)^{(n-K)}\left(y\frac{f(y)}{F(\chi)}-\frac{F(r^{-1}(y))-F(y)}{F(\chi)}\right)dy \\
&= \int_{r(\chi)}^{\chi} g_1^{(n-K+1)}(y,\chi)\cdot\widetilde{\psi}(y,\chi)dy, \qquad (4.48)
\end{aligned}$$

because the additive term in the second line is zero as the inverse of $r$ is given by $r^{-1}(y) = F^{-1}[yf(y)+F(y)]$:

$$\widetilde{\psi}(v,y) = v - \frac{F(y)-F(v)}{f(v)} = 0 \iff y = F^{-1}[vf(v)+F(v)].$$

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