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Econometric analysis of quantile regression models and networks

With empirical applications

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Eidesstattliche Erklärung

Hiermit erkläre ich, dass ich die vorliegende Dissertation selbständig angefertigt und die benutzten Hilfsmittel vollständig und deutlich angegeben habe.

Maria Marchenko 16. Juni 2016

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Contents

Introduction

This thesis consists of three essays that address open research issues in two econometric frameworks: nonparametric quantile regression framework and social networks, supported by empirical applications. Both econometric approaches are used to achieve a deeper understanding of the economic processes and interactions in comparison to the simple mean regression.

Quantile regression, discussed in *Chapter 1*, allows estimating and analysing the whole conditional distribution and, therefore, is able to differentiate the effect for the different quantiles of the outcomes. Quantile regression has various applications and is especially popular in socio-economic problems, analysis of individual and household finances, demand elasticities and many others. In most of this cases, more than one covariates are expected to be included in the model. However, once the nonparametric approach is chosen, the so-called *curse of dimensionality* arises. With the optimal choice of bandwidth h, the dimension of the covariate vector under which the inference and testing are possible might be insufficient for the empirical analysis. *Chapter 1* proposed a possible improvement of the nonparametric quantile regression estimation.

Chapters 2 and 3 explore the research questions in the network analysis. Linked agents are likely to have exhibit similar behaviour, hence, the inclusion of the network information into the analysis improves the understanding of the outcome determinants. Identification of the network effects is usually quite complex due to the *reflection problem* introduced by Manski (1993): the outcomes of the connections that influence one's own outcomes are affected in its turn by the outcomes. Once the network is known, the identification is achieved for the most types of the networks, under assumptions shown in Bramoullé, Djebbari, and Fortin (2009). However, modifications of the classical model may require further thorough analysis. Dynamic network model with endogenous shock (*Chapter 2*) and panel data model with fixed network (*Chapter 3*) are the examples of such modifications, required by the specific empirical examples.

Chapter 1

In the first chapter (based on joint work with Enno Mammen), I consider a problem of studying the asymptotic properties of the quantile regression estimation under increasing dimensions in the nonparametric setting. A classical approach for the analysis of the parametric and nonparametric quantile estimators use Bahadur expansion, which distinguishes two parts of conditional quantile model: the mean regression and the remainder. However, Bahadur expansion requires too restrictive assumptions for the asymptotic analysis. In a lot of interesting cases, the remainder part of the Bahadur expansion has a slower rate of convergence than the main part, and the asymptotic inference and testing is not valid for the models with more than one covariate.

In this chapter, asymptotic properties of marginal averages of kernel quantile estimators are discussed. This estimator arises in some treatment settings, as well as in single-index and partially-linear models and it can be further applied for testing procedures. Asymptotic expansions are developed for higher order terms that allow analysing under which conditions on the dimension of the covariates and on the smoothness of the underlying densities the estimator is consistent. The mathematical approach makes use of higher order Edgeworth expansions that allow calculating moments of the nonparametric kernel quantile estimator.

It was possible to show that the considered weighted average estimator works and achieves $\sqrt{n}$ rates with a normal limit for the dimensions $d = 2$ and $d = 3$ under a certain assumption, but the generalization for all functional forms of the model and for higher dimensions will not work. The first chapter provides the thorough proof of this result.

Chapter 2

In the second chapter, I discuss the dynamic behaviour of connected agents in response to the endogenous shock. I pursue the idea, that the shocks or the treatment happening to one of the players in the network influence not only their future performance but also affect all their network connections. This idea is closely related to the logic behind the spillovers in different settings, in particular, knowledge spillovers via conversational networks. I combine it with the logic used in peer effect literature to develop the dynamic model of network behaviour. Unlike spillovers, the shock on the network considered in this chapter analysis only influence the first-level connections.

Standard peer effect approach explores the co-movement, simultaneous outcomes of connected agents. Manski (1993) distinguishes three effects that determine the similar behaviour of peers. *The endogenous effect* suggests that the performance of an agent will be affected by the average performance of the peer group or network connections. *The exogenous effect* uses mean exogenous characteristics of the peer group to determine the performance. *The correlated effect* appears due to the similar individual characteristics within a group. The most important task of peer effects analysis is to determine the endogenous effect, which can have important policy implications. Some of the examples of the peer effect analysis are Ammermueller and Pischke (2009), who discusses the achievements of the peers in the primary schools, Bruce Sacerdote (2001) and Androushchak, Poldin, and Yudkevich (2013), who look at the exogenously formed groups to study the peer effects in the college or university, Gaviria and Raphael (2001), who study the influence of peers on juvenile behaviour, and many others in various empirical frameworks. The studied outcomes of peers are of the same period. However, the significant individual event is likely to have an importance for the one's connections. Comola and Prina (2014) is the closest to discuss such a network dynamics. They are using the randomized treatment as a shocking event, which is clearly exogenous in the model, although it is not necessarily the case in a more general setting. For example, shocks in educational frameworks, such as exam failures or dropouts are to a big extent determined by the network itself, and hence are endogenous.

This chapter develops the dynamic peer effect model with a shock, accounting for its possible endogeneity as well as for the changes happening to the network as a response to the shock. The model allows to predict the endogenous part of the shock and use the unexpected component to estimate the effect of pure shock. Due to simultaneous influence of connected elements on each other, a model with social interactions alone requires a particular exogenous variation to identify the endogenous effect. The inclusion of the endogenous shock in the model make identification more complex. In this chapter, I derive and prove the identification conditions for both the endogenous effect and the effect of the shock for the network case, which require variability of the network as well as the existence of intransitive triads for the model without correlated effects or the distances of length three for the model with correlated effects. *Intransitive triads* appear when some two nodes of the network are not connected directly, but via the third node, the distances of length three require the existence of two nodes, the shortest distance between which is three links, i.e. there are two nodes between them. The latter identifying assumptions were proposed in Bramoullé, Djebbari, and Fortin (2009), whereas the former assumption is novel for the literature.

I also propose the estimation procedure that uses the exogenous characteristics of the first or the second level of connections, depending on the type of model, as instrumental variables and yields the consistent estimation.

The empirical part of this chapter makes the contribution to the strain of literature analysing peer effects in educational settings. I use the dynamic network data of university students to test the model. I treat exam retakes as endogenous shocks and estimate the effect of the unexpected component of friends' retakes on one's own average grade. It is suggested that the unexpected shock, especially in the important subjects, may have a certain psychological influence on the connections. I apply the estimation procedure to the data on the students in HSE, Nizhniy Novgorod. The results indeed suggest that on average the retake of the friend may have an effect on future performance, and this effect appears to be negative, however, it has a different magnitude for students of different departments, as well as for students with and without own retake.

Chapter 3

In the third chapter, I continue analysing the network environment. I apply the standard peer effect ideology to a rather unusual setting. I assume that the connections in the art world may have an influence on both the development of the art skills and talent and reputation of a particular artist. Art market always attracted a lot of money and attention and is booming in the recent years. Quoting seminal paper by Baumol (1986), "prices [of art objects] can float more or less aimlessly and their unpredictable oscillations are apt to be the exacerbated by the activities of those who treat such art objects as "investments". He suggests that buying art is not likely to deliver any real rate of return different from zero. However, the big strain of the literature come up with a different conclusion either by improving the method or the data used for the analysis. For example, Goetzmann (1993) reports an average annual real return on oil paintings of 3.8% for the period between 1850 and 1986, with returns around 15% after 1940, Mei and Moses (2002) - the return of 4.9% for 1875-1999, with 8.2% after 1950, Renneboog and Spaenjers (2013) - 3.97% over the period 1957-2007.

Among businessmen and collectors, art is indeed often considered as an attractive investment, being one of the possible so-called passion good. Therefore, understanding the price formation is crucial for the potential buyers. As noticed in Baumol (1986), art prices are very unpredictable. Of course, the obvious factors influencing the price are the type of a work, its style, the supply of other works by the same artist, his or her level of recognition and popularity. But sometimes, especially within one particular style, these factors are not enough, and the prices achieve unexpected values. In this chapter, I explore one of the determinants of art price formation, not included into the analysis in previous literature: artists' connections. Connections are suggested to matter due to the already discussed logic of the peer effect analysis as well as by affecting the reputation of particular artists. The more valuable the works of artists' connection, the more likely the connections are to be popular and well-known not only in the art circles. Connecting to more popular artists may result in better reputation. I believe that the potential buyers will react differently, when they learn that the artist was a friend of Pablo Picasso and when they are told that the artist worked together with, for example, Morgan Russel, who was also an important figure in an abstract movement, but who are far less known than Picasso. Abstract art is in general harder to evaluate, since the quality of the work and techniques is not so straightforward, especially for a non-specialist. The network diagram prepared for the "Inventing abstraction" exhibition in MOMA, New York gave an idea to explore the networks further in the art prices setting.

I combine the information about the connections of the artists of the abstract movement with the auction prices of their work and apply the peer effect model to estimate the possible effect of the average price of artists connections on the price of artists' own work. The collected data has a panel structure, however, the network is considered constant. The discussed auctions cover the period of 2000-first half of 2015, whereas the connections were formed in the beginning of the 20th century, mainly in 1910-1925, so the connections are well-known during the auction period. The fixed network makes the usage of the fixed effects model with instrumental variables as discussed in Bramoullé, Djebbari, and Fortin (2009) impossible since the invariant covariates are not identifiable. I propose the adaptation of Hausman and Taylor (1981) approach with additional instrumental variables for the endogenous effect of Bramoullé, Djebbari, and Fortin (2009) type. Combining these two methods allows identifying coefficients for both variant and fixed covariates, including the endogenous effect.

The results of the analysis suggest the presence of endogenous peer effect, however, its direction differs for the final prices achieved on the market and prices expected by the auctioneer. The auctioneer is likely to consider connected as substitutes, alternatively the higher the value of connections' works, the more likely the artist to be "worse" than his peers. The market, however, exhibits similar demand behaviour towards the connected artists leading to the increase in one's own price as the connections' works become more valuable.

Chapter 1

Weighted average estimation in nonparametric higher-dimensional quantile regression

joint with Enno Mammen

1.1 Introduction

For a dataset of n i.i.d. tuples (X_i, Y_i) we consider a nonparametric quantile regression model:

$$Y_i = q_\alpha(X_i) + \varepsilon_{i,\alpha} \ (i = 1, \ldots, n). \tag{1.1}$$

Here, Y_i is a one-dimensional response variable and X_i is a d-dimensional covariate. The function $m_\alpha(X_i)$ is the conditional α-quantile of Y_i given $X_i = x$ for $0 < \alpha < 1$. Thus, the conditional α-quantile of the error variables $\varepsilon_{i,\alpha}$, given X_i is equal to 0. From now on, we fix the value of α and we write also q and ε_i instead of q_α and $\varepsilon_{i,\alpha}$. We are interested in the estimation of a weighted average θ of q:

$$\theta = \int q(x)\omega(x)dx \tag{1.2}$$

for a weight function ω. We discuss the following plug-in estimator

$$\hat{\theta} = \int \hat{q}(x)\omega(x)dx \tag{1.3}$$

where a kernel quantile estimator $\hat{q}$ is plugged into (1.2).

The value of θ can be of direct interest in a statistical analysis. It also arises naturally in single index quantile models where $q(x) = g(x^\intercal\beta)$ for some unknown function $g : \mathbb{R} \to \mathbb{R}$ and unknown parameter $\beta \in \mathbb{R}^d$. For a differentiable function $w : \mathbb{R}^d \to \mathbb{R}$ with compact support one gets that with $\gamma = \int g'(x^\intercal\beta)w(x)\mathrm{d}x$

$$\gamma\beta = \int q'(x)w(x)\mathrm{d}x = -\int q(x)w'(x)\mathrm{d}x.$$

Thus, $\int q(x)w'(x)\mathrm{d}x$ is equal to a multiple of β. Thus, the estimation of β can be reduced to estimation of θ with ω equal to the components of $w'(x)$. This approach has also been called average derivative estimation and was developed in Härdle and Stoker (1989) for mean regression and adapted to quantile regression in Chaudhuri, Doksum, and Samarov (1997).

A classical mathematical approach for the understanding of parametric and nonparametric quantile estimators makes use of Bahadur expansions which transform conditional quantile models to mean regression, see e.g. Chaudhuri (1991) for a discussion of locally polynomial estimators and He and Ng (1999) and He, Ng, and Portnoy (1998) for splines. Other references are El Ghouch and Van Keilegom (2009), Hong (2003), Hoderlein and Mammen (2009), Kong, Linton, and Xia (2010), Y. K. Lee and E. R. Lee (2008), Li and Racine (2008), Koenker, Ng, and Portnoy (1994), Portnoy (1997), Dette and Volgushev (2008), Belloni, Chernozhukov, and Fernández-Val (2011) and others.

It was observed in Mammen, Van Keilegom, and Yu (2015) that the use of Bahadur expansions for the asymptotic analysis of a goodness-of-fit test in nonparametric quantile regression may require too restrictive assumptions, which will not allow developing the asymptotics of the estimator in some interesting cases. In their setting direct application of Bahadur expansion would need the bandwidth h of a kernel regression quantile estimator to fulfill the condition $nh^{3d} \to \infty$ for $n \to \infty$. If the bandwidth is chosen as rate optimal, e.g. $h \sim n^{-1/(4+d)}$ for the estimation of twice differentiable functions, the assumption will only allow one-dimensional covariates. Then, this approach allows no inference and testing results for multidimensional models, $d > 1$.

In this paper we will show an expansion for the estimator $\hat{q}$, see (1.3). In the derivation of the expansion we will make use of the fact that the kernel quantile estimator and its Bahadur expansion are asymptotically independent if they are calculated at points that differ more than a constant times the bandwidth h. Similarly as in Mammen, Van Keilegom, and Yu (2015), we will use Edgeworth expansions for a statistic in a dual problem to get expansions of moments of the kernel quantile estimator and its Bahadur representation.

The paper is organized as follows. In the next section, we will discuss the model, state our main result on the asymptotics of the proposed estimator along with possible applications. Section 3 gives the proof of the main result.

1.2 Problem formulation and theory

We are interested in studying the estimation by the estimator (1.3) in model (1.1). We will use the kernel quantile estimator that is defined by:

$$\hat{q}(x) = \underset{\kappa}{\operatorname{argmin}} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right) \tau(Y_i - \kappa), \tag{1.4}$$

where $K(u_1, \ldots, u_d) = \Pi_{j=1}^{d} k(u_j)$ with a one-dimensional symmetric kernels k and where $\tau = \tau_\alpha$ is the check function $\tau_\alpha(u) = \alpha u_+ - (1-\alpha)u_-$ with $u_+ = uI(u > 0)$ and $u_- = uI(u < 0)$. We assume that the bandwidths $h_1, \ldots, h_d$ are of the same order, and for simplicity, that they are also identical. We write $h = h_1 = \cdots = h_d$. The theoretical discussions in this paper are restricted to the case that k are positive functions. Thus $\hat{k}$ is a Nadaraya-Watson like estimator with kernel of order one. We will comment on Nadaraya-Watson smoothing with higher order kernels and on local polynomial smoothing below but we will state no results for these smoothing methods. An essential argument in our proof cannot be extended to this case. For the following discussion we will assume that q has two derivatives.

Following Chaudhuri (1991), see also Theorem 2 in Guerre and Sabbah (2012), the Bahadur approximation of $\hat{q}$ is given by:

$$\hat{q}(x) - q(x) = \frac{\sum_{i=1}^{n} K\left(\frac{x-X_i}{h}\right)\{I(Y_i - q(X_i) \le 0) - \alpha\}}{\sum_{i=1}^{n} K\left(\frac{x-X_i}{h}\right) f_{\varepsilon|X}(0|X_i)} + O_P(L_n(nh^d)^{-3/4}), \tag{1.5}$$

where $f_{\varepsilon|X}$ is the conditional density of ε_i, given X_i. Here and in the following, we write L_n for sequences that fulfill $L_n = O((\log n)^C)$ for a constant $C > 0$ large enough. This expansion holds uniformly over compact subsets of $\mathbb{R}^d$. Thus, if ω has a compact support we get that

$$\hat{\theta} - \theta = \int \frac{\sum_{i=1}^{n} K\left(\frac{x-X_i}{h}\right)\{I(Y_i - q(X_i) \le 0) - \alpha\}}{\sum_{i=1}^{n} K\left(\frac{x-X_i}{h}\right) f_{\varepsilon|X}(0|X_i)} \omega(x)\mathrm{d}x + O_P(L_n(nh^d)^{-3/4}). \tag{1.6}$$

We now discuss if this expansion can be used to show that $\sqrt{n}(\hat{\theta} - \theta)$ has an asymptotic mean zero normal limiting distribution. The mean of the first term on the right hand side of (1.6) is of order h^2. This follows by standard smoothing theory using our assumption that q has two derivatives. Thus if we want to prove that $\sqrt{n}(\hat{\theta} - \theta)$ has an asymptotic mean zero normal limiting distribution, expansion (1.6) can only be used if $h^2 + (nh^d)^{-3/4} = o(n^{-1/2})$. Such choices of h only exist for $d = 1$.

The aim of this paper is to study if the estimator $\hat{\theta}$ still works for $d > 1$ and achieves $\sqrt{n}$ rates with a normal limit, for an appropriate choice of the bandwidth h. We will show that this is not the case and that the estimator for all choices of h does not has a $\sqrt{n}$ rate. The next question is if for $d > 1$ it is possible to construct an estimator of θ with $\sqrt{n}$-consistency and normal limit that works if we only make the assumption that q has two derivatives. We will give a positive answer to this question for $d = 2$ and $d = 3$. Our estimator of θ is based on the calculation of $\hat{q}$ for several bandwidths h.

For our theory, we need to make the following assumptions. We use the convention that C is a generic strictly positive constant chosen large enough and that c is a generic strictly positive constant chosen small enough. As above, we also write $L_n = (\log n)^C$ for a sequence with $C > 0$ large enough.

(A1) The support R_X of X is a compact convex subset of $\mathbb{R}^d$. The density f_X of X is strictly positive and continuously differentiable on the interior of R_X. The conditional density $f_{\varepsilon|X}(e|x)$ is uniformly bounded over x, e.

(A2) The cumulative distribution function $F_{Y|X}(\cdot|x)$ of the conditional distribution of Y given $X = x$ is twice continuously differentiable with respect to x and has a continuously differentiable density $f_{Y|X}(\cdot|x)$ that satisfies

$$\begin{aligned}
&f_{Y|X}(y|x) > 0, \\
&|f_{Y|X}(y'|x') - f_{Y|X}(y|x)| \leq C(||x' - x|| + |y' - y|)
\end{aligned}$$

for $x, x' \in R_X$ and $y, y' \in \mathbb{R}$, where $||\cdot||$ is the Euclidean norm. The density $f_{\varepsilon|X}(\varepsilon|x)$ and its derivative with respect to ε is twice differentiable in x for ε in a neighborhood of 0.

(A3) The kernel k is a symmetric, continuously differentiable probability density function with compact support (w.l.o.g, equal to $[-1, 1]$). It fulfills a Lipschitz condition and it is monotone strictly increasing on $[-1, 0]$. It holds that $k'(k^{-1}(u)) \geq \min c\{u^\kappa, (k(0)-u)^\kappa$ for some $\kappa > 0$ where k^{-1}: $[0, k(0)] \to [-1, 0]$ denotes the inverse of k: $[-1, 0] \to [0, k(0)]$. The bandwidth h satisfies $h = o(1)$ and $nh^d/L_n \to \infty$.

(A4) The function $\omega(x)$ is Lipschitz-continuous: $|\omega(x' - \omega(x)| \leq C||x' - x||$ for all $x, x' \in R_X$.

Now we can state our main result.

Theorem 1.1 *Assume (A1)-(A4). Then,*

$$\begin{aligned}
\hat{\theta} - \theta \;\; = \;\; & \int \tilde{q}(x)\omega(x)dx + h^2 \int \frac{q_0''(x) + \frac{1}{2} q_0'(x) f_X'}{f_{\varepsilon|X}(0|x) f_X(x)} \omega(x)dx + o(h^2) \\
& + \frac{1}{nh^d} \int \left[f_{\varepsilon|X}(0|x) f_X(x) \int K^2(u)du + \frac{1}{2} \frac{\partial_\varepsilon f_{\varepsilon|X}(0|x)}{f_{\varepsilon|X}(0|x)} \omega(x) \right] dx \\
& + O\Big(L_n (nh^d)^{-3/2}\Big) + O_P(L_n n^{-3/4} h^{-d/4})
\end{aligned}$$

where we put

$$\tilde{q}(x) = \frac{\sum_{i=1}^n K\left(\frac{x - X_i}{h}\right) \{I(Y_i - q_h(x) \leq 0) - \alpha\}}{\sum_{i=1}^n K\left(\frac{x - X_i}{h}\right) f_{\varepsilon|X}(0|X_i)}$$

with $q_h(x)$ such that

$$\mathbb{E}\left[K\left(\frac{x-X_i}{h}\right)\{I(Y_i - q_h(x) \le 0) - \alpha\}\right] = 0.$$

Furthermore, it holds that

$$\sqrt{n}\int \tilde{q}(x)\omega(x)dx \xrightarrow{d} \mathcal{N}(0,V),$$

where

$$V = \alpha(1-\alpha)\int \frac{\omega^2(x)}{f_X(x)f^2_{\varepsilon|X}(0|x)}dx.$$

We now discuss applications of this result. First we will discuss if there exist estimates of θ that are based on $\hat{q}$ and that achieve a parametric $\sqrt{n}$-rate of convergence. For $\sqrt{n}$-consistency we need that the bias of $\hat{q}$ is of order $o(n^{-1/2})$. This requires that $h = o(n^{-1/4})$. Under this assumption the estimator $\hat{q}$ is not consistent if $d \ge 4$. Thus $\sqrt{n}$-consistent estimation of θ based on $\hat{q}$ is not possible for $d \ge 4$. For $d = 1$ we can choose the bandwidth h such that $h = o(n^{-1/4})$, $L_n(nh^d)^{-1} = o(n^{-1/2})$ and $L_n n^{-3/4}h^{-d/4} = o(n^{-1/2})$. For such a choice of h we get that $\hat{\theta} - \theta = \int \tilde{q}(x)\omega(x)dx + o_P(n^{-1/2})$. Thus we have a $\sqrt{n}$-consistent estimator of θ with asymptotic normal limit.

For $d = 2$ there exists no choice of the bandwidth h such that $h = o(n^{-1/4})$, $L_n(nh^d)^{-1} = o(n^{-1/2})$ and $L_n n^{-3/4}h^{-d/4} = o(n^{-1/2})$. Thus we need here another approach. We propose to calculate $\hat{\theta}$ for three choices of bandwidths, h_1, h_2 and h_3, say, which fulfill $h_j = o(n^{-1/4})$, $L_n(nh_j^d)^{-3/2} = o(n^{-1/2})$ and $L_n n^{-3/4}h_j^{-d/4} = o(n^{-1/2})$ for $j = 1,..,3$. This gives three values $\hat{\theta}_1$, $\hat{\theta}_2$ and $\hat{\theta}_3$ for which it holds that

$$\hat{\theta}_j = a_{n,0} + a_{n,1}h_j^2 + a_{n,2}\frac{1}{nh^d} + o_P(n^{-1/2})$$

for $j \in \{1,2,3\}$ where $a_{n,0} = \theta + \int \tilde{q}(x)\omega(x)dx$. The three values $\hat{\theta}_1$, $\hat{\theta}_2$ and $\hat{\theta}_3$ can be used to get least squares fits $\hat{a}_{n,0}$, $\hat{a}_{n,1}$ and $\hat{a}_{n,2}$ of $a_{n,0}$, $a_{n,1}$ and $a_{n,2}$. It holds that $\hat{a}_{n,0} - a_{n,0} = o_P(n^{-1/2})$, $h_j^2(\hat{a}_{n,1} - a_{n,1}) = o_P(n^{-1/2})$ and $(nh^d)^{-1}(\hat{a}_{n,2} - a_{n,2}) = o_P(n^{-1/2})$. We choose $\tilde{\theta} = \hat{a}_{n,0}$ as our estimator of θ. By construction we have that $\tilde{\theta} - \theta = \int \tilde{q}(x)\omega(x)dx + o_P(n^{-1/2})$. Thus, we have a $\sqrt{n}$-consistent estimator of θ with asymptotic normal limit.

For $d = 3$ we need a stronger result than Theorem 1.1. Under slightly stronger smoothness conditions on the conditional density $f_{\varepsilon|X}$ one can show the following higher order expansion:

$$\hat{\theta} - \theta = a_{n,0} + a_{n,1}h^2 + a_{n,2}\frac{1}{nh^d} + a_{n,3}\frac{1}{(nh^d)^{3/2}}o(h^2) + O\Big(L_n(nh^d)^{-2}\Big) + O_P(L_n n^{-3/4}h^{-d/4})$$

with $a_{n,0} = \int \tilde{q}(x)\omega(x)dx$ and appropriate choices of $a_{n,1}$,..., $a_{n,3}$. This can be shown by the same arguments as used in Theorem 1.1 but with a higher order Edgeworth expansion. Now one chooses four bandwidths h_1, ..., h_4 with $h_j = o(n^{-1/4})$, $L_n(nh_j^d)^{-2} = o(n^{-1/2})$ and $L_n n^{-3/4}h_j^{-d/4} = o(n^{-1/2})$ for $j = 1,..,4$. This gives four values $\hat{\theta}_1$, ..., $\hat{\theta}_4$ that can be used to fit $a_{n,0}$,..., $a_{n,3}$. Again, we propose $\tilde{\theta} = \hat{a}_{n,0}$ as our estimator of θ. By construction we have again that $\tilde{\theta} - \theta = \int \tilde{q}(x)\omega(x)dx + o_P(n^{-1/2})$ and thus, we have again a $\sqrt{n}$-consistent estimator of θ with asymptotic normal limit.

We conjecture that similiar expansions as in Theorem 1.1 are valid for kernel quantile estimators with higher order kernels and for local polynomial estimators of q. In such expansions it is expected that the bias term of order h^2 and error bound $o(h^2)$ is replaced by a term of order h^{2k} and error bound $o(h^{2k})$ with an appropriate choice of the order k. Unfortunately, one of our main arguments in the proof cannot be extended to these estimators.

1.3 Proof of Theorem 1.1

For the proof we need some additional notation.

We put

$$\hat{q}^*(x) = \begin{cases} \hat{q}(x), & \text{if } |\hat{q}(x) - q_h(x)| \leq L_n(nh^d)^{-1/2} \\ q(x), & \text{otherwise.} \end{cases}$$

For the proof we also have to define local neighborhoods. For this definition suppose first that X is one-dimensional. Then the support R_X is a compact interval. For arbitrary j and for $k \in \{1, 2, 3\}$, we can then define

$$I_{jk} = [(3j + k - 1)h, (3j + k)h], \qquad \text{and} \qquad I^*_{jk} = [(3j + k - 2)h, (3j + k + 1)h].$$

The set of indices of the X_i $(i = 1, \ldots, n)$ that fall inside the interval I^*_{jk} is denoted by $\mathcal{N}_{jk}$. We write N_{jk} for the number of elements of $\mathcal{N}_{jk}$. An arbitrary $x \in R_X$ belongs to a unique I_{jk} and we define $\mathcal{N}(x) = \mathcal{N}_{jk}$ and $N(x) = N_{jk}$. If the dimension of X is larger than one, this partition of the support into small intervals can be generalized in an obvious way.

We also put $\mathcal{N}^-(x) = \{u : x_j - h \leq u_j \leq x_j + h \text{ for all } j = 1, \ldots, d\}$. This is the support of the kernel $h^{-d}K(h^{-1}[x - \cdot])$. We also write $N^-(x)$ for the random number of X_i's that lie in $\mathcal{N}^-(x)$. Note that $\mathcal{N}^-(x) \subset \mathcal{N}(x)$ and $N^-(x) \leq N(x)$. We use the shorthand notation $m_0 = nh^d$.

For the proof of Theorem 1.1, it is useful to consider the following decomposition:

$$\begin{aligned}
\hat{\theta} - \theta &= \int (\hat{q}(x) - q(x))\omega(x)dx \qquad (1.7) \\
&= \int [\hat{q}(x) - \hat{q}^*(x)]\omega(x)dx \\
&+ \int [E\{\hat{q}^*(x) - \tilde{q}(x) - q_h(x)|N(x)\}]\omega(x)dx \\
&+ \int [\{\hat{q}^*(x) - E[\hat{q}^*(x)|N(x)]\} - \{\tilde{q}(x) - E[\tilde{q}(x)|N(x)]\}]\omega(x)dx \\
&+ \int [\tilde{q}(x) - q(x)]\omega(x)dx \\
&+ \int q_h(x)\omega(x)dx \\
&= \hat{\theta}_{n1} + \ldots + \hat{\theta}_{n5}.
\end{aligned}$$

For the discussion of the terms $\hat{\theta}_{n1}$, ..., $\hat{\theta}_{n5}$ we need the following lemmas.

The following lemma gives a bound on the Bahadur expansion for $\hat{q}$. It can be shown by a small modification of the proof of Theorem 2 in Guerre and Sabbah (2012).

Lemma 1.1 *Suppose that the assumptions of Theorem 1.1 are satisfied. Then,*

$$\sup_{x \in R_x} |\hat{q}(x) - \tilde{q}(x) - q_h(x)| = O_p((nh^d)^{(-3/4)} L_n).$$

The following two lemmas follow by standard smoothing theory.

Lemma 1.2 *Suppose that the assumptions of Theorem 1.1 are satisfied. Then,*

$$\sup_{x \in R_x} |\tilde{q}(x)| = O_p((nh^d)^{(-1/2)} L_n).$$

Lemma 1.3 *Suppose that the assumptions of Theorem 1.1 are satisfied. Then,*

$$\sup_{x \in R_x} \left| q_h(x) - h^2 \frac{q_0''(x) + \frac{1}{2} q_0'(x) f_X'}{f_{\varepsilon|X}(0|x) f_X(x)} \right| = o(h^2).$$

From Lemmas 1.1–1.3 we get that $\hat{q}(x) = \hat{q}^*(x)$ for all $x \in R_x$ with probability tending to one. This gives

$$\hat{\theta}_{n1} = o_P(a_n) \tag{1.8}$$

for any sequence $\{a_n\}$ of positive constants tending to zero as $n \to \infty$. Furthermore, from Lemma 1.3 we get that

$$\hat{\theta}_{n5} = h^2 \int \frac{q_0''(x) + \frac{1}{2} q_0'(x) f_X'}{f_{\varepsilon|X}(0|x) f_X(x)} \omega(x) dx + o(h^2). \tag{1.9}$$

We now consider the third summand $\hat{\theta}_{n3}$.

Lemma 1.4 *Suppose that the assumptions of Theorem 1.1 are satisfied. Then,*

$$\hat{\theta}_{n3} = O_P(L_n n^{-3/4} h^{-d/4}). \tag{1.10}$$

Proof of Lemma 1.4. For simplicity we consider first the one-dimensional case. The term $\hat{\theta}_{n3}$ can be splitted into three summands (for $k = 1, 2, 3$):

$$\hat{\theta}_{n3,k} = \sum_j \int_{I_{jk}} [\{\hat{m}^*(x) - E[\hat{m}^*(x)|N(x)]\} - \{\tilde{m}(x) - E[\tilde{m}(x)|N(x)]\}]\omega(x) dx.$$

The terms $\hat{\theta}_{n3,1}$, $\hat{\theta}_{n3,2}$ and $\hat{\theta}_{n3,3}$ are sums of $O(h^{-1})$ conditionally independent summands. The summands are uniformly bounded by a term of order $O_P(L_n n^{-3/4} h^{-1/4})$. This follows from Lemma 1.1 and from the fact that $\hat{q}(x) = \hat{q}^*(x)$ for all $x \in R_x$ with probability tending to one. Thus for $k \in \{1,2,3\}$ the second conditional moment of $\hat{\theta}_{n3,k}$ is of order $O_P(L_n n^{-3/2} h^{-1/2})$. This shows that $\hat{\theta}_{n3,k} = O_P(L_n n^{-3/4} h^{-1/4})$ for $d=1$ and for $k \in \{1,2,3\}$, which implies the statement of the lemma for $d=1$. For $d>1$ one can use the same approach.

Lemma 1.5 *Suppose that the assumptions of Theorem 1.1 are satisfied. Then,*

$$\begin{aligned}
&E\Big\{\hat{q}^*(x) - q_h(x)\Big|N(x)\Big\} \\
&= m_0^{-1}\frac{2\kappa_{1,1}(h,x)(\kappa_{2,1}(h,x) - \kappa_{1,0}(h,x)\kappa_{1,1}(h,x)) + \kappa_{1,2}(h,x)}{2\kappa_{1,1}(h,x)} + O_P\Big(L_n m_0^{-3/2}\Big),
\end{aligned}$$

uniformly in $x \in R_X$, where for $1 \le k \le 3$

$$\begin{aligned}
\kappa_{k,0}(h,x) &= \mathbb{E}^i\left\{K^k\left(\frac{x-X_i}{h}\right)F_{\varepsilon|X}[q_h(x) - q_0(X_i)|X_i] - \alpha\right\}, \\
\kappa_{k,1}(h,x) &= \mathbb{E}^i\left\{K^k\left(\frac{x-X_i}{h}\right)f_{\varepsilon|X}[q_h(x) - q_0(X_i)|X_i]\right\}, \\
\kappa_{k,2}(h,x) &= \mathbb{E}^i\left\{K^k\left(\frac{x-X_i}{h}\right)\partial_\varepsilon f_{\varepsilon|X}[q_h(x) - q_0(X_i)|X_i]\right\}.
\end{aligned}$$

Thus we have that

$$\hat{\theta}_{n2} = O_P\Big(L_n m_0^{-3/2}\Big). \tag{1.11}$$

Proof of Lemma 1.5. We denote by L_n^* a sequence with $L_n^* = (\log n)^{C^*}$ for some constant $C^* > 0$. Put

$$Z_m(u) = m^{-1/2}\sum_{i\in\mathcal{N}^-(x)} K\left(\frac{x-X_i}{h}\right)\Big[I(Y_i - q_h(x) \le u m_0^{-1/2}) - \alpha\Big]$$

with $m_0 = nh^d$. Note that $\hat{q}(x) - q_h(x) \le u m_0^{-1/2}$ if and only if $Z_m(u) \ge 0$. Denote by $\mathbb{E}_m$ and $\mathbb{E}^i$ the conditional expectation, given that $N^-(x) = m$ or that X_i lies in the support of $K((\cdot - x)/h)$, respectively. Put

$$\begin{aligned}
\mu_m(u) &= -\sigma_m^{-1}(u)\mathbb{E}_m[Z_m(u)], \\
\sigma_m^2(u) &= \mathbb{E}_m\left[\{Z_m(u) - \mathbb{E}_m[Z_m(u)]\}^2\right], \\
\rho_m(u) &= \sigma_m^{-3}(u)\mathbb{E}_m\left[\{Z_m(u) - \mathbb{E}_m[Z_m(u)]\}^3\right].
\end{aligned}$$

By applying Theorem 19.3 in Roy et al. (1976) with $s \ge 4$ we get that

$$\begin{aligned}
&P\Big(\hat{q}(x) - q_h(x) \le u m_0^{-1/2}\Big|\mathcal{N}^-(x), N^-(x) = m\Big) \qquad (1.12) \\
&= 1 - \Phi\Big(\mu_m(u)\Big) + m^{-1/2}\frac{1}{6}\rho_m(u)\Big(1 - \mu_m(u)^2\Big)\phi\Big(\mu_m(u)\Big) + O\Big(m_0^{-1}(1+\mu_m(u)^2)^{-s}\Big),
\end{aligned}$$

uniformly in u and x for $C_1^* m_0 \le m \le C_2^* m_0$ and constants $C_1^* < C_2^*$. This can be seen as in the proof of Theorem 1 in Mammen, Van Keilegom, and Yu (2015).

Now note that for uniformly for $|u| \leq C^* L_n^*$ with $S_{h,i}(v) = F_{\varepsilon|X}[q_h(x) - q_0(X_i) + v|X_i] - \alpha$

$$\begin{aligned}
\sigma_m^2(u) &= \mathbb{E}^i\left\{K^2\left(\frac{x-X_i}{h}\right)S_{h,i}(um_0^{-1/2})\right\} - \mathbb{E}^i\left\{K\left(\frac{x-X_i}{h}\right)S_{h,i}(um_0^{-1/2})\right\}^2 \\
&= \mathbb{E}^i\left\{K^2\left(\frac{x-X_i}{h}\right)S_{h,i}(0)\right\} - \mathbb{E}^i\left\{K\left(\frac{x-X_i}{h}\right)S_{h,i}(0)\right\}^2 \\
&+ \frac{u}{m_0^{1/2}}\left(\mathbb{E}^i\left\{K^2\left(\frac{x-X_i}{h}\right)S'_{h,i}(0)\right\}\right. \\
&- \left. 2\mathbb{E}^i\left\{K\left(\frac{x-X_i}{h}\right)S_{h,i}(0)\right\}\mathbb{E}^i\left\{K\left(\frac{x-X_i}{h}\right)S'_{h,i}(0)\right\}\right) + O(L_n m_0^{-1}) \\
&= \kappa_{2,0}(h) - \kappa_{1,0}^2(h) + \frac{u}{m_0^{1/2}}(\kappa_{2,1}(h) - \kappa_{1,0}(h)\kappa_{1,1}(h)) + O(L_n m_0^{-1}),
\end{aligned}$$

where for $1 \leq k, l \leq 3$ we write $\kappa_{k,l}(h)$ for $\kappa_{k,l}(h,x)$. Thus,

$$\sigma_m(u) = (\kappa_{2,0}(h) - \kappa_{1,0}^2(h))^{1/2} + \frac{u}{m_0^{1/2}}\frac{(\kappa_{2,1}(h) - \kappa_{1,0}(h)\kappa_{1,1}(h))}{(\kappa_{2,0}(h) - \kappa_{1,0}^2(h))^{1/2}} + O(L_n m_0^{-1}).$$

Similarly, one gets that

$$\begin{aligned}
\mu_m(u) &= -\sigma_m^{-1}(u)m^{1/2}\mathbb{E}^i\left\{K\left(\frac{x-X_i}{h}\right)\left[I(Y_i - q_h(x) \leq um_0^{-1/2}) - \alpha\right]\right\} \\
&= -\sigma_m^{-1}(u)m^{1/2}\mathbb{E}^i\left\{K\left(\frac{x-X_i}{h}\right)\left[F_{\varepsilon|X}[q_h(x) - q_0(X_i) + um_0^{-1/2}|X_i] - \alpha\right]\right\} \\
&= -\sigma_m^{-1}(u)m^{1/2}\mathbb{E}^i\left\{K\left(\frac{x-X_i}{h}\right)f_{\varepsilon|X}[q_h(x) - q_0(X_i)|X_i]um_0^{-1/2}\right\} \\
&- \frac{1}{2}\sigma_m^{-1}(u)m^{1/2}\mathbb{E}^i\left\{K\left(\frac{x-X_i}{h}\right)\partial_\varepsilon f_{\varepsilon|X}[q_h(x) - q_0(X_i)|X_i]u^2 m_0^{-1}\right\} + O(L_n m_0^{-1}) \\
&= -\sigma_m^{-1}(u)um^{1/2}m_0^{-1/2}\kappa_{1,1}(h) \\
&- \frac{1}{2}\sigma_m^{-1}(u)u^2 m^{1/2}m_0^{-1}\kappa_{1,2}(h) + O(L_n m_0^{-1}) \\
&= um^{1/2}m_0^{-1/2}(\kappa_{2,0}(h) - \kappa_{1,0}^2(h))^{-1/2}\kappa_{1,1}(h) \\
&- u^2 m^{1/2}m_0^{-1}\left(\frac{2\kappa_{1,1}(h)(\kappa_{2,1}(h) - \kappa_{1,0}(h)\kappa_{1,1}(h)) + \kappa_{1,2}(h)}{2(\kappa_{2,0}(h) - \kappa_{1,0}^2(h))^{1/2}}\right) + O(L_n m_0^{-1}).
\end{aligned}$$

Furthermore,

$$
\begin{aligned}
\rho_m(u) &= \sigma_m^{-3}(u)\mathbb{E}_m\left[\{Z_m(u)-\mathbb{E}_m[Z_m(u)]\}^3\right] \\
&= \sigma_m^{-3}(u)\left(\mathbb{E}^i\left\{K^3\left(\frac{x-X_i}{h}\right)S_{h,i}(0)\right\}\right. \\
&- 3\mathbb{E}^i\left\{K^2\left(\frac{x-X_i}{h}\right)S_{h,i}(0)\right\}\mathbb{E}^i\left\{K\left(\frac{x-X_i}{h}\right)S_{h,i}(0)\right\} \\
&+ \left.2\mathbb{E}^i\left\{K\left(\frac{x-X_i}{h}\right)S_{h,i}(0)\right\}^3\right) + O(L_n m_0^{-1/2}) \\
&= \frac{\kappa_{3,0}(h)-3\kappa_{2,0}(h)\kappa_{1,0}(h)+2\kappa_{1,0}^3(h)}{(\kappa_{2,0}(h)-\kappa_{1,0}^2(h))^{3/2}} + O(L_n m_0^{-1/2}).
\end{aligned}
$$

Now, from the above calculations it follows that

$$
\begin{aligned}
& \mathbb{E}\Big\{\hat{q}^*(x)-q_h(x)\Big|N^-(x)=m\Big\} \\
= \; & \mathbb{E}\Big\{\hat{q}(x)-q_h(x)I(|\hat{q}(x)-q_h(x)|\le L_n^* m_0^{-1/2})\Big|N^-(x)=m\Big\} \\
= \; & m_0^{-1/2}\int_0^{L_n^*} P\Big(\hat{q}(x)-q_h(x)>vm_0^{-1/2}\Big|N^-(x)=m\Big)\,dv \\
& -m_0^{-1/2}\int_{-L_n^*}^{0} P\Big(\hat{q}(x)-q_h(x)\le vm_0^{-1/2}\Big|N^-(x)=m\Big)\,dv \\
= \; & m_0^{-1/2}\int_0^{L_n^*}\Big[P\Big(Z_m(v)\le 0\Big|N^-(x)=m\Big)-P\Big(Z_m(-v)\ge 0\Big|N^-(x)\Big)\Big]\,dv \\
= \; & m_0^{-1/2}\int_0^{L_n^*}\Big[\Phi\Big(\mu_m(u)\Big)+\frac{1}{6}m^{-1/2}\rho_m(u)\Big(1-\mu_m(u)^2\Big)\phi\Big(\mu_m(u)\Big)\Big]\,du \\
& -m_0^{-1/2}\int_0^{L_n^*}\Big[\Big(1-\Phi\Big(\mu_m(-u)\Big)\Big)+\frac{1}{6}m^{-1/2}\rho_m(-u)\Big(1-\mu_m(-u)^2\Big)\phi\Big(\mu_m(-u)\Big)\Big]\,du \\
& +O\Big(L_n m_0^{-3/2}\Big) \\
= \; & m_0^{-1/2}\int_0^{L_n^*}\Big[\Phi\Big(\mu_m(u)\Big)-\Phi\Big(\mu_m(-u)\Big)\Big]\,du+O\Big(L_n m_0^{-3/2}\Big)
\end{aligned}
$$

$$
\begin{aligned}
= \; & 2m_0^{-1/2}\int_0^{L_n^*}\varphi\left(um^{1/2}m_0^{-1/2}(\kappa_{2,0}(h)-\kappa_{1,0}^2(h))^{-1/2}\kappa_{1,1}(h)\right) \\
& \times u^2 m^{1/2}m_0^{-1}\left(\frac{2\kappa_{1,1}(h)(\kappa_{2,1}(h)-\kappa_{1,0}(h)\kappa_{1,1}(h))+\kappa_{1,2}(h)}{2(\kappa_{2,0}(h)-\kappa_{1,0}^2(h))^{1/2}}\right)du+O\Big(L_n m_0^{-3/2}\Big) \\
= \; & m_0^{-1/2}\int_{-\infty}^{\infty}\varphi\left(um^{1/2}m_0^{-1/2}(\kappa_{2,0}(h)-\kappa_{1,0}^2(h))^{-1/2}\kappa_{1,1}(h)\right) \\
& \times u^2 m^{1/2}m_0^{-1}\left(\frac{2\kappa_{1,1}(h)(\kappa_{2,1}(h)-\kappa_{1,0}(h)\kappa_{1,1}(h))+\kappa_{1,2}(h)}{2(\kappa_{2,0}(h)-\kappa_{1,0}^2(h))^{1/2}}\right)du+O\Big(L_n m_0^{-3/2}\Big) \\
= \; & m_0^{-1}\frac{2\kappa_{1,1}(h)(\kappa_{2,1}(h)-\kappa_{1,0}(h)\kappa_{1,1}(h))+\kappa_{1,2}(h)}{2\kappa_{1,1}}+O\Big(L_n m_0^{-3/2}\Big)
\end{aligned}
$$

uniformly in $C_1^* m_0 \le m \le C_2^* m_0$ with constants $C_1^* < C_2^*$.

Thus, we have shown that

$$\begin{aligned}
&\mathbb{E}\Big\{\hat{q}^*(x) - q_h(x)\Big|N^-(x) = m\Big\} \qquad (1.13)\\
&= m_0^{-1}\frac{2\kappa_{1,1}(h)(\kappa_{2,1}(h) - \kappa_{1,0}(h)\kappa_{1,1}(h)) + \kappa_{1,2}(h)}{2\kappa_{1,1}} + O\Big(L_n m_0^{-3/2}\Big),
\end{aligned}$$

uniformly in $x \in R_X$ and $C_1^* m_0 \le m \le C_2^* m_0$. For $m^+ \ge m$ we have by a simple argument that $E\Big\{\hat{q}^*(x) - q_h(x)\Big|N(x) = m^+, N^-(x) = m\Big\} = E\Big\{\hat{q}^*(x) - q_h(x)\Big|N^-(x) = m\Big\}$. Using (1.13) and

$$P\left(N^-(x) \le \frac{m^+}{4}\Big|N(x) = m^+\right) \le C\exp(-cnh^d),$$

uniformly in $m^+ \ge \frac{1}{2}3^d f_X(x)nh^d$ we conclude that

$$\begin{aligned}
&E\Big\{\hat{q}^*(x) - q_h(x)\Big|N(x) = m^+\Big\}\\
&= m_0^{-1}\frac{2\kappa_{1,1}(h)(\kappa_{2,1}(h) - \kappa_{1,0}(h)\kappa_{1,1}(h)) + \kappa_{1,2}(h)}{2\kappa_{1,1}} + O\Big(L_n m_0^{-3/2}\Big),
\end{aligned}$$

uniformly in $x \in R_X$ and $\frac{1}{2}3^d f_X(x)nh^d \le m^+ \le 2\ 3^d f_X(x)nh^d$. Because of

$$P\left(\frac{1}{2}3^d f_X(x)nh^d \le N(x) \le 2\ 3^d f_X(x)nh^d \text{ for all } x \in R_X\right) \to 1,$$

we get the statement of the lemma.

Lemma 1.6 *Suppose that the assumptions of Theorem 1.1 are satisfied. Then, it holds that*

$$\begin{aligned}
&\frac{2\kappa_{1,1}(h,x)(\kappa_{2,1}(h,x) - \kappa_{1,0}(h,x)\kappa_{1,1}(h,x)) + \kappa_{1,2}(h,x)}{2\kappa_{1,1}(h,x)}\\
&= f_{\varepsilon|X}(0|x)f_X(x)\int K^2(u)du + \frac{1}{2}\frac{\partial_\varepsilon f_{\varepsilon|X}(0|x)}{f_{\varepsilon|X}(0|x)} + O(h^2).
\end{aligned}$$

Proof of Lemma 1.6. This follows by standard smoothing theory.

Proof of Theorem 1.1. The theorem follows directly, using the results of the above lemmas.

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Chapter 2

Endogenous Shocks in Social Networks: Effects of Students' Exam Retakes on their Friends' Future Performance

2.1 Introduction

The peer effect, the effect that social connections have on people's behavior and achievements, plays an important role when analyzing educational outcomes. While there are numerous economics papers on peer effects across many fields, from education to juvenile behavior, the effect of shocking events on the friends network is rarely discussed. In particular, in the university framework, students' failures, such as retakes of examinations or dropouts are usually only discussed for the students' results in the same year and not in relation to their friends' future behaviour. However, the shock of a friend's failure influences the future behaviour and outcomes, especially when this failure was not anticipated.

This project contributes to the literature by covering an existing gap in peer effects literature and studying the changes of peers' behaviour and achievement in response to the individual shock. In contrast to some examples in development literature (e.g. Comola and Prina (2014)) considering exogenous individual treatment, I propose the model, allowing the shock to be endogenously formed. Two components of the shock can be disentangled: predicted probability of the shock and unexpected component. The latter is considered to be crucial to the changes of future behavior. I am considering the students' exam failure as the source of the shock and test the model on the sample of students of one cohort at the National Research University - Higher School of Economics, a highly selective university in Russia. The threat of retakes and dropouts may put a lot of pressure on students, and the higher probability of failure may result in lower productivity. Knowing, how these shocks influence the behavior of the students and their friends, can help to understand the whole dynamics of network performance, and maybe help universities to adjust the strategy of setting up the retakes' threshold. Of course, dropouts are likely to influence future behaviour stronger than retakes, since the latter can still be fixed. However, the existing data of the dropouts is not sufficient for proper econometric analysis. I discuss both sources of shock in descriptive analysis but apply the econometric model only to retakes.

The direction of the effect, however, can be twofold. While the unexpected shock may serve as a wake-up call and motivate students to be more dedicated to their studies, the connections can be extremely tight. This can reduce the amount of time spent on one's own studies due to the shared activities with the friend either outside of the university, if the friend left, or helping the friend to prepare for the retake of the exam. The reasons of the retakes during the studies can be different. In the first year, students are more likely to fail due to the lack of the abilities or difficulties with adjustments to the new environment. The fist exams may appear to be too difficult for some of the students, even though they had sufficient abilities to enter the university. Students with lower abilities are either dropping out of the university or adjusting their efforts to improve performance. In the second and higher year, students are more likely to fail due to insufficient efforts. Therefore, the shock during the different time periods may have a different effect on the future performance. This paper discusses only the first year retakes at the moment.

Although I do not study the pure peer effect in this paper, I exploit the general idea of peer effects literature and its methodological fundamentals. Most of the economic literature that analyses peer effects use the framework and the model introduced by Manski (1993). He distinguishes three effects that determine the similar behaviour of peers. *The endogenous effect* explains that the probability of a particular student to drop out of the school or university or to fail an exam will be affected by a number of this student's peers who have already done so. *The exogenous effect* uses mean exogenous characteristics of the peer group, such as parental education, socio-economic status (SES), etc., to determine the probability of the dropout or retake. *The correlated effect* appears due to the similar individual characteristics within a group. The most important task of peer effects analysis is to determine the endogenous effect, which can have important policy implications.

Identification of these three effects in the case of group interactions requires an additional source of exogenous variation, such as exogenous class formation (for example, Carrell, Fullerton, and West, 2009 in military institutions framework and De Giorgi, Pellizzari, and Redaelli (2010) and Androushchak, Poldin, and Yudkevich (2013) in university frameworks with randomly assigned groups) or random assignment of dormmates (for example, B. Sacerdote, 2011). Estimating the endogenous peer effect as an effect of an average group performance obtained some critique, and additional assumptions on the structure or the ranking inside the peer group or even exact links are preferable, but social network data is not always available. Usage of social network data requires other identifying assumptions, which restrict the network. Bramoullé, Djebbari, and Fortin (2009) proved the identification of the peer effect in social networks under rather mild assumptions. Poldin, Valeeva, and Yudkevich (2015) use the same identification result to study the peer effect in the university framework using HSE dataset.

The identification of the direct effect of shock on the friends' future outcome is, however, more challenging, since the changes of the performance are not driven solely by the effects of the shocks. Exogenous and unobserved characteristics of the student and his peers as well as the changes in the network structure are among the other determinants. Moreover, as was already mentioned, the shock itself is not exogenous, and its significant part is driven by the model itself. The paper proposes an econometric model which deals with both problems and estimates the effect of the shock: a two-step dynamic peer effects model. The first step estimates the probability of the shock adopting the instrumental variable 2SLS approach discussed by Bramoullé, Djebbari, and Fortin (2009) after L. Lee (2003). The second step uses the residuals from the first stage to estimate the effect of the unexpected component of the changes in students' performance.

To the best of my knowledge, this project is the first to introduce the dynamic peer effect in social networks model with endogenous shock[1]. Moreover, I provide the identification results for this model and propose estimation procedure. The identification and estimation of the first step are the straightforward adjustments of the Bramoullé, Djebbari, and Fortin (2009) approach, and requires the existence of intransitive triads in the network given the assumption of no correlated effects, i.e. friends of some student's friends not connected to him or her. Hence, the friends of friend affect the student not directly, but via the common friend only. If the assumption of no correlated effects is relaxed, the stricter identifying assumption is necessary. The whole network should include pairs of students with the distance between them of length three or bigger. They are not connected directly, and the shortest path from the one to the other has not less than three links. Friends of friends are used to deal with the correlated effect, therefore, the next level of friends is used as an identifying assumption. The identification of the second step is novel and demonstrates the necessity of the network longitudinal variation. Changes of the network allow comparing the influence of "old" and "new" peer group on the outcome. The presence of the new friends and absence of old ones creates variation in the peer group characteristics and this helps to identify social effects and the effect of the shock. However, it is important that the changes of the network are not driven solely by the shock. Moreover, at the moment, I do not model link formation, and therefore, do not distinguish between different types of network changes and treat them all as equal and given.

The variation of the network is a valid assumption for the students' network setting. The links formed in the first year are highly likely to be revised due to the gradual unveiling of the friends' personal characteristics. Some of the links might be broken, however, due to the exam retakes and dropouts of the friends. The student may seek for a more advantageous peer group or he/she no longer spends much time with the friend preparing for the retakes. But even if the friend fails an exam and the link stays stable in the network, the student may tend to connect to the students with higher results, creating new links. The exam retake is endogenous in the model, and only an unexpected component of the retake probability is

[1]See, for example, a review of the recent econometric literature on networks in Paula (2015)

considered as a shock. The influence of this unexpected component on link formation is not the same as possible channels of influence of retakes on link formation, discussed previously, therefore, the actual importance of the shock for link changes might be lower than the one of exam retakes. The model in the paper is discussed without link formation process and, therefore, under the assumption that changes in the network are exogenously given. This setup is a bit restrictive, and relaxation of this assumption will be considered for future research.

The magnitude of the endogenous effects in different periods is considered to be different, since the unexpected shock may affect performance via the changes of the peer groups, and not only directly. The break of the link itself makes the peer group "better", then the improvement of the results can also be caused by the group's refinement.

Dropouts and retakes are important to study from the university's perspective. Dropouts create the sunk costs for the university. For example, costs of the university dropouts in Germany were estimated at the level of \$11.5 billion in 2007[2] and in Australia at \$1.36 billion[3]. Some of the dropouts are the results of the policies of the university, which can be controlled. In some institutions of higher education, as in the sample used in the analysis, most of the dropouts are directly affected by the retakes. In HSE 3 retakes during the same exam session term will lead to the expulsion of the student. Therefore, understanding the possible mechanisms of retakes' influence on future performance may suggest possible university-level policy improvements in order to reduce sunk costs.

The paper is organized as follows. Section 2 discusses the proposed model, states the identifying assumptions, and proposes the estimation method. Section 3 describes the data used and the institutional environment of the educational system in Russia, as well as results of the descriptive analysis. Section 4 provides the estimation results and evidence of the influence of dropouts and retakes on peers. Section 5 concludes.

2.2 Model

2.2.1 Naïve approach

I propose a two-step model that allows estimating the effect of an unexpected event happening to network connections. Although I do not conduct the pure peer effect estimation, I use the classical peer effect model as a baseline.

A naïve way to write down the dynamic peer effect model without modelling the link formation:

$$y_i^1 = \alpha_1 + \beta_1 \sum_{j \neq i} G_{ij}^1 y_j^1 + \gamma_1 X_i^1 + \delta_1 \sum_{j \neq i} G_{ij}^1 X_j^1 + \xi_i + \epsilon_i^1, \qquad \mathbb{E}[\epsilon_i^1 | X^1] = 0, \tag{2.1}$$

[2] The figures are obtained by the Stifterverband, association of German science and higher education donors. Details can be found on UWN website

[3] According to the report on UWN website

$$y_i^2 = \alpha_2 + \beta_2 \sum_{j \neq i} G_{ij}^2 y_j^2 + \gamma_2 X_i^2 + \delta_2 \sum_{j \neq i} G_{ij}^2 X_j^2 + \xi_i + \epsilon_i^2, \qquad \mathbb{E}[\epsilon_i^2 | X^2] = 0, \tag{2.2}$$

where y_i^1 and y_i^2 are outcome variables of student i in the first wave and the second wave correspondingly. I will consider the average grade in the main specification of the model. Student's rating or grades for some specific subjects, which last more than 1 term, are used for robustness checks;
X_i is a vector of individual characteristics that should be controlled for, such as gender, city of origin, living conditions, some socioeconomic family characteristics. In the discussed empirical example it also includes the results of the high school examination, universal and obligatory for all the students graduating the high school.
G_{ij}^1 and G_{ij}^2 are two adjacency matrices for the first and the second waves correspondingly, weighted by the number of links, and their entries have the value of $1/n_i$ if the link from student i to student j exists. Note that this matrices are not necessarily symmetrical, since the social network can be both directed (as in the sample used later) or undirected.
ξ_i - student-level unobserved fixed characteristics, which may influence students' performance and choice of connections.

Those unobserved individual characteristics also reflect the homophily of the individuals, which may influence both link formation and the network outcomes. In the case of group interactions group fixed effects are often introduced to eliminate correlated effects, whereas in the case of interactions in big networks network fixed effects make little sense. Local differences, proposed by Bramoullé, Djebbari, and Fortin (2009), may be used to address the issue of correlated effects. However, the dynamic structure of the data allows solving this issue differently. The dynamic peer model can be then written in terms of differences, and this will eliminate possible unobserved fixed effect component in the error term, consisting of the common for individual's connections unobservable component and individual's own unobserved fixed characteristics.

$$\Delta y_i = \Delta \alpha + \beta_2 \sum_{j \neq i} G_{ij}^2 y_j^2 - \beta_1 \sum_{j \neq i} G_{ij}^1 y_j^1 + \gamma_2 X_i^2 - \gamma_1 X_i^1 + \delta_2 \sum_{j \neq i} G_{ij}^2 X_j^2 - \delta_1 \sum_{j \neq i} G_{ij}^1 X_j^1 + \Delta \epsilon_i$$

Assumption A. The outcome variable of a single period can be estimated using the one-period model.

This additional assumption allows avoiding the autoregressive component in the second-period model. Assumption A is valid, because the model, including observed and unobserved fixed effects characteristics as well as endogenous and exogenous peer effects, is sufficient to predict the educational achievements. Therefore, it can be claimed that there is no additional mechanism that can influence the outcome via the previous period's outcome.

The proposed model system 2.1 and 2.2, and consequently, the model written in differences, can be further modified in order to catch the desirable effect of shock. In the naïve way, similar to the model of Comola and Prina (2014), the model will now be as follows:

The equation for the first period should remain unchanged:

$$y_i^1 = \alpha_1 + \beta_1 \sum_{j \neq i} G_{ij}^1 y_j^1 + \gamma_1 X_i^1 + \delta_1 \sum_{j \neq i} G_{ij}^1 X_j^1 + \xi_i + \epsilon_i^1,$$

Whereas, the second-period model shall take into account the shock of unexpected retake of the friend. The straightforward way to do it is just to include the binary variable in the vector of controls:

$$y_i^2 = \alpha_2 + \beta_2 \sum_{j \neq i} G_{ij}^2 y_j^2 + \tilde{\delta} D_i + \gamma_2 X_i^2 + \delta_2 \sum_{j \neq i} G_{ij}^2 X_j^2 + \xi_i + \epsilon_i^2$$

where D_i is a dummy for having any friends with a retake in the first period[4].

The system can then be re-written in differences, eliminating the possible individual fixed effect:

$$\Delta y_i = (\alpha_2 - \alpha_1) + \beta_2 \sum_{j \neq i} G_{ij}^2 y_j^2 - \beta_1 \sum_{j \neq i} G_{ij}^1 y_j^1 + \tilde{\gamma} D_i +$$
$$+ \gamma_2 X_i^2 - \gamma_1 X_i^1 + \delta_2 \sum_{j \neq i} G_{ij}^2 X_j^2 - \delta_1 \sum_{j \neq i} G_{ij}^1 X_j^1 + \epsilon_i^2 - \epsilon_i^1$$

However, this type of the equation is only valid if the shock is exogenous, as in the examples of randomized treatment. A big share of the probability of the student's retake can be explained by the observed component of the model, and therefore, the retake itself cannot be considered as unexpected shock. I propose to use the peer effect model of the first period to disentangle predictable and unexpected parts of the probability of the retake, and use the unpredicted part only to estimate the effect of the shock on the performance.

Comola and Prina (2014) also model the changes of the network as a response to the exogenous treatment. At the moment, I am not modelling the link formation. The variation of the network links is assumed and is a crucial identifying assumption. Importantly, a significant part of the changes in the structure of the friendship networks is caused by the individual characteristics and outcome and not solely by the exam retake. The influence of the retake and of the unpredicted component of the retake on the link formation also should be treated and interpreted differently, since the probability of the exam retake is endogenous. The following assumption, therefore, should be made. *Assumption B.* Changes of the network as a response to unexpected shock are neglected, and all changes of the network itself are treated as exogenous.

This assumption can potentially cause overestimation of the direct effect of the shock, and therefore, should be relaxed in the future research.

[4] In general the coefficients in the model with the shock are different from the baseline one-period models (1) and (2), but I left the same notations for simplicity

2.2.2 Proposed model with no correlated effects

The model

Taking into account all above-mentioned argument, I estimate the following model at the first step:

$$P(retake_i) = \alpha + \beta \sum_{j \neq i} G^1_{ij} y^1_j + \gamma X^1_i + \delta \sum_{j \neq i} G^1_{ij} X^1_j + \xi_i + \nu_i, \qquad \mathbb{E}[\nu_i | X^1] = 0 \tag{2.3}$$

In this specification, the error term consists of two parts: unobserved correlated effect, and conditionally independent noise. Dynamic peer effect model will eliminate the correlated effect component at the second step of the model, leading to the conditional independence of the error term. However, on the first step in general $\mathbb{E}[\xi_i + \nu_i | X^1] \neq 0$. I will discuss two cases: assuming no correlated effects and with correlated effect. The latter will be considered in the later subsections. For the former, 2.3 will be transformed as follows :

$$P(retake_i) = \alpha + \beta \sum_{j \neq i} G^1_{ij} y^1_j + \gamma X^1_i + \delta \sum_{j \neq i} G^1_{ij} X^1_j + \nu_i, \qquad \mathbb{E}[\nu_i | X^1] = 0 \tag{2.3a}$$

I then take the residuals of the equation 2.3a, which is the part of the probability of the friends' retake not predicted by the model. I then construct the shock for student i as the combination of the residuals for the students in the network of i. The baseline specification uses the average of the residuals: $UR_i = \sum_{j \neq i} G^1_{ij} \hat{\nu}_j$. However, the other approaches to define UR_i is possible: maximum of friends' unpredicted probability of the exam retakes, residuals for the friends named first, or average weighted according to the order, with which friends are appearing in the answers of the students. The identification results and estimation procedure are not affected by the choice of the approach to defining UR_i. Then I am using it as an unexpected shock to plug-in in the following equation:

$$\Delta y_i = (\alpha_2 - \alpha_1) + \beta_2 \sum_{j \neq i} G^2_{ij} y^2_j - \beta_1 \sum_{j \neq i} G^1_{ij} y^1_j + \tilde{\boldsymbol{\delta}} UR_i + \gamma_2 X^2_i - \gamma_1 X^1_i +$$
$$+ \delta_2 \sum_{j \neq i} G^2_{ij} X^2_j - \delta_1 \sum_{j \neq i} G^1_{ij} X^1_j + \Delta\epsilon_i \tag{2.4}$$

Since the model in differences eliminates possible individual fixed effect component in error term, I am able to make a stricter assumption on the error term: $\mathbb{E}[\Delta\epsilon_i] = 0$, instead of the conditional expectation. This condition will be used to prove the model identification.

Model in differences, additional to the elimination of individual fixed effect, gives a better interpretation of the studied effect. It estimates the changes of own performance in response to the shock additional to the changes of performance in comparison to the classmates, obtained by the single-period model.

Note that the coefficients for the endogenous peer effect and exogenous characteristics are considered to be different in two periods: β_2 and β_1 and δ_2 and δ_1. Students may experience the different magnitude of the effects depending on how advanced they are in their studies, how well they are adjusted to the university environment, etc. Moreover, this also allows to take into account the changes in the network, since the students are experiencing the influence of two different peer groups in two periods.

The own retake of the student is not included explicitly in the model. The unexpected component for the students themselves is close to zero since they can anticipate most of the retakes after writing the exam. Moreover, the outcome of the previous period partially takes care of own retakes. Nonetheless, in the empirical analysis, I will also split the sample and study the effect for those, who were retaking the exams, and for those, who were not, to tackle down possible differences.

Identifying assumptions

The identification results for the first step of the model adopt Bramoullé, Djebbari, and Fortin (2009) approach, whereas the result, obtained for the second stage, is, to the best of my knowledge, a novel result for the literature.

Lemma 2.1 *Let $\gamma_1^2 + \delta_1^2 \neq 0$ and $\beta_1 \neq 0$[5]. If matrices I, G^1, $(G^1)^2$ are linearly independent, coefficients in 2.3a are identified.*

The proof of Lemma 2.1 is given in Appendix A. This is exactly the condition obtained by (Bramoullé, Djebbari, and Fortin, 2009), and can be proven similarly. The identification of the coefficients on the first step, hence, allow using the obtained residuals for the further analysis. The identification is ensured by the existence of intransitive triads in the network, i.e. the existence of a set of three individuals i, j, k such that i is influenced by j, j is influenced by k, but i is not influenced by k. This is a valid assumption for most networks, in particular, for the sample analysed in this paper, which will be discussed in the next section.

Lemma 2.2 *In the case of no correlated effects, if the assumptions of Lemma 2.1 hold, if $\gamma_2^2 + \delta_2^2 \neq 0$ and $\beta_2 \neq 0$[6], if matrices I, G^2, $(G^2)^2$ are linearly independent, and if $G^1 \neq G^2$, with changes not driven by the shock only, coefficients in 2.4 are identified.*

Identification of Step 2 relies heavily on the variation in the network structure. However, it is important that some changes in the network are exogenous. This assumption is quite reasonable for the friendship networks. Students are likely to learn more about their classmates with time, and the friendships, created during the first year, are often unstable.

Once there are new links formed in the next period, the variation between new and old connections help to capture the effect of the changes in the average grade. For example,

[5] These are the coefficients from the baseline peer effect model 2.1.

[6] The coefficients from the baseline peer effect model 2.2

if a student i is no longer connected to student j, and therefore, is not affected by student j, his performance can be evaluating in the two cases and the comparison of two results will result in the effect of not having friend j, and hence, the social effects are easier to catch. The identifying assumptions also put the restriction on the friendship matrix of the second period, as in the first period: the network should include intransitive triads. The proof of Lemma 2.2 can also be found in Appendix A, and the validity of identifying assumptions will be discussed in the next Section.

2.2.3 Model with correlated effects

The model

As was already mentioned, the correlated effect appears due to the similar individual characteristics within a group. The correlated effect is unlikely to be present in big networks, however, once the network may suggest existence of smaller groups or subnetworks in it, the correlated effects are more likely to be present. In the empirical application discussed in this paper, most of the connections are formed inside of the same department, and even inside of the same exogenously formed study group. Therefore, the possible correlated effects could not be ignored and can cause an additional identification issue.

To deal with it and eliminate unobserved variables, I propose taking the local differences, i.e. averaging the equation 2.3 over the friends of i and subtracting this average from 2.3 and noting that ξ_i are the same for the students in one smaller network, and hence, it will vanish after taking the local differences:

$$P(retake_i) - \sum_{j\neq i} G^1_{ij} P(retake_j) = \beta \sum_{j\neq i} G^1_{ij}[y^1_j - \sum_{k\neq j} G^1_{jk} y^1_k] + \gamma[X^1_i - \sum_{j\neq i} G^1_{ij} X^1_j] +$$
$$+ \delta \sum_{j\neq i} G^1_{ij}[X^1_j - \sum_{j\neq k} G^1_{jk} X^1_k] + \eta_i, \qquad \eta_i = [\nu_i - \sum_{j\neq i} G^1_{ij}\nu_j],\ \mathbb{E}[\eta_i|X^1] = 0 \tag{2.5}$$

Similarly to the case without correlated effects, I construct the shock for the student i, taking the average of their networks residuals: $UR_i = \sum_{j\neq i} G^1_{ij}\hat{\eta}_j$. The second stage is then identical to the case with no correlated effects:

$$\Delta y_i = (\alpha_2 - \alpha_1) + \beta_2 \sum_{j\neq i} G^2_{ij} y^2_j - \beta_1 \sum_{j\neq i} G^1_{ij} y^1_j + \tilde{\boldsymbol{\delta}} UR_i + \gamma_2 X^2_i - \gamma_1 X^1_i +$$
$$+ \delta_2 \sum_{j\neq i} G^2_{ij} X^2_j - \delta_1 \sum_{j\neq i} G^1_{ij} X^1_j + \Delta\epsilon_i \tag{2.6}$$

Model in differences, additional to the elimination of individual fixed effect, also gets rid off the correlated effects, therefore, no local differences are needed for the second stage equation.

Identifying assumptions

The identification results for the first step of the model again adopt Bramoullé, Djebbari, and Fortin (2009) approach, whereas the result, obtained for the second stage, is new.

Lemma 2.3 *Let $\gamma_1^2 + \delta_1^2 \neq 0$ and $\beta^1 \neq 0$[7]. If matrices I, G^1, $(G^1)^2$, $(G^1)^3$ are linearly independent, coefficients in 2.5 are identified.*

The proof is given in Appendix A. This condition again follows the result of (Bramoullé, Djebbari, and Fortin, 2009) in the presence of correlated effects, and can be proven in the similar manner. The identification of model with correlated effects is ensured by the existence of distances between two students of length 3 and more, i.e. the existence of a set of at least 4 individuals i, j, k, m such that i is influenced by j, j is influenced by k, k is influenced by m, but i is not influenced by both m and k, and j is not influenced by m. This is a bit more demanding assumption than in the case of no correlated effects, but still valid for a lot of networks' types, and in particular, for the sampled network, which will be discussed in the next section.

Lemma 2.4 *In the case of correlated effects, if the assumptions of Lemma 2.3 hold, if $\gamma_2^2 + \delta_2^2 \neq 0$ and $\beta_2 \neq 0$[8], if matrices I, G^2, $(G^2)^2$, $(G^2)^3$ are linearly independent, and if $G^1 \neq G^2$, with changes not driven by the shock only, coefficients in 2.6 are identified.*

Identification of Step 2 again heavily relies on the variation in the network structure. Moreover, the restrictions are put on the friendship matrix of the second period, requiring the distances between two students of length 3 and more. The proof of Lemma 2.4 is presented in Appendix A.

2.2.4 Estimation strategy

No correlated effects

I first discuss the model that does not take into account correlation effects: 2.3a and 2.4.

Step 1. I partially repeat Bramoullé, Djebbari, and Fortin (2009) for the first step and use the adaptation of Generalized 2SLS strategy proposed by Kelejian and Prucha (1998) and refined by L. Lee (2003). As the identification result suggests, $((\boldsymbol{G^1})^2\boldsymbol{X}, (\boldsymbol{G^1})^3\boldsymbol{X}, \dots)$ can be used as valid instruments to obtain consistent estimators.

[7]The coefficients from the baseline peer effect model 2.1

[8]The coefficients from the baseline peer effect model 2.2

First, recall the peer effect model in reduced form, written in matrix notations, offered in Bramoullé, Djebbari, and Fortin (2009):

$$\boldsymbol{y^1} = \alpha_1 \boldsymbol{i} + \beta_1 \boldsymbol{G^1 y^1} + \gamma_1 \boldsymbol{X^1} + \delta_1 \boldsymbol{G^1 X^1} + \nu^1, \qquad \mathbb{E}[\nu|\boldsymbol{X^1}] = 0,$$

which gives

$$\mathbb{E}[\boldsymbol{G^1 y^1}|\boldsymbol{X^1}] = (\boldsymbol{I} - \beta_1 \boldsymbol{G^1})^{-1} \boldsymbol{G^1} \alpha_1 + (\boldsymbol{I} - \beta_1 \boldsymbol{G^1})^{-1} \boldsymbol{G^1} (\gamma_1 \boldsymbol{I} + \delta_1 \boldsymbol{G^1}) \boldsymbol{X^1}$$

Note that the fist step model can be written as follows:

$$\boldsymbol{PR} = \alpha + \beta \boldsymbol{G^1 Y^1} + \gamma \boldsymbol{X^1} + \delta \boldsymbol{G^1 X^1} + \nu, \qquad \mathbb{E}[\nu|\boldsymbol{X^1}] = 0 \tag{2.7}$$

I propose the following procedure that gives the consistent estimator of $\theta = (\alpha, \beta, \gamma, \delta)$:

First, compute the 2SLS estimator for $\theta^1 = (\alpha_1, \beta_1, \gamma_1, \delta_1)$ of the standard peer effects model, using the following vector of instruments $\boldsymbol{S} = [\boldsymbol{i}, \boldsymbol{X^1}, \boldsymbol{G^1 X^1}, \boldsymbol{(G^1)^2 X^1}]$, and with the vector of covariates $\boldsymbol{\tilde{X}^1} = [\boldsymbol{i}, \boldsymbol{X^1}, \boldsymbol{G^1 X^1}, \boldsymbol{G^1 y^1}]$.
$\hat{\theta}^1_{2SLS} = (\boldsymbol{\tilde{X}^1}^T \boldsymbol{P_S} \boldsymbol{\tilde{X}^1})^{-1} \boldsymbol{\tilde{X}^1}^T \boldsymbol{P_S y^1}$, where $\boldsymbol{P_S} = \boldsymbol{S}(\boldsymbol{S}^T \boldsymbol{S})^{-1} \boldsymbol{S}^T$ is a projection matrix.

Second, define $\boldsymbol{\hat{Z}} = Z(\hat{\theta}^1_{2SLS}) = [\boldsymbol{i}, \boldsymbol{X^1}, \boldsymbol{G^1 X^1}, \mathbb{E}[\boldsymbol{G^1 y^1 (\hat{\theta}^1_{2SLS})}|\boldsymbol{X^1}]]$,
where $\mathbb{E}[\boldsymbol{G^1 y^1 (\hat{\theta}^1_{2SLS})}|\boldsymbol{X^1}] = \boldsymbol{G^1}(\boldsymbol{I} - \hat{\beta}_{1,2SLS} \boldsymbol{G^1})^{-1} \hat{\alpha}_{1,2SLS} + \boldsymbol{G^1}(\boldsymbol{I} - \hat{\beta}_{1,2SLS} \boldsymbol{G^1})^{-1}$ $(\hat{\gamma}_{1,2SLS} \boldsymbol{I} + \hat{\delta}_{1,2SLS} \boldsymbol{G^1}) \boldsymbol{X^1}$

Finally, use $\boldsymbol{\hat{Z}}$ as a vector of instruments to estimate 2.3a. Note that the vector of covariates coincides with the one used at the first step: $\boldsymbol{\tilde{X}^1}$. Then the following consistent estimator is obtained: $\hat{\theta}_{Lee} = (\boldsymbol{\hat{Z}}^T \boldsymbol{\tilde{X}^1})^{-1} \boldsymbol{\hat{Z}}^T \boldsymbol{PR}$.

This procedure is a modification of a procedure proposed in L. Lee (2003), therefore, the consistency result is closely related to his Theorem 1:

Lemma 2.5 *Under regularity conditions defined in Appendix A, the estimator $\hat{\theta}_{Lee}$ is consistent and has the following limiting distribution,*

$$\sqrt{n}(\hat{\theta}_{Lee} - \theta) \xrightarrow{D} \mathcal{N}(0, \Psi), \tag{2.8}$$

with $\Psi = \sigma^2_\nu (lim_{n \to \infty} \frac{1}{n} \boldsymbol{Z}^T \boldsymbol{Z})^{-1}$ *and*
$\boldsymbol{Z} = [\boldsymbol{i}, \boldsymbol{X^1}, \boldsymbol{G^1 X^1}, \boldsymbol{G}^1(\boldsymbol{I} - \beta_1 \boldsymbol{G^1})^{-1} \alpha_1 + (\boldsymbol{I} - \beta_1 \boldsymbol{G^1})^{-1} (\gamma_1 \boldsymbol{I} + \delta_1 \boldsymbol{G^1}) \boldsymbol{X^1}]$

Discussion and detailed proof of the consistency of such estimator are given in Appendix A.

Step 2. I am approaching the estimation of the second step also adopting the 2SLS procedure discussed for the first step. First, the model 2.4 can be rewritten in the following way:

$$\Delta \boldsymbol{y} = (\alpha_2 - \alpha_1)\boldsymbol{i} + \beta_2 \boldsymbol{G}^2 \boldsymbol{y^2} - \beta_1 \boldsymbol{G^1 y^1} + \tilde{\boldsymbol{\delta}} \boldsymbol{UR} + \gamma_2 \boldsymbol{X^2_{TV}} - \gamma_1 \boldsymbol{X^1_{TV}} + \delta_2 \boldsymbol{G^2 X^2} -$$

$$- \delta_1 \boldsymbol{G^1 X^1} + \Delta\epsilon, \qquad \text{with } \boldsymbol{UR} \text{ defined as discussed in Section 2.2} \tag{2.9}$$

By X^1_{TV}, and X^2_{TV} I denote the subset of covariates, which are time-variant to avoid singularity problem of estimation.

Then a vector of covariates is as follows: $\bar{\boldsymbol{X}} = [\boldsymbol{i}, \boldsymbol{X^2_{TV}}, \boldsymbol{X^1_{TV}}, \boldsymbol{G^2 X^2}, \boldsymbol{G^1 X^1}, \boldsymbol{UR}, \boldsymbol{G^1 y^1}, \boldsymbol{G^2 y^2}]$. Following the logic of the first step I use $\boldsymbol{(G^2)^2 X^2}$ as an instrument for $\boldsymbol{G^2 y^2}$. However, $\mathbb{E}[(\boldsymbol{G^1 y^1})^T \Delta\epsilon] \neq 0$, hence the instrument for $\boldsymbol{G^1 y^1}$ is required. I propose to use $\mathbb{E}[\boldsymbol{G^1 y^1 (\hat{\theta}^1_{2SLS})} | \boldsymbol{X^1}]$ as an instrument, as obtained on the step 1. It is obvious that such an instrument is a valid instrument since it is uncorrelated with the second step error term and is clearly correlated with the outcome variable. Then I define $\boldsymbol{M} = [\boldsymbol{i}, \boldsymbol{X^2_{TV}}, \boldsymbol{X^1_{TV}}, \boldsymbol{G^2 X^2}, \boldsymbol{G^1 X^1}, \boldsymbol{UR}, \mathbb{E}[\boldsymbol{G^1 y^1 (\hat{\theta}^1_{2SLS})} | \boldsymbol{X^1}], \boldsymbol{(G^2)^2 X^2}]$ as a vector of instruments.

I modify 2.9, taking expectations given $\boldsymbol{X^2}$ and recalling $\mathbb{E}[\Delta\epsilon] = 0$:

$$(\boldsymbol{I} - \beta_2 \boldsymbol{G^2})\mathbb{E}[\boldsymbol{y^2} | \boldsymbol{X^2}] = (\alpha_2 - \alpha_1)\boldsymbol{i} + (\boldsymbol{I} - \beta_1 \boldsymbol{G^1})\boldsymbol{y^1} + \tilde{\boldsymbol{\delta}} \boldsymbol{UR} + \gamma_2 \boldsymbol{X^2_{TV}} - \gamma_1 \boldsymbol{X^1_{TV}} + \\ + \delta_2 \boldsymbol{G^2 X^2} - \delta_1 \boldsymbol{G^1 X^1}$$

$$\mathbb{E}[\boldsymbol{y^2} | \boldsymbol{X^2}] = (\boldsymbol{I} - \beta_2 \boldsymbol{G^2})^{-1}[(\alpha_2 - \alpha_1)\boldsymbol{i} + (\boldsymbol{I} - \beta_1 \boldsymbol{G^1})\boldsymbol{y^1} + \tilde{\boldsymbol{\delta}} \boldsymbol{UR} + \gamma_2 \boldsymbol{X^2_{TV}} - \gamma_1 \boldsymbol{X^1_{TV}} + \\ + \delta_2 \boldsymbol{G^2 X^2} - \delta_1 \boldsymbol{G^1 X^1}]$$

Let $\mathbb{E}[\boldsymbol{G^2 y^2}(\phi) | \boldsymbol{X^2}, \boldsymbol{X^1}] = \boldsymbol{G^2}(\boldsymbol{I} - \beta_2 \boldsymbol{G^2})^{-1}[(\alpha_2 - \alpha_1)\boldsymbol{i} + (\boldsymbol{I} - \beta_1 \boldsymbol{G^1})\mathbb{E}[\boldsymbol{y^1}(\theta^1) | \boldsymbol{X^1}] + \tilde{\boldsymbol{\delta}} \boldsymbol{UR} + \gamma_2 \boldsymbol{X^2_{TV}} - \gamma_1 \boldsymbol{X^1_{TV}} + \delta_2 \boldsymbol{G^2 X^2} - \delta_1 \boldsymbol{G^1 X^1}]$, where $\mathbb{E}[\boldsymbol{y^1}(\theta^1) | \boldsymbol{X^1}] = \boldsymbol{G^2}(\boldsymbol{I} - \beta_1 \boldsymbol{G^1})^{-1}\alpha_1 + (\boldsymbol{I} - \beta_1 \boldsymbol{G^1})^{-1}(\gamma_1 \boldsymbol{I} + \delta_1 \boldsymbol{G^1})\boldsymbol{X^1}$.

Then I also define the following vector $\bar{\boldsymbol{Z}} = [\boldsymbol{i}, \boldsymbol{X^2_{TV}}, \boldsymbol{X^1_{TV}}, \boldsymbol{G^2 X^2}, \boldsymbol{G^1 X^1}, \boldsymbol{UR}, \mathbb{E}[\boldsymbol{G^1 y^1}(\theta^1) | \boldsymbol{X^1}], \mathbb{E}[\boldsymbol{G^2 y^2}(\phi) | \boldsymbol{X^2}, \boldsymbol{X^1}]$

I propose the following estimation procedure:

First, compute the 2SLS estimator for $\phi = (\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2)$ of the 2.9, using a vector of instruments $\boldsymbol{M}$ and a vector of covariates $\bar{\boldsymbol{X}}^{\boldsymbol{1}}$, as defined above.
$\hat{\phi}^1_{2SLS} = (\bar{\boldsymbol{X}}^T \boldsymbol{P_M} \bar{\boldsymbol{X}})^{-1} \bar{\boldsymbol{X}}^T \boldsymbol{P_M}(\boldsymbol{y^2} - \boldsymbol{y^1})$, where $\boldsymbol{P_M} = \boldsymbol{M}(\boldsymbol{M}^T \boldsymbol{M})^{-1} \boldsymbol{M}^T$ is a projection matrix.

Second, define $\hat{\bar{\boldsymbol{Z}}} = \bar{\boldsymbol{Z}}(\hat{\phi}_{2SLS}) = [\boldsymbol{i}, \boldsymbol{X^2_{TV}}, \boldsymbol{X^1_{TV}}, \boldsymbol{G^2 X^2}, \boldsymbol{G^1 X^1}, \boldsymbol{UR}, \mathbb{E}[\boldsymbol{G^1 y^1}(\hat{\theta}^1_{2SLS}) | \boldsymbol{X^1}]]$, $\mathbb{E}[\boldsymbol{G^2 y^2}(\hat{\phi}_{2SLS}) | \boldsymbol{X^2}, \boldsymbol{X^1}]$,
where $\mathbb{E}[\boldsymbol{G^1 y^1}(\hat{\theta}^1_{2SLS}) | \boldsymbol{X^1}] = (\boldsymbol{I} - \hat{\beta}_{1,2SLS} \boldsymbol{G^1})^{-1} \hat{\alpha}_{1,2SLS} + (\boldsymbol{I} - \hat{\beta}_{1,2SLS} \boldsymbol{G^1})^{-1}(\hat{\gamma}_{1,2SLS} \boldsymbol{I} + \hat{\delta}_{1,2SLS} \boldsymbol{G^1})\boldsymbol{X^1}$, with $\hat{\theta}^1_{2SLS}$ obtained as the estimation of the first stage on the first step.
and $\mathbb{E}[\boldsymbol{G^2 y^2}(\hat{\phi}_{2SLS}) | \boldsymbol{X^2}, \boldsymbol{X^1}] = \boldsymbol{G^2}(\boldsymbol{I} - \hat{\beta}_{2,2SLS} \boldsymbol{G^2})^{-1}[(\hat{\alpha}_{2,2SLS} - \hat{\alpha}_{1,2SLS})\boldsymbol{i} + (\boldsymbol{I} - \hat{\beta}_{1,2SLS} \boldsymbol{G^1})\mathbb{E}[\boldsymbol{y^1}(\hat{\theta}^1_{2SLS}) | \boldsymbol{X^1}] + \hat{\tilde{\boldsymbol{\delta}}}_{2SLS} \boldsymbol{UR} + \hat{\gamma}_{2,2SLS} \boldsymbol{X^2_{TV}} - \hat{\gamma}_{1,2SLS} \boldsymbol{X^1_{TV}} + \hat{\delta}_{2,2SLS} \boldsymbol{G^2 X^2} - \hat{\delta}_{1,2SLS} \boldsymbol{G^1 X^1}]$

Finally, I use $\hat{\bar{\boldsymbol{Z}}}$ as a new vector of instrument to estimate 2.9. Then the following consistent estimator is obtained: $\hat{\phi}_{Lee} = (\hat{\bar{\boldsymbol{Z}}}^T \bar{\boldsymbol{X}})^{-1} \hat{\bar{\boldsymbol{Z}}}^T (\boldsymbol{y^2} - \boldsymbol{y^1})$.

The consistency of this estimator is less straightforward, but it holds under the regularity conditions. The proof of the following Lemma is provided in Appendix A.

Lemma 2.6 *Under regularity conditions defined in Appendix A, the estimator $\hat{\phi}_{Lee}$ is consistent and has the following limiting distribution,*

$$\sqrt{n}(\hat{\phi}_{Lee} - \phi) \xrightarrow{D} \mathcal{N}(0, \Phi),$$

with $\Phi = (\sigma^2_{\epsilon^1} + \sigma^2_{\epsilon^2})(lim_{n\to\infty} \frac{1}{n} \bar{\boldsymbol{Z}}^T \bar{\boldsymbol{Z}})^{-1}$

Correlated effects

If the correlated effects are assumed to be present in the model the fist step model can be written as follows in matrix notation:

$$\begin{aligned}&(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{PR} = \beta(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1}\boldsymbol{Y^1} + \gamma(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{X^1} + \delta(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1}\boldsymbol{X^1} + \eta,\\ &\eta = (\boldsymbol{I} - \boldsymbol{G^1})\nu,\ \mathbb{E}[\eta|\boldsymbol{X^1}] = 0\end{aligned}$$

I then use the peer effect model in local differences proposed in Bramoullé, Djebbari, and Fortin (2009):

$$\begin{aligned}&(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{y^1} = \beta_1(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1}\boldsymbol{y^1} + \gamma_1(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{X^1} + \delta_1(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1}\boldsymbol{X^1} + (\boldsymbol{I} - \boldsymbol{G^1})\nu^1,\\ &\mathbb{E}[\nu^1|\boldsymbol{X^1}] = 0,\end{aligned}$$

which gives

$$\mathbb{E}[(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1}\boldsymbol{y^1}|\boldsymbol{X^1}] = (\boldsymbol{I} - \beta_1\boldsymbol{G^1})^{-1}(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1}(\gamma_1\boldsymbol{I} + \delta_1\boldsymbol{G^1})\boldsymbol{X^1}$$

The proposed estimation procedure, in this case, is close to the first step with no correlated effects. I redo all the steps with the following vectors of instruments and covariates: instruments $\boldsymbol{S} = [(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{X^1}, (\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1}\boldsymbol{X^1}, (\boldsymbol{I} - \boldsymbol{G^1})(\boldsymbol{G^1})^{\boldsymbol{2}}\boldsymbol{X^1}]$ and covariates $\tilde{\boldsymbol{X}}^{\boldsymbol{1}} = [(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{X^1}, (\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1}\boldsymbol{X^1}, (\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1}\boldsymbol{y^1}]$.

Then I find the 2SLS estimator on the first step and use it to get the new vector of instruments: $\hat{\boldsymbol{Z}} = Z(\hat{\theta}^1_{2SLS}) = [(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{X^1}, (\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1}\boldsymbol{X^1}, \mathbb{E}[(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1}\boldsymbol{y^1}(\hat{\boldsymbol{\theta}}^{\boldsymbol{1}}_{\boldsymbol{2SLS}})|\boldsymbol{X^1}]]$, where $\mathbb{E}[(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1}\boldsymbol{y^1}(\hat{\boldsymbol{\theta}}^{\boldsymbol{1}}_{\boldsymbol{2SLS}})|\boldsymbol{X^1}] = \boldsymbol{G^1}(\boldsymbol{I} - \hat{\beta}_{1,2SLS}\boldsymbol{G^1})^{-1}(\boldsymbol{I} - \boldsymbol{G^1})(\hat{\gamma}_{1,2SLS}\boldsymbol{I}+$ $+\hat{\delta}_{1,2SLS}\boldsymbol{G^1})\boldsymbol{X^1}$.

The consistent estimator can then be obtained as follows: $\hat{\theta}_{Lee} = (\hat{\boldsymbol{Z}}^T \tilde{\boldsymbol{X}}^{\boldsymbol{1}})^{-1}\hat{\boldsymbol{Z}}^T\boldsymbol{PR}$. Note that the proof of consistency follows directly by combining the result of L. Lee (2003) and the proof of Lemma 5, which can be found in Appendix A.

Step 2 also requires some adjustments in this case. Due to the presence of correlated effects, $\mathbb{E}[\boldsymbol{G^1 y^1}(\boldsymbol{\hat{\theta}^1_{2SLS}})|\boldsymbol{X^1}]$ is no longer observable since it includes the unobserved fixed effects correlated with covariates and cannot be used as an instrument. Hence, I need to modify both vectors of covariates and instruments in the following way: $\boldsymbol{\bar{X}} = [(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{X^2_{TV}}, (\boldsymbol{I}-\boldsymbol{G^1})\boldsymbol{X^1_{TV}}, (\boldsymbol{I}-\boldsymbol{G^1})\boldsymbol{G^2X^2}, (\boldsymbol{I}-\boldsymbol{G^1})\boldsymbol{G^1X^1}, (\boldsymbol{I}-\boldsymbol{G^1})\boldsymbol{UR}, (\boldsymbol{I}-\boldsymbol{G^1})\boldsymbol{G^1y^1}, (\boldsymbol{I}-\boldsymbol{G^1})\boldsymbol{G^2y^2}]$ is a new vector of covariates. I then use $(\boldsymbol{I} - \boldsymbol{G^1})(\boldsymbol{G^2})^2\boldsymbol{X^2}$ as an instrument for $(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^2y^2}$. I propose to use $\mathbb{E}[(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1y^1}(\boldsymbol{\hat{\theta}^1_{2SLS}})|\boldsymbol{X}]$ as an instrument for $(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1y^1}$. This instrument is clearly a valid instrument since it is uncorrelated with the second step error term and is clearly correlated with the outcome variable. Then I define $\boldsymbol{M} = [(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{X^2_{TV}}, (\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{X^1_{TV}}, (\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^2X^2}, (\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1X^1}, (\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{UR}, \mathbb{E}[(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1y^1}(\boldsymbol{\hat{\theta}^1_{2SLS}})|\boldsymbol{X}], (\boldsymbol{I} - \boldsymbol{G^1})(\boldsymbol{G^2})^2\boldsymbol{X^2}]$ as a vector of instruments.

Applying the same changes to all relevant vectors, I then fully repeat the estimation procedure of the case of no correlated effects, and obtain the consistent estimator. Consistency of the estimator is achieved by the argument similar to the one in Lemma 2.6, proof of which and more detailed discussion on estimation procedure can be found in Appendix A.

2.3 Data and Descriptive analysis

2.3.1 The system of higher education in Russia and specifics of the sampled university.

People with completed full vocational education or completed professional education of non-university level are eligible to enter the university[9]. Most of the places in the state universities are financed by the government: around 65%[10], but it differs among institutions. For example, the analysed in this paper university, National State University - Higher School of Economics, Nizhny Novgorod branch, provided 340 state-financed places out of total 431 in 2012[11]. The tuition fee varies from institution to institution, in our example, it varies between 130000 and 165000 Rubles, which equals to 28-36 times the minimum monthly wage or 18-23 times the minimum cost of living in Russia.

The students are accepted to the universities depending on the scores of the obligatory standardized examination, Unified School Examination, conducted at the end of the last school year. Each high school graduate has to take the exam in several subjects: Mathematics and Russian are mandatory to graduate from the school, the other subjects are chosen by the graduates depending on their preferences and the requirements of the universities they are aiming to apply to. For example, economic department of NRU-HSE requires the USE results in Social Studies (a mixture of basic knowledge about different aspects of society: philosophy, sociology, social psychology, law, political science) and Foreign language addi-

[9] It is more accurate to call them *the institutions of tertiary or post-secondary education*, since not all of these institutions in Russia have the status of the university, however, *the university* will be used for simplicity

[10] According to the Monitoring of education markets and organizations (MEMO), NRU HSE. In Russian

[11] The main dataset uses 2012 cohort of students, details are described in the next subsection

tional to the mandatory to all graduates Mathematics and Russian. However, regional and national level Olympiads can often be used as the second channel to enter some of the universities. These Olympiads are subject-specific and considered to be more sophisticated than the school exams, so they are designed to attract more talented students. In Higher School of Economics, the winners and prize-takers of these competitions are accepted to the university without exams if the major of the Olympiads corresponds to the university department (Economic Olympiads for economics department, Entrepreneurship Olympiads for management department etc.) or automatically given the highest score for the other subjects. However, those students are still required to take the USE and have the scores not lower than the required minimum (65 out of 100 in 2015, significantly lower than the requirement to be accepted). The share of students entering universities using the Olympiads results is around 5-6% overall in Russia, but it is much higher for the Higher School of Economics, around 40%, because of the selective status of HSE. Therefore, in general, the group of students entering HSE is more or less homogeneous and consists of the high-achievers. Even though Nizhniy Novgorod branch of HSE is less selective than the main Moscow branch, the level of the admitted students is still very high. The list of all accepted students is publicly available in the university itself as well as on the website.

Usually, universities in Russia have an exogenous group formation. The students are randomly split into groups of 20-30 people before the beginning of the studies. These groups stay mainly intact for the first three years. Several groups or even all students attend lectures together, whereas each group has separate tutorials. The changes to the group structure may occur if a lot of the students leave the university and the group is too small. Most of the universities have by now adopted the Bologna Process model of 4 years for Bachelor's degree and 1-2 years of Master's degree. In most cases each academic year has 2 terms with exams periods after each, however, HSE has 4 terms per year, with some exams or pass-fail exams after the 1st and the 3rd term and with most exams after the 2nd and the 4th term. The student is not allowed to fail 3 or more exams per half (1+2 or 3+4 term) and the retakes are conducted only after the 2nd and 4th exam periods of each year. All results of all students are publicly available near the students' office in the university and online so that everybody can follow their own performance, compare to the peers, and the tuition students can understand, whether they are eligible for the tuition discount.

2.3.2 Data description

The data is based on two longitudinal studies of the students' network, conducted in the National Research University Higher School of Economics (Nizhny Novgorod branch; state university). The information about the studies is summarized in Table 2.1.

Table 2.1: Studies characteristics

Study	Cohort	Frequency	Departments	Total students

I	2012	Each year	Economics, Management, Law, Computer Science	321
II	2013	Each 3-4 month	Economics, Management	205

Students were asked to indicate three and two networks correspondingly: friends from the university (same cohort), students from the same cohort, whom they ask for help (in the first study this question is divided into two: help with mathematical subjects and help with humanities). The *I study* is of the main interest of the paper due to the longer periods between the surveys that are able to capture a more persistent trend of the network dynamic. The *II study* is used only for the robustness check of the results.

Other data include all exam results, information about retakes and dropouts from the administrative university data, as well as some personal data: gender, high school examination results, type of living (dormitory or not, roommates for those who live in the dormitory), parental education, some indicators of willingness to succeed or efforts (time spent on homework, time spent online on social networks, indicator of having a job parallel to studies).

The typical problems of self-reported data are present in the dataset. There are several observations with partially missing data on the network links. These entries need to be handled with care since they might suggest both the students without friends, indicating the antisocial behavior, or the students that just skipped the questions, while answering the questionnaire. In the *I study* 13 students indicated no friends links, however, two of those provide an information about connections in the help networks, which might demonstrate an antisocial behavior of the students. There is no information on particular friends for 4 more students, who just said they are friends with a lot of students, or even with all students. In the *II study*, there are 9 students without links, however, it is not clear, whether they did not report anybody at all or whether they answered with a sentence, as 4 students from the *I study*, mentioned above.

Sampling is of a slight concern as well. The first survey has 321 observations out of 396 students that entered the 4 departments of the university in 2012, the second has 205 out of 253 students, started in 2013 in 2 departments, that gives approximately 75-80% of the full population of students (Table 2.2). Some of the students could have indicated the link to somebody outside of the sample, which can lead to overestimation of the importance of the observed links. However, the survey was conducted on several occasions, during lecture periods, so those, who did not answer the survey, are likely to attend the university only infrequently, and hence to have less influence on the other students.

Table 2.2: Comparison of samples and population

		Sample	All students	Share
I, year 1	Size	320	432	74.07%

	Retakes	157	203	77.34%
	Dropouts	16	40	40 %
I, year 2	Size	296	393	75.32%
	Retakes	148	190	77.89%
	Dropouts	24	39	62.54%
II	Size	205	254	80.7%
	Retakes	65	*137*	-
	Dropouts	6	21	28.57%

Table 2.2 also demonstrates an inability of the dataset to catch all the information about the dropouts (only 40% are present in the first survey) and their small amount in the network. This makes the econometric analysis of dropouts implausible, and forces to study exam retakes instead.

Note that the *I study* restricted the friends' network to 7 names, whereas the *II study* did not put any restriction. This lead to almost 50% of the students in the first period of *I study* reporting exactly 7 friends, whereas only 13,5% of the students in the *II study* indicated the same number of friends. The second wave of the long study also has space for mentioning the maximum of 7 friends, however, this restriction is not mentioned in the question itself. Therefore, 7 friends are the maximum of the 2nd wave of the long study with only 10% indicating exactly 7 friends. The distribution of the number of the friends for both studies is presented in Table B.1 and Figure B.1 of Appendix B. The average and median number of connections is 6 in both first year of the I sample and in the II sample, whereas it is 4 in the second wave of the I study. It is likely, that in the first wave some of the students had to restrict themselves to exactly 7 names, whereas some felt obliged to include more people than they are actually tightly connected to, which may cause underestimation of the importance of some links and overestimation of the others. Lower average number of friends in the second period may be caused by particularities of the survey construction as well as by the real trends in the network development.

The survey design is different for three networks. The first wave of the I study asks for no more than 7 friends and has 7 lines for the names, which was ignored by approximately 2% of the sample, the second wave of the I study does not put any restriction on the number of friends, although it has 7 lines as well, the short study says explicitly, that a number of friends can be unlimited, but has 15 lines. Therefore, the survey design may influence estimation results from the I survey analysis, hence the analysis of the II study with its unchanged question design can be helpful as a robustness check.

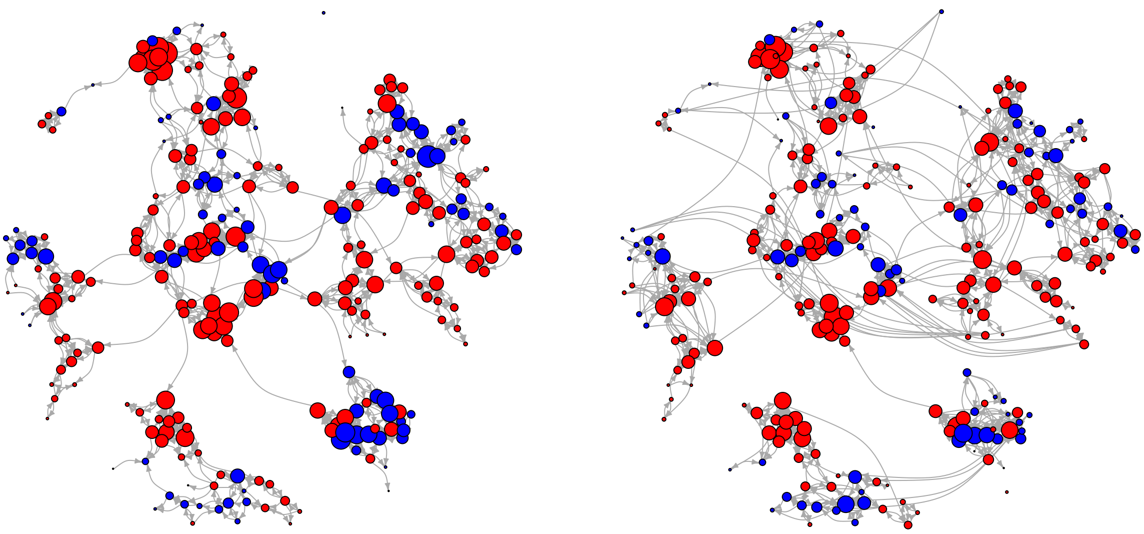
2.3.3 Network characteristics

In this section, I will discuss the validity of the identifying assumptions in a framework of the I survey.

Network stability.

Figure 2.1 visualizes the whole networks for the first wave (left) and for the second wave (right). Red nodes are females, blue - males, the size of the nodes is proportional to the overall degree of the node. It can be observed from this figure that two networks differ. For example, two clusters in the bottom part of the graph are not connected in the first wave, whereas there are several edges between them in the second wave.

More formal justification of the variability of the network is presented in Table 2.3.



(a) Wave 1. (b) Wave 2.

Figure 2.1: Networks.

Quite a lot of variation can be observed: around 11-12% of the students reported exactly the same set of friends. However, the share of completely new networks varies with gender. Females have only 5% of completely new networks. Hence, females tend to be more persistent in forming and retaining the links.

Table 2.3: Overlap of network partners

Network statistics	Full sample	Male	Female
Complete overlap	11.49	11.21	11.89
No new links	24.66	22.43	26.49

Partial overlap	65.20	46.73	77.30
Complete turnover	12.16	24.30	5.41
Observations	296	107	185

Note: Percentages of 1st, 3rd, and 4th rows do not add up to 100%, because there are new observations in the 2nd wave, for which we do not observe the network in the 1st wave

Table 2.4 provides more evidence of the network variation: only around 16% of the links survived after the first period, and around 78% of the links formed in the second period are new.

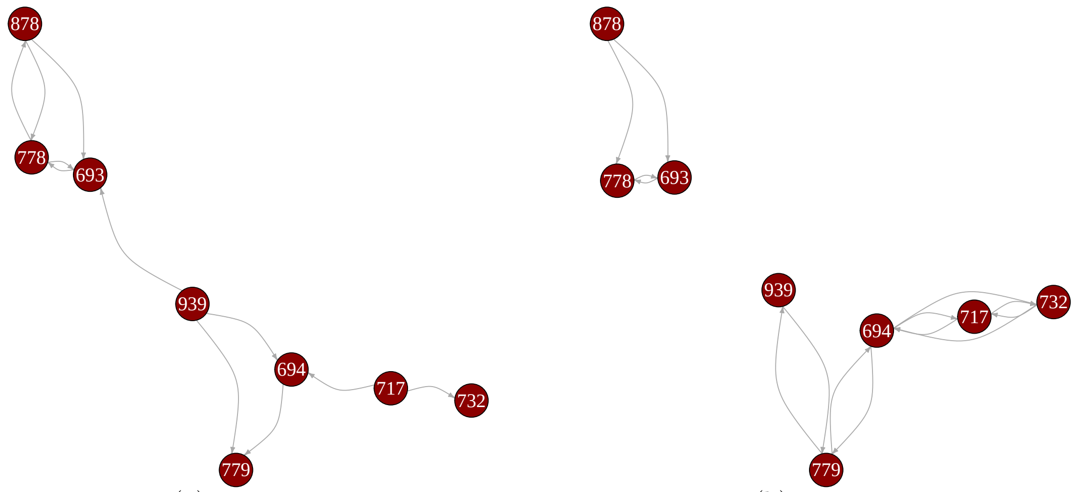
Table 2.4: Some network characteristics

Network statistics	Definition	1 year	2 year
Average indegree	Average number of ingoing ties	4.96 (2.73)	3.93 (2.53)
Average outdegree	Average number of outgoing ties	4.96 (2.01)	3.93 (2.2)
Density	Proportion of existing ties in the network	0.015	0.014
Reciprocity	Proportion of ties which are reciprocated	0.639	0.636
Transitivity	The ratio of the triangles and the connected triples in the graph	0.454	0.443
Share of the links that remained from the 1st wave in total amount of links of the 2nd wave		-	22.61%
Share of the links that remained from the 1st wave in total amount of links of the 1st wave		16.57%	-

Transitivity

. Table 2.4 describes several characteristics of the networks in the sample. The transitivity is measured by the shares total amount of connected triangles in the whole graph. So in more than 50% of all possible sets of three students, at least, one link is missing.

Figure2.2 shows the subgraph of the network to demonstrate the existence of intransitive triads in both of the samples. For example, in wave 1 the following triad is intransitive: $717 \to 694$, $694 \to 779$, but $717 \nrightarrow 779$. Other examples of intransitive triads are: $939 \to 693 \to 778$, $693 \to 778 \to 878$ in the first wave and $939 \to 779 \to 694$, $779 \to 694 \to 717$ in the second wave, and some more.



(a) Wave 1. (b) Wave 2.

Figure 2.2: Subgraph of the network

The characteristics of the networks (see Table2.4) also clearly suggest that the network is directed and cannot be assumed to be indirected since only around 60% of the links are reciprocal. Also, the networks are sparse with the density of the links around 1.5%.

2.3.4 Descriptive analysis

People often tend to connect based on similarities in their observed and unobserved characteristics. Table 2.5 summarizes the findings on the affinity of the peers in the network. Most of the peers are coming from the same group, more than 84% and almost all friends are from the same department. The network can in principle be divided into four smaller networks.

Females are more likely to connect with peers of the same gender, whereas males have more diverse networks. Gender difference also exists in the probability of connecting to the dormitory mates: males are more likely to connect. The share of the friends with the same living conditions is, however, decreasing with time, suggesting that some other characteristics matter more for creating and sustaining the links.

Future plans on average seem not to matter a lot for the link formation: friends with the same plans for the future education are about 50% of the peers. This share could probably be higher, if the students were asked about there plans later in the course of their studies, and not during the first year. However, given the student's willingness to do Master, her peers are as well more oriented on continuing the studies after the Undergraduate level.

Table 2.5: Characteristics of reported networks links by sex

Variables	1st wave			2nd wave		
	Male	Female	All	Male	Female	All

Average size	4.53	5.19	4.96	3.57*	4.18*	3.93*
Average size (with out of sample links)	5.22	5.78	5.58	4.29*	4.96*	4.69*
Study group/department relation (% of network partners)						
Same group	84.17	87.23	86.76	87.21	89.89	88.78
Same department	98.54	99.21	98.99	97.54	99.39	98.95
Individual characteristics of network partners(% of network partners)						
Same gender	64.05	81.97	76.18			
Same working status	62.43	70.33	67.78	50.74	60.95	56.41
Same education of mother	61.75	66.84	65.19	-	-	-
Same education of father	56.45	50.08	52.14	-	-	-
Same living conditions	57.59	46.71	50.23	50.97	39.61	43.33
Same living conditions (dorm/not)	84.14	76.23	78.79	74.27	70.55	72.16
Future plans (% of network partners)						
Same plans for Master	54.44	57.37	56.41	-	-	-
Same plans for Doctorate	47.18	47.32	47.27	-	-	-
Subsample of planning to do Master:						
Same plans for Master	68.34	72.42	74.46	-	-	-

*the network data in the 2nd wave is truncated at 7 friends

More than 1/3 of all links in the first wave are links to the students with retakes (37%). The share of the links to the students with retakes in the first period in the total amount of second wave links is slightly smaller: 33%. It might be caused by the intention of students to improve their peer group and connect to peers with higher outcomes. The average amount of the friends with retakes in the first period is 1.83 while it is lower for the second period: only 1.25. The average amount of peers with exam retakes for the subsample of all students that have at least one peer with retake is higher than the average of the full sample and is equal to 2.5. For the same students in the second wave, the average number of peers who had exam retakes in the first period is now much lower: 1.55. It can be suspected that the decrease in this value may be partially explained by the readjustments of the network towards better connections. Moreover, for the same subsample, the average number of peers with retakes in the second period is even lower: 1.37. Interestingly, some of those, who didn't have any friends with retakes in the first period, connected to new peers that had the retakes in the second period, the average number of such friends is only 0.35 though, but the average number of friends with retakes in the next period is 0.57. So the changes in the network are leading to the improvements as well as worsening of the new peer group. These findings are summarized in the Table 2.6.

Table 2.6: Distribution of retakes

	Links wave 1, retakes wave 1	Links wave 2, retakes wave 1	Links wave 2, retakes wave 2
Share of retakes links in all links	36.99%	32.99%	29.15%
Average amount of friends with retakes	1.83	1.25	1.15
Subsample with retakes of friends	2.5	1.55	1.37

Average amount of friends with retakes *Subsample no retakes of friends* Average amount of friends with retakes	0	0.35	0.57

Observe that students in the studied framework tend to connect to peers, having higher average grades than the students themselves, for the full sample as well as for the samples with and without retake friends. Students, who do not connect to peers with retakes, are performing better than those, whose friends are having retakes. However, the improvements in the performance in the future are not significant, with the changes in the performance of the students without peers' retakes being slightly higher.

Table 2.7: Average grades in samples and subsamples

	Full sample	With retakes of friends	No retakes of friends
Average grade	7.04 (0.99)	6.98 (0.96)	7.37 (0.98)
Average grade of friends	7.18 (0.65)	7.03 (0.63)	7.68 (0.49)
Sample size	320	234	86
Average grade next period	7.13 (1.14)	7.02 (1.15)	7.44 (1.07)
Sample size	297	217	80

It is not possible to distinguish between the predicted and unexpected components of retakes by simply looking at the data. Therefore, the deeper econometric analysis is needed to make conclusions about the existence and the magnitude of the effect of unpredicted shock.

2.4 Results

2.4.1 Main specification

I use the following variables for the main specification of the model:

Outcome: average weighted grade of the student in the corresponding period. The grades are summed up weighted by the amount of the credits assigned to the particular course.

Retakes: indicator of at least one retake in the first period.

Initial ability, measured as the sum of mandatory Unified State Examinations (mathematics and Russian) plus the sum of cross-products between these USE results and a dummy of winning any relevant Olympiads.

Controls: time-invariant, such as gender, socio-economic background like a dummy of parental higher education, a dummy of having a single parent before entering the university and dummy for siblings; and a set of dummies for three departments with law department serving as a base.

Controls: time-varying, such as tuition, which is mostly time-invariant, but some rare students change the type of tuition, working status (dummy for not working versus any type of job) and living conditions (dormitory versus everything else).

Descriptive statistics for these variables is provided in Table B.3 of Appendix B. It can be observed that the average changes in the time-variant variables are rather modest, as well as the changes in the performance. However, the average grade has higher standard deviation and spread in the second period.

Table 2.8 summarizes some of the findings of the estimation of the model without correlated effects. Note that the sample size is smaller than was discussed in the data description, due to the absence of some students in one of the waves. And it is critical to have the information in both waves for each of the students to estimate the effect.

Table 2.8: Estimation of main specification

Variable	(1)	(2)	(3)	(4)	(5)
Constant	-0.1521		-0.1840		-0.0482
Unexpected Retake	-0.2638	-0.2143	-0.3077•	-0.2064	-0.3907*
Endogenous effect, period 1	-0.0307	-0.0425	-0.0317	0.0908*	0.0614*
Endogenous effect, period 2	0.0205	0.0085	0.0218	0.0419	0.0306
Time-variant own controls					
Tuition, w1	0.0208			0.0102	
Tuition, w2	-0.0912			-0.1518	
Working status, w1	-0.0664	-0.0719	-0.0716		
Working status, w2	0.1381•	0.1147*	0.1346•		
Living in dorm, w1					0.1061
Living in dorm, w2					0.1651
Network's controls					
Economics, w1	0.2417	0.1568	0.2692	-0.0732	
Economics, w2	-0.4681**	-0.4367**	-0.4513**	-0.5893***	
Management, w1	0.5409*		0.5712*		
Management, w2	0.1790		0.1996		
Working status, w1				-0.7352*	-0.5420**
Working status, w2				-0.0497	-0.1903
HE of father, w1		0.4010•			
HE of father, w2		0.0006			
Sample size	250	250	250	250	250

BIC	-216.68	-225.24	-226.51	-225.79	-196.71

*** - p-value < 0.01, ** - p-value < 0.05, * - p-value < 0.1, $^{\bullet}$ - p-value < 0.15

It can be observed that for the full sample the estimator of the effect of the unpredicted component of retakes is negative but in most cases insignificant. The magnitude of the effect in specification (5) suggests that if a friend of some student had a retake during the first year, which this student couldn't predict at all, the difference between the average grade of year 2 and the average grade of year 1 of the student will be on average 0.05-0.39 lower, than in the case the student expected the retake of a friend, depending on the total number of friends. For example, the median student on average improves her grades in the second period in comparison to the first by 0.24, which is 2.4% of the maximum grade. The presence of unpredicted retake of the friend, other things equal, may leave the average grade in the second period at the same level or even decrease it up to 1.5% of maximum grade, changing the direction of the dynamics and, moreover, putting the student on average 5-25 positions lower in the overall students' rating, falling lower with less friends.

Note, that there is a highly significant difference between the economics and other departments for most of the specifications. On average, students of economics department have -0.5 lower difference of grades, which suggest the overall lower grades of the economics department in the second year. This evidence indicates the necessity of using the model with correlated effects or treating the departments separately by splitting the full sample.

Discussing the results for those, who had their own retakes, versus those, who did not is the other possible way to improve the estimation results.

The further analysis is given in the next subsections, where I present the results of estimation in the subsamples, of the model with correlated effects, as well as the estimation with a possibly improved network. However, it is worth pointing out, that the sample size for the main specification is 250 students, which may be not sufficiently big to capture the desired effect, and the results of the estimation in the subsamples should be treated with even more care, since with the lower sample size the asymptotic properties of the proposed estimator may suffer.

2.4.2 Connection to one's own retake

I first report the results for the subsamples of students with and without own retakes. It can be suggested that the students that had their own retake may, in general, be connected to worse peers. Therefore, having friends with retakes might lower the performance even further, whereas the friends' retakes are more likely to have either no effect or even positive influence for the better students.

Table 2.9: Presence of own retake

Variable	(1), yes	(2), yes	(1), no	(2), no

Constant	0.2602	0.1027	-0.1001	-0.3734*
Unexpected Retake	-0.2092	-0.1788	0.0246	0.0586
Endogenous effect, period 1	-0.0237	-0.0539	0.0352	-0.0262
Endogenous effect, period 2	0.0756**	0.0671•	0.0429	0.0577•
Time-variant own controls				
Tuition, w1	-0.1612		0.0032	
Tuition, w2	-0.1834		-0.2685	
Working status, w1		-0.1175		-0.0713
Working status, w2		0.1379		0.1192
Network's controls				
Economics, w1	-0.1964	-0.0474	0.0616	0.1081
Economics, w2	-0.9973***	-0.9131***	-0.3932	-0.5344•
Management, w1				0.3098
Management, w2				-0.0338
Working status, w1			-0.5632**	
Working status, w2			-0.2305	
HE of father, w1	0.6306•	0.3753		
HE of father, w2	-0.3627	-0.4972•		
Dummy siblings, w1		0.7689***		
Dummy siblings, w2		0.2113		
Sample size	83	83	167	167
BIC	-336.10	-348.14	-288.34	-290.82

*** - p-value < 0.01, ** - p-value < 0.05, * - p-value < 0.1, • - p-value < 0.15

Table 2.9 gives hints that the outcomes are influenced differently by the peers in case of presence of one's own retake and in the case when the student passed all the exams from the first attempt. First of all, unexpected retake has an insignificant and negative effect of higher magnitude in case of own retake than without own retakes. So, when students in the network have retakes together, they will less likely improve in the future. It may be partially explained by the worse peer group, and partially by the fact that fewer friends are able to help to catch up with the courses after retakes. It can also be observed that the endogenous effect changes the sign, from negative to positive, and is more significant for students with own retakes, which may suggest that the students, especially the ones with their own retake tend to seek for the better peers in the future. However, the data does not provide evidence that the willingness to connect to better peers is coming from the discussed shock, therefore, the changes may be considered as a natural learning process.

2.4.3 Effects in different departments

In this subsection, I discuss the results for subsamples of different departments. I present the results for two departments: economics and management. The economics department showed significantly different results in comparison to the others in the main specification, and the management department is quite similar to the economics in the curriculum and direction of study.

Table 2.10: Departments

Variable	(1), Econ.	(2), Econ.	(3), Man.	(4), Man.
Constant	-0.5228*	-0.2265	0.5614*	0.7032**
Unexpected Retake	-0.4375	-0.4794	0.4043	0.3943
Endogenous effect, period 1	-0.0426	-0.0150	0.2884***	0.3927***
Endogenous effect, period 2	0.0334	0.0544	0.1635**	0.2164**
Time-variant own controls				
Tuition, w1		0.2538		-0.4889
Tuition, w2		-0.0479		-0.7778*
Working status, w1	-0.0888		0.0054	
Working status, w2	0.2119		0.0880	
Network's controls				
Ability, w1			-0.0056*	-0.0062**
Ability, w2			-0.0049**	-0.0048**
Gender, w1		1.0102*		
Gender, w2		0.2889		
Working status, w1		-0.6228		-0.6353*
Working status, w2		-0.5707		-0.4209
HE of mother, w1			-0.4308	-0.7535*
HE of mother, w2			-0.3144	-0.5867*
Dormitory, w1	-1.5248***	-0.8094•		
Dormitory, w1	-0.9558**	-0.8145*		
Dummy siblings, w1	0.4005			
Dummy siblings, w2	-0.3945			
Sample size	82	82	68	68
BIC	-305.45	-300.61	-456.68	-471.57

*** - p-value < 0.01, ** - p-value < 0.05, * - p-value < 0.1, • - p-value < 0.15

As can be seen from Table 2.10, the discussed effect is surprisingly different for two departments. While specification (1) for the economic department have the negative effect of the unexpected retake, the same effect in the specifications for management is positive. However, estimators are not significant. Both subsamples have a small number of observations, which can cause the low significance of the effect of the interest, and the results should be treated with caution. It is possible to eliminate the differences between the departments and estimate the full sample, by exploring the model with correlated effects.

2.4.4 Estimation in presence of correlated effects

In this Subsection, I would like to discuss the results of the estimation proposed in Section 2.4.2. Simple estimation in the presence of correlated effects might lead to the biased results. Next table presents the summary of results, judging from which I can then compare the two specifications: with and without correlated effects.

Table 2.11: Estimation of specification with correlated effects

Variable	(1)	(2)	(3)	(4)	(5)
Unexpected Retake	-0.4144$^{\bullet}$	-0.3899$^{\bullet}$	-0.3817$^{\bullet}$	-0.3817$^{\bullet}$	-0.4616*
Endogenous effect, period 1	-0.0361	-0.0526	-0.0378	-0.0379	0.0186
Endogenous effect, period 2	0.0143	0.0016	0.0544	0.0544	0.0461
Time-variant own controls					
Tuition, w1		0.0834	0.0411		
Tuition, w2		-0.1011	-0.1292		
Working status, w1	-0.0382	0.0266		0.0411	0.0590
Working status, w2	0.1077	0.1355$^{\bullet}$		-0.1292	0.0991
Living conditions, w1	-0.1323				
Living conditions, w2	0.2102				
Network's controls					
HE of mother, w1	-0.6547	-0.5532	-0.5785	-0.5785	-0.6717
HE of mother, w2	-0.2011	-0.1541	-0.3396	-0.3396	-0.3875
HE of father, w1	0.5325	0.5789	0.4663	0.4663	
HE of father, w2	-0.0167	0.0378	-0.0831	0.3817	
Sample size	250	250	250	250	250
BIC	-183.89	-185.56	-195.56	-192.25	-197.99

*** - p-value < 0.01, ** - p-value < 0.05, * - p-value < 0.1, $^{\bullet}$ - p-value < 0.15

Controlling for correlated effects leads to more significant and persistent value of negative effect of unexpected retakes than in the main specification. The magnitude of the effect in specification (5) suggests that if a friend has a retake during the first year, which the student couldn't predict at all, this will make the difference between the average grade of year two and the average grade of year one for this student on average 0.46 lower, if the student has only 1 friend, and approximately 0.065 lower, if the student has 7 friends. The maximum of the grades is 10 so that the person lose almost 5% of the maximum grade when the network includes friends with retakes.

2.4.5 Additional analysis

Improving network

As it was mentioned before, students were asked to name up to 7 friends from their cohort, although some named more than 7. However, it is reasonable to assume that all named friends are not equal for the person. I introduce two possible ways to account for better friends so that the quality of the network can be improved.

First, I assume that the friends named among the first are more important than the others, since they were remembered earlier, and the best friends can't be named last. I reduced the network, only taking up to three named first students. I conducted analysis for both models with and without accounting for correlated effects. The suggested improvement of the network didn't, however, increased the significance of the results[12]. The effect of an unpredicted component of friends' exam retake is not significantly different from zero. Therefore, it might be reasonable to conclude that the unexpected negative or positive performance of the whole network of friends is more important for the future performance of students than the performance of only best friends.

Second, I observe that about 60% of the network is reciprocal, so I conduct similar analysis limiting the network to only reciprocal connections. This again does not bring any improvement in terms of the significance of the studied effect. It seems that the students' performance is shaped not only by their mutual friends, but although by those, who don't consider them as friends, but are considered as friends by the students. These students may be viewed as a sort of role models, and therefore, are important to be taken into account.

Thus, the initial full network is able to capture the effect of unexpected shock better than the versions of the network, considered initially as possible improvements.

Important classes

The further analysis divides the subjects, studied by the students in the sample, into two parts: more important and less important. All subjects have the corresponding amount of ECTS credits, from 0 to 8 with average around 2.5. For the analysis, I set the threshold of 4 ECTS points. However, some subjects have several exams, for example, Mathematical Analysis, and the weight of some of the exam in the series can be lower than 4, but, at least, one exam has ECTS higher than 4. In these cases, I am including all the exams of the series in the sample of important exams. This restricts the set of the students with retakes to 2/3 of the initial set.

Table 2.12 provides the results of the analysis in the new setting for the model without correlated effects.

Table 2.12: Estimation with retakes for classes with ECTS 4 and higher

[12] The detailed results are presented in Tables B.4, B.5 in Appendix B

Variable	(1)	(2)	(3)	(4)	(5)
Constant		-0.2176	-0.1670		-0.2005
Unexpected Retake	-0.4912**	-0.5484***	-0.5158**	-0.4907**	-0.5564**
Endogenous effect, period 1	0.1072*	-0.0211	-0.0160	0.1076*	-0.0158
Endogenous effect, period 2	0.0378	0.0279	0.0284	0.0401	0.0307
Time-variant own controls					
Tuition, w1	0.0417		0.0430	0.0530	
Tuition, w2	-0.0861		-0.0575	-0.0830	
Working status, w1		-0.0568	-0.0570		-0.0616
Working status, w2		0.1488*	0.1494*		0.1469*
Living conditions, w1	-0.0222	-0.2673			
Living conditions, w2	0.0312	-0.1808			
Network's controls					
Economics, w1	-0.1032	0.1652	0.1257	-0.1129	0.1387
Economics, w2	-0.6177***	-0.5279**	-0.5466**	-0.6240***	-0.5411**
Management, w1		0.4322	-0.4049		0.4159
Management, w2		0.1209	0.1045		0.1127
Working status, w1	-0.8120*			-0.8186**	
Working status, w2	-0.0074			-0.0102	
Sample size	250	250	250	250	250
BIC	-215.86	-220.76	-220.81	-226.98	-230.99

*** - p-value < 0.01, ** - p-value < 0.05, * - p-value < 0.1, $^\bullet$ - p-value < 0.15

It can be observed that a lot of the results resemble the results for the model with all retakes, however, the effect of the unpredicted retake is more significant when only important classes are taken into consideration. The sign of the estimator remains negative but it gains much more significance, suggesting the different effect that different classes may have on the future performance of the network. The results also suggest the higher magnitude than in the initial model. Now, the friend's unexpected retake of the important class may make the difference between average grades in two periods bigger and reduce the average grade of the second year additionally by up to 0.5, which equals to 5% of the maximum grade.

This result is expectable. For example, the new set of retakes does not include the class of Discrete Mathematics in the Economics department but includes Mathematical Analysis. These two classes differ not only in the amount of ECTS but also in the length and importance for the further classes. Mathematical Analysis is studied throughout the whole length of the first year, whereas Discrete Mathematics only for one term. Moreover, the former introduces a lot of methods used later in the core classes of the higher years, such as Micro or Macro, while the latter might be considered to contribute less in future studies. The full list of classes, which were retaken at least once and the subset of more important classes are presented in Table B.6 of Appendix B.

The significance of a dummy of the Economics department suggests that the model with correlated effects may be used, as in the model with the full set of retakes. Surprisingly, the estimator of the effect of the unexpected retake in the model with correlated effects loses the significance once I restrict the set of the retakes.

2.5 Conclusion

The paper discusses the spread of the unpredicted shock across the network of friends in the university environment using the newly introduced dynamic peer effect model in the presence of endogenous shock.

Exam retakes play an important role in determining the future of the student. However, it was shown that the unpredicted component of the retake may influence not only the students with a retake but also the whole network of friends. In most of the cases the effect is not very significant, but still should not be ignored. When the threshold of failing the exam is too high, some students, viewed by their friends as high-achievers, are likely to fail. This anticipation mistake leads to the decrease of the average grades of the whole friendship network.

The ideas explored in this paper can be further extended to the analysis of the networks in other settings, not only for educational outcomes. The method is applicable, when the endogenous shocks might have the longitudinal effect on the network outcomes, such as, for example, a treatment that for some reasons cannot be randomized, or conversational networks in developing communities, etc.

I have presented the results for identification of such models, that allow disentangling the effect of unpredicted shock on the future performance. The findings of the paper suggest that it is sufficient to assume time-variability of networks together with the existence of intransitive triads (or distances of length three, depending on the correlated effects assumption) in each of the states of the network for the similar models. Intransitive triads are guaranteed by the presence of two students only connected via the third common friend but not directly. The characteristics of friends of the friends don't influence directly the outcome, and, therefore, can be used as an instrumental variable for the friends' outcome. Such instruments can, therefore, deal with endogeneity issue. The group of new friends, different from the group of old friends, let the model capture the changes, happening due to the shock.

The procedure developed in the paper is shown to yield consistent estimators of the individual characteristics, endogenous peer effect and effect of unpredicted shock.

All theoretical findings are tested on the dataset of university students, connected via the friendship network. Most of the empirical evidence suggest that the unpredicted exam retakes of the friends will have a negative effect on the changes of the performance of students. This effect is more prominent for students with own retakes and for students in the Economics department. The higher significance of the estimators in the model with correlated effects gives evidence of the presence of unobserved homophily that influences link formation. Change of sign of endogenous effect for students with own retakes shows the importance of further exploration of the problem and improvement of the model by inclusion of the link formation mechanism.

Appendix

A. Main proofs

Regularity conditions (adaptation of L. Lee (2003)):

Assumption 1. The matrices $(I - \beta^1 G^1)$ and $(I - \beta_2 G^2)$ are nonsingular

Assumption 2. The row and column sums of the matrices G^1, G^2, $(I - \beta^1 G^1)^{-1}$ and $(I - \beta_2 G^2)^{-1}$ are uniformly bounded in absolute value.

Assumption 3. The elements of the matrices X^1 and X^2 are uniformly bounded in absolute value

Assumption 4. The error terms $\{\nu_i : 1 \leq i \leq n\}$ are identically distributed. Furthermore, they are distributed (jointly) independently with $\mathbb{E}[\nu_i \boldsymbol{X_i^1}] = 0$ and $\mathbb{E}[\nu_i^2] = \sigma_\nu < \infty$. Additionally, they are assumed to possess finite fourth moments. The error terms $\{\Delta\epsilon_i : 1 \leq i \leq n\}$ are identically distributed. Furthermore, they are distributed (jointly) independently with $\mathbb{E}[\Delta\epsilon_i] = 0$ and $\mathbb{E}[\Delta\epsilon_i^2] = \sigma_{\epsilon^1} + \sigma_{\epsilon^2} < \infty$. Additionally, they are assumed to possess finite fourth moments

Assumption 5. The limit $\boldsymbol{J} = \lim_{n\to\infty} \frac{1}{n} \boldsymbol{Z}^T \boldsymbol{Z}$ exists and is nonsingular.

Assumption 6. The limit $\bar{\boldsymbol{J}} = \lim_{n\to\infty} \frac{1}{n} \bar{\boldsymbol{Z}}^T \bar{\boldsymbol{Z}}$ exists and is nonsingular.

Assumption 7. Step 1. The initial estimator β^1_{2SLS} of β_1 is n^a-consistent for some $a > 0$. The initial estimators α^1_{2SLS}, γ^1_{2SLS} and δ^1_{2SLS} are consistent estimators of α^1, γ^1 and δ^1, respectively. **Step 2.** The initial estimators $\beta_{1,2SLS}$ and $\beta_{2,2SLS}$ of β_1 and β_2 are n^b-consistent for some $b > 0$. The initial estimators $\alpha_{1,2SLS}$, $\alpha_{2,2SLS}$, $\gamma_{1,2SLS}$, $\gamma_{2,2SLS}$, $\delta_{1,2SLS}$ and $\delta_{2,2SLS}$ are consistent estimators of α_1, α_2, γ_1, γ_2, δ_1 and δ_2, respectively.

Proof of Lemma 1.

The structural form equation:

$$P(retake_i) = \alpha^1 + \beta^1 \sum_{j\neq i} G^1_{ij} y^1_j + \gamma^1 X^1_i + \delta^1 \sum_{j\neq i} G^1_{ij} X^1_j + \nu_i, \qquad \mathbb{E}[\nu_i|X] = 0$$

can be rewritten in the reduced form in the following manner:

$$\boldsymbol{PR} = \alpha^1 \boldsymbol{i} + \beta^1 \boldsymbol{G^1 y^1} + \gamma^1 \boldsymbol{X^1} + \delta^1 \boldsymbol{G^1 X^1} + \nu, \qquad \mathbb{E}[\nu|\boldsymbol{X^1}] = 0$$

$$\boldsymbol{PR} = \alpha^1 \boldsymbol{i} + \beta^1 \boldsymbol{G^1 y^1} + (\gamma^1 \boldsymbol{I} + \delta^1 \boldsymbol{G^1}) \boldsymbol{X^1} + \nu, \qquad \mathbb{E}[\nu|\boldsymbol{X^1}] = 0$$

Taking conditional expectations:

$$\mathbb{E}[\boldsymbol{PR}|\boldsymbol{X^1}] = \alpha^1 \boldsymbol{i} + \beta^1 \boldsymbol{G^1} \mathbb{E}[\boldsymbol{y^1}|\boldsymbol{X^1}] + (\gamma^1 \boldsymbol{I} + \delta^1 \boldsymbol{G^1}) \boldsymbol{X^1}$$

Note that $\boldsymbol{y}$ can be expressed in terms of peer effect model as the one used for the probability of retakes:

$$y^1_i = \alpha_0 + \beta_0 \sum_{j\neq i} G^1_{ij} y^1_j + \gamma_0 X^1_i + \delta_0 \sum_{j\neq i} G^1_{ij} X^1_j + \xi_i, \qquad \mathbb{E}[\xi_i|X] = 0$$

with reduced form:

$$\boldsymbol{y^1} = \alpha_0 \boldsymbol{i} + \beta_0 \boldsymbol{G^1 y^1} + (\gamma_0 \boldsymbol{I} + \delta_0 \boldsymbol{G^1})\boldsymbol{X^1} + \xi, \qquad \mathbb{E}[\xi|\boldsymbol{X}] = 0$$

Then following steps of Bramoullé, Djebbari, and Fortin (2009):

$$\boldsymbol{y^1} = \alpha_0(\boldsymbol{I} - \beta_0 \boldsymbol{G^1})^{-1} + (\boldsymbol{I} - \beta_0 \boldsymbol{G^1})^{-1}(\gamma_0 \boldsymbol{I} + \delta_0 \boldsymbol{G^1})\boldsymbol{X^1} + (\boldsymbol{I} - \beta_0 \boldsymbol{G^1})^{-1}\xi, \qquad \mathbb{E}[\xi|\boldsymbol{X}] = 0$$

Using $(\boldsymbol{I} - \beta_0 \boldsymbol{G^1})^{-1} = \sum_{k=0}^{\infty} \beta_0^k \boldsymbol{(G^1)}^k$:

$$\boldsymbol{y^1} = \alpha_0(\boldsymbol{I} - \beta_0 \boldsymbol{G^1})^{-1} + \gamma_0 \boldsymbol{X^1} + (\gamma_0\beta_0 + \delta_0)\sum_{k=0}^{\infty} \beta_0^k \boldsymbol{(G^1)}^{k+1}\boldsymbol{X^1} + \sum_{k=0}^{\infty} \beta_0^k \boldsymbol{(G^1)}^k \xi$$

And the expected mean friends' groups' performance conditional on $\boldsymbol{X^1}$ can be written as:

$$\mathbb{E}[\boldsymbol{G^1 y^1}|\boldsymbol{X^1}] = \alpha_0(\boldsymbol{I} - \beta_0 \boldsymbol{G^1})^{-1} + \gamma_0 \boldsymbol{G^1 X^1} + (\gamma_0\beta_0 + \delta_0)\sum_{k=0}^{\infty} \beta_0^k \boldsymbol{(G^1)}^{k+2}\boldsymbol{X^1}$$

As was proven in Bramoullé, Djebbari, and Fortin (2009), if $\gamma_0\beta_0 + \delta_0 \neq 0$ and I, G^1 and $(G^1)^2$ are linearly independent, the social effects are identified. So this expression can be plugged-in into the reduced form of the equation for the probability of retake.

$$\begin{aligned}
&\mathbb{E}[\boldsymbol{PR}|\boldsymbol{X^1}] = \alpha^1 \boldsymbol{i} + \beta^1(\alpha_0(\boldsymbol{I} - \beta_0 \boldsymbol{G^1})^{-1} + \gamma_0 \boldsymbol{G^1 X^1} + (\gamma_0\beta_0 + \delta_0)\sum_{k=0}^{\infty} \beta_0^k \boldsymbol{(G^1)}^{k+2}\boldsymbol{X^1}) + \\
&+(\gamma^1 \boldsymbol{I} + \delta^1 \boldsymbol{G^1})\boldsymbol{X^1} = (\alpha^1 \boldsymbol{I} + \beta^1 \alpha_0(\boldsymbol{I} - \beta_0 \boldsymbol{G^1})^{-1})) + \\
&+\beta^1(\gamma_0\beta_0 + \delta_0)\sum_{k=0}^{\infty} \beta_0^k \boldsymbol{(G^1)}^{k+2}\boldsymbol{X^1} + (\gamma^1 \boldsymbol{I} + (\beta^1\gamma_0 + \delta^1)\boldsymbol{G^1})\boldsymbol{X^1}
\end{aligned}$$

or

$$\mathbb{E}[\boldsymbol{PR}|\boldsymbol{X^1}] = \alpha^1 \boldsymbol{I} + \beta^1(\alpha_0(\boldsymbol{I} - \beta_0 \boldsymbol{G^1})^{-1} + \beta^1(\boldsymbol{I} - \beta_0 \boldsymbol{G^1})^{-1}(\gamma_0 \boldsymbol{I} + \delta_0 \boldsymbol{G^1})\boldsymbol{G^1 X^1} + (\gamma^1 \boldsymbol{I} + \delta^1 \boldsymbol{G^1})\boldsymbol{X^1}$$

Now consider two sets of structural parameters $(\alpha^1, \beta^1, \gamma^1, \delta^1)$ and $(\tilde{\alpha}^1, \tilde{\beta}^1, \tilde{\gamma}^1, \tilde{\delta}^1)$ leading to the same reduced form. It means that:

$$\alpha^1 \boldsymbol{I} + \beta^1 \alpha_0(\boldsymbol{I} - \beta_0 \boldsymbol{G^1})^{-1} = \tilde{\alpha}^1 \boldsymbol{I} + \tilde{\beta}^1 \alpha_0(\boldsymbol{I} - \beta_0 \boldsymbol{G^1})^{-1}$$

$$\alpha^1 \boldsymbol{I} - \alpha^1\beta_0 \boldsymbol{G^1} + \beta^1\alpha_0 \boldsymbol{I} = \tilde{\alpha}^1 \boldsymbol{I} - \tilde{\alpha}^1\beta_0 \boldsymbol{G^1} + \tilde{\beta}^1\alpha_0 \boldsymbol{I}$$

$$(\alpha^1 - \tilde{\alpha}^1)\boldsymbol{I} + (\beta^1\alpha_0 - \tilde{\beta}^1\alpha_0)\boldsymbol{I} - (\alpha^1\beta_0 - \tilde{\alpha}^1\beta_0)\boldsymbol{G^1} = 0$$

$$(\alpha^1 - \tilde{\alpha}^1 + (\beta^1 - \tilde{\beta}^1)\alpha_0)\boldsymbol{I} = (\alpha^1 - \tilde{\alpha}^1)\beta_0 \boldsymbol{G^1}$$

and:

$$\beta^1(\boldsymbol{I} - \beta_0 \boldsymbol{G^1})^{-1}(\gamma_0 \boldsymbol{I} + \delta_0 \boldsymbol{G^1})\boldsymbol{G^1} + (\gamma^1 \boldsymbol{I} + \delta^1 \boldsymbol{G^1}) = \tilde{\beta}^1(\boldsymbol{I} - \beta_0 \boldsymbol{G^1})^{-1}(\gamma_0 \boldsymbol{I} + \delta_0 \boldsymbol{G^1})\boldsymbol{G^1} + (\tilde{\gamma}^1 \boldsymbol{I} + \tilde{\delta}^1 \boldsymbol{G^1})$$

$$\beta^1(\gamma_0 \boldsymbol{I} + \delta_0 \boldsymbol{G^1})\boldsymbol{G^1} + (\boldsymbol{I} - \beta_0 \boldsymbol{G^1})(\gamma^1 \boldsymbol{I} + \delta^1 \boldsymbol{G^1}) = \tilde{\beta}^1(\gamma_0 \boldsymbol{I} + \delta_0 \boldsymbol{G^1})\boldsymbol{G^1} + (\boldsymbol{I} - \beta_0 \boldsymbol{G^1})(\tilde{\gamma}^1 \boldsymbol{I} + \tilde{\delta}^1 \boldsymbol{G^1})$$

$$\begin{aligned}
\beta^1\gamma_0 \boldsymbol{G^1} + \beta^1\delta_0(\boldsymbol{G^1})^2 + (\gamma^1 \boldsymbol{I} - (\beta_0\gamma^1 - \delta^1)\boldsymbol{G^1}) - \beta_0\delta^1\boldsymbol{(G^1)}^2) = \tilde{\beta}^1\gamma_0 \boldsymbol{G^1} + \tilde{\beta}^1\delta_0(\boldsymbol{G^1}^2) + \\
+(\tilde{\gamma}^1 \boldsymbol{I} - (\beta_0\tilde{\gamma}^1 - \tilde{\delta}^1)\boldsymbol{G^1}) - \beta_0\tilde{\delta}^1\boldsymbol{(G^1)}^2)
\end{aligned}$$

$$\gamma^1 \boldsymbol{I} + (\beta^1\gamma_0 + \beta^1\delta_0 - \beta_0\gamma^1 + \delta^1)\boldsymbol{G^1} - \beta_0\delta^1\boldsymbol{(G^1)}^2 = \tilde{\gamma}^1 \boldsymbol{I} + (\tilde{\beta}^1\gamma_0 + \tilde{\beta}^1\delta_0 - \beta_0\tilde{\gamma}^1 + \tilde{\delta}^1)\boldsymbol{G^1} - \beta_0\tilde{\delta}^1\boldsymbol{(G^1)}^2$$

$$(\gamma^1 - \tilde{\gamma}^1)\boldsymbol{I} + ((\beta^1 - \tilde{\beta}^1)\gamma_0 + (\beta^1 - \tilde{\beta}^1)\delta_0 - \beta_0(\gamma^1 - \tilde{\gamma}^1) + \delta^1 - \tilde{\delta}^1)\boldsymbol{G^1} + \beta_0(\tilde{\delta}^1 - \delta^1)(\boldsymbol{G^1})^2 = 0$$

Now let I, G^1 and $(G^1)^2$ be linearly independent. Then the above equality holds only if all three coefficients are 0:

$$\gamma^1 - \tilde{\gamma}^1 = 0$$
$$(\beta^1 - \tilde{\beta}^1)\gamma_0 + (\beta^1 - \tilde{\beta}^1)\delta_0 - \beta_0(\gamma^1 - \tilde{\gamma}^1) + \delta^1 - \tilde{\delta}^1 = 0$$
$$\beta_0(\tilde{\delta}^1 - \delta^1) = 0$$

If $\beta_0 \neq 0$ and $\gamma_0^2 + \delta_0^2 \neq 0$, two sets of coefficients $(\alpha^1, \beta^1, \gamma^1, \delta^1)$ and $(\tilde{\alpha}^1, \tilde{\beta}^1, \tilde{\gamma}^1, \tilde{\delta}^1)$ are equivalent. Note that the restrictions on the coefficients of the peer effect model suggest that the model has an endogenous peer effect and the performance depends on own set of observed characteristics, or on peers observed characteristics, or on both. These requirements are natural for the peer effect model and therefore, the identification result is achieved. ■

Proof of Lemma 2. (Identification, Step 2, no correlated effects)

Recall the second step equation:

$$\Delta y_i = (\alpha_2 - \alpha_1) + \beta_2 \sum_{j \neq i} G_{ij}^2 y_j^2 - \beta_1 \sum_{j \neq i} G_{ij}^1 y_j^1 + \tilde{\boldsymbol{\delta}} UR_i + \gamma_2 X_i^2 - \gamma_1 X_i^1 +$$

$$+\delta_2 \sum_{j \neq i} G_{ij}^2 X_j^2 - \delta_1 \sum_{j \neq i} G_{ij}^1 X_j^1 + \Delta\epsilon_i$$

It can be rewritten in the reduced form as following:

$$\Delta \boldsymbol{y} = (\alpha_2 - \alpha_1)\boldsymbol{i} + \beta_2 \boldsymbol{G}^2 \boldsymbol{y^2} - \beta_1 \boldsymbol{G^1} \boldsymbol{y^1} + \tilde{\boldsymbol{\delta}} \boldsymbol{UR} + \gamma_2 \boldsymbol{X^2_{TV}} - \gamma_1 \boldsymbol{X^1_{TV}} + \delta_2 \boldsymbol{G^2} \boldsymbol{X^2} -$$

$$-\delta_1 \boldsymbol{G^1} \boldsymbol{X^1} + \Delta\epsilon, \qquad \text{with } \boldsymbol{UR} \text{ defined as discussed in Section 2 and } \mathbb{E}[\Delta\epsilon] = 0$$

This can be further modified in the following manner:

$$\mathbb{E}[\Delta \boldsymbol{y} | \boldsymbol{X^2}] = (\alpha_2 - \alpha_1)\boldsymbol{i} + \beta_2 \boldsymbol{G^2} \mathbb{E}[\boldsymbol{y^2} | \boldsymbol{X^2}] - \beta_1 \boldsymbol{G^1} \mathbb{E}[\boldsymbol{y^1} | \boldsymbol{X^2}] + \tilde{\boldsymbol{\delta}} \mathbb{E}[\boldsymbol{UR} | \boldsymbol{X^2}] +$$

$$+\gamma_2 \boldsymbol{X^2_{TV}} - \gamma_1 \boldsymbol{X^1_{TV}} + \delta_2 \boldsymbol{G^2} \boldsymbol{X^2} - \delta_1 \boldsymbol{G^1} \boldsymbol{X^1}$$

with

$$\mathbb{E}[\boldsymbol{y^1} | \boldsymbol{X^2}] = (\boldsymbol{I} - \beta_{0,1} \boldsymbol{G^1})^{-1} \alpha_{0,1} + (\boldsymbol{I} - \beta_{0,1} \boldsymbol{G^1})^{-1} (\gamma_{0,1} \boldsymbol{I} + \delta_{0,1} \boldsymbol{G^1}) \mathbb{E}[\boldsymbol{X^1} | \boldsymbol{X^2}] =$$

$$= (\boldsymbol{I} - \beta_{0,1} \boldsymbol{G^1})^{-1} \alpha_{0,1} + (\boldsymbol{I} - \beta_{0,1} \boldsymbol{G^1})^{-1} (\gamma_{0,1} \boldsymbol{I} + \delta_{0,1} \boldsymbol{G^1}) \boldsymbol{X^1},$$

since $\boldsymbol{X^1}$ is already known by the time $\boldsymbol{X^2}$ is revealed, therefore, the latter cannot add any new information.

Also:

$$\mathbb{E}[\boldsymbol{y^2} | \boldsymbol{X^2}] = (\boldsymbol{I} - \beta_{0,2} \boldsymbol{G^2})^{-1} \alpha_{0,2} + (\boldsymbol{I} - \beta_{0,2} \boldsymbol{G^2})^{-1} (\gamma_{0,2} \boldsymbol{I} + \delta_{0,2} \boldsymbol{G^2}) \boldsymbol{X^2}$$

Note that $\boldsymbol{UR}$ is also defined at the first period, hence, the new information in $\boldsymbol{X^2}$ will not anything new for the expected value of the $\boldsymbol{UR}$, hence $\mathbb{E}[\boldsymbol{UR} | \boldsymbol{X^2}] = \boldsymbol{UR}$.

Also notice than in principle coefficients in the model in differences $\alpha_1, \beta_1, \gamma_1, \delta_1$, $\alpha_2, \beta_2, \gamma_2, \delta_2$ can be different from the corresponding coefficients in the single period peer effect models $\alpha_{0,1}, \beta_{0,1}, \gamma_{0,1}, \delta_{0,1}, \alpha_{0,2}, \beta_{0,2}, \gamma_{0,2}, \delta_{0,2}$. This can be due to the unaccounted in single period model fixed effects that can be eliminated in the model in differences and due to the presence of the shock in the model, which can take some of the effect, that would be otherwise attributed towards endogenous or exogenous effect.

Then, letting $\alpha = \alpha_2 - \alpha_1$

$$\mathbb{E}[\Delta \boldsymbol{y}|\boldsymbol{X^2}] = \alpha \boldsymbol{i} + \beta_2 \boldsymbol{G^2}(\boldsymbol{I} - \beta_{0,2}\boldsymbol{G^2})^{-1}(\alpha_{0,2} + (\gamma_{0,2}\boldsymbol{I} + \delta_{0,2}\boldsymbol{G^2})\boldsymbol{X^2}) - \beta_1 \boldsymbol{G^1}(\boldsymbol{I} - \beta_{0,1}\boldsymbol{G^1})^{-1}(\alpha_{0,1} + (\gamma_{0,1}\boldsymbol{I} + \delta_{0,1}\boldsymbol{G^1})\boldsymbol{X^1}) + \tilde{\boldsymbol{\delta}}\boldsymbol{UR} + \gamma_2 \boldsymbol{X^2_{TV}} - \gamma_1 \boldsymbol{X^1_{TV}} + \delta_2 \boldsymbol{G^2}\boldsymbol{X^2} - \delta_1 \boldsymbol{G^1}\boldsymbol{X^1}$$

First, if $\boldsymbol{G^1} = \boldsymbol{G^1}$, then δ_2 and δ_1 are identified only partially, for time-variant variables of $\boldsymbol{X^1}$ and $\boldsymbol{X^2}$ respectively. This assumption can be relaxed, if we let the coefficients of the single period coincide with the coefficients of the coefficients of the model in differences. Then, however, the following assumption need to be made $\tilde{\delta} = 0$, meaning that the shock has no effect on the outcome, which is not true in the setting of the model of the paper. Hence, $\boldsymbol{G^1} = \boldsymbol{G^1}$ is one of the identifying assumptions for the second step model.

Next, I follow similar steps to the proof of Lemma 1. Consider two sets of the parameters leading to the same reduced form, $(\alpha_1, \beta_1, \gamma_1, \delta_1,\ \alpha_2, \beta_2, \gamma_2, \delta_2,\ \tilde{\delta})$ and $(\tilde{\alpha_1}, \tilde{\beta_1}, \tilde{\gamma_1}, \tilde{\delta_1}, \tilde{\alpha_2}, \tilde{\beta_2}, \tilde{\gamma_2}, \tilde{\delta_2},\ \tilde{\tilde{\delta}})$. I do not include the single-period parameters, since their identification is achieved separately, if $\boldsymbol{I}, \boldsymbol{G^1}, (\boldsymbol{G^1})^2$ are linearly independent and if $\boldsymbol{I}, \boldsymbol{G^2}, (\boldsymbol{G^2})^2$ are also linearly independent. Then:

$$\begin{gathered}
\alpha \boldsymbol{I} + \beta_2 \boldsymbol{G^2}(\boldsymbol{I} - \beta_{0,2}\boldsymbol{G^2})^{-1}\alpha_{0,2} - \beta_1 \boldsymbol{G^1}(\boldsymbol{I} - \beta_{0,1}\boldsymbol{G^1})^{-1}\alpha_{0,1} = \\
= \tilde{\alpha} \boldsymbol{I} + \tilde{\beta_2} \boldsymbol{G^2}(\boldsymbol{I} - \beta_{0,2}\boldsymbol{G^2})^{-1}\alpha_{0,2} - \tilde{\beta_1} \boldsymbol{G^1}(\boldsymbol{I} - \beta_{0,1}\boldsymbol{G^1})^{-1}\alpha_{0,1} \\
\beta_2 \boldsymbol{G^2}(\boldsymbol{I} - \beta_{0,2}\boldsymbol{G^2})^{-1}(\gamma_{0,2}\boldsymbol{I} + \delta_{0,2}\boldsymbol{G^2}) + (\gamma_2 \boldsymbol{I} + \delta_2 \boldsymbol{G^2}) = \\
= \tilde{\beta_2} \boldsymbol{G^2}(\boldsymbol{I} - \beta_{0,2}\boldsymbol{G^2})^{-1}(\gamma_{0,2}\boldsymbol{I} + \delta_{0,2}\boldsymbol{G^2}) + (\tilde{\gamma_2} \boldsymbol{I} + \tilde{\delta_2} \boldsymbol{G^2}) \\
\beta_1 \boldsymbol{G^1}(\boldsymbol{I} - \beta_{0,1}\boldsymbol{G^1})^{-1}(\gamma_{0,1}\boldsymbol{I} + \delta_{0,1}\boldsymbol{G^1}) + (\gamma_1 \boldsymbol{I} + \delta_1 \boldsymbol{G^1}) = \\
= \tilde{\beta_1} \boldsymbol{G^1}(\boldsymbol{I} - \beta_{0,1}\boldsymbol{G^1})^{-1}(\gamma_{0,1}\boldsymbol{I} + \delta_{0,1}\boldsymbol{G^1}) + (\tilde{\gamma_1} \boldsymbol{I} + \tilde{\delta_1} \boldsymbol{G^1}) \\
\tilde{\delta} = \tilde{\tilde{\delta}}
\end{gathered}$$

Note, that I added time invariant own exogenous variables to the vectors $\boldsymbol{X^1_{TV}}$ and $\boldsymbol{X^2_{TV}}$. Since they are not in the model, zeros are assumed on the additional elements of γ_1 and γ_2.

The third equation can be further simplified as following:

$$\begin{gathered}
\gamma_1 \boldsymbol{I} + (\delta_1 - \gamma_1\beta_{0,1} - \beta_1\gamma_{0,1})\boldsymbol{G^1} + (\beta_1\delta_{0,1} - \delta_1\beta_{0,1})\boldsymbol{G^1})^2 = \\
= \tilde{\gamma_1} \boldsymbol{I} + (\tilde{\delta_1} - \tilde{\gamma_1}\beta_{0,1} - \tilde{\beta_1}\gamma_{0,1})\boldsymbol{G^1} + (\tilde{\beta_1}\delta_{0,1} - \tilde{\delta_1}\beta_{0,1})\boldsymbol{G^1})^2
\end{gathered}$$

Then, if $\boldsymbol{I}, \boldsymbol{G^1}, (\boldsymbol{G^1})^2$ are linearly independent, the coefficients in front of these three matrices are 0:

$$\begin{aligned}
&\gamma_1 - \tilde{\gamma_1} = 0 \\
&\delta_1 - \gamma_1\beta_{0,1} - \beta_1\gamma_{0,1} = \tilde{\delta_1} - \tilde{\gamma_1}\beta_{0,1} - \tilde{\beta_1}\gamma_{0,1}, \text{ or} \\
&(\delta_1 - \tilde{\delta_1}) - (\gamma_1 - \tilde{\gamma_1})\beta_{0,1} + (\beta_1 - \tilde{\beta_1})\gamma_{0,1} = 0 \\
&\beta_1\delta_{0,1} - \delta_1\beta_{0,1} = \tilde{\beta_1}\delta_{0,1} - \tilde{\delta_1}\beta_{0,1}, \text{ or} \\
&(\beta_1 - \tilde{\beta_1})\delta_{0,1} - (\delta_1 - \tilde{\delta_1})\beta_{0,1} = 0
\end{aligned}$$

Now, if $\beta_{0,1} \neq 0$ and $\gamma_{0,1}^2 + \delta_{0,1}^2 \neq 0$, the two sets of the coefficients, $(\gamma_1, \delta_1, \beta_1)$ and $(\tilde{\gamma_1}, \tilde{\delta_1}, \tilde{\beta_1})$, coincide.

Similar argument is valid for the coefficient in front of $\boldsymbol{X^2}$, hence $(\gamma_2, \delta_2, \beta_2)$ and $(\tilde{\gamma_2}, \tilde{\delta_2}, \tilde{\beta_2})$, also coincide, when $\boldsymbol{I}, \boldsymbol{G^2}, (\boldsymbol{G^2})^2$ are linearly independent and $\beta_{0,2} \neq 0$ and $\gamma_{0,2}^2 + \delta_{0,2}^2 \neq 0$.

The other two equalities lead then automatically to $\alpha = \tilde{\alpha}$ and $\tilde{\delta} = \tilde{\tilde{\delta}}$ without any additional assumptions. Hence, the identification is achieved under the conditions of linear independence of $\boldsymbol{I}, \boldsymbol{G^1}, (\boldsymbol{G^1})^2$ and $\boldsymbol{I}, \boldsymbol{G^2}, (\boldsymbol{G^2})^2$ and $\boldsymbol{G^1} \neq \boldsymbol{G^2}$ and mentioned assumptions on the coefficients. ■

Proof of Lemma 3.

The structural form equation:

$$P(retake_i) - \sum_{j \neq i} G^1_{ij} P(retake_j) = \beta \sum_{j \neq i} G^1_{ij} [y^1_j - \sum_{k \neq j} G^1_{jk} y^1_k] + \gamma [X^1_i - \sum_{j \neq i} G^1_{ij} X^1_j] +$$
$$+ \delta \sum_{j \neq i} G^1_{ij} [X^1_j - \sum_{k \neq j} G^1_{jk} X^1_k] + [\eta_i - \sum_{j \neq i} G^1_{ij} \eta_j], \qquad \mathbb{E}[\eta_i | X^1] = 0$$

can be rewritten in the reduced form in the following manner:

$$(\boldsymbol{I} - \boldsymbol{G^1}) \boldsymbol{PR} = \beta (\boldsymbol{I} - \boldsymbol{G^1}) \boldsymbol{G^1} \boldsymbol{y^1} + \gamma (\boldsymbol{I} - \boldsymbol{G^1}) \boldsymbol{X^1} + \delta (\boldsymbol{I} - \boldsymbol{G^1}) \boldsymbol{G^1} \boldsymbol{X^1} + (\boldsymbol{I} - \boldsymbol{G^1}) \eta, \ \mathbb{E}[\eta | \boldsymbol{X^1}] = 0$$

$$(\boldsymbol{I} - \boldsymbol{G^1}) \boldsymbol{PR} = \beta (\boldsymbol{I} - \boldsymbol{G^1}) \boldsymbol{G^1} \boldsymbol{y^1} + (\gamma \boldsymbol{I} + \delta \boldsymbol{G^1}) (\boldsymbol{I} - \boldsymbol{G^1}) \boldsymbol{X^1} + (\boldsymbol{I} - \boldsymbol{G^1}) \eta, \qquad \mathbb{E}[\eta | \boldsymbol{X^1}] = 0$$

Taking conditional expectations:

$$\mathbb{E}[(\boldsymbol{I} - \boldsymbol{G^1}) \boldsymbol{PR} | \boldsymbol{X^1}] = \beta (\boldsymbol{I} - \boldsymbol{G^1}) \boldsymbol{G^1} \mathbb{E}[\boldsymbol{y^1} | \boldsymbol{X^1}] + (\gamma \boldsymbol{I} + \delta \boldsymbol{G^1}) (\boldsymbol{I} - \boldsymbol{G^1}) \boldsymbol{X^1}$$

Note that $\boldsymbol{y}$ can be expressed in terms of peer effect model as the one used for the probability of retakes:

$$y^1_i - \sum_{j \neq i} G^1_{ij} y^1_j = \beta_0 \sum_{j \neq i} G^1_{ij} [y^1_j - \sum_{k \neq j} G^1_{jk} y^1_k] + \gamma_0 [X^1_i - \sum_{j \neq i} G^1_{ij} X^1_j] + \delta_0 \sum_{j \neq i} G^1_{ij} [X^1_j -$$
$$- \sum_{k \neq j} G^1_{jk} X^1_k] + [\xi_i - \sum_{j \neq i} G^1_{ij} \xi_j], \quad \mathbb{E}[\xi_i | X^1] = 0$$

with reduced form:

$$(\boldsymbol{I} - \boldsymbol{G^1}) \boldsymbol{y^1} = \beta_0 (\boldsymbol{I} - \boldsymbol{G^1}) \boldsymbol{G^1} \boldsymbol{y^1} + (\gamma_0 \boldsymbol{I} + \delta_0 \boldsymbol{G^1}) (\boldsymbol{I} - \boldsymbol{G^1}) \boldsymbol{X^1} + (\boldsymbol{I} - \boldsymbol{G^1}) \xi, \quad \mathbb{E}[\xi | \boldsymbol{X^1}] = 0$$

Then following steps of Bramoullé, Djebbari, and Fortin (2009):

$$(\boldsymbol{I} - \boldsymbol{G^1}) \boldsymbol{y^1} = (\boldsymbol{I} - \beta_0 \boldsymbol{G^1})^{-1} (\gamma_0 \boldsymbol{I} + \delta_0 \boldsymbol{G^1}) (\boldsymbol{I} - \boldsymbol{G^1}) \boldsymbol{X^1} + (\boldsymbol{I} - \beta_0 \boldsymbol{G^1})^{-1} (\boldsymbol{I} - \boldsymbol{G^1}) \xi, \quad \mathbb{E}[\xi | \boldsymbol{X}] = 0$$

And:

$$\mathbb{E}[(\boldsymbol{I} - \boldsymbol{G^1}) \boldsymbol{G^1} \boldsymbol{y^1} | \boldsymbol{X^1}] = (\boldsymbol{I} - \beta_0 \boldsymbol{G^1})^{-1} (\gamma_0 \boldsymbol{I} + \delta_0 \boldsymbol{G^1}) (\boldsymbol{I} - \boldsymbol{G^1}) \boldsymbol{G^1} \boldsymbol{X^1}$$

As was proven in Bramoullé, Djebbari, and Fortin (2009), if $\gamma_0 \beta_0 + \delta_0 \neq 0$ and $I, G^1, (G^1)^2$ and $(G^1)^3$ are linearly independent, the social effects are identified. So this expression can be plugged-in into the reduced form of the equation for the probability of retake.

$$\mathbb{E}[(\boldsymbol{I}-\boldsymbol{G^1})\boldsymbol{PR}|\boldsymbol{X^1}] = \beta((\boldsymbol{I}-\beta_0\boldsymbol{G^1})^{-1}(\gamma_0\boldsymbol{I}+\delta_0\boldsymbol{G^1})(\boldsymbol{I}-\boldsymbol{G^1})\boldsymbol{G^1})\boldsymbol{X^1}+(\gamma\boldsymbol{I}+\delta\boldsymbol{G^1})(\boldsymbol{I}-\boldsymbol{G^1})\boldsymbol{X^1}$$

Now consider two sets of structural parameters (β,γ,δ) and $(\tilde{\beta},\tilde{\gamma},\tilde{\delta})$ leading to the same reduced form. It means that:

$$\begin{aligned}&\beta(\boldsymbol{I}-\beta_0\boldsymbol{G^1})^{-1}(\gamma_0\boldsymbol{I}+\delta_0\boldsymbol{G^1})(\boldsymbol{I}-\boldsymbol{G^1})\boldsymbol{G^1}+(\gamma\boldsymbol{I}+\delta\boldsymbol{G^1})(\boldsymbol{I}-\boldsymbol{G^1})=\\&=\tilde{\beta}(\boldsymbol{I}-\beta_0\boldsymbol{G^1})^{-1}(\gamma_0\boldsymbol{I}+\delta_0\boldsymbol{G^1})(\boldsymbol{I}-\boldsymbol{G^1})\boldsymbol{G^1}+(\tilde{\gamma}\boldsymbol{I}+\tilde{\delta}\boldsymbol{G^1})(\boldsymbol{I}-\boldsymbol{G^1})\end{aligned}$$

$$\begin{aligned}&\beta(\gamma_0\boldsymbol{I}+\delta_0\boldsymbol{G^1})(\boldsymbol{I}-\boldsymbol{G^1})\boldsymbol{G^1}+(\boldsymbol{I}-\beta_0\boldsymbol{G^1})(\gamma\boldsymbol{I}+\delta\boldsymbol{G^1})(\boldsymbol{I}-\boldsymbol{G^1})=\\&=\tilde{\beta}(\gamma_0\boldsymbol{I}+\delta_0\boldsymbol{G^1})(\boldsymbol{I}-\boldsymbol{G^1})\boldsymbol{G^1}+(\boldsymbol{I}-\beta_0\boldsymbol{G^1})(\tilde{\gamma}\boldsymbol{I}+\tilde{\delta}\boldsymbol{G^1})(\boldsymbol{I}-\boldsymbol{G^1})\end{aligned}$$

$$\begin{aligned}&\beta\gamma_0\boldsymbol{G^1}+(\beta\delta_0-\beta\gamma_0)(\boldsymbol{G^1})^2-\beta\delta_0(\boldsymbol{G^1})^3+(\gamma\boldsymbol{I}-(\beta_0\gamma-\delta+\gamma)\boldsymbol{G^1})-\\&\qquad-(\beta_0\delta-\gamma\beta_0+\delta)(\boldsymbol{G^1})^2+\beta_0\delta(\boldsymbol{G^1})^3)=\\&=\tilde{\beta}\gamma_0\boldsymbol{G^1}+(\tilde{\beta}\delta_0-\tilde{\beta}\gamma_0)(\boldsymbol{G^1})^2-\tilde{\beta}\delta_0(\boldsymbol{G^1})^3+(\tilde{\gamma}\boldsymbol{I}-(\beta_0\tilde{\gamma}-\tilde{\delta}+\tilde{\gamma})\boldsymbol{G^1})-\\&\qquad-(\beta_0\tilde{\delta}-\tilde{\gamma}\beta_0+\tilde{\delta})(\boldsymbol{G^1})^2+\beta_0\tilde{\delta}(\boldsymbol{G^1})^3)\end{aligned}$$

$$\begin{aligned}&\gamma\boldsymbol{I}+(\beta\gamma_0-\beta_0\gamma+\delta-\gamma)\boldsymbol{G^1}+(\beta\delta_0-\beta\gamma_0-\beta_0\delta+\beta_0\gamma-\delta)\left(\boldsymbol{G^1}\right)^2+(\beta_0\delta-\beta\delta_0)\left(\boldsymbol{G^1}\right)^3=\\&=\tilde{\gamma}\boldsymbol{I}+(\tilde{\beta}\gamma_0-\beta_0\tilde{\gamma}+\tilde{\delta}-\tilde{\gamma})\boldsymbol{G^1}+(\tilde{\beta}\delta_0-\tilde{\beta}\gamma_0-\beta_0\tilde{\delta}+\beta_0\tilde{\gamma}-\tilde{\delta})\left(\boldsymbol{G^1}\right)^2+(\beta_0\tilde{\delta}-\tilde{\beta}\delta_0)\left(\boldsymbol{G^1}\right)^3\end{aligned}$$

$$\begin{aligned}&(\gamma^1-\tilde{\gamma}^1)\boldsymbol{I}+((\beta-\tilde{\beta})\gamma_0-(\gamma-\tilde{\gamma})\beta_0+(\delta-\tilde{\delta})-(\gamma-\tilde{\gamma}))\boldsymbol{G^1}+((\beta-\tilde{\beta})\delta_0-(\beta-\tilde{\beta})\gamma_0-\\&\qquad-(\delta-\tilde{\delta})\beta_0+(\gamma-\tilde{\gamma})\beta_0-(\delta-\tilde{\delta}))\left(\boldsymbol{G^1}\right)^2+((\delta-\tilde{\delta})\beta_0-(\beta-\tilde{\beta})\delta_0)\left(\boldsymbol{G^1}\right)^3=0\end{aligned}$$

Now let I, G^1, $(G^1)^2$ and $(G^1)^3$ be linearly independent. Then the above equality holds only if all three coefficients are 0:

$$\begin{aligned}&\gamma-\tilde{\gamma}=0\\&(\beta-\tilde{\beta})\gamma_0-(\gamma-\tilde{\gamma})\beta_0+(\delta-\tilde{\delta})-(\gamma-\tilde{\gamma})=0\\&(\beta-\tilde{\beta})\delta_0-(\beta-\tilde{\beta})\gamma_0-(\delta-\tilde{\delta})\beta_0+(\gamma-\tilde{\gamma})\beta_0-(\delta-\tilde{\delta})=0\\&(\delta-\tilde{\delta})\beta_0-(\beta-\tilde{\beta})\delta_0=0\end{aligned}$$

If $\beta_0\neq 0$ and $\gamma_0^2+\delta_0^2\neq 0$, two sets of coefficients (β,γ,δ) and $(\tilde{\beta},\tilde{\gamma},\tilde{\delta})$ are equivalent. Note that the restrictions on the coefficients of the peer effect model suggest that the model has an endogenous peer effect and the performance depends on own set of observed characteristics, or on peers observed characteristics, or on both. These requirements are natural for the peer effect model and therefore, the identification result is achieved. ■

Proof of Lemma 4. (Identification, Step 2, correlated effects)

The proof for Lemma 4 follows directly by applying similar arguments to the proofs of Lemma 2 and Lemma 3. Then, the identification is achieved under the conditions of linear independence of $\boldsymbol{I}, \boldsymbol{G^1}, (\boldsymbol{G^1})^2, (\boldsymbol{G^1})^3$ and $\boldsymbol{I}, \boldsymbol{G^2}, (\boldsymbol{G^2})^2, (\boldsymbol{G^2})^3$ and $\boldsymbol{G^1} \neq \boldsymbol{G^2}$ and the following assumptions on the coefficients: $\beta_{0,1} \neq 0$, $\gamma_{0,1}^2 + \delta_{0,1}^2 \neq 0$, $\beta_{0,2} \neq 0$ and $\gamma_{0,2}^2 + \delta_{0,2}^2 \neq 0$. ■

Proof of Lemma 5. Consistency of $\hat{\theta}_{Lee}$ of Step 1

$$\sqrt{n}(\hat{\theta}_{Lee} - \theta) = (\frac{1}{n}\boldsymbol{\hat{Z}}^T\boldsymbol{\tilde{X}^1})^{-1}\frac{1}{\sqrt{n}}\boldsymbol{\hat{Z}}^T\boldsymbol{PR} - \sqrt{n}\theta = (\frac{1}{n}\boldsymbol{\hat{Z}}^T\boldsymbol{\tilde{X}^1})^{-1}(\frac{1}{\sqrt{n}}\boldsymbol{\hat{Z}}^T\boldsymbol{PR} - \frac{1}{\sqrt{n}}\boldsymbol{\hat{Z}}^T\boldsymbol{\tilde{X}^1}\theta)$$

Then we can rewrite the last term:

$$\frac{1}{\sqrt{n}}\boldsymbol{\hat{Z}}^T\boldsymbol{PR} - \frac{1}{\sqrt{n}}\boldsymbol{\hat{Z}}^T\boldsymbol{\tilde{X}^1}\theta = \frac{1}{\sqrt{n}}\boldsymbol{\hat{Z}}^T(\boldsymbol{PR} - \boldsymbol{\tilde{X}^1}\theta) = \frac{1}{\sqrt{n}}\boldsymbol{\hat{Z}}^T(\alpha\boldsymbol{i} + \beta\boldsymbol{G}^1\boldsymbol{y}^1 + (\gamma\boldsymbol{I} + \delta\boldsymbol{G^1})\boldsymbol{X^1} + \nu -$$

$$-(\alpha\boldsymbol{i} + (\gamma\boldsymbol{I} + \delta\boldsymbol{G^1})\boldsymbol{X^1} + \beta\boldsymbol{G}^1\boldsymbol{y}^1)) = \frac{1}{\sqrt{n}}\boldsymbol{\hat{Z}}^T\nu$$

Hence,

$$\sqrt{n}(\hat{\theta}_{Lee} - \theta) = (\frac{1}{n}\boldsymbol{\hat{Z}}^T\boldsymbol{\tilde{X}^1})^{-1}\frac{1}{\sqrt{n}}\boldsymbol{\hat{Z}}^T\nu$$

Then the following two statements can be shown under the assumed regularity conditions and by direct application of Lemmas A.7, A.8 and A.9 in L. Lee (2003):

$$plim\frac{1}{n}\boldsymbol{\hat{Z}}^T\boldsymbol{\tilde{X}^1} = plim\frac{1}{n}\boldsymbol{Z}^T\boldsymbol{Z} = \boldsymbol{J}$$
$$\frac{1}{\sqrt{n}}\boldsymbol{\hat{Z}}^T\nu \xrightarrow{D} \mathcal{N}(0, \sigma_\nu^2\boldsymbol{J})$$

which will yield the desired result. ■

Proof of Lemma 6. Consistency of $\hat{\phi}_{Lee}$ of Step 2

$$\sqrt{n}(\hat{\phi}_{Lee} - \phi) = (\frac{1}{n}\boldsymbol{\hat{\bar{Z}}}^T\boldsymbol{\bar{X}})^{-1}\frac{1}{\sqrt{n}}\boldsymbol{\hat{\bar{Z}}}^T(\boldsymbol{y^2} - \boldsymbol{y^1}) - \sqrt{n}\phi = (\frac{1}{n}\boldsymbol{\hat{\bar{Z}}}^T\boldsymbol{\bar{X}})^{-1}(\frac{1}{\sqrt{n}}\boldsymbol{\hat{\bar{Z}}}^T(\boldsymbol{y^2} - \boldsymbol{y^1}) - \frac{1}{\sqrt{n}}\boldsymbol{\hat{\bar{Z}}}^T\boldsymbol{\bar{X}}\phi)$$

Then we can rewrite the last term:

$$\frac{1}{\sqrt{n}}\boldsymbol{\hat{\bar{Z}}}^T(\boldsymbol{y^2} - \boldsymbol{y^1}) - \frac{1}{\sqrt{n}}\boldsymbol{\hat{\bar{Z}}}^T\boldsymbol{\bar{X}}\phi = \frac{1}{\sqrt{n}}\boldsymbol{\hat{\bar{Z}}}^T(\boldsymbol{y^2} - \boldsymbol{y^1} - \boldsymbol{\bar{X}}\phi) = \frac{1}{\sqrt{n}}\boldsymbol{\hat{\bar{Z}}}^T((\alpha_2 - \alpha_1)\boldsymbol{i} + \beta_2\boldsymbol{G}^2\boldsymbol{y^2} -$$

$$-\beta_1\boldsymbol{G^1}\boldsymbol{y^1} + \boldsymbol{\tilde{\delta}UR} + \gamma_2\boldsymbol{X^2_{TV}} - \gamma_1\boldsymbol{X^1_{TV}} + \delta_2\boldsymbol{G^2}\boldsymbol{X^2} - \delta_1\boldsymbol{G^1}\boldsymbol{X^1} + \Delta\epsilon - ((\alpha_2 - \alpha_1)\boldsymbol{i} + \beta_2\boldsymbol{G}^2\boldsymbol{y^2} - \beta_1\boldsymbol{G^1}\boldsymbol{y^1} +$$

$$+\boldsymbol{\tilde{\delta}UR} + \gamma_2\boldsymbol{X^2_{TV}} - \gamma_1\boldsymbol{X^1_{TV}} + \delta_2\boldsymbol{G^2}\boldsymbol{X^2} - \delta_1\boldsymbol{G^1}\boldsymbol{X^1}) = \frac{1}{\sqrt{n}}\boldsymbol{\hat{\bar{Z}}}^T\Delta\epsilon$$

Hence,

$$\sqrt{n}(\hat{\phi}_{Lee} - \phi) = (\frac{1}{n}\hat{\bar{\boldsymbol{Z}}}^T\bar{\boldsymbol{X}})^{-1}\frac{1}{\sqrt{n}}\hat{\bar{\boldsymbol{Z}}}^T\Delta\epsilon$$

The following two statements have to hold to get the desired result:

$$plim\frac{1}{n}\hat{\bar{\boldsymbol{Z}}}^T\bar{\boldsymbol{X}} = plim\frac{1}{n}\bar{\boldsymbol{Z}}^T\bar{\boldsymbol{Z}} = \bar{\boldsymbol{J}}$$
$$\frac{1}{\sqrt{n}}\hat{\bar{\boldsymbol{Z}}}^T\Delta\epsilon \xrightarrow{D} \mathcal{N}(0,(\sigma^2_{\epsilon^1}+\sigma^2_{\epsilon^2})\bar{\boldsymbol{J}})$$

First, let's consider $\frac{1}{n}\hat{\bar{\boldsymbol{Z}}}^T\bar{\boldsymbol{X}}$. It is equivalent to $\frac{1}{n}[\boldsymbol{i},\boldsymbol{X^2_{TV}},\boldsymbol{X^1_{TV}},\boldsymbol{G^2X^2},\boldsymbol{G^1X^1},\boldsymbol{UR}$, $\mathbb{E}[\boldsymbol{G^1y^1}(\hat{\theta}^1_{2SLS})|\boldsymbol{X^1}]$, $\mathbb{E}[\boldsymbol{G^2y^2}(\hat{\phi}_{2SLS})|\boldsymbol{X^2},\boldsymbol{X^1}]]^T[\boldsymbol{i},\boldsymbol{X^2_{TV}},\boldsymbol{X^1_{TV}},\boldsymbol{G^2X^2},\boldsymbol{G^1X^1},\boldsymbol{UR}$, $\boldsymbol{G^1y^1}$, $\boldsymbol{G^2y^2}]$
First six rows do not consist any element of estimated vector of coefficients, and therefore, will not matter for the consistency argument.

Notice also that $\boldsymbol{G^1y^1} = \boldsymbol{G^1}(\boldsymbol{I}-\beta^1\boldsymbol{G^1})^{-1}\alpha^1 + \boldsymbol{G^1}(\boldsymbol{I}-\beta^1\boldsymbol{G^1})^{-1}(\gamma^1\boldsymbol{I}+\delta^1\boldsymbol{G^1})\boldsymbol{X^1} + \boldsymbol{G^1}(\boldsymbol{I}-\beta^1\boldsymbol{G^1})^{-1}\epsilon_1$ and $\boldsymbol{G^2y^2} = \boldsymbol{G^2}(\boldsymbol{I}-\beta_2\boldsymbol{G^2})^{-1}[(\alpha_2-\alpha_1)\boldsymbol{i}+(\boldsymbol{I}-\beta_1\boldsymbol{G^1})((\boldsymbol{I}-\beta^1\boldsymbol{G^1})^{-1}\alpha^1 + (\boldsymbol{I}-\beta^1\boldsymbol{G^1})^{-1}(\gamma^1\boldsymbol{I}+\delta^1\boldsymbol{G^1})\boldsymbol{X^1}) + \tilde{\boldsymbol{\delta}}\boldsymbol{UR} + \gamma_2\boldsymbol{X^2_{TV}} - \gamma_1\boldsymbol{X^1_{TV}} + \delta_2\boldsymbol{G^2X^2} - \delta_1\boldsymbol{G^1X^1}] + \boldsymbol{G^2}(\boldsymbol{I}-\beta_2\boldsymbol{G^2})^{-1}\Delta\epsilon$ can be both split into two part: with and without error term.

Define $\mathbb{E}[\boldsymbol{G^1y^1}] \equiv \boldsymbol{G^1y^1} - \boldsymbol{G^1}(\boldsymbol{I}-\beta^1\boldsymbol{G^1})^{-1}\epsilon_1$ and $\mathbb{E}[\boldsymbol{G^2y^2}] \equiv \boldsymbol{G^2y^2} - \boldsymbol{G^2}(\boldsymbol{I}-\beta_2\boldsymbol{G^2})^{-1}\Delta\epsilon$

Consider now row six: $\frac{1}{n}(\mathbb{E}[\boldsymbol{G^1y^1}(\hat{\theta}^1_{2SLS})|\boldsymbol{X^1}])^T[\boldsymbol{i},\boldsymbol{X^2_{TV}},\boldsymbol{X^1_{TV}},\boldsymbol{G^2X^2},\boldsymbol{G^1X^1},\boldsymbol{UR}$, $\mathbb{E}[\boldsymbol{G^1y^1}]$, $\mathbb{E}[\boldsymbol{G^2y^2}]]$ $+\frac{1}{n}(\mathbb{E}[\boldsymbol{G^1y^1}(\hat{\theta}^1_{2SLS})|\boldsymbol{X^1}])^T[0,0,0,0,0,0,$ $\boldsymbol{G^1}(\boldsymbol{I}-\beta^1\boldsymbol{G^1})^{-1}\epsilon_1$, $\boldsymbol{G^2}(\boldsymbol{I}-\beta_2\boldsymbol{G^2})^{-1}\Delta\epsilon]$.

By the assumed uniform boundedness of X^1, X^2 in absolute values as well as by the uniform boundness of the row and column sums of the matrices G^1, G^2, $(I-\beta^1G^1)^{-1}$ and $(I-\beta_2G^2)^{-1}$, by $\mathbb{E}[\Delta\epsilon]=0$ and by Lemmas A.6, A.7 and A.8 in L. Lee (2003), it can be shown that this row will have a limit in probability, which equals to corresponding row of $\bar{\boldsymbol{J}}$.

Similar argument holds for the row seven: $\frac{1}{n}(\mathbb{E}[\boldsymbol{G^2y^2}(\hat{\phi}_{2SLS})|\boldsymbol{X^1},\boldsymbol{X^2}])^T[\boldsymbol{i},\boldsymbol{X^2_{TV}},\boldsymbol{X^1_{TV}}$, $\boldsymbol{G^2X^2},\boldsymbol{G^1X^1},\boldsymbol{UR},\mathbb{E}[\boldsymbol{G^1y^1}],\mathbb{E}[\boldsymbol{G^2y^2}]]$ $+\frac{1}{n}(\mathbb{E}[\boldsymbol{G^2y^2}(\hat{\phi}_{2SLS})|\boldsymbol{X^1},\boldsymbol{X^2}])^T[0,0,0,0,0,0,\boldsymbol{G^1}(\boldsymbol{I}-\beta^1\boldsymbol{G^1})^{-1}\epsilon_1$, $\boldsymbol{G^2}(\boldsymbol{I}-\beta_2\boldsymbol{G^2})^{-1}\Delta\epsilon]$. Therefore, the first statement is correct.

For the second statement consider $\frac{1}{\sqrt{n}}\hat{\bar{\boldsymbol{Z}}}^T\Delta\epsilon = [\boldsymbol{i},\boldsymbol{X^2_{TV}},\boldsymbol{X^1_{TV}},\boldsymbol{G^2X^2},\boldsymbol{G^1X^1},\boldsymbol{UR}$, $\mathbb{E}[\boldsymbol{G^1y^1}(\hat{\theta}^1_{2SLS})|\boldsymbol{X^1}]$, $\mathbb{E}[\boldsymbol{G^2y^2}(\hat{\phi}_{2SLS})|\boldsymbol{X^2},\boldsymbol{X^1}]]^T\Delta\epsilon$.

None of the elements in $\hat{\bar{\boldsymbol{Z}}}$ consist $\Delta\epsilon$, therefore, since $\mathbb{E}[\Delta\epsilon]=0$, the expectation of the whole term gives 0, which concludes the consistency part of the proof.

Moreover, the variance can be written as $(\sigma_{\epsilon^1}+\sigma_{\epsilon^2})\mathbb{E}[\frac{1}{n}\hat{\bar{\boldsymbol{Z}}}^T\hat{\bar{\boldsymbol{Z}}}]$. By the same Lemmas as before, it can be shown that $plim\mathbb{E}[\frac{1}{n}\hat{\bar{\boldsymbol{Z}}}^T\hat{\bar{\boldsymbol{Z}}}] = plim\frac{1}{n}\bar{\boldsymbol{Z}}^T\bar{\boldsymbol{Z}} = \bar{\boldsymbol{J}}$, which concludes the proof of normality.$\blacksquare$.

Discussion of 2.4.2, step 2.

I am approaching the estimation of the second step also adopting the 2SLS procedure discussed for the first step. First, the model (5) can be rewritten in the following way:

$$\Delta\boldsymbol{y} = (\alpha_2-\alpha_1)\boldsymbol{i} + \beta_2\boldsymbol{G}^2\boldsymbol{y^2} - \beta_1\boldsymbol{G^1y^1} + \tilde{\boldsymbol{\delta}}\boldsymbol{UR} + \gamma_2\boldsymbol{X^2_{TV}} - \gamma_1\boldsymbol{X^1_{TV}} + \delta_2\boldsymbol{G^2X^2} -$$

$$-\delta_1 \boldsymbol{G^1 X^1} + \Delta\epsilon$$

Then:

$$(\boldsymbol{I} - \boldsymbol{G^1})\Delta \boldsymbol{y} = \beta_2(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G}^2\boldsymbol{y^2} - \beta_1(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1 y^1} + \tilde{\boldsymbol{\delta}}(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{UR} + \gamma_2(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{X^2_{TV}} -$$

$$- \gamma_1(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{X^1_{TV}} + \delta_2(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^2 X^2} - \delta_1(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1 X^1} + (\boldsymbol{I} - \boldsymbol{G^1})\Delta\epsilon \qquad (2.10)$$

Recall: $\bar{\boldsymbol{X}} = [(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{X^2_{TV}}, (\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{X^1_{TV}}, (\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^2 X^2}, (\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1 X^1}, (\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{UR}, (\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1 y^1}, (\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^2 y^2}]$.

And $\boldsymbol{M} = [(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{X^2_{TV}}, (\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{X^1_{TV}}, (\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^2 X^2}, (\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1 X^1}, (\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{UR}, \mathbb{E}[(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1 y^1}(\hat{\boldsymbol{\theta}}^1_{\boldsymbol{2SLS}})|\boldsymbol{X^1}], (\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{(G^2)^2 X^2}]$.

I modify 2.10, taking expectations given $\boldsymbol{X^2}$ and recalling $\mathbb{E}[\Delta\epsilon] = 0$:

$$(\boldsymbol{I} - \beta_2\boldsymbol{G^2})\mathbb{E}[(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{y^2}|\boldsymbol{X^2}] = (\boldsymbol{I} - \beta_1\boldsymbol{G^1})(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{y^1} + \tilde{\boldsymbol{\delta}}(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{UR} +$$
$$+\gamma_2(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{X^2_{TV}} - \gamma_1(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{X^1_{TV}} + \delta_2(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^2 X^2} - \delta_1(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1 X^1}$$

$$\mathbb{E}[(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{y^2}|\boldsymbol{X^2}] = (\boldsymbol{I} - \beta_2\boldsymbol{G^2})^{-1}[(\boldsymbol{I} - \beta_1\boldsymbol{G^1})(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{y^1} + \tilde{\boldsymbol{\delta}}(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{UR} +$$
$$+\gamma_2(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{X^2_{TV}} - \gamma_1(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{X^1_{TV}} + \delta_2(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^2 X^2} - \delta_1(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1 X^1}]$$

Let $\mathbb{E}[(\boldsymbol{I}-\boldsymbol{G^1})\boldsymbol{G^2 y^2}(\phi)|\boldsymbol{X^2}, \boldsymbol{X^1}] = \boldsymbol{G^2}(\boldsymbol{I}-\beta_2\boldsymbol{G^2})^{-1}[(\boldsymbol{I}-\beta_1\boldsymbol{G^1})\mathbb{E}[(\boldsymbol{I}-\boldsymbol{G^1})\boldsymbol{y^1}(\theta^1)|\boldsymbol{X^1}] + \tilde{\boldsymbol{\delta}}(\boldsymbol{I}-\boldsymbol{G^1})\boldsymbol{UR} + \gamma_2(\boldsymbol{I}-\boldsymbol{G^1})\boldsymbol{X^2_{TV}} - \gamma_1(\boldsymbol{I}-\boldsymbol{G^1})\boldsymbol{X^1_{TV}} + \delta_2(\boldsymbol{I}-\boldsymbol{G^1})\boldsymbol{G^2 X^2} - \delta_1(\boldsymbol{I}-\boldsymbol{G^1})\boldsymbol{G^1 X^1}]$, where $\mathbb{E}[(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{y^1}(\theta^1)|\boldsymbol{X^1}] = (\boldsymbol{I} - \beta_1\boldsymbol{G^1})^{-1}(\boldsymbol{I} - \boldsymbol{G^1})(\gamma_1\boldsymbol{I} + \delta_1\boldsymbol{G^1})\boldsymbol{X^1}$.

Then I also define the following vector $\bar{\boldsymbol{Z}} = [(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{X^2_{TV}}, (\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{X^1_{TV}}, (\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^2 X^2}, (\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1 X^1}, (\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{UR}, \mathbb{E}[(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1 y^1}(\theta^1)|\boldsymbol{X^1}], \mathbb{E}[(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^2 y^2}(\phi)|\boldsymbol{X^2}, \boldsymbol{X^1}]$

I propose the following estimation procedure:

First, compute the 2SLS estimator for $\phi = (\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2)$ of the (7), using vector of instruments $\boldsymbol{M}$ and vector of covariates $\bar{\boldsymbol{X}}^{\boldsymbol{1}}$, as defined above.
$\hat{\phi}^1_{2SLS} = (\bar{\boldsymbol{X}}^T\boldsymbol{P_M}\bar{\boldsymbol{X}})^{-1}\bar{\boldsymbol{X}}^T\boldsymbol{P_M}(\boldsymbol{y^2} - \boldsymbol{y^1})$, where $\boldsymbol{P_M} = \boldsymbol{M}(\boldsymbol{M}^T\boldsymbol{M})^{-1}\boldsymbol{M}^T$ is a projection matrix.

Second, define $\hat{\bar{\boldsymbol{Z}}} = \bar{\boldsymbol{Z}}(\hat{\phi}_{2SLS}) = [(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{X^2_{TV}}, (\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{X^1_{TV}}, (\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^2 X^2}, (\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1 X^1}, (\boldsymbol{I}-\boldsymbol{G^1})\boldsymbol{UR}, \mathbb{E}[(\boldsymbol{I}-\boldsymbol{G^1})\boldsymbol{G^1 y^1}(\hat{\theta}^1_{2SLS})|\boldsymbol{X^1}]], \mathbb{E}[(\boldsymbol{I}-\boldsymbol{G^1})\boldsymbol{G^2 y^2}(\hat{\phi}_{2SLS})|\boldsymbol{X^2}, \boldsymbol{X^1}]$, where $\mathbb{E}[(\boldsymbol{I}-\boldsymbol{G^1})\boldsymbol{G^1 y^1}(\hat{\theta}^1_{2SLS})|\boldsymbol{X^1}] = (\boldsymbol{I}-\hat{\beta}_{1,2SLS}\boldsymbol{G^1})^{-1}(\boldsymbol{I}-\boldsymbol{G^1})(\hat{\gamma}_{1,2SLS}\boldsymbol{I}+\hat{\delta}_{1,2SLS}\boldsymbol{G^1})\boldsymbol{X^1}$, with $\hat{\theta}^1_{2SLS}$ obtained as the estimation of the first stage on the first step.
and $\mathbb{E}[(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^2 y^2}(\hat{\phi}_{2SLS})|\boldsymbol{X^2}, \boldsymbol{X^1}] = \boldsymbol{G^2}(\boldsymbol{I} - \hat{\beta}_{2,2SLS}\boldsymbol{G^2})^{-1}[(\boldsymbol{I} - \hat{\beta}_{1,2SLS}\boldsymbol{G^1})\mathbb{E}[(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{y^1}(\hat{\theta}^1_{2SLS})|\boldsymbol{X^1}] + \hat{\tilde{\boldsymbol{\delta}}}_{2SLS}(\boldsymbol{I}-\boldsymbol{G^1})\boldsymbol{UR} + \hat{\gamma}_{2,2SLS}(\boldsymbol{I}-\boldsymbol{G^1})\boldsymbol{X^2_{TV}} - \hat{\gamma}_{1,2SLS}(\boldsymbol{I}-\boldsymbol{G^1})\boldsymbol{X^1_{TV}} + \hat{\delta}_{2,2SLS}(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^2 X^2} - \hat{\delta}_{1,2SLS}(\boldsymbol{I} - \boldsymbol{G^1})\boldsymbol{G^1 X^1}]$

Finally, we use $\hat{\bar{\boldsymbol{Z}}}$ as a new vector of instrument to estimate (7). Then the following consistent estimator is obtained: $\hat{\phi}_{Lee} = (\hat{\bar{\boldsymbol{Z}}}^T\bar{\boldsymbol{X}})^{-1}\hat{\bar{\boldsymbol{Z}}}^T(\boldsymbol{y^2} - \boldsymbol{y^1})$.

B. Additional tables and figures

Table B.1: Distribution of the number of friends in samples

# of friends	Long study, year 1	Long study, year2	Short study

0	17	5.29%	26	8.12 %	9	4.39 %
1	1	0.31%	14	4.37%	3	1.46 %
2	5	1.56%	34	10.62%	4	1.95 %
3	28	8.72%	39	12.19%	17	8.29 %
4	32	9.97%	56	17.5%	28	13.66 %
5	39	12.15%	55	17.19%	34	16.59 %
6	41	12.77%	39	12.19%	34	16.59 %
7	150	46.73%	33	10.31%	28	13.66 %
8	2	0.62%	0	0.00%	21	10.24 %
9	1	0.31%	0	0.00%	14	6.83 %
10	3	0.93%	0	0.00%	4	1.95 %
11	0	0.00%	0	0.00%	3	1.46 %
12	0	0.00%	0	0.00%	1	0.49 %
13	1	0.31%	0	0.00%	3	1.46 %
14	0	0.00%	0	0.00%	2	0.98 %

Table B.2: Unified State Exams statistics

Subject	Number of participated	Average grade
Mathematics	305	59.87
Russian	305	79.85
Biology	2	71.5
Chemistry	1	80
Computer Science	49	76.96
Economics	27	32.52
Foreign Language	272	70.64
Geography	4	67
History	78	70.94
Law	20	69.4
Literature	20	69.35
Orientalism	2	75
Physics	49	58.45
Social Studies	269	71.01

Table B.3: Descriptive statistics

Variable	Mean	St.Dev.	Min	Max
Average grade, wave 1	7.20	0.94	4.58	9.35
Average grade, wave 2	7.23	1.13	4.50	9.86
Retakes (dummy)	0.33	0.47	0	1
Retakes (number)	0.684	1.25	0	6
Ability	183.6	70.09	106	355
Gender (f)	0.67	0.47	0	1
Tuition, wave 1 (private)	0.18	0.38	0	1

Tuition, wave 2 (private)	0.184	0.39	0	1
Economics department	0.328	0.47	0	1
Management department	0.272	0.45	0	1
Computer Science department	0.26	0.44	0	1
Working status, wave1 (not working)	0.804	0.39	0	1
Working status, wave2 (not working)	0.74	0.44	0	1
Higher Education of mother	0.796	0.4	0	1
Higher Education of father	0.624	0.49	0	1
Single parent family	0.2	0.40	0	1
Family with more than 1 kid	0.54	0.50	0	1
Living conditions, wave 1 (dormitory)	0.16	0.37	0	1
Living conditions, wave 2 (dormitory)	0.172	0.38	0	1

Figure B.1: Distribution of friends in Short and Long surveys

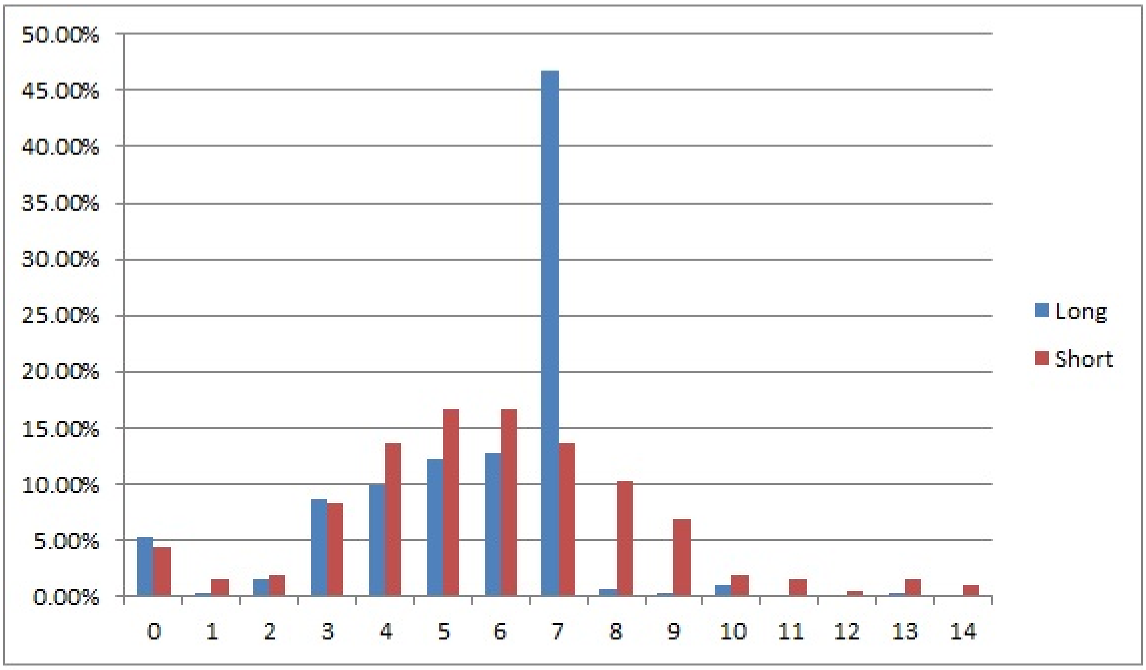


Table B.4: Results for the models with reciprocal links and best friends, no correlated effects

Variable	Recipr., (1)	Recipr., (2)	Best, (3)	Best, (4)
Constant		-0.2318$^\bullet$		-0.3127**
Unexpected Retake	0.1320	-0.0097	0.0469	0.0768
Endogenous effect, period 1	0.0180	0.0111	0.0231	-0.0467**
Endogenous effect, period 2	0.0480*	0.0407*	0.0818***	0.0215
Time-variant own controls				
Tuition, w1	0.0317		-0.0771	
Tuition, w2	-0.1547		-0.2518	
Working status, w1		-0.0909		-0.1235
Working status, w2		0.1483*		0.1510*
Network's controls				
Economics, w1	0.0778	-0.0214	0.0082	0.1953
Economics, w2	-0.4701**	-0.5337**	-0.4869**	-0.4096**
Computer Science, w1		-0.4977**		
Computer Science, w2		-0.3899$^\bullet$		
Working status, w1	-0.2272		-0.1881	
Working status, w2	-0.1623		-0.4340**	
Siblings, w1				0.2611*
Siblings, w2				-0.0080
Sample size	250	250	250	250
BIC	-224.67	-221.77	-221.41	-218.07

*** - p-value < 0.01, ** - p-value < 0.05, * - p-value < 0.1, $^\bullet$ - p-value < 0.15

Table B.5: Results for the models with best friends and reciprocal links, with correlated effects

Variable	Recipr.,(1)	Recipr.,(2)	Best,(3)	Best,(4)
Unexpected Retake	-0.1212	-0.0913	0.0406	-0.1365
Endogenous effect, period 1	-0.0081	-0.1188	0.0235	-0.0404
Endogenous effect, period 2	0.0498	0.0016	0.0811	-0.0262
Time-variant own controls				
Tuition, w1		0.0394		0.1946
Tuition, w2		-0.1933		-0.0608
Working status, w1	0.0296		-0.0248	
Working status, w2	0.0493		0.1437	
Network's controls				
Abilities, w1		-0.0004		
Abilities, w2		-0.0029		
Tuition, w1	-0.5923			0.2977
Tuition, w2	-0.4462*			0.1171
Economics, w1			1.0059	
Economics, w2			-0.2755	
HE of mother, w1				-0.3578
HE of mother, w2				-0.2662
Single Parent, w1	-0.0083			
Single Parent, w2	-0.1985			
Siblings, w1			0.2174	
Siblings, w2			0.1179	

Sample size	250	250	250	250
BIC	-191.02	-192.71	-189.98	-183.06

*** - p-value < 0.01, ** - p-value < 0.05, * - p-value < 0.1, • - p-value < 0.15

Table B.6: List of classes with retakes in the sample

Class	Department	Total No. of retakes	Important classes
Algebra	Computer Science	21	No
Architecture of Computer Systems	Computer Science	2	No
Architecture of ECM	Computer Science	1	No
Basics of computer technology and programming	Computer Science	3	Yes
Discrete Mathematic	Computer Science	8	No
Discrete Mathematic	Economics	2	No
Economic Theory and Institutional Analysis	Management	28 (in 2 terms)	Yes
Economic Theory and Institutional Analysis	Computer Science	12	No
Economic Theory Basics	Economics	27 (in 3 terms)	Yes
Economics	Computer Science	3	No
English and other languages	All departments	9	No
Geometry and Algebra	Computer Science	8	Yes
History of economic thoughts	Economics	1	Yes
History of foreign state and law	Law	2	Yes
Introduction to software engineering	Computer Science	3	Yes
Judicial power and law enforcement	Law	1	No
Life safety	All departments	3	No
Linear Algebra	Economics	28	No
Mathematical Analysis	Computer Science	68 (in 2 terms)	Yes
Mathematical Analysis	Economics	12	Yes
Mathematics	Management	31 (in 2 terms)	Yes
Methods of financial and economic computations	Economics	1	No
Microeconomics	Computer Science	18 (in 2 terms)	Yes
Philosophy	Management	6	Yes
Roman Law	Law	1	No
Socio-Economic Statistics	Economics	2	No
Sociology	Management	1	Yes
Theoretical basics of computer technology	Computer Science	9 (in 2 terms)	No
Theory of state and law	Law	4	Yes

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Chapter 3

Peer effects in art prices

3.1 Introduction

Art is receiving an increased attention in recent years as a possible investment. Art is often attributed to the category of so-called passion investment, which also includes jewelry, antiques, classic car, wine, etc. Passion investments are assessed to amount to 6% of total wealth (The Wall Street Journal, 2010), and the high-net-worth investors allocate globally around 17% of their cash to art (World Wealth Report 2013). According to Knight Frank Luxury Investment Index 2014, the 10-year capital appreciation of art is among highest and equal to 226%[1]. The estimation of the real returns in existing literature varies, depending on the methodology, time period and data used. For example, Goetzmann (1993) reports an average annual real return on oil paintings of 3.8% for the period between 1850 and 1986, with returns around 15% after 1940. Mei and Moses (2002) calculate the return of 4.9% for 1875-1999, with 8.2% after 1950. Renneboog and Spaenjers (2013) are more careful in their estimation with 3.97% over the period 1957-2007.

However, some of these results show the underperformance of art in comparison to the other types of investments. Moreover, the volatility of the prices on the art market is rather high. The attractiveness of the art market is, therefore, cannot be explained by the investment purposes. Global art and antique sales are steadily high in the last several years and are close to 50 billion Euro (with the exception of 2009 crisis)[2]. Buying pieces of art is also a popular tendency among high earners. Owning the work by a famous artist may help to strengthen a status in the society. Creating one's own collection can yield additional respect for their owner. The attractiveness of the art market in comparison to the other luxury items, as stated in the World Wealth Report 2013, is likely "driven by auction house sales, the art market is lively compared to other categories that are characterized more by inheritance and private sales". Approximately half of the art sales are made via auctions (McAndrew 2010), the rest are privately traded, mostly via art dealers. The auction prices

[1] Consult here for more details.
[2] European Fine Art Foundation report

are the ones who set the benchmark for the art prices in general.

In the literature, discussing art pricing, the following standard set of variables is usually used as determinants of the price: artists' characteristics, works' characteristics, such as medium, authenticity, attribution, size, and topic, as well as the sales characteristics, such as a date and a place. Artists' reputation is usually included, however, it is not always clear, how to account for it. Renneboog and Spanjers (2012) include a dummy for mentioning of the artist in the classic art history textbook "Gardner's Art Through Ages" and a dummy of exhibiting at Dokumenta in Kassel.

This paper explores one important determinant of the art price formation that is potentially missing in the existing analysis, namely the artists' connections. I believe that connections can influence in two ways. First of all, following the classical peer effect logic, artists' links are influencing the development of artists' style and quality of the works. But also the prices of the work of one artist may be driven by the prices of the connected artists. If the artists worked together or were connected by the same movement, it is likely that their works will resemble some similarities and may get similar prices on the market. Alternatively, the demand for some artist's works may increase, increasing the price, if the works of connected to them artist are not available or too expensive. The famous film about street artist Banksy, "Exit through the Gift Shop", also depicts, how connections can attract high demand and big money to quite mediocre pieces of art. The paper uses the data on abstract movement artists and their works, collected especially for this project from the open resources. Abstract art is, along with contemporary art, among the movements, for which the price is especially difficult to determine, and hence, exploring the new channels of price formation may help to understand it better.

I apply the peer effect model introduced by Manski (1993) to the panel data on the prices of abstract artists' work sold at Sotheby's in 2000-first half of 2015. Although the model is modified to be applied to the panel data, and to use price as an individual outcome variable and average price as an endogenous variable of the connections, the identifying assumptions of Bramoullé, Djebbari, and Fortin (2009) are still valid. The network is required to have intransitive triads for the model without the correlated effect and the connections of length 3 for the model with correlated effects. This means that there should exist two artists that are not connected directly, but via one more artist (or two). These assumptions are plausible for most of the networks, and for the network of abstract artists in particular.

The paper is organized as follows. Section 2 discusses the proposed peer effect model and suggested estimation method. Section 3 introduces the data and provides some of the descriptive analysis. Section 4 provides the empirical results. Section 5 concludes.

3.2 Model

I have the sample of all works sold at Sotheby's for all of the artists in the sample, for $i \in [1, I]$ being an artwork from the whole set, $j \in [1, J]$ being an artist, created the work. In order to estimate the effect of being connected to other artists on prices of own work, I am proposing the model of peer effect, similar to Manski (1993) and Bramoullé, Djebbari, and Fortin (2009), with the slight appropriate modification:

$$P_i^j = \alpha_0 + \alpha_1 t_i + \beta_0 Z_i + \beta_1 X_j + \beta_2 \sum_{l \neq j} G_{jl} X_l + \gamma \sum_{l \neq j} G_{jl} \bar{P}_l + \nu_j + \epsilon_i^j \tag{3.1}$$

Some comments on the model ingredients are necessary.
P_i^j - price of the piece i by artist j. The prices are normalized as will be discussed in the next section.
Z_i - are the characteristics of the painting, that may include type of the work, size, possibly attribution to an artists' important period, provenance or exhibition history[3].
X_j, X_l - are the characteristics of the artists, such as particular style, major work type (paintings, sculpture etc.), country of birth and living, years active, possibly the total amount of works produced.
G_{jl} - is adjacency matrix with $1/n_j$ in the jl cell, if an artist j is connected with an artist l, with n_j being the total number of connections of an artist j. Note that, although all the links are reciprocal in this particular setting, the matrix is not symmetrical, since
$nu._j$ - are the unobserved effects of the artists.

Here β_2 represents exogenous effect, how the similar characteristics of the connections influence the outcomes, γ is endogenous effect, showing how the outcomes of the connections may influence the outcome of the individual, ν_j are the unobserved characteristics of particular painter in the panel.

The presence of individual unobservable effect creates additional issue for the identification of the endogenous effects.

Potentially, the correlated effects can be also present in the model, making the smaller group within the network to behave similarly due to the unobserved similarities of the group. However, the network used in this paper is rather small, and the potential similarities of the subgroup can be captured by observed characteristics, such as the country of origin or/and work, the group affiliation etc. In more general setting the correlated effects problem can be dealt with by applying the local differences, averaging over the first level connections' outcome variables. The identification of endogenous effect is then achieved by the presence of links of the length 3 or more, and so $G_{jl}^3 X_l$ can be used as identifying instruments for the endogenous covariates.

Since the network is held constant in the panel, the fixed effects model, that is more suitable from the empirical point of view, is not applicable, as then the endogenous effect cannot be identified. So either the unobserved individual effects should be treated by random effects analysis or Hausman and Taylor type models should be used.

Both possibilities are plausible, depending on the explanation of assumptions one believes in. The unobserved individual effects in our empirical could represent the level of talent of the particular artist, his/her popularity, as well as characteristics not included or

[3] The variables will be discussed in details in later sections

missing in the vector of covariates attributed to the artist that could influence the outcome variable. The level of talent and some potential covariates are more likely to be uncorrelated with explanatory variables. The popularity is, however, possibly correlated with one of the explanatory variables, in particular, with the characteristics and outcomes of the connections. In this case, the random effects will produce inconsistent estimators of all parameters, and the Hausman and Taylor type models are more suitable for this particular setting. It is, however, more computationally demanding, and once the correlation is absent, the random effects are preferable.

3.2.1 Hausman and Taylor type models

If it is suspected that individual unobservables are correlated with the explanatory variables and time-constant[4] variables are of interest, as was already mentioned, neither fixed effects nor random effects are suitable for the analysis.

For now, I will ignore the correlated effects for simplicity of explaining the estimation approach. I will give a note on adding the correlated effects in the model later in the paper. First, I divide all explanatory variables into two vectors: time-variant $\boldsymbol{Z_i^j} = \{t_i, Z_i^j\}$ and time-constant $\boldsymbol{X_j} = \{1, X_j, \sum_{l\neq j} G_{jl}X_l, \sum_{l\neq j} G_{jl}\bar{P}_l\}$. I follow Hausman and Taylor (1981) approach and partition both vectors as follows: $\boldsymbol{Z_i^j} = (\boldsymbol{Z_{i1}^j}, \boldsymbol{Z_{i2}^j})$ and $\boldsymbol{X_j} = (\boldsymbol{X_{j1}}, \boldsymbol{X_{j2}})$ where $\boldsymbol{Z_{i1}^j}$ is $1 \times K_1$, $\boldsymbol{Z_{i2}^j}$ is $1 \times K_2$, $\boldsymbol{X_{j1}}$ is $1 \times J_1$, $\boldsymbol{X_{j2}}$ is $1 \times J_2$ and the following assumptions are fulfilled:

$$\mathbb{E}(\boldsymbol{X_{j1}}\nu_j) = 0 \quad \text{and} \quad \mathbb{E}(\boldsymbol{Z_{i1}^j}\nu_j) = 0$$

Additionally, the following assumption is necessary:

$$\mathbb{E}(\epsilon_i^j|\boldsymbol{X_j}, \boldsymbol{Z_1^j}, \ldots, \boldsymbol{Z_{w_j}^j}, \nu_j) = 0, \quad i = 1, \ldots, w_j$$

with w_j being the number of works of artist j in the sample.

We can then rewrite the original model in a simpler way:

$$P_i^j = \beta Z_i^j + \alpha X_j + \nu_j + \epsilon_i^j \tag{3.2}$$

where $\beta = (\alpha_1, \beta_0)$ and $\alpha = (\alpha_0, \beta_1, \beta_2, \gamma)$.

Note that under assumptions in this subsections the time-constant variables are likely to include the outcome variable of friends, which are endogenous in the model. Therefore, the vector of instruments necessary to apply HT approach should additionally include an identifying instrument for the endogenous peer effect, which is the exogenous characteristics of friends of friends, following Bramoullé, Djebbari, and Fortin (2009). The estimation procedure is as follows:

- First, the fixed effects approach is applied, which gives consistent estimation of the

[4] Note, that notion of time is used here not in the direct sense, but rather follows the conventional terminology of panel data analysis. Time here denotes each time one of the works was auctioned.

coefficients of the time-varying variables $\hat{\beta}_{FE}$.

- Using the estimator of the first step, calculate the residuals as follows:

$$\hat{d}_j = \bar{P}^j - \hat{\beta}_{FE}\bar{Z}^j = \alpha X_j + \nu_j + \bar{\epsilon}^j \tag{3.3}$$

- Now, estimate 3.3 with a 2SLS approach. The standard vector of instruments in Hausman and Taylor approach is $[\boldsymbol{Z^j_{i1}}, \boldsymbol{X_{j1}}]$. To be able to identify the endogenous peer effect, I suggest adding $\sum_{l\neq j} G_{jl} \sum_{k\neq l} G_{lk}X_k$ to vector of instruments.

- Using the residual variance σ^{*2} from the previous step and estimator of σ^2_ϵ from the first step, calculate $\sigma^2_\nu = \sigma^{*2} - \sigma^2_\epsilon/\bar{T}$, where $\bar{T}$ is a harmonic mean of T_j's. And compute the weighting coefficients for GLS as:

$$\theta_j = 1 - (\frac{\sigma^2_\epsilon}{\sigma^2_\epsilon + T_j\sigma^2_\nu})^{0.5}$$

- Finally, the following transformations are made: $P^{j*}_i = P^j_i - \theta_j\bar{P^j}$, $Z^{j*}_i = Z^j_i - \theta_j\bar{Z^j}$, $X_j* = X_j - \theta_j\bar{X}$. IV regression of P^{j*}_i on Z^{j*}_i, X_j* are performed, using the vector of instruments $[Z^j_i - \bar{Z^j}, Z^j_{i1}, X_{j1}, \sum_{l\neq j} G_{jl} \sum_{k\neq l} G_{lk}X_k]$

3.2.2 Alternative approach

The Hausman and Taylor approach is computationally challenging, and the number of artists in the sample is not very high with the number of observations for each of them varies a lot, therefore, the alternative approach, avoiding usage of panel data structure is also proposed. The artists dummies are used in this case, combined with the standard 2SLS proposed by Bramoullé, Djebbari, and Fortin (2009) following L. Lee (2003) approach. The model is then as follows:

$$P^j_i = \alpha_0 + \alpha_1 t_i + \alpha_2 AD_i + \beta_0 Z_i + \beta_1 X_j + \beta_2 \sum_{l\neq j} G_{jl}X_l + \gamma \sum_{l\neq j} G_{jl}\bar{P}_l + \epsilon^j_i \tag{3.4}$$

with AD_i - set of artists' dummies.

The 2SLS is then conducted with $\sum_{l\neq j} G_{jl} \sum_{k\neq l} G_{lk}X_k$ as an instrument for an endogenous variable $\sum_{l\neq j} G_{jl}\bar{P}_l$.

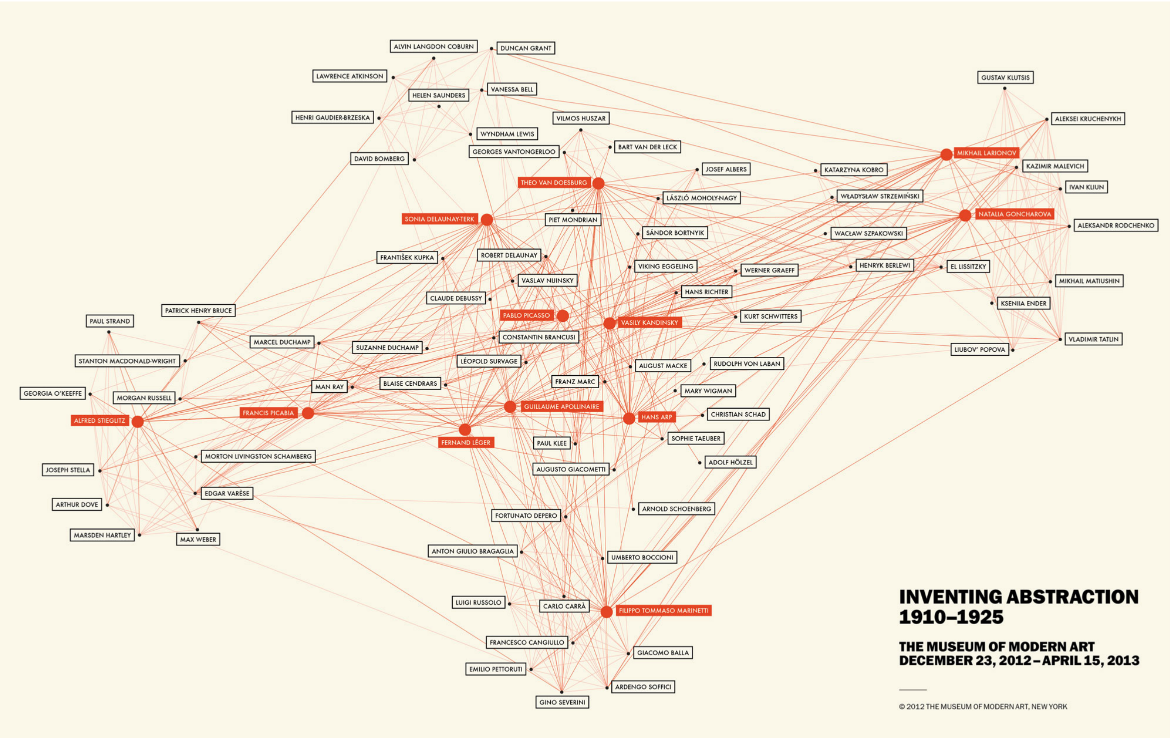
The two proposed approaches should yield comparable results since both of them provide consistent estimators. I do not prove the consistency of the resulting estimators in this paper, but it logically follows from the proofs in Hausman and Taylor (1981) and L. Lee (2003).

3.3 Data description

The data in the paper consists of several parts: the data on the artists' connections, the data on the artists' characteristics and the collection of the prices for the artists' work sold at the auctions at Sotheby's since 2000.

The network data is taken from the diagram, prepared by the group of researchers for the exhibition "Inventing Abstraction, 1910-1925" at the Museum of Modern Art, New York in December 2012-April 2013.[5]

Figure 3.1: Network of abstract artists



This diagram represents documented relationships among the artists, who played significant roles in the development of the new art language. The whole list of the artists can be found in the Appendix C. The diagram was manually transformed into adjacency matrix, with $1/n_i$ at the ij cell, if the artist i is connected to the artist j, and n_i is the total number of connections of the artist i. Moreover, I collected additional information about art groups at the time, such as Der Blaue Reiter (The Blue Rider), De Stijl, etc., to be able to distinguish between the links of different intensity. Table 1 shows the number of artists affiliated with the groups and the number of artists worked in the particular country. Some of the artists worked in more than one country, I included each of them if it was mentioned either on the official website of the MOMA exhibition or in "A Dictionary of Twentieth-Century Art"(1999) by Ian Chilvers.

[5]An interactive detailed network can be found online at the MOMA website.

Table 3.1: Countries and groups allocation

Countries	Groups
France (30), USA (20), Germany (18), Russia (18), Italy (15), Switzerland (9), England (8), Poland (5), Spain (4), The Netherlands (4), Hungary (3), Romania (2)	Der Blaue Reiter (7), Puteaux Group (6), De Stijl (4), Union of Youth (4), Donkey's Tail (3), Supremus (3), 291 Gallery (3), Jack of Diamonds (2), Societe Anonyme (2), Bloomsbury group (2)

The existing network determined the sample of the artists used for the analysis. However, several of the names from the list were not included in the analysis. Those are people important for the abstraction movement, but they were not creative artists, such as, for example, Guillaume Apollinaire, who was a writer and an art critic, or Claude Debussy, who was a composer. Therefore, the auctioned items related to these people are more likely to be personal items or similar. There are 83 artists in the initial list, 11 of which were eliminated.

The set of artists' characteristics were collected from different biographical sources. Among those characteristics are the following: years of life and years active, the country of birth and the country, where the artist was more active, main artistic mode (paintings, sculpture, photographs etc.), particular major style inside of abstractionism; if available, approximate amount of known works and most valuable periods to be able to determine the rarity of the works.

The price dataset was collected specially for this paper from the Sotheby's auction house website. The special program was written to obtain all the lots for each of the artists, auctioned at Sotheby's and both sold and not. The data were available for the auctions that took place from the year 2000, earlier lots are not available online. For each lot the following information is usually available: an estimated price of the work, whether it was sold and the price, which lot was sold, date of creation, type of work, size, provenance, history of exhibitions, although the descriptive information is missing in quite a lot of cases. There might also be some additional catalogue notes, including conditions, authenticity information, etc. However, not all of this information can be included in the model. Part of the description is very difficult to re-translate into quantitative variables.

There are several data issues that should be pointed out. **First of all**, for now only the data from one auction house is used. This may cause some distortion of the results since I am not looking at the whole market situation, i.e. I am not controlling for the availability of the works of a particular artist in the other auction house at the same time. The full dataset will be collected in the future research, however, new programs should be written for each of the auction houses to scrap the data from their websites due to the different layouts.

Secondly, the types of works included in the sample are rather different, such as oil paintings, watercolours, lithographs, different types of sculptures, photographs etc. Including dummies for each of the type of the work is possible. However, the description is missing for almost 30% of the cases, hence, not all of the works can be attributed to a particular medium. Since most of the artists in the sample used different media in their work, so only some lots can be attributed to a specific medium used by the artists[6]. I am treating these lots as not attributed. However, buyers at the auction were aware of the type of work on sale. This missing information is likely to result in biased estimation. One of the solutions is to obtain the better dataset. However, it can be observed that most of the missing information corresponds to the sales in years 2000-2003. So the missing information problem can be partially dealt with by restricting the sales sample for the years after 2003.

Thirdly, the prices are given in the local currency of the auction, which requires price adjustment not only for the inflation but also for the exchange rates. Most of the prices are in USD (43,3% of the lots), in GBP (42,3%), or EUR (13,6%). Several lots are in Swiss Franks, Australian, and Hong Kong Dollars. I use daily historical exchange rates to convert all of the prices into US Dollars. I then adjust prices by CPI of USD, taking the beginning of the sample, January of 2000 as a baseline of 1. In both nominal and real terms the most expensive transaction in my sample is "Garcon á la Pipe" by Pablo Picasso sold at Sotheby's New York in May 2004 for 104 million USD.

The total sample consists of about 12000 observation, however, not all of the observed lots were sold, and the sample used for the analysis is, therefore, smaller, and consists of 9857 lots.

Description of the sales

Table 3.2: Average prices

	Median	Mean	Standard deviation	Number of observations
All sold	20570	384000	2701223	9857
Sold after 2003	25550	463620	3041266	7686
Oil paintings	195800	1919583	6727491	1637
Watercolors	86040	396300	1238530	351
Drawings	27270	179300	824219	4166
Sculptures	43810	541300	3144502	225
Photographs	21900	91480	281245	389
Not attributed	11000	83191	545287	4103

It can be noticed from Table 3.2 that the prices are indeed different for the different media. The oil paintings are expectedly the most expensive, and the photographs have the lowest prices. Drawings form the biggest group, with almost half of all the sales. Not attributed works amount to more than 40% of all sold lots, which is more than in a total sample. Judging from the distribution of the values of unattributed works, most of them are likely to be either drawings or photographs, and do not belong to the categories with

[6]The online database The Art Sales Index could possibly provide a better quality data, I am currently writing the program to obtain it, and the new dataset will be used in the further extension of the paper.

higher prices. Hence, I can still analyse the full sample, including dummies for oil paintings, watercolors, and sculptures only.

Figure 3.2: Distribution of sales over the years

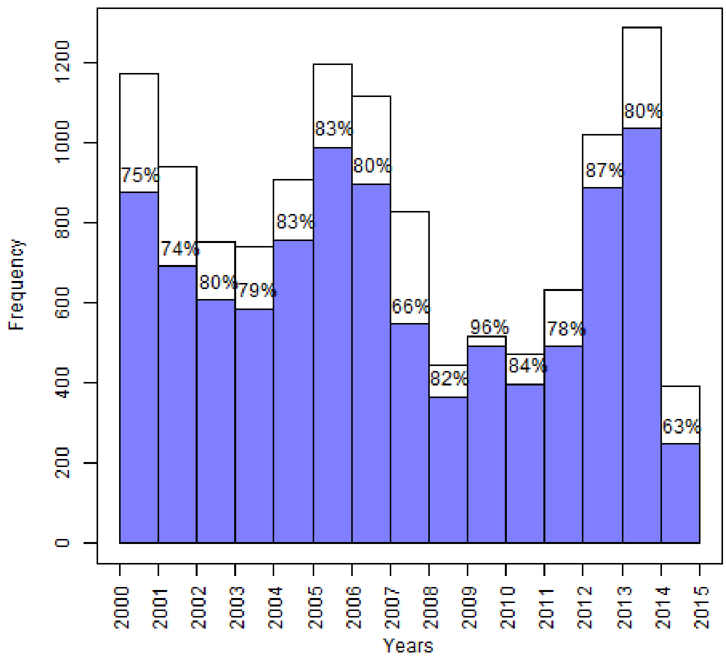


Figure 3.2 shows the distribution of the auctioned and sold lots of the observed time period. The number of lots in abstraction art varies during the observed time period with the biggest decrease around the 2008 crisis. The percentage of the lots sold in the first years after the crisis is, however, among the highest (82% in 2009 and 96% in 2010). In the recent years, the level of sales recovered to the pre-crisis year. Note, that the data was collected in the middle of 2015, determining the low amount of sales in this years.

Table 3.3: Prices over time (in 2000 prices)

	Average	Sum	Maximum	Sold lots	Total lots
2000	181 916.44	57 121 761	10 064 195	314	421
2001	55 504.83	31 138 212	3 016 847	561	749
2002	97 233.27	67 090 953	5 634 195	690	938
2003	109 052.58	66 085 862	10 659 028	606	753
2004	497 636.84	291 117 551	116 695 313	585	740
2005	179 560.06	135 747 408	21 430 066	756	906
2006	300 164.66	296 562 685	114 225 355	988	1195
2007	430 133.71	385 399 804	36 309 073	896	1114
2008	704 978.33	385 623 149	50 360 523	547	826

2009	424 994.75	155 548 077	14 712 432	366	446
2010	374 195.48	184 478 373	12 049 552	493	515
2011	803 838.78	319 123 997	53 231 183	397	471
2012	700 985.01	343 482 657	56 631 229	490	632
2013	549 474.89	486 834 749	61 579 591	886	1018
2014	464 436.80	480 227 654	44 430 080	1034	1285
2015	400 162.95	99 240 411	18 688 892	248	392

Table 3.3 provides more details on the dynamics of the prices and sales of abstract art over the discussed period of time. The first four years of the observations show the relatively low total value of sales, as well as the average price of the lots. After that, the sales increased significantly with slight decline around after 2008. Demand and for abstract art auctioned during that period cannot be considered as homogeneous, therefore, year dummies should be included in the analysis.

Data also include the minimal and maximal expected prices of each lot, which is the auction house estimate of a potential price before the beginning of the auction. In the cases, when the lots were sold, these prices are quite an accurate estimate of the final price with a correlation of around 95%. However, these prices don't help to predict, whether the lot will achieve the reserve price, set by auction and seller together. One of the expected prices can be used as an outcome variable to estimate the effect on the price formation, however, it will have a different meaning than the effect on the final prices of the lots, representing the market response. I will consider both possibilities in the empirical analysis.

Network characteristics

The network, as was already mentioned, consists of 83 artists. Table 4 gives some of the characteristics of the network.

Table 3.4: Network characteristics

Network statistics	Definition	Value
Average indegree	Average number of ingoing ties	12.84 (6.44)
Minimum indegree	Minimal number of ties	2
Maximum indegree	Maximal number of ties	28
Density	Proportion of existing ties in the network	0.1566
Transitivity	The ratio of the triangles and the connected triples in the graph	0.4629

Links from same country		0.8435
Links inside the group		0.0999
Links inside the group, if belonged to a group		0.2807

The network has quite a high number of average connections, more than 12. It is highly likely, that not all of the connections have an effect on the outcome variables, however, it is almost impossible to restrict the number of connections. One option is to put higher weight on those, working in the same country or affiliated with the same group. Most of the links formed between artists lived or worked in the same country for a significant amount of time. The share of the links in the same group is, however, rather small. Not all of the artists belonged to official groups, even though they belonged to a particular movement, whereas some of the artists had affiliations to several groups. Therefore, the group affiliation might be useful as a control variable of artists' characteristics, but not as an indicator of the tightness of connections.

The transitivity of the network is 46%, which is sufficient for the identifying assumptions to hold.

3.4 Empirical analysis

I am considering several scenarios in my empirical analysis. Recall, that not all of the lots were sold during the auctions. Unsold lots don't necessarily indicate the quality or importance of the work, but more likely to be a characteristic of the market situation. First, I am analysing the sample, excluding these observations. The results characterize the price formation determinants, but not describe the market situation in general. Unsold works are very relevant to understand, how the buyers value the works with particular characteristics. Therefore, I also consider the full sample, letting the price of unsold lots be equal to zero. Additionally, I am conducting the analysis of the full sample, using the minimal and maximal estimated prices, established by the auction house, and, therefore, representing a possibly more objective value of the work. I expected the latter results to be similar to the results for the sample of only sold lots. However, the empirical analysis proved otherwise, which I will explain in more details in the course of this section.

3.4.1 Subsample of sold lots

As was described in Section 2, I am mostly relying on the modified version of Hausman-Taylor Type models. However, I also provide the results for the 2SLS procedure with artists' dummies, as described in 2.2. The estimation results for the subsample of sold lots can be found in Table 3.5. Note that the some of the exogenous characteristics of friends have slightly different meaning than own exogenous characteristics. The characteristics related to the works or the auction are not artist-specific, therefore, the average is used as artists' characteristics, which in a lot of cases has the meaning of shares. For example, averaging the dummy variable of oil paintings as a type of work gives the share of oil paintings among the works of the artist. Since most of the variables are either dummy variables or shares, use of logarithm of prices is more appropriate. To avoid the problem with the logarithm of zero, I take a logarithm of $Price + 1$.

Table 3.5: Results for sold lots

	HT	IV
Constant		3.4121$^{\bullet}$
Av.log price of friends' works	**0.8645*** **	**0.4538*** **
Artists' dummies		incl.
Work's characteristics		
Oil painting	2.1485***	2.1892***
Watercolor	0.9453***	0.8754***
Drawings	0.1946***	0.1923***
Sculpture	0.6152***	0.5325***
Signed	0.2005***	0.1944***
Sale's characteristics		
2008	0.3578***	0.3463***
2009	0.5409***	0.5436***
2010	0.7089***	0.6915***
London	0.5862***	0.5302***
New York	0.6299***	0.5802***
Artist's characteristics		
Germany	-0.0271	0.4131*
USA	-0.0671	-0.2108
Russia	-0.3955*	-0.7322***
Link's characteristics		
Share of oil paintings	3.1866	-2.9127
Share of watercolors	-0.5486	3.60977$^{\bullet}$
Share of drawings	3.6839***	1.4016$^{\bullet}$
Share of sculptures	-11.2321	9.0579
Share of signed	-0.6691	0.7366
Share of 2008	-1.2717	-8.6464**
Share of 2009	0.8975	15.8423**
Share of 2010	3.7417	-36.3713***
Share of London	-1.7103	0.2385
Share of New York	-1.0710	1.4047

Germany	0.0634	-0.5304
USA	0.8562**	0.8710***
Russia	-1.7922***	-1.2580***

*** - p-value $<$ 0.001, ** - p-value $<$ 0.01, * - p-value $<$ 0.05, • - p-value $<$ 0.1

Both models detect the highly significant positive effect of the average prices of friends' works on the price of one's own work. The magnitude is, however, not very high. For example, the increase of friends' average price by 10000 USD will on average result in increase of 3000 USD. Most of the results are consistent across two methods for own characteristics, whereas the effect of the exogenous characteristics of connections differs. In particular, the IV regression detects a very high negative effect of share of friends' work sold in 2010. One possible explanation is the general market situation in 2010. The art marked didn't not recover completely after the 2008 crisis, and the number of sales went down, as was discussed in Section 3. The average price of sales in 2010 is lower than in following years, hence the higher share of 2010 sales suggest the cheaper set of works among friends. Hence, this effect is related to the endogenous peer effect. However, the Hausman-Taylor model does not support this finding. Note, that both models show the positive effect of the one's own work sale in all the crisis years I am controlling for. It probably suggests, that once the work got on the auction in these years it is more likely to be less risky since the share of sold lots in these years is very high, hence, the prices are slightly higher.

In Section 3 it was noted that most of the missing information in the sample is for lots that were auctioned in the period 2000-2003. I am, therefore, restrict my sample to the after 2003 sales to check, whether the results are stable in the better subsample. The estimation results are presented in Table 3.6.

Table 3.6: Results for sold lots, after 2003

	HT	IV
Constant		3.8531•
Av.log price of friends' works	**0.8179*** **	**0.3926*** **
Artists' dummies		incl.
Work's characteristics		
Oil painting	2.1385***	2.1845***
Watercolor	0.9267***	0.9034***
Drawings	0.1339***	0.1350***
Sculpture	0.4762***	0.3970***
Signed	-0.0073	-0.0045
Sale's characteristics		
2008	0.2160**	0.2011**
2009	0.4253***	0.4008***
2010	0.5629***	0.5436***
London	0.5929***	0.5507***

New York	0.5482***	0.5235***
Artist's characteristics		
Germany	0.1372	0.1486
USA	-0.1825	-0.3001•
Russia	-0.1161	-0.6072**
Link's characteristics		
Share of oil paintings	0.7454	-0.4430
Share of watercolors	2.4584	6.7982**
Share of drawings	3.0220***	0.6620
Share of sculptures	-11.5981	9.4014
Share of signed	2.5989	-0.1220
Share of 2008	-9.6161*	-9.4391**
Share of 2009	20.3077**	13.6115*
Share of 2010	-0.9464	-28.0462**
Share of London	-1.9372	0.7853
Share of New York	-1.4191	1.3778
Germany	0.4970	0.6033
USA	1.2969***	0.7846**
Russia	-2.1823***	-1.2936***

*** - p-value < 0.001, ** - p-value < 0.01, * - p-value < 0.05, • - p-value < 0.1

It can be observed, that most of the results hold for the restricted sample. First of all, the positive effect of the average price of works of friends exists. The magnitude is slightly smaller than in the sample of all sold lots.

The Hausman-Taylor model is able to catch the same effect for the share of works of friends auctioned in 2009, as was observed in the sample of all sold lots for IV estimator. However, the respective effect for 2010 is not detected. IV estimator, on the contrary, provides the same evidence as before for these variables. In general, restriction of the sample is not changing the results of the estimation significantly.

3.4.2 Full sample

Letting the price of unsold lots be zero, I am now repeating the analysis for the full sample. The correlation of the final market price and the maximal estimated price is no longer as high as it is in the sample of sold lots, therefore, this price can be included in the analysis, as one of the covariates. It can serve as a proxy for the quality of the work and capture some of the work's and artist's characteristics, determining the price. The estimated effect of the other variables will have slightly different meaning than in the previous case, and will catch the effect on the success of the sale.

Table 3.7: Results for the full sample

	HT	IV
Constant		-6.8033
Log Max EP	0.6182***	0.6169***
Av.log price of friends' works	**1.2115****	**0.4886**
Artists' dummies		incl.
Work's characteristics		
Oil painting	0.1743	0.1958
Watercolor	0.0345	0.0292
Drawings	-0.1311**	-0.1268
Sculpture	0.3112	0.2799
Signed	0.6158***	0.6106***
Sale's characteristics		
2008	-1.4809***	-1.3565***
2009	$0.4002^{\bullet}$	0.312*
2010	1.8663***	1.9270***
London	-0.0038	0.0315
New York	-0.4963***	-0.5004***
Artist's characteristics		
Germany	-0.2545	-0.5176
USA	0.7821*	0.3759
Russia	0.0035	-0.5119
Link's characteristics		
Share of oil paintings	6.2693	3.4184
Share of watercolors	12.3529**	4.2451
Share of drawings	-0.0392	0.5824
Share of sculptures	45.1176**	38.1053**
Share of signed	-3.6614	-2.7254
Share of 2008	24.1230**	7.0740
Share of 2009	-18.2541	-2.0767
Share of 2010	25.3134	25.3024
Share of London	-3.8261	-0.3059
Share of New York	-3.6135	-1.9739
Germany	1.6356	1.2606
USA	$0.9896^{\bullet}$	0.3583
Russia	-0.0174	-0.5003

*** - p-value < 0.001, ** - p-value < 0.01, * - p-value < 0.05, $^{\bullet}$ - p-value < 0.1

It can be observed from Table 3.7, that presence of maximal estimated price causes the insignificance of some of the coefficients, significant in the sample of sold lots. For example, for both models, the type of the work is no longer relevant (with the exception of drawings in the Hausman-Taylor model). The maximal estimated price is, as expected, highly significant.

The endogenous effect, in which I am mostly interested, is present only in Hausman-Taylor model analysis. The magnitude is even bigger than in the sample of the sold lots, suggesting that the average prices of friends' work might be even more relevant for the market responses although the two models are different, and the direct comparison of the coefficient not totally correct. I also run the regressions without maximal estimated price, which repeats the model from the previous subsection completely. The results are reported in Table C.3 in Appendix. This model is a worse fit than the one above, and the endogenous effects are insignificant.

The analysis of the sample of all lots auctioned after 2003 does not detect the endogenous effect at all, whereas the other results are quite similar to the full sample (see Table C.2 in Appendix).

3.4.3 Full sample, Maximal and Minimal Estimated Prices

I continue the analysis of the full sample of the lots with the new outcome variable: estimated by auctioneer price of the lot. The obtained endogenous effect is opposite from the one obtained for the final market price of sold lots.

Table 3.8: Results for the full sample, max and min EP

	HT, Max	IV, Max	HT, Min
Constant		18.5371***	
Av.log price of friends' works	**-1.0643*** **	**-0.6299*** **	**-1.0925*** **
Artists' dummies		incl.	
Work's characteristics			
Oil painting	2.1836***	2.2183***	2.1838***
Watercolor	0.7829***	0.7663***	0.7708***
Drawings	0.2199***	0.2258***	0.2227***
Sculpture	0.4816***	0.4512***	0.4826***
Signed	0.1162***	0.1123***	0.1118**
Sale's characteristics			
2008	0.4195***	0.3920***	0.4204***
2009	0.5097***	0.5069*	0.5125***
2010	0.6793***	0.6968***	0.6680***
London	0.5233***	0.4869***	0.5203***
New York	0.7845***	0.7376***	0.7716***
Artist's characteristics			
Germany	1.0920***	0.7959***	1.0947***
USA	-0.1613	0.0797	-0.1674
Russia	-0.3871*	-0.7442***	-0.3933*
Link's characteristics			

Share of oil paintings	-1.5643	-5.4842***	-1.6907
Share of watercolors	-9.6956***	3.8806*	-9.9808***
Share of drawings	5.1483***	1.0577•	5.3086***
Share of sculptures	-12.6471*	-4.2252	-13.9513*
Share of signed	2.3945	2.1634•	2.5370•
Share of 2008	-11.8377***	-17.6327***	-12.7042***
Share of 2009	35.3523***	14.6875**	35.4269***
Share of 2010	-32.9338***	-50.5332***	-33.0736***
Share of London	-2.9070**	-0.3744	-3.1158**
Share of New York	0.8093	1.9496	0.5636
Germany	-1.1076*	-0.9735***	-1.1208**
USA	0.0061	0.7295***	-0.0076
Russia	-1.1539***	-0.5717*	-1.1208***

*** - p-value $<$ 0.001, ** - p-value $<$ 0.01, * - p-value $<$ 0.05, • - p-value $<$ 0.1

Results, presented in Table 8, suggest, that the higher average price is expected by the auctioneer for the artists' connections, the lower the expected price of artist's own work. This is opposite effect in comparison to what I observed for the final price of sold lots. This possibly reveals two different trends. On one hand, the market treats works of connected artists as substitutes. Once the auction house expects the high demand, and hence, high price for the works of artist's connections, it logically expects lower demand for the artist's own works. However, conditional on the sale being successful the prices of connected artists are likely to move in one direction. The increased interest in one artist leads to raising interest towards artist's connections, setting higher prices for all of them.

The other coefficient exhibit rather similar behavior as in Section 4.1. The most noticeable difference is in the estimation of the share of 2008, 2009 and 2010 in the friends' works. The effects are close to the ones from the IV estimation and are similar to the IV estimates of the sample of sold lots.

General observations about other determinants

- All of the models are able to capture the differences in the prices for different media. The types of work dummies are only insignificant in the full sample model, which includes the maximum estimated by auction house price. In this case, the effect of differences in the medium is captured by the effect of estimated price.
- The price of sold lots and the expected prices are estimated to be higher in 2008 crisis year, and two years after it when art market still did not recover from the crisis. 2009 and 2010 also have the positive effect on the price of the full sample (with 0 prices for unsold), reflecting the caution behavior of the sellers in these years. The total amount of sales went down in these two years, but the share of sold lots are higher than in the other years. It means that the lots auctioned in these two years are less risky, the lots

with less uncertainty about their quality or value, hence "better" works were auctioned in these difficult for the market years. The dummy for 2008 for the full sample with unsold lots exhibits opposite behavior to the one in the sold lot sample. The share of sold lots in 2008 is very low due to the crisis, therefore, the amount of zero-priced works is higher in 2008, resulting in the negative highly significant coefficient.

- Sotheby's London and New York are the main auction locations and attract more buyers, hence, the more valuable lots are likely to be auctioned there. The price of sold lots is, therefore, likely to be higher for sales on one of the two locations. Once I am analysing the sample with unsold lots and including the maximal expected price, that accounts for a lot of lots characteristics, the prices in New York are likely to be lower. It reflects the higher probability of having unsold lots, once the volume of sales is bigger than at the other locations.
- Russian art is very popular in the recent decades[7] with every auction house having their own Russian Art Auction couple of times per year, and "works by Russian Avant-Garde are among the most sought-after on the international market" (Hewitt, 2014). The result of my analysis suggests that the prices of the Russian artists are on average lower, and being connected to more Russian artists lower the price as well. So the high demand for Russian art does not result in higher prices in comparison to the other nationalities. It is reasonable to assume that the affordability and availability of Russian abstract art is one of the determinants of its high demand.

3.5 Conclusion

This paper adopts the peer effect logic to analyse the price formation on the art market. I explore the auction results for abstract art from Sotheby's auction house for the period of 2000-first half of 2015 and the connections between the abstract artists as reported for the MOMA exhibition "Inventing abstraction". The connections between the artists can be an important determinant of the artists' style and quality of works, as well as of the resulting price of the works. It can be caused by collaborations, joint exhibitions or the particular reputation of one of the artist's connections.

I am proposing the model, combining Hausman and Taylor (1981) approach for the panel data with the Manski(1993) peer effect model and Bramoulle et al. (2009) instruments. I am also using the alternative model, that uses artists' dummies instead of the panel structure.

I am analysing both the sample of only sold lots and the full sample setting the price of unsold lots to zero. Both settings exhibit the positive peer effect of connections' works average prices, with more prominent effect in the sold lots sample. The market is, therefore, rather responsive to the artist's connections performance and reputations and the buyers are

[7]See, for example, the report on London Russian Weeks Auction sales

willing to pay more if the artist is connected to the "better" artist.

The auctioneers behaviour towards price formation is, however, likely to be different. The more valuable the average works of the connected artists are, the smaller is the expected minimal and maximal prices set by the auction house before the sales begin. The auction house probably views the connected artists as substitutes. Moreover, the more "valuable" peer group of particular artist suggests that the best artists among the group and the artist him/herself are probably among the peer group and not the artist. Hence, this artist's work are valued lower by the specialists.

The data used in the analysis have several limitations described in Section 3, which can distort some of the results. However, the paper shows clear evidence of the importance of connections in the price formation and market outcomes at the art market.

Appendix C. Additional figures

Table C.1: Artists in the sample

Name	Years	Main mode	Country born/active	Group belonging
Josef Albers	1888-1976	Paintings	Germany/ USA	-
Guillaume Apollinaire (not incl.)	1880-1918	Art critic	Poland, Italy/ France	-
Hans Arp	1886-1966	Sculpture	Germany, France	Der Blaue Reiter
Lawrence Atkinson	1873-1931	Paintings (graphical)	England	-
Giacomo Balla	1871-1958	Paintings	Italy	-
Vanessa Bell	1879-1961	Paintings	England	Bloomsbury group
Henryk Berlewi	1894-1967	Paintings	Poland/ Germany, France	-
Umberto Boccioni	1882-1916	Paintings (Sculpture)	Italy	-
David Bomberg	1890-1957	Paintings	England	Whitechapel Boys
Anton Giulio Bragaglia	1890-1960	Photography	Italy	-
Constantin Brancusi	1876-1957	Sculpture	Romania/ France	-
Patrick Henry Bruce	1881-1936	Paintings	USA	-
Francesco Cangiullo	1884-1977	Paintings	Italy	-
Carlo Carra	1881-1966	Paintings	Italy	-
Blaise Cendrars (not incl.)	1887-1961	Writer	Switzerland/ France	-
Alvin Langdon Coburn	1882-1966	Photographt	USA	Linked Ring
Claude Debussy (not incl.)	1862-1918	Composer	France	-
Robert Delaunay	1885-1941	Painings	France/ Spain, France	Der Blaue Reiter, Puteaux Group
Sonia Delaunay-Terk	1885-1979	Paintings	Ukraine/ Spain, France	-
Fortunato Depero	1892-1960	Paintings (Sculpture)	Italy/ Italy, USA	-
Theo van Doesburg	1883-1931	Paintings	The Netherlands	De Stijl
Arthur Dove	1880-1946	Paintings	USA	-

Marcel Duchamp	1887-1968	Paintings (Sculpture)	France/ France, USA	Societe Anonyme, Puteaux Group
Suzanne Duchamp	1889-1963	Paintings	France	-
Viking Eggeling	1880-1925	Various media	Sweden/ Germany	Das Neue Leben, Novembergruppe
Kseniia Ender	1894-1955	Paintings	Russia	-
Henri Gaudier-Brzeska	1891-1915	Sculpture	France/ France, England	-
Augusto Giacometti	1877-1947	Paintings	Switzerland/ Switzerland, Italy	-
Natalia Goncharova	1881-1962	Paintings	Russia/ Russia, France	Puteaux Group, Donkey's Tail
Duncan Grant	1885-1978	Paintings	England	Bloomsbury Group
Marsden Hartley	1877-1943	Paintings	USA/ USA, Germany	-
Vilmos Huszar	1884-1960	Paintings	Hungary/ The Netherlands	De Stijl
Vasily Kandinsky	1866-1944	Paintings	Russia/ Russia, Germany	Der Blaue Reiter
Paul Klee	1879-1940	Paintings	Switzerland/ Germany, Switzerland	Der Blaue Reiter
Ivan Kliun	1873-1943	Sculpture	Russia	Union of Youth, Supremus
Gustav Klutsis	1895-1938	Photography	Latvia/ Latvia, Russia	-
Katarzyna Kobro	1898-1951	Sculpture	Russia/ Russia, Poland	-
Aleksei Kruchenykh (not incl.)	1886-1968	Poet	Russia	-
Frantisek Kupka	1871-1957	Paintings (Graphics)	Czech Republic/ France	Puteaux Group

Rudolph von Laban (not incl.)	1879-1958	Dancer	Hungary/ Germany, Switzerland	-
Mikhail Larionov	1881-1964	Paintings	Russia/ Russia, France	Jack of Diamonds, Donkey's Tail
Fernand Leger	1881-1955	Paintings	France	Puteaux Group
Wyndham Lewis	1882-1957	Paintings	Canada/ England	-
El Lissitzky	1890-1941	Graphics (Photography)	Russia/ Russia, Germany	UNOVIS
Stanton Macdonald-Wright	1890-1973	Paintings	USA	Synchromism
August Macke	1887-1914	Paintings	Germany	Der Blaue Reiter
Kazimir Malevich	1878-1935	Paintings	Russia	Union of Youth, Supremus, UNOVIS, Donkey's Tail
Franz Marc	1880-1916	Paintings (Printmaker)	Germany	Der Blaue Reiter
Filippo Tommaso Marinetti (not incl.)	1876-1944	Poet	Italy/ Italy, France	-
Mikhail Matiushin	1861-1934	Paintings	Russia	Union of Youth
Laszlo Moholy-Nagy	1895-1946	Photography (various media)	Hungary/ Germany, England, USA	-
Piet Mondrian	1972-1944	Paintings	The Netherlands/ France, The Netherlands	De Stijl
Vaslav Nijinsky (not incl.)	1889-1950	Dancer	Russia/ France, Russia	-
Georgia O'Keeffe	1887-1986	Paintings	USA	291 Gallery
Emilio Pettoruti	1892-1971	Paintings (Drawings)	Argentina/ Italy, Argentina	-
Francis Picabia	1879-1953	Paintings	France/ France, USA, Spain	Puteaux Group
Pablo Picasso	1881-1973	Paintings (Sculpture)	Spain/ France, Spain	-

Liubov Popova	1889-1924	Paintings	Russia/ Russia, France	Knave of Diamonds, Supremus
Man Ray	1890-1976	Photography (Various media)	USA/ USA, France	-
Hans Richter	1888-1976	Paintings (Filmmaker)	Germany/ Germany, Switzerland, USA	Artistes Radicaux
Aleksandr Rodchenko	1891-1956	Photography (Sculpture)	Russia	October Circle
Morgan Russel	1886-1953	Paintings	USA/ USA, France	Synchromism
Luigi Russolo	1885-1947	Paintings (Composer)	Italy/ Italy, Argentina	-
Helen Saunders	1885-1963	Paintings	England	-
Christian Schad	1894-1982	Paintings	Germany/ Germany, Switzerland, Italy	-
Morton Livingston Schamberg	1881-1918	Paintings (Photography)	USA	Societe Anonyme
Arnold Schoenberg (not incl.)	1874-1951	Composer	Austria/ Germany	Der Blaue Reiter
Kurt Schwitters	1887-1948	Various media	Germany	-
Gino Severini	1883-1966	Paintings	Italy/ Italy, France	-
Ardengo Soffici	1879-1964	Paintings (Writer)	Italy/ Italy, France	-
Joseph Stella	1877-1946	Paintings	Italy/ USA, France, Italy	-
Alfred Stieglitz	1864-1946	Photography	USA	291 Gallery
Paul Strand	1890-1976	Photography	USA	291 Gallery
Wladyslaw Strzeminski	1893-1953	Paintings	Poland/ Russia	Blok
Leopold Survage	1879-1968	Paintings	Russia/ France	-
Waclaw Szpakowski	1883-1973	Graphics	Poland	-
Sophie Taeuber-Arp	1889-1943	Paintings (Graphics)	Switzerland	Cercle el Carre
Vladimir Tatlin	1885-1953	Paintings (Architecture)	Russia	Union of Youth
Tristan Tzara (not incl.)	1896-1963	Poet	Romania/ Romania, Switzerland, France	-
Georges Vantongerloo	1886-1965	Sculpture	Belgium/ The Netherlands, France	De Stijl

Edgar Varese (not incl.)	1883-1965	Composer	France/ Germany, USA	-
Max Weber	1881-1961	Paintings	Russia/ USA	-
Mary Wigman (not incl.)	1886-1973	Dancer	Germany/ Germany, Switzerland	-

Table C.2: Results for full sample, after 2003

	HT	IV
Constant		-5.606
Log Max EP	0.6164***	0.6265***
Av.log price of friends' works	**0.2332**	**0.3782**
Artists' dummies		incl.
Work's characteristics		
Oil painting	0.1828	0.1103
Watercolor	0.1940	0.0753
Drawings	-0.1988***	-0.1968***
Sculpture	0.2632	0.2958
Signed	0.5584***	0.5464***
Sale's characteristics		
2008	-1.5539***	-1.486***
2009	0.2594	0.3253
2010	1.7851***	1.804***
London	0.0750	0.0666
New York	-0.6135***	-0.5949***
Artist's characteristics		
Germany	-0.7793	-0.7548
USA	0.6151	0.2056
Russia	0.0853	-0.0111
Link's characteristics		
Share of oil paintings	6.7206	6.465
Share of watercolors	5.9433	2.887
Share of drawings	4.0074	3.352•
Share of sculptures	22.5245	42.94*
Share of signed	-0.8956	-3.304
Share of 2008	1.5815	9.164
Share of 2009	-4.6728	1.596
Share of 2010	30.2939	10.38
Share of London	-4.0689	-3.300
Share of New York	-6.1573	-4.396•
Germany	2.6296•	2.522**
USA	-0.6291	-0.1124
Russia	-0.8737	-1.067

*** - p-value < 0.001, ** - p-value < 0.01, * - p-value < 0.05, • - p-value < 0.1

Table C.3: Results for full sample, first model

	HT	IV
Constant		5.0217
Av.log price of friends' works	**0.5152**	**0.0674**
Artists' dummies		incl.
Work's characteristics		
Oil painting	1.5214***	1.5643***
Watercolor	0.5148*	0.5036*
Drawings	0.0044	0.0127
Sculpture	0.6098*	0.5586*

Signed	0.6869***	0.6794***
Sale's characteristics		
2008	-1.2225***	-1.1155***
2009	0.7140***	0.7433***
2010	2.2849***	2.3561***
London	0.3230**	0.2696*
New York	-0.0103	-0.0466
Artist's characteristics		
Germany	0.4028	0.0228
USA	0.6813*	0.3854
Russia	-0.2602	-1.0098*
Link's characteristics		
Share of oil paintings	5.4281	0.1631
Share of watercolors	5.0402	6.0087
Share of drawings	3.3378	1.2951
Share of sculptures	37.9046*	35.5186*
Share of signed	-2.3242	-1.6292
Share of 2008	16.7121•	-3.7773
Share of 2009	0.8907	4.1055
Share of 2010	3.4210	-6.4842
Share of London	-5.3984*	-0.2770
Share of New York	-2.9385	-0.5461
Germany	0.9276	0.5358
USA	1.0146•	0.8755•
Russia	-0.7959	-0.8468

*** - p-value < 0.001, ** - p-value < 0.01, * - p-value < 0.05, • - p-value < 0.1

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