Brownian Motions on Metric Graphs

Inauguraldissertation
zur Erlangung des akademischen Grades
eines Doktors der Naturwissenschaften
der Universität Mannheim

vorgelegt von

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aus Ludwigshafen/Rhein

Mannheim, 2016

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Tag der mündlichen Prüfung: 24. November 2016

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Abstract

In this work, Brownian motions on metric graphs are defined as right continuous, strong Markov processes which, while inside an edge, are equivalent to the one-dimensional Brownian motion. Their generators are identified as Laplace operators on the graph subject to non-local Feller–Wentzell boundary conditions at the vertices. Conversely, a pathwise construction is achieved for any set of admissible boundary conditions.

This thesis generalizes the recent works of Kostrykin, Potthoff and Schrader, who examined Brownian motions on metric graphs which are continuous up to their lifetime. The theory is significantly complicated by the extension to the discontinuous setting. Here, the processes in question might feature jumps of infinite activity in the vicinity of any vertex, and their excursions from a vertex are not limited to adjacent edges.

To overcome the challenges, transformation methods for Markov processes are surveyed and expanded in the modern context of Meyer–Getoor–Sharpe's right processes. A universal revival method is established in order to concatenate various processes and to implement jump discontinuities. Probabilistic properties of Brownian motions on a metric graph are obtained, and their generators and resolvents are analyzed with the help of Dynkin's formulas. By extending the results and the constructions of Itô–McKean's fundamental paper on Brownian motions on the half line to the star graph case, the local description of all Brownian paths is achieved. By applying the transformation techniques and the Brownian properties, the local solutions are pasted together to obtain the process on the complete graph.

Zusammenfassung

In dieser Arbeit werden Brownsche Bewegungen auf metrischen Graphen untersucht. Die Hauptresultate umfassen die Identifizierung der Erzeuger als Laplace-Operatoren auf den Graphen mit nicht-lokalen Feller–Wentzell Randbedingungen an deren Knoten, sowie die pfadweise Konstruktion solcher Prozesse für jede zulässige Kombination von Randbedingungen.

Hierbei werden stochastische Prozesse, der klassischen Definition von Itô–McKean folgend, als Brownsche Bewegung auf metrischen Graphen bezeichnet, falls sie rechtsseitig stetige, starke Markovprozesse sind, welche sich auf den Kanten des Graphen wie eindimensionale Brownsche Bewegungen verhalten. Diese Arbeit verallgemeinert damit die Resultate früherer Arbeiten von Kostrykin, Potthoff und Schrader, welche Brownsche

Bewegungen mit ausschließlich stetigen Pfaden (bis zur Todeszeit des Prozesses) behandeln. Die Erweiterung auf den Kontext unstetiger Prozesse erlaubt es den Brownschen Bewegungen, neben zeitlich anordenbarer Sprünge und Exkursionen zu entfernten Teilgraphen auch Häufungspunkte von Sprüngen an den Knoten des Graphen zu besitzen, welche die Pfadanalyse und -konstruktion deutlich erschweren.

Die möglichen Randbedingungen der Prozesse werden mit Hilfe der Dynkinschen Formel für den Erzeuger abgeleitet. Einem Ansatz Itô–McKeans für die Halbachse folgend werden alle möglichen Brownschen Bewegungen auf Sternengraphen konstruiert. Diese lokalen Lösungen werden sodann kombiniert zu globalen Lösungen auf allgemeinen metrischen Graphen. Hierzu wird die allgemeine Theorie der Markovschen Prozesse, insbesondere im Hinblick auf Pfadtransformationen und Prozesswiederbelebungen, im modernen Kontext von Meyer–Getoor–Sharpes „right processes“ erweitert und angewandt.

Acknowledgements

In the first place, I would like to thank my supervisor Prof. Dr. Jürgen Potthoff for his continuous support and reassurance over all the years. At all times, he shared his insight into the topic of this thesis with me, helped me with good hints on literature and ideas for new techniques, and always had some tricks up his sleeve. He gave me the freedom to try different paths (and to fail on some of them) and when necessary, guided me—as well as possible—back on track.

I am thankful to Prof. Dr. Max von Renesse for kindly agreeing to co-referee this work and for giving me the opportunity to visit University of Leipzig.

I thank the colleagues and students of the School of Business Informatics and Mathematics of the University of Mannheim. They were always open to discussions, a source of helpful advice, and provided good opportunities for short-term or longer-term breaks with activities on other mathematical and non-mathematical topics.

While mathematics could not exist without its logical clarity and rigorousness, it could neither advance without our compassion and eagerness for knowledge. For me, the origin of both lies in the love and encouragement from the people that surround me. I wholeheartedly thank my family, who has supported and shaped me for my whole life, and who enabled me to find and maintain the strength and endurance to succeed in this project.

My special thanks go to my close friends who I have the pleasure to share my days with. They have brought me here, give me the wrong and right perspectives, carry me through the downs of life and really enjoy the ups with me.

I am especially grateful for the valuable suggestions by Claudia Ridinger, who took up the monstrous task of proof-reading the whole thesis. Due to her efforts, the probability of my train of thought being understood increased by a significant amount.

This dissertation is the sum of all this, none of which I take for granted. Thank you!

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Introduction

The goal of this thesis is the classification and pathwise construction of all Brownian motions on a metric graph. We now clarify the underlying definitions illustratively, for rigorous definitions the reader may consult the beginnings of sections 18 and 20:

A metric graph $\mathcal{G}$ is a mathematical description of a set of locally one-dimensional structures, "edges" $l \in \mathcal{L}$, which are "glued together" at "vertices" $v \in \mathcal{V}$ by the graph's combinatorial structure, and every edge $l \in \mathcal{L}$ is isomorphic to a finite interval or half line of length $\rho_l \in (0, +\infty]$. The metric graph is then represented by the set

$$\mathcal{G} = \mathcal{V} \cup \bigcup_{l \in \mathcal{L}} \big(\{l\} \times [0, \rho_l] \big)$$

(with $[0, \rho_l] := [0, \infty)$ if $\rho_l = +\infty$), where the finite endpoint(s) of the edges are identified with appropriate vertices, see figure 18.1. The length of the edges introduces the notion of the length of paths on the graph along edges via adjacent vertices, while the Euclidean metric induces a distance inside the edges. The canonical metric of $\mathcal{G}$ is then defined by the length of the shortest possible path connecting two points on $\mathcal{G}$. We will only consider graphs with finite sets of edges and vertices.

A Brownian motion on a metric graph $\mathcal{G}$ is a right continuous, strong Markov process on $\mathcal{G}$ which behaves on every edge like the standard one-dimensional Brownian motion, more accurately: If a Brownian motion X on $\mathcal{G}$ is started inside some edge $\{l\} \times (0, \rho_l)$, then the process X, stopped at leaving its initial edge, must be equivalent to the one-dimensional Brownian motion, stopped when leaving the interval $(0, \rho_l)$.

The context of metric graphs generalizes the class of Brownian motions on half lines, which has been studied extensively in the past: Started by first path considerations by Kac in [Kac51] and Feller's and Wentzell's analytic examinations of semigroups in [Fel52], [Wen56] and [Wen59], the complete, pathwise description of all Brownian motions on $\mathbb{R}_+$ was obtained by Itô and McKean in [IM63]; for a more detailed historical overview, we would like to refer the reader to subsection 16.4 and to [Pes15]. The interval setting has been further examined by Weber in [Web94], Favini et al. in [FGGR00], and Xiao and Liang in [XL08]. Recently, there is a growing interest in metric graphs, networks and quantum graphs, and stochastic processes thereon. They arise in many areas of physics, chemistry and engineering applications, for an elaborate survey the reader may consult [Kuc02] and Kuchment's introductory article [Kuc04]. A collection of recent developments is found in the proceedings [MS08] and Mugnolo's monograph [Mug14]. The research of continuous processes on graph-like structures seems to be started by Baxter and Chacon in [BC84], who introduced the notion of diffusions on graphs and transferred some classical one-dimensional results to this setting. Since then, a wide variety of results and techniques evolved: Freidlin and Wentzell investigated an averaging principle for processes on graphs in [FW93], which was further developed by Barret and von Renesse with the help of Dirichlet methods in [BR14]. Processes on special tree structures have been examined by Dean and Jansons in [DJ93] via excursion theory and by Krebs in [Kre95] via Dirichlet forms. With the help of graphs, Walsh [Wal78] and Eisenbaum and Kaspi [EK96] studied and extended classical one-dimensional results like local time properties. Particular Brownian motions on graphs have been constructed and studied by Barlow, Pitman and Yor in [BPY89] via semigroup considerations, by Enriquez and Kifer in [EK01] as weak limits of Markov chains, and by Georgakopoulos and Kolesko in [GK14] as weak limits of graph approximations. In [Lej03], Lejay develops simulation methods for diffusions on graphs, which can also be applied in the Brownian context. Diverse results for continuous Brownian motions on star graphs have been researched by Najnudel in [Naj07] and Papanicolaou et al. in [PPL12]. Fitzsimmons and Kuter conducted potential theoretic investigations in the star graph setting in [Jeh09] and [FK14], and extended their findings to general metric graphs in [FK15].

In [KPS12b], [KPS12c] and [KPS12a], Kostrykin, Potthoff and Schrader achieved the classification and pathwise construction of all Brownian motions on a metric graph which are continuous up to their lifetime. Their works mark the starting point of this thesis, in which we weaken the condition of continuity to right continuity. By extending the findings and the construction approaches of the above-mentioned works by Kostrykin, Potthoff and Schrader, and of Itô–McKean's extensive analysis of the half-line case in [IM63], employing the techniques of the modern "general theory" of Markov (right) processes given in Sharpe's monograph [Sha88], we will obtain the classification and a complete pathwise construction for right continuous Brownian motions on metric graphs.

Classification of Brownian Motions

By its very definition, the behavior of a Brownian motion on a metric graph is already fixed inside the edges, where it must act like the standard one-dimensional Brownian motion. Therefore, the "non-Brownian" effects can only take place at the vertices of the graph and still must respect (strongly) Markovian "characteristics". Thus, it is feasible to classify a Brownian motion by its local behavior, which is reflected in its generator:

As mentioned above, the classical case of a "metric graph" with only one vertex and one edge—that is the half line $\mathbb{R}_+$—is completely understood. Here, the generator A of a Brownian motion is a contraction of $\frac{1}{2}\,\Delta$, with Δ being the Laplacian on $\mathbb{R}_+$. Its domain is then uniquely characterized by a set of constants $p_1 \geq 0$, $p_2 \geq 0$, $p_3 \geq 0$ and a measure p_4 on $(0,\infty)$, normalized by

$$p_1 + p_2 + p_3 + \int_{(0,\infty)} (1 \wedge x)\, p_4(dx) = 1,$$

which constitute the following "non-local Wentzell boundary condition" of the generator:

$$\begin{aligned} \mathscr{D}(A) = \Big\{ & f \in \mathcal{C}_0^2(\mathbb{R}_+) : \\ & p_1\, f(0) - p_2\, f'(0+) + \frac{p_3}{2} f''(0+) - \int_{(0,\infty)} \big(f(x) - f(0)\big)\, p_4(dx) = 0 \Big\}. \end{aligned} \tag{I.1}$$

This result is easily extended to the case of a general metric graph $\mathcal{G}$. Just like in the case of the half line, the generator of a Brownian motion reads $A = \frac{1}{2}\Delta$, with Δ now being the Laplacian on $\mathcal{G}$. For every vertex $v \in \mathcal{V}$ there exist constants $p_1^v \geq 0$, $p_2^{v,l} \geq 0$ for each $l \in \mathcal{L}(v)$, $p_3^v \geq 0$ and a measure p_4^v on $\mathcal{G} \backslash \{v\}$ with

$$p_1^v + \sum_{l \in \mathcal{L}(v)} p_2^{v,l} + p_3^v + \int \left(1 - e^{-d(v,g)}\right) p_4^v(dg) = 1,$$

such that the domain of A satisfies

$$\tag{I.2} \begin{aligned} \mathscr{D}(A) \subseteq \Big\{ & f \in \mathcal{C}_0^2(\mathcal{G}) : \forall v \in \mathcal{V} : \\ & p_1^v\, f(v) - \sum_{l \in \mathcal{L}(v)} p_2^{v,l}\, f_l'(v) + \frac{p_3^v}{2} f''(v) - \int \big(f(g) - f(v)\big)\, p_4^v(dg) = 0 \Big\}, \end{aligned}$$

where $\mathcal{L}(v)$ is the set of edges incident with a vertex v, and $f_l'(v)$ is the directional derivative of f at v along the edge l.

These results can be derived through various techniques: The classical proofs of [Wen56] and [Fel57b] are based on the analysis of the underlying semigroup, which then were extended giving special attention on non-local boundaries in [Man68] and [LPS71]. Other approaches are possible by analytic analysis of the resolvent in [Rog83] or of the Dirichlet form such as in [KKVW09] and [Fuk14], or by probabilistic methods via Dynkin's formulas like in [Kni81] and [IM63], or by the excursion theory of [Itô72]. As our goal is a pathwise construction, we will use a method which obtains the generator via a probabilistic method rather than by analytic means: Dynkin's formula gives access to the generator directly through the local exit behavior of the process. It states that, under certain conditions, the generator A of a strong Markov process X on a state space E can be computed by

$$Af(x) = \lim_{n \to \infty} \frac{\mathbb{E}_v\big(f\big(X(\tau_{\varepsilon_n})\big)\big) - f(x)}{\mathbb{E}_x(\tau_{\varepsilon_n})}, \quad f \in \mathscr{D}(A),\ x \in E,$$

with $(\varepsilon_n, n \in \mathbb{N})$ being a sequence of positive numbers converging to 0 and τ_{ε_n} being the first exit time of X from the closed ball $\overline{B_x(\varepsilon_n)}$.

Surprisingly, the components of the "generator data" given in equation (I.2)

$$\tag{I.3} \big(p_1^v, (p_2^{v,l})_{l \in \mathcal{L}(v)}, p_3^v, p_4^v\big)_{v \in \mathcal{V}}$$

have, for the most part, easy probabilistic interpretations. We briefly explain their effects for Brownian motions on the half line $\mathbb{R}_+$, where their set (I.3) of defining boundary weights reduces to (p_1, p_2, p_3, p_4) of equation (I.1): If $B = (B_t, t \geq 0)$ is the Brownian motion on $\mathbb{R}$, then the reflecting Brownian motion $|B| = (|B_t|, t \geq 0)$ is a Brownian motion on $\mathbb{R}_+$ which is characterized by its boundary set $(p_1, p_2, p_3, p_4) = (0, 1, 0, 0)$. If instead we consider the "absorbed" process $(B_{t \wedge H_0}, t \geq 0)$ which results from stopping B at the time $H_0 := \inf\{B_t = 0\}$ of B hitting 0 for the first time, it turns out that this

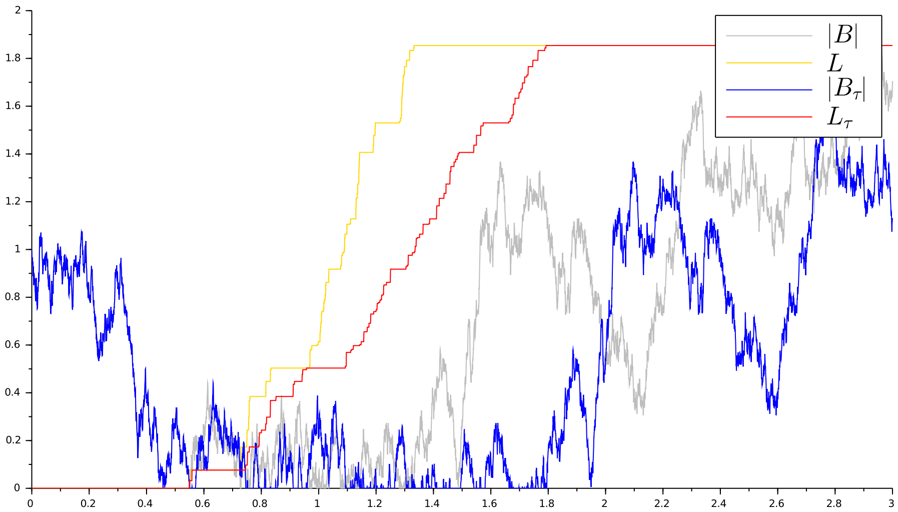


Figure I.1: The "sticky" Brownian motion on $\mathbb{R}_+$: By "slowing down" the reflecting Brownian motion $\big(|B_t|, t \geq 0\big)$ at the origin via a time change with respect to its local time $(L_t, t \geq 0)$, the resulting "sticky" Brownian motion $\big(|B_{\tau(t)}|, t \geq 0\big)$ realizes the boundary weights $(0, p_2, p_3, 0)$. Its local time turns out to be the time changed original local time $(L_{\tau(t)}, t \geq 0)$.

is a Brownian motion on $\mathbb{R}_+$ with $(p_1, p_2, p_3, p_4) = (0, 0, 1, 0)$. On the other hand, the boundary set $(p_1, p_2, p_3, p_4) = (1, 0, 0, 0)$ is implemented by the "Dirichlet" process B^D,

$$B_t^D := \begin{cases} B_t, & t < H_0, \\ \Delta, & t \geq H_0, \end{cases}$$

constructed by killing B at H_0 (this is not a Brownian motion in the sense of our definition, as Markov processes will always be assumed to be normal in this work).

Thus, p_1, p_2, p_3 can be interpreted as the "weights" governing the *killing*, *reflection* and *stickiness* at the origin. These effects are especially illuminated when examining the following "mixed" cases, as surveyed in [KPS10]: The "quasi absorbed case" $(p_1, p_2, p_3, p_4) = (\neq 0, 0, \neq 0, 0)$ can be realized by stopping the Brownian motion B at the origin for an exponentially distributed random time, independent of B, and then killing it. The "elastic case" $(p_1, p_2, p_3, p_4) = (\neq 0, \neq 0, 0, 0)$ is obtained by killing the reflecting Brownian motion $|B|$ when its local time at the origin exceeds some exponentially distributed random time, independent of $|B|$. Finally, the "sticky case" $(p_1, p_2, p_3, p_4) = (0, \neq 0, \neq 0, 0)$ is achieved by "slowing down" the reflecting Brownian motion $|B|$ at the origin: With $(L_t, t \geq 0)$ being the local time of $|B|$ at the origin, define the function $\tau^{-1} \colon t \mapsto t + \frac{p_3}{p_2} L_t$. Then the "sticky" boundary condition is realized by the time changed Brownian motion $\big(|B_{\tau(t)}|, t \geq 0\big)$, see figure I.1. The complete "local" case $(p_1, p_2, p_3, p_4) = (\neq 0, \neq 0, \neq 0, 0)$ is a mixture of the sticky and the elastic case: It is achieved by killing the sticky Brownian motion $\big(|B_{\tau(t)}|, t \geq 0\big)$ once its local time $(L_{\tau(t)}, t \geq 0)$ at the origin exceeds some exponentially distributed, independent random time.

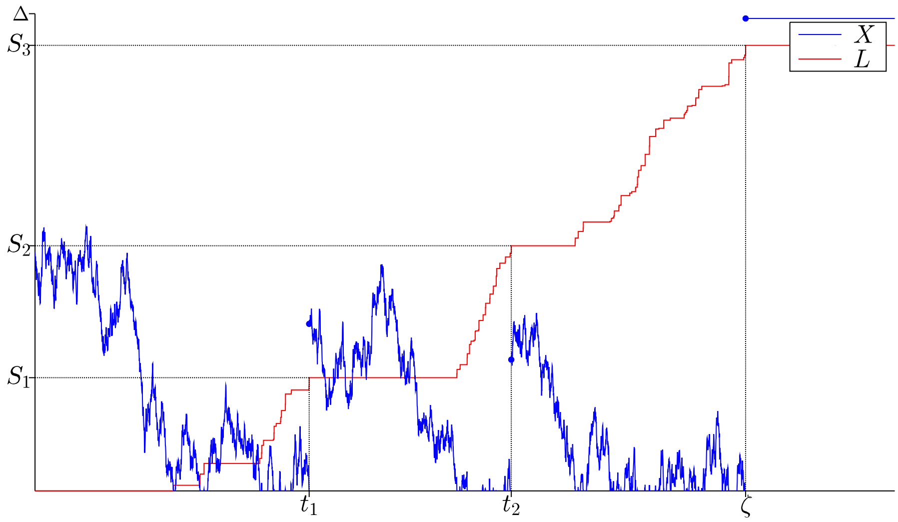


Figure I.2: Implementation of jumps for Brownian motions on $\mathbb{R}_+$: Starting with the "sticky" Brownian motion, as given in figure I.1, restart the process whenever its local time exceeds some level S_n at a point chosen by $p_4 + p_1\,\varepsilon_\Delta$, resulting in the Brownian motion X with local time L which implements the boundary weights (p_1, p_2, p_3, p_4).

The measure p_4 now introduces jumps of the resulting process from the origin to points other than the absorbing cemetery point Δ. If p_4 is finite, then this *jump measure* can be implemented just like the jumps of a compound Poisson process: Starting with the Brownian motion realizing the local boundary condition $(p_1, p_2, p_3, 0)$, we restart this process—if it has not been killed already—whenever its local time at the origin exceeds some independent, exponentially distributed random time with rate proportional to $p_4((0,\infty))$, at some point chosen independently by the probability measure $\frac{p_4}{p_4((0,\infty))}$, see figure I.2. In the case of an infinite measure p_4, the description of the complete process is not as easy: As the finite case already suggests, the resulting process will be a Brownian motion which implements the local boundary conditions and jumps out of the origin like a subordinator with Lévy measure p_4, run on the time axis of the local time. A detailed construction of such paths will be given in this thesis.

These results can be transferred directly to the case of a metric graph, where the set of boundary conditions $\big(p_1^v, (p_2^{v,l})_{l\in\mathcal{L}(v)}, p_3^v, p_4^v\big)_{v\in\mathcal{V}}$ governs the local behavior at any vertex $v \in \mathcal{V}$. The only additional effect which arises here is that the process can usually leave a vertex v on more than one edge. Thus, the reflection weight p_2^v is split up into partial weights $p_2^{v,l}$, $l \in \mathcal{L}(v)$, where $\mathcal{L}(v)$ is the set of edges incident with v. For any excursion which exits the vertex v continuously, the starting edge of this excursion is then chosen independently by the distribution $\big(p_2^{v,l}/p_2^v, l \in \mathcal{L}(v)\big)$, with $p_2^v := \sum_{l\in\mathcal{L}(v)} p_2^{v,l}$.

Accepting these rather illustrative descriptions for the moment, it is clear that in absence of the jumping measure p_4, the Brownian motion may be realized by a process which is continuous up to its lifetime. On the other hand, the case $p_4 \neq 0$ can only be achieved by a discontinuous process.

Construction Approach

As already mentioned, the boundary conditions on the edges can be implemented via path transformations of an easy prototype process like the reflecting Brownian motion: The killing parameter is introduced by killing with respect to the pseudo inverse of the local time, which turns out to be a terminal time, or equivalently, by killing with respect to a multiplicative functional. Stickiness can be implemented by the time change relative to the local time, which is an additive functional. These transformations are classical and well understood. However, the implementation of jumps seems to be a non-standard problem. Here, we will mainly use the technique of "killing and reviving" a (strong) Markov process, which proceeds as follows: We define the concatenation of a sequence of Markov processes $(X^n, n \in \mathbb{N})$, which forms a new Markov process that behaves like X^1 until this process dies, afterwards is "revived" as X^2 at some point chosen by a probability kernel which takes "Markovian information of X^1 until its death" into account, then behaves like X^2 until it dies, and so on. Having this general concept of concatenation at our disposal, we now take independent copies of one basis process X^0 which dies "conveniently", and revive them with appropriate kernels in order to introduce the required jumps.

These techniques enable us to implement the boundary conditions at one vertex point. As all effects, except for the "large" jumps, appear locally at the vertices of the graph, the construction on the whole graph can be achieved by building the Brownian motion locally on star graphs (that is, on simple graphs with only one vertex), and then "glue" them together. This idea is already mentioned by Itô and McKean in their fundamental paper [IM63] on Brownian motions on the half line $\mathbb{R}_+$: They suggest that, in order to solve the problem of constructing all Brownian motions on the interval $[0,1]$, one should choose two independent Brownian motions on the half lines $[0,\infty)$ and $(-\infty,1]$ that implement the correct boundary conditions on 0, 1 respectively, and then "switch" from one process to another whenever they hit their corresponding boundary 1 or 0. Itô and McKean leave the details and verification of the construction to the "industrious reader", and it seems that, while this idea is mentioned in several works, up to now only Kostrykin, Potthoff and Schrader have taken up this task in the continuous case with rather elaborate computations in [KPS10] and [KPS12a]. Therefore, one of our main goals is the establishment of a rigorous method for connecting the subprocesses on various subgraphs to a complete process on the whole graph, which maintains both the (strong) Markov property and the boundary conditions induced by the subprocesses. We will solve this problem with the help of the technique which is already employed for the implementation of jumps: We build up Brownian motions on star graphs, which realize the correct "local" boundary conditions at the corresponding vertex, with techniques similar to the half-line case as explained above. Then, whenever one of the partial processes exits a specific neighborhood of "its" vertex, it is killed and revived at its exit point on another star graph as the corresponding subprocess on this new subgraph. After all partial subgraphs have been glued together, it is still necessary to implement the "global" jumps to edges not incident with the originating vertex, as these jump destinations did not exist when the subprocesses on the star graphs were originally constructed.

Technical Difficulties

In order to guarantee that the processes constructed with the above techniques are strongly Markovian and to analyze their generators and resolvents via probabilistic methods, we may only apply transformations that preserve the strong Markov property. For classical procedures such as killing or time-changing, standard conditions which assert it are well known. However, we need to ensure that the above-mentioned, essential technique of “killing and reviving” also respects the Markov property. To this end, a detailed analysis of the notion of concatenation will be necessary. While the Markov property of concatenated processes seems to be considered as easily provable and thus negligible by some authors such as Itô and McKean [IM63] or Knight [Kni81], the present author will approach this result rather doubtfully, particularly with regard to the extensive computations which were already needed by Kostrykin, Potthoff and Schrader in the continuous case, and therefore is going to present a complete and rigorous treatment.

Due to the graph’s combinatorial structure and the admission of path discontinuities, the semigroups and resolvents of the Brownian motions are typically only known implicitly. Thus, and as we are concerned with pathwise constructions, Dynkin’s formulas [Dyn65] will be an essential tool for the classification of the generators. However, this approach merely gives necessary conditions on their domains (cf. theorem (3.17)): While a full description of the generator is achieved in the continuous setting by Kostrykin, Potthoff and Schrader in [KPS12a, Section 3], their technique is not easily applicable in the discontinuous case. Here, we are only able to achieve knowledge of the complete domain for Brownian motions on half lines, intervals or star graphs. In the general case, we will have to work mostly with “incomplete” information about intermediate processes and their generators. Therefore, we will need to retrace the paths after every step in the process’ construction and try to find “processable” invariants of the rather unwieldy formulas for its boundary data (cf. theorem (20.16)) in order to obtain sufficient information about the final process. Furthermore, due to the possible jumps needed for the non-local boundary conditions of the generator, any path analysis will be much more involved than for the classical continuous Brownian motions with vanishing jump measures.

The pathwise solution for infinite jump measures will pose a completely different challenge. In this case, just as in the context of a general Lévy process, the resulting process needs to feature infinitely many “small” jumps in arbitrarily small time intervals, so the jumps will not be arrangeable in time and the process cannot be constructed by the concatenation of a countable product of independent subprocesses. Here, we will employ a local, “bare hand” construction, utilizing the ingenious ideas of Itô and McKean which are described in subsections 16.4 and 21.2. Again, the proof of the (strong) Markov property of the resulting process will be highly non-trivial, and we will only succeed by utilizing Galmarino’s results [Gal63] on the characterization of stopped σ-algebras.

Overview

Chapter I summarizes the fundamental results on Markov processes which will be needed in our work. Its functional-analytic basis, namely the semigroup theory on Banach spaces, will be introduced together with its probabilistic counterpart of (strong) Markov processes, full of seemingly cumbersome but necessary structures. Two special types of Markov processes are treated, at the two extremes of the spectrum: Right processes form one of the most general classes and provide a suitable context for path transformations. Conversely, Feller processes have the easiest structural properties and are well suited for the analysis of processes, such as Brownian motions, via their generators. The chapter ends with a short summary concerning Lévy processes and their associated Poisson point processes, which will be used for the implementation of local jumps. Of course, the reader may skip this chapter entirely and only consult its content when needed.

The first main part of this work, chapter II, is a compilation of all Markovian path transformations needed for our construction later. The "classical" transformations, such as mapping of a process to its path space version, stopping, time change and curtailment of lifetime (also known as "killing") are only briefly recalled, as we can employ the well-known results given in the existing literature. The concatenation of processes with disjoint state spaces is then completely discussed, followed by a collection of results on state space transformations. These two types of transformations form the basis for the main vehicle of our upcoming constructions, namely the technique of concatenation of independent identical copies of one underlying process or of alternating copies of two processes, which is treated in the last part of this chapter.

We are then ready to turn to Brownian motions on metric graphs in chapter III: After some basic properties of the "standard" one-dimensional Brownian motion have been collected, we explain Itô–McKean's construction of all Brownian motions on the half line in order to put the reader in the position to understand our generalization to the star graph. To give an insight into the problems that arise in the general graph case, we will briefly consider the easiest graph with two vertices, namely the interval case. The rigorous definitions of metric graphs and Brownian motion thereon mark the start of the second main part of the thesis: After having examined the basic properties of Brownian motions on metric graphs and classified their generators, we construct all Brownian motions on a star graph "bare-handedly" by extending Itô–McKean's ideas of the half-line case. Then, by using the techniques developed in chapter II, we complete the construction by gluing the partial graphs together, implementing the missing jumps, and verifying the boundary conditions of the resulting process.

Open Problems

As already mentioned, while we succeed in constructing Brownian motions for any admissible set of boundary conditions on a metric graph, the question on the completeness of the generator's domain remains unsolved. That is, we are not able to prove that the relation (I.2) holds true identically in the general case of a metric graph. As the equality has been proved for the continuous case in [KPS12a, Lemma 3.3], and in the non-local setting for intervals and star graphs in theorem (17.2) and lemma (20.25), we expect it to hold in general as well. A potential approach might employ a combination of the methods of [KPS12a, Section 3], [IM63, Section 16] and subsection 17.2.

Index of Notation

Sectioning and Referencing

The symbol ■ marks the end of remarks and examples, proofs are finished by □. The logical numbering and referencing proceeds as follows: Sections are numbered independently of their chapter, and the numbering of references is only based on the section they are located in, neither on the subsection nor their context. This means that a theorem (2.13) in section 2 may include a referenced equation (2.14), and is then followed by a lemma (2.15). This way, we try to avoid references like (II.5.3.1) and (hopefully!) simplify the look-up of references. Chapters and subsections are merely used for a contextual distinction.

Standard Results

The Borel σ-algebra $\mathscr{B}(E)$ on some topological space E is defined to be the smallest σ-algebra on E which contains all the open sets of E. If the topology of E is induced by a metric, $\mathscr{B}(E)$ is also generated by the set $b\mathcal{C}(E)$ of bounded, continuous functions on E (see [Par67, Theorem I.1.7]), and thus, also by the set $b\mathcal{C}_d(E)$ of bounded, uniformly continuous functions on E (see [Sha88, Proposition (A2.1)]). Most of the theory will be built up for Radon spaces, sometimes we will restrict ourselves to Lusin spaces, LCCB (locally compact spaces with countable base) or Polish spaces. A quick summary concerning these spaces can be found at the beginning of [Sha88, Appendix A2].

We will frequently use the two fundamental limit theorems of Lebesgue integration theory (see, e.g., [Kal02, Theorems 1.19, 1.21]): Lebesgue's dominated convergence theorem will be named *LDCT*, Levi's monotone convergence theorem is abbreviated by *LMCT*. Basic properties of conditional expectation will be used without special mention (see, e.g., [RW00a, Section II.41] for a short summary), LDCT and LMCT will be named *cLDCT* and *cLMCT* in the context of conditional expectation.

Any form of the monotone class theorem (see, e.g., [Sha88, Appendix A0] and [BG69, Section 0.2] for a collection of results) will be cited as *MCT*, with a *MVS* $\mathscr{H}$ (signifying a monotone vector space) being a vector space of bounded, real functions on some set, such that $\mathscr{H}$ contains the constant functions and is closed under monotone convergence:

$$\forall (f_n, n \in \mathbb{N}) \subseteq \mathscr{H}, 0 \leq f_1 \leq \cdots \leq f_n \uparrow f, f \text{ bounded} \quad \Rightarrow \quad f \in \mathscr{H}.$$

Frequently Used Notations

Before we introduce the notations which will frequently be used in this work, we already would like to apologize to the reader by citing [RW00a, p. 132]: *"There are never* *enough symbols to round in mathematics. When we combine different ideas, we often find conflict of commonly used notations."* We tried to retain notations and symbols that are commonly used, as long as they are not already reserved in another context.

Additionally, we noticed that there is never enough space in the super- and subscripts of a symbol. The basic parameter of a structure, such as the time parameter t of a stochastic process $(X_t, t \geq 0)$, will always be in the subscript. If necessary, we will distinct different entities of a structure via the superscript, for instance we use $(X_t^i, t \geq 0)$, i in some index-set I, for different stochastic processes X^i, $i \in I$.

Standard Abbreviations

càdlàg	continue à gauche, limite à droite (right continuous with left limits)
HD2	hypothèse droite 2, **27**
LCCB	locally compact space with countable base, **xxi**
LDCT	Lebesgue's dominated convergence theorem, **xxi**
LMCT	Levi's monotone convergence theorem, **xxi**
MCT	monotone class theorem, **xxi**
MVS	monotone vector space, **xxi**

Basic Notations

$B_r(x)$	open ball with radius $r > 0$ and center x
$\mathscr{B}(E)$	Borel σ-algebra on E, **xxi**
$\mathcal{C}_0(E)$	continuous functions on E, vanishing at infinity
$\mathcal{C}_d(E)$	uniformly continuous functions on E
d	metric inducing the topology of E
E	state space of a stochastic process
$\mathscr{E}$	σ-algebra on E, typically: Borel σ-algebra
$\mathscr{E}^u$	universally measurable sets on E, **14**
$b\mathscr{E}$	bounded, $\mathscr{E}$-measurable functions
$p\mathscr{E}$	non-negative, $\mathscr{E}$-measurable functions
ε	Dirac measure
λ	Lebesgue measure
$\mathbb{N}$	natural numbers
$\mathbb{Q}$, $\mathbb{Q}_+$	rational numbers, non-negative rational numbers
$\mathbb{R}$, $\mathbb{R}_+$	real numbers, non-negative real numbers
$\sigma(\mathscr{H})$	smallest σ-algebra containing $\mathscr{H}$
$a \wedge b$	minimum of a, b
$a \vee b$	maximum of a, b
$\complement A$, $A^\complement$	complement of a set A

Semigroup Theory

A, $\mathscr{D}(A)$	generator, and its domain, **3**
$(T_t, t \geq 0)$	semigroup, **2**
$(U_\alpha, \alpha > 0)$	resolvent, **3**

$\mathbb{X}$, $\mathbb{Y}$	Banach space, **1**
$\mathbb{X}'$, $\mathbb{Y}'$	dual space of $\mathbb{X}$, $\mathbb{Y}$, **1**

Theory of Markov Processes

$(\alpha_t, t \geq 0)$	stopping operators, **25**
B	one-dimensional Brownian motion, **87**
$c_1, \ldots, c_4$	Feller–Wentzell boundary conditions of a Brownian motion, **142**
Δ	cemetery point, **28**
$\mathbb{E}$, $\mathbb{E}_x^i$	(conditional) expectation relative to $\mathbb{P}$, $\mathbb{P}_x^i$
Φ	mapping operator, typically: canonical coordinate process mapping, **49**
$\mathscr{F}$	σ-algebra, typically: generated by a stochastic process, **15**
$(\mathscr{F}_t, t \geq 0)$	natural filtration, **15**
$(\mathscr{F}_t^0, t \geq 0)$	filtration generated by a stochastic process, **14**
$\mathscr{G}$	σ-algebra
$(\mathscr{G}_t, t \geq 0)$	filtration
$(\gamma_t, t \geq 0)$	translation operators, **43**
Γ	centering operator, **43**
H	first entry time, **20**
ι	reflection operator, **46**
K	kernel, typically: transfer kernel, **10**, **56**
L	local time, **90**
N	Poisson random measure, **35**
$\mathscr{N}$	null sets
$p_1, \ldots, p_4$	boundary conditions of a Brownian motion, **143**
$\mathbb{P}$, $\mathbb{P}_x^i$	probability measure, usually: initial measure of a Markov process, **12**
P, Q	subordinator, **40**
ψ	transformation mapping, **74**
T	terminal time, **19**
$(\Theta_t, t \geq 0)$	shift operators, **12**
W	Walsh process, **126**
X	stochastic process, typically: Markov process, **12**
Y	stochastic process, typically: coordinate process, **49**
ζ	lifetime, **29**
Ω	sample space

Metric Graphs

$\mathcal{E}$	set of external edges, **112**
$\mathcal{G}$	metric graph and its geometric representation, **112**, **116**
$\mathcal{I}$	set of internal edges, **112**
$\mathcal{L}$	set of all edges, **112**
$\mathcal{V}$	set of vertices, **112**
ρ	length of the edges, **112**
∂	endpoint(s) of the edges, **112**

Chapter I.

Markov Processes

This chapter gives a summary on all of the basic results on Markov processes and their related fields which we will base our work on. Most of the results given here are well-known (although sometimes a little hard to find in the literature), so we recommend any reader familiar with the topics below to omit the respective sections.

Fundamental results on semigroups and Markov processes, as well as their connections, are collected in sections 1 and 2. The probabilistic basis for the study of Markov processes, namely the strong Markov property, together with Dynkin's formulas and Galmarino's theorem, which will be crucial for our computations later, is given in section 3. Then various types of Markov processes are introduced: Right processes, which constitute the most general class and provide the suitable context for process transformations, are recalled in section 4, followed by some classical results on Feller processes and Lévy processes (with some non-standard, but easy extensions) in sections 5 and 6.

1. Semigroups, Generators and Resolvents

In this section, we give a brief reminder on the theory of semigroups on Banach spaces. This is not supposed to be a complete treatment of the theory (e.g., the Hille–Yosida theory is missing entirely), as we only collect basic results that will be used later on, together some more detailed coverage where needed.

1.1. Banach Spaces

We start with the fundamental definitions:

(1.1) Definition. *The pair $(\mathbb{X}, \|\cdot\|)$ is a Banach space, if $\mathbb{X}$ is a vector space, $\|\cdot\|$ is a norm on $\mathbb{X}$, and $\mathbb{X}$ is complete with respect to the metric induced by $\|\cdot\|$.*

(1.2) Definition. *The* dual space *of a Banach space $\mathbb{X}$ is*

$$\mathbb{X}' := \{x' \colon \mathbb{X} \to \mathbb{R} \mid x' \text{ is linear and bounded}\}.$$

The first basic result on dual spaces is found, e.g., in [Yos78, Theorem IV.7.1]):

(1.3) Lemma. *Let $\mathbb{X}$ be a Banach space. The dual space $\mathbb{X}'$ of $\mathbb{X}$ equipped with the operator norm $\|x'\| := \sup_{x\in\mathbb{X}, \|x\|\le 1} |x'(x)|$, $x' \in \mathbb{X}'$, is a Banach space.*

Throughout this section we assume that $\mathbb{Y}$ is a Banach space with dual space $\mathbb{Y}'$ and $\mathbb{X}$ is a subspace of $\mathbb{Y}'$, equipped with the operator norm. In this context we have two types of convergence on $\mathbb{X}$ at our disposal:

(1.4) Definition. *Let $(x_n, n \in \mathbb{N})$ be a sequence in $\mathbb{X}$ and $x \in \mathbb{X}$.*

(i) $(x_n, n \in \mathbb{N})$ converges (strongly) to x (notation: $\lim_n x_n = x$), if $\lim_n \|x_n - x\| = 0$.

(ii) $(x_n, n \in \mathbb{N})$ converges weakly to x (notation: $\mathrm{w}^*\lim_n x_n = x$), if for all $y \in \mathbb{Y}$, $\lim_n \|x_n(y) - x(y)\| = 0$.*

Clearly, strong convergence implies weak convergence.

(1.5) Example. As discussed in [Dyn65, Section 2.4], our situation will typically be as follows: Let E be a topological space with the Borel σ-algebra $\mathscr{E} = \mathscr{B}(E)$, and $\mathbb{Y} = \nu\mathscr{E}$ be the space of all finite measures on $\mathscr{E}$, endowed with the norm of total variation. We consider the space $\mathbb{X} = b\mathscr{E}$ of all bounded, $\mathscr{E}$-measurable functions or some subspace $\mathbb{X} \subseteq b\mathcal{C}(E)$ of all bounded, continuous (thus $\mathscr{E}$-measurable) functions. Then $\mathbb{X}$ is isomorphic to a subspace of $\mathbb{Y}' = \nu\mathscr{E}'$, because for all $f \in \mathbb{X}$, the functional

$$l_f \colon \nu\mathscr{E} \to \mathbb{R},\ \mu \mapsto l_f(\mu) := \int f \, d\mu$$

defines a linear functional on $\mathbb{Y}$ with $\|l_f\| = \|f\|$. Thus, $\mathbb{X}$ is isometrically embedded in $\nu\mathscr{E}'$. The strong convergence of $\mathbb{X}$ in the subspace topology coincides with the uniform convergence of bounded (continuous) functions, whereas $\mathrm{w}^*\lim_n x_n = x$ if and only if $\lim_n x_n(e) = x(e)$ for all $e \in E$ and the sequence $(\|x_n\|, n \in \mathbb{N})$ is bounded. ■

1.2. Definitions and Basic Results

(1.6) Definition. *Let $(T_t, t \geq 0)$ be a family of bounded linear operators on $\mathbb{X}$. $(T_t, t \geq 0)$ is a semigroup, if it possesses the semigroup property*

$$\forall s, t \geq 0: \quad T_{t+s} = T_t \circ T_s.$$

If $T_0 = \mathrm{id}$, the semigroup is normal. $(T_t, t \geq 0)$ is uniformly continuous, if

$$\lim_{t \downarrow 0} \|T_t - \mathrm{id}\| = 0,$$

strongly continuous, if

$$\forall x \in \mathbb{X}: \quad \lim_{t \downarrow 0} \|T_t x - x\| = 0,$$

and a contraction semigroup, if $\|T_t\| \leq 1$ for all $t \geq 0$.

For a given semigroup $(T_t, t \geq 0)$ on $\mathbb{X}$, denote the strong and weak continuity set by

$$\mathbb{X}_0^s := \{x \in \mathbb{X} : \lim_{t \downarrow 0} T_t x = x\},$$

$$\mathbb{X}_0^w := \{x \in \mathbb{X} : \mathrm{w}^*\lim_{t \downarrow 0} T_t x = x\}.$$

Due to the semigroup property, the mapping $t \mapsto T_t x$ is strongly continuous for every $x \in \mathbb{X}_0^s$ and weakly* continuous for every $x \in \mathbb{X}_0^w$, see e.g. [Dyn65, 1.3.A].

For the rest of this section we assume that we are given a normal contraction semigroup $(T_t, t \geq 0)$ on $\mathbb{X}$.

(1.7) Definition. *The strong generator A^s and the weak generator A^w of $(T_t, t \geq 0)$ are*

$$A^s\colon \mathscr{D}(A^s) \to \mathbb{X}, \qquad x \mapsto A^s x := \lim_{t \downarrow 0} \frac{T_t x - x}{t},$$
$$A^w\colon \mathscr{D}(A^w) \to \mathbb{X}, \qquad x \mapsto A^w x := \mathrm{w}^*\lim_{t \downarrow 0} \frac{T_t x - x}{t},$$

with the domains $\mathscr{D}(A^s)$, $\mathscr{D}(A^w)$ being the sets of all $x \in \mathbb{X}$ for which the right-hand limits exist.

An immediate consequence of the definitions is that $\mathscr{D}(A^s) \subseteq \mathscr{D}(A^w) \subseteq \mathbb{X}_0^w$ and $\mathbb{X}_0^s \subseteq \mathbb{X}_0^w$ hold true. As most of the basic properties hold in the strong as well as in the weak context, we will only cite them in the latter case.

(1.8) Definition. *The resolvent $(U_\alpha, \alpha > 0)$ of $(T_t, t \geq 0)$ is a family of mappings $(U_\alpha, \alpha > 0)$, defined by*

$$U_\alpha\colon \mathbb{X}_0^w \to \mathbb{X}, \quad x \mapsto \int_0^\infty e^{-\alpha t}\, T_t x\, dt.$$

The resolvent $(U_\alpha, \alpha > 0)$ is just the Laplace transform of the semigroup $(T_t, t \geq 0)$. The fundamental connection of the resolvent to the generator A is given now (see, e.g., [Dyn65, Theorems 1.1, 1.7]):

(1.9) Theorem. *For every $\alpha > 0$, the mappings*

$$\alpha - A^s\colon \mathscr{D}(A^s) \to \mathbb{X}_0^s \quad \text{and} \quad \alpha - A^w\colon \mathscr{D}(A^w) \to \mathbb{X}_0^w$$

are bijective with inverse U_α, defined on the respective space $\mathbb{X}_0^s$, $\mathbb{X}_0^w$.

The resolvent possesses many useful properties. The most common ones, which will be used quite frequently, are summarized next:

(1.10) Corollary. *Let $(U_\alpha, \alpha > 0)$ be the resolvent of $(T_t, t \geq 0)$. Then*

(i) the resolvent equation holds true:

$$\forall x \in \mathbb{X}_0^w, 0 < \alpha \leq \beta: \quad U_\alpha x = U_\beta x + (\beta - \alpha) U_\alpha U_\beta x;$$

(ii) the resolvent family is commutative, that is,

$$\forall \alpha, \beta > 0: \quad U_\alpha\, U_\beta = U_\beta\, U_\alpha;$$

(iii) the range of U_α is independent of $\alpha > 0$, that is,

$$\forall \alpha, \beta > 0 : \quad U_\alpha(\mathbb{X}_0^w) = U_\beta(\mathbb{X}_0^w).$$

Here, (i) can be shown with theorem (1.9), see e.g. [Yos78, Theorem VIII.2.2], and (i) then implies the properties (ii) and (iii).

1.3. Uniqueness and Existence

In all that follows, let $(T_t, t \geq 0)$ be a contraction semigroup on $\mathbb{X}$ with weak generator A and weak continuity set $\mathbb{X}_0$.

As the Laplace transform uniquely determines any right continuous function (see, e.g., [Dyn65, Lemma 1.1, Theorem 1.2]), the following uniqueness theorem is immediate:

(1.11) Theorem. *The semigroup $(T_t, t \geq 0)$ is uniquely determined on $\mathbb{X}_0$ by its resolvent $(U_\alpha, \alpha > 0)$ or by its generator A.*

The next lemma gives a sufficient condition for the domain of a generator to be maximal. It is a slight generalization of [Dyn65, Corollary of Theorem 1.1].

(1.12) Lemma. *Let $(T_t, t \geq 0)$ be a semigroup on $\mathbb{X}$ with generator A and resolvent $(U_\alpha, \alpha > 0)$, as well as continuity set $\mathbb{X}_0$. Let the linear operator $\big(A^\bullet, \mathscr{D}(A^\bullet)\big)$ on $\mathbb{X}$ be an extension of A, and let $\mathscr{D} \subseteq \mathbb{X}$ be a linear subspace, satisfying*

(i) $\mathscr{D}(A) \subseteq \mathscr{D} \subseteq \mathbb{X}_0$,

(ii) $\mathscr{D} \subseteq \mathscr{D}(A^\bullet)$ and $A^\bullet(\mathscr{D}) \subseteq \mathbb{X}_0$, and

(iii) there is an $\alpha > 0$ such that the following implication holds true:

$$A^\bullet u = \alpha u, \ u \in \mathscr{D} \quad \Rightarrow \quad u = 0.$$

Then $\mathscr{D}(A) = \mathscr{D}$.

Proof. By theorem (1.9), the mapping

$$U_\alpha = (\alpha - A)^{-1} \colon \ \mathbb{X}_0 \to \mathscr{D}(A)$$

is a bijection for any $\alpha > 0$. As $A^\bullet$ is an extension of A, we have

$$(\alpha - A^\bullet)\, U_\alpha = (\alpha - A)\, U_\alpha = \mathrm{id} \quad \text{on } \mathbb{X}_0.$$

Now let $u \in \mathscr{D}$, and set for every $\alpha > 0$

$$u_\alpha^\bullet := U_\alpha\, (\alpha - A^\bullet)\, u - u.$$

Then $u_\alpha^\bullet \in \mathscr{D}$ and

$$\begin{aligned} A^\bullet u_\alpha^\bullet &= \alpha\, U_\alpha\,(\alpha - A^\bullet)\, u - (\alpha - A^\bullet)\, U_\alpha\,(\alpha - A^\bullet)\, u - A^\bullet u \\ &= \alpha\big(U_\alpha\,(\alpha - A^\bullet)\, u - u\big) \\ &= \alpha\, u_\alpha^\bullet. \end{aligned}$$

But then condition (iii) implies $u_\alpha^\bullet = 0$, that is,

$$u = U_\alpha\,(\alpha - A^\bullet)\, u \in \mathscr{D}(A). \qquad \square$$

1.4. On the Laplace Transform

We give an inversion formula for the Laplace transform which will be very useful later:

(1.13) Theorem. *Let $g\colon \mathbb{R}_+ \to \mathbb{R}$ be bounded and right continuous, and*

$$\varphi(\alpha) := \int_0^\infty e^{-\alpha t}\, g(t)\, dt, \quad \alpha > 0,$$

be the Laplace transform of g. Then, for every $t > 0$,

$$g(t) = \lim_{\varepsilon \downarrow\downarrow 0} \lim_{\alpha\to\infty} \frac{1}{\varepsilon} \sum_{\alpha t < k \le (\alpha+\varepsilon)t} \frac{(-1)^k}{k!}\, \alpha^k\, \varphi^{(k)}(\alpha).$$

This formula is given in [Sha88, Formula (4.14)] with a reference to [Fel71, p. 232, (6.4)]. However, Feller considers Laplace transforms of probability measures, so we are only able to apply his results if g is integrable. In the general case, the justification of the interchange of limits and integration, which is essential in Feller's proof, is much harder. Therefore, we first need to prepare for our proof of the above theorem.

(1.14) Lemma. *For every $\alpha > 0$, $x > 0$, $t > 0$, consider*

$$\begin{aligned} \psi_t^x(\alpha) &:= e^{-\alpha t} \sum_{k \le \alpha x} \frac{(\alpha t)^k}{k!}, \\ \Psi_t^x(\alpha) &:= e^{-\alpha(t-x)} \left(\frac{2t}{x}\right)^{\alpha x}, \end{aligned}$$

with $\sum_{k\le\alpha x}$ denoting $\sum_{k=0}^{\lfloor \alpha x\rfloor}$. Then

(i) for all $\alpha > 0$, $x > 0$, $t > 0$, it is $\psi_t^x(\alpha) = \mathbb{P}(X \le \alpha x)$, with X being a Poisson-distributed random variable with mean αt; so especially $\psi_t^x(\alpha) \in [0,1]$;

(ii) for all $x > 0$, $t > 0$, it holds that $\lim_{\alpha\to\infty} \psi_t^x(\alpha) = \begin{cases} 1, & t < x, \\ 0, & t > x; \end{cases}$

(iii) for all $x > 0$, $t > 0$ with $\alpha t > \lfloor \alpha x \rfloor \ge 1$, it is $\psi_t^x(\alpha) \le \Psi_t^x(\alpha)$;

(iv) for all $x > 0$, $\alpha > 0$, the function $t \mapsto \Psi_t^x(\alpha)$ is integrable on $[0, +\infty)$;

(v) for all $x > 0$, there exists $T_x > 0$ such that for all $t > T_x$, the function $\alpha \mapsto \Psi_t^x(\alpha)$ is decreasing.

Combination of these properties yields that for every $x > 0$, there exist $T_x > 0$ and $A_x > 0$ such that for all $\alpha > A_x$,

$$\psi_t^x(\alpha) \leq \mathbb{1}_{[0,T_x]}(t) + \Psi_t^x(A_x)\,\mathbb{1}_{(T_x,\infty)}(t) =: h(t) \tag{1.15}$$

holds true, and the dominating function h on the right-hand side is integrable on $[0, +\infty)$.

Proof. (i) This is clear.

(ii) We use (i) together with Chebyshev's inequality: As $\mathbb{E}(X) = \operatorname{Var}(X) = \alpha t$, we conclude for $t < x$ that

$$\begin{aligned} \mathbb{P}(X \leq \alpha x) &= \mathbb{P}\big(X - \mathbb{E}(X) \leq \alpha(x-t)\big) \\ &\geq 1 - \mathbb{P}\big(|X - \mathbb{E}(X)| > \alpha(x-t)\big) \\ &\geq 1 - \frac{\alpha t}{(\alpha(x-t))^2}, \end{aligned}$$

while for $t > x$, we have

$$\begin{aligned} \mathbb{P}(X \leq \alpha x) &= \mathbb{P}\big(-(X - \mathbb{E}(X)) \geq \alpha(t-x)\big) \\ &\leq \mathbb{P}\big(|X - \mathbb{E}(X)| \geq \alpha(t-x)\big) \\ &\leq \frac{\alpha t}{(\alpha(t-x))^2}. \end{aligned}$$

Letting $\alpha \to \infty$ yields the result.

(iii) We will employ the following bounds for the tail probabilities of the Poisson distribution: For any Poisson-distributed random variable X with mean μ, it is

$$\forall x < \mu : \quad \mathbb{P}(X \leq x) \leq \frac{e^{-\mu}\,(e\,\mu)^x}{x^x}.$$

This inequality can be found, e.g., in [MU05, p. 97, Theorem 5.4], it follows immediately from $\mathbb{P}(X \leq x) \leq \mathbb{P}(e^{tX} \geq e^{tx})$ with $t := \ln(x/\mu) < 0$, by using Chebyshev's inequality and the well-known formula of the Laplace transform of the Poisson distribution.

Using this for the first inequality, $\alpha t > \lfloor \alpha x \rfloor$ for the second inequality, as well as that $\lfloor \alpha x \rfloor \geq 1$ ensures $\lfloor \alpha x \rfloor \geq \frac{\alpha x}{2}$ for the last step, gives

$$\begin{aligned} \mathbb{P}(X \leq \alpha x) &\leq \frac{e^{-\alpha t}(e\,\alpha t)^{\lfloor \alpha x \rfloor}}{\lfloor \alpha x \rfloor^{\lfloor \alpha x \rfloor}} \\ &\leq e^{-\alpha(t-x)} \Big(\frac{\alpha t}{\lfloor \alpha x \rfloor}\Big)^{\alpha x} \\ &\leq e^{-\alpha(t-x)} \Big(\frac{2t}{x}\Big)^{\alpha x}. \end{aligned}$$

(iv) For all $x > 0$, $\alpha > 0$, we have

$$\int_0^\infty \Psi_t^x(\alpha)\,dt = \frac{1}{\alpha}\Big(\frac{e}{\alpha x}\Big)^{\alpha x} \int_0^\infty e^{-s} s^{\alpha x}\,ds < +\infty.$$

(v) The function $\alpha \mapsto \Psi_t^x(\alpha)$ is obviously differentiable on $(0,\infty)$ with derivative

$$\begin{aligned}(\Psi_t^x)'(\alpha) &= -(t-x)\,\Psi_t^x(\alpha) + e^{-\alpha(t-x)}\,x\ln\big(\frac{2t}{x}\big)\Big(\frac{2t}{x}\Big)^{\alpha x}\\ &= \Psi_t^x(\alpha)\Big(x\ln\big(\frac{2t}{x}\big) - (t-x)\Big).\end{aligned}$$

Because $\Psi_t^x(\alpha) > 0$ holds for every choice of parameters $t > 0$, $x > 0$, $\alpha > 0$, and the term in the parentheses above is independent of α and tends to $-\infty$ with $t \to +\infty$, there exists $T_x > 0$ such that for all $t > T_x$, $\Psi_t^{x\prime}(\alpha) < 0$ holds true.

□

We are now ready to prove the inversion formula for the Laplace transform:

Proof of theorem (1.13). The right continuous function g is Borel measurable. This and the boundedness of g guarantee the existence and finiteness of the function φ on $(0,\infty)$. Indeed, by using LDCT together with the mean value theorem (see, e.g., [BB01, Lemma 16.2]), it is obvious that φ is infinitely differentiable on $(0,\infty)$, and that for every $k \in \mathbb{N}_0$, the derivatives read

$$\varphi^{(k)}(\alpha) = \int_0^\infty e^{-\alpha t}\,(-t)^k\,g(t)\,dt, \quad \alpha > 0.$$

Fix $x > 0$, and consider

$$\sum_{k\le\alpha x} \frac{(-1)^k}{k!}\,\alpha^k\,\varphi^{(k)}(\alpha) = \int_0^\infty e^{-\alpha t} \sum_{k\le\alpha x} \frac{(\alpha t)^k}{k!}\,g(t)\,dt.$$

Then letting $\alpha \to \infty$ and interchanging the limit and integration with LDCT, using $h\,\|g\|$ with h as given in equation (1.15) as integrable majorant, and taking (ii) of lemma (1.14) into account, yield

$$\lim_{\alpha\to\infty} \sum_{k\le\alpha x} \frac{(-1)^k}{k!}\,\alpha^k\,\varphi^{(k)}(\alpha) = \int_0^x g(t)\,dt =: G(x).$$

By the right continuity of g, for any $x > 0$, $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall h \in [0,\delta): \quad \big|g(x+h) - g(x)\big| < \varepsilon.$$

But then, for all $h \in [0,\delta)$, we have

$$\left|\frac{G(x+h)-G(x)}{h} - g(x)\right| = \left|\frac{1}{h}\int_x^{x+h} g(t)\,dt - g(x)\right| < \varepsilon,$$

so we have shown that

$$\lim_{\varepsilon\downarrow 0}\frac{G(x+\varepsilon)-G(x)}{\varepsilon}=g(x).$$

This completes the proof, as

$$\begin{aligned}G(x+\varepsilon)-G(x)&=\lim_{\alpha\to\infty}\Big(\sum_{k\le\alpha(x+\varepsilon)}\frac{(-1)^k}{k!}\,\alpha^k\,\varphi^{(k)}(\alpha)-\sum_{k\le\alpha x}\frac{(-1)^k}{k!}\,\alpha^k\,\varphi^{(k)}(\alpha)\Big)\\&=\lim_{\alpha\to\infty}\sum_{\alpha x<k\le\alpha(x+\varepsilon)}\frac{(-1)^k}{k!}\,\alpha^k\,\varphi^{(k)}(\alpha).\end{aligned}$$ □

Later, we will cope with measures μ on $(0,+\infty)$ that might be infinite, but fulfill $\int(1\wedge x)\,\mu(dx)<+\infty$. We are going to analyze them with the help of a modified Laplace transform. We prepare some basic results on such measures first:

(1.16) Lemma.

(i) For all $\alpha\ge 1$, $x\in\mathbb{R}_+$,

$$e^{-\alpha}(1\wedge x)\le 1-e^{-\alpha x}\le\alpha(1\wedge x).$$

(ii) For all $\alpha>0$, $x\in\mathbb{R}_+$,

$$1-e^{-\alpha x}\le(1\vee\alpha)\,(1\wedge x).$$

(iii) For all $\alpha>0$, $x\in\mathbb{R}_+$,

$$1-e^{-\alpha x}\le 1\wedge\alpha x.$$

Proof. (i) For $x=0$, this is trivial. For $x\ge 1$, we have

$$e^{-\alpha}(1\wedge x)=e^{-\alpha}\le 1-e^{-\alpha x},$$

as $e^{-\alpha}\big(1+e^{-\alpha(x-1)}\big)\le e^{-1}\cdot 2\le 1$, and the inequality

$$1-e^{-\alpha x}\le\alpha=\alpha(1\wedge x)$$

is obvious. For $x\in(0,1)$, consider the difference quotient

$$\frac{1-e^{-\alpha x}}{1\wedge x}=\frac{\big(1-e^{-\alpha x}\big)-0}{x-0},$$

of the function

$$f\colon[0,1]\to\mathbb{R},\ \xi\mapsto f(\xi)=1-e^{-\alpha\xi}.$$

Then the mean value theorem implies that there exists $\xi\in(0,x)\subseteq(0,1)$ with

$$\frac{1-e^{-\alpha x}}{1\wedge x}=f'(\xi)=\alpha e^{-\alpha\xi}\ \in(\alpha\,e^{-\alpha},\alpha)\subseteq(e^{-\alpha},\alpha).$$

(ii) In view of (i), it remains to check that for $\alpha \in (0,1)$, $x \geq 1$,

$$1 - e^{-\alpha x} \leq 1 = (1 \vee \alpha)\,(1 \wedge x)$$

holds, but this is trivial.

(iii) For all $\alpha > 0$, $x \in \mathbb{R}_+$, obviously $1 - e^{-\alpha x} \leq 1$ holds true. Furthermore, we have

$$1 - e^{-\alpha x} = \int_{-\alpha x}^{0} e^t \, dt \leq \alpha x,$$

as $e^t \leq 1$ for all $t \in [-\alpha x, 0]$.

□

(1.17) Corollary. *For any measure μ on $\mathbb{R}_+$,*

$$\int_0^\infty (1 \wedge x)\,\mu(dx) < +\infty, \quad \textit{if and only if} \quad \int_0^\infty (1 - e^{-x})\,\mu(dx) < +\infty.$$

(1.18) Lemma. *For any measure μ on $\mathbb{R}_+$ with $\int_0^\infty (1 \wedge x)\,\mu(dx) < +\infty$, it is*

$$\lim_{\alpha \downarrow 0} \int_0^\infty \left| e^{-\alpha x} - 1 \right| \, \mu(dx) = 0$$

and

$$\lim_{\alpha \to +\infty} \frac{1}{\alpha} \int_0^\infty \left| e^{-\alpha x} - 1 \right| \, \mu(dx) = 0.$$

Proof. For $0 < \alpha \leq 1$, by using (iii) of lemma (1.16), we obtain

$$\int_0^\infty \left| e^{-\alpha x} - 1 \right| \, \mu(dx) \leq \int_0^\infty (1 \wedge \alpha x)\,\mu(dx) \leq \int_0^\infty (1 \wedge x)\,\mu(dx) < +\infty.$$

Then LDCT yields the first claim, as for all $x \geq 0$, $1 \wedge \alpha x$ tends to 0 for $\alpha \downarrow 0$.

For $\alpha \geq 1$, property (iii) of lemma (1.16) gives

$$\frac{1}{\alpha} \int_0^\infty \left| e^{-\alpha x} - 1 \right| \, \mu(dx) \leq \int_0^\infty (\frac{1}{\alpha} \wedge x)\,\mu(dx) \leq \int_0^\infty (1 \wedge x)\,\mu(dx) < +\infty,$$

so LDCT yields the second claim as well, because for all $x \geq 0$, $\frac{1}{\alpha} \wedge x$ tends to 0 for $\alpha \to +\infty$. □

In the proof above, we have also shown the following:

(1.19) Corollary. *Let μ be a measure on $\mathbb{R}_+$ with $\int_0^\infty (1 \wedge x)\,\mu(dx) < +\infty$. Then,*

$$\forall \alpha > 0: \quad \int_0^\infty \left| e^{-\alpha x} - 1 \right| \, \mu(dx) < +\infty.$$

(1.20) Theorem. *Let μ be a measure on $(0,+\infty)$ with $\int (1\wedge x)\,\mu(dx) < +\infty$. Then*

$$\varphi(\alpha) := \int_0^\infty \left(1 - e^{-\alpha x}\right) \mu(dx)$$

is finite for all $\alpha > 0$, and the function $\varphi\colon (0,\infty) \to \mathbb{R}_+$ uniquely determines μ.

Proof. Finiteness was just observed in corollary (1.19). Let μ, ν be measures on $(0,+\infty)$ with $\int (1\wedge x)\,\mu(dx) < +\infty$, $\int (1\wedge x)\,\nu(dx) < +\infty$, satisfying

$$\forall \alpha > 0: \quad \int_0^\infty \left(1 - e^{-\alpha x}\right) \mu(dx) = \int_0^\infty \left(1 - e^{-\alpha x}\right) \nu(dx).$$

Consider the vector space $\mathcal{A}$ of $\mathcal{C}_0(\mathbb{R}_+)$-functions spanned by the functions $x \mapsto e^{-\alpha x}$, $\alpha > 0$. Then, due to the equality $e^{-\alpha x}\left(1 - e^{-x}\right) = \left(1 - e^{-(\alpha+1)x}\right) - \left(1 - e^{-\alpha x}\right)$, linearity of the integral yields

$$\forall f \in \mathcal{A}: \quad \int_0^\infty f(x)\left(1 - e^{-x}\right) \mu(dx) = \int_0^\infty f(x)\left(1 - e^{-x}\right) \nu(dx).$$

This shows that the Laplace transforms of the finite measures $(1 - e^{-x})\mu(dx)$ and $(1 - e^{-x})\nu(dx)$ coincide, and therefore these measures must be equal.[1] □

1.5. Markov Transition Semigroups

We are going to consider semigroups originating from Markov kernels:

(1.21) Definition. *Let $(E_1, \mathscr{E}_1)$ and $(E_2, \mathscr{E}_2)$ be measurable spaces. A kernel K from $(E_1, \mathscr{E}_1)$ to $(E_2, \mathscr{E}_2)$ is a function $K\colon E_1 \times \mathscr{E}_2 \to [0,\infty]$ such that*

(i) for all $A \in \mathscr{E}_2$, $x \mapsto K(x,A)$ is $\mathscr{E}_1$-measurable, and

(ii) for all $x \in E_1$, $A \mapsto K(x,A)$ is a measure on $(E_2, \mathscr{E}_2)$.

[1] The implication just mentioned is somehow not easily available in the literature, so we will quickly repeat the standard argument proving it: Obviously, $\mathcal{A}$ is an algebra that separates the points, so by the Stone–Weierstrass theorem, $\mathcal{A}$ is dense in $\mathcal{C}_0(\mathbb{R}_+)$ (w.r.t. the topology of uniform convergence). As $\left(1 - e^{-x}\right)\mu(dx)$ and $\left(1 - e^{-x}\right)\nu(dx)$ are finite measures, LDCT (using, e.g., $x \mapsto (\|f\| + 1)\left(1 - e^{-x}\right)$ as integrable majorant) shows that

$$\forall f \in \mathcal{C}_0(\mathbb{R}_+): \quad \int_0^\infty f(x)\left(1 - e^{-x}\right) \mu(dx) = \int_0^\infty f(x)\left(1 - e^{-x}\right) \nu(dx).$$

Approximation of the indicator functions of intervals by $\mathcal{C}_0(\mathbb{R}_+)$-functions (see, e.g., [IW89, Proposition I.2.2]) yields

$$\forall 0 < a < b < +\infty: \quad \int_0^\infty \mathbb{1}_{(a,b)}(x)\left(1 - e^{-x}\right) \mu(dx) = \int_0^\infty \mathbb{1}_{(a,b)}(x)\left(1 - e^{-x}\right) \nu(dx),$$

thus the measures $\left(1 - e^{-x}\right)\mu(dx)$ and $\left(1 - e^{-x}\right)\nu(dx)$ are equal (cf. [Kal02, Lemma 1.17]). As $x \mapsto 1 - e^{-x}$ is a bijective, bimeasurable map on $(0,\infty)$, μ and ν coincide as well.

A kernel K is sub-Markov if $K(x, E_2) \leq 1$ for all $x \in E_1$, and Markov if $K(x, E_2) = 1$ for all $x \in E_1$. A kernel from $(E, \mathscr{E})$ to $(E, \mathscr{E})$ is called kernel on $(E, \mathscr{E})$.

Every sub-Markov kernel K on $(E, \mathscr{E})$ gives rise to linear operators on $p\mathscr{E}$ and on $b\mathscr{E}$. Both will be named K again, that is, with a slight abuse of notation, we set

$$\forall f \in p\mathscr{E} \cup b\mathscr{E}, x \in E: \quad Kf(x) := \int f(y)\, K(x, dy).$$

Besides being linear, this operator is also positive and respects positive monotone convergence. Conversely, every mapping with these properties is induced by a kernel (see, e.g., [Sha88, Theorem (A3.3)]).

(1.22) Definition. *A family $(T_t, t \geq 0)$ of Markov kernels on $(E, \mathscr{E})$ is called a Markov transition semigroup, if the induced family of linear operators on $b\mathscr{E}$ is a semigroup, that is, if it satisfies*

$$\forall s, t \geq 0, x \in E, f \in b\mathscr{E}: \quad T_t(T_s f)(x) = T_{t+s} f(x).$$

Inserting indicator functions into the equality above, it is easily seen to be equivalent to the *Chapman-Kolmogorov equation*

$$\forall s, t \geq 0, x \in E, A \in \mathscr{E}: \quad \int T_s(y, A)\, T_t(x, dy) = T_{t+s}(x, A).$$

2. Basic Theory of Markov Processes

Given a Markov transition semigroup $(T_t, t \geq 0)$ on a space $(E, \mathscr{E})$, one can define a projective system of probability measures for every $x \in E$ by setting

$$\begin{aligned} &\mathbb{P}_x^{t_1,\dots,t_n}(A_1 \times \cdots \times A_n) \\ &:= \int_{A_1} \int_{A_2} \cdots \int_{A_n} T_{t_n - t_{n-1}}(x_{n-1}, dx_n) \cdots T_{t_2 - t_1}(x_1, dx_2)\, T_{t_1}(x, dx_1). \end{aligned} \tag{2.1}$$

for $n \in \mathbb{N}$, $0 \leq t_1 < \cdots < t_n$, $A_1, \cdots, A_n \in \mathscr{E}$. Given that the underlying space E has a sufficiently "nice" structure, the Kolmogorov extension theorem provides us with a stochastic process $(X_t, t \geq 0)$ on E and a set of probability measures $(\mathbb{P}_x, x \in E)$ on $(E, \mathscr{E})$ such that for every $x \in E$, the finite dimensional distributions of X under $\mathbb{P}_x$ coincide with the projective system above. Thus, the resulting process X admits Huygens' principle of wave propagation (see e.g. [End09]), namely, for all $n \in \mathbb{N}$, $0 \leq t_1 < t_2 < \cdots < t_n$, it holds that

$$\begin{aligned} &\mathbb{P}_x(X_{t_1} \in dx_1, X_{t_2} \in dx_2, \dots, X_{t_n} \in dx_n) \\ &= T_{t_1}(x, dx_1)\, T_{t_2 - t_1}(x_1, dx_2) \cdots T_{t_n - t_{n-1}}(x_{n-1}, dx_n). \end{aligned}$$

In particular, we have

$$\forall t \geq 0, x \in E, f \in b\mathscr{E}: \quad T_t f(x) = \mathbb{E}_x(f(X_t)),$$

that is $T_t(x, A) = \mathbb{P}_x(X_t \in A)$ for all $x \in E$, $A \in \mathscr{E}$. The measures $(\mathbb{P}_x, x \in E)$ can be seen as "starting measures" for the stochastic process X, which is apparent if the semigroup is normal in the sense that $\mathbb{P}_x(X_0 \in A) = T_0(x, A) = \varepsilon_x(A)$. Then $T_t(x, \cdot\,)$ is just the distribution of the process evolved until time t when started at x.

In fact, even more is true: "Huygen's principle" for the finite dimensional distributions gives rise to a special property of the emerging process X (see, e.g., [BB96, Theorem 42.3]): It is "memoryless", that is, it "starts anew" at every (fixed) time. This behavior will be described formally by the Markov property below.

While the possibility of studying an analytic object (at least in the classical sense) like a semigroup in probabilistic context already gives a good motivation for the definition of Markov processes, it turns out that for a profound study of this new object, its definition should be ranked among a probabilistic context more than just be derived from the analytic context of a semigroup. That is, its definition will impose regularity conditions on a stochastic process and probabilistic conditions rather than conditions on the semigroup. The former is necessary to reasonably study the process in a probabilistic context: For example, Kolmogorov's extension theorem traditionally gives a process only on the path space, and quite some work is needed to modify this process in order to obtain a right continuous version. However, right continuity (or a similar regularity hypothesis) is needed to overcome technical problems which are always encountered when working with uncountable sets of random variables. Another reason to shift into a probabilistic setting is that the semigroup of the process, while it of course always exists, is not always known explicitly or easy to study, for instance when the process in question is constructed by transformations of some other processes. Still, the construction (and analysis) of a Markov process via its semigroup is an important field in special settings and even in very general ones (e.g. in the context of Feller semigroups, see section 5, or Ray semigroups, cf. [Sha88, Section 9]), and especially needed when constructing prototypes like the Brownian motion.

2.1. Fundamental Definitions

There are indefinitely many ways to define a "Markov process", which makes it hard to compare results when switching from one part of literature to another one. We will follow the modern context of [BG69] and [Sha88].

For all that follows, let E be a Radon space and $\mathscr{E}$ be a σ-algebra over E such that all bounded, continuous functions on E are measurable with respect to $\mathscr{E}$ (this is the case if $\mathscr{E}$ contains or is equal to the Borel σ-algebra $\mathscr{B}(E)$ over E).

(2.2) Definition. $X = (\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ *is a ($\mathscr{E}$-)Markov process with state space E, if the following conditions are satisfied:*

(i) $(\Omega, \mathscr{G})$ *is a measurable space with filtration* $(\mathscr{G}_t, t \geq 0)$, $\mathbb{P}_x$ *is a probability measure on* $(\Omega, \mathscr{G})$ *for every* $x \in E$, *and* $x \mapsto \mathbb{P}_x(X_t \in B)$ *is* $\mathscr{E}$*-measurable for all* $t \geq 0$, $B \in \mathscr{E}$*;*

(ii) $(X_t, t \geq 0)$ *is an* E*-valued stochastic process and* $\mathscr{E}$*-adapted to* $(\mathscr{G}_t, t \geq 0)$, *that is,* X_t *is* $\mathscr{G}_t/\mathscr{E}$*-measurable for every* $t \geq 0$*;*

(iii) $(\Theta_t, t \geq 0)$ *are shift operators for* X, *that is,* $(\Theta_t, t \geq 0)$ *is a collection of mappings* $\Theta_t \colon \Omega \to \Omega$, $t \geq 0$, *satisfying*

$$\forall s, t \geq 0: \quad \Theta_{s+t} = \Theta_s \circ \Theta_t, \quad X_t \circ \Theta_s = X_{s+t};$$

(iv) X *has the Markov property, that is,* X *fulfills*

$$\forall s, t \geq 0, f \in b\mathscr{E}: \quad \mathbb{E}_x\big(f(X_{s+t}) \,|\, \mathscr{G}_s\big) = \mathbb{E}_{X_s}\big(f(X_t)\big);$$

(v) X *is normal, that is,* $\mathbb{P}_x(X_0 = x) = 1$ *holds true for every* $x \in E$.

The Markov process X *is called right continuous, if the stochastic process* $(X_t, t \geq 0)$ *is right continuous, that is, if*

(vi) *for every* $\omega \in \Omega$, *the path* $\mathbb{R}_+ \to E$, $t \mapsto X_t(\omega)$ *is right continuous.*

We deliberately included the normality into the definition of a Markov process, as all classes of Markov processes considered in this work will require condition (v) to hold. However, many results concerning Markov processes are still valid (possibly in a weaker form) when normality is dropped.

It is well-known (see, e.g., [BG69, Proposition I.3.5]) that every Markov process X on $(E, \mathscr{E})$ gives rise to a semigroup $(T_t, t \geq 0)$ on $\mathscr{E}$ through

$$T_t f(x) = \mathbb{E}_x\big(f(X_t)\big), \quad t \geq 0, f \in b\mathscr{E}, x \in E, \tag{2.3}$$

where the semigroup property follows directly from the Markov property. One of the most important features of a semigroup associated with a right continuous Markov process will be the following:

(2.4) Lemma. *Let* X *be a right continuous Markov process with semigroup* $(T_t, t \geq 0)$. *Then the mapping* $\mathbb{R}_+ \to \mathbb{R}, t \mapsto T_t f(x)$ *is right continuous for every* $x \in E$, $f \in b\mathcal{C}(E)$.

Proof. This follows directly from the right continuity of $t \mapsto X_t$, together with LDCT. □

The following two fundamental definitions summarize all properties of definition (2.2) and implement the connection to the associated semigroup:

(2.5) Definition. *Let* $(T_t, t \geq 0)$ *be a Markov semigroup on* $(E, \mathscr{E})$. *The tuple* $X = (\Omega, \mathscr{G}, (\mathscr{G}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (\Theta_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in E})$ *is a right continuous simple* ($\mathscr{E}$-)*Markov process with transition semigroup* $(T_t, t \geq 0)$, *if properties (i), (ii), (iii), (v) and (vi) of definition* (2.2) *are fulfilled and if*

(iv') X *has the Markov property with respect to the semigroup* $(T_t, t \geq 0)$:

$$\forall s, t \geq 0, f \in b\mathscr{E}: \quad \mathbb{E}_x\big(f(X_{s+t}) \,|\, \mathscr{G}_s\big) = T_t f(X_s).$$

(2.6) Definition. $X = (\Omega, \mathscr{G}, (\mathscr{G}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (\Theta_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in E})$ *is a right continuous simple* ($\mathscr{E}$-)*Markov process, if it is a right continuous simple* ($\mathscr{E}$-)*Markov process with transition semigroup* $(T_t, t \geq 0)$ *given by equation* (2.3).

We close this introduction by recalling that, with help of the shift operators, it is possible to lift the Markov property to general bounded functions which are measurable with respect to the σ-algebra generated by the process. The following theorem is a standard result which can be proved using MCT (see, e.g., [BG69, Theorems I.1.3, I.3.6]):

(2.7) Theorem. *Let X be a right continuous Markov process on $(E, \mathscr{E})$, and consider the σ-algebra $\mathscr{F}^0 := \sigma(X_t, t \geq 0)$ generated by X. Then for all $Y \in b\mathscr{F}^0$, the mapping $x \mapsto \mathbb{E}_x(Y)$ is $\mathscr{E}$-measurable, and for all $x \in E$, $t \geq 0$,*

$$\mathbb{E}_x(Y \circ \Theta_t \,|\, \mathscr{G}_t) = \mathbb{E}_{X_t}(Y).$$

2.2. The Usual Hypotheses

We briefly summarize the "usual hypotheses", which will be in force for the majority of our work, and the standard technique leading to them. These hypotheses will ensure the right continuity of the underlying filtration, that is

$$\forall t \geq 0: \quad \mathscr{F}_t = \cap_{s>t} \mathscr{F}_s =: \mathscr{F}_{t+},$$

as well as the measurability of most basic random times, like the first hitting times introduced in section 3. It turns out that the proper method to achieve above-mentioned features is to complete the entire σ-algebra $\sigma(\mathscr{F}_t, t \geq 0)$, and then augment the filtration by the null sets of this completed entire σ-algebra; it does not suffice to solely complete every single σ-algebra $\mathscr{F}_t$ of the filtration. Furthermore, as every Markov process has a whole set of associated measures $(\mathbb{P}_x, x \in E)$, completions and augmentations must be "universal", that is relative to this whole set. This results in the following procedure, which is completely laid out, e.g., in [BG69, Section I.5] and [Sha88, Sections 3, 6]. Before we start, we would like to remind the reader that this procedure is even necessary in the most basic cases such as in the setting of continuous stochastic processes like the Brownian motion (see, e.g., [KS91, Problem 2.7.4]).

Let E be a Radon space equipped with the Borel σ-algebra $\mathscr{E} := \mathscr{B}(E)$, and define the σ-algebra $\mathscr{E}^u$ of *universally measurable subsets* of E by

$$\mathscr{E}^u := \bigcap \{\mathscr{E}^\mu : \mu \text{ finite measure on } E\},$$

where $\mathscr{E}^\mu$ is the μ-completion of $\mathscr{E}$ (for basic results concerning universal completions, see, e.g., [Sha88, Appendices A1–A2]).

Consider an intermediate σ-algebra $\mathscr{E} \subseteq \mathscr{E}^\bullet \subseteq \mathscr{E}^u$ (typically, $\bullet = u$ or $\bullet = 0$, with the latter case being $\mathscr{E}^0 := \mathscr{E}$). Let $X = \big(\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E}\big)$ be a right continuous simple $\mathscr{E}^\bullet$-Markov process with transition semigroup $(T_t, t \geq 0)$ and state space E. Define the "raw" natural filtration $(\mathscr{F}^\bullet_t, t \geq 0)$ by

$$\mathscr{F}^\bullet_t := \sigma\big(f(X_s), s \leq t, f \in b\mathscr{E}^\bullet\big), \quad t \geq 0,$$

and the σ-algebra generated by the process by

$$\mathscr{F}^\bullet := \sigma\big(f(X_t), t \geq 0, f \in b\mathscr{E}^\bullet\big).$$

Because $(T_t, t \geq 0)$ is a family of kernels with $\mathbb{P}_x \circ X_t^{-1} = T_t(x, \cdot)$, the mapping $x \mapsto \mathbb{P}_x(f(X_t))$ is $\mathscr{E}^\bullet$-measurable for $f \in b\mathscr{E}^\bullet$, $t \geq 0$. Then, by MCT, the mapping $x \mapsto \mathbb{E}_x(Y)$ is measurable for every $Y \in b\mathscr{F}^\bullet$ (see, e.g., [BG69, Theorem I.3.6], [Sha88, Lemma (2.6)]). Thus, we can define for every finite measure μ on $\mathscr{E}^\bullet$ a measure $\mathbb{P}_\mu$ by

$$\mathbb{P}_\mu(A) := \int \mathbb{P}_x(A)\,\mu(dx), \quad A \in \mathscr{F}^\bullet.$$

Following [Sha88, Section 3], we consider the *usual augmentations*:

(2.8) Definition. *For every probability measure μ on $(E, \mathscr{E}^\bullet)$, let $\mathscr{F}^\mu$ denote the completion of $\mathscr{F}^u$ relative to $\mathbb{P}_\mu$, and let $\mathscr{N}^\mu$ denote the set of all $\mathbb{P}_\mu$-null sets in $\mathscr{F}^\mu$. For any $t \geq 0$, set*

- *(i)* $\mathscr{F} := \bigcap\{\mathscr{F}^\mu : \mu$ *probability measure on* $E\}$,
- *(ii)* $\mathscr{N} := \bigcap\{\mathscr{N}^\mu : \mu$ *probability measure on* $E\}$,
- *(iii)* $\mathscr{F}_t^\mu := \mathscr{F}_t^u \vee \mathscr{N}^\mu$, μ *probability measure on* E,
- *(iv)* $\mathscr{F}_t := \bigcap\{\mathscr{F}_t^\mu : \mu$ *probability measure on* $E\}$.

As [Sha88] points out, this definition "is not the one most common in the literature", which however is fixed by [Sha88, Proposition (3.8)]:

(2.9) Lemma. *For every probability measure μ on E,*

- *(i)* $\mathscr{F}^\mu$ *is the* $\mathbb{P}_\mu$*-completion of* $\mathscr{F}^0$,
- *(ii) for every* $t \geq 0$, $\mathscr{F}_t^\mu = \mathscr{F}_t^0 \vee \mathscr{N}^\mu$ *holds true.*

The following theorem [Sha88, Theorem (3.9)] simplifies the work with the augmented filtration $(\mathscr{F}_t, t \geq 0)$:

(2.10) Theorem. *For every $t \geq 0$, $\mathscr{F}_t = \mathscr{F}_t^0 \vee \mathscr{F}_0^u \vee \mathscr{N} = \mathscr{F}_t^u \vee \mathscr{N}$ holds true. That is, $\mathscr{F}_t$ is generated by random variables of the form*

$$f(X_0)\,f_1(X_{t_1}) \cdots f_n(X_{t_n}) + H,$$

with $0 < t_1 < \cdots < t_n$, $f \in b\mathscr{E}^u$, $f_1, \ldots, f_n \in b\mathscr{E}$ and $H \in b\mathscr{F}$ with $\{H \neq 0\} \in \mathscr{N}$.

We define the *augmentation of the filtration* $(\mathscr{G}_t, t \geq 0)$ analogously to definition (2.8):

(2.11) Definition. *For every $x \in E$, let $\mathscr{N}^x(\mathscr{G})$ denote the set of all $\mathbb{P}_x$-null sets in the completion $\mathscr{G}^x$ of $\mathscr{G}$ relative to $\mathbb{P}_x$. For any $t \geq 0$, set*

- *(i)* $\overline{\mathscr{G}} := \bigcap_{x \in E} \mathscr{G}^x$,
- *(ii)* $\mathscr{N}(\mathscr{G}) := \bigcap_{x \in E} \mathscr{N}^x(\mathscr{G})$,

(iii) $\mathscr{G}_t^x := \mathscr{G}_t \vee \mathscr{N}^x(\mathscr{G})$, $x \in E$,

(iv) $\overline{\mathscr{G}}_t := \bigcap_{x \in E} \mathscr{G}_t^x$.

The basic principle is that, "*roughly speaking, one can replace the σ-algebras $\mathscr{G}_t$ and $\mathscr{F}_t^0$ [in the definitions and results of subsection 2.1] by $\overline{\mathscr{G}}_t$ and $\mathscr{F}_t$, provided one replaces $\mathscr{E}$ by $\mathscr{E}^u$*" (cf. [BG69, p. 28]). We summarize [BG69, Propositions I.5.8–I.5.12]:

(2.12) Theorem. *For all $F \in b\mathscr{F}$, $s, t \geq 0$, the mapping $x \mapsto \mathbb{E}_x(F)$ is $\mathscr{E}^u$-measurable, X_t is $\mathscr{F}_t/\mathscr{E}^u$-measurable, Θ_t is $\mathscr{F}_{s+t}/\mathscr{F}_s$-measurable, and for any $x \in E$,*

$$\mathbb{E}_x(F \circ \Theta_t \mid \overline{\mathscr{G}}_t) = \mathbb{E}_x(F).$$

In many cases, right continuity of the augmented natural filtration $(\mathscr{F}_t, t \geq 0)$ is ensured by the following result (cf. [BG69, Proposition I.8.12]):

(2.13) Lemma. *If X admits the Markov property relative to the filtration $(\mathscr{F}_{t+}^0, t \geq 0)$, then $\mathscr{F}_t = \mathscr{F}_{t+}$ holds for all $t \geq 0$.*

Therefore, we usually can (and will) assume the filtrations $(\mathscr{F}_t, t \geq 0)$ and $(\mathscr{G}_t, t \geq 0)$ of a Markov process to be augmented, and the natural filtration $(\mathscr{F}_t, t \geq 0)$ to be right continuous. These conditions are called the *usual hypotheses*.

We end this section by citing *Blumenthal's zero–one law* [BG69, Proposition I.5.17], which really gains its power through the augmentation (the same result for $\mathscr{F}_0^0$ instead of $\mathscr{F}_0$ would be trivial due to the normality of the process):

(2.14) Corollary. *Let X be a Markov process. Then for all $x \in E$, $A \in \mathscr{F}_0$,*

$$\mathbb{P}_x(A) \in \{0, 1\}.$$

2.3. Connection to the Theory of Semigroups

Let $X = (\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geq 0}, (X_t)_{t \geq 0}, (\Theta_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in E})$ be a right continuous simple Markov process with transition semigroup $(T_t, t \geq 0)$ and state space E.

(2.15) Definition. *The resolvent of X is the family of linear operators $(U_\alpha, \alpha \geq 0)$ on $(E, \mathscr{E})$, defined for all $\alpha \geq 0, f \in p\mathscr{E}^u$ or $\alpha > 0, f \in b\mathscr{E}^u$ by*

$$\forall x \in E: \quad U_\alpha^X f(x) = \mathbb{E}_x\Big(\int_0^\infty e^{-\alpha t} f(X_t)\, dt \Big).$$

An interchange of the order of integration (justified by [Sha88, Proposition 4.3]) gives

(2.16) Theorem. *For $\alpha \geq 0, f \in p\mathscr{E}^u$ or $\alpha > 0, f \in b\mathscr{E}^u$,*

$$U_\alpha^X f(x) = \int_0^\infty e^{-\alpha t}\, T_t f(x)\, dt.$$

Thus, the resolvent $(U^X, \alpha > 0)$ of the Markov process X coincides with the resolvent $(U_\alpha, \alpha > 0)$ of the corresponding semigroup $(T_t, t \geq 0)$ (on their shared domain), and we will omit the superscript X of U^X. All properties of the resolvent U thus hold for U^X as well. Especially, as the semigroup is uniquely characterized by its restriction to $b\mathcal{C}(E)$ and $t \mapsto T_t f(x) = \mathbb{E}_x(f(X_t))$ is right continuous for every $f \in b\mathcal{C}(E)$ by lemma (2.4), theorem (1.11) immediately yields:

(2.17) Corollary. *The resolvent of the semigroup $(T_t, t \geq 0)$ of a right continuous Markov process completely determines $(T_t, t \geq 0)$.*

As the semigroup property of $(T_t, t \geq 0)$ is the reflection of the Markov property of the underlying process X, it is not surprising that the Markov property can be equivalently characterized by a condition on the resolvent:

(2.18) Theorem. *Let $X = (\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ satisfy the properties (i), (ii), (iii), (v) and (vi) of definition (2.2) with respect to $\mathscr{E} = \mathscr{E}^u$. Set $T_t f(x) := \mathbb{E}_x(f(X_t))$ for all $t \geq 0$, $f \in b\mathscr{E}^u$, $x \in E$. Then X is a right continuous simple $\mathscr{E}^u$-Markov process with transition semigroup $(T_t, t \geq 0)$ on the state space E, if and only if*

(iv") for all $\alpha > 0$, $s \geq 0$, $f \in b\mathcal{C}(E)$, $J \in b\mathscr{G}_s$,

$$\mathbb{E}_x\Big(\int_0^\infty e^{-\alpha t} f(X_{s+t})\, dt \cdot J \Big) = \mathbb{E}_x(U_\alpha f(X_s) \cdot J).$$

Proof. We need to show the equivalence of (iv") and (iv'). We note that, by the MCT, (iv') is equivalent to its restriction on $f \in b\mathcal{C}(E)$, as

$$\mathscr{H} := \{ f \in b\mathscr{E}^u : \; \mathbb{E}_x(f(X_{s+t}) \,|\, \mathscr{G}_s) = T_t f(X_s) \}$$

is a MVS and the Borel σ-algebra $\mathscr{E}$ is generated by $b\mathcal{C}(E)$; so if $b\mathcal{C}(E) \subseteq \mathscr{H}$, then $b\mathscr{E} \subseteq \mathscr{H}$, and, by using sandwiching, $\mathscr{H} = b\mathscr{E}^u$ holds true. As $T_t f(X_s)$ is $\mathscr{G}_s$-measurable, condition (iv') holds, if and only if for all $s, t \geq 0$, $f \in b\mathcal{C}(E)$, $J \in b\mathscr{G}_s$:

$$\mathbb{E}_x(f(X_{s+t})\, J) = \mathbb{E}_x(T_t f(X_s)\, J). \tag{2.19}$$

But then (iv") is just the Laplace transform of (iv'): Both sides of above equation (2.19) are right continuous in t (see lemma (2.4) and its proof), so it is equivalent to its Laplace-transformed version. That is, (iv') holds, if and only if we have for all $\alpha > 0$, $s \geq 0$, $f \in b\mathcal{C}(E)$, $J \in b\mathscr{G}_s$:

$$\int_0^\infty e^{-\alpha t}\, \mathbb{E}_x(f(X_{s+t})\, J)\, dt = \int_0^\infty e^{-\alpha t}\, \mathbb{E}_x(T_t f(X_s)\, J)\, dt,$$

which, after an interchange of the order of integration with Fubini–Tonelli's theorem (see [Sha88, Proposition 4.3]), is just condition (iv"). $\square$

While resolvent techniques will be fundamental throughout our whole work, the generator will mostly be used in the Feller context. We only give its definition here (for the functional-analytic context, see example (1.5)), and postpone the important results to section 5.

(2.20) Definition. *The generator $(A, \mathscr{D}(A))$ of X is the weak generator of its transition semigroup $(T_t, t \geq 0)$, that is,*

$$\forall f \in \mathscr{D}(A), x \in E: \quad Af(x) = \lim_{t \downarrow 0} \frac{\mathbb{E}_x\big(f(X_t)\big) - f(x)}{t},$$

with $\mathscr{D}(A)$ being the set of all $f \in b\mathscr{E}$, for which the right-hand limit exists pointwise and is uniformly bounded.

3. Strong Markov Processes

In this section, we are going to recall a stronger version of the Markov property, which ensures the Markovian behavior of a stochastic process not only at any deterministic time, but also at so called stopping times, that is, at random times which "do not depend on the process' future". Introduced rigorously by [Hun56], this property is strictly stronger than the "deterministic" Markov property (see, e.g., example [CW05, Example 8.14], which offers an insight into what usually "goes wrong" and how it can be fixed), and is nowadays an indispensable tool for the study of Markov processes, with famous applications such as the rigorous proof of André's reflection principle for the Brownian motion. It is therefore no surprise that this "strong Markov property" constitutes a basic requirement for most of the modern classes of Markov processes.

While the Markov property has a direct analytic reflection in the semigroup property, the strong Markov property will lead to more refined, probabilistic results which have no direct equivalent in semigroup theory. They will be essential in the analysis of our constructions later.

3.1. Stopping Times

Filtrations, that is families of σ-algebras $(\mathscr{G}_t, t \geq 0)$ with $\mathscr{G}_s \subseteq \mathscr{G}_t$ for all $s \leq t$, can be understood as the representation of "information" gained by an observer over time. With this interpretation at hand, a stopping time represents the point in time when a random event occurs, with the property that, at any time, the observer can determine with the current information whether this event already occurred or not:

(3.1) Definition. *Let $(\Omega, \mathscr{G})$ be a measurable space with filtration $(\mathscr{G}_t, t \geq 0)$. A random variable $\tau \colon \Omega \to [0, +\infty]$ is called a stopping time over $(\mathscr{G}_t, t \geq 0)$, if*

$$\forall t \geq 0: \quad \{\tau \leq t\} \in \mathscr{G}_t.$$

Predictable times are a special type of stopping times, whose occurrence can be "announced" in the following sense:

(3.2) Definition. *A stopping time τ over $(\mathscr{G}_t, t \geq 0)$ is predictable, if there exists an increasing sequence of stopping times $(\tau_n, n \in \mathbb{N})$ over $(\mathscr{G}_t, t \geq 0)$ such that*

$$\forall n \in \mathbb{N}: \ \tau_n < \tau \text{ on } \{\tau > 0\} \quad \text{and} \quad \lim_{n \in \mathbb{N}} \tau_n = \tau.$$

Stopping times and predictable times are stable under a wide variety of operations, such as under summation, infima and suprema, see [DM78, Chapter IV, 50–73] for a collection of results. We will use these properties without special mention.

While $\mathscr{F}_t$ represents the insight of an observer up to a fixed time $t \geq 0$, the following σ-algebra $\mathscr{F}_\tau$ ($\mathscr{F}_{\tau-}$, $\mathscr{F}_{\tau+}$) "collects all information" up to the random time τ (infinitesimally before τ, after τ, respectively). This statement is not really apparent from the following definition, but it will be justified in subsection 3.5.

(3.3) Definition. *Let $(\Omega, \mathscr{G})$ be a measurable space with filtration $(\mathscr{G}_t, t \geq 0)$ and $\tau \colon \Omega \to [0, +\infty]$ be a mapping. Set*

$$\begin{aligned}
\mathscr{F}_\tau &:= \big\{A \in \mathscr{F}_\infty \,\big|\, \forall t \geq 0: \ A \cap \{\tau \leq t\} \in \mathscr{F}_t\big\}, \\
\mathscr{F}_{\tau-} &:= \sigma\big(\{A \cap \{t < \tau\} \,|\, t \geq 0, A \in \mathscr{F}_t\}\big), \\
\mathscr{F}_{\tau+} &:= \big\{A \in \mathscr{F}_\infty \,\big|\, \forall t \geq 0: \ A \cap \{\tau \leq t\} \in \mathscr{F}_{t+}\big\}.
\end{aligned}$$

If τ is a stopping time over $(\mathscr{F}_t, t \geq 0)$, then $\mathscr{F}_\tau$, $\mathscr{F}_{\tau-}$, $\mathscr{F}_{\tau+}$ are σ-algebras, and τ is measurable with respect to each of them. For an in-depth analysis of their relationship to each other, see, e.g., [CW05, Section 1.3] or [BG69, Section I.6]. Basic properties stated there will be used without special mention. The following result will be helpful later:

(3.4) Theorem. *Let X be a right continuous stochastic process on a measurable space $(\Omega, \mathscr{G})$ and adapted to a filtration $(\mathscr{G}_t, t \geq 0)$.*

(i) If τ is a stopping time over $(\mathscr{G}_t, t \geq 0)$, then $X_\tau \, \mathbb{1}_{\{\tau < +\infty\}}$ is $\mathscr{G}_\tau$-measurable and $X_{\tau+} \, \mathbb{1}_{\{\tau < +\infty\}}$ is $\mathscr{G}_{\tau+}$-measurable.

(ii) If X has left limits and τ is predictable, then $X_{\tau-} \, \mathbb{1}_{\{\tau < +\infty\}}$ is $\mathscr{G}_{\tau-}$-measurable, with $X_{0-} := X_0$.

Proof. As X is right continuous, it is progressively measurable, so X_τ is a composition of measurable functions and thus admits $\mathscr{F}_\tau$-measurability, see [CW05, Theorem 1.5.2]. The rest is provided by [CW05, Theorem 1.3.10]. □

While stopping times cannot "look into the future", they may still have some kind of "memory". When transforming a Markov process with the help of a stopping time (e.g. with methods treated in chapter II), this "memory" may destroy the Markov property of the resulting process. Therefore, it is necessary to examine a property of "memorylessness" for stopping times:

(3.5) Definition. *A stopping time T over $(\mathscr{F}_t, t \geq 0)$ is a terminal time for the Markov process $X = (\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$, provided that for every stopping time R over $(\mathscr{G}_t, t \geq 0)$,*

$$R + T \circ \Theta_R = T \quad \text{holds a.s. on } \{R < T\}.$$

A terminal time T is exact, if for every sequence $(t_n, n \in \mathbb{N}_0)$ with $t_n \downdownarrows 0$,

$$\lim_{n\to\infty} t_n + T \circ \Theta_{t_n} = T \quad \text{holds a.s.}.$$

Let

$$\Lambda := \{\omega \in \Omega : t + T(\omega) = T(\omega) \text{ for all } t < T(\omega)\}.$$

T is almost perfect, if $\Lambda^c \in \mathscr{N}$, and perfect, if $\Lambda^c = \emptyset$.

Terminal times typically "*represent the first time the path [of a process] exhibits some particular geometric behavior*" (cf. [Sha88, p. 66]). The most important example is the first entry time into a set:

(3.6) Definition. *Let $(X_t, t \geq 0)$ be a stochastic process with state space $(E, \mathscr{E})$. For $A \in \mathscr{E}$, the first entry time of X into A (or debut of A) is the random time*

$$H_A := \inf\{t \geq 0 : X_t \in A\}.$$

It is easy to show that, for any stochastic process X adapted to a filtration $(\mathscr{G}_t, t \geq 0)$, the first entry time into an open set is a stopping time over $(\mathscr{G}_{t+}, t \geq 0)$ given that X is right continuous, and that the first entry time into a closed set is a stopping time over $(\mathscr{G}_t, t \geq 0)$ if X is continuous (see [CW05, Theorem 2.4.5], [BB96, Theorems 49.4, 49.5]). Indeed, continuity is only needed up to the first entry time, which can be seen by a detailed examination of the proof of [BB96, Theorem 49.5]:

(3.7) Lemma. *Let $(X_t, t \geq 0)$ be a right continuous stochastic process with state space $(E, \mathscr{E})$, and $A \in \mathscr{E}$ be a closed set. If X is left continuous on $(0, H_A]$, then the first entry time H_A of X into A is a stopping time over the filtration $(\mathscr{F}_t^0 = \sigma(X_s, s \leq t), t \geq 0)$ generated by the process X.*

The general result on first entry times reads:

(3.8) Theorem. *Let X be a right continuous process on E, adapted to an augmented filtration $(\mathscr{F}_t, t \geq 0)$. Then the first entry time H_A of X into an universally measurable Borel set $A \in \mathscr{E}^u$ is a stopping time over $(\mathscr{F}_t, t \geq 0)$. If X is equipped with shift operators $(\Theta_t, t \geq 0)$, then H_A satisfies*

$$\forall t < H_A : \quad t + H_A \circ \Theta_t = H_A.$$

Regarding the above theorem, the proof of the stopping time property is hard and uses deep results of Choquet capacity theory, see, e.g., [RW00a, Sections II.75–II.76], [Sha88, Sections 10, A.5], or [DM78, Theorem IV.50]. In general, the requirement of the usual hypotheses, which especially ensure the right continuity of the natural filtration, cannot be weakened. On the other hand, the terminal time property of H_A follows directly from its definition, as for all $0 \le t < H_A$, we have

$$\begin{aligned} H_A &= \inf\{s \ge t : X_s \in A\} \\ &= \inf\{s \ge 0 : X_s \circ \Theta_t \in A\} + t \\ &= H_A \circ \Theta_t + t. \end{aligned}$$

The theorem above shows that, for any right continuous Markov process X satisfying the usual hypotheses, the first entry time H_A of X into any set $A \in \mathscr{E}^u$ is a perfect terminal time. Observe that, in general, the first entry time into a set A is not exact, as

$$\lim_{t \downarrow\downarrow 0} t + H_A \circ \Theta_t = \inf\{t > 0 : X_t \in A\}.$$

For more refined explanations and examples considering terminal times, see, e.g., [Sha88, pp. 65f].

3.2. Strong Markov Property

The following definition extends the Markov property to stopping times:

(3.9) Definition. *X has the strong Markov property relative to $((\mathscr{G}_t)_{t\ge0}, \tau)$, if for all $f \in b\mathscr{E}$, $t \ge 0$, μ probability measure on $(E, \mathscr{E})$,*

$$\mathbb{E}_\mu\big(f(X_{t+\tau})\, \mathbb{1}_{\{\tau<\infty\}} \,\big|\, \mathscr{G}_\tau\big) = \mathbb{E}_{X_\tau}\big(f(X_t)\big)\, \mathbb{1}_{\{\tau<\infty\}}.$$

X has the strong Markov property relative to a filtration $(\mathscr{G}_t, t \ge 0)$, if it has the strong Markov property relative to $((\mathscr{G}_t)_{t\ge0}, \tau)$ for every stopping time τ over $(\mathscr{G}_t, t \ge 0)$. A Markov process $X = \big(\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\ge0}, (X_t)_{t\ge0}, (\Theta_t)_{t\ge0}, (\mathbb{P}_x)_{x\in E}\big)$ is a strong Markov process, if it has the strong Markov property relative to $(\mathscr{G}_t, t \ge 0)$.

The strong Markov property incorporates the "normal" one, because every deterministic random time $\tau := t$, $t \ge 0$, is a stopping time with $\mathscr{F}_\tau = \mathscr{F}_t$ and $\mathscr{F}_{\tau+} = \mathscr{F}_{t+}$. As mentioned above, this stronger property is not automatically fulfilled when a Markov process is derived from its semigroup (for instance, when using the approach given in the beginning of section 2); it is then usually necessary to impose some regularity conditions on the paths and on the semigroup or resolvent in order to ensure the strong Markov property (see, e.g., [BG69, Theorem I.8.11]). This property is a basic tool needed for many deep results, therefore most of the studied classes of Markov processes already entail it in their very definition. Usually, one gains or requires the strong Markov property relative to $(\mathscr{G}_{t+}, t \ge 0)$, with $(\mathscr{G}_t, t \ge 0)$ being an augmented filtration equal to or larger than the natural filtration $(\mathscr{F}_t, t \ge 0)$. In this case, lemma (2.13) asserts the right continuity of the natural filtration.

As usual, the strong Markov property can be lifted to general functions on the augmented natural filtration (see, e.g., [BG69, Corollary I.8.6]):

(3.10) Theorem. *Let X be a Markov process admitting the strong Markov property relative to $(\mathscr{G}_t, t \geq 0)$. Then, for any $Y \in b\mathscr{F}$, $x \in E$, τ stopping time over $(\mathscr{G}_t, t \geq 0)$,*

$$\mathbb{E}_x(Y \circ \Theta_\tau \,|\, \mathscr{G}_\tau) = \mathbb{E}_{X_\tau}(Y).$$

With the help of some properties of the conditional expectation, the above theorem can be further refined to more general functions which also depend on "information" of $\mathscr{G}_\tau$. As usual, the application of conditional expectation on $\mathscr{G}_\tau$ will leave the $\mathscr{G}_\tau$-measurable part invariant, while the time shifted process part is affected by the strong Markov property. The next result can be found in [Sha88, Exercise 6.12]:

(3.11) Lemma. *Let the Markov process $X = (\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ admit the strong Markov property relative to $((\mathscr{G}_t)_{t\geq 0}, \tau)$ for some stopping time τ over $(\mathscr{G}_t, t \geq 0)$. Then, for every function $G\colon \Omega \times \mathbb{R}_+ \times \Omega \to \mathbb{R}$ with $G \in b(\mathscr{G}_\tau \otimes \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F})$, and for every $x \in E$,*

$$\mathbb{E}_x\big(G(\,\cdot\,, \tau(\cdot), \Theta_{\tau(\cdot)}(\cdot))\, \mathbb{1}_{\{\tau<\infty\}} \,|\, \mathscr{G}_\tau\big)(\omega) = \int G(\omega, \tau(\omega), \omega')\, \mathbb{P}_{X_{\tau(\omega)}}(d\omega')$$

holds a.s. on $\{\tau < +\infty\}$.

Proof. For $G \in b\big(\mathscr{G}_\tau \otimes \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F}\big)$ of the form

$$G(\omega, t, \omega') := F(\omega)\, \psi(t)\, H(\omega'), \quad F \in b\mathscr{G}_{\tau+},\ \psi \in b\mathscr{B}(\mathbb{R}_+),\ H \in b\mathscr{F},$$

we have a.s. on $\{\tau < +\infty\}$:

$$\begin{aligned} &\mathbb{E}_\mu\big(G(\,\cdot\,, \tau(\cdot), \Theta_{\tau(\cdot)}(\cdot))\, \mathbb{1}_{\{\tau<\infty\}} \,|\, \mathscr{G}_\tau\big)(\omega) \\ &= \mathbb{E}_\mu\big(F \cdot \psi \circ \tau \cdot H \circ \Theta_\tau\, \mathbb{1}_{\{\tau<\infty\}} \,|\, \mathscr{G}_\tau\big)(\omega) \\ &= F(\omega)\, \psi \circ \tau(\omega)\, \mathbb{E}_\mu\big(H \circ \Theta_\tau\, \mathbb{1}_{\{\tau<\infty\}} \,|\, \mathscr{G}_\tau\big)(\omega) \\ &= F(\omega)\, \psi \circ \tau(\omega)\, \mathbb{E}_{X_{\tau(\omega)}}\big(H\big)(\omega) \\ &= \int G\big(\omega, \tau(\omega), \omega'\big)\, \mathbb{P}_{X_{\tau(\omega)}}(d\omega'). \end{aligned}$$

Due to the cLMCT, the set of all functions of the above form is a MVS generating the set $b\big(\mathscr{G}_\tau \otimes \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F}\big)$, and the claim now follows from the MCT. ☐

We close the discussion of stopping times and the strong Markov property by giving a special property for terminal times, which will turn out to be helpful later. The following result can be found in [BG69, Corollary I.8.5]:

(3.12) Lemma. *Let X be a right continuous strong Markov process, and S be a stopping time over $(\mathscr{F}_t, t \geq 0)$. Then for any $t \geq 0$, Θ_S is $\mathscr{F}_{t+S}/\mathscr{F}_t$-measurable.*

(3.13) Lemma. *Let T be a perfect terminal time for a right continuous strong Markov process X, and S be a stopping time over $(\mathscr{F}_t, t \geq 0)$ with $S < T$. Then Θ_S is $\mathscr{F}_{T-}/\mathscr{F}_{T-}$-measurable.*

Proof. Let $t \geq 0$ and $A \in \mathscr{F}_t$. Then the terminal time property of T yields

$$\begin{aligned}\Theta_S^{-1}(A \cap \{t < T\}) &= \Theta_S^{-1}(A) \cap \{t < T \circ \Theta_S\} \\ &= \Theta_S^{-1}(A) \cap \{t + S < T\} \\ &= \bigcup_{q \in \mathbb{Q}_+} \Big((\Theta_S^{-1}(A) \cap \{S < q - t\}) \cap \{q < T\} \Big).\end{aligned}$$

As $\Theta_S^{-1}(A) \in \mathscr{F}_{t+S}$ by lemma (3.12), we see that the inner term satisfies

$$\Theta_S^{-1}(A) \cap \{t + S < q\} \in \mathscr{F}_q$$

by the definition of $\mathscr{F}_{t+S}$ for every $q \in \mathbb{Q}_+$. So every set of the countable union above is an element of $\mathscr{F}_{T-}$ by its definition, therefore the set $\Theta_S^{-1}\big(A \cap \{t < T\}\big)$ is as well. □

3.3. Holding Points

Let X be a right continuous Markov process on $(E, \mathscr{E})$ and $x \in E$. Consider the first exit time of X from $\{x\}$, that is,

$$\tau_x := H_{\complement\{x\}} = \inf\{t \geq 0 : X_t \neq x\}.$$

Then, as $\complement\{x\}$ is open, the event $\{\tau_x = 0\} \in \mathscr{F}_{0+}$, and by the Blumenthal zero–one law,

$$\mathbb{P}_x(\tau_x = 0) \in \{0, 1\}.$$

(3.14) Definition. *For a right continuous Markov process X on $(E, \mathscr{E})$, a point $x \in E$ is a holding point, if $\mathbb{P}_x(\tau_x = 0) = 0$.*

It is well-known (see, e.g., [Çın11, Theorem 9.4.22]) that, when the process is started at a holding point x, the holding time τ_x is exponentially distributed and the exit point X_{τ_x} is independent of τ_x:

(3.15) Theorem. *Let X be a right continuous Markov process on $(E, \mathscr{E})$, and $x \in E$ be a holding point of X. Then there exists a number $\lambda(x) \in [0, +\infty)$ and a measure $B \mapsto K(x, B)$ on $(E, \mathscr{E})$, such that*

$$\forall t \geq 0, B \in \mathscr{E} : \quad \mathbb{P}_x(\tau_x > t, X_{\tau_x} \in B) = e^{-\lambda(x)t} K(x, B).$$

Furthermore, a right continuous, strong Markov process can only exit a holding point by a jump (see, e.g., [Çın11, Proposition 9.5.23], [Sha88, Exercise (6.16)]).

3.4. Dynkin's Formulas

The following formulas, which can be found in [Dyn65, Section 5.1], give probabilistic representations and decompositions for the resolvent and generator of a strong Markov process. They are a direct consequence of the strong Markov property and have no equivalent in the analytic representation via the process' semigroup. They can also be gained by martingale techniques, see, e.g., [RW00a, Section III.10]. We will call any of the following results *Dynkin's formula*:

(3.16) Theorem. *Let X be a right continuous Markov process on $(E, \mathscr{E})$, admitting the strong Markov property relative to $\big((\mathscr{G}_t)_{t\geq 0}, \tau\big)$. Then, for every $\alpha > 0$, $f \in b\mathscr{E}$, $x \in E$,*

$$U_\alpha f(x) = \mathbb{E}_x\Big(\int_0^\tau e^{-\alpha t} f(X_t)\,dt\Big) + \mathbb{E}_x\big(e^{-\alpha\tau}\, U_\alpha f(X_\tau)\big).$$

(3.17) Theorem. *Let X be a right continuous Markov process on $(E, \mathscr{E})$, admitting the strong Markov property relative to $\big((\mathscr{G}_t)_{t\geq 0}, \tau\big)$. Let $x \in E$ be such that $\mathbb{E}_x(\tau) < +\infty$. Then, for every $f \in \mathscr{D}(A)$,*

$$\mathbb{E}_x\Big(\int_0^\tau Af(X_t)\,dt\Big) = \mathbb{E}_x\big(f(X_\tau)\big) - f(x).$$

(3.18) Theorem. *Let X be a right continuous, strong Markov process on $(E, \mathscr{E})$. Let $x \in E$, and $(\varepsilon_n, n \in \mathbb{N})$ be a sequence of positive numbers converging to zero, such that*

$$\tau_\varepsilon := \inf\big\{t \geq 0 : X_t \in \complement\overline{B_\varepsilon(x)}\big\}, \quad \varepsilon > 0,$$

are stopping times and fulfill $0 < \mathbb{E}_x(\tau_{\varepsilon_n}) < +\infty$ for all $\varepsilon := \varepsilon_n$, $n \in \mathbb{N}$. Then, for every $f \in \mathscr{D}(A)$ for which Af is continuous at x,

$$Af(x) = \lim_{n\to\infty} \frac{\mathbb{E}_v\big(f\big(X(\tau_{\varepsilon_n})\big)\big) - f(x)}{\mathbb{E}_x(\tau_{\varepsilon_n})}.$$

This theorem can be stated for more general choices of sequences of stopping times $(\tau_{\varepsilon_n}, n \in \mathbb{N})$, see [Dyn65, Theorem 5.2]. For our applications, the first entry times into the open sets $\complement\overline{B_\varepsilon(x)} = \{y \in E : d(x,y) > \varepsilon\}$ turn out to be sufficient. They are always stopping times over the right continuous extension $(\mathscr{F}^0_{t+}, t \geq 0)$ of the filtration generated by any right continuous process $(X_t, t \geq 0)$, as seen in subsection 3.1.

3.5. Galmarino's Theorem

We are going to provide a different approach to stopping times and the filtrations generated by them, which will extend the definitions of subsection 3.1. It seems that the following results are mentioned to a broad audience for the first time in [IM74, p. 86], cited as private communication of the authors with Galmarino. The original source is probably [Gal63]. Since then, results around Galmarino's test are scattered in the literature in various forms, cf. [Kni81, pp. 44–45], [Rao77, pp. 2.13–2.15], [Sha88, Proposition (23.16)], [DM78, Chapter IV, 99–102], or [RY94, Exercise (4.21)].

As the results of this subsection hold true for an arbitrary stochastic process $X = (X_t, t \geq 0)$, we will not assume the process X to be Markovian. The only condition needed here will be that X can be "stopped" in the following sense:

(3.19) Definition. *Let $X = (X_t, t \geq 0)$ be a stochastic process on $(\Omega, \mathscr{F})$. A family $(\alpha_t, t \geq 0)$ of mappings $\alpha_t \colon \Omega \to \Omega$, $t \geq 0$, is called stopping operators for X, if*

$$\forall s, t \geq 0: \quad X_s \circ \alpha_t = X_{s \wedge t}.$$

(3.20) Example. If $(X_t, t \geq 0)$ is the canonical coordinate process, that is $X_t(\omega) = \omega(t)$ for all $\omega \in \Omega$, $t \geq 0$, on the path space

$$\Omega := \{\omega \colon \mathbb{R}_+ \to E \,|\, \omega \text{ right continuous}\},$$

then there exist canonical stopping operators for X, namely

$$\alpha_t \colon \Omega \to \Omega,\ \omega \mapsto \alpha_t(\omega) := \omega(\,\cdot\, \wedge t), \quad t \geq 0. \qquad \blacksquare$$

The first result, called *Galmarino's test*, gives a characterization of stopping times via the stopped paths of a stochastic process. It can be slightly adjusted to fit predictable times or weak stopping times as well, see, e.g., [DM78, Chapter IV, 99–101].

(3.21) Theorem. *Let τ be a non-negative, $\mathscr{F}^0_\infty$-measurable function. Then τ is an*

(i) $(\mathscr{F}^0_{t+})$-stopping time, if and only if for all $t \geq 0$,

$$\alpha_t(\omega_1) = \alpha_t(\omega_2), \tau(\omega_1) < t \quad \textit{implies} \quad \tau(\omega_1) = \tau(\omega_2);$$

(ii) $(\mathscr{F}^0_t)$-stopping time, if and only if for all $t \geq 0$,

$$\alpha_t(\omega_1) = \alpha_t(\omega_2), \tau(\omega_1) \leq t \quad \textit{implies} \quad \tau(\omega_1) = \tau(\omega_2).$$

In [DM78, Chapter IV, 102], Dellacherie and Meyer remark that "*one shouldn't live with too many illusions about the practical value of the test*", as the uncompleted σ-algebra $\mathscr{F}^0_\infty$ is too restrictive in most cases, and even if it is not, the actual proof of the $\mathscr{F}^0_\infty$-measurability of a random time often also exhibits its stopping time property.

Nonetheless, this "test" leads to *Galmarino's theorem*, which provides the natural characterization of the stopped filtration $\mathscr{F}_\tau$ for an $\mathscr{F}^0$-stopping time τ, and which will be essential for us later. The following form is a slight generalization of [Kni81, Theorem 3.2.13], whose proof exactly carries over to this case:

(3.22) Theorem. *Let X be a right continuous stochastic process with stopping operators and τ be a stopping time over $(\mathscr{F}^0_t, t \geq 0)$. Then*

$$\mathscr{F}^0_\tau = \sigma(X_{t \wedge \tau}, t \geq 0).$$

4. Right Processes

We are going to introduce the fundamental class of Markov processes for our work. First set down by Meyer in [WM71], who also established their name *processus de Markov satisfaisant aux hypothèses droites* (that is, "process having the right properties"), the hypotheses of this class were further refined by Getoor in [Get75] and Sharpe in [Sha88].

The class of right processes is one of the most general classes in the study of Markov processes. As pointed out in the summary given in [Get75, pp. 55f], "*one has the following inclusions among these various classes of processes: (Feller) ⊆ (Hunt) ⊆ (special standard) ⊆ (standard) ⊆ (right). [...] it seems to me that [all the subclasses] are now mainly of historical interest.*" For a short survey of the history on the development of Markov processes which led to right processes, we recommend [Mey89] to the reader. Nowadays, this class seems to be coined *Borel right process*, see, e.g., [CF11] or [MR06]. We will stick to the term *right process*, following our main sources [Sha88] and [Get75].

4.1. Excessive Functions

A main interest in the theory of Markov processes is the class of excessive functions. They are a generalization of harmonic functions evolving from potential theoretical considerations (see, e.g., [Rao77, Section 5.1]) and will be essential in the definition of right processes. In this subsection, we assume that $(T_t, t \geq 0)$ is a Markov semigroup on a Radon space E with associated resolvent $(U_\alpha, \alpha > 0)$.

(4.1) Definition. *Let $\alpha \geq 0$. A function $f \in p\mathscr{E}$ is α-super-mean-valued, if*

$$\forall t \geq 0: \quad e^{-\alpha t}\, T_t f \leq f.$$

f is α-excessive, if f is α-super-mean-valued and

$$\lim_{t \downarrow 0} e^{-\alpha t}\, T_t f = f.$$

We set $\mathscr{S}_\alpha := \{f \in p\mathscr{E} : f \text{ is } \alpha\text{-excessive}\}$.

The class of excessive functions has many nice properties (see, e.g., [BG69, Chapter II]). It might be understood as an effort to extend the reach of resolvent methods as far as possible, as many important techniques and studies in the field of Markov processes are based on the work with potentials. The basic connection between excessive functions and potential functions is the following: (cf. [BG69, Propositions II.2.2, II.2.6])

(4.2) Lemma. *For every $\alpha > 0$, the following properties hold:*

(i) $\mathscr{S}_\alpha$ is closed under monotone increasing limits.

(ii) $U_\alpha f \in \mathscr{S}_\alpha$ for all $f \in p\mathscr{E}$.

(iii) For every $f \in \mathscr{S}_\alpha$, there exists a sequence $(h_n, n \in \mathbb{N})$ in $bp\mathscr{E}^u$ with $U_\alpha h_n \uparrow f$.

In many cases, this result will allow us to reduce the analysis of excessive functions to the study of potentials.

4.2. Definition and Basic Results

The second fundamental hypothesis for the "right" study of Markov processes—the first one basically being the existence of a right continuous Markov process for a given semigroup, see [Sha88, Definition (2.1)]—is the following "hypothèse droite", which immediately shows the role of the excessive functions:

(4.3) Definition. *Let* $X = (\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ *be a right continuous simple Markov process with transition semigroup* $(T_t, t \geq 0)$ *on a Radon space* E. *It satisfies HD2, if for every* $\alpha > 0$ *and every* $f \in \mathscr{S}_\alpha$, *the process* $t \mapsto f(X_t)$ *is a.s. right continuous.*

(4.4) Definition. $X = (\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ *is a right process on the Radon space* E *with transition semigroup* $(T_t, t \geq 0)$, *if*

(i) X *is a right continuous simple Markov process with transition semigroup* $(T_t, t \geq 0)$ *and state space* E,

(ii) X *satisfies HD2 relative to* $(\mathscr{G}_t, t \geq 0)$, *and*

(iii) $(\mathscr{G}_t, t \geq 0)$ *is augmented and right continuous.*

(4.5) Definition. $(T_t, t \geq 0)$ *is a right semigroup, if there exists a right process* $X = (\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ *with transition semigroup* $(T_t, t \geq 0)$.

Normally, there is no need to check HD2 directly. Instead, we have a collection of equivalent conditions at hand (see [Sha88, Theorem (7.4)]), which we call the *portmanteau of right processes*. This result also hints that the correct setting for the study of right processes is the "general theory of stochastic processes" by Dellacherie and Meyer:

(4.6) Theorem. *Let* $X = (\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\theta_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ *satisfy (i) and (iii) of definition* (4.4). *Then the following conditions on* X *are equivalent:*

(i) X *satisfies HD2 relative to* $(\mathscr{G}_t, t \geq 0)$.

(ii) For all $f \in b\mathcal{C}_d(E)$, *the path* $s \mapsto U_\alpha f(X_s)$ *is a.s. right continuous.*

(iii) X *admits the strong Markov property relative to* $(\mathscr{G}_{t+}, t \geq 0)$ *and for all* $\alpha > 0$, $f \in b\mathcal{C}_d(E)$, *the potential* $U_\alpha f$ *is nearly optional with respect to* X.

(iv) X *admits the strong Markov property relative to* $(\mathscr{G}_{t+}, t \geq 0)$ *and for all* $t \geq 0$, $f \in b\mathcal{C}_d(E)$, *the function* $T_t f$ *is nearly optional with respect to* X.

(v) For all $t \geq 0$, $f \in b\mathcal{C}_d(E)$, *the path* $s \mapsto T_t f(X_s)$ *is a.s. right continuous.*

In particular, condition (iii) implies that any right process is strongly Markovian.

It can be shown that if a semigroup or a "resolvent like" family of kernels satisfies certain regularity conditions, it gives rise to a right process. A general existence theorem for so called *Ray resolvents* is discussed in [Sha88, Section 9]. In section 5, we introduce

Feller semigroups in order to construct prototypes of right processes, which turns out to be sufficient for our needs. We will then transform them with the help of various techniques exposed in the following chapter II.

4.3. Lifetime Formalisms

Following [Sha88, Section 11], we recall the notion of an absorbing *cemetery point* Δ and the conventions regarding it, which are going to be in place for the rest of our work.

Let E be a Radon space and $(T_t, t \geq 0)$ be a sub-Markovian transition semigroup on $(E, \mathscr{E}^u)$. We extend $(T_t, t \geq 0)$—even if it is already Markovian—to a semigroup $(\widetilde{T}_t, t \geq 0)$ by adjoining a new point $\Delta \notin E$ to E, forming the Radon space $E_\Delta := E \cup \{\Delta\}$ equipped with the universally measurable sets $\mathscr{E}^u_\Delta$, and setting

$$\widetilde{T}_t(x, A) := \begin{cases} T_t(x, A), & x \in E, A \in \mathscr{E}^u_\Delta \text{ with } A \subseteq E, \\ 1 - T_t(x, A), & x \in E, A = \{\Delta\}, \\ \varepsilon_\Delta(A), & x = \Delta. \end{cases}$$

Then $(\widetilde{T}_t, t \geq 0)$ is a Markov semigroup on $(E_\Delta, \mathscr{E}^u_\Delta)$. We will now call $(T_t, t \geq 0)$ a *right semigroup*, if $(\widetilde{T}_t, t \geq 0)$ is a right semigroup in the sense of definition (4.5).

Assume that we are given a right semigroup $(T_t, t \geq 0)$ with a right process $\widetilde{X} = (\Omega, \widetilde{\mathscr{G}}, (\widetilde{\mathscr{G}}_t)_{t \geq 0}, (\widetilde{X}_t)_{t \geq 0}, (\Theta_t)_{t \geq 0}, (\widetilde{\mathbb{P}}_x)_{x \in E_\Delta})$ on E_Δ realizing it. Then, by the definition of its semigroup and the right continuity of $\widetilde{X}$, Δ is a trap in the sense of definition (5.14), that is

$$\widetilde{\mathbb{P}}_\Delta(\forall t \geq 0 : \widetilde{X}_t = \Delta) = 1.$$

An application of the strong Markov property of $\widetilde{X}$ at the first entry time into Δ $\widetilde{\zeta} := \inf\{t \geq 0 : \widetilde{X}_t = \Delta\}$, which is a stopping time by theorem (3.8), yields

$$\widetilde{\mathbb{P}}_x(\forall t \geq \widetilde{\zeta} : \widetilde{X}_t = \Delta) = 1.$$

Following the axioms of [BG69], we require Δ to be absorbing for every path, that is

$$\forall \omega \in \Omega : \quad \widetilde{X}_t(\omega) = \Delta \quad \Rightarrow \quad \forall s \geq t : \widetilde{X}_s(\omega) = \Delta,$$

which can be achieved, if necessary, by restriction of the sample space Ω (see, e.g., [BB96, Section 38]).

As one is mainly interested in the behavior of the process while it is in E, we de-emphasize the role of $(\widetilde{T}, t \geq 0)$ and E_Δ as follows: Introducing the convention that every function f on E is extended to E_Δ by

$$f(\Delta) := 0, \tag{4.7}$$

we define the E_Δ-valued process $X = (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (\Theta_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in E})$ on E by setting $X_t = \widetilde{X}_t$, $t \geq 0$, on Ω, with $\mathbb{P}_x = \widetilde{\mathbb{P}}_x$, $x \in E$, and natural filtration $(\mathscr{F}_t, t \geq 0)$.

With the convention (4.7) it then follows (see [Sha88, Exercise (11.14)]) that $\mathscr{F}$ is generated by the constant function $\mathbb{1}_\Omega$ and functions of the form

$$f_1(X_{t_1})\cdots f_n(X_{t_n}), \quad n\in\mathbb{N},\ 0\le t_1<\cdots<t_n,\ f_1,\dots,f_n\in b\mathscr{E}^u.$$

Then, the strong Markov property of $\widetilde{X}$ immediately transfers to X, that is, we have for any stopping time τ

$$\mathbb{E}_x\big(f(X_t)\circ\Theta_\tau\,\mathbb{1}_{\{\tau<\infty\}}\,\big|\,\mathscr{F}_\tau\big)=T_tf(X_\tau)\,\mathbb{1}_{\{\tau<\infty\}},$$

where the term $\mathbb{1}_{\{\tau<\infty\}}$ can be dropped when defining $X_\infty:=\Delta$.

(4.8) Definition. *The lifetime of a right process X is*

$$\zeta:=\inf\{t\ge 0: X_t=\Delta\}.$$

The process X above will then be called the *right process on E with lifetime ζ and transition semigroup* $(T_t, t\ge 0)$. For such a process, we will always assume that Ω contains a *dead path* $[\Delta]$ with

$$\forall t\ge 0:\quad X_t([\Delta])=\Delta.$$

We end this section with an easy, but valuable result concerning the lifetime ζ:

(4.9) Lemma. *The lifetime ζ of a right process X is a terminal time. For any random time $R\colon\Omega\to[0,\infty]$, it satisfies*

$$\zeta\circ\Theta_R=(\zeta-R)^+.$$

Proof. By definition, $\zeta=\inf\{t\ge 0: X_t=\Delta\}$ is the first entry time into a closed set and therefore a terminal time by theorem (3.8). For any random time R, we have

$$\zeta\circ\Theta_R=\inf\{t\ge 0: X_t\circ\Theta_R=\Delta\}=\inf\{t\ge R: X_t=\Delta\}-R.$$

Thus, if $X_t\ne\Delta$ for all $t\le R$, then $\zeta=\zeta-R$ holds. On the other hand, if $X_t=\Delta$ for some $t\le R$, then $X_R=\Delta$ holds (as Δ is absorbing for every path), so $\zeta\circ\Theta_R=0$. □

5. Feller Processes

When constructing Markov processes from a given Markov transition semigroup, the hypotheses of section 4 for right processes are directly verifiable only in special cases, for instance if this semigroup satisfies strong regularity conditions. We are summarizing some results on a “nice” class of Markov processes which originate from such a class of semigroups. Their “Feller” properties are based on analytic conditions on the semigroup or resolvent rather than on a probabilistic context. Still, due to strong regularity of their semigroups, the resulting “Feller processes” admit all of the important probabilistic properties of a Markov process we could wish for.

Feller processes are therefore the most advisable basis for the study of a Markov process, in case its semigroup is readily available and the needed regularity can actually be verified. They are a special case of right processes and encompass the big class of Lévy processes (see section 6), which includes important prototypes such as the Brownian motion. The class of Feller processes can be deeply studied on its own. However, we will mainly use them to simplify various proofs, as many probabilistic properties and their analytic counterparts are automatically fulfilled in the Feller context.

As noted by [RW00a, p. 241], "*every author has his or her own definition of 'Feller semigroup'*", the main divergency being whether regularity conditions should hold on $\mathcal{C}_0(E)$ or on $b\mathcal{C}(E)$. We are considering $\mathcal{C}_0(E)$-Feller semigroups in our work, as here $\mathcal{C}_0(E)$ (in contrast to $b\mathcal{C}(E)$) turns out to be separable, which will be essential for a fundamental result later.

5.1. Definition

Let E be an LCCB. Following subsection 4.3, we begin by adjoining a new point Δ as follows: If E is not compact, there exists a *one-point compactification* $E_\Delta := E \cup \{\Delta\}$ with Δ being the *point at infinity* (see, e.g., [Mun00, Theorem 29.1]). If E is compact, adjoin a new point Δ, isolated from E, forming $E_\Delta := E \cup \{\Delta\}$. Just like in subsection 4.3 (see also [CW05, p. 9]), we extend every Markov transition semigroup $(T_t, t \geq 0)$ on E to a Markov transition semigroup $(\widetilde{T}_t, t \geq 0)$ on E_Δ by setting for $t \geq 0$, $x \in E$, $A \in \mathscr{E}$,

$$\begin{aligned} \widetilde{T}_t(x, A) &:= T_t(x, A), & \widetilde{T}_t(x, \Delta) &:= 0, \\ \widetilde{T}_t(\Delta, E) &:= 0, & \widetilde{T}_t(\Delta, \Delta) &:= 1. \end{aligned}$$

Since E_Δ is compact, $b\mathcal{C}(E_\Delta) = \mathcal{C}(E_\Delta)$ holds, and every function $f \in \mathcal{C}(E_\Delta)$ is the sum of a function in $\mathcal{C}_0(E)$ and a constant $f(\Delta)$ (see [CW05, Section 2.2]), and the latter vanishes in the context of subsection 4.3. As usual, we rename $(\widetilde{T}_t, t \geq 0)$ to $(T_t, t \geq 0)$, and E_Δ to E. Following [RY94, Sections III.2 and VII.1], we define:

(5.1) Definition. *A Feller semigroup $(T_t, t \geq 0)$ is a family of positive operators on $\mathcal{C}_0(E)$, which satisfies the following conditions:*

(i) $T_0 = \mathrm{id}$ *and* $\|T_t\| \leq 1$ *for all* $t \geq 0$*;*

(ii) $T_{t+s} = T_t \circ T_s$ *for all* $s, t \geq 0$*;*

(iii) $\lim_{t\downarrow 0} \|T_t f - f\| = 0$ *for every* $f \in \mathcal{C}_0(E)$.

One can show (see, e.g., [RY94, Proposition III.2.4]) that under conditions (i) and (ii), property (iii) is equivalent to the weaker condition

$$\forall f \in \mathcal{C}_0(E), x \in E : \quad \lim_{t\downarrow 0} T_t f(x) = f(x).$$

In order to apply a Feller semigroup $(T_t, t \geq 0)$ to the context of Markov process, we first need to extend it from $\mathcal{C}_0(E)$ to $b\mathscr{E}$. The existence of such an extension is asserted by [RY94, Proposition III.2.2]:

(5.2) Theorem. *For every Feller semigroup $(T_t, t \geq 0)$ on E, there exists a unique Markov transition semigroup $(\widetilde{T}_t, t \geq 0)$ on $(E, \mathscr{E})$ with*

$$\forall f \in \mathcal{C}_0(E), x \in E : \quad T_t f(x) = \widetilde{T}_t f(x).$$

(5.3) Definition. *A Markov transition semigroup $(\widetilde{T}_t, t \geq 0)$ which is associated to a Feller semigroup $(T_t, t \geq 0)$ via* (5.2) *is called Feller transition semigroup.*

In the following, we will name this extension $(\widetilde{T}_t, t \geq 0)$ of $(T_t, t \geq 0)$ again $(T_t, t \geq 0)$.

(5.4) Definition. *A right continuous Markov process having a Feller transition semigroup is called Feller process.*

The basic example of a Feller process is the Brownian motion, which will be introduced in section 14. The constant process is a trivially a Feller process:

(5.5) Example. Construct the *constant process* X on an LCCB E by defining the stochastic process $(X_t, t \geq 0)$ on $\Omega := E$ with $X_t(\omega) := \omega$, $t \geq 0$, and measures $\mathbb{P}_x := \varepsilon_x$, $x \in E$. It is trivial to show that X is a Markov process with respect to its natural filtration. Its semigroup reads

$$T_t f(x) = \mathbb{E}_x\big(f(X_t)\big) = f(x), \quad f \in b\mathscr{B}(E),\ x \in E,\ t \geq 0,$$

and is obviously a Feller semigroup. Therefore, X is a Feller process. ■

5.2. Basic Results

For Feller semigroups, canonical processes can be readily constructed with the help of Kolmogorov's extension theorem, followed by some technique of path regularization (e.g. via supermartingale regularization, as given in [RY94, Theorem III.2.7]):

(5.6) Theorem. *Let $(T_t, t \geq 0)$ be a Feller transition semigroup on $(E, \mathscr{E})$. Then there exists a simple Markov process with semigroup $(T_t, t \geq 0)$ which admits càdlàg paths.*

As already mentioned above, many properties of Feller processes directly follow from the regularity of their semigroups. The most important one will be the following (see, e.g., [RW00a, Sections III.8–III.9]):

(5.7) Theorem. *Every Feller process admits the strong Markov property relative to its right continuous (raw) natural filtration $(\mathscr{F}^0_{t+}, t \geq 0)$.*

Then, after the usual completions, every Feller process is also strongly Markovian relative to its augmented, right continuous natural filtration $(\mathscr{F}_t, t \geq 0)$, and an examination of the resolvent (see, e.g., [MR06, Corollary 4.1.4]) shows:

(5.8) Theorem. *Every Feller process satisfying the usual hypotheses is a right process.*

As every Feller semigroup preserves $\mathcal{C}_0(E)$, we can analyze it as a semigroup restricted to this Banach space. It turns out that it is sufficient to consider this restriction:

(5.9) Theorem. *Either one of the resolvent and the generator of a Feller transition semigroup $(T_t, t \geq 0)$ on $\mathcal{C}_0(E)$ completely determines $(T_t, t \geq 0)$ (as semigroup on $b\mathscr{E}$).*

Proof. Setting $\mathbb{X} = \mathcal{C}_0(E)$, the strong continuity of $(T_t, t \geq 0)$ implies that $\lim_{t\downarrow 0} T_t f = f$ for all $f \in \mathbb{X}$. Thus, $\mathbb{X}_0 = \mathbb{X}$ holds true. Theorem (1.11) then shows that the Feller semigroup on $\mathcal{C}_0(E)$ is uniquely determined by the resolvent or the generator. This completes the proof, because the Feller transition semigroup on E is uniquely determined by its restriction to $\mathcal{C}_0(E)$, as seen in theorem (5.2). □

For the study of the semigroup, resolvent and generator of a Feller process, the above theorem implies that it suffices to restrict the analysis to $\mathcal{C}_0(E)$. Therefore, we will always work with the semigroup restricted to $\mathcal{C}_0(E)$ when speaking about the resolvent or generator of a semigroup in the Feller context. In particular, theorem (1.9) then yields

$$\mathscr{D}(A) = U\mathcal{C}_0(E). \tag{5.10}$$

This simplifies some results of sections 1–3:

(5.11) Remark. Dynkin's formula (3.18) for the generator is always applicable in the Feller context: Because every first entry time into an open set is a stopping time over $(\mathscr{F}^0_{t+}, t \geq 0)$, every Feller process is strongly Markovian with respect to the stopping times τ_ε, $\varepsilon > 0$, as asserted by theorem (5.7). Additionally, the continuity condition is always fulfilled, because $\mathscr{D}(A) \subseteq \mathbb{X}_0 = \mathcal{C}_0(E)$. The "trap condition" $\mathbb{E}_x(\tau_\varepsilon) < +\infty$ is further examined in subsection 5.3 below. ■

(5.12) Theorem. *Let $(T_t, t \geq 0)$ be a Feller semigroup on E with generator A and the linear operator $A^\bullet$ be an extension of A to $\mathscr{D}(A^\bullet)$. Let $\mathscr{D} \subseteq \mathcal{C}_0(E)$ be a linear subspace, satisfying*

(i) $\mathscr{D}(A) \subseteq \mathscr{D} \subseteq \mathscr{D}(A^\bullet)$, *and*

(ii) there is an $\alpha > 0$ such that the following implication holds true:

$$A^\bullet u = \alpha u,\ u \in \mathscr{D} \quad \Rightarrow \quad u = 0.$$

Then $\mathscr{D}(A) = \mathscr{D}$.

Proof. The rest of the requirements of lemma (1.12) are fulfilled, because we have $\mathbb{X}_0 = \mathbb{X} = \mathcal{C}_0(E)$ in the Feller case. □

The Feller properties of definition (5.1) are conditions on the semigroup, which can be verified directly only in special cases. The following theorem, as given in [KPS12a, Appendix B], shows that these defining properties are perfectly reflected in the resolvent, which will give us more flexibility when examining whether a given process is Feller. In the following theorem, "$T\mathcal{C}_0(E) \subseteq \mathcal{C}_0(E)$" is an abbreviation for "$T_t\mathcal{C}_0(E) \subseteq \mathcal{C}_0(E)$ for all $t \geq 0$", and $U\mathcal{C}_0(E) \subseteq \mathcal{C}_0(E)$ stands for "$U_\alpha\mathcal{C}_0(E) \subseteq \mathcal{C}_0(E)$ for all $\alpha > 0$":

(5.13) Theorem. *Let X be a right continuous simple Markov process with semigroup $(T_t, t \geq 0)$ and resolvent $(U_\alpha, \alpha > 0)$. Then the following statements are equivalent:*

(i) $(T_t, t \geq 0)$ *is Feller.*

(ii) $T\mathcal{C}_0(E) \subseteq \mathcal{C}_0(E)$*, and for all* $f \in \mathcal{C}_0(E)$*,* $x \in E$*,* $\lim_{t\downarrow 0} T_t f(x) = f(x)$*.*

(iii) $T\mathcal{C}_0(E) \subseteq \mathcal{C}_0(E)$*, and for all* $f \in \mathcal{C}_0(E)$*,* $x \in E$*,* $\lim_{\alpha\to\infty} \alpha U_\alpha f(x) = f(x)$*.*

(iv) $U\mathcal{C}_0(E) \subseteq \mathcal{C}_0(E)$*, and for all* $f \in \mathcal{C}_0(E)$*,* $x \in E$*,* $\lim_{t\downarrow 0} T_t f(x) = f(x)$*.*

(v) $U\mathcal{C}_0(E) \subseteq \mathcal{C}_0(E)$*, and for all* $f \in \mathcal{C}_0(E)$*,* $x \in E$*,* $\lim_{\alpha\to\infty} \alpha U_\alpha f(x) = f(x)$*.*

5.3. On Traps

Holding points have already been discussed in subsection 3.3. We collect some additional results on absorbing holding points, which are also called traps, in the Feller context.

(5.14) Definition. *For a right continuous Markov process on $(E, \mathscr{E})$, a point $x \in E$ is called trap, if it satisfies*

$$\mathbb{P}_x(\forall t \geq 0 : X_t = x) = 1.$$

The following equivalent characterization is well known, see e.g. [Dyn65, pp. 135ff]:

(5.15) Lemma. *Let X be a right continuous Markov process on $(E, \mathscr{E})$. A point $x \in E$ is a trap, if and only if*

$$\forall f \in \mathscr{D}(A) : \quad Af(x) = 0.$$

If x is not a trap, then it is immediate from theorem (3.15) that $\mathbb{E}_x(\tau_0) < +\infty$. But even more is true for Feller processes:

(5.16) Theorem. *Let X be a Feller process on a metric space E, $x \in E$, and consider the first exit times*

$$\tau_\varepsilon := \inf\{t \geq 0 : d(X_t, X_0) > \varepsilon\}, \quad \varepsilon > 0.$$

If x is not a trap for X, then there exists $\delta > 0$ such that

$$\forall \varepsilon \in (0, \delta) : \quad \mathbb{E}_x(\tau_\varepsilon) < +\infty.$$

As will be seen now, the Feller context is not really necessary for this result to hold, but it simplifies the argument. Our proof follows [Kni81, p. 53] quite closely:

Proof. As x is not a trap, there exists $\tilde{f} \in \mathscr{D}(A)$ with $A\tilde{f}(x) \neq 0$. The semigroup is Feller, so its domain satisfies $\mathscr{D}(A) \subseteq \mathcal{C}_0(E)$, and we can rescale $\tilde{f}$ to $f \in \mathscr{D}(A)$ such that

$$\exists \delta > 0 : \forall y \in B_\delta(x) : \quad Af(y) \geq 1.$$

Let $\varepsilon \in (0,\delta)$. For any $t \geq 0$ consider the stopping time $\tau_\varepsilon \wedge t$. Then $\mathbb{E}_x(\tau_\varepsilon \wedge t) < +\infty$, and Dynkin's formula (3.17) yields

$$\mathbb{E}_x\big(f(X_{\tau_\varepsilon \wedge t}) - f(X_0)\big) = \mathbb{E}_x\Big(\int_0^{\tau_\varepsilon \wedge t} Af(X_s)\,dt\Big) \geq \mathbb{E}_x(\tau_\varepsilon \wedge t),$$

as $X_s \in \overline{B_\varepsilon(x)} \subseteq B_\delta(x)$ holds $\mathbb{P}_x$-a.s. for all $s < \tau_\varepsilon$. Then, by LDCT,

$$\mathbb{E}_x(\tau_\varepsilon) \leq \limsup_{t\to\infty} \mathbb{E}_x(\tau_\varepsilon \wedge t) \leq 2\,\|f\|_\infty < \infty. \qquad \square$$

6. Lévy Processes

We are going to give a quick reminder on the theory of Lévy processes and their connection to Poisson random measures. This is not a complete treatise of the theory, we will only collect the basic results which will be needed in our construction of Brownian motions on a star graph. Our summary is based on the standard literature [Itô72], [Itô06], [Itô10], [App09], [Sat13], where the reader may find more detailed information.

6.1. Definitions

(6.1) Definition. *A Markov process $X = \big(\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}^d}\big)$ on $\mathbb{R}^d$ is called Lévy Markov process, if $(X_t, t \geq 0)$*

(i) is right continuous,

(ii) has independent increments: for all $n \in \mathbb{N}$, $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables $\{X_{t_0}, X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}\}$ are independent, and

(iii) has stationary increments: for all $s, t \geq 0$, $X_{s+t} - X_t$ has the same distribution as $X_s - X_0$.

In the definition above, the conditions of independence and stationarity are understood to hold under every initial measure $\mathbb{P}_x$, $x \in \mathbb{R}^d$.

(6.2) Definition. *A semigroup $(T_t, t \geq 0)$ on $\mathbb{R}^d$ is spatially homogeneous, if for all $x, y \in \mathbb{R}^d$, $t \geq 0$, $A \in \mathscr{B}(\mathbb{R}^d)$,*

$$T_t(x, A) = T_t(x + y, A + y).$$

The Lévy Markov property of a Markov process is reflected in the spatial homogeneity of the semigroup: (see, e.g., [BB96, Theorems 37.2, 37.3])

(6.3) Theorem. *Every Lévy Markov process X has a spatially homogeneous semigroup $(T_t, t \geq 0)$, and every right continuous Markov process X with spatially homogeneous semigroup $(T_t, t \geq 0)$ is a Lévy Markov process.*

An examination of the spatially homogeneous semigroup (see, e.g., [App09, Theorem 3.1.9]) quickly shows the following:

(6.4) Theorem. *Every Lévy Markov process is a Feller process.*

It is rather unfortunate for us that most of the study of Lévy processes is only done for the initial measure $\mathbb{P}_0$, because most results can be extended to other initial laws by the translation of the process (see subsection 6.5 below). However, without the proper context of Markov processes, it is complicated to perform Markovian techniques rigorously. In order to distinguish between both contexts, we give a second definition which will be more suitable for the basic results given in the literature, following [Sat13, Definition 1.6]:

(6.5) Definition. *A stochastic process $(X_t, t \geq 0)$ on some probability space $(\Omega, \mathscr{G}, \mathbb{P})$ is called Lévy process, if $(X_t, t \geq 0)$*

(i) admits $X_0 = 0$ a.s.,

(ii) is right continuous for $t \geq 0$ and has left limits for $t > 0$ on some set $\Omega_0 \in \mathscr{G}$ with $\mathbb{P}(\Omega_0) = 1$,

(iii) is stochastically continuous, and

(iv) has independent and stationary increments.

Both theories can be connected as follows: Every stochastically continuous Lévy Markov process admitting a.s. left limits is a Lévy process for $\mathbb{P} = \mathbb{P}_0$. On the other hand, given a Lévy process $(X_t, t \geq 0)$ on $(\Omega, \mathscr{G}, \mathbb{P})$, one can define the measures $(\mathbb{P}_x, x \in \mathbb{R}^d)$ on $\mathscr{F}^0 = \sigma(X_t, t \geq 0)$ by setting for any $x \in \mathbb{R}^d$, $A_1, \ldots, A_n \in \mathscr{B}(\mathbb{R}^d)$:

$$\mathbb{P}_x\big(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n\big) := \mathbb{P}\big(X_{t_1} + x \in A_1, \ldots, X_{t_n} + x \in A_n\big).$$

Then, the process $X = \big(\Omega, \mathscr{F}^0, (\mathscr{F}^0_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}^d}\big)$ is a strong Markov process on $\mathbb{R}^d$, if some shift operators $(\Theta_t, t \geq 0)$ exist for X (cf. [Sat13, Section 40]).

6.2. Poisson Point Processes

The Lévy–Itô decomposition (6.17), which will be stated in the next subsection, shows that Lévy processes have a close connection to Poisson random measures. We are going to introduce the latter now, following [IW89, Sections I.8–9]:

Let $(E, \mathscr{E})$ be a measurable space. Consider the set M of all $\mathbb{N}_0 \cup \{\infty\}$-valued measures on $(E, \mathscr{E})$, endowed with the smallest σ-algebra $\mathscr{M}$ on M for which the mappings $M \to \mathbb{N}_0 \cup \{\infty\}$, $\mu \mapsto \mu(B)$, are measurable for all $B \in \mathscr{E}$.

(6.6) Definition. *An $(M, \mathscr{M})$-valued random variable μ on a probability space $(\Omega, \mathscr{G}, \mathbb{P})$ is a Poisson random measure, if*

(i) for each $B \in \mathscr{E}$, $\mu(B)$ is Poisson-distributed or a.s. equals $+\infty$, and

(ii) if $B_1, \ldots, B_n \in \mathscr{E}$, $n \in \mathbb{N}$, are pairwise disjoint, then $\mu(B_1), \ldots, \mu(B_n)$ are mutually independent.

We define the *intensity measure* λ of a Poisson random measure μ by

$$\lambda(B) := \mathbb{E}(\mu(B)), \quad B \in \mathscr{E}.$$

It uniquely determines the law of μ (see, e.g., [Kal02, Lemma 12.1]). On the other hand, for any σ-finite measure λ, there exists a Poisson measure μ realizing it as intensity measure, as proved in [IW89, Theorem 8.1] or [Sat13, Proposition 19.4].

Another representation of Poisson random measures are Poisson point processes. These have been introduced in [Itô72] as follows:

(6.7) Definition. *Let $(E, \mathscr{E})$ be a measurable space.*

(i) A point function p on E is a mapping $p\colon D_p \mapsto E$, where the domain D_p of p is a countable subset of $(0, \infty)$.

(ii) For any point function p, let the counting measure N_p on $(0, \infty) \times E$ with σ-algebra $\mathscr{B}((0, \infty)) \otimes \mathscr{E}$ be defined by

$$N_p\big((0, t] \times B\big) = \#\{s \in D_p : s \leq t, p(s) \in B\}, \quad t > 0, B \in \mathscr{E}.$$

(iii) Let Π_E be the set of all point functions on E, and $\mathscr{B}(\Pi_E)$ be the smallest σ-algebra on Π_E such that for all $t > 0$, $B \in \mathscr{E}$, the mapping $p \mapsto N_p\big((0, t] \times B\big)$ is measurable.

(6.8) Definition. *Let p be a point function on E.*

(i) For $t \geq 0$, the shifted point function $\Theta_t p$ is defined by

$$\Theta_t p\colon D_{\Theta_t p} \to E, \quad s \mapsto (\Theta_t p)(s) := p(s + t),$$

with $D_{\Theta_t p} := \{s > 0 : s + t \in D_p\}$.

(ii) For $s \geq 0$, the stopped point function $\alpha_s p$ is defined by

$$\alpha_s p\colon D_{\alpha_s p} \to E, \quad t \mapsto (\alpha_s p)(t) := p(t),$$

with $D_{\alpha_s p} := \{t \leq s : t \in D_p\}$.

(6.9) Definition.

(i) A $(\Pi_E, \mathscr{B}(\Pi_E))$-valued random variable p on a probability space $(\Omega, \mathscr{G}, \mathbb{P})$ is a point process on E.

(ii) A point process p is σ-finite, if there exists an increasing sequence $(B_n, n \in \mathbb{N})$ in $\mathscr{E}$ with $\bigcup_n B_n = E$, such that $N_p\big((0, t) \times B_n\big) < +\infty$ for every $t \geq 0$, $n \in \mathbb{N}$.

(iii) A point process p is stationary, if p and $\Theta_t p$ have the same law for every $t \geq 0$.

(iv) A point process p is renewal, if p is stationary and for every $t \geq 0$, $\alpha_t p$ and $\Theta_t p$ are independent.

(v) *A point process p is a Poisson point process, if N_p is a Poisson random measure on $(0, \infty) \times E$.*

There is a natural connection between the renewal property of a point process and its associated random measure being Poisson, as explained in [Itô72, Theorem 3.1]:

(6.10) Theorem. *Let p be a σ-finite, renewal point process. Then p is a Poisson point process.*

6.3. Lévy–Khintchine representation and Lévy–Itô decomposition

We are ready to give the fundamental properties of Lévy processes, which will provide an insight into their characteristics and their path structure. Due to their additive structure, the distributions associated to a Lévy process are regular in the following sense:

(6.11) Definition. *A probability measure μ on $\mathbb{R}^d$ is infinitely divisible, if for every $n \in \mathbb{N}$, there exists a probability measure μ_n on $\mathbb{R}^d$ such that ${\mu_n}^n = \mu$ holds.*[2]

Because of the increments being stationary and independent (and $\mathbb{P}_{X_0}$ being trivial), the distributions of a Lévy process X are already characterized by the one-dimensional law X_t for any $t > 0$. The Lévy property is then reflected in the infinitely divisibility of this law, as seen in [Sat13, Theorem 7.10, Corollary 11.6]:

(6.12) Theorem.

(i) *If X is a Lévy process, then for any $t \geq 0$, $\mathbb{P}_{X_t}$ is infinitely divisible, and with $\mu := \mathbb{P}_{X_1}$, $\mathbb{P}_{X_t} = \mu^t$ holds true.*[3]

(ii) *If μ is an infinitely divisible measure on $\mathbb{R}^d$, there exists a Lévy process X with $\mathbb{P}_{X_1} = \mu$.*

Infinitely divisible measures, and thus the distributions of Lévy processes, can be characterized by the *Lévy–Khintchine representation*, as given in [Sat13, Theorem 8.1]:

(6.13) Theorem.

(i) *If μ is an infinitely divisible measure on $\mathbb{R}^d$, then the characteristic function of μ reads*

$$
\begin{aligned}
\widehat{\mu}(z) = \exp\Big(&-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle \\
&+ \int \big(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle\, \mathbb{1}_{B_1(0)}(x)\big)\nu(dx)\Big), \quad z \in \mathbb{R}^d,
\end{aligned}
\tag{6.14}
$$

[2] For $n \in \mathbb{N}$, the n-fold convolution of a probability measure μ is denoted by μ^n.

[3] For $t \geq 0$, the t-th power μ^t of an infinitely divisible measure μ is defined to be the probability measure with the characteristic function $\hat{\mu}^t$, see [Sat13, Lemmas 7.6–7.9]. By [Sat13, Lemma 2.5 (iii)], this is consistent with footnote 2 for $t \in \mathbb{N}$.

where A is a symmetric, nonnegative-definite $d \times d$-matrix, $\gamma \in \mathbb{R}^d$, and ν is a measure on $\mathbb{R}^d$ satisfying

$$\nu(\{0\}) = 0 \quad \textit{and} \quad \int (|x|^2 \wedge 1)\nu(dx) < +\infty. \tag{6.15}$$

(ii) The representation of $\widehat{\mu}$ in (i) by A, ν, and γ is unique.

(iii) If A is a symmetric, nonnegative-definite $d \times d$-matrix, $\gamma \in \mathbb{R}^d$, and ν is a measure on $\mathbb{R}^d$ satisfying (6.15), *then there exists an infinitely divisible measure μ on $\mathbb{R}^d$ with characteristic function as given in* (6.14).

If $\int_{|x|\leq 1} |x| \ \nu(dx) < +\infty$, equation (6.14) can be rewritten (cf. [Sat13, Remark 8.3]) to

$$\widehat{\mu}(z) = \exp\Big(-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma_0, z\rangle + \int (e^{i\langle z,x\rangle} - 1)\nu(dx)\Big), \quad z \in \mathbb{R}^d, \tag{6.16}$$

for some $\gamma_0 \in \mathbb{R}^d$.

For an infinitely divisible measure μ, the set (A, ν, γ) or $(A, \nu, \gamma_0)_0$, as given in (6.14) or (6.16) respectively, is called *generating triplet* of μ. For any Lévy process X, the measure $\mu := \mathbb{P}_{X_1}$ satisfies $\mathbb{P}_{X_t} = \mu^t$, so the generating triplet of the infinitely divisible measure $\mathbb{P}_{X_t}$ reads (cf. [Sat13, Corollary 8.3])

$$(A_t, \nu_t, \gamma_t) = (tA, t\nu, t\gamma), \quad t \geq 0.$$

Thus, in the Lévy case, it is sufficient to know the generating triplet (A, ν, γ) or $(A, \nu, \gamma_0)_0$ of $\mu = \mathbb{P}_{X_1}$. This is called the *generating triplet of the Lévy process.*

Summarizing all of the above results, we see that for every choice (A, ν, γ) as in (iii) of theorem (6.13), there exists an infinitely divisible measure μ, giving rise (by theorem (6.12)) to a Lévy process X with generating triplet (A, ν, γ), and this process is uniquely determined in law by μ or equivalently by (A, ν, γ) (see [Sat13, Theorem 7.10(iii), Theorem 9.8(ii)]).

The fundamental theorem for studying the sample path behavior of an additive process is the *Lévy–Itô decomposition* [Sat13, Theorems 19.2–19.3], which, roughly speaking, enables us to decompose any additive process into its jump part and its continuous part. We will only cite this result for the Lévy case. In the following context, let $H := (0, \infty) \times (\mathbb{R}^d \backslash \{0\})$ be equipped with its Borel σ-algebra $\mathscr{B}(H)$, and set $D(a, b] := \{x \in \mathbb{R}^d : a < |x| \leq b\}$ for $0 \leq a < b \leq \infty$.

(6.17) Theorem. *Let X be a Lévy process on $(\Omega, \mathscr{G}, \mathbb{P})$ with generating triplet (A, ν, γ), and define the measure $\widetilde{\nu}$ on H by*

$$\widetilde{\nu}((0, t] \times B) := \nu_t(B) = t\,\nu(B), \quad t > 0, \ B \in \mathscr{B}(\mathbb{R}^d \backslash \{0\}).$$

With Ω_0 as given in definition (6.5), *define for $B \in \mathscr{B}(H)$,*

$$J(B, \omega) := \begin{cases} \#\{s \geq 0 : (s, X_s(\omega) - X_{s-}(\omega)) \in B\}, & \omega \in \Omega_0, \\ 0, & \omega \notin \Omega_0. \end{cases}$$

Then the following holds:

(i) *J is a Poisson random measure on H with intensity measure $\widetilde{\nu}$.*

(ii) *There exists $\Omega_1 \in \mathscr{G}$ with $\mathbb{P}(\Omega_1) = 1$ such that for any $\omega \in \Omega_1$,*

$$X_t^1(\omega) := \lim_{\varepsilon \downarrow 0} \int_{(0,t]\times D(\varepsilon,1]} \big(x\, J\big(d(s,x),\omega\big) - x\, \widetilde{\nu}\big(d(s,x)\big)\big) + \int_{(0,t]\times D(1,\infty)} x\, J\big(d(s,x),\omega\big)$$

is defined for all $t \geq 0$, and the convergence is uniform in t on any bounded interval. The process $(X_t^1, t \geq 0)$ is a Lévy process with generating triplet $(0, \nu, 0)$.

(iii) *Define for $\omega \in \Omega_1$*

$$X_t^2(\omega) := X_t(\omega) - X_t^1(\omega), \quad t \geq 0.$$

There exists $\Omega_2 \in \mathscr{G}$ with $\mathbb{P}(\Omega_2) = 1$ such that for any $\omega \in \Omega_2$, $t \mapsto X_t^2(\omega)$ is continuous. The process $(X_t^2, t \geq 0)$ is a Lévy process with generating triplet $(A, 0, \gamma)$.

(iv) *The processes $(X_t^1, t \geq 0)$ and $(X_t^2, t \geq 0)$ are independent.*

If the “small jumps” have a finite mean, then the decomposition above can be simplified by omitting the “compensated sum of jumps” (namely the first part of X^1):

(6.18) Theorem. *Suppose the Lévy process X of theorem (6.17) satisfies*

$$\int_{|x|\leq 1} |x|\ \nu(dx) < +\infty.$$

Then there exists $\Omega_3 \in \mathscr{G}$ with $\mathbb{P}(\Omega_3) = 1$ such that for any $\omega \in \Omega_3$,

$$X_t^3(\omega) := \int_{(0,t]\times D(0,\infty)} x\, J\big(d(s,x),\omega\big)$$

is defined for all $t \geq 0$. Then $(X_t^3, t \geq 0)$ is a Lévy process with characteristic function

$$\mathbb{E}\big(e^{i\langle z, X_t^3\rangle}\big) = \exp\Big(t \int_{\mathbb{R}^d} \big(e^{i\langle z,x\rangle} - 1\big)\nu(dx)\Big), \quad t \geq 0,\ z \in \mathbb{R}^d.$$

Define for $\omega \in \Omega_3$

$$X_t^4(\omega) := X_t(\omega) - X_t^3(\omega), \quad t \geq 0.$$

Then, for any $\omega \in \Omega_2 \cap \Omega_3$, $t \mapsto X_t^4(\omega)$ is continuous, and $(X_t^4, t \geq 0)$ is a Lévy process with characteristic function

$$\mathbb{E}\big(e^{i\langle z, X_t^4\rangle}\big) = \exp\big(-\frac{1}{2}\langle z, tAz\rangle + i\langle t\gamma_0, z\rangle\big), \quad t \geq 0,\ z \in \mathbb{R}^d.$$

The processes $(X_t^3, t \geq 0)$ and $(X_t^4, t \geq 0)$ are independent.

It is also possible to assemble Lévy processes by realizing the prescriptions given in the above Lévy–Itô decomposition in a process construction. This is done, e.g., in [Sat13, Section 20]. We will not need this approach, but recall some basic properties of Lévy processes following from the Lévy–Itô decomposition. They are given in [Sat13, Theorems 21.1–21.5]:

(6.19) Theorem. *Let X be a Lévy process with generating triplet (A, ν, γ).*

(i) X has a.s. continuous paths, if and only if $\nu = 0$.

(ii) X has a.s. piecewise constant paths, if and only if $A = 0$, $\nu(\mathbb{R}^d) < +\infty$ and $\gamma_0 = 0$.

(iii) If $\nu(\mathbb{R}^d) = +\infty$, then, a.s., the jumping times are countable and dense in $[0, \infty)$. If $\nu(\mathbb{R}^d) < +\infty$, then, a.s., jumping times are infinitely many and countable in increasing order, and the first jumping time has exponential distribution with mean $1/\nu(\mathbb{R}^d)$.

(iv) Let $d = 1$. X has a.s. increasing paths, if and only if $A = 0$, $\nu\big((-\infty, 0)\big) = 0$, $\int_{(0,1]} x\,\nu(dx) < +\infty$ and $\gamma_0 \geq 0$.

We will mainly work with the following special type of Lévy processes:

(6.20) Definition. *A Lévy process on $\mathbb{R}$ with a.s. increasing paths is a* subordinator.

For subordinators, the Lévy–Itô decomposition (6.18) takes its easiest form. In this case, it is also more convenient to work with the Laplace transform instead of the characteristic function. It reads (see [Sat13, Remark 21.6], [RW00a, Section II.37])

$$\mathbb{E}(e^{-\alpha X_t}) = \exp\Big(t\Big(\int_{(0,\infty)} (e^{-\alpha x} - 1)\nu(dx) - \alpha\gamma_0\Big)\Big), \quad t \geq 0,\ \alpha \geq 0. \tag{6.21}$$

Lévy–Khintchine representation and Lévy–Itô decomposition present powerful tools in the study of Lévy processes, especially when used combined. We will end this section by employing them to derive a proof of lemma (6.25), which we prepare first by the following examinations:

(6.22) Remark. Let $X^1, \ldots, X^d$ be independent, $\mathbb{R}$-valued Lévy processes on a common probability space with generating triplets (a^k, ν^k, γ^k), $k \in \{1, \ldots, d\}$. Then, by definition, the process $X := (X^1, \ldots, X^d)$ is an $\mathbb{R}^d$-valued Lévy processes, and its Lévy–Khintchine representation reads, for $z = (z^1, \ldots, z^d) \in \mathbb{R}^d$,

$$\begin{aligned}
\widehat{\mu}(z) = \mathbb{E}\Big(e^{i\langle z, X\rangle}\Big) &= \prod_{k=1}^{d} \mathbb{E}\Big(e^{iz^k X^k}\Big) \\
&= \prod_{k=1}^{d} \exp\Big(-\frac{1}{2} z^k a^k z^k + i\,\gamma^k z^k + \int (e^{i\,z^k x^k} - 1 - i\,z^k x^k\,\mathbb{1}_{B_1(0)}(x^k))\nu^k(dx^k)\Big) \\
&= \exp\Big(-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int (e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle\,\mathbb{1}_{B_1(0)}(x))\nu(dx)\Big),
\end{aligned}$$

with $\gamma = (\gamma^1, \dots, \gamma^d)$, $A = \operatorname{diag}(a^1, \dots, a^d)$ and

$$\nu = \nu^1 \otimes \varepsilon_0 \otimes \cdots \otimes \varepsilon_0 + \varepsilon_0 \otimes \nu^2 \otimes \varepsilon_0 \otimes \cdots \otimes \varepsilon_0 + \cdots + \varepsilon_0 \otimes \cdots \otimes \varepsilon_0 \otimes \nu^d. \qquad \blacksquare$$

(6.23) Remark. Furthermore, in the context of remark (6.22), $X^\Sigma := X^1 + \cdots + X^d$ is an $\mathbb{R}$-valued Lévy processes with its generating triplet $(a^\Sigma, \nu^\Sigma, \gamma^\Sigma)$ being given by $a^\Sigma = a^1 + \cdots + a^d$, $\nu^\Sigma = \nu^1 + \cdots + \nu^d$, and $\gamma^\Sigma = \gamma^1 + \cdots + \gamma^d$, as its Lévy–Khintchine representation reads, for $z \in \mathbb{R}$,

$$\begin{aligned}\widehat{\mu}^\Sigma(z) &= \mathbb{E}\Big(e^{i(zX^1 + \cdots + zX^d)}\Big) = \prod_{k=1}^d \mathbb{E}\Big(e^{izX^k}\Big) \\ &= \prod_{k=1}^d \exp\Big(-\frac{1}{2} z\, a^k z + i\,\gamma^k z + \int (e^{i\,z\,x} - 1 - i\,z\,x\,\mathbb{1}_{B_1(0)}(x))\nu^k(dx)\Big). \qquad \blacksquare\end{aligned}$$

The stochastic continuity of a Lévy process X implies that X has no fixed times of discontinuity (see, e.g., [Sat13, Equation (1.10)]), that is,

$$\forall t > 0: \quad X_t = X_{t-} \quad \text{a.s.}. \tag{6.24}$$

For independent Lévy processes, even more is true: They have a.s. no simultaneous jumps. This result is well known, but a complete proof is somewhat difficult to find in the literature. We are following the argument indicated in [MS12, p. 106]:

(6.25) Lemma. *Let $X^1, \dots, X^k$ be independent, $\mathbb{R}$-valued Lévy processes on $(\Omega, \mathscr{G}, \mathbb{P})$. Then, a.s., not more than one of the processes $X^1, \cdots, X^k$ jumps at the same time, that is, with*

$$\Delta X_t^k := X_t^k - X_{t-}^k, \quad t > 0,\ k \in \{1, \dots, d\},$$

the set $\big\{\exists t > 0,\ i, j \in \{1, \dots, d\}, i \neq j : \Delta X_t^i \neq 0 \wedge X_t^j \neq 0\big\}$ is a null set.

Proof. As the above set is the union of simultaneous jumps of any two component processes, it suffices to consider the set S of simultaneous jumps of $X := (X^1, X^2)$, which reads

$$S := \{t > 0 : \Delta X_t^1 \neq 0 \wedge X_t^2 \neq 0\} = \{t > 0 : (t, X_t - X_{t-}) \in B\}$$

for $B := (0, \infty) \times A \in \mathscr{B}(H)$ with $A := \{(x^1, x^2) \in \mathbb{R}^2 : x^1 = 0 \vee x^2 = 0\}^\complement$ (as $x^1 \neq 0 \wedge x^2 \neq 0$ is equivalent to $\neg(x^1 = 0 \vee x^2 = 0)$).

Then, by the Lévy–Itô decomposition (6.17), the random variable $J(B) := |S|$ is Poisson distributed with mean $\widetilde{\nu}(B)$, and $\{\exists t > 0 : \Delta X_t^1 \neq 0 \wedge X_t^2 \neq 0\} = \{J(B) > 0\}$ is measurable. But the continuity of the measure $\widetilde{\nu}$ yields

$$\widetilde{\nu}(B) = \lim_{t \upuparrows \infty} \widetilde{\nu}\big((0, t) \times A\big) = \lim_{t \upuparrows \infty} t\, \nu(A) = 0,$$

because remark (6.23) gives, with A as above,

$$\nu(A) = \int_A d(\nu^1 \otimes \varepsilon_0 + \varepsilon_0 \otimes \nu^2) = 0.$$

Therefore, $J(B) = 0$ a.s.. $\square$

6.4. Integration with respect to a Poisson Point Processes

Later, we will need to evaluate an integral with respect to a Poisson point process. We quickly establish just as much theory as necessary for the derivation of result (6.29) by following [IW89, Section II.3]:

(6.26) Definition. *For a point process p on a probability space $(\Omega, \mathscr{G}, \mathbb{P})$, let*

$$N_p(t, B) := N_p\big((0,t] \times B\big) = \sum_{s \in D_p,\, s \le t} \mathbb{1}_B(p(s)), \quad t > 0,\ B \in \mathscr{E}.$$

The point process is adapted to a filtration $(\mathscr{G}_t, t \ge 0)$ of $\mathscr{G}$, if for every $B \in \mathscr{E}$, $t > 0$, $N_p(t, B)$ is $\mathscr{G}_t$-measurable. Define $\Gamma_p := \big\{B \in \mathscr{E} : \mathbb{E}(N_p(t, B)) < +\infty$ for all $t > 0\big\}$.

Let p be a $(\mathscr{G}_t, t \ge 0)$-adapted point process. Then, for every $B \in \Gamma_p$, $t \mapsto N_p(t, B)$ is an adapted, integrable, càdlàg increasing process, thus a càdlàg submartingale. Therefore, by the Doob–Meyer decomposition [IW89, Theorem I.6.12], there exists a natural, integrable, increasing process $t \mapsto \hat{N}_p(t, B)$ such that $t \mapsto N_p(t, B) - \hat{N}_p(t, B)$ is a martingale. This leads to:

(6.27) Definition. *A random measure $\hat{N}_p$ is called compensator of a $(\mathscr{G}_t, t \ge 0)$-adapted point process p (or its associated random measure N_p), if it satisfies the following:*

(i) for each $B \in \Gamma_p$, $t \mapsto \hat{N}_p(t, B)$ is a continuous, $(\mathscr{G}_t, t \ge 0)$-adapted, increasing process;

(ii) for each $t \ge 0$, $\hat{N}_p(t, \cdot\,)$ is a.s. a σ-finite measure on $\big(E, \mathscr{B}(E)\big)$;

(iii) for each $B \in \Gamma_p$, $t \mapsto N_p(t, B) - \hat{N}_p(t, B)$ is a $(\mathscr{G}_t, t \ge 0)$-adapted martingale.

For a Poisson random measure, $N_p(s+t, B) - N_p(t, B) = N_p\big((t, s+t] \times B\big)$ is independent of $N_p\big((0,t] \times B\big)$, so it is immediate (see also [IW89, p. 60]) that the compensator of a Poisson point process p with respect to its natural filtration is given by

$$\hat{N}_p(t, B) = \mathbb{E}\big(N_p(t, B)\big), \quad t > 0,\ B \in \Gamma_p. \tag{6.28}$$

(6.29) Theorem. *Let P be a subordinator with Lévy measure ν and N be the Poisson random measure of P as given in the Lévy–Itô decomposition. Then, for all $\alpha > 0$, $\beta > 0$ and $f \in p\mathscr{B}(\mathbb{R}_+)$ satisfying $\int_{0+}^{\infty} f(l)\, \nu(dl) < +\infty$, the following formula holds true:*

$$\mathbb{E}\Big(\int_0^\infty \int_{0+}^\infty e^{-\alpha P(t-)}\, e^{-\beta t}\, f(l)\, N(dt \times dl)\Big) = \int_0^\infty \mathbb{E}\Big(e^{-\alpha P(t-)}\Big)\, e^{-\beta t}\, dt \cdot \int_{0+}^\infty f(l)\, \nu(dl).$$

Proof. As seen in equation (6.28) in combination with theorem (6.17) (or [IW89, Example II.4.1]), the compensator of the Poisson point process of a Lévy process with Lévy measure ν reads $\hat{N} = \lambda \otimes \nu$. The process $(t, l) \mapsto e^{-\alpha P(t-)}\, e^{-\beta t}\, f(l)$ is predictable and

$$\begin{aligned} &\mathbb{E}\Big(\int_0^\infty \int_{(0,\infty)} \Big|e^{-\alpha P(t-)}\, e^{-\beta t}\, f(l)\Big|\, \hat{N}(dt \times dl)\Big) \\ &= \int_0^\infty \mathbb{E}\big(e^{-\alpha P(t-)}\big) e^{-\beta t}\, dt \cdot \int_{(0,\infty)} f(l)\, \nu(dl) < +\infty \end{aligned}$$

holds true, thus $(t,l) \mapsto e^{-\alpha P(t-)} e^{-\beta t} f(l)$ is in F_p^1 in the sense of [IW89, Definition (II.3.3)], and the result follows from [IW89, Equation between (II.3.7) and (II.3.8)] (see also [Çın11, Theorem 6.2]). □

The reader may observe that for $\alpha = 0$, that is without the term $e^{-\alpha P(t-)}$, this theorem reduces to Campbell's theorem, see, e.g., [CSKM13, Theorem 4.1]. However, in the case $\alpha > 0$, the part $P(t-)$ depends on all "marks" of the point process up to t, so the integrated function does not only depend on the current "mark" at t, and we had to apply the theory of stochastic integration with respect to Poisson point processes.

6.5. Translation, Centering and Reflection

Let $X = (\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ be a Lévy Markov process on $E = \mathbb{R}^d$. The Markov property is a regularity condition on the process when translated in time, which is best described with the help of the (time) shift operators $(\Theta_t, t \geq 0)$. Lévy Markov processes additionally feature regularity on space translations, so we introduce transformation operators which capture the "spatial shift" the Lévy process X:

(6.30) Definition. *A family $(\gamma_x, x \in E)$ is called translation operators for X, if it is a collection of mappings $\gamma_x\colon \Omega \to \Omega$, $x \in E$, satisfying*

$$\begin{aligned} \forall x \in E, t \geq 0: &\quad X_t \circ \gamma_x = X_t + x, \\ \forall x, y \in E: &\quad \gamma_x \circ \gamma_y = \gamma_{x+y}. \end{aligned}$$

(6.31) Definition. *A mapping $\Gamma\colon \Omega \to \Omega$ is called centering operator for X, if*

$$\forall t \geq 0: \quad X_t \circ \Gamma = X_t - X_0.$$

We assume the existence of translation operators $(\gamma_x, x \in E)$ and of a centering operator Γ for X for the rest of this subsection.

(6.32) Example. If we are in the context of the canonical coordinate process $(X_t, t \geq 0)$, that is $X_t(\omega) = \omega(t)$ for all $\omega \in \Omega$, $t \geq 0$, on the path space

$$\Omega = \{\omega\colon \mathbb{R}_+ \to E \,|\, \omega \text{ càdlàg}\},$$

then the existence of these operators is trivial: We can choose for all $\omega \in \Omega$

$$\begin{aligned} \gamma_x(\omega) &:= \omega + x, \quad x \in E, \\ \Gamma(\omega) &:= \omega - \omega(0), \end{aligned}$$

which exist, as translated càdlàg functions remain càdlàg. When also using the canonical shift operators $(\Theta_t, t \geq 0)$, namely

$$\Theta_t(\omega) := \omega(t + \cdot\,), \quad t \geq 0,\ \omega \in \Omega,$$

then the chosen translation and shift operators commute, as for all $t \geq 0$, $x \in E$, $\omega \in \Omega$,

$$\gamma_x \circ \Theta_t(\omega) = \gamma_x\big(\omega(t + \cdot\,)\big) = \omega(t + \cdot\,) + x = \Theta_t(\omega + x) = \Theta_t \circ \gamma_x(\omega). \qquad \blacksquare$$

(6.33) Remark. Translation and shift operators always commute on process level, as

$$X_s \circ \gamma_x \circ \Theta_t = X_{s+t} + x = X_s \circ \Theta_t \circ \gamma_x,$$

whereas translating a centered processes has no effect:

$$X_s \circ \Gamma \circ \gamma_x = (X_s + x) - (X_0 + x) = X_s \circ \Gamma. \qquad \blacksquare$$

We are going to show that, due to the spatial homogeneity, a translated Lévy Markov process behaves just like the original process with its starting point being translated, while centering lets it start at the origin. Of course, the following results only hold true if translation and centering operators exist for the Lévy Markov process X.

(6.34) Lemma. *For all $x, y \in E$, $t \geq 0$, $f \in b\mathscr{E}$,*

$$\mathbb{E}_x\big(f(X_t) \circ \gamma_y\big) = \mathbb{E}_{x+y}\big(f(X_t)\big).$$

Proof. The set of all $f \in b\mathscr{E}$ for which the above identity holds true forms a MVS. By the MCT, it is therefore sufficient to prove that for all $f = \mathbb{1}_A$, $A \in \mathscr{E}$,

$$\begin{aligned} \mathbb{E}_x\big(f(X_t) \circ \gamma_y\big) &= \mathbb{E}_x\big(f(X_t + y)\big) = T_t\big(f(\,\cdot\, + y)\big)(x) = T_t(x, A - y) \\ &= T_t(x + y, A) = T_t f(x + y) = \mathbb{E}_{x+y}\big(f(X_t)\big) \end{aligned}$$

holds, where we used $\mathbb{1}_A(x + y) = \mathbb{1}_{A-y}(x)$ for the third identity and the translation invariance of $(T_t, t \geq 0)$ for the forth identity. $\square$

As seen in subsection 2.1, the Markov property can be lifted from its standard definition with the help of shift operators to general functions, which are measurable with respect to the σ-algebra generated by the process. By employing the same standard techniques, we are able to show that this generalization also holds true for the spatial homogeneity, represented by translation and centering operators:

(6.35) Theorem. *For all $x, y \in E$, $F \in b\mathscr{F}$, the mapping $F \circ \gamma_y$ is in $b\mathscr{F}$ and satisfies*

$$\mathbb{E}_x(F \circ \gamma_y) = \mathbb{E}_{x+y}(F).$$

Proof. As usual, it is sufficient to prove the above claim for functions of the form

$$F = f_1(X_{t_1}) \cdots f_n(X_{t_n}),$$

with $n \in \mathbb{N}$, $0 \leq t_1 < \cdots < t_n$, $f_1, \ldots, f_n \in b\mathscr{E}$, which we are showing inductively over $n \in \mathbb{N}$. The case $n = 1$ was already done in lemma (6.34). Assuming the assertion holds true for an $n \in \mathbb{N}$, we are computing for $n + 1$ the expectation

$$\begin{aligned} \mathbb{E}_x(F \circ \gamma_y) &= \mathbb{E}_x\big(f_1(X_{t_1} + y) \cdots f_{n+1}(X_{t_{n+1}} + y)\big) \\ &= \mathbb{E}_x\big(f_1(X_{t_1} + y) \cdots f_n(X_{t_n} + y)\, \mathbb{E}_x\big(f_{n+1}(X_{t_{n+1}} + y) \,\big|\, \mathscr{F}_{t_n}\big)\big). \end{aligned}$$

The Markov property of X and lemma (6.34) applied on the last term yield

$$\begin{aligned}\mathbb{E}_x\big(f_{n+1}(X_{t_{n+1}}+y)\,\big|\,\mathscr{F}_{t_n}\big) &= \mathbb{E}_{X_{t_n}}\big(f_{n+1}(X_{t_{n+1}-t_n}+y)\big)\\ &= \mathbb{E}_{X_{t_n}}\big(f_{n+1}(X_{t_{n+1}-t_n})\circ\gamma_y\big)\\ &= \mathbb{E}_{X_{t_n}+y}\big(f_{n+1}(X_{t_{n+1}-t_n})\big),\end{aligned}$$

so we get by renaming $\widetilde{f}_i := f_i$ for $i \in \{1,\dots,n-1\}$ and $\widetilde{f}_n := f_n\,\mathbb{E}_\cdot\big(f_{n+1}(X_{t_{n+1}-t_n})\big)$:

$$\mathbb{E}_x(F\circ\gamma_y) = \mathbb{E}_x\big((\widetilde{f}_1(X_{t_1})\cdots\widetilde{f}_n(X_{t_n}))\circ\gamma_y\big).$$

By using the inductive basis for n, which is applicable because $\widetilde{f}_1,\dots,\widetilde{f}_n \in b\mathscr{E}$ as well, we conclude that

$$\begin{aligned}\mathbb{E}_x(F\circ\gamma_y) &= \mathbb{E}_{x+y}\big(\widetilde{f}_1(X_{t_1})\cdots\widetilde{f}_n(X_{t_n})\big)\\ &= \mathbb{E}_{x+y}\big(f_1(X_{t_1})\cdots f_{n+1}(X_{t_n})\,\mathbb{E}_{X_{t_n}}\big(f_{n+1}(X_{t_{n+1}-t_n})\big)\big)\\ &= \mathbb{E}_{x+y}\big(f_1(X_{t_1})\cdots f_n(X_{t_n})\,f_{n+1}(X_{t_{n+1}})\big)\\ &= \mathbb{E}_{x+y}(F).\end{aligned}$$

□

(6.36) Theorem. *For all $x \in E$, $F \in b\mathscr{F}$, the mapping $F\circ\Gamma$ is in $b\mathscr{F}$ and satisfies*

$$\mathbb{E}_x(F\circ\Gamma) = \mathbb{E}_0(F).$$

Proof. Again, we only need to prove this for

$$F = f_1(X_{t_1})\cdots f_n(X_{t_n}),$$

with $n \in \mathbb{N}$, $0 \le t_1 < \dots < t_n$, $f_1,\dots,f_n \in b\mathscr{E}$. As $X_0 = x$ holds $\mathbb{P}_x$-a.s., we can directly use theorem (6.35) to compute

$$\begin{aligned}\mathbb{E}_x(F\circ\Gamma) &= \mathbb{E}_x\big(f_1(X_{t_1}-X_0)\cdots f_n(X_{t_n}-X_0)\big)\\ &= \mathbb{E}_x\big(f_1(X_{t_1}-x)\cdots f_n(X_{t_n}-x)\big)\\ &= \mathbb{E}_x\big((f_1(X_{t_1})\cdots f_n(X_{t_n}))\circ\gamma_{-x}\big)\\ &= \mathbb{E}_{x+(-x)}\big(f_1(X_{t_1})\cdots f_n(X_{t_n})\big)\\ &= \mathbb{E}_0(F).\end{aligned}$$

□

Some Lévy Markov processes are not only spatial homogeneous, but also invariant under the reflection at the origin. Just as with (time) shifts, (spatial) translation and centering, we can lift this reflection property from the semigroup up to the process level:

(6.37) Definition. *X is called reflection invariant, if for all $t \ge 0$, $x \in E$, $A \in \mathscr{E}$,*

$$T_t(x,A) = T_t(-x,-A).$$

(6.38) Example. The Brownian motion on $\mathbb{R}$ is reflection invariant, as for all $t \geq 0$, $x \in \mathbb{R}$, $A \in \mathscr{B}(\mathbb{R})$, its semigroup satisfies

$$\begin{aligned} T_t(x, A) &= \int_A \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{(y-x)^2}{2t}}\, dy \\ &= \int_{-A} \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{(y+x)^2}{2t}}\, dy \\ &= T_t(-x, -A). \end{aligned}$$ ■

(6.39) Definition. *A mapping $\iota\colon \Omega \to \Omega$ is called reflection operator for X, if it satisfies*

$$\forall t \geq 0: \quad X_t \circ \iota = -X_t.$$

(6.40) Example. If the process X is the canonical coordinate process on the path space as given in example (6.32), then a reflection operator ι exists. It can be defined by

$$\iota(\omega) := -\omega, \quad \omega \in \Omega.$$ ■

The same course of discussion as for the centering operator also applies to the reflection operator. If the sample space admits a reflection operator ι for X, the following results hold true:

(6.41) Lemma. *If X is reflection invariant, then for all $x \in E$, $t \geq 0$, $f \in b\mathscr{E}$,*

$$\mathbb{E}_x\big(f(X_t) \circ \iota\big) = \mathbb{E}_{-x}\big(f(X_t)\big).$$

Proof. Again using the MCT, it is sufficient to prove this for $f = \mathbb{1}_A$, $A \in \mathscr{E}$. We have

$$\begin{aligned} \mathbb{E}_x\big(f(X_t) \circ \iota\big) &= \mathbb{E}_x\big(f(-X_t)\big) = T_t\big(f(-\,\cdot\,)\big)(x) = T_t(x, -A) \\ &= T_t(-x, A) = T_t f(-x) = \mathbb{E}_{-x}\big(f(X_t)\big), \end{aligned}$$

where we used $\mathbb{1}_A(-x) = \mathbb{1}_{-A}(x)$ for the third identity and the reflection invariance for the forth identity. □

The proof of the following theorem proceeds exactly like the proof of theorem (6.35):

(6.42) Theorem. *If X is reflection invariant, then for all $x \in E$, $F \in b\mathscr{F}$, the mapping $F \circ \iota$ is in $b\mathscr{F}$ and satisfies*

$$\mathbb{E}_x\big(F \circ \iota\big) = \mathbb{E}_{-x}\big(F\big).$$

Proof. As usual, it is sufficient to prove the above claim for

$$F = f_1(X_{t_1}) \cdots f_n(X_{t_n}),$$

with $n \in \mathbb{N}$, $0 \leq t_1 < \cdots < t_n$, $f_1, \ldots, f_n \in b\mathscr{E}$, which we are showing inductively over $n \in \mathbb{N}$. The case $n = 1$ was already done in lemma (6.41). Assuming the assertion holds true for an $n \in \mathbb{N}$, we are computing for $n + 1$ the expectation

$$\begin{aligned} \mathbb{E}_x\big(F \circ \iota\big) &= \mathbb{E}_x\big(f_1(-X_{t_1}) \cdots f_{n+1}(-X_{t_{n+1}})\big) \\ &= \mathbb{E}_x\big(f_1(-X_{t_1}) \cdots f_n(-X_{t_n})\, \mathbb{E}_x\big(f_{n+1}(-X_{t_{n+1}}) \,\big|\, \mathscr{F}_{t_n}\big)\big). \end{aligned}$$

The Markov property of X and lemma (6.41) applied on the last term yield

$$\begin{aligned}
\mathbb{E}_x\big(f_{n+1}(-X_{t_{n+1}}) \,\big|\, \mathscr{F}_{t_n}\big) &= \mathbb{E}_{X_{t_n}}\big(f_{n+1}(-X_{t_{n+1}-t_n})\big) \\
&= \mathbb{E}_{X_{t_n}}\big(f_{n+1}(X_{t_{n+1}-t_n}) \circ \iota\big) \\
&= \mathbb{E}_{-X_{t_n}}\big(f_{n+1}(X_{t_{n+1}-t_n})\big),
\end{aligned}$$

so we get by renaming $\widetilde{f}_i := f_i$ for $i \in \{1, \dots, n-1\}$ and $\widetilde{f}_n := f_n\,\mathbb{E}_{\cdot}\big(f_{n+1}(X_{t_{n+1}-t_n})\big)$:

$$\mathbb{E}_x(F \circ \iota) = \mathbb{E}_x\big((\widetilde{f}_1(X_{t_1}) \cdots \widetilde{f}_n(X_{t_n})) \circ \iota\big).$$

By using the inductive basis for n, which is applicable because $\widetilde{f}_1, \dots, \widetilde{f}_n \in b\mathscr{E}$ as well, we conclude that

$$\begin{aligned}
\mathbb{E}_x(F \circ \iota) &= \mathbb{E}_{-x}\big(\widetilde{f}_1(X_{t_1}) \cdots \widetilde{f}_n(X_{t_n})\big) \\
&= \mathbb{E}_{-x}\big(f_1(X_{t_1}) \cdots f_{n+1}(X_{t_n})\,\mathbb{E}_{X_{t_n}}\big(f_{n+1}(X_{t_{n+1}-t_n})\big)\big) \\
&= \mathbb{E}_{-x}\big(f_1(X_{t_1}) \cdots f_n(X_{t_n})\, f_{n+1}(X_{t_{n+1}})\big) \\
&= \mathbb{E}_{-x}(F).
\end{aligned}$$

□

Chapter II.

Transformations

We prepare the different transformation methods for Markov processes which we will employ later for the characterization and construction of Brownian motions. We focus our attention on transformations which (under certain conditions) preserve the (strong) Markov property. Fortunately, most of the required methods are commonly known and we can resort to elaborate results in the literature, which we only adjust or extend slightly.

In section 7, path-space realizations, which are a standard "ad-hoc" technique in existence theorems, are treated. In section 8, we name special conditions which ensure that stopping a Markov process at a random time maintains its Markovian structure. The well-known transformations of time substitution via additive functionals and of killing via multiplicative functionals are merely reminded in sections 9 and 10. On the other hand, the concatenation of various Markov processes on different state spaces (forming a joint process which behaves like the first process until it dies, is revived as the second process, etc.) typically is treated not at all or only in a very specific fashion in the literature. In section 11, we assemble an extensive basis on this method, in order to establish, with the help of state space transformations given in section 12, a technique allowing us to concatenate alternating, independent copies of two underlying processes in section 13. This will be the main vehicle in the construction of chapter III, where we will join processes on different subgraphs to a Brownian motion on the complete graph.

7. Versions on the Path Space

At times, it turns out to be helpful to change the underlying sample space Ω of a stochastic process X to a better structured, more controllable set. This is already the case in existence theorems for right continuous Markov processes (such as [Sha88, Theorem 2.7]), where one switches from a possibly unknown or too large space (e.g. the general path space, being the space of all mappings from $\mathbb{R}_+$ to a given state space) to the common path space of all right continuous maps. We will give a short reminder on this technique and then apply it in order to compare processes which have the same laws, but are not necessarily defined on the same sample space.

7.1. Markov Processes on Path Space

Let $X = (\Omega^X, \mathscr{G}^X, (\mathscr{G}_t^X)_{t\geq 0}, (X_t)_{t\geq 0}, (\Theta_t^X)_{t\geq 0}, (\mathbb{P}_x^X)_{x\in E})$ be a right continuous Markov process with values in $(E, \mathscr{E})$, and assume that every path $t \mapsto X_t(\omega)$, $\omega \in \Omega$, satisfies a property Π (e.g., the property Π of being right continuous).

Define the path space $\Omega^Y := \{\omega\colon \mathbb{R}_+ \to E \,|\, \omega \text{ fulfills } \Pi\}$ and the path mapping

$$\Phi\colon \Omega^X \to \Omega^Y, \quad \omega^X \mapsto \Phi(\omega^X) \quad \text{with} \quad \Phi(\omega^X)(t) := X_t(\omega^X),\ t \geq 0.$$

Consider the canonical coordinate process $(Y_t, t \geq 0)$ on Ω^Y, that is, set for all $t \geq 0$

$$Y_t\colon \Omega^Y \to E, \quad \omega \mapsto Y_t(\omega) := \omega(t),$$

equipped with the σ-algebra $\mathscr{F}^Y = \sigma(Y_t, t \geq 0)$ and filtration $\mathscr{F}^Y_t = \sigma(Y_s, s \leq t)$, $t \geq 0$, generated by Y. Then $Y_t(\Phi(\omega^X)) = Y_t(X_{\cdot}(\omega^X)) = X_t(\omega^X)$ holds for all $\omega^X \in \Omega^X$, thus

$$\forall t \geq 0: \quad Y_t \circ \Phi = X_t \quad \text{on } \Omega^X. \tag{7.1}$$

Therefore, Φ is $\mathscr{F}^0_t/\mathscr{F}^Y_t$-measurable for all $t \geq 0$ and $\mathscr{F}^0/\mathscr{F}^Y$-measurable. We are then able to define, for every $x \in E$, the measure $\mathbb{P}^Y_x$ on $\mathscr{F}^Y$ as the image of $\mathbb{P}^X_x$ under the map Φ. Equation (7.1) gives for all $n \in \mathbb{N}$, $f_1, \ldots, f_n \in b\mathscr{E}$, $t_1, \ldots, t_n \in \mathbb{R}_+$,

$$\begin{aligned}\mathbb{E}^Y_x\big(f_1(Y_{t_1}) \cdots f_n(Y_{t_n})\big) &= \mathbb{E}^X_x\big((f_1(Y_{t_1}) \cdots f_n(Y_{t_n})) \circ \Phi\big) \\ &= \mathbb{E}^X_x\big(f_1(X_{t_1}) \cdots f_n(X_{t_n})\big),\end{aligned}$$

and as $\{f_1(Y_{t_1}) \cdots f_n(Y_{t_n}); n \in \mathbb{N}, f_1, \ldots, f_n \in b\mathscr{E}, t_1, \ldots, t_n \in \mathbb{R}_+\}$ is an $\cap$-stable generator of $\mathscr{F}^Y$, we have by MCT

$$\forall G \in b\mathscr{F}^Y: \quad \mathbb{E}^Y_x(G) = \mathbb{E}^X_x(G \circ \Phi). \tag{7.2}$$

Thus, we have found a canonical process Y, equivalent to the original process X, whose new sample space is the path space restricted to paths admitting the property Π, and which can be "pulled back" to the original measures by equation (7.2). These results can be further extended to fit into the context of right processes, see, e.g., [Sha88, Proposition 19.6 and Theorem 19.7].

This construction of switching to the (restricted) path space is strictly easier than other procedures presented in the literature, such as in [BB96, Section 38], which are used to restrict the original sample space and the underlying σ-algebra to a (not necessarily measurable) subset having full outer measure. We can resort to the easier method above, as the σ-algebra $\mathscr{F}^Y = \sigma(Y_t, t \geq 0)$ generated by the coordinate process will turn out sufficient for our applications and all paths—in contrast to a subset of paths with full (outer) measure—will feature the desired property Π.

7.2. Comparison of Processes

The procedure above can be used to establish a common basis on which we are then able to compare two equivalent processes which are defined on different probability spaces: Let X, Z be two right continuous, $(E, \mathscr{E})$-valued stochastic processes on $(\Omega^X, \mathscr{F}^X, \mathbb{P}^X)$, $(\Omega^Z, \mathscr{F}^Z, \mathbb{P}^Z)$ respectively, having the same finite dimensional distributions, that is, satisfying for all $n \in \mathbb{N}$, $f_1, \ldots, f_n \in b\mathscr{E}$, $t_1, \ldots, t_n \in \mathbb{R}_+$:

$$\mathbb{E}^X\big(f_1(X_{t_1}) \cdots f_n(X_{t_n})\big) = \mathbb{E}^Z\big(f_1(Z_{t_1}) \cdots f_n(Z_{t_n})\big).$$

Define $\Omega := \{\omega\colon \mathbb{R}_+ \to E \,|\, \omega \text{ is right continuous}\}$, together with the path mappings

$$\Phi^X\colon \Omega^X \to \Omega, \quad \omega^X \mapsto \Phi^X(\omega^X) \quad \text{with} \quad \Phi^X(\omega^X)(t) := X_t(\omega^X),\ t \geq 0,$$
$$\Phi^Z\colon \Omega^Z \to \Omega, \quad \omega^Z \mapsto \Phi^Z(\omega^Z) \quad \text{with} \quad \Phi^Z(\omega^Z)(t) := Z_t(\omega^Z),\ t \geq 0,$$

and consider the right continuous canonical coordinate process Y on Ω with its generated σ-algebra $\mathscr{F}^Y$ as above. Then, as seen in equation (7.1), we have

$$Y_t \circ \Phi^X = X_t \quad \text{on } \Omega^X,$$
$$Y_t \circ \Phi^Z = Z_t \quad \text{on } \Omega^Z,$$

so the equivalence of the processes X and Z yields that for all $n \in \mathbb{N}$, $f_1, \dots, f_n \in b\mathscr{E}$, $t_1, \dots, t_n \in \mathbb{R}_+$,

$$\mathbb{E}^X\big(f_1(Y_{t_1} \circ \Phi^X) \cdots f_n(Y_{t_n} \circ \Phi^X)\big) = \mathbb{E}^Z\big(f_1(Y_{t_1} \circ \Phi^Z) \cdots f_n(Y_{t_n} \circ \Phi^Z)\big).$$

As $\{f_1(Y_{t_1}) \cdots f_n(Y_{t_n}); n \in \mathbb{N}, f_1, \dots, f_n \in b\mathscr{E}, t_1, \dots, t_n \in \mathbb{R}_+\}$ is an $\cap$-stable generator of $\mathscr{F}^Y = \sigma(Y_t, t \geq 0)$, the MCT then concludes that

$$\forall G \in b\mathscr{F}^Y: \quad \mathbb{E}^X(G \circ \Phi^X) = \mathbb{E}^Z(G \circ \Phi^Z).$$

8. Stopping

We are going to consider the following question: Let X be a (strong) Markov process and τ be a random time. Is the stopped process $X_{\cdot \wedge \tau}$ still a (strong) Markov process? It seems that this problem is not commonly treated in the literature, the only source known to us is [Dyn65, Section X.2]. One may suspect that the random time τ must be a stopping time which admits the "memoryless" property, encoded in the concept of terminal times, in order to prevent a "memory structure" to be introduced by the transformation of stopping. Furthermore, if the transformed process possesses the Markov property, it must stop immediately again when restarted at the stopping point, that is, the stopping time τ should "trigger instantly" when the original process (re-)starts at X_τ.

Indeed, a rigorous refinement of these heuristic conditions ensures the stopped process to be (strongly) Markovian: Let $X = \big(\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E}\big)$ be a right continuous (strong) Markov process on $(E, \mathscr{E})$, and assume that there exists a constant path in x for every $x \in E$, that is,

$$\exists \overline{\omega}_x \in \Omega: \ \forall t \geq 0: \quad X_t(\overline{\omega}_x) = x. \tag{8.1}$$

Furthermore, let τ be a terminal time for X, which satisfies for all $x \in E$ the condition

$$\mathbb{P}_{X_\tau}(\tau = 0) = 1 \quad \mathbb{P}_x\text{-a.s.}. \tag{8.2}$$

Define the process $\widetilde{X}$ resulting from *stopping X at τ* by the process $\widetilde{X}_t = X_{t\wedge\tau}$, $t \geq 0$, on $(\Omega, \mathscr{G})$, equipped with the filtration $\widetilde{\mathscr{F}_t} := \{A \in \mathscr{F}_\infty : A \cap \{\tau > t\} \in \mathscr{F}_t\}$, $t \geq 0$, and shift operators

$$\widetilde{\Theta}_t(\omega) := \begin{cases} \Theta_t(\omega), & t < \tau(\omega), \\ \bar{\omega}_{X_{\tau(\omega)}}, & t \geq \tau(\omega), \end{cases} \quad \omega \in \Omega,\ t \geq 0.$$

Then the stopped process $\widetilde{X}$ is (strongly) Markovian:

(8.3) Theorem. *Let $X = (\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ be a right continuous Markov process on $(E, \mathscr{E})$ and τ be a terminal time for X, which satisfy conditions* (8.1) *and* (8.2)*. Then the stopped process $\widetilde{X} = (\Omega, \mathscr{G}, (\widetilde{\mathscr{F}}_t)_{t\geq 0}, (\widetilde{X}_t)_{t\geq 0}, (\widetilde{\Theta}_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ is a right continuous Markov process on $(E, \mathscr{E})$. If X is a strong Markov process, then $\widetilde{X}$ is a strong Markov process as well.*

This theorem is a slight generalization of [Dyn65, Theorem 10.2], its proof proceeds completely analogously and can be found in [Wer10]. It has also been shown in [Wer10] that the conditions of the theorem above cannot be weakened in general.

The conditions on τ are always satisfied by the first hitting time $H_A = \inf\{t \geq 0 : X_t \in A\}$ of any closed set A: It is a terminal time by theorem (3.8), right continuity of X implies that $X_{H_A} \in \bar{A} = A$ holds, and normality then ensures (8.2). The existence of constant paths (8.1) is not very restrictive and usually can be achieved by adjoining a null set of the needed points to the sample space Ω, see, e.g., [Dyn65, footnote on p. 79].

9. Time Change

We briefly remind the technique of time-changing a Markov process. This technique seems to date back to [Boc55] and [Vol58], and found extensive applications in potential theory, see, e.g., [BG69, Chapter V] or [Sha88, Chapter IV]. Nowadays, results on this topic can fill up libraries alone (cf. [CF11] for a modern treatment). We only give one result concerning the time change of a right process with respect to a perfect continuous additive functional, which we will employ later.

We try to motivate this technique by giving the following basic idea, fitting to our context: If the time scale of a stochastic process $t \mapsto X_t$ is changed by some increasing function τ tending to infinity, then the time changed process $t \mapsto X_{\tau(t)}$ will assume the same hitting distributions as the original process, only the hitting times of any given set will be adjusted. Thus, in regard to Dynkin's formula for the generator (3.18), it is natural to expect that the generator of a time scaled Markov process equals the original generator, rescaled at every point subject to the time changing function τ (for a rigorous result, see [Dyn65, Theorem 10.12]). On the other hand, under some regularity conditions, it can be shown that any two Markov processes with the same hitting distributions are equivalent up to a time change (see [BG69, Theorem 5.1]).

9.1. Additive Functionals

There is a finely tuned classification for additive functionals, see [Sha88, Section IV.35, Chapter VIII]. We only consider a special case, following [RW00a, Definition III.16.3]:

(9.1) Definition. *A perfect continuous additive functional with respect to some Markov process* $X = (\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ *is an* $(\mathscr{F}_t, t \geq 0)$*-adapted process* $A = (A_t, t \geq 0)$ *with values in* $\mathbb{R}_+$*, which satisfies the following properties on some set* $\Omega_0 \in \mathscr{G}$ *with* $\mathbb{P}_x(\Omega_0) = 1$ *for all* $x \in E$*:*

(i) $A_0 = 0$*;*

(ii) $t \mapsto A_t$ *is monotone increasing and continuous;*

(iii) $A_{s+t} = A_s + A_t \circ \Theta_s$ *holds for all* $s, t \geq 0$.

Let $R_A := \inf\{t \geq 0 : A_t > 0\}$*. The fine support of* A *is given by*

$$\operatorname{supp}(A) := \operatorname{reg}(R_A) = \{x \in E : \mathbb{P}_x(R_A = 0) = 1\}.$$

(9.2) Example. Let B be the standard Brownian motion on $\mathbb{R}$ and $L = (L_t, t \geq 0)$ be its local time at the origin (cf. section 15). Then L is a perfect continuous additive functional with respect to B with fine support $\operatorname{supp}(L) = \{0\}$. Furthermore,

$$A_t := t + c\, L_t, \quad t \geq 0,$$

is a perfect continuous additive functional for any $c \geq 0$, with $\operatorname{supp}\big((A_t, t \geq 0)\big) = \mathbb{R}$. ■

9.2. Basic Result

We will only need one well-known result, which can be found, e.g., in [Sha88, Theorem 65.9] or [CF11, Theorem A.3.11]:

(9.3) Theorem. *Let* $X = (\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ *be a right process and* $A = (A_t, t \geq 0)$ *be a perfect continuous additive functional with respect to* X*. Define the right continuous "pseudoinverse process"* $(\tau_t, t \geq 0)$ *of* A *by*

$$\tau_t := \inf\{s \geq 0 : A_s > t\}, \quad t \geq 0,$$

and the time changed process Y *with its shift operators by*

$$Y_t := X_{\tau(t)}, \quad \hat{\Theta}_t := \Theta_{\tau(t)}, \quad t \geq 0.$$

Then $Y = (\Omega, \mathscr{G}, (\mathscr{G}_{\tau(t)})_{t\geq 0}, (Y_t)_{t\geq 0}, (\hat{\Theta}_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in \operatorname{supp}(A)})$ *is a right process on* $\operatorname{supp}(A)$.

10. Killing

Construction of subprocesses by curtailing the lifetime of a Markov process is mainly done by killing with respect to multiplicative functionals. This is a classic, well-understood field, which has deep applications in potential theory (see [BG69, Chapter III]), and which is also applicable to right processes, see [Sha88, Chapter VII]. However, we will not need these results in their full generality, so we restrict our attention to two easier methods:

10.1. Killing at a Terminal Time

Let $X = (\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ be a right process and T be an almost perfect terminal time for X with $T \leq \zeta$. Following [Sha88, p. 70], we define the process $\hat{X}$ obtained by *killing X at time T* by

$$\hat{X} = (\hat{\Omega}, \hat{\mathscr{G}}, (\hat{\mathscr{G}}_t)_{t\geq 0}, (\hat{X}_t)_{t\geq 0}, (\hat{\Theta}_t)_{t\geq 0}, (\hat{\mathbb{P}}_x)_{x\in E}),$$

with

(i) $\hat{\Omega} := \{\omega \in \Omega : t + T \circ \Theta_t(\omega) = T(\omega) \text{ for all } t < T(\omega)\}$,

(ii) for all $t \geq 0$, $\hat{X}_t := \begin{cases} X_t, & t < T, \\ \Delta, & t \geq T, \end{cases}$ and $\hat{\Theta}_t := \begin{cases} \Theta_t, & t < T, \\ [\Delta], & t \geq T, \end{cases}$ on $\hat{\Omega}$,

(iii) $\hat{\mathscr{G}}$, $(\hat{\mathscr{G}}_t, t \geq 0)$ being the traces of $\mathscr{G}$, $(\mathscr{G}_t, t \geq 0)$ on $\hat{\Omega}$,

(iv) $\hat{\mathbb{P}}_x$ being the trace of $\mathbb{P}_x$ on $\hat{\mathscr{G}}$ for every $x \in E$, and $\hat{\mathbb{P}}_\Delta := \varepsilon_{[\Delta]}$.

Then [Sha88, Theorem 12.23] analyzes conditions on X which assert that $\hat{X}$ is strongly Markovian or even a right process on a possibly restricted state space F. For our applications, the more specialized results of [Sha88, Corollary 12.24] will be sufficient:

(10.1) Theorem. *Let $F := \operatorname{reg}(T) = \{x \in E : \mathbb{P}_x(T = 0) = 1\}$. If*

(i) T is exact, or

(ii) T is the debut of a nearly optional set,

then $\hat{X}$ is a right process on $E \backslash F$ with lifetime T.

In particular, (ii) is applicable to the first entry time $T := H_A$ of any measurable set $A \in \mathscr{E}$, because $H_A = \inf\{t \geq 0 : X_t \in A\}$ is just the debut of the optional set $\{(s, \omega) \in \mathbb{R}_+ \times \Omega : \mathbb{1}_A(X_s(\omega)) = 1\}$, and F then equals the set of regular points for A.

10.2. Killing via Additive Functional with Exponential Rate

Let $X = (\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ be a right process and $(A_t, t \geq 0)$ be a perfect continuous additive functional with respect to X. The objective is to kill the process once the additive functional surpasses some level given by an independent, exponential variable, that is, to kill the process on basis of $(A_t, t \geq 0)$ with a given rate.

As mentioned in [Sha88, p. 74], theorem (10.1) is not applicable in every case of killing which arises in practice, as, by its definition, the terminal time needs to be a stopping time with respect to the process' natural filtration. Therefore, we need to extend the sample space for the above transformation. We follow the construction of [KPS10, Appendix A], which is along the lines of the standard procedure for killing with respect to a multiplicative functional (see, e.g., [BG69, Section III.3]):

Set $\hat{\Omega} := \Omega \times [0,\infty]$, endowed with the σ-algebra $\hat{\mathscr{G}} := \mathscr{G} \otimes \mathscr{B}([0,\infty])$, and let $\hat{\zeta}\big((\omega,s)\big) := \inf\{t \geq 0 : A_t(\omega) > s\}$. Introduce the process $(\hat{X}_t, t \geq 0)$ on $\hat{\Omega}$ by

$$\hat{X}_t\big((\omega,s)\big) := \begin{cases} X_t(\omega), & t < \hat{\zeta}, \\ \Delta, & t \geq \hat{\zeta}, \end{cases}$$

with shift operators $(\hat{\Theta}_t, t \geq 0)$ given by $\hat{\Theta}_t\big((\omega,s)\big) := \big(\Theta_t(\omega), s - A_t(\omega) \wedge s\big)$ for $\omega \in \Omega$, $s \in [0,\infty]$. Define the filtration $(\hat{\mathscr{G}}_t, t \geq 0)$ by

$$\hat{\mathscr{G}}_t := \{\hat{B} \in \hat{\mathscr{G}} \,|\, \exists B \in \mathscr{G}_t : \hat{B} \cap \{\hat{\zeta} > t\} = (B \times [0,\infty]) \cap \{\hat{\zeta} > t\}\}, \quad t \geq 0.$$

Set $\hat{\mathbb{P}}_x := \mathbb{P}_x \otimes \widetilde{\mathbb{P}}$ for every $x \in E$, with $\widetilde{\mathbb{P}}$ being the exponential law on $\big([0,\infty], \mathscr{B}([0,\infty])\big)$ with mean 1, and consider the random variable S on $\hat{\Omega}$ defined by $S\big((\omega,s)\big) := s$.

Then it can be shown that $\hat{X} = \big(\hat{\Omega}, \hat{\mathscr{G}}, (\hat{\mathscr{G}}_t)_{t\geq 0}, (\hat{X}_t)_{t\geq 0}, (\hat{\Theta}_t)_{t \geq 0}, (\hat{\mathbb{P}}_x)_{x\in E}\big)$ is a strong Markov process with semigroup given by (see [KPS10, Corollary A.8, Theorem A.12])

$$\hat{\mathbb{E}}_x\big(f(\hat{X}_t)\big) = \mathbb{E}_x\big(e^{-A_t} f(X_t)\big), \quad t \geq 0, f \in b\mathscr{E}.$$

An examination of its resolvent, as done in [CF11, Theorem A.3.13], leads to:

(10.2) Theorem. $\hat{X}$ *is a right process.*

11. Concatenation

Let $(E^j, j \in \mathbb{N})$ be a collection of pairwise disjoint Radon spaces and, for each $j \in \mathbb{N}$, let $X^j = \big(\Omega^j, \mathscr{F}^j, (\mathscr{F}^j_t)_{t\geq 0}, (X^j_t)_{t\geq 0}, (\Theta^j_t)_{t\geq 0}, (\mathbb{P}^j_x)_{x \in E^j}\big)$ be a right process on E^j with lifetime ζ^j. We assume (cf. subsection 4.3) that each Ω^j contains an element $[\Delta^j]$ with

$$\forall t \geq 0 : \quad X^j_t\big([\Delta^j]\big) = \Delta^j.$$

Our goal is to have a technique at hand which concatenates the processes $(X^j, j \in \mathbb{N})$, forming a right process X on $\bigcup_j E^j$ with the following behavior: If started in E^j, the process X should "behave like" X^j until it dies at ζ^j, and then is "revived" in E^{j+1}, where it "behaves like" X^{j+1} until the next death ζ^{j+1}, etc., see figure 11.1. Such techniques of "revival" or "pasting of processes" have been treated before, mostly in an "ad-hoc" fashion for special cases, e.g. by [Nag76], [Mey75], and [INW68] (which seems to be one of the first appearances). We follow [Sha88, Section 14] for a general treatment in the context of right processes, which will give us enough freedom for our constructions later. Sharpe only considers the concatenation of two right processes X^1, X^2 via a transfer kernel $K^1 = K$, which we will recall in the following subsection 11.1 in order to prepare its generalization to the concatenation of finite and countably many processes in subsections 11.2 and 11.3. The foundation for Sharpe's approach forms the so called "transfer kernels", which we will introduce now (see [Sha88, Definition 14.2] and [Pit81]):

In order to define initial measures $(\mathbb{P}_x, x \in E)$ for the concatenated process X, we need to constitute a "transfer mechanism" between the "subprocesses" $(X^j, j \in \mathbb{N})$,

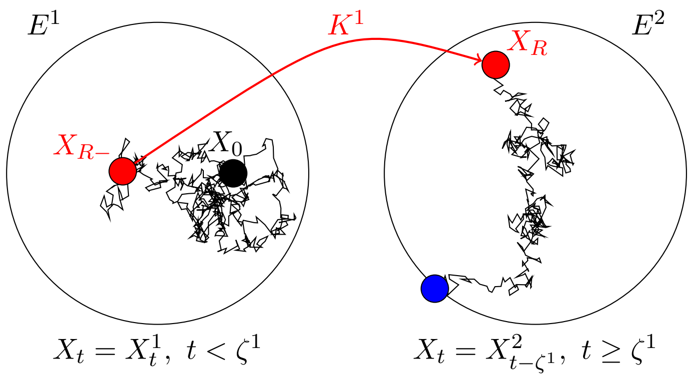


Figure 11.1: Concatenation of two processes X^1 and X^2 on E^1, E^2, resulting in the process X, which, if started in E^1, behaves like X^1 until $R = \zeta^1$, afterwards is revived on some point in E^2 (chosen by a transfer kernel K^1), where it then runs like X^2.

more precisely: a law on how the process X^{j+1} initiates in E^{j+1} after X^j died. This "mechanism" can depend on all "information" until the terminal time ζ^j of the subprocess X^j, but it should admit a "memoryless" property in the following sense in order to ensure the Markov property of the resulting process X:

(11.1) Definition. *Let X be a right process on E and T be a terminal time for X. The left germ field $\mathscr{F}_{[T-]}$ for X at T consists of all $\mathscr{F}_{T-}$-measurable random variables H which satisfy*

$$\forall t \geq 0 : \quad H \circ \Theta_t = H \text{ a.s. on } \{t < T\}.$$

(11.2) Example. Let X be a right process on E and T be a predictable terminal time for X, such that T is finite and X has a left limit in E at T. Then $X_{T-} \in \mathscr{F}_{[T-]}$, because X_{T-} is $\mathscr{F}_{T-}$-measurable by theorem (3.4), and, by using the terminal time property of T, we have for all $t \geq 0$ a.s. on $\{t < T\}$:

$$X_{T-} \circ \Theta_t = X_{(T \circ \Theta_t + t)-} = X_{T-}. \qquad \blacksquare$$

(11.3) Example. Let X be a right process on E with lifetime ζ, such that $X_{\zeta-}$ exists a.s. in E. Because $f(X_t) = f(X_t)\,\mathbb{1}_{\{t<\zeta\}}$ for all $t \geq 0$, $f \in b\mathscr{E}^u$, $\mathscr{F}_{\zeta-} = \mathscr{F}$ holds true. Then $X_{\zeta-} \in \mathscr{F}_{\zeta-}$, and as ζ is a terminal time (see lemma (4.9)), we have

$$X_{\zeta-} \circ \Theta_t = X_{(\zeta \circ \Theta_t + t)-} = X_{\zeta-} \quad \text{a.s. on } \{t < \zeta\}.$$

Thus, we see that $X_{\zeta-} \in \mathscr{F}_{[\zeta-]}$. $\blacksquare$

In the following, we will write $\mathscr{E}_u^j := (\mathscr{E}^j)^u$ for the universally measurable sets over E^j.

(11.4) Definition. *Let X^1, X^2 be right processes on E^1, E^2 respectively. A probability kernel K from $(\Omega^1, \mathscr{F}^1_{[\zeta^1-]})$ to $(E^2, \mathscr{E}^2_u)$ is a transfer kernel from X^1 to (X^2, E^2).*

Our definition is slightly stricter than the definition given in [Sha88], as we will not allow jumps to Δ^2 in order to ensure a clean construction when concatenating more than two subprocesses. A standard method of constructing transfer kernels is by imposing conditional distributions $k^1(x,\,\cdot\,)$ for the transfer point (that is the "revival point" of X^2) given the "exit point" $X^1_{\zeta^1-} = x$ of X^1:

(11.5) Lemma. *Let X^1, X^2 be right processes on E^1, E^2 respectively, such that $X^1_{\zeta^1-}$ exists a.s. in E^1, and let $k^1\colon E^1 \times \mathscr{E}^2_u \to [0,1]$ be a probability kernel from $(E^1, \mathscr{E}^1)$ to $(E^2, \mathscr{E}^2_u)$. Then the map $K^1\colon \Omega^1 \times \mathscr{E}^2_u \to [0,1]$ with*

$$K^1(\omega^1, A) := k^1(X^1_{\zeta^1-}(\omega^1), A), \quad \omega \in \Omega^1, A \in \mathscr{E}^2_u,$$

defines a transfer kernel from X^1 to (X^2, E^2).

Proof. For almost every $\omega^1 \in \Omega^1$, $K^1(\omega^1,\,\cdot\,) := k^1(X^1_{\zeta^1-}(\omega^1),\,\cdot\,)$ is a probability kernel on $(E^2, \mathscr{E}^2)$. Furthermore $K^1(\,\cdot\,, dx^2) = k^1(\,\cdot\,, dx^2) \circ X^1_{\zeta^1-}$ is $\mathscr{F}^1_{[\zeta^1-]}/\mathscr{E}^2_u$-measurable, as $X^1_{\zeta^1-}$ is an element of $\mathscr{F}^1_{[\zeta^1-]}$ by example (11.3), which gives for all $f \in b\mathscr{E}^2_u$:

$$K^1 f \circ \Theta^1_t = \int f(x^2)\, k^1(X^1_{\zeta^1-} \circ \Theta^1_t(\omega^1), dx^2) = K^1 f \quad \text{a.s. on } \{t < \zeta^1\}. \qquad \square$$

For the rest of our developments we assume that we are given a transfer kernel K^j from X^j to (X^{j+1}, E^{j+1}) for each $j \in \mathbb{N}$.

Before we begin, we would like to remark that the assumption of disjoint subspaces $(E^j, j \in \mathbb{N})$ of the subprocesses $(X^j, j \in \mathbb{N})$ can be weakened, which will be done in section 13. However, we then need to impose additional conditions on the subprocesses, namely, they need to coincide on the shared state space, and their entry and exit distributions into this subset must be equal irrespective of the mode of entry or exit (namely by either subprocess behavior or revival). Therefore, we maintain the assumption of disjointedness in the following constructions of this section in order to keep our results as universal as possible.

11.1. Concatenation of Two Processes

We set the concatenated process X of X^1 and X^2 via the transfer kernel $K := K^1$ on the sample space $\Omega := \Omega^1 \times \Omega^2$ with σ-algebra $\mathscr{F} := \mathscr{F}^1 \otimes \mathscr{F}^2$ to be $X_t\colon \Omega \to E$, defined for each $t \geq 0$, $\omega = (\omega^1, \omega^2) \in \Omega$ by

$$X_t\big((\omega^1, \omega^2)\big) := \begin{cases} X^1_t(\omega^1), & t < \zeta^1(\omega^1), \\ X^2_{t-\zeta^1(\omega^1)}(\omega^2), & t \geq \zeta^1(\omega^1), \end{cases}$$

as well as introduce a family of operators $(\Theta_t, t \geq 0)$ on Ω, defined by

$$\Theta_t\big((\omega^1, \omega^2)\big) := \begin{cases} (\Theta^1_t(\omega^1), \omega^2), & t < \zeta^1(\omega^1), \\ ([\Delta^1], \Theta^2_{t-\zeta^1(\omega^1)}(\omega^2)), & t \geq \zeta^1(\omega^1). \end{cases}$$

We use the transfer kernel K to concatenate the processes X^1 and X^2 probabilistically by giving a transition between the distributions $(\mathbb{P}^1_x, x \in E^1)$ and $(\mathbb{P}^2_x, x \in E^2)$. To this end, define measures $(\mathbb{P}_x, x \in E)$ on $\mathscr{F}$ by setting for $x \in E^1$, $H \in b(\mathscr{F}^1 \otimes \mathscr{F}^2)$:

$$\mathbb{E}_x(H) = \begin{cases} \int H(\omega^1, \omega^2)\, \mathbb{P}^2_y(d\omega^2)\, K(\omega^1, dy)\, \mathbb{P}^1_x(d\omega^1), & x \in E^1, \\ \int H(\omega^1, \omega^2)\, \mathbb{P}^2_x(d\omega^2)\, \varepsilon_{[\Delta^1]}(\omega^1), & x \in E^2. \end{cases}$$

That is, if the process starts in $x \in E^1$ (the state space of X^1), the first part of the process evolves exactly as the original process X^1 under $\mathbb{P}^1_x$, and then, depending on this path and the transfer kernel K, the entrance point y for the second partial process is selected, which now evolves as the original process X^2 under $\mathbb{P}^2_y$. If, on the other hand, the process starts in $x \in E^2$ (the state space of X^2), the initial law $\mathbb{P}_x$ is just the image of $\mathbb{P}^2_x$ under $\omega^2 \mapsto ([\Delta^1], \omega^2)$, so the first partial process dies immediately and the process completely evolves like the original process X^2 under $\mathbb{P}^2_x$.

The main result for the concatenation X of two processes X_1 and X_2 via the transfer kernel K is:

(11.6) Theorem. *X is a right process. For $R := \inf\{t \geq 0 : X_t \in E^2\}$, $x \in E^1$, $f \in b\mathscr{E}^2_u$,*

$$\mathbb{E}_x\big(f(X_R)\, \mathbb{1}_{\{R<\infty\}} \,\big|\, \mathscr{F}_{R-}\big) = Kf \circ \pi^1\, \mathbb{1}_{\{R<\infty\}}.$$

This theorem is proved in detail in [Sha88, Theorem (14.8)] by an examination of the resolvent and of the excessive functions of the resulting concatenated process X. We give a short sketch:

Using Dynkin's formula (3.16) for decomposing the resolvent $(U_\alpha, \alpha > 0)$ of X at the revival time R (which coincides with the terminal time ζ^1 of X^1), one obtains for $\alpha > 0$, $f \in b\mathcal{C}(E)$, $x \in E = E^1 \cup E^2$,

$$U_\alpha f(x) = \mathbb{1}_{E^1}(x)\big(U^1_\alpha f^1(x) + \mathbb{E}^1_x(e^{-\alpha\zeta^1}\, KU^2_\alpha f^2)\big) + \mathbb{1}_{E^2}(x)\, U^2_\alpha f^2(x),$$

with $f^j := f\big|_{E^j}$, and U^j being the resolvent of X^j, $j \in \{1, 2\}$. An extensive analysis of the above components under the utilization of the strong Markov property of X^1 and X^2 as well as the properties of the transfer kernel K then shows that the conditions of theorem (2.18) are fulfilled, thus proving the Markov property of X. But $U^2_\alpha f^2$ is α-excessive for X^2, and both $U^1_\alpha f^1$ and, by the shift properties of the transfer kernel K, the function $x \mapsto \mathbb{E}^1_x(e^{-\alpha\zeta^1}\, KU^2_\alpha f^2)$ are α-excessive for X^1. As X^1 and X^2 satisfy HD2, it is immediate from the above decomposition that $t \mapsto U_\alpha f(X_t)$ is a.s. right continuous, which by (ii) of the portmanteau (4.6) for right processes implies that X satisfies HD2.

11.2. Concatenation of Finitely Many Processes

We now consider the concatenation of $m \in \mathbb{N}$ right processes $X^1, \ldots, X^m$ via the transfer kernels $K^1, \ldots, K^{m-1}$: For every $n \in \{1, \ldots, m\}$ set $E^{(n)} := \bigcup_{j=1}^n E^j$ as topological union of the spaces $(E^j, j \in \{1, \ldots, n\})$, as well as $E := E^{(m)}$. Directly extending the construction of subsection 11.1, we define the concatenated process X on the sample space $\Omega := \Omega^1 \times \cdots \times \Omega^m$ with σ-algebra $\mathscr{F} := \mathscr{F}^1 \otimes \cdots \otimes \mathscr{F}^m$ to be $X_t \colon \Omega \to E$, defined for each $t \geq 0$, $\omega = (\omega^1, \ldots, \omega^m) \in \Omega$ by

$$X_t(\omega) := \begin{cases} X^1_t(\omega^1), & t < \zeta^1(\omega^1), \\ X^2_{t-\zeta^1(\omega^1)}(\omega^2), & \zeta^1(\omega^1) \leq t < \zeta^1(\omega^1) + \zeta^2(\omega^2), \\ \vdots & \vdots \\ X^m_{t-(\zeta^1(\omega^1)+\cdots+\zeta^{m-1}(\omega^{m-1}))}(\omega^m), & t \geq \zeta^1(\omega^1) + \cdots + \zeta^{m-1}(\omega^{m-1}), \end{cases}$$

that is, for all $\zeta^1(\omega^1) + \cdots + \zeta^{n-1}(\omega^{n-1}) \leq t < \zeta^1(\omega^1) + \cdots + \zeta^{n-1}(\omega^{n-1}) + \zeta^n(\omega^n)$, $n \in \{0, \ldots, m-1\}$, we have:

$$X_t\Big((\omega^1, \ldots, \omega^m)\Big) = X^n_{t-(\zeta^1(\omega^1)+\cdots+\zeta^{n-1}(\omega^{n-1}))}(\omega^n).$$

Furthermore, we introduce a family of operators $(\Theta_t, t \geq 0)$ on Ω by setting for each $t \geq 0$, $\omega = (\omega^1, \ldots, \omega^m) \in \Omega$:

$$\Theta_t(\omega) :=$$

$$\begin{cases} (\Theta^1_t(\omega^1), \omega^2, \ldots, \omega^m), & t < \zeta^1(\omega^1), \\ ([\Delta^1], \Theta^2_{t-\zeta^1(\omega^1)}(\omega^2), \omega^3, \ldots, \omega^m), & \zeta^1(\omega^1) \leq t < \zeta^1(\omega^1) + \zeta^2(\omega^2), \\ \vdots & \vdots \\ ([\Delta^1], \ldots, [\Delta^{m-1}], \Theta^n_{t-(\zeta^1(\omega^1)+\cdots+\zeta^{m-1}(\omega^{m-1}))}(\omega^m)), & t \geq \zeta^1(\omega^1) + \cdots + \zeta^{m-1}(\omega^{m-1}), \end{cases}$$

that is, for all $\zeta^1(\omega^1) + \cdots + \zeta^{n-1}(\omega^{n-1}) \leq t < \zeta^1(\omega^1) + \cdots + \zeta^{n-1}(\omega^{n-1}) + \zeta^n(\omega^n)$, $n \in \{0, \ldots, m-1\}$, it is

$$\Theta_t\Big((\omega^1, \ldots, \omega^m)\Big) = ([\Delta^1], \ldots, [\Delta^{n-1}], \Theta^n_{t-(\zeta^1(\omega^1)+\cdots+\zeta^{n-1}(\omega^{n-1}))}(\omega^n), \omega^{n+1}, \ldots, \omega^m).$$

The formal proof that $(\Theta_t, t \geq 0)$ is indeed a family of shift operators for $(X_t, t \geq 0)$ will be postponed to the next subsection, as it proceeds completely analogously (but easier) to the countable case, see lemma (11.10).

Like in the construction for two processes in above subsection 11.1, we use the transfer kernels $(K^n, n \in \{1, \ldots, m-1\})$ to concatenate the separate measures $(\mathbb{P}^j_x, x \in E^j)$, $j \in \{1, \ldots, m\}$, of the partial processes $(X^j, j \in \{1, \ldots, m\})$. For every $x \in E$, we define the measure $\mathbb{P}_x$ on $\mathscr{F}$ by setting for $x \in E^n$, $H \in b(\mathscr{F}^1 \otimes \cdots \otimes \mathscr{F}^m)$:

$$\begin{aligned} \mathbb{E}_x(H) := \int H(\omega^1, \ldots, \omega^n)\, \mathbb{P}^n_{x^m}(d\omega^m)\, K^{m-1}(\omega^{m-1}, dx^m)\, \mathbb{P}^{m-1}_{x^{m-1}}(d\omega^{m-1}) \\ \cdots\ \mathbb{P}^{n+1}_{x^{n+1}}(d\omega^{m+1})\, K^n(\omega^n, dx^{n+1})\, \mathbb{P}^n_x(d\omega^n) \\ \varepsilon_{[\Delta^{n-1}]}(d\omega^{n-1}) \cdots \varepsilon_{[\Delta^1]}(d\omega^1). \end{aligned}$$

Furthermore, we consider the *n-th revival time*

$$R^n := \inf\{t \geq 0 : X_t \in E^{n+1}\}, \quad n \in \{1, \ldots, m-1\},$$

which is terminal time, as X is right continuous by construction, and every subspace E^{n+1} is isolated in E.

The extension of theorem (11.6) to the above defined process X, resulting from the finite concatenation of $X^1, \ldots, X^m$ via the transfer kernels $K^1, \ldots, K^{m-1}$, then reads:

(11.7) Theorem. *X is a right process. For all $n \in \{1, \ldots, m-1\}$, $x \in E^{(n)}$, $f \in b\mathscr{E}^{n+1}$,*

$$\mathbb{E}_x\big(f(X_{R^n})\, \mathbb{1}_{\{R^n<\infty\}} \,\big|\, \mathscr{F}_{R^n-}\big) = K^n f \circ \pi^n\, \mathbb{1}_{\{R^n<\infty\}}.$$

We will prove this theorem iteratively, that is, by assuming that the concatenation $X^{(n)}$ of the processes $X^1, \ldots, X^n$ via the transfer kernels $K^1, \ldots, K^{n-1}$ is already a right process for any fixed $n \in \{1, \ldots, m-1\}$, and then applying Sharpe's result (11.6) in order to concatenate $X^{(n)}$ with X^{n+1} via the transfer kernel K^n. Before doing this, we need to lift the transfer kernels K^n from X^n (to (X^{n+1}, E^{n+1})) to transfer kernels from $X^{(n)}$ (to (X^{n+1}, E^{n+1})):

(11.8) Lemma. *For every $n \in \{1, \ldots, n-1\}$, $K^n \circ \pi^n$ defined by*

$$K^n \circ \pi^n\big((\omega^1, \ldots, \omega^n), dy\big) := K^n(\omega^n, dy)$$

is a transfer kernel from $X^{(n)}$ to (X^{n+1}, E^{n+1}).

Proof. Obviously, $K^n \circ \pi^n$ is a probability measure in the second argument, because K^n is a Markov kernel. In order to show the $\mathscr{F}^{(n)}_{[R^n-]}$-measurability of $K^n \circ \pi^n(\,\cdot\,, dy)$, we start by observing that

$$(\pi^n)^{-1}\big(\mathscr{F}^n_{\zeta^n-}\big) = \Omega^1 \times \cdots \times \Omega^{n-1} \times \mathscr{F}^n_{\zeta^n-} \subseteq \mathscr{F}^{(n)}_{R^n-}.$$

This can be seen by the following argument: By definition (3.3), $\mathscr{F}^n_{\zeta^n-}$ is generated by

$$f(X^n_t)\, \mathbb{1}_{\{t<\zeta^n\}}, \quad f \in b\mathscr{E}^n_u,$$

and these functions, extended to $\Omega^{(n)}$, fulfill for $\omega = (\omega^1, \ldots, \omega^n)$

$$\begin{aligned}
&\big(f(X^n_t)\, \mathbb{1}_{\{t<\zeta^n\}}\big) \circ \pi^n(\omega) \\
&= f\big(X^n_t(\omega^n)\big)\, \mathbb{1}_{\{t<\zeta^n(\omega^n)\}} \\
&= f\big(X^{(n)}_{t+(\zeta^1(\omega^1)+\ldots+\zeta^{n-1}(\omega^{n-1}))}(\omega)\big)\, \mathbb{1}_{\{t+(\zeta^1(\omega^1)+\ldots+\zeta^{n-1}(\omega^{n-1}))<R^n(\omega)\}} \\
&= f\big(X^{(n)}_t\big) \circ \Theta_{R^{n-1}}\, \mathbb{1}_{\{t+R^{n-1}<R^n\}}(\omega) \\
&= \big(f(X^{(n)}_t)\, \mathbb{1}_{\{t<R^n\}}\big) \circ \Theta_{R^{n-1}}(\omega).
\end{aligned}$$

Because $X^{(n)}_t\, \mathbb{1}_{\{t<R^n\}}$ is $\mathscr{F}^{(n)}_{R^n-}$-measurable and the terminal time R^{n-1} is strictly smaller than the terminal time R^n on $\{t < \zeta^n\}$ (as $\zeta^n = 0$ otherwise), lemma (3.13) shows that the above function is indeed $\mathscr{F}^{(n)}_{R^n-}$-measurable. Therefore, $(\pi^n)^{-1}\big(\mathscr{F}^n_{\zeta^n-}\big) \subseteq \mathscr{F}^{(n)}_{R^n-}$, and as $K^n(\,\cdot\,, dy)$ is $\mathscr{F}^n_{[\zeta^n-]}$-measurable and π^n is a projection, $K^n \circ \pi^n$ is $\mathscr{F}^{(n)}_{R^n-}$-measurable.

It remains to prove that the shift invariance also lifts from K^n to $K^n \circ \pi^n$: Fix $t \geq 0$ and let N^n be a null set on $\mathscr{F}^n$ such that, for all $\omega^n \in \complement N^n$,

$$K^n \circ \Theta^n_t(\omega^n) = K^n(\omega^n), \quad \text{if } t < \zeta^n(\omega^n).$$

But then $N^{(n)} := (\pi^n)^{-1}(N^n)$ is a null set on $\mathscr{F}^{(n)}$ (as $\mathbb{P}^{(n)}\big((\pi^n)^{-1}(N^n)\big) = \mathbb{P}^n(N^n) = 0$), and for all $\omega = (\omega^1, \dots, \omega^n) \in \complement N^{(n)}$ (thus, $\omega^n \in \complement N^n$), we have for $t < R^n(\omega)$:

$$\begin{aligned}
(K^n \circ \pi^n) \circ \Theta^{(n)}_t(\omega) &= \begin{cases} K^n \circ \pi^n\big(\cdots, \omega^n\big), & t < R^{n-1}(\omega), \\ K^n \circ \pi^n\big(\cdots, \Theta^n_{t-R^{n-1}(\omega)}(\omega^n)\big), & R^{n-1}(\omega) \leq t < R^n(\omega) \end{cases} \\
&= \begin{cases} K^n(\omega^n), & t < R^{n-1}(\omega), \\ K^n \circ \Theta^n_{t-R^{n-1}(\omega)}(\omega^n), & 0 \leq t - R^{n-1}(\omega) < \zeta^n(\omega) \end{cases} \\
&= (K^n \circ \pi^n)(\omega),
\end{aligned}$$

where we used the shift invariance of K^n for the last identity. □

We are ready to prove the main result on the concatenation of finitely many processes:

Proof of theorem (11.7). For $m = 2$, this is already done in theorem (11.6).

Assume now that, for some $m \in \mathbb{N}$, the process $X^{(m)}$ resulting from the concatenation of $X^1, \dots, X^m$ via the transfer kernels $K^1, \dots, K^{m-1}$ is a right process and satisfies for all $n \in \{1, \dots, m-1\}$, $x \in E^{(n)}$, $f \in b\mathscr{E}^{n+1}$, with $R^{(n)} := \inf\{t \geq 0 : X^{(m)} \in E^{(n+1)}\}$:

$$\mathbb{E}_x\big(f(X^{(m)}_{R^{(n)}})\,\mathbb{1}_{\{R^{(n)}<\infty\}} \,\big|\, \mathscr{F}^{(m)}_{R^{(n)}-}\big) = K^n f \circ \pi^n\,\mathbb{1}_{\{R^{(n)}<\infty\}}. \tag{11.9}$$

Define $X^{(m+1)}$ to be the concatenation of $X^{(m)}$ and X^{m+1} via the transfer kernel $K^{(m)} := K^m \circ \pi^m$. Then $X^{(m+1)}$ is equal to the process X arising from the concatenation of $X^1, \dots, X^m, X^{m+1}$ via the transfer kernels $K^1, \dots, K^{m-1}, K^m$, because $E^{(m+1)} = \Big(\bigcup_{n=1}^m E^n\Big) \cup E^{m+1} = E$, $\Omega^{(m+1)} = \Big(\prod_{n=1}^m \Omega^n\Big) \times \Omega^{m+1} = \Omega$, $\mathscr{F}^{(m+1)} = \Big(\bigotimes_{n=1}^m \mathscr{F}^n\Big) \otimes \mathscr{F}^{m+1} = \mathscr{F}$, and for all $t \geq 0$, $\omega = (\omega^1, \dots, \omega^{m+1}) \in \Omega^{(m+1)} = \Omega$, we have

$$\begin{aligned}
X^{(m+1)}_t(\omega) &= \begin{cases} X^{(m)}_t(\omega^1, \dots, \omega^m), & t < \zeta^{(m)}(\omega), \\ X^{m+1}_{t-\zeta^{(m)}(\omega)}(\omega^{m+1}), & t \geq \zeta^{(m)}(\omega) \end{cases} \\
&= X_t(\omega)
\end{aligned}$$

and

$$\begin{aligned}
\Theta^{(m+1)}_t(\omega) &= \begin{cases} \big(\Theta^{(m)}_t(\omega^1, \dots, \omega^m), \omega^{m+1}\big), & t < \zeta^{(m)}(\omega), \\ \big([\Delta^1], \dots, [\Delta^m], \Theta^{m+1}_{t-\zeta^{(m)}(\omega)}(\omega^{m+1})\big), & t \geq \zeta^{(m)}(\omega) \end{cases} \\
&= \Theta_t(\omega),
\end{aligned}$$

as $\zeta^{(m)}(\omega) = \zeta^1(\omega_1) + \cdots + \zeta^m(\omega)$, so $\mathscr{F}^{(m+1)}_t = \sigma\big(X^{(m+1)}_s, s \leq t\big) = \mathscr{F}_t$ for all $t \geq 0$.

Furthermore, for all $H \in b\mathscr{F}^{(m+1)}$ and $x \in E^n \subseteq E^{(m)}$, we have

$$\begin{aligned}
&\mathbb{E}_x^{(m+1)}(H)\\
&= \int H(\omega^1, \dots, \omega^{m+1})\, \mathbb{P}^{m+1}_{x^{m+1}}(d\omega^{m+1})\, K^m \circ \pi^m(\omega^1, \dots, \omega^m, dx^{m+1})\, \mathbb{P}_x^{(m)}(d\omega^1, \dots, d\omega^m)\\
&= \int H(\omega^1, \dots, \omega^{m+1})\, \mathbb{P}^{m+1}_{x^{m+1}}(d\omega^{m+1})\, K^m(\omega^m, dx^{m+1})\, \mathbb{P}^n_{x^m}(d\omega^m)\, K^{m-1}(\omega^{m-1}, dx^m)\\
&\qquad\qquad \mathbb{P}^{m-1}_{x^{m-1}}(d\omega^{m-1}) \cdots\ \mathbb{P}^{n+1}_{x^{n+1}}(d\omega^{m+1})\, K^n(\omega^n, dx^{n+1})\, \mathbb{P}^n_x(d\omega^n)\\
&\qquad\qquad \varepsilon_{[\Delta^{n-1}]}(d\omega^{n-1}) \cdots \varepsilon_{[\Delta^1]}(d\omega^1)\\
&= \mathbb{E}_x(H),
\end{aligned}$$

and for $x \in E^{m+1}$,

$$\begin{aligned}
\mathbb{E}_x^{(m+1)}(H) &= \int H(\omega^1, \dots, \omega^{m+1})\, \mathbb{P}^{m+1}_x(d\omega^{m+1})\, \varepsilon_{([\Delta^1], \dots, [\Delta^m])}(d\omega^1, \dots, d\omega^m)\\
&= \mathbb{E}_x(H),
\end{aligned}$$

so $\mathbb{P}_x^{(m+1)} = \mathbb{P}_x$ for all $x \in E^{(m+1)}$.

Now theorem (11.6) states that $X = X^{(m+1)}$ is a right process, and that, with the revival time $R^m = \inf\{t \geq 0 : X_t \in E^{m+1}\} =: R^{(m)}$, it satisfies

$$\begin{aligned}
\mathbb{E}_x\big(f(X_{R^m})\, \mathbb{1}_{\{R^m<\infty\}} \,\big|\, \mathscr{F}_{R^m-}\big) &= \mathbb{E}_x^{(m+1)}\big(f(X^{(m+1)}_{R^{(m)}})\, \mathbb{1}_{\{R^{(m)}<\infty\}} \,\big|\, \mathscr{F}^{(m+1)}_{R^{(m)}-}\big)\\
&= (K^m \circ \pi^m) f \circ \pi^{(m)}\, \mathbb{1}_{\{R^{(m)}<\infty\}}\\
&= (K^m f) \circ \pi^m\, \mathbb{1}_{\{R^m<\infty\}}.
\end{aligned}$$

Then assumption (11.9) for $X^{(m)}$ concludes the proof, as we get for $n \in \{1, \dots, m-1\}$:

$$\begin{aligned}
(K^n f) \circ \pi^n\, \mathbb{1}_{\{R^n<\infty\}} &= \mathbb{E}_x^{(m)}\big(f(X^{(m)}_{R^{(n)}})\, \mathbb{1}_{\{R^{(n)}<\infty\}} \,\big|\, \mathscr{F}^{(m)}_{R^{(n)}-}\big) \circ \pi^{(m)}\\
&= \mathbb{E}_x\big(f(X_{R^n})\, \mathbb{1}_{\{R^n<\infty\}} \,\big|\, \mathscr{F}_{R^n-}\big).
\end{aligned}$$

Here, the equality of both conditional expectations is seen as follows: Because $R^n = R^{(n)} \circ \pi^{(m)}$ and $X_t = X_t^{(m)} \circ \pi^{(m)}$ hold for all $t < R^{(m)}$, we have $X_{R^n} = X^{(m)}_{R^n} \circ \pi^{(m)}$. The σ-algebras $\mathscr{F}_{R^n-}$ and $\mathscr{F}^{(m)}_{R^{(n)}-}$ are generated by MVS of functions

$$\begin{aligned}
J &:= f_1(X_{t_1}) \cdots f_k(X_{t_k})\, \mathbb{1}_{\{t<R^n\}},\\
J^{(m)} &:= f_1(X^{(m)}_{t_1}) \cdots f_k(X^{(m)}_{t_k})\, \mathbb{1}_{\{t<R^{(n)}\}},
\end{aligned}$$

with $0 \leq t_1 < \cdots < t_k \leq t$, $f_1, \dots, f_k \in b\mathscr{E}^u$, and it is immediate that $J = J^{(m)} \circ \pi^{(m)}$. Therefore, the integrals over both functions are the same (in their respective spaces), that is, we obtain

$$\begin{aligned}
\mathbb{E}_x\big(f(X_{R^m})\, \mathbb{1}_{\{R^m<\infty\}}\, J\big) &= \mathbb{E}_x\big(\big(f(X_{R^{(m)}})\, \mathbb{1}_{\{R^{(m)}<\infty\}}\, J^{(m)}\big) \circ \pi^{(m)}\big)\\
&= \mathbb{E}_x^{(m)}\big(f(X_{R^{(m)}})\, \mathbb{1}_{\{R^{(m)}<\infty\}}\, J^{(m)}\big)\\
&= \mathbb{E}_x^{(m)}\big(\mathbb{E}_x^{(m)}\big(f(X^{(m)}_{R^{(n)}})\, \mathbb{1}_{\{R^{(n)}<\infty\}} \,\big|\, \mathscr{F}^{(m)}_{R^{(n)}-}\big)\, J^{(m)}\big)\\
&= \mathbb{E}_x\big(\mathbb{E}_x^{(m)}\big(f(X^{(m)}_{R^{(n)}})\, \mathbb{1}_{\{R^{(n)}<\infty\}} \,\big|\, \mathscr{F}^{(m)}_{R^{(n)}-}\big) \circ \pi^{(m)}\, J\big).
\end{aligned}$$

On the other hand, $\pi^{(m)}$ is $\mathscr{F}_{R^n-}/\mathscr{F}^{(m)}_{R^{(m)}-}$-measurable, because for all $f \in b\mathscr{E}^u$,

$$f(X_t)\,\mathbb{1}_{\{t<R^n\}} = f(X_t^{(m)})\,\mathbb{1}_{\{t<R^{(n)}\}} \circ \pi^{(m)},$$

which yields the $\mathscr{F}_{R^n-}$-measurability of $\mathbb{E}_x^{(m)}\big(f(X^{(m)}_{R^{(n)}})\,\mathbb{1}_{\{R^{(n)}<\infty\}} \,\big|\, \mathscr{F}^{(m)}_{R^{(n)}-}\big) \circ \pi^{(m)}$. □

11.3. Concatenation of Countably Many Processes

We are ready to turn to the concatenation of the processes $(X^j, j \in \mathbb{N})$ via the transfer kernels $(K^j, j \in \mathbb{N})$: Define $E = \bigcup_{j\in\mathbb{N}} E^j$ as topological union of the disjoint spaces $(E^j, j \in \mathbb{N})$, that is, E is equipped with the topology

$$\mathscr{O} := \{O \subseteq E \,|\, \forall n \in \mathbb{N} :\ O \cap E^n \text{ is open in the topology of } E^n\},$$

and assume that E is a Radon space. This is the case, for instance (and this will suffice for our applications), if the spaces E^n, $n \in \mathbb{N}$, are Lusin, see [Sch73, Corollary to Lemma II.5]. Adjoin a point $\Delta \notin E$ as a new, isolated point and form $E_\Delta := E \cup \{\Delta\}$.

Set $\Omega := \prod_{n\in\mathbb{N}} \Omega^n$, and define $X_t \colon \Omega \to E_\Delta$ for each $t \geq 0$, $\omega = (\omega^1, \omega^2, \ldots) \in \Omega$ by

$$X_t(\omega) := \begin{cases} X_t^1(\omega^1), & t < \zeta^1(\omega^1), \\ X^2_{t-\zeta^1(\omega^1)}(\omega^2), & \zeta^1(\omega^1) \leq t < \zeta^1(\omega^1) + \zeta^2(\omega^2), \\ X^3_{t-(\zeta^1(\omega^1)+\zeta^2(\omega^2))}(\omega^3), & \zeta^1(\omega^1) + \zeta^2(\omega^2) \leq t < \zeta^1(\omega^1) + \zeta^2(\omega^2) + \zeta^3(\omega^3), \\ \vdots & \vdots \\ \Delta, & t \geq \sum_{n\in\mathbb{N}} \zeta^n(\omega^n), \end{cases}$$

that is, for all $\zeta^1(\omega^1) + \cdots + \zeta^{n-1}(\omega^{n-1}) \leq t < \zeta^1(\omega^1) + \cdots + \zeta^{n-1}(\omega^{n-1}) + \zeta^n(\omega^n)$, $n \in \mathbb{N}$, we constitute

$$X_t\Big((\omega^1, \omega^2, \ldots)\Big) = X^n_{t-(\zeta^1(\omega^1)+\cdots+\zeta^{n-1}(\omega^{n-1}))}(\omega^n).$$

The reader may easily observe that, due to the right continuity of all underlying processes X^n, $n \in \mathbb{N}$, the process X is right continuous as well.

Set $\mathscr{F} := \bigotimes_{n\in\mathbb{N}} \mathscr{F}^n$, and introduce the measures $(\mathbb{P}_x, x \in E)$ on $(\Omega, \mathscr{F})$, following the construction of subsection 11.2, by giving a transition between the subprocesses' distributions $(\mathbb{P}^n_x, x \in E^n)$, $n \in \mathbb{N}$, via the transfer kernels $(K^n, n \in \mathbb{N})$. We define the measures $(\mathbb{P}_x, x \in E)$ as projective limits of the following prescriptions: For any $m \in \mathbb{N}$ and $H \in b(\mathscr{F}^1 \otimes \cdots \otimes \mathscr{F}^m)$, we set for $x \in E^1$:

$$\begin{aligned} \mathbb{E}_x(H) := \int H(\omega^1, \ldots, \omega^m)\, &\mathbb{P}^m_{x^m}(d\omega^m)\, K^{m-1}(\omega^{m-1}, dx^m)\, \mathbb{P}^{m-1}_{x^{m-1}}(d\omega^{m-1}) \\ &\cdots\ \mathbb{P}^2_{x^2}(d\omega^2)\, K^1(\omega^1, dx^2)\, \mathbb{P}^1_x(d\omega^1), \end{aligned}$$

while for $x \in E^n$, $n \geq 2$, we set

$$\begin{aligned} \mathbb{E}_x(H) := \int H(\omega^1, \ldots, \omega^m)\, &\mathbb{P}^m_{x^m}(d\omega^m)\, K^{m-1}(\omega^{m-1}, dx^m)\, \mathbb{P}^{m-1}_{x^{m-1}}(d\omega^{m-1}) \\ &\cdots\ \mathbb{P}^{n+1}_{x^{n+1}}(d\omega^{n+1})\, K^n(\omega^n, dx^{n+1})\, \mathbb{P}^n_x(d\omega^n) \\ &\varepsilon_{[\Delta^{n-1}]}(d\omega^{n-1}) \cdots \varepsilon_{[\Delta^1]}(d\omega^1). \end{aligned}$$

An easy calculation shows that the above definitions admit consistency and therefore, by the Kolmogorov existence theorem, exist as measures on $(\Omega, \mathscr{F})$.

Finally, let $(\Theta_t, t \geq 0)$ be a family of operators on Ω defined for each $t \geq 0$, $\omega = (\omega^1, \omega^2, \dots) \in \Omega$ by

$$\Theta_t(\omega) := \begin{cases} (\Theta_t^1(\omega^1), \omega^2, \omega^3, \dots), & t < \zeta^1(\omega^1), \\ ([\Delta^1], \Theta^2_{t-\zeta^1(\omega^1)}(\omega^2), \omega^3, \dots), & \zeta^1(\omega^1) \leq t < \zeta^1(\omega^1) + \zeta^2(\omega^2), \\ \vdots & \vdots \\ ([\Delta^1], [\Delta^2], [\Delta^3], \dots), & t \geq \sum_{n \in \mathbb{N}} \zeta^n(\omega^n), \end{cases}$$

that is, for all $\zeta^1(\omega^1) + \dots + \zeta^{n-1}(\omega^{n-1}) \leq t < \zeta^1(\omega^1) + \dots + \zeta^{n-1}(\omega^{n-1}) + \zeta^n(\omega^n)$, $n \in \mathbb{N}$, it is

$$\Theta_t\Big((\omega^1, \omega^2, \dots)\Big) = ([\Delta^1], \dots, [\Delta^{n-1}], \Theta^n_{t-(\zeta^1(\omega^1)+\dots+\zeta^{n-1}(\omega^{n-1}))}(\omega^n), \omega^{n+1}, \omega^{n+2}, \dots).$$

(11.10) Lemma. *$(\Theta_t, t \geq 0)$ is a family of shift operators for X.*

Proof. As $\Theta^n_s(\omega^n) \in \Omega^n$ for all $n \in \mathbb{N}$, $s \geq 0$, $\omega^n \in \Omega^n$, it is evident that $\Theta_t \colon \Omega \to \Omega$ for each $t \geq 0$. In order to keep the following computations readable, we define for all $n \in \mathbb{N}$, $\omega = (\omega^1, \omega^2, \dots) \in \Omega$:

$$R^n(\omega) := \zeta^1(\omega^1) + \dots + \zeta^n(\omega^n), \quad R^0(\omega) := 0.$$

It is useful to notice that $R^n(\omega)$ only depends on $(\omega^1, \dots, \omega^n)$.

Now let $\omega = (\omega^1, \omega^2, \dots) \in \Omega$ and $s, t \geq 0$. We need to show that

$$X_s(\Theta_t(\omega)) = X_{s+t}(\omega) \quad \text{and} \quad \Theta_s(\Theta_t(\omega)) = \Theta_{s+t}(\omega).$$

Set $n \in \mathbb{N}$ such that $R^{n-1}(\omega) \leq t < R^n(\omega)$. By definition of Θ_t, we have

$$\Theta_t(\omega) = ([\Delta^1], \dots, [\Delta^{n-1}], \Theta^n_{t-R^{n-1}(\omega)}(\omega^n), \omega^{n+1}, \omega^{n+2}, \dots).$$

We need to examine the revival times $R^m(\Theta_t(\omega))$, $m \in \mathbb{N}$, of the shifted process: By definition of the shift operator, we get (with n as above)

$$\forall m < n: \quad R^m(\Theta_t(\omega)) = \zeta^1([\Delta^1]) + \dots + \zeta^m([\Delta^m]) = 0. \tag{11.11}$$

For all $m \geq n$, on the other hand, the definition yields

$$\begin{aligned} R^m(\Theta_t(\omega)) = \zeta^1([\Delta^1]) + \dots + \zeta^{n-1}([\Delta^{n-1}]) + \zeta^n(\Theta^n_{t-R^{n-1}(\omega)}(\omega^n)) \\ + \zeta^{n+1}(\omega^{n+1}) + \dots + \zeta^m(\omega^m). \end{aligned}$$

As $t < R^n(\omega)$, that is $\zeta^1(\omega^1) + \dots + \zeta^n(\omega^n) - t > 0$, lemma (4.9) yields

$$\begin{aligned} \zeta^n(\Theta^n_{t-R^{n-1}(\omega)}(\omega^n)) &= \big(\zeta^n(\omega^n) - (t - R^{n-1}(\omega))\big)^+ \\ &= \zeta^1(\omega^1) + \dots + \zeta^n(\omega^n) - t, \end{aligned}$$

and we get

$$\forall m \geq n: \quad R^m \circ \Theta_t = R^m - t. \tag{11.12}$$

Now set $m \in \mathbb{N}$ such that $R^{m-1}(\Theta_t(\omega)) \leq s < R^m(\Theta_t(\omega))$. We need to distinguish the following cases:

(i) $m < n$:
Because $s \geq 0$ and $R^m(\Theta_t(\omega)) = 0$ by equation (11.11), this is impossible.

(ii) $m > n$:
Using equation (11.12), we obtain $R^{m-1}(\omega) - t \leq s < R^m(\omega) - t$, so

$$R^{m-1}(\omega) \leq s + t < R^m(\omega). \tag{11.13}$$

Then, by employing the definition of X for the first identity, (11.12) for the second identity and (11.13) for the last identity, it follows that

$$\begin{aligned} X_s(\Theta_t(\omega)) &= X^m_{s-R^{m-1}(\Theta_t(\omega))}(\omega^m) \\ &= X^m_{s+t-R^{m-1}(\omega)}(\omega^m) \\ &= X_{s+t}(\omega). \end{aligned}$$

Using the definition of Θ_s, we get, with $\Theta_t(\omega)^n$ being the n-th coordinate of $\Theta_t(\omega)$,

$$\Theta_s(\Theta_t(\omega)) = ([\Delta^1], \ldots, [\Delta^{m-1}], \Theta^m_{t-R^{m-1}(\Theta_t(\omega))}(\Theta_t(\omega)^m), \Theta_t(\omega)^{m+1}, \ldots).$$

Now, as $\Theta_t(\omega)^m = \omega^m$ for $m > n$, we have

$$\begin{aligned} \Theta_s(\Theta_t(\omega)) &= ([\Delta^1], \ldots, [\Delta^{m-1}], \Theta^m_{t-R^{m-1}(\Theta_t(\omega))}(\omega^m), \omega^{m+1}, \ldots) \\ &= ([\Delta^1], \ldots, [\Delta^{m-1}], \Theta^m_{s+t-R^{m-1}(\omega)}(\omega^m), \omega^{m+1}, \ldots) \\ &= \Theta_{s+t}(\omega), \end{aligned}$$

where we again used (11.12) for the second to last and (11.13) for the last identity.

(iii) $m = n$:
This means that $R^{n-1}(\Theta_t(\omega)) \leq s < R^n(\Theta_t(\omega))$ holds as well, which implies that

$$0 \leq s < R^n(\omega) - t$$

by (11.12). Thus, by implementing $R^{n-1}(\omega) \leq t$ in the inequality above, we get

$$R^{n-1}(\omega) \leq t \leq s + t < R^n(\omega). \tag{11.14}$$

Using $m = n$ together with the definition of X and Θ for the first, $R^{n-1}(\Theta_t(\omega)) = 0$ by (11.11) and the shift operation on the process X^n for the second, and again the definition of X together with (11.14) for the last identity, we obtain

$$\begin{aligned} X_s(\Theta_t(\omega)) &= X^n_{s-R^{n-1}(\Theta_t(\omega))}(\Theta^n_{t-R^{n-1}(\omega)}(\omega^n)) \\ &= X^n_{s+t-R^{n-1}(\omega)}(\omega^n) \\ &= X_{s+t}(\omega). \end{aligned}$$

Using the same course of observations, we also get

$$\begin{aligned}\Theta_s(\Theta_t(\omega)) &= \Theta_s\big(\,([\Delta^1],\dots,[\Delta^{n-1}],\Theta^n_{t-R^{n-1}(\omega)}(\omega^n),\omega^{n+1},\dots)\,\big)\\ &= ([\Delta^1],\dots,[\Delta^{n-1}],\Theta^n_{s-R^{n-1}(\Theta_t(\omega))}(\Theta^n_{t-R^{n-1}(\omega)}(\omega^n)),\omega^{n+1},\dots)\\ &= ([\Delta^1],\dots,[\Delta^{n-1}],\Theta^n_{s+t-R^{n-1}(\omega)}(\omega^n),\omega^{n+1},\dots)\\ &= \Theta_{s+t}(\omega).\end{aligned}$$ □

As already seen in the above proof, the *n-th revival time*, $n \in \mathbb{N}$,

$$R^n(\omega) := \zeta^1(\omega^1) + \dots + \zeta^{n-1}(\omega^{n-1}), \quad \omega = (\omega^1, \omega^2, \dots) \in \Omega,$$

plays an essential role for the study of X. It follows immediately from the definition of the process X that, for any $n \in \mathbb{N}$,

$$R^n = \inf\{t \geq 0 : \ X_t \in E^{n+1}\} \quad \mathbb{P}_x\text{-a.s. for } x \in \textstyle\bigcup_{m \leq n} E^m.$$

As X is right continuous by definition and the spaces E^{n+1} are isolated, $(R^n, n \geq m)$ is a sequence of terminal times, $\mathbb{P}_x$-a.s. strictly increasing for $x \in E^m$, $m \in \mathbb{N}$.

We are going to prepare the main vehicle for the proof of X being a right process. A general result, which will be made rigorous in lemma (11.15) below, states the following: Assume we are given a right continuous process X and an increasing sequence of terminal times $(R^n, n \in \mathbb{N})$. If process X killed at R^n is a right process for every $n \in \mathbb{N}$, then X killed at $R := \lim_n R^n$ is a right process as well.

This result is then directly applicable in our context, because, for every $n \in \mathbb{N}$, the concatenated process X killed at the n-th revival time R^n is just the finite concatenation of $X^1, \dots, X^n$ via $K^1, \dots, K^{n-1}$, which is a right process by the results of subsection 11.2. Thus, X killed at $\lim_n R^n = \sum_n \zeta^n$ (which equals X by construction) is proved to be a right process.

(11.15) Lemma. *Let $(X_t, t \geq 0)$ be a right continuous stochastic process with shift operators $(\Theta_t, t \geq 0)$, $(\mathbb{P}_x, x \in E)$ be a family of probability measures on a measurable space $(\Omega, \mathscr{F})$, $(R^n, n \in \mathbb{N})$ be an increasing sequence of random times with $R := \lim_{n\in\mathbb{N}} R^n$, and $(E^{R,n}, n \in \mathbb{N})$ be an increasing sequence of Radon spaces. Define the processes $(X^{R,n}_t, t \geq 0)$, $n \in \mathbb{N}$, and $(X^R_t, t \geq 0)$ on Ω by*

$$X^{R,n}_t = \begin{cases} X_t, & t < R^n,\\ \Delta, & t \geq R^n,\end{cases} \quad \text{and} \quad X^R_t = \begin{cases} X_t, & t < R,\\ \Delta, & t \geq R,\end{cases} \quad t \geq 0.$$

Then $X^R = (\Omega, \mathscr{F}, (\mathscr{F}^R_t)_{t\geq 0}, (X^R_t)_{t\geq 0}, (\Theta^R_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$, with $(\mathscr{F}^R_t, t \geq 0)$ being the natural filtration of X^R, is a right process on E, if the following conditions are fulfilled:

(i) $(R^n, n \in \mathbb{N})$ *is a sequence of stopping times over* $(\mathscr{F}^R_t, t \geq 0)$*;*

(ii) $\bigcup_{n\in\mathbb{N}} E^{R,n} = E$*;*

(iii) for each $n \in \mathbb{N}$, there exist a filtration $(\mathscr{F}_t^{R,n}, t \geq 0)$ on $(\Omega, \mathscr{F})$ and a family of operators $(\Theta_t^{R,n}, t \geq 0)$ on Ω, such that

$$X^{R,n} := \big(\Omega, \mathscr{F}, (\mathscr{F}_t^{R,n})_{t\geq 0}, (X_t^{R,n})_{t\geq 0}, (\Theta_t^{R,n})_{t\geq 0}, (\mathbb{P}_x)_{x\in E^{R,n}}\big)$$

is a right process on $E^{R,n}$;

(iv) for each $n \in \mathbb{N}$, R^n is a terminal time for the process $X^{R,n}$, satisfying $R^n > 0$ $\mathbb{P}_x$-a.s. for all $x \in E^{R,n}$.

Proof.

(i) X^R *is normal:* Let $x \in E$ and choose $n \in \mathbb{N}$ such that $x \in E^{R,n}$. By the definition of X, we have for all $f \in \mathscr{E}^u$:

$$\mathbb{E}_x\big(f(X_t^R)\,;\, t < R^n\big) = \mathbb{E}_x\big(f(X_t^{R,n})\,;\, t < R^n\big).$$

Inserting $t = 0$, the normality of X^R follows from the normality of $X^{R,n}$ together with the assumption $\mathbb{P}_x(R^n > 0) = 1$.

(ii) X^R *is a Markov process:* Let $s, t \geq 0$ and $f \in b\mathscr{E}^u$. For any $k \in \mathbb{N}$, $0 < t_1 < t_2 < \cdots < t_k \leq t$, $g_0 \in b\mathscr{E}^u$, $g_1, \ldots, g_k \in b\mathscr{E}$, set

$$\begin{aligned} J &:= g_0(X_{t_0})\, g_1(X_{t_1}) \cdots g_k(X_{t_k}), \\ J^{R,n} &:= g_0(X_{t_0}^{R,n})\, g_1(X_{t_1}^{R,n}) \cdots g_k(X_{t_k}^{R,n}), \quad n \in \mathbb{N}. \end{aligned}$$

Then, by LDCT together with $\{s + t < R\} = \bigcup_n \{s + t < R^n\}$ and $X_{s+t} = X_{s+t}^{R,n}$ on $\{s + t < R^n\}$, we get

$$\begin{aligned} \mathbb{E}_x\Big(f(X_{s+t}^R) \cdot J\Big) &= \mathbb{E}_x\Big(f(X_{s+t}) \cdot J;\, s + t < R\Big) \\ &= \lim_n \mathbb{E}_x\Big(f(X_{s+t}^{R,n}) \cdot J^{R,n};\, s + t < R^n\Big) \\ &= \mathbb{E}_x\Big(\mathbb{E}_{X_t^R}\big(f(X_s^R)\big) \cdot J\Big). \end{aligned}$$

By employing both the terminal time property and the stopping time property of R^n with respect to $X^{R,n}$ next, we obtain

$$\begin{aligned} &\lim_n \mathbb{E}_x\Big(f(X_{s+t}^{R,n}) \cdot J^{R,n};\, s + t < R^n\Big) \\ &= \lim_n \mathbb{E}_x\Big(f(X_s^{R,n}) \circ \Theta_t^{R,n} \cdot J^{R,n};\, s < R^n \circ \Theta_t^{R,n}, t < R^n\Big) \\ &= \lim_n \mathbb{E}_x\Big(\mathbb{E}_x\big(f(X_s^{R,n}) \circ \Theta_t^{R,n};\, s < R^n \circ \Theta_t^{R,n} \,\big|\, \mathscr{F}_t^{R,n}\big) \cdot J^{R,n};\, t < R^n\Big). \end{aligned}$$

Now, we are able to apply the Markov property of $X^{R,n}$, which yields

$$\begin{aligned} &\lim_n \mathbb{E}_x\Big(\mathbb{E}_x\big(f(X_s^{R,n}) \circ \Theta_t^{R,n};\, s < R^n \circ \Theta_t^{R,n} \,\big|\, \mathscr{F}_t^{R,n}\big) \cdot J^{R,n};\, t < R^n\Big) \\ &= \lim_n \mathbb{E}_x\Big(\mathbb{E}_{X_t^{R,n}}\big(f(X_s^{R,n});\, s < R^n\big) \cdot J^{R,n};\, t < R^n\Big), \end{aligned}$$

and by carrying out the above steps in reverse order, we conclude that

$$\begin{aligned}
&\lim_n \mathbb{E}_x\Big(\mathbb{E}_{X_t^{R,n}}\big(f(X_s^{R,n});\, s < R^n\big)\cdot J^{R,n};\, t < R^n\Big)\\
&= \lim_n \mathbb{E}_x\Big(\mathbb{E}_{X_t}\big(f(X_s);\, s < R^n\big)\cdot J;\, t < R^n\Big)\\
&= \mathbb{E}_x\Big(\mathbb{E}_{X_t}\big(f(X_s);\, s < R\big)\cdot J;\, t < R\Big)\\
&= \mathbb{E}_x\Big(\mathbb{E}_{X_t^R}\big(f(X_s^R)\big)\cdot J\Big).
\end{aligned}$$

We have thus shown that $\mathbb{E}_x\big(f(X_{s+t}^R)\cdot J\big) = \mathbb{E}_x\Big(\mathbb{E}_{X_t^R}\big(f(X_s^R)\big)\cdot J\Big)$ holds true for all functions J which form a generating MVS of $b\mathscr{F}_t^R$. This yields the Markov property by MCT, as the measurability of $\mathbb{E}_{X_t^R}\big(f(X_s^R)\big)$ with respect to the natural filtration $(\mathscr{F}_t^R, t \geq 0)$ is trivially fulfilled.

(iii) *Every f which is α-excessive for X is also α-excessive for $X^{R,n}$ for each $n \in \mathbb{N}$:* Let $\mathscr{S}_\alpha(X^{R,n})$, $\mathscr{S}_\alpha(X^R)$ be the sets of all α-excessive functions, $T_t^{R,n}$, T_t^R, $t \geq 0$, be the transition operators, and $U_\alpha^{R,n}$, U_α^R, $\alpha > 0$, be the α-potential operators with respect to $X^{R,n}$, X^R respectively, that is,

$$U_\alpha^{R,n}h(x) = \mathbb{E}_x\Big(\int_0^\infty e^{-\alpha s}\, h(X_s^{R,n})\, ds\Big), \quad h \in p\mathscr{E}^u, n \in \mathbb{N}.$$

Now let $f \in \mathscr{S}_\alpha(X^R)$. Then there exists a sequence $(h_m, m \in \mathbb{N})$ in $bp\mathscr{E}^u$ such that

$$f = \sup_m U_\alpha^R h_m.$$

Because (see, e.g., [CW05, Proposition 2.2])

$$e^{-\alpha t}\, T_t^R\, U_\alpha^R h_m = \mathbb{E}\Big(\int_t^\infty e^{-\alpha s}\, h_m(X_s^R)\, ds\Big),$$

every $U_\alpha^R h_m$ is in $\mathscr{S}_\alpha(X^R)$.

However, we are going to show that $U_\alpha^R h_m \in \mathscr{S}_\alpha(X^{R,n})$ holds as well. Firstly,

$$\begin{aligned}
e^{-\alpha t}\, T_t^{R,n}\, U_\alpha^R h_m &= \mathbb{E}\Big(e^{-\alpha t}\, U_\alpha^R h_m(X_t^{R,n})\Big)\\
&= \mathbb{E}\Big(e^{-\alpha t}\, U_\alpha^R h_m(X_t^R);\, t < R^n\Big)\\
&= \mathbb{E}\Big(e^{-\alpha t}\, \mathbb{E}_{X_t^R}\Big(\int_0^\infty e^{-\alpha s}\, h_m(X_s^R)\, ds\Big);\, t < R^n\Big)\\
&= \mathbb{E}\Big(\mathbb{E}\Big(\int_t^\infty e^{-\alpha s}\, h_m(X_s^R)\, ds \,\big|\, \mathscr{F}_t^R\Big);\, t < R^n\Big)\\
&= \mathbb{E}\Big(\int_t^\infty e^{-\alpha s}\, h_m(X_s^R)\, ds;\, t < R^n\Big).
\end{aligned}$$

Here, the second identity follows from the fact that $X^{R,n}$ is a subprocess of X^R and the forth identity from the Markov property of X^R. The stopping time property of R^n with respect to X^R gives the last identity.

Therefore, we have $e^{-\alpha t}\, T_t^{R,n}\, U_\alpha^R h_m \le U_\alpha^R h_m$, and because $R^n > 0$ holds $\mathbb{P}_x$-a.s. for all $x \in E^{R,n}$, we get with LMCT that, on $E^{R,n}$,

$$\begin{aligned}\lim_{t\downarrow 0} e^{-\alpha t}\, T_t^{R,n}\, U_\alpha^R h_m &= \mathbb{E}\Big(\int_0^\infty e^{-\alpha s}\, h_m(X_s^R)\, ds\Big)\\ &= U_\alpha^R h_m.\end{aligned}$$

Thus $U_\alpha^R h_m\Big|_{E^{R,n}} \in \mathscr{S}^\alpha(X^{R,n})$ for each $m \in \mathbb{N}$, and as the set of excessive functions is closed under suprema, we have

$$f\big|_{E^{R,n}} = \sup_m \Big(U_\alpha^R h_m\Big|_{E^{R,n}}\Big) \in \mathscr{S}_\alpha(X^{R,n}).$$

(iv) X^R *is a right process:* It remains to show that X satisfies HD2. Let $f \in \mathscr{S}_\alpha(X^R)$. We need to prove that $t \mapsto f(X_t^R)$ is a.s. right continuous. But as seen in (iii), $f \in \mathscr{S}_\alpha(X^{R,n})$ for any $n \in \mathbb{N}$. As $X^{R,n}$ is a right process, the map $t \mapsto f(X_t^{R,n})$ therefore is a.s. right continuous for all $n \in \mathbb{N}$. With $X_t^R = X_t^{R,n}$ for $t < R^n$, $\lim_n R^n = R$ and $f(\Delta) = 0$, we immediately get that $t \mapsto f(X_t^R)$ is a.s. right continuous.

□

Let X be the concatenation of the right processes $(X^j, j \in \mathbb{N})$ via the transfer kernels $(K^j, j \in N)$, as constructed above, and $(R^n, n \in \mathbb{N})$ be the revival times of X. As announced, we are going to apply lemma (11.15) with $X^{R,n}$ being the subprocesses of X killed at the revival times R^n, that is, we consider for all $\omega = (\omega^1, \omega^2, \ldots) \in \Omega$, $t \ge 0$,

$$\begin{aligned} X_t^{R,n}(\omega) &:= \begin{cases} X_t(\omega), & t < R^n,\\ \Delta, & t \ge R^n \end{cases}\\ &= \begin{cases} X_t^1(\omega^1), & t < \zeta^1(\omega^1),\\ X^2_{t-\zeta^1(\omega^1)}(\omega^2), & \zeta^1(\omega^1) \le t < \zeta^1(\omega^1) + \zeta^2(\omega^2),\\ \vdots & \vdots\\ X^n_{t-(\zeta^1(\omega^1)+\cdots+\zeta^{n-1}(\omega^{n-1}))}(\omega^n), & \zeta^1(\omega^1) + \cdots + \zeta^{n-1}(\omega^{n-1}) \le t\\ & \wedge\, t \le \zeta^1(\omega^1) + \cdots + \zeta^n(\omega^n),\\ \Delta, & t \ge \zeta^1(\omega^1) + \cdots + \zeta^n(\omega^n) \end{cases} \end{aligned} \tag{11.16}$$

equipped with shift operators

$$\Theta_t^{R,n}(\omega) := \begin{cases} (\Theta_t^1(\omega^1), \omega^2, \dots, \omega^n), & t < \zeta^1(\omega^1), \\ ([\Delta^1], \Theta^2_{t-\zeta^1(\omega^1)}(\omega^2), \omega^3, \dots, \omega^n), & \zeta^1(\omega^1) \le t < \zeta^1(\omega^1) + \zeta^2(\omega^2), \\ \vdots & \vdots \\ ([\Delta^1], \dots, [\Delta^{n-1}], \Theta^n_{t-(\zeta^1(\omega^1)+\dots+\zeta^{n-1}(\omega^{n-1}))}(\omega^n)), & \zeta^1(\omega^1) + \dots + \zeta^{n-1}(\omega^{n-1}) \le t \\ & \wedge\, t \le \zeta^1(\omega^1) + \dots + \zeta^n(\omega^n), \\ ([\Delta^1], \dots, [\Delta^{n-1}], [\Delta^{n-1}]), & t \ge \zeta^1(\omega^1) + \dots + \zeta^n(\omega^n) \end{cases}$$

We first need to show that the subprocesses $X^{R,n}$, $n \in \mathbb{N}$, fulfill the requirements of lemma (11.15), especially that they are right processes:

(11.17) Lemma. *For every $n \in \mathbb{N}$, the process*

$$X^{R,n} = \big(\Omega, \mathscr{F}, (\mathscr{F}_t^{R,n})_{t\ge0}, (X_t^{R,n})_{t\ge0}, (\Theta_t^{R,n})_{t\ge0}, (\mathbb{P}_x)_{x\in E^{R,n}}\big),$$

with $(\mathscr{F}_t^{R,n}, t \ge 0)$ being its natural filtration, is a right process on $E^{(n)} := \bigcup_{j=1}^n E^j$.

Proof. Let $X^{(n)} = \big(\Omega^{(n)}, \mathscr{F}^{(n)}, (\mathscr{F}_t^{(n)})_{t\ge0}, (X_t^{(n)})_{t\ge0}, (\Theta_t^{(n)})_{t\ge0}, (\mathbb{P}_x^{(n)})_{x\in E^{(n)}}\big)$ be the concatenation of $X^1, \dots, X^n$ with the transfer kernels $K^1, \dots, K^{n-1}$. Then $X^{(n)}$ is a right process on $E^{(n)}$ by theorem (11.7). One could argue that this already completes the proof, as "$X^{R,n} = X^{(n)}$". However, the processes are defined on different spaces, so a little bit more care is needed:

Let $\pi^{(n)} \colon \Omega \to \Omega^{(n)}$ be the canonical projection. By looking at the decomposition (11.16), it is evident that, for all $t \ge 0$,

$$X_t^{R,n} = X_t^{(n)} \circ \pi^{(n)} \quad \text{a.s. on } \Omega.$$

By checking the definitions of the measures $\mathbb{P}_x$, $\mathbb{P}_x^{(n)}$ for the countable and finite concatenations, we also observe that, for all $x \in E^{(n)}$,

$$\mathbb{P}_x \circ (\pi^{(n)})^{-1} = \mathbb{P}_x^{(n)} \quad \text{on } \mathscr{F}^{(n)} = \mathscr{F}^1 \otimes \dots \otimes \mathscr{F}^n.$$

It follows that, for all $k \in \mathbb{N}$, $0 \le t_1 < \dots < t_k$, $g_1, \dots, g_k \in b\mathscr{E}_u^{(n)}$,

$$\begin{aligned} \mathbb{E}_x\big(g_1(X_{t_1}^{R,n}) \cdots g_k(X_{t_k}^{R,n})\big) &= \int g_1(X_{t_1}^{(n)} \circ \pi^{(n)}) \cdots g_k(X_{t_k}^{(n)} \circ \pi^{(n)})\, d\mathbb{P}_x \\ &= \int g_1(X_{t_1}^{(n)}) \cdots g_k(X_{t_k}^{(n)})\, d\mathbb{P}_x^{(n)} \end{aligned}$$

holds true, that is, $X^{R,n}$ and $X^{(n)}$ have the same finite dimensional distributions (with respect to their corresponding measures $\mathbb{P}$ and $\mathbb{P}^{(n)}$):

$$\mathbb{P}_x \circ \big(X_{t_1}^{R,n}, \dots, X_{t_k}^{R,n}\big)^{-1} = \mathbb{P}_x^{(n)} \circ \big(X_{t_1}^{(n)}, \dots, X_{t_k}^{(n)}\big)^{-1}. \tag{11.18}$$

With this, we can easily transfer the right process properties from $X^{(n)}$ to $X^{R,n}$:

(i) It is immediate from the construction that $X^{R,n}$ is right continuous and, as $R^n > 0$ on $E^{(n)}$, admits normality. Furthermore, $X^{R,n}$ is $E^{(n)}$-valued and adapted to its natural filtration $(\mathscr{F}_t^{R,n}, t \geq 0)$.

(ii) $X^{R,n}$ *is a Markov process:* For all $x \in E^{(n)}$, $s, t \geq 0$, $f \in b\mathscr{E}_u^{(n)}$, $g_1, \dots, g_k \in b\mathscr{E}_u^{(n)}$, $0 \leq t_1 < \dots < t_k \leq t$, we get with (11.18) and the Markov property of $X^{(n)}$:

$$\begin{aligned}
\mathbb{E}_x\big(f(X_{s+t}^{R,n})\; g_1(X_{t_1}^{R,n}) \cdots g_k(X_{t_k}^{R,n})\big) &= \mathbb{E}_x^{(n)}\big(f(X_{s+t}^{(n)})\; g_1(X_{t_1}^{(n)}) \cdots g_k(X_{t_k}^{(n)})\big)\\
&= \mathbb{E}_x^{(n)}\big(\mathbb{E}_{X_t^{(n)}}^{(n)}\big(f(X_s^{(n)})\big)\; g_1(X_{t_1}^{(n)}) \cdots g_k(X_{t_k}^{(n)})\big)\\
&= \mathbb{E}_x^{(n)}\big(\mathbb{E}_{X_t^{(n)}}\big(f(X_s^{R,n})\big)\; g_1(X_{t_1}^{(n)}) \cdots g_k(X_{t_k}^{(n)})\big)\\
&= \mathbb{E}_x\big(\mathbb{E}_{X_t^{R,n}}\big(f(X_s^{R,n})\big)\; g_1(X_{t_1}^{R,n}) \cdots g_k(X_{t_k}^{R,n})\big).
\end{aligned}$$

(iii) $X^{R,n}$ *fulfills HD2:* Let $\mathscr{S}_\alpha(X^{(n)})$, $\mathscr{S}_\alpha(X^{R,n})$, $\alpha > 0$, be the sets of all α-excessive functions of $X^{(n)}$, $X^{R,n}$, and $T_t^{X^{(n)}}$, $T_t^{X^{R,n}}$, $t \geq 0$, be the transition operators of $X^{(n)}$, $X^{R,n}$ respectively.

By definition, we have $f \in \mathscr{S}_\alpha$ if and only if $e^{-\alpha t} T_t f \leq f$ for all $t \geq 0$ and $\lim_{t\downarrow 0} e^{-\alpha t} T_t f = f$. However, the semigroups of $X^{R,n}$ and $X^{(n)}$ coincide, that is, for all $f \in p\mathscr{E}^{(n)}$, $x \in E^{(n)}$, we have

$$T_t^{X^{R,n}} f(x) = \mathbb{E}_x\big(f(X_t^{R,n})\big) = \mathbb{E}_x^{(n)}\big(f(X_t^{(n)})\big) = T_t^{(n)} f(x),$$

so we can immediately conclude that $\mathscr{S}_\alpha(X^{R,n}) = \mathscr{S}_\alpha(X^{(n)})$.

Now let $f \in \mathscr{S}_\alpha(X^{R,n})$. It remains to show that the mapping $t \mapsto f(X_t^{R,n})$ is a.s. right continuous. But as $f \in \mathscr{S}_\alpha(X^{(n)})$ and $X^{(n)}$ is a right process, $t \mapsto f(X_t^{(n)})$ is a.s. right continuous, more precisely, there exists $N^{(n)} \in \mathscr{F}^{(n)}$ with $\mathbb{P}^{(n)}(N^{(n)}) = 0$ such that

$$\forall \omega \in \complement N^{(n)}: \quad t \mapsto f\big(X_t^{(n)}(\omega)\big) \text{ is right continuous.}$$

Set $N := (\pi^{(n)})^{-1}(N^{(n)}) \in \mathscr{F}$. Then

$$\mathbb{P}(N) = \mathbb{P}\big(\,(\pi^{(n)})^{-1}(N^{(n)})\,\big) = \mathbb{P}^{(n)}(N^{(n)}) = 0,$$

and as $\omega \in \complement N$ if and only if $\pi^{(n)}(\omega) \in \complement N^{(n)}$, we conclude that

$$\forall \omega \in \complement N: \quad t \mapsto f\big(X_t^{R,n}(\omega)\big) = f\big(X_t^{(n)} \circ \pi^{(n)}(\omega)\big) \text{ is right continuous.}$$

$\square$

We are ready to prove the generalization of theorems (11.6) and (11.7) to the concatenation of countably many processes:

(11.19) Theorem. *X is a right process. For all $n \in \mathbb{N}$, $x \in E^{(n)}$, $f \in b\mathscr{E}^{n+1}$,*

$$\mathbb{E}_x\big(f(X_{R^n})\,\mathbb{1}_{\{R^n<\infty\}}\,\big|\,\mathscr{F}_{R^n-}\big) = K^n f\circ\pi^n\,\mathbb{1}_{\{R^n<\infty\}}.$$

Proof. Let $X^{R,n}$ be the processes as defined in lemma (11.15) for the revival times R^n, $n\in\mathbb{N}$, equipped with their natural filtrations, on their state spaces $E^{R,n} := E^{(n)}$. Then the sequence $(R^n, n\in\mathbb{N})$ increases to the lifetime of X, and the sequence $(E^{R,n}, n\in\mathbb{N})$ increases to $E = \bigcup_n E^n$. Furthermore, by lemma (11.17), the process $X^{R,n}$ is a right processes on $E^{R,n}$ for every $n\in\mathbb{N}$, and being a subprocess of X, its natural filtration satisfies $\mathscr{F}^{R,n} \subseteq \mathscr{F}^R$. Finally, R^n coincides with its lifetime, so it is a terminal time for $X^{R,n}$, and being the first entry time of X into a closed set, it is also a stopping time for X. Thus, lemma (11.15) is applicable, which shows that $X = X^R$ is a right process.

It only remains to prove the formula given in theorem (11.19). To this end, we compare once again the processes $X^{R,n}$ and $X^{(n)}$ like in the proof of lemma (11.17):

As $X^{(n+1)}$ is the concatenation of $X^{(n)}$ and X^{n+1} with transfer kernel $K^n\circ\pi^n$ (see subsection 11.2), theorem (11.7) yields, with $R^{(n)} = \inf\{t\geq 0 : X_t^{(n+1)} \in E^{n+1}\}$:

$$\mathbb{E}_x^{(n+1)}\big(f(X^{(n+1)}_{R^{(n)}})\,\mathbb{1}_{\{R^{(n)}<\infty\}}\,\big|\,\mathscr{F}^{(n+1)}_{R^{(n)}-}\big) = K^n f\circ\pi^n\,\mathbb{1}_{\{R^{(n)}<\infty\}}.$$

It is evident from the definitions that

$$R^{(n)}\circ\pi^{(n+1)} = R^n \quad\text{and}\quad X^{(n+1)}_{R^{(n)}}\circ\pi^{(n+1)} = X^{R,n+1}_{R^n} \quad \text{a.s. on } \Omega.$$

By definition, $\mathscr{F}_{R^n-} = \sigma\big(\{A\cap\{t<R^n\} : t\geq 0, A\in\mathscr{F}_t\}\big)$, and this generator is $\cap$-stable, because for all $s,t\geq 0$, $A_s\in\mathscr{F}_s$, $A_t\in\mathscr{F}_t$, with $s\leq t$:

$$\big(A_s\cap\{s<R^n\}\big)\cap\big(A_t\cap\{t<R^n\}\big) = \big(A_s\cap A_t\big)\cap\{t<R^n\},$$

and $A_s\cap A_t \in \mathscr{F}_t$. Thus, it suffices to show that for all $t\geq 0$, $f\in b\mathscr{E}^u$, $k\in\mathbb{N}$, $0\leq t_1<\cdots<t_k\leq t$, $g_1,\dots,g_k\in b\mathscr{E}^u$ with

$$\begin{aligned} J &:= g_1(X_{t_1})\cdots g_k(X_{t_k})\cdot\mathbb{1}_{\{t<R^n\}},\\ J^{R,n+1} &:= g_1(X^{R,n+1}_{t_1})\cdots g_k(X^{R,n+1}_{t_k})\cdot\mathbb{1}_{\{t<R^n\}},\\ J^{(n+1)} &:= g_1(X^{(n+1)}_{t_1})\cdots g_k(X^{(n+1)}_{t_k})\cdot\mathbb{1}_{\{t<R^{(n)}\}} \end{aligned}$$

the following holds true, as $X_{R^n} = X^{R,n+1}_{R^n}$ a.s.:

$$\begin{aligned} \mathbb{E}_x\big(f(X_{R^n})\,\mathbb{1}_{\{R^n<\infty\}}\cdot J\big) &= \mathbb{E}_x\big(f(X^{R,n+1}_{R^n})\,\mathbb{1}_{\{R^n<\infty\}}\cdot J^{R,n+1}\big)\\ &= \mathbb{E}_x^{(n+1)}\big(f(X^{(n+1)}_{R^{(n)}})\,\mathbb{1}_{\{R^{(n)}<\infty\}}\cdot J^{(n+1)}\big)\\ &= \mathbb{E}_x^{(n+1)}\big(K^n f\circ\pi^n\,\mathbb{1}_{\{R^{(n)}<\infty\}}\cdot J^{(n+1)}\big)\\ &= \mathbb{E}_x\big(K^n f\circ\pi^n\,\mathbb{1}_{\{R^n<\infty\}}\cdot J\big). \end{aligned}$$

This completes the proof, as $K^n f\circ\pi^n$ is $\mathscr{F}_{R^n-}$-measurable by lemma (11.8). $\square$

11.4. Disjoint Union of Processes with Infinite Lifetime

The above-established technique of concatenation covers a simple special case, in which all of the concatenated subprocesses feature infinite lifetime. Here, the resulting process will not have any revivals at all, and every partial process evolves secluded without ever being extended by another process. Therefore, we can use our construction of subsection 11.3 to merge various right processes on disjoint spaces, each with infinite lifetime, into a combined state space. The combined process will evolve as any partial process on its designated partial space, which can be chosen by using an appropriate initial distribution.

Of course, the whole revival mechanism of the above constructions is completely unnecessary in this context and one could simplify the proofs considerably, as it is done, e.g., in [Sha88, p. 83]. We are not going to offer another proof, as we have already elaborately examined the general concatenation technique which encompasses this case. The following application will be used later:

(11.20) Corollary. *Let X be a right process with infinite lifetime on a Radon space E, and F be a Radon space disjoint from E. Then there exists a right process Y on $E \uplus F$ such that the restriction of Y to E is X and Y is the constant process on F. If X is a Feller process, then Y is a Feller process.*

Proof. Let X^2 be the constant process on F (see example (5.5)) in the right process setting, and define Y as the concatenation of $X^1 := X$ and X^2 with some arbitrary transfer kernel, for example derived from the transfer distribution $k(x, \cdot\,) = \varepsilon_y$ for some $y \in F$ and all $x \in E$. Then theorem (11.6) implies that Y is a right process on $E \uplus F$. As the first revival time R^1 is $\mathbb{P}_x$-a.s. infinite or zero, depending on the initial measure lying in $x \in E$ or $x \in F$, we have for all $n \in \mathbb{N}$, $0 \le t_1 < \cdots < t_n$, $f_1, \ldots, f_n \in b\sigma\big(\mathscr{E} \cup \mathscr{B}(F)\big)$:

$$\mathbb{E}_x\big(f_1(X_{t_1}) \cdots f_n(X_{t_n})\big) = \begin{cases} \mathbb{E}^1_x\big(f_1(X_{t_1}) \cdots f_n(X_{t_n})\big), & x \in E, \\ \mathbb{E}^2_x\big(f_1(Y_{t_1}) \cdots f_n(Y_{t_n})\big), & x \in F, \end{cases}$$

with the functions $f_1, \ldots, f_n$ on the right-hand side being the restrictions to E, F respectively. This also implies that the semigroup of Y reads

$$T_t f(x) = \mathbb{E}_x\big(f(Y_t)\big) = \begin{cases} \mathbb{E}^1_x\big(f(X_t)\big) = T^X_t f(x), & x \in E, \\ \mathbb{E}^2_x\big(f(x)\big) = f(x), & x \in F. \end{cases}$$

Thus, $(T_t, t \ge 0)$ is a Feller semigroup, if the semigroup $(T^X_t, t \ge 0)$ of X is Feller. □

12. Mapping of the State Space

Let X be a (strong) Markov process (or a right process) on E and $\psi\colon E \to \hat{E}$ be a surjective mapping. In this section we will be concerned with the question on whether $\psi(X)$ is again a (strong) Markov process (or even a right process) on $\hat{E}$.

Heuristically, one expects that this should be true if—conditions on measurability taken aside for a moment—the original process X "behaves identically" on points of E

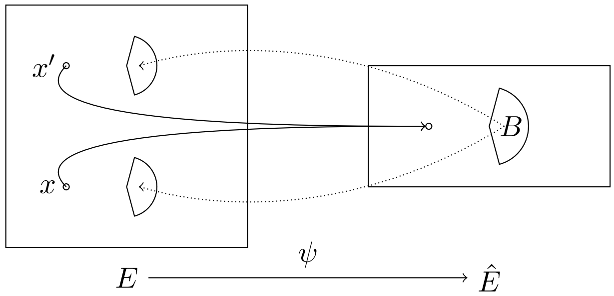


Figure 12.1: Consistency condition for state space transformations: If the transformed process $\psi(X)$ starts at $\psi(x) = \psi(x')$ and the original process X shows same transition behavior for x and x' with respect to preimages of ψ, then the Markov property lifts from X to $\psi(X)$.

that are mapped together by ψ, see also figure 12.1:

$$\forall B \in \hat{\mathscr{E}}, x, x' \in E \text{ with } \psi(x) = \psi(x') : \quad T_t(x, \psi^{-1}(B)) = T_t(x', \psi^{-1}(B)).$$

Indeed, this consistency condition is given in [Dyn65, Theorem 10.13] in order to retrieve the (strong) Markov property of $\psi(X)$ from a (strong) Markov process X. A similar characterization via semigroups can be found, e.g., in [RW00a, Lemma I.14.1], and in a more sophisticated form in [RP81].

12.1. Basic Results

In the context of right processes the result is almost the same, flavored only by some measurability conditions. It is found in [Sha88, Theorem (13.5)]:

(12.1) Theorem. *Let $X = \big(\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E}\big)$ be a right process on a Radon space E with semigroup $(T_t, t \geq 0)$ and resolvent $(U_\alpha, \alpha > 0)$. Let $(\hat{E}, \hat{\mathscr{E}})$ be a Radon space and $\psi\colon E \to \hat{E}$ be a mapping, satisfying the following conditions:*

(i) ψ is $\mathscr{E}^u/\hat{\mathscr{E}}^u$-measurable and $\psi(E) = \hat{E}$;

(ii) $t \mapsto \psi(X_t)$ is a.s. right continuous in $\hat{E}$;

(iii) for all $f \in b\mathcal{C}_d(\hat{E})$ and all $t \geq 0$, there exists $g_t \in b\hat{\mathscr{E}}^u$ such that $T_t(f \circ \psi) = g_t \circ \psi$.

Define the transformed process $Y_t := \psi(X_t)$, $t \geq 0$, on

$$\hat{\Omega} := \big\{\omega \in \Omega : t \mapsto \psi\big(X_t(\omega)\big) \text{ is right continuous in } \hat{E}\big\},$$

equipped with shift operators $\hat{\Theta}_t := \Theta_t$, $t \geq 0$, on $\hat{\Omega}$, and σ-algebras generated by Y

$$\hat{\mathscr{F}}^u := \sigma\big(\{f(Y_t) : f \in \hat{\mathscr{E}}^u, t \geq 0\}\big),$$
$$\hat{\mathscr{F}}^u_t := \sigma\big(\{f(Y_s) : f \in \hat{\mathscr{E}}^u, s \leq t\}\big), \quad t \geq 0,$$

and choose measures for $\hat{\mathbb{P}}_y$, $y \in \hat{E}$, by

$$\hat{\mathbb{P}}_y := \mathbb{P}_x \text{ on } \hat{\mathscr{F}}^u, \quad \text{for } x \in E \text{ with } \psi(x) = y \in \hat{E}. \tag{12.2}$$

Furthermore, let $\hat{\mathscr{F}}$, $(\hat{\mathscr{F}}_t, t \geq 0)$ be the usual completion and augmentations of $\hat{\mathscr{F}}^u$, $(\hat{\mathscr{F}}^u_t, t \geq 0)$ respectively, relative to the family $(\hat{\mathbb{P}}_y, y \in \hat{E})$.

Then $Y = (\hat{\Omega}, \hat{\mathscr{F}}, (\hat{\mathscr{F}}_t)_{t\geq 0}, (Y_t)_{t\geq 0}, (\hat{\Theta}_t)_{t\geq 0}, (\hat{\mathbb{P}}_y)_{y\in\hat{E}}) =: \psi(X)$ is a right process on $\hat{E}$.

As usual, property (iii) can be extended to all functions $f \in b\hat{\mathscr{E}}^u$ by using the MCT and standard completion arguments (see [Sha88, Remarks (13.6)]). Because of this property, the definition of the measures $\mathbb{P}_y$ on $\hat{\mathscr{F}}^u$ in (12.2) is independent of the representatives chosen for $y = \psi(x)$, $x \in E$: For any $f \in b\hat{\mathscr{E}}^u$, $t \geq 0$, we have

$$\hat{\mathbb{E}}_y(f(Y_t)) = \mathbb{E}_x(f(\psi(X_t))) = T_t(f \circ \psi)(x) = g_t \circ \psi(x) = g_t(y).$$

Usually, the fundamental condition (iii) must be verified manually. However, it follows automatically if the transformation ψ is bijective, as seen in [Sha88, Corollary (13.7)]):

(12.3) Theorem. *Let $\psi \colon E \to \hat{E}$ be injective and satisfy (i) and (ii) of theorem* (12.1)*. If $\mathscr{E} \subseteq \psi^{-1}(\hat{\mathscr{E}})$, then $\psi(X)$ as defined in theorem* (12.1) *is a right process on $\hat{E}$.*

Fortunately, just like in the case of the Markov property (cf. theorem (2.18)), there is also a Laplace transformed version of this condition, which sometimes is easier to control:

(12.4) Theorem. *In theorem* (12.1)*, under (i) and (ii), condition (iii) is equivalent to*

(iii') for all $f \in b\mathcal{C}_d(\hat{E})$ and all $\alpha > 0$, there exists $f_\alpha \in b\hat{\mathscr{E}}^u$ such that $U_\alpha(f \circ \psi) = f_\alpha \circ \psi$.

Proof. Assume that (i), (ii) and (iii) hold. Then for $f \in b\mathcal{C}_d(\hat{E})$, $\alpha > 0$, $x \in E_\Delta$,

$$\begin{aligned} U_\alpha(f \circ \psi)(x) &= \int_0^\infty e^{-\alpha t}\, T_t(f \circ \psi)(x)\, dt \\ &= \int_0^\infty e^{-\alpha t}\, g_t \circ \psi(x)\, dt \\ &= f_\alpha \circ \psi(x) \end{aligned}$$

holds with $g_t \in b\hat{\mathscr{E}}^u$ as given by (iii), and thus $f_\alpha := \int_0^\infty e^{-\alpha t}\, g_t\, dt \in b\hat{\mathscr{E}}^u$ fulfills the condition (iii').

Now assume that (i), (ii) and (iii') hold. Let $f \in b\mathcal{C}_d(\hat{E})$ and consider for every $\alpha > 0$ the function $f_\alpha \in b\hat{\mathscr{E}}^u$ as given by (iii') with $U_\alpha(f \circ \psi) = f_\alpha \circ \psi$. For $t = 0$, the function $g_0 = f$ satisfies $T_0(f \circ \psi) = g_0 \circ \psi$. For $t > 0$, we need to invert the Laplace transform, which is encoded in f_α, $\alpha > 0$. We first observe that $f_\alpha^{(k)} := \frac{\partial^k}{\partial \alpha^k} f_\alpha$ exists for all $k \in \mathbb{N}_0$, because for each $y \in \hat{E}$, there is $x \in E$ with $\psi(x) = y$, so

$$f_\alpha(y) = f_\alpha(\psi(x)) = U_\alpha(f \circ \psi)(x)$$

holds and $\alpha \mapsto U_\alpha(f \circ \psi)(x)$ is in $\mathcal{C}^\infty(\mathbb{R}_{>0})$ (see [DM88, Theorem XII.20]). Furthermore, for any $x \in E$, the function

$$t \mapsto T_t(f \circ \psi)(x) = \mathbb{E}_x\big(f(\psi(X_t))\big)$$

is a bounded and right continuous, because f is bounded and continuous and $t \mapsto \psi(X_t)$ is right continuous by (ii). Let $y \in \hat{E}$, and choose any $x \in E$ with $\psi(x) = y$. Then the general inversion formula (1.13) of the Laplace transform of $t \mapsto T_t(f \circ \psi)$ yields

$$\begin{aligned} T_t(f \circ \psi)(x) &= \lim_{\varepsilon \downarrow\downarrow 0} \lim_{\alpha \to \infty} \frac{1}{\varepsilon} \sum_{\alpha t < k \leq (\alpha+\varepsilon)t} \frac{(-1)^k}{k!} \alpha^k U_\alpha^{(k)}(f \circ \psi)(x) \\ &= \lim_{\varepsilon \downarrow\downarrow 0} \lim_{\alpha \to \infty} \frac{1}{\varepsilon} \sum_{\alpha t < k \leq (\alpha+\varepsilon)t} \frac{(-1)^k}{k!} \alpha^k f_\alpha^{(k)}(y) \\ &= g_t(y), \end{aligned}$$

with the function $g_t \colon \hat{E} \to \mathbb{R}$ being defined by

$$g_t := \lim_{\varepsilon \downarrow\downarrow 0} \lim_{\alpha \to \infty} \frac{1}{\varepsilon} \sum_{\alpha t < k \leq (\alpha+\varepsilon)t} \frac{(-1)^k}{k!} \alpha^k f_\alpha^{(k)},$$

which is bounded as $\|g_t\| = \|T_t(f \circ \psi)\|$ and measurable due to the measurability of all f_α^k, $\alpha > 0$, $k \in \mathbb{N}_0$. We have thus shown that there exists $g_t \in b\hat{\mathscr{E}}^u$ with $g_t \circ \psi = T_t(f \circ \psi)$. $\square$

12.2. Killing on an Absorbing Set

Let $\widetilde{E} = E_\Delta \uplus F$ be the topological union of two disjoint Radon spaces E_Δ and F, and consider a right process X on $\widetilde{E}$ such that F is an absorbing set for X (see [Sha88, Definition 12.27]). Our goal is to map the set F to Δ, thus killing the process on this absorbing set. To this end, define the mapping

$$\psi \colon \widetilde{E} \to E_\Delta,\ x \mapsto \psi(x) := \begin{cases} x, & x \in E_\Delta, \\ \Delta, & x \in F. \end{cases}$$

(12.5) Theorem. *$\psi(X)$ is a right process on E_Δ.*

Proof. We are using theorem (12.1). ψ is clearly surjective and $\widetilde{\mathscr{E}}^u/\mathscr{E}_\Delta$-measurable, as

$$\forall B \in \mathscr{E}^u : \quad \psi^{-1}(B) = \begin{cases} B, & \Delta \notin B, \\ B \cup F, & \Delta \in B. \end{cases}$$

We have $X_t \in F$ for all $t \geq H_F$ a.s., because the strong Markov property at H_F yields

$$\mathbb{P}\big(X_{H_F+t} \in F \text{ for all } t \geq 0\big) = \mathbb{E}\big(\mathbb{P}_{X_{H_F}}(X_t \in F \text{ for all } t \geq 0)\big) = 1,$$

where we also used that $X_{H_F} \in F$, following from the isolation of F in $\widetilde{E}$, and that F is absorbing for X. Furthermore, it is evident that $X_t \notin F$ for all $t < H_F$, so

$$t \mapsto \psi(X_t) = \begin{cases} X_t, & t < H_F, \\ \Delta, & t \geq H_F \end{cases}$$

is a.s. right continuous.

For all $f \in b\mathscr{E}^u$, $x \in \widetilde{E}$, we have

$$\begin{aligned} T_t(f \circ \psi)(x) &= \mathbb{E}_x\big(f \circ \psi(X_t)\big) \\ &= \mathbb{E}_x\big(f(X_t)\,;\, t < H_F\big) + f(\Delta)\,\mathbb{P}_x\big(t \geq H_F\big) \\ &= g \circ \psi(x), \end{aligned}$$

with $g \in b\mathscr{E}^u$ being defined by

$$g(x) := \begin{cases} \mathbb{E}_x\big(f(X_t)\,;\, t < H_F\big) + f(\Delta)\,\mathbb{P}_x\big(t \geq H_F\big), & x \in E, \\ f(\Delta), & x = \Delta, \end{cases}$$

because $H_F = 0$ holds $\mathbb{P}_x$-a.s. for all $x \in F$. □

13. Copies of Processes

The main limitation of the concatenation results presented in section 11 is the required disjointedness of the partial processes' state spaces. In order to apply this technique in our work, we will overcome this restriction in two special cases: In this section, we consider the concatenation of independent copies of one process X on E, and of alternating independent copies of two processes X^{-1} and X^{+1} on not necessarily disjoint spaces E^{-1} and E^{+1}.

In both cases, we will render the spaces of the independent copies disjoint by introducing a new coordinate which "counts" the current process iteration. Thus, we realize the concatenated process on the state space $\mathbb{N} \times E$ or $\mathbb{N} \times (E^{-1} \cup E^{+1})$, and then project it onto E, $E^{-1} \cup E^{+1}$ respectively, see figures 13.1 and 13.2, with the help of the mapping techniques of section 12. As we have seen there, some consistency conditions on the state space transformation ψ and the underlying Markov process X are needed to achieve the Markov property of the mapped process, namely, the original process should "behave identically" on points which are mapped together by the transformation. It is therefore necessary to examine these consistency conditions for the projection mapping $\psi(X) = \pi(X)$ of the concatenated process X; plainly stated, we need to ensure that the concatenated process X behaves identically irrespective of the iteration it is started as.

It will turn out that these conditions are always satisfied if X is the concatenation of identical copies. We thus obtain a method to construct an "instant return process" via a given "revival kernel", similar to the findings of [INW68] and [Mey75]. This process will behave like a given subprocess until it dies and is then immediately "revived" by a given kernel as the same subprocess again.

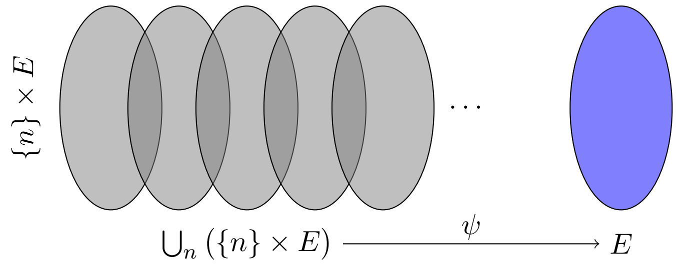


Figure 13.1: Construction of an instant revival process on E via concatenation of copies of one subprocess X on $\{n\} \times E$, $n \in \mathbb{N}$, and subsequent projection onto E.

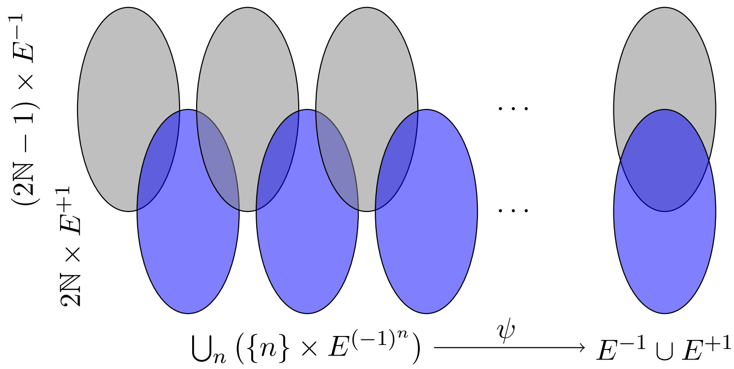


Figure 13.2: Construction of a process "pasting" of two subprocesses X^{-1}, X^{+1} on E^{-1}, E^{+1}, via concatenation of alternating subprocess copies on $(2\mathbb{N}-1) \times E^{-1}$, $2\mathbb{N} \times E^{+1}$ respectively, and subsequent projection onto $E^{-1} \cup E^{+1}$.

The concatenation of alternating copies of two processes results in a method of "pasting them together", as also done by [Nag76]. In this case however, we need to impose the following consistency conditions on the partial processes: They need to coincide on their shared state space, and the exit behavior of the concatenated process from this subspace must be irrespective of the mode of exit, which can either be realized by one subprocess simply exiting it, or by being killed and the next subprocess being revived outside it, see figure 13.3. These conditions will be made rigorous in subsection 13.2.

13.1. Identical Copies of One Process

Let X^0 be a right process on E, and K^0 be a transfer kernel from X^0 to (X^0, E). For each $n \in \mathbb{N}$, consider the following process with transfer kernel:

$$X^n := \{n\} \times X^0, \quad K^n := \varepsilon_{n+1} \otimes K^0,$$

that is, define the process $X^n := \big(\Omega^0, \mathscr{F}^0, (\mathscr{F}^0_t)_{t\geq 0}, (X^n_t)_{t\geq 0}, (\Theta^0_t)_{t\geq 0}, (\mathbb{P}^n_{(n,x)})_{x\in E}\big)$ with $X^n_t(\omega) := \big(n, X^0_t(\omega)\big)$ for $t \geq 0$, $\omega \in \Omega^0$, probability measures $\mathbb{P}^n_{(n,x)} := \varepsilon_n \otimes \mathbb{P}^0_x$ for $x \in E$,

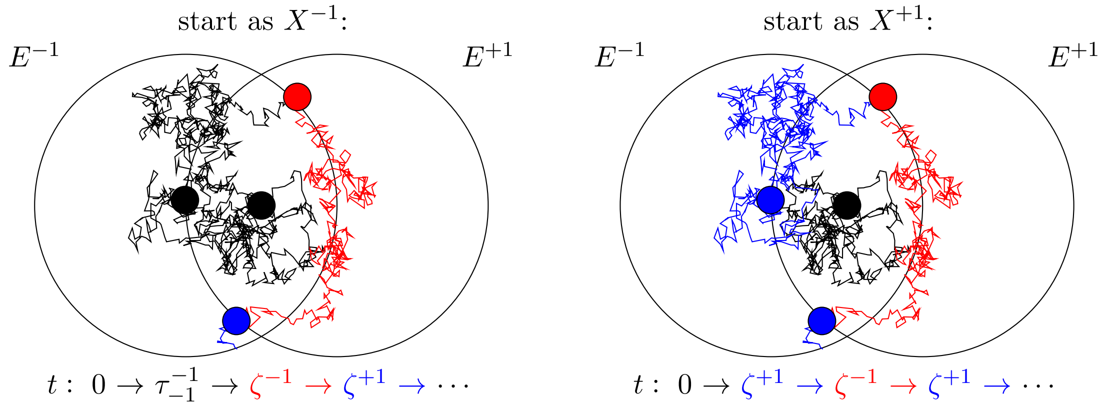


Figure 13.3: Consistency condition for pasting two processes together on a common state space: The process behavior must be independent of the chosen starting process X^{-1}, X^{+1}. The left-hand picture shows a path behavior if the concatenated process is started as X^{-1} (black), which is then revived after its death at ζ^{-1} as X^{+1} (red), afterwards revived as X^{-1} at ζ^{+1} (blue), etc. The concatenated process must show the same behavior if started as X^{+1}, as illustrated in the right-hand picture.

and the kernel $K^n(\,\cdot\,, \{n+1\} \times B) := K^0(\,\cdot\,, B)$ for $B \in \mathscr{E}^u$. Then X^n is a right process on $E^n := \{n\} \times E$, $\mathscr{E}^n_u = \{n\} \otimes \mathscr{E}^u$, and K^n is a transfer kernel from X^n to (X^{n+1}, E^{n+1}).

Let X be the concatenation of $(X^n, n \in \mathbb{N})$ via the transfer kernels $(K^n, n \in \mathbb{N})$, as constructed in subsection 11.3. By theorem (11.19), X is a right process on $\widetilde{E} = \bigcup_{n\in\mathbb{N}} E^n = \mathbb{N} \times E$, equipped with the universal measurable sets $\widetilde{\mathscr{E}}^u$.

Consider the canonical projection $\pi\colon \mathbb{N} \times E \to E$ onto the second coordinate. We then obtain the following result for the *instant revival process* $\pi(X)$, constructed of X^0 with *revival kernel* K^0:

(13.1) Theorem. *$\pi(X)$ is a right process on E.*

Proof. π is clearly surjective. As $\pi^{-1}(B) = \mathbb{N} \times B \in \widetilde{\mathscr{E}}^u$ holds for all $B \in \mathscr{E}^u$, π is $\widetilde{\mathscr{E}}^u/\mathscr{E}^u$-measurable. Because the right process X is right continuous and the projection π is continuous, the transformed process $\pi(X)$ is also right continuous.

Therefore, following theorem (12.4), it suffices to show that for all $\alpha > 0$, $f \in b\mathscr{E}^u$, there exists an $f_\alpha \in b\mathscr{E}^u$ such that $U_\alpha(f \circ \pi) = f_\alpha \circ \pi$ holds, which basically results in proving that the function $U_\alpha(f \circ \pi)(\,\cdot\,, n)$ is independent of the iteration $n \in \mathbb{N}$.

By decomposing the concatenated process X into its partial processes with the help of the revival times R^n, $n \in \mathbb{N}$, as defined in section 13, we have for $(n, x) \in \widetilde{E} = \mathbb{N} \times E$:

$$\begin{aligned} U_\alpha(f \circ \pi)\,(n,x) &= \mathbb{E}_{(n,x)}\Big(\sum_{m=0}^{\infty} \mathbb{1}_{\{R^{n+m-1}<\infty\}} \int_{R^{n+m-1}}^{R^{n+m}} e^{-\alpha t}\, f \circ \pi(X_t)\, dt \Big) \\ &= \sum_{m=0}^{\infty} \mathbb{E}_{(n,x)}\Big(\mathbb{1}_{\{R^{n+m-1}<\infty\}} \int_{R^{n+m-1}}^{R^{n+m}} e^{-\alpha t}\, f \circ \pi(X_t)\, dt \Big). \end{aligned}$$

It is $X_t = X_t^0 \circ \pi$ for all $t < R^n = \zeta^0 \circ \pi$ $\mathbb{P}_{(n,x)}$-a.s., with ζ^0 being the lifetime of X^0, so we get for $m = 0$:

$$\begin{aligned}\mathbb{E}_{(n,x)}\Big(\int_0^{R^n} e^{-\alpha t} f \circ \pi(X_t)\, dt\Big) &= \mathbb{E}_x^0\Big(\int_0^{\zeta^0} e^{-\alpha t} f(X_t^0)\, dt\Big)\\ &=: g_0(x).\end{aligned}$$

For $m = 1$, we use $R^{n+1} - R^n = R^{n+1} \circ \Theta_{R^n}$ on $\{R^n < \infty\}$ and the strong Markov property of the right process X at the terminal time R^n in order to obtain

$$\begin{aligned}&\mathbb{E}_{(n,x)}\Big(\mathbb{1}_{\{R^n<\infty\}} \int_{R^n}^{R^{n+1}} e^{-\alpha t} f \circ \pi(X_t)\, dt\Big)\\ &= \mathbb{E}_{(n,x)}\Big(\mathbb{E}_{(n,x)}\Big(\mathbb{1}_{\{R^n<\infty\}}\, e^{-\alpha R^n} \int_0^{R^{n+1}\circ\Theta_{R^n}} e^{-\alpha t} f \circ \pi(X_t \circ \Theta_{R^n})\, dt \,\big|\, \mathscr{F}_{R^n}\Big)\Big)\\ &= \mathbb{E}_{(n,x)}\Big(\mathbb{1}_{\{R^n<\infty\}}\, e^{-\alpha R^n}\, \mathbb{E}_{X_{R^n}}\Big(\int_0^{R^{n+1}} e^{-\alpha t} f \circ \pi(X_t)\, dt\Big)\Big).\end{aligned}$$

Now, $e^{-\alpha R^n} \in \mathscr{F}_{R^n-}$ and $\mathbb{E}_\cdot\big(\int_0^{R^{n+1}} \cdots\, dt\big)\Big|_{E^{n+1}} \in b\mathscr{E}_u^{n+1}$, so theorem (11.19) yields

$$\begin{aligned}&\mathbb{E}_{(n,x)}\Big(\mathbb{1}_{\{R^n<\infty\}}\, \mathbb{E}_{X_{R^n}}\Big(\int_0^{R^{n+1}} e^{-\alpha t} f \circ \pi(X_t)\, dt\Big) \,\big|\, \mathscr{F}_{R^n-}\Big)\\ &= \mathbb{E}_{(n,x)}\Big(\mathbb{1}_{\{R^n<\infty\}}\, K^n \mathbb{E}_\cdot\Big(\int_0^{R^{n+1}} e^{-\alpha t} f \circ \pi(X_t)\, dt\Big) \circ \pi^n\Big),\end{aligned}$$

which results in

$$\begin{aligned}&\mathbb{E}_{(n,x)}\Big(\mathbb{1}_{\{R^n<\infty\}} \int_{R^n}^{R^{n+1}} e^{-\alpha t} f \circ \pi(X_t)\, dt\Big)\\ &= \mathbb{E}_{(n,x)}\Big(\mathbb{1}_{\{R^n<\infty\}}\, e^{-\alpha R^n}\, K^n \mathbb{E}_\cdot\Big(\int_0^{R^{n+1}} e^{-\alpha t} f \circ \pi(X_t)\, dt\Big) \circ \pi^n\Big)\\ &= \mathbb{E}_x^0\Big(\mathbb{1}_{\{\zeta^0<\infty\}}\, e^{-\alpha\zeta^0}\, K^0 \mathbb{E}_{(n+1,\,\cdot\,)}\Big(\int_0^{R^{n+1}} e^{-\alpha t} f \circ \pi(X_t)\, dt\Big)\Big)\\ &= \mathbb{E}_x^0\Big(\mathbb{1}_{\{\zeta^0<\infty\}}\, e^{-\alpha\zeta^0}\, K^0 g_0\Big)\\ &=: g_1(x).\end{aligned}$$

For general $m \in \mathbb{N}_0$, we inductively show that

$$\mathbb{E}_{(n,x)}\Big(\mathbb{1}_{\{R^{n+m-1}<\infty\}} \int_{R^{n+m-1}}^{R^{n+m}} e^{-\alpha t} f \circ \pi(X_t)\, dt\Big) = g_m(x)$$

holds true with $g_m \in b\mathscr{E}^u$ being independent of $n \in \mathbb{N}$. The cases $m = 0$ and $m = 1$ are already done. Suppose that the assertion is shown for an $m \in \mathbb{N}$. We then calculate for

$m+1$, by using exactly the same techniques as in the case $m=1$:

$$\begin{aligned}
&\mathbb{E}_{(n,x)}\Big(\mathbb{1}_{\{R^{n+m}<\infty\}} \int_{R^{n+m}}^{R^{n+m+1}} e^{-\alpha t} f\circ\pi(X_t)\,dt\Big)\Big)\\
&= \mathbb{E}_{(n,x)}\Big(\mathbb{1}_{\{R^n<\infty\}}\, e^{-\alpha R^n}\mathbb{E}_{X_{R^n}}\Big(\mathbb{1}_{\{R^{n+m}<\infty\}} \int_{R^{n+m}}^{R^{n+m+1}} e^{-\alpha t} f\circ\pi(X_t)\,dt\Big)\Big)\\
&= \mathbb{E}^0_x\Big(\mathbb{1}_{\{R^n<\infty\}}\, e^{-\alpha\zeta^0}\, K^0\mathbb{E}_{(n+1,\,\cdot\,)}\Big(\mathbb{1}_{\{R^{n+m}<\infty\}} \int_{R^{(n+1)+m-1}}^{R^{(n+1)+m}} e^{-\alpha t} f\circ\pi(X_t)\,dt\Big)\Big)\\
&= \mathbb{E}^0_x\Big(\mathbb{1}_{\{\zeta^0<\infty\}}\, e^{-\alpha\zeta^0}\, K^0 g_m\Big)\\
&=: g_{m+1}(x).
\end{aligned}$$

We have thus shown that, for all $(n,x)\in\widetilde{E}$,

$$U_\alpha(f\circ\pi)\,(n,x) = \sum_{m=0}^\infty g_m(x) = \sum_{m=0}^\infty g_m\circ\pi(n,x) = f_\alpha\circ\pi(n,x)$$

holds with $f_\alpha := \sum_{m=0}^\infty g_m$. It is $f_\alpha\in b\mathscr{E}^u$, as $g_m\in b\mathscr{E}^u$ for all $m\in\mathbb{N}$ and for all $x\in E$,

$$\begin{aligned}
\big|f_\alpha(x)\big| &= \Big|\sum_{m=0}^\infty g_m(x)\Big|\\
&= \Big|\sum_{m=0}^\infty \mathbb{E}_{(n,x)}\Big(\mathbb{1}_{\{R^{n+m-1}<\infty\}} \int_{R^{n+m-1}}^{R^{n+m}} e^{-\alpha t} f\circ\pi(X_t)\,dt\Big)\Big|\\
&= \Big|\mathbb{E}_{(n,x)}\Big(\int_0^\infty e^{-\alpha t} f\big(\pi(X_t)\big)\,dt\Big)\Big|\\
&\le \frac{1}{\alpha}\,\|f\|_\infty\,.
\end{aligned}$$

□

13.2. Alternating Copies of Two Processes

Let X^{-1}, X^{+1} be two right processes with lifetimes ζ^{-1}, ζ^{+1} on E^{-1}, E^{+1} respectively, and K^{-1}, K^{+1} be transfer kernels from X^{-1} to (X^{+1},E^{+1}) and from X^{+1} to (X^{-1},E^{-1}). Following our construction in the previous subsection, we define for each $n\in\mathbb{N}$

$$X^n := \{n\}\times X^{(-1)^n}, \quad K^n := \varepsilon_{n+1}\otimes K^{(-1)^n},$$

where now and in all that follows, $(-1)^n$ really means the n-th power of the number -1, that is $(-1)^n\in\{-1,+1\}$, and the exponent -1 will index the odd-numbered process and not the preimage of a mapping, if nothing else is said. Then again, X^n is a right process on $E^n := \{n\}\times E^{(-1)^n}$, $\mathscr{E}^n_u = \{n\}\otimes\mathscr{E}_u^{(-1)^n}$, and K^n is a transfer kernel from X^n to (X^{n+1},E^{n+1}). Let X be the concatenation of $(X^n, n\in\mathbb{N})$ via the transfer kernels $(K^n, n\in\mathbb{N})$. By theorem (11.19), it is a right process on $\widetilde{E} = \bigcup_{n\in\mathbb{N}} E^n$, equipped with the universal measurable sets $\widetilde{\mathscr{E}}^u$.

Set $E := E^{-1}\cup E^{+1}$, and let $\pi\colon \widetilde{E}\to E$ be the canonical projection onto the second coordinate.

(13.2) Theorem. *Let τ_{-1}^{-1} be the first entry time of X^{-1} into $E^{-1}\backslash E^{+1}$, and τ_{+1}^{+1} be the first entry time of X^{+1} into $E^{+1}\backslash E^{-1}$. If for all $x \in E^{-1} \cap E^{+1}$, $f \in b\mathscr{E}^u$, the equalities*

(i) $\mathbb{E}_x^{-1}\big(\int_0^{\tau_{-1}^{-1}} e^{-\alpha t} f(X_t^{-1})\,dt\big) = \mathbb{E}_x^{+1}\big(\int_0^{\tau_{+1}^{+1}} e^{-\alpha t} f(X_t^{+1})\,dt\big)$,

(ii) $\mathbb{E}_x^{-1}\big(e^{-\alpha\tau_{-1}^{-1}}\, g^{-1}(X_{\tau_{-1}^{-1}}^{-1});\ \tau_{-1}^{-1} < \zeta^{-1}\big) = \mathbb{E}_x^{+1}\big(e^{-\alpha\zeta^{+1}}\, K^{+1}g^{-1};\ \zeta^{+1} < \tau_{+1}^{+1}\big)$,
$\mathbb{E}_x^{+1}\big(e^{-\alpha\tau_{+1}^{+1}}\, g^{+1}(X_{\tau_{+1}^{+1}}^{+1});\ \tau_{+1}^{+1} < \zeta^{+1}\big) = \mathbb{E}_x^{-1}\big(e^{-\alpha\zeta^{-1}}\, K^{-1}g^{+1};\ \zeta^{-1} < \tau_{-1}^{-1}\big)$

hold true, then $\pi(X)$ is a right process on E.

Proof. π is clearly surjective. It is $\widetilde{\mathscr{E}}^u/\mathscr{E}^u$-measurable, as the preimage of π reads

$$\pi^{-1}(B) = \big((2\mathbb{N}-1) \times (B \cap E^{-1})\big) \cup \big(2\mathbb{N} \times (B \cap E^{+1})\big), \quad B \in \mathscr{E}^u.$$

The right process X is right continuous and the projection π is continuous, so $\pi(X)$ is right continuous as well. By theorem (12.4), it therefore suffices to prove that for all $\alpha > 0$, $f \in b\mathscr{E}^u$, there exists $f_\alpha \in b\mathscr{E}^u$ such that $U_\alpha(f \circ \pi) = f_\alpha \circ \pi$ holds true:

We will follow the same course as in the proof of theorem (13.1). However, as the underlying subprocesses differ for odd-numbered and even-numbered revival times, we need to look at cycles of two revivals, that is, we examine for $(n,x) \in \widetilde{E}$:

$$U_\alpha(f \circ \pi)\,(n,x) = \sum_{m=0}^{\infty} \mathbb{E}_{(n,x)}\Big(\mathbb{1}_{\{R^{n+2m-1}<\infty\}} \int_{R^{n+2m-1}}^{R^{n+2m+1}} e^{-\alpha t}\, f \circ \pi(X_t)\,dt\Big).$$

For $m = 0$, we decompose the partial resolvent at R^{n+1} and get with theorem (11.19):

$$\begin{aligned}
&\mathbb{E}_{(n,x)}\Big(\int_0^{R^{n+1}} e^{-\alpha t}\, f \circ \pi(X_t)\,dt\Big)\\
&= \mathbb{E}_x^{(-1)^n}\Big(\int_0^{\zeta^{(-1)^n}} e^{-\alpha t}\, f\big(X_t^{(-1)^n}\big)\,dt\Big)\\
&\quad + \mathbb{E}_x^{(-1)^n}\Big(\mathbb{1}_{\{\zeta^{(-1)^n}<\infty\}}\, e^{-\alpha\zeta^{(-1)^n}}\, K^{(-1)^n}\mathbb{E}_\cdot^{(-1)^{n+1}}\Big(\int_0^{\zeta^{(-1)^{n+1}}} e^{-\alpha t}\, f\big(X_t^{(-1)^{n+1}}\big)\,dt\Big)\Big)\\
&=: g_0^{(-1)^n}(x).
\end{aligned}$$

For general $m \in \mathbb{N}_0$, we will show inductively that

$$\mathbb{E}_{(n,x)}\Big(\mathbb{1}_{\{R^{n+2m-1}<\infty\}} \int_{R^{n+2m-1}}^{R^{n+2m+1}} e^{-\alpha t}\, f \circ \pi(X_t)\,dt\Big) = g_m^{(-1)^n}(x)$$

holds with $g_m^{-1}, g_m^{+1} \in b\mathscr{E}^u$ being independent of $n \in \mathbb{N}$. The case $m = 0$ is already done. Assuming that the assertion is shown for an $m \in \mathbb{N}$, we calculate for $m+1$, by applying exactly the same techniques as in the proof of theorem (13.1):

$$
\begin{aligned}
&\mathbb{E}_{(n,x)}\Big(\mathbb{1}_{\{R^{n+2(m+1)-1}<\infty\}}\int_{R^{n+2(m+1)-1}}^{R^{n+2(m+1)+1}} e^{-\alpha t}\, f\circ\pi(X_t)\,dt\Big)\\
&= \mathbb{E}_{(n,x)}\Big(\mathbb{1}_{\{R^n<\infty\}}\,e^{-\alpha R^n}\,K^n\mathbb{E}_{\cdot}\Big(\mathbb{1}_{\{R^{n+1}<\infty\}}\,e^{-\alpha R^{n+1}}\\
&\qquad\qquad K^{n+1}\mathbb{E}_{\cdot}\Big(\mathbb{1}_{\{R^{n+2m+1}<\infty\}}\int_{R^{n+2m+1}}^{R^{n+2m+3}} e^{-\alpha t}\, f\circ\pi(X_t)\,dt\Big)\circ\pi^{n+1}\Big)\circ\pi^n\Big)\\
&= \mathbb{E}_x^{(-1)^n}\Big(\mathbb{1}_{\{\zeta^{(-1)^n}<\infty\}}\,e^{-\alpha\zeta^{(-1)^n}}\,K^{(-1)^n}\mathbb{E}_{\cdot}^{(-1)^{n+1}}\Big(\mathbb{1}_{\{\zeta^{(-1)^{n+1}}<\infty\}}\,e^{-\alpha\zeta^{(-1)^{n+1}}}\\
&\qquad\qquad K^{(-1)^{n+1}}\,\mathbb{E}_{(n+2,\,\cdot\,)}\Big(\mathbb{1}_{\{R^{n+2m+1}<\infty\}}\int_{R^{n+2m+1}}^{R^{n+2m+3}} e^{-\alpha t}\, f\circ\pi(X_t)\,dt\Big)\Big)\Big)\\
&= \mathbb{E}_x^{(-1)^n}\Big(\mathbb{1}_{\{\zeta^{(-1)^n}<\infty\}}\,e^{-\alpha\zeta^{(-1)^n}}\,K^{(-1)^n}\mathbb{E}_{\cdot}^{(-1)^{n+1}}\Big(\mathbb{1}_{\{\zeta^{(-1)^{n+1}}<\infty\}}\,e^{-\alpha\zeta^{(-1)^{n+1}}}\\
&\qquad\qquad\qquad\qquad K^{(-1)^{n+1}}\,g_m^{(-1)^n}\Big)\Big)\\
&=: g_{m+1}^{(-1)^n}(x),
\end{aligned}
$$

where we used in the next-to-last identity the inductive assumption and that trivially $g_m^{(-1)^{n+2}} = g_m^{(-1)^n}$ holds true.

Setting $g^{-1} := \sum_{m=0}^\infty g_m^{-1}$ and $g^{+1} := \sum_{m=0}^\infty g_m^{+1} \in b\mathscr{E}^u$, we thus have shown that

$$
U_\alpha(f\circ\pi)\,(n,x) = \begin{cases} g^{-1}(x), & n \text{ odd-numbered},\\ g^{+1}(x), & n \text{ even-numbered}\end{cases}
$$

holds for all $(n,x)\in\widetilde{E}$, so the value of the resolvent $U_\alpha(f\circ\pi)\,(n,x)$ is independent of n for all odd-numbered n, and for all even-numbered n.

It remains to prove $g^{-1} = g^{+1}$, which is equivalent to showing that

$$
U_\alpha(f\circ\pi)\,(n_o,x) = U_\alpha(f\circ\pi)\,(n_e,x)
$$

holds true for all $n_o\in(2\mathbb{N}-1)$, $n_e\in 2\mathbb{N}$, $x\in E^{-1}\cap E^{+1}$ (because $(n_0,x)\notin E$ for $x\in E^{+1}\backslash E^{-1}$, and $(n_e,x)\notin E$ for $x\in E^{-1}\backslash E^{+1}$).

Let τ_{-1} be the first entry time of $\pi(X)$ into $E^{-1}\backslash E^{+1}$, and τ_{+1} be the first entry time of $\pi(X)$ into $E^{+1}\backslash E^{-1}$. We synchronize the start of both processes by decomposing the resolvent at the stopping time $\tau_{-1}\wedge\tau_{+1}$ with the help of Dynkin's formula (3.16):

$$
\begin{aligned}
U_\alpha(f\circ\pi)\,(n,x) &= \mathbb{E}_{(n,x)}\Big(\int_0^{\tau_{-1}\wedge\tau_{+1}} e^{-\alpha t}\, f\circ\pi(X_t)\,dt\Big)\\
&\quad + \mathbb{E}_{(n,x)}\big(e^{-\alpha(\tau_{-1}\wedge\tau_{+1})}\,U_\alpha(f\circ\pi)(X_{\tau_{-1}\wedge\tau_{+1}})\big).
\end{aligned}
$$

$\tau_{-1}\wedge\tau_{+1}$ is the exit time of the process X from $E^{-1}\cap E^{+1}$. The above formula will turn out to be independent of n if the process' behavior on $E^{-1}\cap E^{+1}$ and its exit/entry behavior into $E\backslash(E^{-1}\cap E^{+1})$ (represented by $e^{-\alpha(\tau_{-1}\wedge\tau_{+1})}$ and $X_{\tau_{-1}\wedge\tau_{+1}}$) are independent of n. It has already been shown that this is the case for all odd-numbered n, and for all even-numbered n. It remains to compare the odd-numbered and even-numbered starting processes, that is, the behavior of the original processes X^{-1} and X^{+1} together with their transfer kernels K^{-1} and K^{+1}:

For odd-numbered $n_o \in (2\mathbb{N}-1)$, the starting process is $X^{(-1)^{n_o}} = X^{-1}$, living on E^{-1}, so the process $\pi(X)$ starting at (n_o, x) only enters $E^{+1}\backslash E^{-1}$ when the first subprocess dies. Therefore, $\tau_{-1} \wedge \tau_{+1} = \tau_{-1} \wedge R^{n_o}$ holds true in this case, and using Dynkin's formula (3.16) again, we get

$$\begin{aligned} U_\alpha(f \circ \pi)\,(n_o, x) = \mathbb{E}_{(n_o,x)}\Big(\int_0^{\tau_{-1}\wedge R^{n_o}} e^{-\alpha t}\, f \circ \pi(X_t)\, dt \Big) \\ + \mathbb{E}_{(n_o,x)}\big(e^{-\alpha \tau_{-1}}\, U_\alpha(f \circ \pi)(X_{\tau_{-1}});\, \tau_{-1} < R^{n_o}\big) \\ + \mathbb{E}_{(n_o,x)}\big(e^{-\alpha R^{n0}}\, U_\alpha(f \circ \pi)(X_{R^{n_o}});\, R^{n_o} \le \tau_{-1}\big). \end{aligned}$$

We have, $\mathbb{P}_{(n_o,x)}$-a.s., $X_t = \big(n_o, X_t^{(-1)^{n_o}} \circ \pi^{n_o}\big)$ for all $t < R^{n_o} = \zeta^{(-1)^{n_o}} \circ \pi^{n_o} = \zeta^{-1} \circ \pi^{n_o}$, and $\tau_{-1}^{-1} \circ \pi^{n_o} < \zeta^{-1} \circ \pi^{n_o}$ if and only if $\tau_{-1} < R^{n_o}$, and in this case $\tau_{-1} = \tau_{-1}^{-1} \circ \pi^{n_o}$ holds true. Thus, the first part of the above decomposition reads

$$\begin{aligned} &\mathbb{E}_{(n_o,x)}\Big(\int_0^{\tau_{-1}\wedge R^{n_o}} e^{-\alpha t}\, f \circ \pi(X_t)\, dt \Big) \\ &= \mathbb{E}_{(n_o,x)}\Big(\Big(\int_0^{\tau_{-1}^{-1}} e^{-\alpha t}\, f(X_t^{-1})\, dt \Big) \circ \pi^{n_o};\, \tau_{-1} < R^{n_o} \Big) \\ &\quad + \mathbb{E}_{(n_o,x)}\Big(\Big(\int_0^{\zeta^{-1}} e^{-\alpha t}\, f(X_t^{-1})\, dt \Big) \circ \pi^{n_o};\, R^{n_o} \le \tau_{-1} \Big). \end{aligned}$$

As $f(X_t^{-1}) = f(\Delta) = 0$ holds for all $t > \zeta^{-1}$, we can replace the upper limit of the latter integration by $\tau_{-1}^{-1} \ge \zeta^{-1}$, in order to obtain

$$\begin{aligned} &\mathbb{E}_{(n_o,x)}\Big(\int_0^{\tau_{-1}\wedge R^{n_o}} e^{-\alpha t}\, f \circ \pi(X_t)\, dt \Big) \\ &= \mathbb{E}_{(n_o,x)}\Big(\Big(\int_0^{\tau_{-1}^{-1}} e^{-\alpha t}\, f(X_t^{-1})\, dt \Big) \circ \pi^{n_o};\, \tau_{-1} < R^{n_o} \Big) \\ &\quad + \mathbb{E}_{(n_o,x)}\Big(\Big(\int_0^{\tau_{-1}^{-1}} e^{-\alpha t}\, f(X_t^{-1})\, dt \Big) \circ \pi^{n_o};\, R^{n_o} \le \tau_{-1} \Big) \\ &= \mathbb{E}_{(n_o,x)}\Big(\Big(\int_0^{\tau_{-1}^{-1}} e^{-\alpha t}\, f(X_t^{-1})\, dt \Big) \circ \pi^{n_o} \Big). \end{aligned}$$

Together with the process transfer at R^{n_o} via $K^{(-1)^{n_o}} = K^{-1}$, and recalling that we already showed $U_\alpha(f \circ \pi)\,(n_o, \cdot\,) = g^{-1}$ and $U_\alpha(f \circ \pi)\,(n_o + 1, \cdot\,) = g^{+1}$, we get

$$\begin{aligned} U_\alpha(f \circ \pi)\,(n_o, x) = \mathbb{E}_x^{-1}\Big(\int_0^{\tau_{-1}^{-1}} e^{-\alpha t}\, f(X_t^{-1})\, dt \Big) \\ + \mathbb{E}_x^{-1}\big(e^{-\alpha \tau_{-1}^{-1}}\, g^{-1}(X^{-1}_{\tau_{-1}^{-1}});\, \tau_{-1}^{-1} < \zeta^{-1}\big) \\ + \mathbb{E}_x^{-1}\big(e^{-\alpha \zeta^{-1}}\, K^{-1} g^{+1};\, \zeta^{-1} \le \tau_{-1}^{-1}\big). \end{aligned}$$

Analogously, we find that for any even-numbered $n_e \in 2\mathbb{N}$,

$$\begin{aligned} U_\alpha(f \circ \pi)(n_e, x) = \mathbb{E}_x^{+1}\Big(\int_0^{\tau_{+1}^{+1}} e^{-\alpha t} f(X_t^{+1})\, dt\Big) \\ + \mathbb{E}_x^{+1}\big(e^{-\alpha\tau_{+1}^{+1}} g^{+1}(X_{\tau_{+1}^{+1}}^{+1});\, \tau_{+1}^{+1} < \zeta^{+1}\big) \\ + \mathbb{E}_x^{+1}\big(e^{-\alpha\zeta^{+1}} K^{+1} g^{-1};\, \zeta^{+1} \le \tau_{+1}^{+1}\big) \end{aligned}$$

holds. Using the assumptions (i) and (ii) of the theorem, we conclude that

$$U_\alpha(f \circ \pi)\,(n_o, x) = U_\alpha(f \circ \pi)\,(n_e, x),$$

proving $U_\alpha(f \circ \pi)\,(n, x) = g^{\pm 1} \circ \pi(x)$ for all $x \in E$, $n \in \mathbb{N}$. $\square$

Chapter III.

Brownian Motions on Metric Graphs

In this chapter, we examine, characterize and construct Brownian motions on metric graphs. We start by collecting some standard results on the one-dimensional Brownian motion and its local time in sections 14 and 15. In sections 16, 17 and 19, we give a summary of classical results for the half-line, interval and "skew" cases (with some extensions to the non-continuous context), which form the basis for the generalizations to the graph setting. Metric graphs and Brownian motions thereon are introduced and examined in sections 18 and 20. By extending Itô and McKean's ideas for the half-line case, which are elaborately studied in subsection 16.4, we achieve a pathwise construction for all possible Brownian motions on a star graph in section 21. By applying the techniques of chapter II, we then "glue" Brownian motions on various star graphs together in order to obtain Brownian motions on general metric graphs in section 22.

14. Brownian Motion on the Real Line

In this and the following section, we collect the results concerning the "standard" one-dimensional Brownian motion which will be needed in our work. Considering the vast theory on this process, we will use surprisingly few properties: Its actual existence, the closed forms for its resolvent and generator, and some basic knowledge of the well-known passage time formulas and of Brownian local time will turn out to be sufficient.

14.1. Definition and Existence

Brownian motion can be studied (and thus defined) in several contexts, such as via the theories of Lévy processes, Gaussian processes, or martingales. For our purposes, the adequate approach obviously lies in the setting of Markov processes:

(14.1) Definition. *A continuous, strong Markov process*

$$B = (\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geq 0}, (B_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}})$$

on $\mathbb{R}$ *with transition semigroup*

$$T^B_t f(x) = \int_{\mathbb{R}} f(y)\, \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{(y-x)^2}{2t}}\, dt, \quad x \in \mathbb{R},\ t \geq 0,\ f \in b\mathscr{B}(\mathbb{R}),$$

is called (standard) Brownian motion on $\mathbb{R}$*.*

There are various ways to construct such a process, [Kni81, Chapter 1] summarizes and explains some methods. The quickest proof of existence in our context might be the following: Show that the defining Gaussian linear operators $(T^B_t, t \geq 0)$ indeed form a semigroup, use them to constitute the projective family of probability measures as given in section 2, and gain a path space version of the "Brownian motion" with the help of Kolmogorov's extension theorem. Then use a path regularization mechanism such as the Kolmogorov–Chentsov theorem in order to obtain a continuous version, see, e.g., [KS91, Section 2.2].

It is immediate from the very definition that the Brownian motion on $\mathbb{R}$ is a Lévy Markov process and thus a Feller process. We will assume, if needed, that B is realized on a sample space Ω which offers stopping, translation, centering and reflection operators (see subsections 3.5 and 6.5), for instance we can choose the Wiener space $\Omega = \mathcal{C}(\mathbb{R})$.

14.2. Generator and Resolvent

As the Brownian motion on $\mathbb{R}$ is a Feller process, its semigroup satisfies $T^B\mathcal{C}_0(\mathbb{R}) \subseteq \mathcal{C}_0(\mathbb{R})$, and an application of LDCT shows that $U^B\mathcal{C}_0(\mathbb{R}) \subseteq \mathcal{C}_0(\mathbb{R})$ for its resolvent $(U^B_\alpha, \alpha > 0)$ (see also theorem (5.13)). The generator and resolvent of a Brownian motion are well known (see, e.g., [Dyn65, Section 2.16] and [RW00a, Exercise III.3.13, Example III.6.9]), the resolvent is given by

$$
\begin{aligned}
U^B_\alpha f(x) &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\alpha}}\, e^{-\sqrt{2\alpha}\,|y-x|}\, f(y)\, dy \\
&= \frac{1}{\sqrt{2\alpha}}\, e^{-\sqrt{2\alpha}\,x} \int_{-\infty}^{x} e^{\sqrt{2\alpha}\,y}\, f(y)\, dy + \frac{1}{\sqrt{2\alpha}}\, e^{\sqrt{2\alpha}\,x} \int_{x}^{\infty} e^{-\sqrt{2\alpha}\,y}\, f(y)\, dy
\end{aligned}
\tag{14.2}
$$

for any $f \in b\mathscr{B}(\mathbb{R})$, $x \in \mathbb{R}$. An easy analysis of the resolvent shows the following:

(14.3) Lemma. *The resolvent of B admits $U^B b\mathcal{C}(\mathbb{R}) \subseteq b\mathcal{C}(\mathbb{R})$ and $U^B\mathcal{C}_0(\mathbb{R}) \subseteq \mathcal{C}^2_0(\mathbb{R})$.*

Differentiating (14.2) twice yields

$$
U^B_\alpha f'(x) = -e^{-\sqrt{2\alpha}\,x} \int_{-\infty}^{x} e^{\sqrt{2\alpha}\,y}\, f(y)\, dy + e^{\sqrt{2\alpha}\,x} \int_{x}^{\infty} e^{-\sqrt{2\alpha}\,y}\, f(y)\, dy,
$$

and

$$
\begin{aligned}
U^B_\alpha f''(x) &= \sqrt{2\alpha}\, e^{-\sqrt{2\alpha}\,x} \int_{-\infty}^{x} e^{\sqrt{2\alpha}\,y}\, f(y)\, dy - f(x) \\
&\quad + \sqrt{2\alpha}\, e^{\sqrt{2\alpha}\,x} \int_{x}^{\infty} e^{-\sqrt{2\alpha}\,y}\, f(y)\, dy - f(x) \\
&= 2\big(\alpha\, U^B_\alpha f(x) - f(x)\big).
\end{aligned}
$$

Together with theorem (1.9), this gives (with a brief consideration in order to gain the complete generator domain, see, e.g., [RY94, Proposition VII.1.10]) the following fundamental result:

(14.4) Theorem. *The Brownian motion B on $\mathbb{R}$ is a Feller process with generator $A = \frac{\Delta}{2}$ on $\mathscr{D}(A) = \mathcal{C}^2_0(\mathbb{R})$.*

14.3. Passage Times

Fortunately, a vast amount of explicit formulas are available for various data of the Brownian motion. In particular, the following formulas for the passage times will turn out to be quite useful. They can be found, e.g., in [IM74, Section 1.7]:

(14.5) Lemma. *Let B be a Brownian motion on $\mathbb{R}$, and for every $x \in \mathbb{R}$, let*

$$H_x := \inf\{t \geq 0 : X_t = x\}$$

be the first hitting time of x. Then,

(i) for $x \in \mathbb{R}$,

$$\mathbb{E}_0\big(e^{-\alpha H_x}\big) = e^{-\sqrt{2\alpha}x};$$

(ii) for $x \in \mathbb{R}$, $a < x < b$,

$$\mathbb{E}_x\big(e^{-\alpha H_a}\,;\, H_a < H_b\big) = \frac{\sinh\big(\sqrt{2\alpha}\,(b-x)\big)}{\sinh\big(\sqrt{2\alpha}\,(b-a)\big)},$$

$$\mathbb{E}_x\big(e^{-\alpha H_b}\,;\, H_b < H_a\big) = \frac{\sinh\big(\sqrt{2\alpha}\,(x-a)\big)}{\sinh\big(\sqrt{2\alpha}\,(b-a)\big)},$$

$$\mathbb{E}_x\big(e^{-\alpha H_a \wedge H_b}\big) = \frac{\cosh\big(\sqrt{2\alpha}\,d\big)}{\cosh\big(\sqrt{2\alpha}\,(b-a)/2\big)}, \quad d := \left|x - \frac{a+b}{2}\right|.$$

As we are in the context of Lévy Markov processes, we can derive analogous results for other starting points with the help of translation operators $(\gamma_x, x \in \mathbb{R})$, which were established in subsection 6.5. For instance, we have for $x, y \in \mathbb{R}$,

$$H_x \circ \gamma_y = \inf\{t \geq 0 : X_t \circ \gamma_y = x\} = \inf\{t \geq 0 : X_t + y = x\} = H_{x-y},$$

which shows with the help of theorem (6.35):

$$\mathbb{E}_y\big(e^{-\alpha H_x}\big) = \mathbb{E}_y\big(e^{-\alpha H_{x-y}} \circ \gamma_y\big) = \mathbb{E}_{y-y}\big(e^{-\alpha H_{x-y}}\big) = e^{-\sqrt{2\alpha}(x-y)}.$$

15. Local Time of Brownian Motion

An essential tool in the study of the Brownian sample paths (and of course in the theory of stochastic integration, which we will not need here) is the Brownian local time, or "*mesure du voisinage*", as it was coined when first introduced by Lévy in [Lév48].

Brownian local time is the source of many deep and outstanding results, such as the Ray–Knight theorems. However, we will "only" resort to one main result by Lévy in our work and therefore try to keep this summary as brief as possible.

15.1. Definition and Existence

We are going to use the approach offered by [KS91, Section 3.6], but choose the normalization used in [IM63] in order to extend their results later without too much confusion.

(15.1) Definition. *The family of random variables $\big(L_t(x), t \geq 0, x \in \mathbb{R}\big)$ with values in $[0, \infty)$ is called local time for the Brownian motion B, if for every $t \geq 0$, $x \in \mathbb{R}$, $L_t(x)$ is $\mathscr{F}_t$-measurable, and a.s., $(t, x) \mapsto L_t(x)$ is continuous and satisfies*

$$\forall t \geq 0, A \in \mathscr{B}(\mathbb{R}) : \qquad \int_A L_t(x)\, dx = \lambda\big(\{s \leq t : B_s \in A\}\big).$$

The local time at the origin is denoted by $L = (L_t, t \geq 0)$ with $L_t := L_t(0)$.

Existence of the local time is not trivial, its proof is usually named Trotter's theorem [Tro58], see also [KS91, Theorem 3.6.11] for a modern approach.

The following properties of the Brownian local time are immediate from its definition:

(15.2) Lemma. *For every $x \in \mathbb{R}$, $(L_t(x), t \geq 0)$ is a perfect continuous additive functional, and satisfies*

$$\forall t \geq 0 : \qquad L_t(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon}\, \lambda\big(\{s \leq t : |B_s - x| \leq \varepsilon\}\big).$$

15.2. Lévy's Characterization

Tanaka's formulas [KS91, Proposition 3.6.8] show an intimate connection between the reflecting Brownian motion $|B|$ and the local time L at zero of a Brownian motion. We will mainly use the celebrated characterization by Lévy, as given in [KS91, Theorem 3.6.17]:

(15.3) Theorem. *Let B be a Brownian motion with local time L at the origin. Then the process $\widetilde{B}_t := -\int_0^t \operatorname{sgn}(B_s)\, dB_s$, $t \geq 0$, is a Brownian motion. Define its running maximum process $\widetilde{M}_t := \max_{s \leq t} \widetilde{B}_s$, $t \geq 0$. Then,*

$$\mathbb{P}_0\big(\forall t \geq 0 : |B_t| = \widetilde{M}_t - \widetilde{B}_t, L_t = \widetilde{M}_t\big) = 1.$$

In particular, for a Brownian motion B with local time L at the origin and running maximum process $M_t := \max_{s \leq t} B_t$, $t \geq 0$, the processes $\big((M_t - B_t, M_t), t \geq 0\big)$ and $\big((|B_t|, L_t), t \geq 0\big)$ have the same law under $\mathbb{P}_0$.

By Lévy's characterization (15.3) of the local time, $(|B_t|, L_t)$ has the same distribution as $(M_t - B_t, M_t)$. As the joint distribution of (B_t, M_t) is well known (see, e.g., [IM74, Problem 1.7.1] or [KS91, Proposition 2.8.15]), we can deduce:

(15.4) Theorem. *The joint distribution of $\big(|B_t|, L_t\big)$, $t \geq 0$, is given by*

$$\mathbb{P}_0\big(|B_t| \in dx, L_t \in dy\big) = 2\frac{x+y}{\sqrt{2\pi t^3}}\, e^{-\frac{(x+y)^2}{2t}}\, dx\, dy, \quad dx, dy \geq 0.$$

Another representation of the local time is achieved by *Lévy's downcrossing theorem*, [KS91, Theorem 6.2.23]:

(15.5) Theorem. *Let B be a Brownian motion with local time L at zero. Then, a.s.,*

$$\forall t \geq 0: \quad L_t = \lim_{\varepsilon \downarrow 0} \varepsilon\, D_t(\varepsilon),$$

with $D_t(\varepsilon)$ being the number of downcrossings of the interval $[0, \varepsilon]$ by the reflecting Brownian motion $\left(|B_s|, s \leq t\right)$ (see, e.g., [KS91, p. 13]).

15.3. Extensions to Lévy's Characterization

An immediate consequence of Lévy's characterization (15.3) is that, because

$$|B_t| - L_t = \widetilde{M}_t - \widetilde{B}_t - \widetilde{M}_t = -\widetilde{B}_t, \quad t \geq 0, \quad \mathbb{P}_0\text{-a.s.},$$

the process $(|B_t| - L_t, t \geq 0)$ is a Brownian motion under $\mathbb{P}_0$. We will extend this result to initial laws other than $\mathbb{P}_0$.

We start by examining the pseudo-inverses of the local time $(L_t, t \geq 0)$:

$$\begin{aligned} L^{-1}(a) &= \inf\{t \geq 0 : L_t > a\}, \quad a \geq 0, \\ L_-^{-1}(a) &= \inf\{t \geq 0 : L_t \geq a\}, \quad a \geq 0. \end{aligned}$$

The following basic properties will be very helpful later:

(15.6) Lemma. *For every $a \in \mathbb{R}_+$, $L^{-1}(a)$ is an $(\mathscr{F}_{t+}, t \geq 0)$-stopping time and $L_-^{-1}(a)$ is an $(\mathscr{F}_t, t \geq 0)$-stopping time.*

Proof. By (ii) and (iv) of lemma (21.6), we have for all $t \geq 0$,

$$\begin{aligned} \{L^{-1}(a) < t\} &= \{a < L_t\} \in \mathscr{F}_t, \\ \{L_-^{-1}(a) \leq t\} &= \{a \leq L_t\} \in \mathscr{F}_t. \end{aligned}$$

□

(15.7) Lemma. *For $a \in \mathbb{R}_+$ and any random time $\tau \leq L^{-1}(a)$ with $L(\tau) = 0$ a.s.,*

$$\begin{aligned} L^{-1}(a) \circ \Theta_\tau &= L^{-1}(a) - \tau, \\ L_-^{-1}(a) \circ \Theta_\tau &= L_-^{-1}(a) - \tau \end{aligned}$$

hold a.s. true.

Proof. Let τ be as above. Then, a.s.,

$$\begin{aligned} L^{-1}(a) - \tau &= \inf\{u \geq 0 : L(u) > a\} - \tau \\ &= \inf\{u \geq \tau : L(u) > a\} - \tau \\ &= \inf\{u \geq 0 : L(u + \tau) - L(\tau) > a\} \\ &= L^{-1}(a) \circ \Theta_\tau, \end{aligned}$$

where we used $L_t \leq a$ for all $t \leq L^{-1}(a)$ for the second identity.

The computation for $L_-^{-1}(a)$ proceeds completely analogously. □

(15.8) Lemma. *For any $a > 0$, $L_-^{-1}(a) = L^{-1}(a)$ a.s. holds true.*

Proof. [MR06, Lemma 3.6.18] shows that $L_-^{-1}(a) = L^{-1}(a)$ holds $\mathbb{P}_0$-a.s. for any $a > 0$. For a general initial law $\mathbb{P}$, we compute by using lemma (15.7) (as $H_0 < L^{-1}(a)$ for any $a > 0$) and the strong Markov property of B:

$$\begin{aligned}\mathbb{P}\big(L_-^{-1}(a) = L^{-1}(a)\big) &= \mathbb{P}\big(L_-^{-1}(a) \circ \Theta_{H_0} = L^{-1}(a) \circ \Theta_{H_0}\big)\\ &= \mathbb{P}\big(\mathbb{P}_{H_0}\big(L_-^{-1}(a) = L^{-1}(a)\big)\big)\\ &= \mathbb{P}_0\big(L_-^{-1}(a) = L^{-1}(a)\big) = 1.\end{aligned}$$ □

The inverses of the local time have a close relation to the first hitting times H_a of points $a \geq 0$ (see, e.g., [Çın11, Theorem 5.9]), which appears natural in view of Lévy's characterization (15.3). We will only note the following formula for later use:

(15.9) Lemma. *For all $x, a \in \mathbb{R}_+$,*

$$\mathbb{E}_x\big(e^{-\alpha L^{-1}(a)}\big) = e^{-\sqrt{2\alpha}(x+a)}.$$

Proof. For $x = 0$, this is proved in [KPS10, Lemma B.1] or found in the collection of results of [KS91, Theorem 6.2.1]. For $x \neq 0$, by using $L^{-1}(a) = H_0 + L^{-1}(a) \circ \Theta_{H_0}$ of lemma (15.7), we get

$$\begin{aligned}\mathbb{E}_x\big(e^{-\alpha L^{-1}(a)}\big) &= \mathbb{E}_x\big(e^{-\alpha H_0}\,\mathbb{E}_x\big(e^{-\alpha L^{-1}(a)} \circ \Theta_{H_0} \big| \mathscr{F}_{H_0}\big)\big)\\ &= \mathbb{E}_x\big(e^{-\alpha H_0}\big)\,\mathbb{E}_0\big(e^{-\alpha L^{-1}(a)}\big),\end{aligned}$$

and insertion of the values for both expectations completes the proof. □

(15.10) Lemma. *For all $x, a \in \mathbb{R}_+$, $\alpha > 0$, $f \in b\mathscr{B}(\mathbb{R})$,*

$$\mathbb{E}_0\Big(\int_0^{L^{-1}(a)} e^{-\alpha t} f\big(|B_t| - L_t + a\big)\,dt\Big) = \mathbb{E}_a\Big(\int_0^{H_0} e^{-\alpha t} f\big(|B_t|\big)\,dt\Big).$$

Proof. It is $B_{L^{-1}(a)} = 0$, as L only grows when B is at the origin, and using the additive functional property and the continuity of L, we get

$$L_{t+L^{-1}(a)} = L_t \circ \Theta_{L^{-1}(a)} + L(L^{-1}(a)) \quad \text{with} \quad L(L^{-1}(a)) = a.$$

Thus, Dynkin's formula (3.16) applied for the stopping time $L^{-1}(a)$ yields

$$\begin{aligned}&\mathbb{E}_0\Big(\int_0^\infty e^{-\alpha t} f\big(|B_t| - L_t + a\big)\,dt\Big)\\ &= \mathbb{E}_0\Big(\int_0^{L^{-1}(a)} e^{-\alpha t} f\big(|B_t| - L_t + a\big)\,dt\Big)\\ &\quad + \mathbb{E}_0\Big(e^{-\alpha L^{-1}(a)}\,\mathbb{E}_0\Big(\int_0^\infty e^{-\alpha t} f\big(\big|B_{t+L^{-1}(a)}\big| - L_{t+L^{-1}(a)} + a\big)\,dt \,\Big|\, \mathscr{F}_{L^{-1}(a)}\Big)\Big)\\ &= \mathbb{E}_0\Big(\int_0^{L^{-1}(a)} e^{-\alpha t} f\big(|B_t| - L_t + a\big)\,dt\Big)\\ &\quad + \mathbb{E}_0\Big(e^{-\alpha L^{-1}(a)}\,\mathbb{E}_0\Big(\int_0^\infty e^{-\alpha t} f\big(|B_t| - L_t\big)\,dt\Big)\Big).\end{aligned}$$

With theorem (15.3), theorem (6.35) and lemma (15.9), we conclude that

$$
\begin{aligned}
&\mathbb{E}_0\Big(\int_0^{L^{-1}(a)} e^{-\alpha t}\, f\big(\,|B_t| - L_t + a\big)\, dt\Big) \\
&= \mathbb{E}_a\Big(\int_0^{\infty} e^{-\alpha t}\, f(B_t)\, dt\Big) - \mathbb{E}_a\Big(e^{-\alpha H_0}\, \mathbb{E}_0\Big(\int_0^{\infty} e^{-\alpha t}\, f(B_t)\, dt\Big)\Big),
\end{aligned}
$$

and an application of Dynkin's formula (3.16) for H_0 yields the result, as $B_{H_0} = 0$ by the continuity of B, and $B_t = |B_t|$ $\mathbb{P}_a$-a.s. for all $t \leq H_0$. $\square$

(15.11) Theorem. *For all $x, a \in \mathbb{R}_+$, $\alpha > 0$, $f \in b\mathscr{B}(\mathbb{R}^2)$,*

$$
\mathbb{E}_x\Big(\int_0^{\infty} e^{-\alpha t}\, f\big(\,|B_t| - L_t + a, (L_t - a)^+\big)\, dt\Big) = \mathbb{E}_{x+a}\Big(\int_0^{\infty} e^{-\alpha t}\, f\big(\,|B_t| - L_t, L_t\big)\, dt\Big).
$$

Proof. We decompose both sides of the claimed identity separately via Dynkin's formula (3.16) with respect to the stopping times $L^{-1}(a)$ and H_0, using the same techniques as in the proof of lemma (15.10), as well as $L(t+H_0) = L_t \circ \Theta_{H_0} + L(H_0)$ with $L(H_0) = 0$ by lemma (15.2), and $L^{-1}(a) - H_0 = L^{-1}(a) \circ \Theta_{H_0}$ by lemma (15.7). Then, the left-hand side of the above claim reads

$$
\begin{aligned}
&\mathbb{E}_x\Big(\int_0^{\infty} e^{-\alpha t}\, f\big(\,|B_t| - L_t + a, (L_t - a)^+\big)\, dt\Big) \\
&= \mathbb{E}_x\Big(\int_0^{L^{-1}(a)} e^{-\alpha t}\, f\big(\,|B_t| - L_t + a, 0\big)\, dt\Big) \\
&\quad + \mathbb{E}_x\big(e^{-\alpha L^{-1}(a)}\, \mathbb{E}_0\Big(\int_0^{\infty} e^{-\alpha t}\, f\big(\,|B_t| - L_t, L_t\big)\, dt\Big)\Big) \\
&= \mathbb{E}_x\Big(\int_0^{H_0} e^{-\alpha t}\, f\big(\,|B_t| + a, 0\big)\, dt\Big) \\
&\quad + \mathbb{E}_x\Big(e^{-\alpha H_0}\, \mathbb{E}_0\Big(\int_0^{L^{-1}(a)} e^{-\alpha t}\, f\big(\,|B_t| - L_t + a, 0\big)\, dt\Big)\Big) \\
&\quad + \mathbb{E}_x\big(e^{-\alpha L^{-1}(a)}\, \mathbb{E}_0\Big(\int_0^{\infty} e^{-\alpha t}\, f\big(\,|B_t| - L_t, L_t\big)\, dt\Big)\Big),
\end{aligned}
$$

while the right-hand side is transformed to

$$
\begin{aligned}
&\mathbb{E}_{x+a}\Big(\int_0^{\infty} e^{-\alpha t}\, f\big(\,|B_t| - L_t, L_t\big)\, dt\Big) \\
&= \mathbb{E}_{x+a}\Big(\int_0^{H_0} e^{-\alpha t}\, f\big(\,|B_t|\,, 0\big)\, dt\Big) \\
&\quad + \mathbb{E}_{x+a}\big(e^{-\alpha H_0}\, \mathbb{E}_0\Big(\int_0^{\infty} e^{-\alpha t}\, f\big(\,|B_t| - L_t, L_t\big)\, dt\Big)\Big).
\end{aligned}
$$

Another decomposition of the first integral at H_a, employing the terminal time property

$H_0 - H_a = H_0 \circ \Theta_{H_a}$ $\mathbb{P}_{x+a}$-a.s. by the continuity of B, gives

$$\begin{aligned}
&\mathbb{E}_{x+a}\Big(\int_0^\infty e^{-\alpha t}\, f\big(|B_t| - L_t, L_t\big)\, dt\Big) \\
&= \mathbb{E}_{x+a}\Big(\int_0^{H_a} e^{-\alpha t}\, f\big(|B_t|\,, 0\big)\, dt\Big) \\
&\quad + \mathbb{E}_{x+a}\Big(e^{-\alpha H_a}\, \mathbb{E}_a\Big(\int_0^{H_0} e^{-\alpha t}\, f\big(|B_t|\,, 0\big)\, dt\Big)\Big) \\
&\quad + \mathbb{E}_{x+a}\big(e^{-\alpha H_0}\, \mathbb{E}_0\Big(\int_0^\infty e^{-\alpha t}\, f\big(|B_t| - L_t, L_t\big)\, dt\Big)\Big).
\end{aligned}$$

A comparison of the particular summands with the help of lemmas (14.5), (15.9) and (15.10) yields the result. □

16. Brownian Motions on a Half Line

We are ready to examine Brownian motions on the easiest non-trivial "metric graph", namely on a metric graph with only one vertex and one (external) edge, which is equivalent to the setting of the half line $\mathbb{R}_+$. Brownian motions on $\mathbb{R}_+$ are well understood, the main reference is [IM63] (a short historical summary will follow in subsection 16.4). We are going to recall the basic definition and results in this case as well as an approach for the pathwise construction, in order to extend them later to the setting of a general metric graph.

16.1. Definition

We call a right continuous, strong Markov process on $\mathbb{R}_+$ a Brownian motion on the half line, if this process, stopped at the origin, is equivalent to the one-dimensional Brownian motion on $\mathbb{R}$, stopped at the origin:

(16.1) Definition. *Let $X = \big(\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}_+}\big)$ be a right continuous, strong Markov process on $\mathbb{R}_+$. X is a Brownian motion on $\mathbb{R}_+$, if*

$$H_X := \inf\{t \geq 0 : X_t = 0\}$$

is a stopping time over $(\mathscr{G}_t, t \geq 0)$, and for all $x \geq 0$, $n \in \mathbb{N}$, $f_1, \ldots, f_n \in b\mathscr{B}(\mathbb{R}_+)$, $t_1, \ldots, t_n \in \mathbb{R}_+$,

$$\mathbb{E}_x\big(f_1(X_{t_1\wedge H_X}) \cdots f_n(X_{t_n\wedge H_X})\big) = \mathbb{E}^B_x\big(f_1(B_{t_1\wedge H_B}) \cdots f_n(B_{t_n\wedge H_B})\big)$$

holds, with B being the Brownian motion on $\mathbb{R}$ and $H_B := \inf\{t \geq 0 : B_t = 0\}$.

This definition follows [Kni81, Definition 6.2] and the definition of [IM63, Section 5]. It is however a generalization of Knight's definition (which only allows continuous paths up to the process' lifetime) and a slight specialization of Itô–McKean's context: They do not require the process to be normal at the origin and consider the time of the "first approach to 0" given by

$$H_{X+} := \lim_{\varepsilon \downarrow 0} \inf\{t \geq 0 : X_t < \varepsilon\}$$

instead of the first hitting time H_X. As the analysis of the measures

$$\begin{aligned} p(A) &= \mathbb{P}_0(X_0 \in A), \\ p_+(A) &= \mathbb{P}_x(X_{H_{X+}} \in A), \quad x > 0, \end{aligned}$$

for $A \in \mathscr{B}(\mathbb{R}_+)$ in [IM63, Section 6] shows (the measure p_+ turns out to be independent of $x > 0$ due to the Markov property), Itô–McKean's definition allows that, if

$$p(\{0\}) = p_+(\{0\}) = 0, \quad \text{or} \quad p(\{0\}) = 1 > p_+(\{0\}),$$

the point of origin becomes a branching point or decomposes into a holding point "$0-$" and a branching point "$0+$" in the sense of Ray processes (see, e.g., [CW05, Section 8.2]). The "normal" case

$$p(\{0\}) = p_+(\{0\}) = 1$$

then occupies most of [IM63] and reduces to our definition (16.1) of a Brownian motion on $\mathbb{R}_+$. This definition will be generalized in section 20 to the general case of Brownian motions on a metric graph.

16.2. Some Prototypes

In the introduction of this thesis we already described the possible behaviors at the origin together with a list of easy prototypes of Brownian motions on $\mathbb{R}_+$. Two of them are going to be useful auxiliary processes later, so we take a closer look at them:

(16.2) Example. Mapping the Brownian motion B on $\mathbb{R}$ to $\mathbb{R}_+$ by the absolute-value norm $|\cdot|$ results in the *reflecting Brownian motion* $(|B_t|, t \geq 0)$ on $\mathbb{R}_+$, which is a Brownian motion on $\mathbb{R}_+$ in the sense of definition (16.1). This is done rigorously, for example, with the help of theorem (12.1) (see [Dyn65, Example 10.26] or [RW00a, Section I.14]), and we then get for $t \geq 0$, $x \in \mathbb{R}_+$, $A \in \mathscr{B}(\mathbb{R}_+)$:

$$\begin{aligned} \mathbb{P}_x(|B_t| \in A) &= \mathbb{P}_x(B_t \in A) + \mathbb{P}_x(B_t \in -A) \\ &= \mathbb{P}_x(B_t \in A) + \mathbb{P}_{-x}(B_t \in A). \end{aligned}$$

With this, the resolvent of $|B|$ can be derived from the resolvent of B. For $f \in b\mathscr{B}(\mathbb{R}_+)$, it reads at the origin

$$\begin{aligned} U_\alpha^{|B|} f(0) &= 2\, U_\alpha^B f^+(0) \\ &= 2 \int_0^\infty \frac{1}{\sqrt{2\alpha}}\, e^{-\sqrt{2\alpha} y}\, f(y)\, dy, \end{aligned}$$

where we used an auxiliary function $f^+ \colon \mathbb{R} \to \mathbb{R}$, defined by $f^+(y) = f(y)$ for $y \geq 0$ and $f^+(y) = 0$ otherwise, as well as the closed formula (14.2) for the resolvent U^B of B. ■

(16.3) Example. Instead of reflecting the "Brownian particle" at the origin, we can just let it "disappear" there, which results in the *killed Brownian motion* on $\mathbb{R}_+$. To this end, let $(|B_t|, t \geq 0)$ be the reflecting Brownian motion on $\mathbb{R}_+$ with its first hitting time H_0 of the origin, and consider the process resulting from killing $|B|$ at H_0:

$$B_t^{[0,\infty)} := \begin{cases} |B_t|, & t < H_0, \\ \Delta, & t \geq H_0. \end{cases}$$

$B^{[0,\infty)}$ is not a Brownian motion on $\mathbb{R}_+$ in the sense of our definition, because it is not normal at $0 \in \mathbb{R}_+$:

$$\mathbb{P}_0(B_t^{[0,\infty)} = \Delta) = 1.$$

However, it is certainly a right process on $\mathbb{R}_{>0} = (0, \infty)$ by theorem (10.1). Thus, it is not really in the scope of our work and will not be treated extensively (for more results on killed Brownian motions, the reader may consult, e.g., [CZ95, Chapter 2]). Nonetheless, it is going to be a supporting process in some of our computations, so we will examine this process a bit further: As $f(\Delta) = 0$ holds for all functions f, the resolvent of $B^{[0,\infty)}$ can be computed with the help of Dynkin's formula (3.17). The decomposition of the one-dimensional Brownian motion B at the stopping time H_0 gives for $x \geq 0$

$$U_\alpha^B f(x) = \mathbb{E}_x\Big(\int_0^{H_0} e^{-\alpha t} f(|B_t|)\, dt\Big) + \mathbb{E}_x(e^{-\alpha H_0})\, U_\alpha^B f(0),$$

which is equivalent to

$$\begin{aligned} U_\alpha^{[0,\infty)} f(x) &= \mathbb{E}_x\Big(\int_0^{H_0} e^{-\alpha t} f(|B_t|)\, dt\Big) \\ &= U_\alpha^B f(x) - \mathbb{E}_x(e^{-\alpha H_0})\, U_\alpha^B f(0), \end{aligned} \tag{16.4}$$

where we interpret the function f in $U_\alpha^B f$ as an arbitrary continuation of $f \in b\mathscr{B}([0,\infty))$ to $b\mathscr{B}(\mathbb{R})$. With lemma (14.3) and $\mathbb{E}_x(e^{-\alpha H_0}) = e^{-\sqrt{2\alpha x}}$ (see lemma (14.5)), we get

$$U^{[0,\infty)} b\mathcal{C}(\mathbb{R}_+) \subseteq b\mathcal{C}(\mathbb{R}_+) \quad \text{and} \quad U^{[0,\infty)} \mathcal{C}_0(\mathbb{R}_+) \subseteq \mathcal{C}_0^2(\mathbb{R}_+).$$

Differentiating (16.4) twice yields

$$\begin{aligned} U_\alpha^{[0,\infty)} f'(x) &= U_\alpha^B f'(x) + \sqrt{2\alpha}\, e^{-\sqrt{2\alpha}\, x}\, U_\alpha^B f(0), \\ U_\alpha^{[0,\infty)} f''(x) &= U_\alpha^B f''(x) - 2\alpha\, e^{-\sqrt{2\alpha}\, x}\, U_\alpha^B f(0) \\ &= 2\big(\alpha\, U_\alpha^{[0,\infty)} f(x) - f(x)\big), \end{aligned}$$

and using the derivatives of U^B, calculated in subsection 14.2, results in

$$\begin{aligned} U_\alpha^{[0,\infty)} f(0) &= 0, \\ U_\alpha^{[0,\infty)} f'(0) &= 2 \int_0^\infty e^{-\sqrt{2\alpha}\, y}\, f(y)\, dy, \\ U_\alpha^{[0,\infty)} f''(0) &= -2 f(0). \end{aligned}$$

Another review of (16.4) gives, by using the closed form (14.2) for U^B,

$$U_\alpha^{[0,\infty)} f(x) = \frac{1}{\sqrt{2\alpha}} \int_0^\infty \Big(e^{-\sqrt{2\alpha}\,|x-y|} - e^{-\sqrt{2\alpha}\,(x+y)} \Big)\, f(y)\, dy,$$

which is in accordance with André's reflection principle:

$$\mathbb{P}_x(B_t^{[0,\infty)} \in A) = \mathbb{P}_x(B_t \in A) - \mathbb{P}_x(B_t \in -A), \quad A \in \mathscr{B}(\mathbb{R}_+). \qquad \blacksquare$$

(16.5) Remark. For later use, we compute the resolvent of the killed Brownian motion $B^{[0,\infty)}$ for the functions $f(x) = e^{-\beta x}$, $\beta > 0$:

$$\begin{aligned}
&U_\alpha^{[0,\infty)} f(x)\\
&= \int_0^x e^{-\sqrt{2\alpha}\,(x-y)}\, e^{-\beta y}\, dy + \int_x^\infty e^{-\sqrt{2\alpha}\,(y-x)}\, e^{-\beta y}\, dy - \int_0^\infty e^{-\sqrt{2\alpha}\,(x+y)}\, e^{-\beta y}\, dy\\
&= e^{-\sqrt{2\alpha} x} \int_0^x e^{(\sqrt{2\alpha}-\beta)\,y}\, dy + e^{\sqrt{2\alpha} x} \int_x^\infty e^{-(\sqrt{2\alpha}+\beta)\,y}\, dy - e^{-\sqrt{2\alpha} x} \int_0^\infty e^{-(\sqrt{2\alpha}+\beta)\,y}\, dy\\
&= \frac{e^{-\sqrt{2\alpha} x}}{\sqrt{2\alpha}-\beta}\, (e^{(\sqrt{2\alpha}-\beta)\,x} - 1) + \frac{e^{\sqrt{2\alpha} x}}{\sqrt{2\alpha}+\beta}\, e^{-(\sqrt{2\alpha}+\beta)\,x} - \frac{e^{-\sqrt{2\alpha} x}}{\sqrt{2\alpha}+\beta}\\
&= \frac{2\sqrt{2\alpha}}{2\alpha-\beta^2}\, (e^{-\beta x} - e^{-\sqrt{2\alpha} x}). \qquad \blacksquare
\end{aligned}$$

These examples depict the easiest boundary behaviors at the origin. Closed forms for the resolvents and semigroups are known for any possible local boundary condition in the half-line case, see [KPS10, Section 4] and [Tai14, Section 9.1].

16.3. Feller's Theorem

[Kni81, Theorem 6.1] or, by a different approach via the resolvent of the process, [IM63, Section 7] yields the main result on the characterization of Brownian motions on the half line:

(16.6) Theorem. *Let X be a Brownian motion on $\mathbb{R}_+$. Then X is a Feller process and is uniquely determined by its generator $A = \frac{1}{2}\Delta$, with $\mathscr{D}(A) \subseteq \mathcal{C}_0^2(\mathbb{R}_+)$.*

In this case, the generator is completely analyzed in [Kni81, Lemma 6.2] or [IM63, Section 8]. Its domain—and thus the underlying Markov process—can be uniquely characterized with the help of the following theorem, which we will call *Feller's theorem*:

(16.7) Theorem. *Let X be a Brownian motion on $\mathbb{R}_+$. Then there exist constants $c_1 \geq 0$, $c_2 \geq 0$, $c_3 \geq 0$ and a measure c_4 on $(0,\infty)$, satisfying*

$$c_1 + c_2 + c_3 + \int_{(0,\infty)} (1 \wedge x)\, c_4(dx) = 1$$

and

$$c_4\big((0,\infty)\big) = +\infty, \quad \text{if} \quad c_2 = c_3 = 0,$$

such that the domain of the generator $A = \frac{1}{2}\Delta$ *of* X *reads*

$$\mathscr{D}(A) = \Big\{ f \in \mathcal{C}_0^2(\mathbb{R}_+) : \\ c_1\, f(0) - c_2\, f'(0+) + c_3\, Af(0) - \int_{(0,\infty)} \big(f(x) - f(0)\big)\, c_4(dx) = 0 \Big\}.$$

The constants and the measure only depend on the exit behavior of the process from any arbitrarily small neighborhood of the origin. They can be explicitly specified, as seen in the proof of [Kni81, Lemma 6.2] or later in theorem (20.16) for a general metric graph. As the reader will notice then (see theorem (20.16) and the results following it), we will consider the equivalent normalization $1 - e^{-x}$ rather than $1 \wedge x$ for the measure c_4, which turns out to be more appropriate in our context and will simplify some computations.

16.4. Itô–McKean's Construction

It seems surprising to the present author that the characterizing "data" (c_1, c_2, c_3, c_4), as given in theorem (16.7), of any Brownian motion on $\mathbb{R}_+$ has an easy probabilistic interpretation, which was already briefly explained in the introduction. This allowed Itô and McKean in [IM63] to obtain a complete pathwise construction of a Brownian motion on $\mathbb{R}_+$ for any given set (c_1, c_2, c_3, c_4) of boundary conditions. Before explaining their solution, we feel more than obligated to remind the reader of their words in [IM63, Section 2] on the evolution of the whole theory and on some of the persons who were directly involved:

"M. Kac [Kac51] cited the problem of describing the sample paths of the elastic Brownian motion ($c_3 = c_4 = 0 < c_1\, c_2$), and it was W. Feller's (private) suggestion that these should be the reflecting Brownian sample paths, killed at the instant some increasing function $t \mapsto \mathfrak{t}^+(\mathfrak{B}^+ \cap [0,t])$ of the visiting set $\mathfrak{B}^+ = \{t \geq 0 : |B_t| = 0\}$ hits a certain level, and that was the starting point of this paper [IM63]. P. Lévy's profound studies [Lév48] had clarified the fine structure of the standard and reflecting Brownian motions and their local times, the papers of E. B. Dynkin [Dyn56] and G. Hunt [Hun56] on Markov times provided an indispensable tool, H. Trotter [Tro58] proved a deep result about local times, and W. Feller [Fel54] hat presented a (partial) description of the sample paths of the Brownian motion associated with A in the special case $c_4((0,\infty)) < +\infty$ (the case $c_4((0,\infty)) = +\infty$ was not discovered in Feller's original proof [of theorem (16.7)], but this error was corrected by W. Feller [Fel57a] and A. D. Wentzell [Wen56]). It was left to use these ideas (and some new ones) to build up the sample paths of Feller's Brownian motions from the reflecting Brownian motion and its local time and (independent) exponential holding times and differential processes [...]."

If c_4 is finite, then it is easy to see (cf. [IM63, Sections 9], and also remark (21.2)) that the adequate jumps for achieving the measure c_4 can be implemented just like the killing parameter c_1, namely by introducing jumps whenever the original process' local time at the origin exceeds some independent, exponentially distributed random time. The only difference is that the process is not necessarily transfered to the absorbing cemetery point Δ, as done for c_1, but is restarted at some point chosen with respect to the distribution c_4. This also appears natural when displaying the domain of the generator in a slightly different form, using the convention $f(\Delta) = 0$ of subsection 4.3:

$$\begin{aligned}\mathscr{D}(A) &= \Big\{ f \in \mathcal{C}_0^2(\mathbb{R}_+) : \\ &\qquad c_1\, f(0) - c_2\, f'(0+) + c_3\, Af(0) - \int_{(0,\infty)} \big(f(x) - f(0)\big)\, c_4(dx) = 0 \Big\} \\ &= \Big\{ f \in \mathcal{C}_0^2(\mathbb{R}_+) : \\ &\qquad - c_2\, f'(0+) + c_3\, Af(0) - \int_{(0,\infty)} \big(f(x) - f(0)\big)\, \big(c_1 \varepsilon_\Delta + c_4\big)(dx) = 0 \Big\}.\end{aligned}$$

The essential "new idea" of [IM63] is the solution on the implementation of jumps for an infinite measure c_4. In this case, the Brownian motion, when started at or hitting the origin, needs to perform infinitely many small jumps in some arbitrarily small time interval. Thus, just like when considering excursions of the one-dimensional Brownian motion from any point, it is not possible to enumerate them in temporal order to construct the complete process via successive independent copies of killed Brownian motions. However, Itô and McKean managed to transform the paths of one underlying reflecting Brownian motion in a suitable way in order to implement the correct excursions from the origin. As the present author considers their solution to be remarkably ingenious, he will happily share their story told by McKean in [GMM15, Section 4.7]:

"But what if $c_4((0,\infty)) = +\infty$? That was mysterious. Luckily, Itô saw at once that it must describe "jumps" of a new kind, produced by the increasing "differential" process [...], and as we were flying one day, to Fukuoka I think, Itô kept drawing pictures, one after another, trying to see how these jumps could be interlaced with the Brownian path. After a while he got it; after a longer while I got it, too, and the rest was plain sailing [...]."

Without being able to verify whether the following chain of logic really led them to their solution, we try to motivate their approach: As jumps are only possible if the process is at the origin, they appear on the timeline of the local time at the origin. Furthermore, there is at most one jump at a time, and jumps need to be independent, in the sense that they need to exhibit a Markovian character, as any Brownian motion on $\mathbb{R}_+$ is strongly Markovian. By characterization (6.10), it is therefore natural to expect that the jumps are guided by a Poisson point process (or equivalently, by a subordinator) with Lévy measure c_4, on the timeline of the local time. Thus, starting with a reflecting Brownian motion $|B|$, we try to superpose $|B|$ with a subordinator P: The naïve approach of considering the process $t \mapsto |B_t| + P(L_t)$ fails, as shown in figure 16.1, because, after every jump, the process must behave like a standard Brownian motion—in contrast to a reflecting one—until the next hit of the origin.

Therefore, the goal is to find a way to toggle between reflecting Brownian motion and standard Brownian motion on the level of paths. As seen in Lévy's characterization of the local time (15.3), for a reflecting Brownian motion $|B|$ with local time L at the origin, the process $|B| - L$ behaves like a standard Brownian motion. So the main idea is to toggle the paths $|B|$ and $|B| - L$, more accurately: Start with $|B|$ until the first

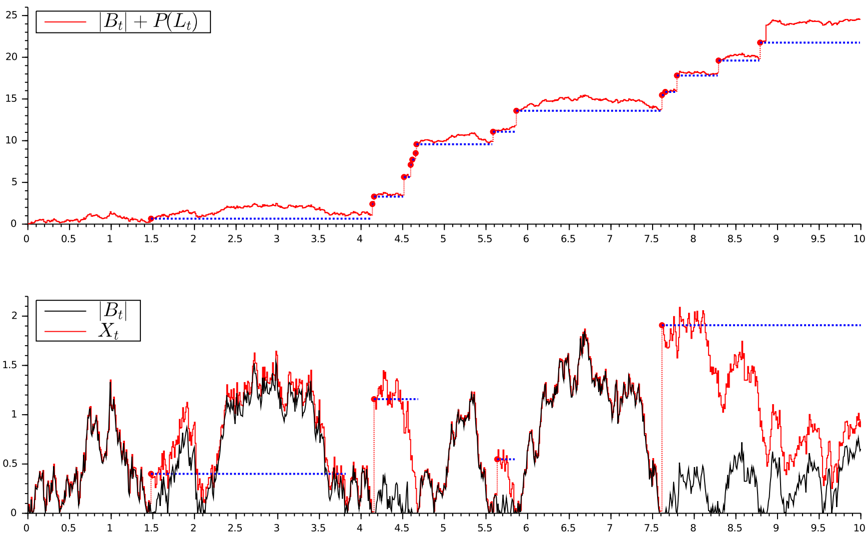


Figure 16.1: Construction approach for Brownian motions on $\mathbb{R}_+$: The idea of $t \mapsto |B_t| + P(L_t)$, illustrated in the above graph, fails, as the process must switch to a standard Brownian motion after every jump until the next hit of zero, as shown in the graph of X below. The blue lines mark the starting heights of every "jump excursion", as well as (in the graph for X) the time elapsed until the switchover back to the reflecting Brownian motion.

jump is introduced by P, say of height $h > 0$, then switch to the "jump excursion" $h+|B|-L$ until this part hits the origin again, then toggle back to $|B|$, and so on. As $|B|$ is non-negative and L only grows when $|B|$ is at the origin, the partial process $h+|B|-L$ hits zero exactly when L is increased by h. Following this thought, the prototype of the process should be of the form $t \mapsto |B_t| - L_t + F(L_t)$ for some random function F which is the identity while the reflecting Brownian motion needs to be in place, jumps by h whenever a jump excursion with jump height $h > 0$ needs to be started, and then is constant for h units of time. Such a function is gained by the choice $F = P \circ P^{-1}$ with P^{-1} being the right continuous pseudo-inverse of the subordinator P:

$$P^{-1}\colon [0,\infty) \to [0,\infty], \quad P^{-1}(t) := \inf\{s \geq 0 : P(s) > t\}.$$

Pseudo-inverses and functions of the form $P\circ P^{-1}$ are examined in detail in subsection 21.2. For now, we recommend the graphs of figure 16.2 to the reader: The upper right hand graph contains the jumps of the Poisson point process (in black) and its associated subordinator P with an additional deterministic drift (in red), the lower left hand graph shows the resulting process $P \circ P^{-1}$ which exactly features the properties stated above, that is, being a diagonal, interrupted by upper isosceles triangles. In summary, Itô–McKean's solution is the process

$$X_t = |B_t| - L_t + P\big(P^{-1}(L_t)\big), \quad t \geq 0,$$

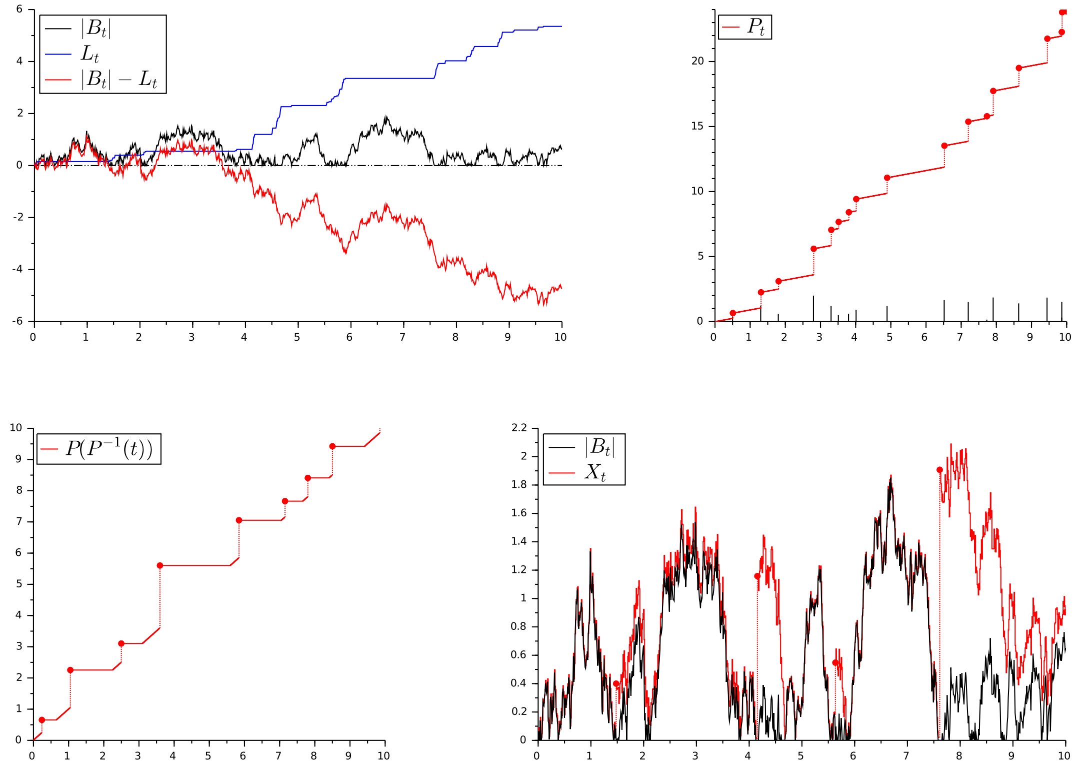


Figure 16.2: Itô–McKean's construction of Brownian motions on $\mathbb{R}_+$

which is shown in the lower right hand graph of figure 16.2. Proving that this process is a (strong) Markov process and indeed introduces the correct jump measure c_4 is not an easy task and is done in [IM63, Sections 13–15]. We will take up this challenge in section 21 when we extend Itô–McKean's construction to the star graph. Afterwards, the missing killing and stickiness parameters c_1 and c_3 can be introduced by the standard procedure of "slowing down" the process X by time changing it with respect to its local time at the origin, and then kill it once its new local time exceeds some independent, exponentially distributed random time, see [IM63, Sections 10, 15] or subsection 21.10.

We end the treatment of the half-line case by noting that, of course, there are other ways to analyze and construct Brownian motions on $\mathbb{R}_+$. A natural approach is via Itô excursion theory, see, e.g., [Rog89], [RW00b, Section VI.57], or [Blu92].

17. Brownian Motions on an Interval

We will briefly consider the case of a "metric graph" with one edge and two endpoints. Thus, we are examining Brownian motions on an interval $[a, b]$, that is, right continuous strong Markov processes on $[a, b]$, which, if stopped at the endpoints, are equivalent to the one-dimensional Brownian motion on $\mathbb{R}$, stopped when leaving this interval.

Expectedly, any Brownian motion on $[a,b]$ behaves locally at the boundary points a and b like Brownian motions on the half lines $[a,+\infty)$, $(-\infty,b]$ respectively. Therefore, its characterization and construction can be deduced from the half-line case by obtaining the boundary conditions and the corresponding boundary behavior at both endpoints.

It turns out that most of the results and constructions in this case are not particularly simpler than in the context of general metric graphs, so we will postpone most findings to later sections. In contrast to the case of general graphs, however, we are still able to attain the complete description of the generator in the interval case. The laborious proof, necessary due to the non-local boundary conditions, is given in subsection 17.2.

17.1. Definition and Basic Results

We are extending definition (16.1) in the obvious way, which again follows the definitions of [Kni81, Section 6.3] and [IM63, Section 16]:

(17.1) Definition. *Let* $X = \big(\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in[a,b]}\big)$ *be a right continuous, strong Markov process on* $[a,b]$. X *is a Brownian motion on* $[a,b]$, *if*

$$H_X := \inf\big\{t \geq 0 : X_t \in \{a,b\}\big\}$$

is a stopping time over $(\mathscr{G}_t, t\geq 0)$, *and for all* $x\in[a,b]$, $n\in\mathbb{N}$, $f_1,\dots,f_n \in b\mathscr{B}(\mathbb{R}_+)$, $t_1,\dots,t_n\in\mathbb{R}_+$,

$$\mathbb{E}_x\big(f_1(X_{t_1\wedge H_X})\cdots f_n(X_{t_n\wedge H_X})\big) = \mathbb{E}_x^B\big(f_1(B_{t_1\wedge H_B})\cdots f_n(B_{t_n\wedge H_B})\big)$$

holds, with B *being the Brownian motion on* $\mathbb{R}$ *and* $H_B := \inf\big\{t\geq 0 : B_t\in\{a,b\}\big\}$.

Just like in the half-line case, one can characterize any Brownian motion by the boundary conditions of the generator via *Feller's theorem*. The assertions of the following theorem can be found in [IM63, Section 16] without proof. We will prove them in the context of a general metric graph in section 20, extended by the computations of subsection 17.2.

(17.2) Theorem. *Let* X *be a Brownian motion on* $[a,b]$. *Then* X *is a Feller process with generator* $A = \frac{1}{2}\Delta$. *There exist constants* $c_1^a\geq 0$, $c_2^a\geq 0$, $c_3^a\geq 0$ *and a measure* c_4^a *on* $(a,b]$ *as well as* $c_1^b\geq 0$, $c_2^b\geq 0$, $c_3^b\geq 0$ *and a measure* c_4^b *on* $[a,b)$, *satisfying*

$$c_1^a + c_2^a + c_3^a + \int_{(a,b]} (1\wedge x)\, c_4^a(dx) = 1,$$
$$c_1^b + c_2^b + c_3^b + \int_{[a,b)} (1\wedge x)\, c_4^b(dx) = 1,$$

and

$$c_4^a\big((a,b]\big) = +\infty, \quad \text{if} \quad c_2^a = c_3^a = 0,$$
$$c_4^b\big([a,b)\big) = +\infty, \quad \text{if} \quad c_2^b = c_3^b = 0,$$

such that the domain of the generator of X reads

$$\begin{aligned}\mathscr{D}(A) = \Big\{ & f \in \mathcal{C}_0^2(\mathbb{R}_+) : \\ & c_1^a\, f(a) - c_2^a\, f'(a+) + \frac{c_3^a}{2}\, f''(a+) - \int_{(a,b]} \big(f(x) - f(a)\big)\, c_4^a(dx) = 0, \\ & c_1^b\, f(b) + c_2^b\, f'(b-) + \frac{c_3^b}{2}\, f''(b-) - \int_{[a,b)} \big(f(x) - f(b)\big)\, c_4^b(dx) = 0 \ \Big\}.\end{aligned}$$

(17.3) Example. Like in example (16.3) for the half line, we consider the Brownian motion on $[a, b]$ killed when it reaches the boundary, that is, the process $B^{[a,b]}$ defined by

$$B_t^{[a,b]} := \begin{cases} B_t, & t < H_a \wedge H_b, \\ \Delta, & t \geq H_a \wedge H_b. \end{cases}$$

We compute its resolvent by using the decomposition of the standard Brownian motion at $H_a \wedge H_b$ with the help of Dynkin's formula (3.16). For all $f \in b\mathscr{B}([a, b])$, $x \in [a, b]$, this gives (see also lemma (14.5) for the passage time formulas)

$$\begin{aligned} U_\alpha^{[a,b]} f(x) &= \mathbb{E}_x\Big(\int_0^{H_a \wedge H_b} e^{-\alpha t}\, f(B_t)\, dt \Big) \\ &= U_\alpha^B f(x) - \mathbb{E}_x\big(e^{-\alpha H_a}\,;\, H_a < H_b\big)\, U_\alpha^B f(a) - \mathbb{E}_x\big(e^{-\alpha H_b}\,;\, H_b < H_a\big)\, U_\alpha^B f(b) \\ &= U_\alpha^B f(x) - \frac{\sinh\big(\sqrt{2\alpha}\,(b-x)\big)}{\sinh\big(\sqrt{2\alpha}\,(b-a)\big)}\, U_\alpha^B f(a) - \frac{\sinh\big(\sqrt{2\alpha}\,(x-a)\big)}{\sinh\big(\sqrt{2\alpha}\,(b-a)\big)}\, U_\alpha^B f(b), \end{aligned}$$

with the boundary values

$$U_\alpha^{[a,b]} f(a) = 0, \quad U_\alpha^{[a,b]} f(b) = 0.$$

As U^B maps $\mathcal{C}_0(\mathbb{R})$ to $\mathcal{C}_0^2(\mathbb{R})$ (see lemma (14.3)), $U^{[a,b]}$ maps $\mathcal{C}([a, b])$ to $\mathcal{C}^2([a, b])$. Differentiation of the above formula then yields, for all $x \in [a, b]$,

$$\begin{aligned} & U_\alpha^{[a,b]} f'(x) \\ &= U_\alpha^B f'(x) + \sqrt{2\alpha}\, \frac{\cosh\big(\sqrt{2\alpha}\,(b-x)\big)}{\sinh\big(\sqrt{2\alpha}\,(b-a)\big)}\, U_\alpha^B f(a) - \sqrt{2\alpha}\, \frac{\cosh\big(\sqrt{2\alpha}\,(x-a)\big)}{\sinh\big(\sqrt{2\alpha}\,(b-a)\big)}\, U_\alpha^B f(b), \\ & U_\alpha^{[a,b]} f''(x) \\ &= U_\alpha^B f''(x) - 2\alpha\, \frac{\sinh\big(\sqrt{2\alpha}\,(b-x)\big)}{\sinh\big(\sqrt{2\alpha}\,(b-a)\big)}\, U_\alpha^B f(a) - 2\alpha\, \frac{\sinh\big(\sqrt{2\alpha}\,(x-a)\big)}{\sinh\big(\sqrt{2\alpha}\,(b-a)\big)}\, U_\alpha^B f(b) \\ &= 2\big(\alpha\, U_\alpha^{[a,b]} f(x) - f(x)\big). \end{aligned}$$

■

17.2. Proof of Feller's Theorem

In Feller's theorem in the context of a general metric graph (cf. theorems (20.16) and (20.21)) we are only able to show that the boundary conditions are necessary for functions to lie inside the domain $\mathscr{D}(A)$ of the generator A. We are going to prove now that they are also sufficient in the interval case, thus showing that identity holds between $\mathscr{D}(A)$ and the right-hand set of boundary conditions in theorem (17.2). The following proof is an attempt to transfer the approach of [Kni81, Proof of Theorem 6.6], which only considers the continuous case (that is $c_4^a = c_4^b = 0$), to our setting. It will show that the permission of discontinuity of the underlying process, which introduces non-local boundary conditions through the jump measures c_4^a and c_4^b, exceedingly complicates results. It seems unlikely to us that this approach is still feasible in the case of a general metric graph.

Proof. Without loss of generality, we only consider the case $[a,b] = [-1,1]$. Furthermore, we rename $c_i^{-1} = p_{-i}$, $c_i^{+1} = p_{+i}$, $i \in \{1,\dots,4\}$, for this proof.

We are using lemma (5.12), that is, we need to show that, with $\mathscr{D}$ being the right-hand set of boundary conditions in theorem (17.2), there is an $\alpha > 0$ such that the differential equation

$$\frac{\Delta}{2} f = \alpha f,\ \ f \in \mathscr{D},$$

is only solved by $f = 0$. We will demonstrate this by using the approach of [Kni81, Theorem 6.6]. However, due to the possible jumps, our proof will be much more involved.

For each $\alpha > 0$, all solutions of $\frac{\Delta}{2} f_\alpha = \alpha f_\alpha$, $f_\alpha \in \mathcal{C}_0^2([-1,1])$, are given by

$$f_\alpha(x) = \tilde{c}_1^\alpha\, e^{-\sqrt{2\alpha}x} + \tilde{c}_2^\alpha\, e^{\sqrt{2\alpha}x}, \quad \tilde{c}_1^\alpha, \tilde{c}_2^\alpha \in \mathbb{R},$$

or, what will be more convenient in our context, by

$$f_\alpha(x) = c_1^\alpha \sinh(\sqrt{2\alpha}x) + c_2^\alpha \cosh(\sqrt{2\alpha}x), \quad c_1^\alpha, c_2^\alpha \in \mathbb{R}.$$

For all solutions with $c_2^\alpha \neq 0$, the boundary conditions of $f_\alpha \in \mathscr{D}$ give

$$\begin{aligned}
-1 = \frac{c_1^\alpha}{c_2^\alpha}\, & \frac{p_{-1} \sinh(-\sqrt{2\alpha}) - p_{-2}\sqrt{2\alpha}\cosh(-\sqrt{2\alpha}) + p_{-3}\,\alpha \sinh(-\sqrt{2\alpha}) + \cdots}{p_{-1} \cosh(-\sqrt{2\alpha}) - p_{-2}\sqrt{2\alpha}\sinh(-\sqrt{2\alpha}) + p_{-3}\,\alpha \cosh(-\sqrt{2\alpha}) + \cdots} \\
& \frac{\cdots - \int_{(-1,1)} \big(\sinh(\sqrt{2\alpha}x) - \sinh(-\sqrt{2\alpha})\big)\, p_{-4}(dx) + \cdots}{\cdots - \int_{(-1,1)} \big(\cosh(\sqrt{2\alpha}x) - \cosh(-\sqrt{2\alpha})\big)\, p_{-4}(dx) + \cdots} \\
& \frac{\cdots - p_{-4}(\{+1\})\big(\sinh(\sqrt{2\alpha}) - \sinh(-\sqrt{2\alpha})\big)}{\cdots - p_{-4}(\{+1\})\big(\cosh(\sqrt{2\alpha}) - \cosh(-\sqrt{2\alpha})\big)},
\end{aligned} \tag{17.4}$$

and

$$\begin{aligned}
-1 = \frac{c_1^\alpha}{c_2^\alpha}\, & \frac{p_{+1} \sinh(\sqrt{2\alpha}) + p_{+2}\sqrt{2\alpha}\cosh(\sqrt{2\alpha}) + p_{+3}\,\alpha \sinh(\sqrt{2\alpha}) + \cdots}{p_{+1} \cosh(\sqrt{2\alpha}) + p_{+2}\sqrt{2\alpha}\sinh(\sqrt{2\alpha}) + p_{+3}\,\alpha \cosh(\sqrt{2\alpha}) + \cdots} \\
& \frac{\cdots - \int_{(-1,1)} \big(\sinh(\sqrt{2\alpha}x) - \sinh(\sqrt{2\alpha})\big)\, p_{+4}(dx) + \cdots}{\cdots - \int_{(-1,1)} \big(\cosh(\sqrt{2\alpha}x) - \cosh(\sqrt{2\alpha})\big)\, p_{+4}(dx) + \cdots} \\
& \frac{\cdots - p_{+4}(\{-1\})\big(\sinh(-\sqrt{2\alpha}) - \sinh(\sqrt{2\alpha})\big)}{\cdots - p_{+4}(\{-1\})\big(\cosh(-\sqrt{2\alpha}) - \cosh(\sqrt{2\alpha})\big)},
\end{aligned} \tag{17.5}$$

where the "$\cdots$" indicate that both nominator and denominator of the fractions are continued in the following line.

We are going to show that this cannot be true for any sequence $(\alpha_n, n \in \mathbb{N})$ of positive numbers tending to infinity, because for $\alpha \to +\infty$, $\sinh(-\sqrt{2\alpha}) \to -\infty$ and $\cosh(-\sqrt{2\alpha}) \to +\infty$, while $\sinh(\sqrt{2\alpha}) \to +\infty$ and $\cosh(\sqrt{2\alpha}) \to +\infty$. Thus, the latter fraction of (17.4) converges to -1, which implies that c_1^α / c_2^α must converge to -1, while the latter fraction of (17.5) converges to $+1$, which can only be true if c_1^α / c_2^α converges to $+1$. To avoid technical problems when carrying out this argument rigorously, we will rather examine finite limits by switching from sinh and cosh to tanh and employing that $\tanh(-\sqrt{2\alpha}) \to -1$ and $\tanh(\sqrt{2\alpha}) \to +1$ for $\alpha \to +\infty$.

Some preparations are necessary so that we do not need to interrupt our argument later whenever taking limits on the integral terms: For all $x \in [-1, 1]$, we have

$$\begin{aligned}
\left|\sinh(\sqrt{2\alpha}x) - \sinh(\sqrt{2\alpha})\right| &= \left| - \int_{\sqrt{2\alpha}x}^{\sqrt{2\alpha}} \cosh(x)\, dx \right| \leq \sqrt{2\alpha} \cosh(\sqrt{2\alpha})\,(1-x), \\
\left|\cosh(\sqrt{2\alpha}x) - \cosh(\sqrt{2\alpha})\right| &= \left| - \int_{\sqrt{2\alpha}x}^{\sqrt{2\alpha}} \sinh(x)\, dx \right| \leq \sqrt{2\alpha} \sinh(\sqrt{2\alpha})\,(1-x),
\end{aligned}$$

and, analogously,

$$\begin{aligned}
\left|\sinh(\sqrt{2\alpha}x) - \sinh(-\sqrt{2\alpha})\right| &\leq \sqrt{2\alpha} \cosh(-\sqrt{2\alpha})\,(1+x) = \sqrt{2\alpha}\cosh(\sqrt{2\alpha})\,(1+x), \\
\left|\cosh(\sqrt{2\alpha}x) - \cosh(-\sqrt{2\alpha})\right| &\leq \sqrt{2\alpha} \left|\sinh(-\sqrt{2\alpha})\right|\,(1-x) = \sqrt{2\alpha}\sinh(\sqrt{2\alpha})\,(1-x).
\end{aligned}$$

Therefore, the integrals in (17.4) and (17.5) with respect to $p_{\pm 4}$ are always finite. Additionally, they diverge slower than $\sqrt{2\alpha}\sinh(\pm\sqrt{2\alpha})$, because

$$\begin{aligned}
&\left| \frac{1}{\sqrt{2\alpha}\sinh(\pm\sqrt{2\alpha})} \int_{(-1,1)} \big(\sinh(\sqrt{2\alpha}x) - \sinh(\pm\sqrt{2\alpha})\big)\, p_{\pm 4}(dx) \right| \\
&\leq \coth(\sqrt{2\alpha}) \int_{(-1,1)} (1 \mp x)\, p_{\pm 4}(dx),
\end{aligned}$$

and as $\coth(\sqrt{2\alpha}) \to +1$ for $\alpha \to +\infty$, we get by LDCT

$$\begin{aligned}
&\lim_{\alpha\to+\infty} \frac{1}{\sqrt{2\alpha}\sinh(\pm\sqrt{2\alpha})} \int_{(-1,1)} \big(\sinh(\sqrt{2\alpha}x) - \sinh(\pm\sqrt{2\alpha})\big)\, p_{\pm 4}(dx) \\
&= \int_{(-1,1)} \lim_{\alpha\to+\infty} \frac{1}{\sqrt{2\alpha}} \Big(\frac{\sinh(\sqrt{2\alpha}x)}{\sinh(\pm\sqrt{2\alpha})} - \frac{\sinh(\pm\sqrt{2\alpha})}{\sinh(\pm\sqrt{2\alpha})} \Big)\, p_{\pm 4}(dx) \\
&= 0.
\end{aligned}$$

By exactly the same argument, this is also true for the integrals with respect to cosh:

$$\begin{aligned}
&\lim_{\alpha\to+\infty} \frac{1}{\sqrt{2\alpha}\sinh(\pm\sqrt{2\alpha})} \int_{(-1,1)} \big(\cosh(\sqrt{2\alpha}x) - \cosh(\pm\sqrt{2\alpha})\big)\, p_{\pm 4}(dx) \\
&= \int_{(-1,1)} \lim_{\alpha\to+\infty} \frac{1}{\sqrt{2\alpha}} \Big(\frac{\cosh(\sqrt{2\alpha}x)}{\sinh(\pm\sqrt{2\alpha})} - \coth(\pm\sqrt{2\alpha}) \Big)\, p_{\pm 4}(dx) \\
&= 0.
\end{aligned}$$

We are ready to examine (17.4) and (17.5) as announced above. The objective is to show that (17.4) implies

$$\lim_{\alpha \to +\infty} \frac{c_1^\alpha}{c_2^\alpha} = -1,$$

while (17.5) implies

$$\lim_{\alpha \to +\infty} \frac{c_1^\alpha}{c_2^\alpha} = 1.$$

To this end, we will always compare the terms of highest rate of divergence in the nominator to the ones in the denominator of the terms in (17.4) and (17.5), so the examination depends on the boundary weights. Mostly we will treat (17.4) and (17.5) at the same time, but always separately. This means, for instance, that the following case (i) "$p_{\pm 3} \neq 0$" is employed for (17.4) if $p_{-3} \neq 0$, and for (17.5) if $p_{+3} \neq 0$, choosing another of below cases for the other equation if necessary. It does not mean that both $p_{-3} \neq 0$ and $p_{+3} \neq 0$ must be true at the same time.

(i) $p_{\pm 3} \neq 0$:
By dividing both nominator and denominator of the latter fraction of (17.4) or (17.5) by $\alpha \sinh(\pm\sqrt{2\alpha})$, and letting $\alpha \to +\infty$, we immediately see that the latter fraction converges to -1 for (17.4), or to $+1$ for (17.5).

(ii) $p_{\pm 3} = 0$, $p_{\pm 2} \neq 0$:
We use exactly the same approach as in case (i), but divide by $\sqrt{2\alpha}\sinh(\pm\sqrt{2\alpha})$, to show that the latter fraction of (17.4), (17.5) converges to -1, $+1$ respectively.

(iii) $p_{+3} = 0$, $p_{+2} = 0$, $p_{+4}([-1,1]) = +\infty$:
We need to further examine the rate of divergence of the integral terms, therefore we introduce for $\alpha > 0$:

$$\begin{aligned} a^\alpha &:= \int_{[-1,1)} \big(\sinh(\sqrt{2\alpha}x) - \sinh(\sqrt{2\alpha})\big)\, p_{+4}(dx), \\ b^\alpha &:= \int_{[-1,1)} \big(\cosh(\sqrt{2\alpha}x) - \cosh(\sqrt{2\alpha})\big)\, p_{+4}(dx) \\ &= \int_{(-1,1)} \big(\cosh(\sqrt{2\alpha}x) - \cosh(\sqrt{2\alpha})\big)\, p_{+4}(dx). \end{aligned}$$

Then, we have

$$\begin{aligned} \frac{b^\alpha}{e^{\sqrt{2\alpha}}} &= -\int_{(-1,1)} \Big(\frac{e^{\sqrt{2\alpha}} - e^{-\sqrt{2\alpha}}}{e^{\sqrt{2\alpha}}} - \frac{e^{\sqrt{2\alpha}x} - e^{-\sqrt{2\alpha}x}}{e^{\sqrt{2\alpha}}}\Big)\, p_{+4}(dx) \\ &\leq -\int_{(-1,1)} \Big(1 - e^{\sqrt{2\alpha}(x-1)} - e^{-\sqrt{2\alpha}(x+1)}\Big)\, p_{+4}(dx), \end{aligned}$$

and because both exponentials functions decrease pointwise to 0 with $\alpha \to +\infty$ and the complete integrand is non-negative for every $\alpha > 0$, as the integrand function of

b^α is non-negative, LMCT yields that $\int_{(-1,1)} \left(1 - e^{\sqrt{2\alpha}(x-1)} - e^{-\sqrt{2\alpha}(x+1)}\right) p_{+4}(dx)$ diverges to $\int_{(-1,1)} 1\, p_{+4}(dx) = +\infty$. Therefore, we have proved that

$$\lim_{\alpha\to+\infty} \frac{b^\alpha}{e^{\sqrt{2\alpha}}} = -\infty. \tag{17.6}$$

Next, we will show that

$$\frac{|a^\alpha - b^\alpha|}{e^{\sqrt{2\alpha}}} \quad \text{is bounded for } \alpha \to +\infty. \tag{17.7}$$

This follows from decomposing the numerator into

$$\begin{aligned}
&|a^\alpha - b^\alpha| \\
&= \left| - \int_{[-1,1)} \Big(\big(\sinh(\sqrt{2\alpha}) - \cosh(\sqrt{2\alpha})\big) - \big(\sinh(\sqrt{2\alpha}x) - \cosh(\sqrt{2\alpha}x)\big) \Big)\, p_{+4}(dx) \right| \\
&\le \int_{1-\varepsilon}^{1-} |\cdots|\, p_{+4}(dx) + \int_{(-1)+}^{1-\varepsilon} |\cdots|\, p_{+4}(dx) + 2\sinh(\sqrt{2\alpha})\, p_{+4}(\{-1\}) \\
&=: d_1^\alpha + d_2^\alpha + d_3^\alpha,
\end{aligned}$$

for any $\varepsilon \in (0,1)$. Here, we have

$$\begin{aligned}
d_1^\alpha &= \int_{1-\varepsilon}^{1-} \left| e^{-\sqrt{2\alpha}} - e^{-\sqrt{2\alpha}x} \right| p_{+4}(dx) \\
&\le \int_{1-\varepsilon}^{1-} e^{-\sqrt{2\alpha}x}\, \sqrt{2\alpha}\, (1-x)\, p_{+4}(dx) \\
&\le \sqrt{2\alpha} e^{-\sqrt{2\alpha}(1-\varepsilon)} \int_{1-\varepsilon}^{1-} (1-x)\, p_{+4}(dx),
\end{aligned}$$

so $\lim_{\alpha\to+\infty} d_1^\alpha = 0$, which yields

$$\lim_{\alpha\to+\infty} \frac{d_1^\alpha}{e^{\sqrt{2\alpha}}} = 0.$$

The two remaining terms are easier: As p_{+4} is a finite measure on $(-1, 1-\varepsilon)$, it is

$$\begin{aligned}
\frac{d_2^\alpha}{e^{\sqrt{2\alpha}}} &= \int_{(-1)+}^{1-\varepsilon} \left| \frac{e^{-\sqrt{2\alpha}} - e^{-\sqrt{2\alpha}x}}{e^{\sqrt{2\alpha}}} \right| p_{+4}(dx) \\
&= \int_{(-1)+}^{1-\varepsilon} \left| e^{-2\sqrt{2\alpha}} - e^{-\sqrt{2\alpha}(1+x)} \right| p_{+4}(dx),
\end{aligned}$$

so LDCT (with $x \mapsto 2$ being a dominating integrable function) yields

$$\lim_{\alpha\to+\infty} \frac{d_2^\alpha}{e^{\sqrt{2\alpha}}} = 0.$$

Finally, we have

$$\lim_{\alpha\to+\infty}\frac{d_3^\alpha}{e^{\sqrt{2\alpha}}}=\lim_{\alpha\to+\infty}2\,\frac{\sinh(\sqrt{2\alpha})}{e^{\sqrt{2\alpha}}}\,p_{+4}(\{-1\})=p_{+4}(\{-1\}).$$

This shows (17.7), which together with (17.6) implies

$$\lim_{\alpha\to+\infty}\frac{a^\alpha}{b^\alpha}=\lim_{\alpha\to+\infty}\Big(\frac{\frac{a^\alpha-b^\alpha}{e^{\sqrt{2\alpha}}}}{\frac{b^\alpha}{e^{\sqrt{2\alpha}}}}+\frac{b^\alpha}{b^\alpha}\Big)=1,$$

as well as

$$\lim_{\alpha\to+\infty}\frac{\sinh(\sqrt{2\alpha})}{b^\alpha}=\lim_{\alpha\to+\infty}\frac{\frac{\sinh(\sqrt{2\alpha})}{e^{\sqrt{2\alpha}}}}{\frac{b^\alpha}{e^{\sqrt{2\alpha}}}}=0,$$
$$\lim_{\alpha\to+\infty}\frac{\cosh(\sqrt{2\alpha})}{b^\alpha}=\lim_{\alpha\to+\infty}\frac{\frac{\cosh(\sqrt{2\alpha})}{e^{\sqrt{2\alpha}}}}{\frac{b^\alpha}{e^{\sqrt{2\alpha}}}}=0.$$

By dividing both the nominator and the denominator of (17.5) by b^α, and using the above limits, we get

$$\lim_{\alpha\to+\infty}\frac{c_1^\alpha}{c_2^\alpha}=1.$$

(iv) $p_{-3}=0$, $p_{-2}=0$, $p_{-4}([-1,1])=+\infty$:
This case proceeds similar to the case (iii), however we need to adjust it at some steps to fit it to (17.4). Consider

$$\begin{aligned}a^\alpha&:=\int_{(-1,1]}\big(\sinh(\sqrt{2\alpha}x)-\sinh(-\sqrt{2\alpha})\big)\,p_{-4}(dx),\\ b^\alpha&:=\int_{(-1,1]}\big(\cosh(\sqrt{2\alpha}x)-\cosh(-\sqrt{2\alpha})\big)\,p_{-4}(dx)\\ &=\int_{(-1,1)}\big(\cosh(\sqrt{2\alpha}x)-\cosh(-\sqrt{2\alpha})\big)\,p_{-4}(dx).\end{aligned}$$

Exactly as in the case (iii), we obtain

$$\lim_{\alpha\to+\infty}\frac{b^\alpha}{e^{\sqrt{2\alpha}}}=-\infty. \tag{17.8}$$

Next, we observe that

$$\frac{|a^\alpha+b^\alpha|}{e^{\sqrt{2\alpha}}}\quad\text{is bounded for }\alpha\to+\infty, \tag{17.9}$$

which follows from a similar decomposition: We have

$$|a^\alpha+b^\alpha|\le d_1^\alpha+d_2^\alpha+d_3^\alpha,$$

with

$$\begin{aligned}
d_1^\alpha &= \int_{(-1)+}^{-1+\varepsilon} \left| e^{-\sqrt{2\alpha}} - e^{\sqrt{2\alpha}x} \right| p_{-4}(dx) \\
&\leq \int_{(-1)+}^{-1+\varepsilon} e^{\sqrt{2\alpha}x} \sqrt{2\alpha}\,(1+x)\,p_{-4}(dx) \\
&\leq \sqrt{2\alpha} e^{\sqrt{2\alpha}(-1+\varepsilon)} \int_{(-1)+}^{-1+\varepsilon} (1+x)\,p_{-4}(dx), \\
\frac{d_2^\alpha}{e^{\sqrt{2\alpha}}} &= \int_{-1+\varepsilon}^{1-} \left| \frac{e^{-\sqrt{2\alpha}} - e^{\sqrt{2\alpha}x}}{e^{\sqrt{2\alpha}}} \right| p_{-4}(dx) \\
&= \int_{-1+\varepsilon}^{1-} \left| e^{-2\sqrt{2\alpha}} - e^{\sqrt{2\alpha}(x-1)} \right| p_{-4}(dx), \\
\frac{d_3^\alpha}{e^{\sqrt{2\alpha}}} &= 2\,\frac{\sinh(\sqrt{2\alpha})}{e^{\sqrt{2\alpha}}}\,p_{-4}(\{+1\})
\end{aligned}$$

for any $\varepsilon \in (0,1)$. As above, we then easily see that all three components converge. This proves (17.9), which together with (17.8) implies

$$\lim_{\alpha\to+\infty} \frac{a^\alpha}{b^\alpha} = \lim_{\alpha\to+\infty} \left(\frac{\frac{a^\alpha+b^\alpha}{e^{\sqrt{2\alpha}}}}{\frac{b^\alpha}{e^{\sqrt{2\alpha}}}} - \frac{b^\alpha}{b^\alpha} \right) = -1,$$

as well as $\lim_{\alpha\to+\infty} \frac{\sinh(-\sqrt{2\alpha})}{b^\alpha} = 0$ and $\lim_{\alpha\to+\infty} \frac{\cosh(-\sqrt{2\alpha})}{b^\alpha} = 0$.

By dividing both the nominator and the denominator of (17.4) by b^α, and using the above limits, we get

$$\lim_{\alpha\to+\infty} \frac{c_1^\alpha}{c_2^\alpha} = -1.$$

We have shown that for any sequence of positive numbers $(\alpha_n, n \in \mathbb{N})$ converging to infinity with $c_2^{\alpha_n} \neq 0$ for all $n \in \mathbb{N}$, (17.4) and (17.5) imply that $(c_1^{\alpha_n}/c_2^{\alpha_n}, n \in \mathbb{N})$ converges to two different values, which is impossible.

Therefore, we can find a sequence $(\alpha_n, n \in \mathbb{N})$, converging to infinity, such that $c_2^{\alpha_n} = 0$ for all $n \in \mathbb{N}$. But then (17.4) and (17.5) reduce for these values of α to

$$\begin{aligned}
0 = c_1^\alpha \Big(& p_{-1} \sinh(-\sqrt{2\alpha}) - p_{-2} \sqrt{2\alpha} \cosh(-\sqrt{2\alpha}) + p_{-3}\, \alpha \sinh(-\sqrt{2\alpha}) \\
& - \int_{(-1,1]} \big(\sinh(\sqrt{2\alpha}x) - \sinh(-\sqrt{2\alpha}) \big)\, p_{-4}(dx) \Big)
\end{aligned} \tag{17.10}$$

and

$$\begin{aligned}
0 = c_1^\alpha \Big(& p_{+1} \sinh(\sqrt{2\alpha}) + p_{+2} \sqrt{2\alpha} \cosh(\sqrt{2\alpha}) + p_{+3}\, \alpha \sinh(\sqrt{2\alpha}) \\
& - \int_{(-1,1)} \big(\sinh(\sqrt{2\alpha}x) - \sinh(\sqrt{2\alpha}) \big)\, p_{+4}(dx) \Big).
\end{aligned} \tag{17.11}$$

Dividing any of both equations by the term of highest order, that is, if

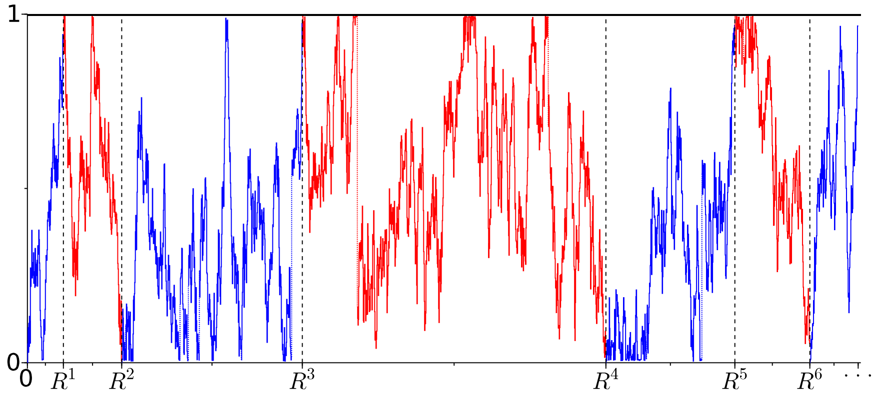


Figure 17.1: Construction of a Brownian motion on $[0,1]$ via toggling

(i) $p_{\pm 3} \neq 0$:
dividing by $\alpha \sinh(\pm\sqrt{2\alpha})$,

(ii) $p_{\pm 3} = 0$, $p_{\pm 2} \neq 0$:
dividing by $\sqrt{2\alpha} \sinh(\pm\sqrt{2\alpha})$,

(iii) $p_{\pm 3} = 0$, $p_{\pm 2} = 0$, $p_{\pm 4}([-1,1]) = +\infty$:
dividing by b^α,

(iv) $p_{\pm 3} = 0$, $p_{\pm 2} = 0$, $p_{\pm 4}([-1,1]) < +\infty$:
dividing by $\sinh(\pm\sqrt{2\alpha})$,

we see that the terms in the brackets of (17.10) and (17.11) diverge to $\pm\infty$ for $\alpha \to +\infty$. In particular, there are α_n, $n \in \mathbb{N}$, such that the term in one of the brackets does not vanish, and for these values of α, it must be $c_1^\alpha = 0$.

Thus, we have shown that there is an $\alpha > 0$ such that $\frac{\Delta}{2} f = \alpha f$, $f \in \mathscr{D}$, is only solved by $f = c_1^\alpha \sinh(\sqrt{2\alpha}\,\cdot\,) + c_2^\alpha \cosh(\sqrt{2\alpha}\,\cdot\,) = 0$. □

17.3. Construction

[IM63, Section 16] and [Kni81, Section 6.3] give instructions on how to construct a Brownian motion on $[0,1]$ which realizes a set of given boundary conditions $(c_1^i, c_2^i, c_3^i, c_4^i)_{i\in\{0,1\}}$: The basic idea is to consider two Brownian motions X^0, X^1 on the half-lines $[0,\infty)$, $(-\infty,1]$ which implement the correct boundary conditions $(c_1^0, c_2^0, c_3^0, c_4^0)$, $(c_1^1, c_2^1, c_3^1, c_4^1)$ at 0, 1 respectively, constructed by the techniques for the half-line case as stated in section 16. Now take independent copies of these processes, start, for instance, with the first copy of X^0 until it hits 1, then switch to the first copy of X^1, on returning to 0 switch "back" to the second copy of X^0, and so on (Itô and McKean only consider two processes which are toggled whenever they hit one of the boundary, which is basically the same idea). Figure 17.1 shows the resulting process, where the blue subgraphs are paths of the respective copies of X^0, and the red ones are the paths of the copies of X^1. Then, Knight argues that "*it is not hard to see that [the resulting process] is a homogeneous Markov process [...]*", thus, as Itô and McKean put it, "*leaving the proofs to the industrious reader*". The authors of [KPS10] took up this task in the continuous case (and then extended it to the graph context in [KPS12a]), and their extensive proofs seem to show that this problem is not as trivial as the above authors suggest. We will solve this problem in the general setting of metric graphs in subsection 22.4 by employing the technique of alternating copies, as introduced in subsection 13.2.

18. Metric Graphs

Following the common notion, a graph is a collection of two (disjoint) entities, called the set of vertices $\mathcal{V}$ and the set of edges $\mathcal{L}$, whereby one vertex $\partial(l)$ or two vertices $(\partial_-(l), \partial_+(l))$ are assigned to each edge $l \in \mathcal{L}$ as its "endpoint(s)", building up the graph's combinatorial structure. When also assigning to each edge $l \in \mathcal{L}$ a positive length $\rho(l)$ (being $+\infty$ in case of l having only one "endpoint") and thus identifying l with some interval $[0, \rho(l)]$ ($[0, +\infty)$ in the case $\rho(l) = +\infty$), it is possible to examine the resulting metric graph as a locally one-dimensional structure of subintervals of $\mathbb{R}_+$, which are "glued together" at their respective endpoints. This introduces the metric of "shortest paths" on this graph: Inside an edge, the metric will conform locally to the Euclidean distance on $\mathbb{R}$, while the distance between points on different edges will be measured by the shortest path along the edges of the graph leading from one point to the other.

By the identification of edges with intervals, the order of $\mathbb{R}_+$ introduces a "orientation" on the graph, which we will implement in the following way: For an "internal" edge $l \in \mathcal{L}$ with two endpoints $(\partial_-(l), \partial_+(l))$, the "initial point" 0 of the respective edge interval $[0, \rho(l)]$ will be identified with $\partial_-(l)$, and the "final point" $\rho(l)$ with $\partial_+(l)$. For an "external" edge $l \in \mathcal{L}$ with only one endpoint $\partial(l)$, the "initial point" 0 of its edge interval $[0, +\infty)$ will be equal to $\partial(l)$. Despite of this "orientation" of the underlying intervals, we will only consider "undirected graphs" in the classical sense of this term, that is, paths along the edges are always allowed in both directions.

In this section, we give a full, rigorous definition of metric graphs and functions thereon, followed by the discussion of tadpoles and by a method of compactification, which will be needed for a main result on the characterization of Brownian motions, theorem (20.16).

18.1. Basic Definitions

An unified definition or notation for metric graphs does not appear to exist. Classically, they originate in the context of "quantum graphs", see, e.g., [BK13]. We follow a similar notationally basis established in [KS06], which Kostrykin, Potthoff and Schrader also use in their works [KPS12b], [KPS12c], [KPS12a] on (continuous) Brownian motions on metric graphs. Observe that we will only consider *finite* graphs, in the sense that the sets of vertices and edges will always be finite sets:

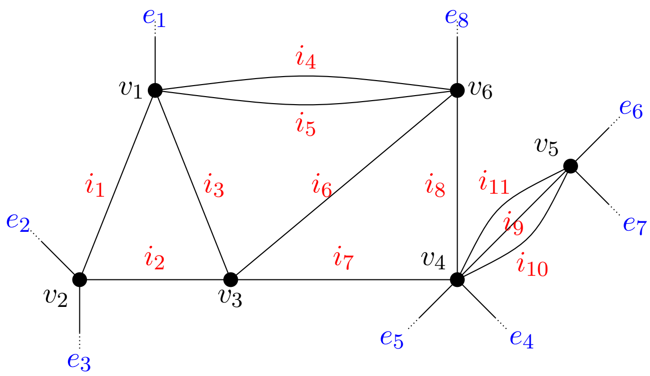


Figure 18.1: A metric graph with 6 vertices, 11 internal edges, 8 external edges. Here the curved lines are only used for illustrative reasons, they should still be considered as "straight lines" $[0, \rho(i)]$, $i \in \mathcal{I}$. The "orientation" of the edges is not depicted here: e.g., if $\partial(i_1) = (\partial_-(i_1), \partial_+(i_1)) = (v_1, v_2)$, then $(i_1, 0) \equiv v_1$ and $(i_1, \rho(i_1)) \equiv v_2$, whereas otherwise $(i_1, 0) \equiv v_2$ and $(i_1, \rho(i_1)) \equiv v_1$.

(18.1) Definition. *A tuple $\mathcal{G} = (\mathcal{V}, \mathcal{I}, \mathcal{E}, \partial)$ is a graph, if $\mathcal{V} \neq \emptyset$, $\mathcal{I}$ and $\mathcal{E}$ are finite, pairwise disjoint sets, and ∂ is a map from the set $\mathcal{L} := \mathcal{I} \cup \mathcal{E}$ into $(\mathcal{V} \times \mathcal{V}) \cup \mathcal{V}$, such that $\partial(e) \in \mathcal{V}$ for all $e \in \mathcal{E}$ and $\partial(i) = (\partial_-(i), \partial_+(i)) \in \mathcal{V} \times \mathcal{V}$ for all $i \in \mathcal{I}$. $\mathcal{V}$ is called the set of vertices, elements of $\mathcal{I}$ and $\mathcal{E}$ are called internal edges and external edges, $\mathcal{L}$ is the set of all edges. For an internal edge i, $\partial_-(i)$ and $\partial_+(i)$ are called the initial vertex and final vertex of i, while for an external edge e, $\partial(e)$ is the initial vertex of e. An internal edge i is called tadpole, if $\partial_-(i) = \partial_+(i)$.*

For a vertex $v \in \mathcal{V}$, we define the sets

$$
\begin{aligned}
\mathcal{I}_-(v) &:= \{i \in \mathcal{I} : \partial_-(i) = v\}, \quad \mathcal{I}_+(v) := \{i \in \mathcal{I} : \partial_+(i) = v\}, \\
\mathcal{I}(v) &:= \mathcal{I}_-(v) \cup \mathcal{I}_+(v), \\
\mathcal{E}(v) &:= \{e \in \mathcal{E} : \partial(e) = v\}, \\
\mathcal{L}(v) &:= \mathcal{I}(v) \cup \mathcal{E}(v)
\end{aligned}
$$

of (initial, final) internal edges, external edges, all edges respectively, incident with v.

Whenever it is notationally convenient, we will also write $\partial(l)$ for the set containing the vertex/vertices incident with the edge $l \in \mathcal{L}$, that is, $v \in \partial(l)$ means $v \in \{\partial_-(l), \partial_+(l)\}$ for an internal edge l, and $v \in \{\partial(l)\}$ for an external edge l.

(18.2) Definition. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{I}, \mathcal{E}, \partial)$ be a graph and $\rho \colon \mathcal{L} \to (0, +\infty]$ be a map, such that $\rho(i) < +\infty$ for all $i \in \mathcal{I}$, and $\rho(e) = +\infty$ for all $e \in \mathcal{E}$. Then $(\mathcal{G}, \rho)$ is called metric graph. For every edge $l \in \mathcal{L}$, $\rho_l := \rho(l)$ is called length of l.*

The lengths of the edges and the graph's combinatorial structure induce the metric of the shortest paths on a metric graph $(\mathcal{V}, \mathcal{I}, \mathcal{E}, \partial, \rho)$, which we will introduce rigorously next. To this end, consider

$$\widetilde{\mathcal{G}} = V \cup \bigcup_{i \in \mathcal{I}} (\{i\} \times [0, \rho_i]) \cup \bigcup_{e \in \mathcal{E}} (\{e\} \times [0, +\infty)).$$

We extend the mapping ∂ to $\widetilde{\mathcal{G}}$ by setting $\partial(v) := v$ for all $v \in \mathcal{V}$ and $\partial((l, x)) := \partial(l)$ for all $(l, x) \in \bigcup_{l \in \mathcal{L}} (\{l\} \times [0, \rho_l])$.

The distance between two points inside the same edge can be measured by the Euclidean distance on $\mathbb{R}$, while the distance of vertices can be measured by the length of the shortest possible path along the edges of the graph. In order to distinguish both modes, we first define an auxiliary metric which only measures the direct distance inside the same edge:

(18.3) Definition. *The internal length* $d^{\text{int}} : \widetilde{\mathcal{G}} \to [0, +\infty]$ *is defined by*

$$\begin{aligned}
\forall e \in \mathcal{E},\ x, y \in [0, +\infty): &\quad d^{\text{int}}((e, x), (e, y)) := |x - y|\,, \\
\forall i \in \mathcal{I},\ x, y \in [0, \rho_i]: &\quad d^{\text{int}}((i, x), (i, y)) := |x - y|\,, \\
\forall e \in \mathcal{E},\ x \in [0, +\infty): &\quad d^{\text{int}}((e, x), \partial(e)) := d^{\text{int}}(\partial(e), (e, x)) := x, \\
\forall i \in \mathcal{I},\ x \in [0, \rho_i]: &\quad d^{\text{int}}((i, x), \partial_-(i)) := d^{\text{int}}(\partial_-(i), (i, x)) := x, \\
&\quad d^{\text{int}}((i, x), \partial_+(i)) := d^{\text{int}}(\partial_+(i), (i, x)) := \rho_i - x, \\
\forall v \in \mathcal{V}: &\quad d^{\text{int}}(v, v) := 0,
\end{aligned}$$

and $d^{\text{int}}(g_1, g_2) := +\infty$ *for all other* $g_1, g_2 \in \widetilde{\mathcal{G}}$.

The metric properties of d^{int} are immediate from its definition:

(18.4) Lemma. *The following assertions hold true:*

(i) $d^{\text{int}}(g, g) = 0$ *for all* $g \in \widetilde{\mathcal{G}}$.

(ii) $d^{\text{int}}(g_1, g_2) = d^{\text{int}}(g_2, g_1)$ *for all* $g_1, g_2 \in \widetilde{\mathcal{G}}$.

(iii) $d^{\text{int}}(g_1, g_3) \leq d^{\text{int}}(g_1, g_2) + d^{\text{int}}(g_2, g_3)$ *for all* $g_1, g_2, g_3 \in \widetilde{\mathcal{G}}$.

Proof. (i) and (ii) are obvious. Turning to (iii), we note that the choices of $g_1, g_2, g_3 \in \widetilde{\mathcal{G}}$ involving vertices, namely $g_k = \partial(e)$, $e \in \mathcal{E}$, or $g_k = \partial_\pm(i)$, $i \in \mathcal{I}$, for some $k \in \{1, 2, 3\}$, can instead be verified for $g_k = (e, 0)$, $g_k = (i, 0)$, $g_k = (i, \rho_i)$, respectively. Thus, let $g_k = (l_k, x_k)$, $k \in \{1, 2, 3\}$. It is sufficient to check the case $l_1 = l_2 = l_3 = l$, as otherwise $d^{\text{int}}(g_1, g_2) = +\infty$ or $d^{\text{int}}(g_2, g_3) = +\infty$. But this case directly follows from the triangle inequality of the Euclidean norm. □

In order to measure the distance between points on different edges, we need to consider the possible paths along the edges of the graph, leading from the initial or final vertices of their respective edges:

(18.5) Definition. *For $n \in \mathbb{N}_0$, $v_0, \ldots, v_n \in \mathcal{V}$, $i_1, \ldots, i_n \in \mathcal{I}$, $(v_0, i_1, v_1, \ldots, v_{n-1}, i_n, v_n)$ is called path from v_0 to v_n of length n across $(v_0, \ldots, v_n)$ via $(i_1, \ldots, i_n)$, if*

$$\forall k \in \{1, \ldots, n\}: \quad v_{k-1} \in \partial(i_k), v_k \in \partial(i_k).$$

For $v, w \in \mathcal{V}$, $\mathcal{P}(v, w)$ is the set of all paths from v to w, and $\mathcal{P} = \bigcup_{v,w \in \mathcal{V}} \mathcal{P}(v, w)$ is the set of all possible paths.

Notice that there is always a path from a vertex v_0 to itself, namely the path (v_0), and every path can be reversed: If $(v_0, i_1, v_1, \ldots, v_{n-1}, i_n, v_n)$ is a path from v_0 to v_n, then $(v_n, i_n, v_{n-1}, \ldots, v_1, i_1, v_0)$ is a path from v_n to v_0. In particular, $\mathcal{P}(v, v)$ is not empty and $\mathcal{P}(v, w) = \mathcal{P}(w, v)$ holds for any vertices $v, w \in \mathcal{V}$. It also follows directly from the definition that paths can be concatenated: If

$$(v, i_1, v_1, \ldots, v_{n-1}, i_n, w) \quad \text{and} \quad (w, j_1, w_1, \ldots, w_{n-1}, j_n, u)$$

are paths from v to w, from w to u respectively, then

$$(v, i_1, v_1, \ldots, v_{n-1}, i_n, w, j_1, w_1, \ldots, w_{n-1}, j_n, u)$$

is a path from v to u. Thus, the relation of being connected by a path is an equivalence relation on $\mathcal{V}$.

(18.6) Definition. *The length of a path $d^{\mathcal{P},\rho} \colon \mathcal{P} \to [0, +\infty]$ is defined by*

$$d^{\mathcal{P},\rho}\big((v_0, i_1, v_1, \ldots, v_{n-1}, i_n, v_n)\big) := \rho_{i_1} + \cdots + \rho_{i_n}.$$

We are now able to define a metric on the metric graph, induced by its combinatorial structure and its edge lengths:

(18.7) Definition. *The metric of the shortest paths $d \colon \widetilde{\mathcal{G}} \times \widetilde{\mathcal{G}} \to [0, +\infty]$ on a metric graph $(\mathcal{V}, \mathcal{I}, \mathcal{E}, \partial, \rho)$ is defined for $v, w \in \mathcal{V}$ by*

$$d(v, w) := \inf_{(v,\ldots,w) \in \mathcal{P}(v,w)} d^{\mathcal{P},\rho}\big((v, \ldots, w)\big),$$

as well as for $(g_1, g_2) \in (\widetilde{\mathcal{G}} \times \widetilde{\mathcal{G}}) \backslash (\mathcal{V} \times \mathcal{V})$ by

$$\begin{aligned} d(g_1, g_2) := \inf \big\{ & d^{\text{int}}(g_1, g_2), \\ & \inf_{\substack{v_1 \in \partial(g_1), \\ v_2 \in \partial(g_2)}} \{ d^{\text{int}}(g_1, v_1) + d(v_1, v_2) + d^{\text{int}}(v_2, g_2) \} \big\}. \end{aligned}$$

Here, as usual, we set $\inf \emptyset := +\infty$. Therefore, $d(g_1, g_2) = +\infty$ holds if and only if there is no path from g_1 to g_2 along the edges of $\mathcal{G}$.

The reader should observe that the "shortest path" (and thus the distance) of two points inside the same edge must not equal the Euclidean distance of their local coordinates, cf. figure 18.2 for an example. However, this will not cause any problems, because the neighborhoods of points of the interior of an edge can always be chosen small enough in order to completely lie inside the corresponding edge.

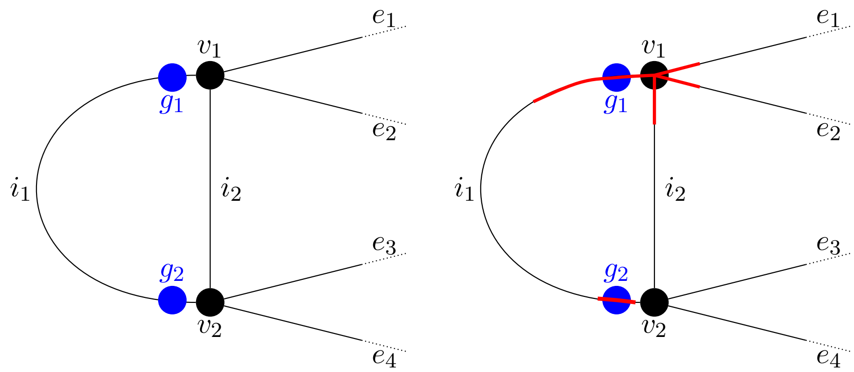


Figure 18.2: Shortest distance and neighborhoods in a metric graph: In the metric graph depicted above, assume the edge lengths $\rho(i_1) = 10$, $\rho(i_2) = 5$ and the points $g_1 = (i_1, 1)$, $g_2 = (i_2, 9)$. Then the "internal" distance inside the edge i_1 is given by $d^{\text{int}}(g_1, g_2) = 8$, while the path across (v_1, v_2) via i_2 realizes the shortest distance $d(g_1, g_2) = 7$. On the right-hand graph, two neighborhoods of g_1 and g_2 are illustrated.

(18.8) Lemma. *The following assertions hold true:*

(i) $d(g, g) = 0$ *for all* $g \in \widetilde{\mathcal{G}}$.

(ii) $d(g_1, g_2) = d(g_2, g_1)$ *for all* $g_1, g_2 \in \widetilde{\mathcal{G}}$.

(iii) $d(g_1, g_3) \le d(g_1, g_2) + d(g_2, g_3)$ *for all* $g_1, g_2, g_3 \in \widetilde{\mathcal{G}}$.

Proof. (i) This follows directly from the definition of d^{int} and from $d^{\mathcal{P},\rho}((v)) = 0$.

(ii) The symmetry of d is inherited from the symmetry of d^{int}, shown in property (ii) of lemma (18.4), and of d on $\mathcal{V} \times \mathcal{V}$, which is easily seen as all paths are reversible.

(iii) We start with the triangle inequality for vertices $v, w, u \in \mathcal{V}$: As for all paths $(v, \dots, w) \in \mathcal{P}(v, w)$, $(w, \dots, u) \in \mathcal{P}(w, u)$, the concatenated path $(v, \dots, w, \dots, u)$ is a path from v to u and therefore lies in $\mathcal{P}(v, u)$, we have

$$d(v, u) \le d^{\mathcal{P},\rho}((v, \dots, w, \dots, u)) = d^{\mathcal{P},\rho}((v, \dots, w)) + d^{\mathcal{P},\rho}((w, \dots, u)).$$

Taking the infima over all possible paths from v to w and from w to u yields

$$d(v, u) \le d(v, w) + d(w, u).$$

Now, let $g_k \in \widetilde{\mathcal{G}}$, $k \in \{1, 2, 3\}$. Then, for all $v_1 \in \partial(g_1)$, $v_2 \in \partial(g_2)$, $w_1 \in \partial(g_2)$, $w_2 \in \partial(g_3)$, the triangle inequalities of d^{int} on $\widetilde{\mathcal{G}} \times \widetilde{\mathcal{G}}$ (see (iii) of lemma (18.4)) and of d on $\mathcal{V} \times \mathcal{V}$ imply

- $d^{\text{int}}(g_1, g_3) \le d^{\text{int}}(g_1, g_2) + d^{\text{int}}(g_2, g_3)$,
- $d^{\text{int}}(g_1, v_1) + d(v_1, v_2) + d^{\text{int}}(v_2, g_3) \le d^{\text{int}}(g_1, v_1) + d(v_1, v_2) + d^{\text{int}}(v_2, g_2) + d^{\text{int}}(g_2, g_3)$, with both sides being $+\infty$ if $v_2 \notin \partial(g_3)$,

- $d^{\text{int}}(g_1, w_1) + d(w_1, w_2) + d^{\text{int}}(w_2, g_3) \le d^{\text{int}}(g_1, g_2) + d^{\text{int}}(g_2, w_1) + d(w_1, w_2) + d^{\text{int}}(w_2, g_3)$, with both sides being $+\infty$ if $w_1 \notin \partial(g_1)$,
- $d^{\text{int}}(g_1, v_1) + d(v_1, w_2) + d^{\text{int}}(w_2, g_3) \le d^{\text{int}}(g_1, v_1) + d(v_1, v_2) + d^{\text{int}}(v_2, g_2) + d^{\text{int}}(g_2, w_1) + d(w_1, w_2) + d^{\text{int}}(w_2, g_3)$, as $d(v_1, v_2) + d^{\text{int}}(v_2, g_2) + d^{\text{int}}(g_2, w_1) + d(w_1, w_2)$ on the right-hand side has at least the length of a path from v_1 to w_2 over the vertices $(v_1, v_2, \dots, w_1, \dots, w_2)$, and thus is an upper bound for $d(v_1, w_2)$.

Therefore, we have

$$\begin{aligned} &\inf\{d^{\text{int}}(g_1, g_3), (g_1, v_1) + d(v_1, w_2) + d^{\text{int}}(w_2, g_3)\} \\ &\le \inf\{d^{\text{int}}(g_1, g_2), (g_1, v_1) + d(v_1, v_2) + d^{\text{int}}(v_2, g_2)\} \\ &\quad + \inf\{d^{\text{int}}(g_2, g_3), (g_2, w_1) + d(w_1, w_2) + d^{\text{int}}(w_2, g_3)\}, \end{aligned}$$

and taking infima over $v_1 \in \partial(g_1)$, $v_2 \in \partial(g_2)$, $w_1 \in \partial(g_2)$, $w_2 \in \partial(g_3)$ yields $d(g_1, g_3) \le d(g_1, g_2) + d(g_2, g_3)$.

The mixed case follows from this, as for all $v \in \mathcal{V}$, $g \in \widetilde{\mathcal{G}}$, it is $d(v, g) = d\big((l, 0), g\big)$ with $l \in \mathcal{L}$ such that $v = \partial_-(l)$, or $d(v, g) = d\big((l, \rho_l), g\big)$ with $l \in \mathcal{L}$, $v = \partial_+(l)$. □

We introduce the *geometric representation* of the metric graph $(\mathcal{V}, \mathcal{I}, \mathcal{E}, \partial, \rho)$ by identifying the points which have zero distance:

$$\mathcal{G} := \widetilde{\mathcal{G}}/_{\{(g_1, g_2) \in \widetilde{\mathcal{G}} : d(g_1, g_2) = 0\}}. \tag{18.9}$$

The equivalence sets of $\mathcal{G}$ are very simple here, as only the vertices are identified with the endpoints of their respective edges, that is, we have the following classes of points:

- vertex points: $\{v\} \cup \big\{(e, 0) : e \in \mathcal{E}, v = \partial(e)\big\} \cup \big\{(i_-, 0) : i_- \in \mathcal{I}, v = \partial_-(i_-)\big\} \cup \big\{(i_+, \rho_{i_+}) : i_+ \in \mathcal{I}, v = \partial_+(i_+)\big\}$ for $v \in \mathcal{V}$;
- inner points: $\big\{(l, x)\big\}$ for $l \in \mathcal{L}$, $x \in (0, \rho_l)$.

Thus, $\mathcal{G}$ can be seen as a collection of closed intervals and half lines of $\mathbb{R}$ of lengths given by ρ, with some of their endpoints being "glued together" by the graph's combinatorial structure ∂. We will call the "position" on these intervals $\{l\} \times [0, \rho_l]$ (with $[0, \rho_l] := [0, +\infty)$ if $\rho_l = +\infty$) *local coordinate*, that is, a point $g = (l, x)$ has the local coordinate x. Of course, this coordinate is only meaningful in the context of its relative edge l, as the identification may "glue together" an "initial" coordinate 0 of some edge with a "final" coordinate ρ_i of some other edge i at their mutual vertex.

From time to time, we will also identify any edge $l \in \mathcal{L}$ with the set of its corresponding points $\{l\} \times [0, \rho_l]$. For later use, we define the *open interior* of an edge $l \in \mathcal{L}$ to be

$$l^0 := \{l\} \times (0, \rho_l),$$

as well as the set $\mathcal{G}^0$ of all inner points of $\mathcal{G}$ by

$$\mathcal{G}^0 := \bigcup_{l \in \mathcal{L}} \left(\{l\} \times (0, \rho_l) \right).$$

Owing to the triangle inequality of d on $\widetilde{\mathcal{G}}$, d assumes the same value on all representants of an equivalence class and thus can be extended to a mapping $d \colon \mathcal{G} \times \mathcal{G} \to [0, +\infty]$. It follows from lemma (18.8) that d is a metric on $\mathcal{G}$. Here we allow a metric to take values in $[0, +\infty]$. This is a slight extension of the standard definition of a "metric", which does not impact any topological results which will be needed later (see [BBI01, Chapter 1]).

The topology on $\mathcal{G}$ induced by d is structured as follows: Inside $\mathcal{G}^0$, it locally "looks" like the topology of some interval of $\mathbb{R}_+$, as for $(l, x) \in \mathcal{G}^0$, $\varepsilon \in \big(0, \min\{x, \rho_l - x\}\big)$,

$$B_\varepsilon\big((l,x)\big) = \{g \in \mathcal{G} : d((l,x), g) < \varepsilon\} = \big\{(l,y) : |x-y| < \varepsilon\big\} = \{l\} \times (x - \varepsilon, x + \varepsilon),$$

which is "glued together" at the vertices by ∂, as for $v \in \mathcal{V}$, $\varepsilon \in \big(0, \min\{\rho_l, l \in \mathcal{L}(v)\}\big)$,

$$B_\varepsilon(v) = \{g \in \mathcal{G} : d(v, g) < \varepsilon\} = \bigcup_{\substack{l \in \mathcal{L}(v) \\ \partial_-(l) = v}} (\{l\} \times [0, \varepsilon)) \cup \bigcup_{\substack{l \in \mathcal{I}(v) \\ \partial_+(l) = v}} (\{l\} \times (\rho_l - \varepsilon, \rho_l]).$$

(18.10) Theorem. *d defines a complete, separable metric on $\mathcal{G}$.*

Proof. As every sequence in $\mathcal{G}$ can be identified with a sequence in

$$\bigcup_{i \in \mathcal{I}} \left(\{i\} \times [0, \rho_i] \right) \cup \bigcup_{e \in \mathcal{E}} \left(\{e\} \times [0, +\infty) \right),$$

and each of the intervals $[0, \rho_i]$, $[0, +\infty)$ is complete, every Cauchy sequence in $\mathcal{G}$ converges. Furthermore, every edge is homeomorphic to an interval, which contains a countable, dense subset (take, e.g., the rational points), and the topology of $\mathcal{G}$ inside $\mathcal{G}^0$ locally coincides with the internal topology induced on the edges, so using the (finite) union of these countable separability sets for all edges $l \in \mathcal{L}$ together with the (finite) set of vertices gives a separability set for $\mathcal{G}$. $\square$

18.2. Discussion of Tadpoles

Tadpoles, that is, internal edges $i \in \mathcal{I}$ with the same initial and final vertex $\partial_-(i) = \partial_+(i)$, will provide a nuisance in our constructions. The following technique, as explained in [KPS12a, Section VI], will allow us to eliminate the tadpoles while maintaining the graph's topological structure (and thus, when applied in the context of Brownian motions, will not alter the description of the processes on the graph, see remark (20.24)).

Assume we are given a metric graph $\mathcal{G} = (\mathcal{V}, \mathcal{I}, \mathcal{E}, \partial, \rho)$ with a non-empty set of tadpoles $\mathcal{I}_t = \{i \in \mathcal{I} : \partial_-(i) = \partial_+(i)\}$. We "split" every tadpole into two "regular" internal edges by introducing, for each $i \in \mathcal{I}_t$, a new vertex v_t^i and two new internal edges i^+, i^-, each with edge length $\rho(i)/2$, thus defining a new metric graph $\widetilde{\mathcal{G}} = (\widetilde{\mathcal{V}}, \widetilde{\mathcal{I}}, \widetilde{\mathcal{E}}, \widetilde{\partial}, \widetilde{\rho})$ with $\widetilde{\mathcal{V}} := \mathcal{V} \cup \{v_t^i : i \in \mathcal{I}_t\}$, $\widetilde{\mathcal{I}} := (\mathcal{I} \backslash \mathcal{I}_t) \cup \{i^+, i^- : i \in \mathcal{I}_t\}$, and $\widetilde{\mathcal{E}} := \mathcal{E}$. The edge lengths $\widetilde{\rho}$ and

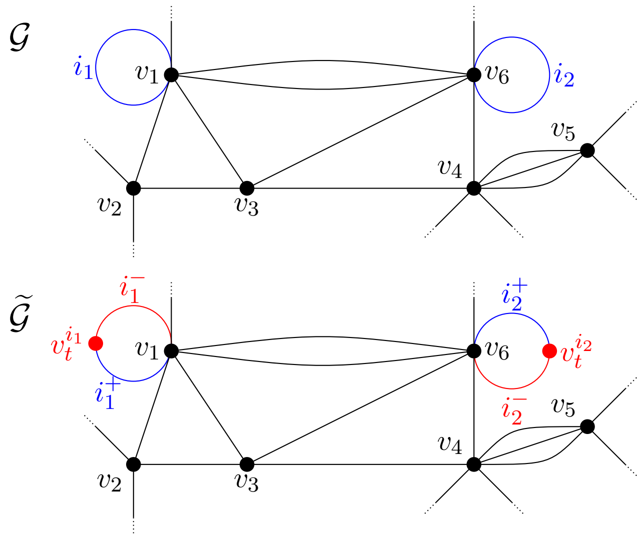


Figure 18.3: Extension of a metric graph for elimination of tadpoles: Pictured above is a metric graph $\mathcal{G}$ with two tadpoles i_1, i_2 at v_1, v_6. By splitting each tadpole i up into two new internal edges i^-, i^+, connected via the original vertex and a newly adjoined vertex v_t^i, we obtain the resulting graph $\widetilde{\mathcal{G}}$ below, which does not possess tadpoles anymore.

the new graph's combinatorial structure $\widetilde{\partial}$ are chosen to be equal to the old ones ρ, ∂ respectively, on the remaining original set $(\mathcal{I}\backslash\mathcal{I}_t) \cup \mathcal{E}$, and are extended to the new edges by $\widetilde{\rho}(i^-) := \widetilde{\rho}(i^+) := \rho(i)/2$ and $\widetilde{\partial}(i^-) := (\partial_-(i), v_t^i)$, $\widetilde{\partial}(i^+) := (v_t^i, \partial_+(i))$, for $i \in \mathcal{I}_t$, see figure 18.3.

Due to the identification of the new edges' endpoints with the adjoined vertices, and to the graphs' metric only being dependent on the length of paths, the induced topology on the new metric graph $\widetilde{\mathcal{G}}$ equals the topology on $\mathcal{G}$. $\widetilde{\mathcal{G}}$ does not possess any tadpoles. Therefore, we will always be able to restrict our attention to metric graphs without tadpoles in the sequel, as all our examinations will solely be based on the topological structure of the underlying graph, but not on its representation.

18.3. Functions on a Metric Graph

Any real valued function f on a metric graph $\mathcal{G}$ can be represented by collections of real values $(f_v, v \in \mathcal{V})$ and of functions $(f_l, l \in \mathcal{L})$ with $f_l \colon [0, \rho_l] \to \mathbb{R}$, satisfying $f_l(x) = f\big((l,x)\big)$, $x \in [0, \rho_l]$ (where we set in the following for notationally convenience $[0, \rho_l] := [0, +\infty)$ for $l \in \mathcal{E}$), and $f_v = f(v)$, $v \in \mathcal{V}$. As the endpoints of the edges are identified by the graph's combinatorial structure, the values

$$f_e(0) = f\big((e,0)\big), \;\; f_v = f(v), \;\; f_{i_-}(0) = f\big((i_-,0)\big), \;\; f_{i_+}(\rho_{i_+}) = f\big((i_+, \rho_{i_+})\big),$$

must coincide in case $e \in \mathcal{E}$, $v = \partial(e)$, and $i_- \in \mathcal{I}$, $v = \partial_-(i_-)$, and $i_+ \in \mathcal{I}$, $v = \partial_+(i_+)$.

In every small neighborhood of a non-vertex point $g \in \mathcal{G}^0$, a real valued function f on $\mathcal{G}$ can locally be interpreted as a function on some interval of $\mathbb{R}$. Thus, the differentiability of f_l at x induces the notion of differentiability of f at $g = (l, x) \in \mathcal{G}^0$. In order to define differentiability at the vertices, we must take care of the edges' "orientation":

(18.11) Definition. *Let $f\colon \mathcal{G} \to \mathbb{R}$ be a function on $\mathcal{G}$, $v \in \mathcal{V}$ and $l \in \mathcal{L}(v)$. Then the directional derivative of f at v along l is defined by*

$$f_l'(v) := \begin{cases} \lim_{\xi \to v, \xi \in l^0} f'(\xi), & v = \partial_-(l), \\ -\lim_{\xi \to v, \xi \in l^0} f'(\xi), & v = \partial_+(l), \end{cases}$$

whenever the right-hand side exists.

(18.12) Definition. *Let $\mathcal{C}_0^{0,2}(\mathcal{G})$ be the subspace of all functions f in $\mathcal{C}_0(\mathcal{G})$, which are twice continuously differentiable on $\mathcal{G}^0$, such that for every $v \in \mathcal{V}$, $l \in \mathcal{L}(v)$, the limit*

$$f_l''(v) := \lim_{\xi \to v, \xi \in l^0} f''(\xi)$$

exists, and for every $e \in \mathcal{E}$, f_e'' vanishes at infinity. Let $\mathcal{C}_0^2(\mathcal{G})$ be the subset of those functions f in $\mathcal{C}_0^{0,2}(\mathcal{G})$, for which f'' extends from $\mathcal{G}^0$ to a function in $\mathcal{C}_0(\mathcal{G})$.

By definition, a function $f \in \mathcal{C}_0^{0,2}(\mathcal{G})$ lies in $\mathcal{C}_0^2(\mathcal{G})$, if and only if, for every $v \in \mathcal{V}$, the second derivatives at v coincide, that is, if $f_k''(v) = f_l''(v)$ holds for all $k, l \in \mathcal{L}(v)$, and in this case, we will just write $f''(v)$ for this value. If $f \in \mathcal{C}_0^2(\mathcal{G})$, then, for any edge $l \in \mathcal{L}$, the limits of the first derivatives at its endpoint(s) $\lim_{x \downdownarrows 0} f_l'(x)$ (and $\lim_{x \upuparrows \rho_l} f_l'(x)$, if $l \in \mathcal{I}$) must exist, which can easily be seen by the fundamental theorem of calculus:

$$f_l'(x) = -\int_x^b f_l''(t)\, dt + f_l'(b), \quad x, b \in (0, \rho_l).$$

However, these limits on various edges do not need to coincide at their mutual vertex: In general, the first derivate f' of $f \in \mathcal{C}_0^2(\mathcal{G})$ does not extend from $\mathcal{C}_0(\mathcal{G}^0)$ to a function in $\mathcal{C}_0(\mathcal{G})$.

We will mainly be concerned with the following operator on $\mathcal{C}_0^2(\mathcal{G})$:

(18.13) Definition. *The Laplacian Δ on $\mathcal{G}$ is defined by*

$$\Delta\colon \mathcal{C}_0^2(\mathcal{G}) \to \mathcal{C}_0(\mathcal{G}),\ f \mapsto \Delta(f) := f''.$$

18.4. Compactification of a Metric Graph

For later purposes, we introduce the following method of "cutting out" vertex points from an existing graph and compactifying the resulting set: Let $(\mathcal{V}, \mathcal{I}, \mathcal{E}, \partial, \rho)$ be a metric graph with geometric representation

$$\widetilde{\mathcal{G}} = V \cup \bigcup_{i \in \mathcal{I}} (\{i\} \times [0, \rho_i]) \cup \bigcup_{e \in \mathcal{E}} (\{e\} \times [0, +\infty)),$$

and $\mathcal{G}$ be the set $\widetilde{\mathcal{G}}$ with vertex points and endpoints of edges identified by its canonical metric d, as introduced in subsection 18.1. Let $\mathcal{V}_0 \subsetneq \mathcal{V}$, and $\widetilde{\mathcal{G}}_1$ be the subset of $\mathcal{G}$ which results from removing the vertices $\mathcal{V}_0$ together with their identified edge points from $\mathcal{G}$, that is, consider

$$\begin{aligned}\widetilde{\mathcal{G}}_1 &:= \widetilde{\mathcal{G}} \backslash \Big(\mathcal{V}_0 \cup \bigcup_{i_- \in \mathcal{I}_-(\mathcal{V}_0)} \{(i_-, 0)\} \cup \bigcup_{i_+ \in \mathcal{I}_+(\mathcal{V}_0)} \{(i_+, \rho_i)\} \cup \bigcup_{e \in \mathcal{E}(\mathcal{V}_0)} \{(e, 0)\} \Big) \\ &= (\mathcal{V} \backslash \mathcal{V}_0) \cup \bigcup_{i \in \mathcal{I}} (\{i\} \times I_i) \cup \bigcup_{e \in \mathcal{E}} (\{e\} \times E_e)\end{aligned}$$

with

$$I_i := \begin{cases} [0, \rho_i], & i \in \mathcal{I} \backslash \mathcal{I}(\mathcal{V}_0), \\ (0, \rho_i], & i \in \mathcal{I}_-(\mathcal{V}_0) \backslash \mathcal{I}_+(\mathcal{V}_0), \\ [0, \rho_i), & i \in \mathcal{I}_+(\mathcal{V}_0) \backslash \mathcal{I}_-(\mathcal{V}_0), \\ (0, \rho_i), & i \in \mathcal{I}_-(\mathcal{V}_0) \cap \mathcal{I}_+(\mathcal{V}_0), \end{cases} \qquad E_e := \begin{cases} [0, +\infty), & e \in \mathcal{E} \backslash \mathcal{E}(\mathcal{V}_0), \\ (0, +\infty), & e \in \mathcal{E}(\mathcal{V}_0). \end{cases}$$

We compactify $\widetilde{\mathcal{G}}_1$ by adjoining the missing interval endpoints 0, ρ_i, $+\infty$, where needed. For convenience (and for staying in the context of a metric graph as much as possible), we also add new vertices for newly adjoined finite endpoints. Altogether, we set

$$\overline{\widetilde{\mathcal{G}}_1} := \mathcal{V}_1 \cup \bigcup_{i \in \mathcal{I}} (\{i\} \times [0, \rho_i]) \cup \bigcup_{e \in \mathcal{E}} (\{e\} \times [0, +\infty]),$$

with

$$\mathcal{V}_1 := (\mathcal{V} \backslash \mathcal{V}_0) \cup \{v_-^i, i \in \mathcal{I}_-(\mathcal{V}_0)\} \cup \{v_+^i, i \in \mathcal{I}_+(\mathcal{V}_0)\} \cup \{v^e, e \in \mathcal{E}(\mathcal{V}_0)\},$$

where all new vertices v_-^i, v_+^i, v^e are distinct points which are not in $\mathcal{G}$. We adapt the combinatorial structure of the original graph to $\overline{\widetilde{\mathcal{G}}_1}$ by defining $\partial_1 \colon \mathcal{L} \to (\mathcal{V}_1 \times \mathcal{V}_1) \cup \mathcal{V}_1$ by

$$\partial_1(i) = \begin{cases} (\partial_-(i), \partial_+(i)), & i \in \mathcal{I} \backslash \mathcal{I}(\mathcal{V}_0), \\ (v_-^i, \partial_+(i)), & i \in \mathcal{I}_-(\mathcal{V}_0) \backslash \mathcal{I}_+(\mathcal{V}_0), \\ (\partial_-(i), v_+^i), & i \in \mathcal{I}_+(\mathcal{V}_0) \backslash \mathcal{I}_-(\mathcal{V}_0), \\ (v_-^i, v_+^i), & i \in \mathcal{I}_-(\mathcal{V}_0) \cap \mathcal{I}_+(\mathcal{V}_0), \end{cases} \qquad \partial_1(e) = \begin{cases} \partial(e), & e \in \mathcal{E} \backslash \mathcal{E}(\mathcal{V}_0), \\ v^e, & e \in \mathcal{E}(\mathcal{V}_0). \end{cases}$$

Thus, by removing vertices from the original graph $\mathcal{G}$, we disconnected some edges which needed new initial or final vertices. We added these, and additionally compactified the non-compact external edges $\{e\} \times [0, +\infty)$ to $\{e\} \times [0, +\infty]$. Observe that the latter causes the "compactified graph" $\overline{\widetilde{\mathcal{G}}_1}$ not to be a metric graph in the sense of our definition anymore.

Let d_1 be the metric of shortest paths, as defined in subsection 18.1, for the just constructed metric graph $((\mathcal{V}_1, \mathcal{I}, \mathcal{E}, \partial_1), \rho)$. We extend the metric d_1 to $\overline{\widetilde{\mathcal{G}}_1}$ by defining the distance of a point "at infinity" $(e, +\infty)$, $e \in \mathcal{E}$, to any other point to be $+\infty$. Then, as usual, we identify the points $g_1, g_2 \in \overline{\widetilde{\mathcal{G}}_1}$ for which $d_1(g_1, g_2) = 0$ holds true, naming the resulting set of equivalence sets $\overline{\mathcal{G}_1}$.

In order to be able to distinguish between the original vertex points of $\mathcal{G}$ and the newly introduced ones of $\overline{\mathcal{G}_1}$ in the local representation, we set

- if $i \in \mathcal{I}_-(\mathcal{V}_0)$: $(i, 0+)$ for $(i, 0) = v_-^i$,
- if $i \in \mathcal{I}_+(\mathcal{V}_0)$: (i, ρ_i-) for $(i, \rho_i) = v_+^i$,
- if $e \in \mathcal{E}(\mathcal{V}_0)$: $(e, 0+)$ for $(e, 0) = v^e$.

Let the topology inside $\overline{\mathcal{G}_1} \backslash \{(e, +\infty), e \in \mathcal{E}\}$ be induced by d_1, while all $(e, +\infty)$, $e \in \mathcal{E}$, are distinct points in the topology, topological inserted as the points at infinity of each $\{e\} \times [0, +\infty)$ by the same technique the "point at infinity" $+\infty$ is embedded in $[0, +\infty)$ by the Alexandroff one-point compactification, that is, as a point outside every compact set.

Observe that by removing a vertex point v and compactifying the resulting graph, the "connection" of all edges incident with v is removed and a new endpoint is adjoint for each disconnected edge. Furthermore, every external edge $\{e\} \times [0, +\infty)$ is compactified to $\{e\} \times [0, +\infty]$, thus adding points $(e, +\infty)$ for all external edges $e \in \mathcal{E}$, see figure 18.4.

(18.14) Definition. *$\mathcal{C}(\overline{\mathcal{G}_1})$ is the set of all continuous, real valued functions on $\overline{\mathcal{G}_1}$, that is, the set of all functions $f \colon \overline{\mathcal{G}_1} \to \mathbb{R}$ which are continuous in $\overline{\mathcal{G}_1} \backslash \{(e, +\infty), e \in \mathcal{E}\}$ with respect to d_1 and for which*

$$f\big((e, +\infty)\big) = \lim_{x \to +\infty} f\big((e, x)\big)$$

exists for all $e \in \mathcal{E}$. We endow $\mathcal{C}(\overline{\mathcal{G}_1})$ with its natural norm

$$\|f\|_\infty := \sup_{x \in \overline{\mathcal{G}_1}} |f(x)|, \quad f \in \mathcal{C}(\overline{\mathcal{G}_1}).$$

We did not show that $\mathcal{C}(\overline{\mathcal{G}_1})$ is compact, so we need to prove the next result manually:

(18.15) Lemma. *For all $f \in \mathcal{C}(\overline{\mathcal{G}_1})$,*

$$\|f\|_\infty = \max_{x \in \overline{\mathcal{G}_1}} |f(x)| < +\infty.$$

Proof. Assume that there exists a sequence $(g_n, n \in \mathbb{N})$ in $\overline{\mathcal{G}_1}$, such that the sequence of its values $\big(f(g_n), n \in \mathbb{N}\big)$ tends to infinity. Then $(g_n, n \in \mathbb{N})$ cannot have an accumulation point inside $\overline{\mathcal{G}_1} \backslash \{(e, +\infty), e \in \mathcal{E}\}$, as f is continuous in any neighborhood of such a point and thus cannot tend to infinity there. So $(g_n, n \in \mathbb{N})$ needs to converge to $+\infty$ on some external edges, and we can decompose $(g_n, n \in \mathbb{N})$ into subsequences $(g_{n_k^e}, k \in \mathbb{N})$, $e \in \mathcal{E}^g \subseteq \mathcal{E}$, with $g_{n_k^e} = (e, x_{n_k^e})$ and $\lim_{k \to \infty} x_{n_k^e} = +\infty$. But here, because $f \in \mathcal{C}(\overline{\mathcal{G}_1})$, we have $\lim_{k \to \infty} f(g_{n_k^e}) = \lim_{k \to \infty} f\big((e, x_{n_k^e})\big) = f\big((e, +\infty)\big) < +\infty$.

This also shows that the supremum is attained, as it is obviously attained if the accumulation point lies inside $\overline{\mathcal{G}_1} \backslash \{(e, +\infty), e \in \mathcal{E}\}$ by continuity, and also if it accumulates at some $(e, +\infty)$, $e \in \mathcal{E}$, as just shown. $\square$

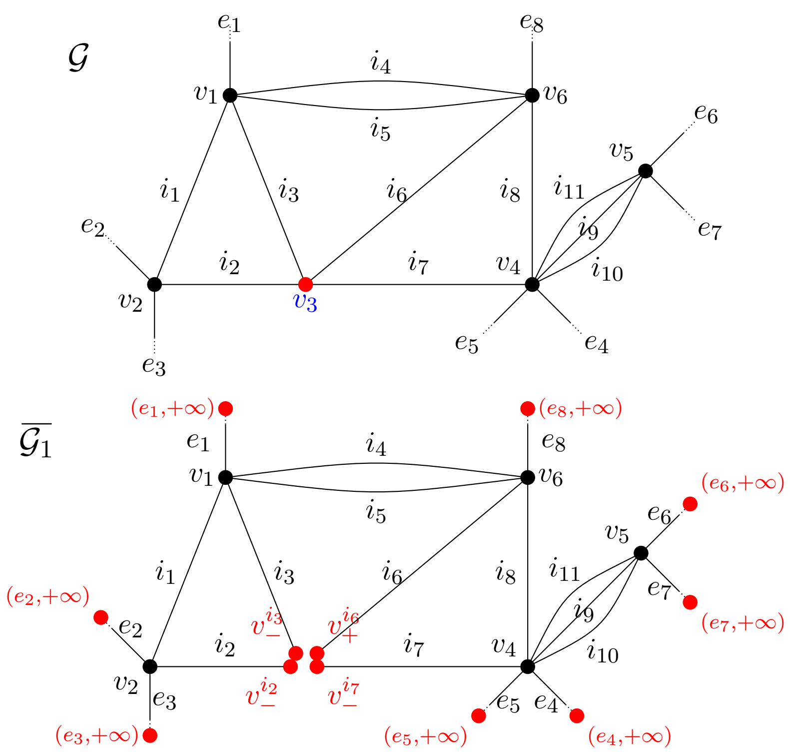


Figure 18.4: A metric graph $\mathcal{G}$ and its resulting compactification $\overline{\mathcal{G}_1}$ when the vertex set $\mathcal{V}_0 := \{v_3\}$ is removed from $\mathcal{G}$. Here, $\mathcal{I}_-(v_3) = \{i_2, i_3, i_7\}$, $\mathcal{I}_+(v_3) = \{i_6\}$, $\mathcal{E}(v_3) = \emptyset$. The new points introduced by the compactification are depicted in red.

Our main interest is the separability of $\mathcal{C}(\overline{\mathcal{G}_1})$. We start with the well-known result for $(\mathcal{C}([0,\infty]), \|\cdot\|_\infty)$ (which is, of course, incorrect for $(\mathcal{C}([0,\infty)), \|\cdot\|_\infty)$).

(18.16) Lemma. $(\mathcal{C}([0,\infty]), \|\cdot\|_\infty)$ *is separable.*

Proof. The classical Weierstrass approximation theorem yields that, for every $N \in \mathbb{N}$, there exists a countable, dense subset $\tilde{\mathscr{S}}_N$ of $\mathcal{C}([0,N])$. We extend all functions in this set constantly to $\mathcal{C}([0,\infty])$ by defining

$$\mathscr{S}_N := \{f\colon [0,\infty] \to \mathbb{R} \,|\, f|_{[0,N]} \in \tilde{\mathscr{S}}_N, \forall x > N : f(x) = f(N)\}, \quad N \in \mathbb{N}.$$

Of course, $\mathscr{S}_N$ has the same cardinality as $\tilde{\mathscr{S}}_N$, therefore

$$\mathscr{S} := \bigcup_{N\in\mathbb{N}} \mathscr{S}_N$$

is a countable subset of $\mathcal{C}([0,\infty])$. It remains to show that $\mathscr{S}$ is dense in $\mathcal{C}([0,\infty])$. To this end, let $h \in \mathcal{C}([0,\infty])$ and $\varepsilon > 0$. Then, as the limit at infinity exists, there is an

$N \in \mathbb{N}$ such that

$$\sup_{x\in[N,\infty]} \big|h(x) - h(\infty)\big| < \frac{\varepsilon}{4},$$

and as $h\big|_{[0,N]} \in \mathcal{C}([0,N])$, we can also find $f \in \mathscr{S}_N \subseteq \mathscr{S}$ with

$$\sup_{x\in[0,N]} \big|h(x) - f(x)\big| < \frac{\varepsilon}{4}.$$

But then

$$\begin{aligned}
\|h - f\|_\infty &\leq \sup_{x\in[0,N]} \big|h(x) - f(x)\big| + \sup_{x\in[N,\infty]} \big|h(x) - f(x)\big| \\
&\leq \sup_{x\in[0,N]} \big|h(x) - f(x)\big| + \sup_{x\in[N,\infty]} \big|h(x) - h(\infty)\big| \\
&\quad + \big|h(\infty) - h(N)\big| + \big|h(N) - f(N)\big| \\
&< \varepsilon,
\end{aligned}$$

where we used $f(x) = f(N)$ for all $x \geq N$ and the approximation properties established above. □

For the following result and proof, we will be naming $\overline{\mathcal{G}}$ for $\overline{\mathcal{G}_1}$, together with ∂ for ∂_1:

(18.17) Theorem. $(\mathcal{C}(\overline{\mathcal{G}}), \|\cdot\|_\infty)$ *is separable.*

Proof. We are able to approximate every continuous function on each separate edge $l \in \mathcal{L}$ of $\overline{\mathcal{G}}$ by functions in the respective separability set $\mathscr{S}^l$ of $\mathcal{C}([0,\rho_l])$. Thus, we only need to connect these functions continuously on the entire graph. It turns out sufficient to connect them by an easy linear interpolation.

To this end, we define for every choice of data $\delta > 0$, $(y^v \in \mathbb{R}, v \in \mathcal{V})$, $(f^l \in \mathscr{S}^l, l \in \mathcal{L})$ the function $f = (\delta, (y^v)_{v\in\mathcal{V}}, (f^l)_{l\in\mathcal{L}}) \colon \overline{\mathcal{G}} \to \mathbb{R}$ as follows: For $l \in \mathcal{E}$, we set

$$f(l,x) := \begin{cases} y^{\partial_-(l)}, & x = 0, \\ y^{\partial_-(l)} + \frac{x}{\delta}\left(f^l(\delta) - y^{\partial_-(l)}\right), & 0 < x \leq \delta, \\ f^l(x), & \delta < x, \end{cases}$$

whereas for $l \in \mathcal{I}$, we set

$$f(l,x) := \begin{cases} y^{\partial_-(l)}, & x = 0, \\ y^{\partial_-(l)} + \frac{x}{\delta}\left(f^l(\delta) - y^{\partial_-(l)}\right), & 0 < x \leq \delta, \\ f^l(x), & \delta < x \leq \rho_l - \delta, \\ y^{\partial_+(l)} + \frac{\rho_l - x}{\delta}\left(f^l(\rho_l - \delta) - y^{\partial_+(l)}\right), & \rho_l - \delta < x < \rho_l \\ y^{\partial_+(l)}, & x = \rho_l. \end{cases}$$

We collect all these functions in the set

$$\mathscr{S} := \{f = \big(\delta, (y^v)_{v\in\mathcal{V}}, (f^l)_{l\in\mathcal{L}}\big) \,|\, \delta \in \mathbb{Q}_{>0},\ y^v \in \mathbb{Q}, v \in \mathcal{V},\ f^l \in \mathscr{S}^l, l \in \mathcal{L}\}.$$

Being determined by a product of finitely many countable sets, $\mathscr{S}$ is countable and by construction a subset of $\mathcal{C}(\overline{\mathcal{G}})$. We show that it is dense in $\mathcal{C}(\overline{\mathcal{G}})$: Let $h \in \mathcal{C}(\overline{\mathcal{G}})$ and $\varepsilon > 0$. For each $l \in \mathcal{L}$, choose $f^l \in \mathscr{S}^l$ with

$$\sup_{g\in l} |h(g) - f^l(g)| < \frac{\varepsilon}{5},$$

for each $v \in \mathcal{V}$, let $y^v \in \mathbb{Q}$ satisfy

$$|h(v) - y^v| < \frac{\varepsilon}{5},$$

and choose $\delta \in \mathbb{Q}_{>0}$ such that

$$\max_{v\in\mathcal{V}} \sup_{g\in\overline{B_\delta(v)}} |h(g) - h(v)| < \frac{\varepsilon}{5}.$$

Then $f = \big(\delta, (y^v)_{v\in\mathcal{V}}, (f^l)_{l\in\mathcal{L}}\big) \in \mathscr{S}$ satisfies $\|h - f\| < \varepsilon$, because for all $l \in \mathcal{L}$, we have

- for $x = 0$:

$$\big|h(l,x) - f(l,x)\big| = \big|h\big(\partial_-(l)\big) - f\big(\partial_-(l)\big)\big| < \varepsilon,$$

- for $0 \le x \le \delta$: with $v = \partial_-(l)$,

$$\begin{aligned}\big|h(l,x) - f(l,x)\big| &= \big|h(l,x) - h(v)\big| + \big|h(v) - y^v\big| \\ &\qquad + \frac{x}{\delta}\left(\big|f(\delta) - h(l,\delta)\big| + \big|h(l,\delta) - h(v)\big| + \big|h(v) - y^v\big|\right) \\ &< \varepsilon,\end{aligned}$$

- if $l \in \mathcal{I}$, for $\delta < x \le \rho_l - \delta$, or if $l \in \mathcal{E}$, for $\delta < x$:

$$\big|h(l,x) - f(l,x)\big| = \big|f^l(x) - f(l,x)\big| < \varepsilon,$$

- if $l \in \mathcal{I}$, for $\rho_l - \delta < x < \rho_l$: with $v = \partial_+(l)$,

$$\begin{aligned}\big|h(l,x) - f(l,x)\big| &= \big|h(l,x) - h(v)\big| + \big|h(v) - y^v\big| \\ &\qquad + \frac{\rho_l - x}{\delta}\left(\big|f(\rho_l - \delta) - h(l,\rho_l - \delta)\big| + \big|h(l,\rho_l - \delta) - h(v)\big|\right. \\ &\qquad\qquad \left. + \big|h(v) - y^v\big|\right) \\ &< \varepsilon,\end{aligned}$$

- if $l \in \mathcal{I}$, for $x = \rho_l$:

$$\big|h(l,x) - f(l,x)\big| = \big|h\big(\partial_+(l)\big) - f\big(\partial_+(l)\big)\big| < \varepsilon.$$

□

19. Walsh's Brownian Motions on a Star Graph

Following the half-line case of section 16, the next step would be to turn to a metric graph with one vertex and two external edges, which is equivalent to consider the real line $\mathbb{R}$ with vertex point 0. Brownian motions have been studied for this setting in [IM63, Section 17], where Itô and McKean construct "skew Brownian motions" by taking the excursions from the origin of a reflecting Brownian motion and then choosing the sign of each excursion independently relative to the distribution $p^{-1}\,\varepsilon_{-1} + p^{+1}\,\varepsilon_{+1}$ with weights $p^{-1}, p^{+1} \geq 0$ satisfying $p^{-1} + p^{+1} = 1$. This results in a Brownian motion in the sense of definition (20.1): On both edges $(-\infty, 0)$, $(0, +\infty)$, the process will behave just like a reflecting Brownian motion, while "drifting" at the origin to either of the edges depending on the weights p_{-1}, p_{+1}. It is then seen that the domain of the generator $A = \frac{1}{2}\,\Delta$ reads[1]

$$\mathscr{D}(A) = \{ f \in \mathcal{C}_0^2(\mathbb{R}) : -p^{-1}\,f'(0-) + p^{+1}\,f'(0+) = 0 \},$$

which reduces to the standard one-dimensional Brownian motion on $\mathbb{R}$ in the symmetric case $p_{-1} = p_{+1} = \frac{1}{2}$.

In the general case of a star graph $\mathcal{G}$, that is a metric graph with a single vertex v and finitely many external edges $\mathcal{E}$, the conception of a "skew Brownian motion" $W = (W^1, W^2)$ with weights $(p_2^e, e \in \mathcal{E})$ proceeds completely analogously: Here, the edge W^1 of any excursion of the reflecting Brownian motion W^2 is chosen independently relative to a given distribution $\mu := \sum_{e \in \mathcal{E}} p_2^e\,\varepsilon_e$, see figure (19.1).

By embedding the geometric representation of the star graph as a subspace of $\mathbb{R}^2$, such a process turns out to be a specialization of the so called "Walsh processes", which are defined to be stochastic processes $W = (W^1, W^2)$, expressed in polar coordinates of $\mathbb{R}^2$, for which the "radial part" W^2 is a reflecting Brownian motion, and for any excursion of W^2 from the origin, the "ray" W^1 is constant and chosen independently of W^2 by a general distribution μ on $[0, 2\pi)$. Such processes were first proposed by Walsh in [Wal78], who introduced them for examinations on Brownian local time. Since then, they have been applied in various fields, such as in studies on Brownian filtrations and on the generalization of typically one-dimensional results on Brownian motion, like local time characterizations and arcsine laws, to higher dimensions. For a survey we refer the reader to [BPY89], which still seems to be a main reference for Walsh processes and which also provides most of the results needed here.

Nowadays, these "skew Brownian motions" are especially used as a prototype class for Brownian motions on metric graphs. For instance, [Jeh09] and [FK14] analyze their harmonic functions in order to extend their characteristics to general metric graphs in [FK15]. We are going to use them as a main building block for general Brownian motions on star graphs in section 21, which are then "glued together" to metric graphs in section 22. As we will only consider processes on graphs, there will be no confusion when we use the term "Walsh Brownian motions" for the restriction of the "general Walsh processes" on $\mathbb{R}^2$ to the star graph case.

[1] With $\mathcal{C}_0^2(\mathbb{R})$ being defined in this case in the sense of definition (18.12) for the metric graph $\mathbb{R} = \{0\} \cup (-\infty, 0) \cup (0, +\infty)$ with vertex 0.

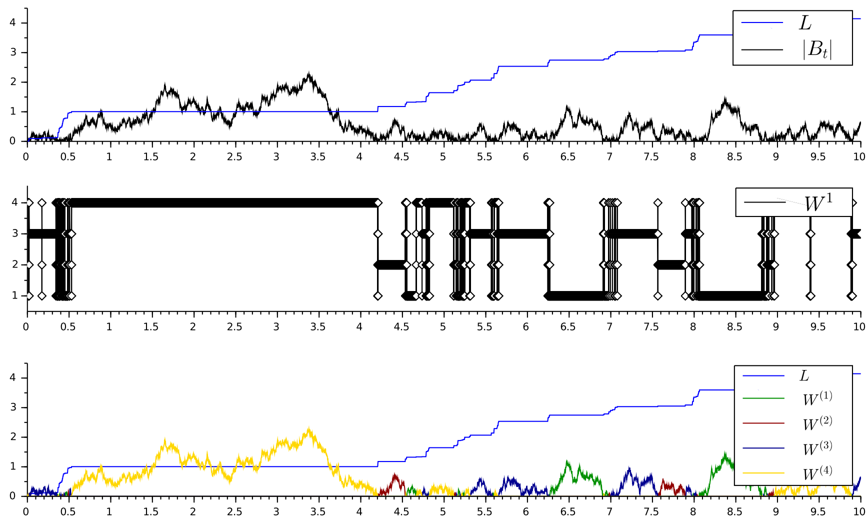


Figure 19.1: Construction of a Walsh Brownian motion on a star graph: Starting with a reflecting Brownian motion $|B|$ on $\mathbb{R}_+$ with local time L at the origin, choose for each excursion of $|B|$ an edge independently with respect to some distribution μ, resulting in the edge process W^1. Then $\big((W^1_t, |B_t|), t \geq 0\big)$ is a Walsh Brownian motion with local time L at the star vertex. The parts $W^{(e)}$, $e \in \mathcal{E}$, in the above graph indicate on which of the edges $\mathcal{E} = \{1, 2, 3, 4\}$ the Walsh Brownian motion is currently running.

19.1. Definition

Let $\mathcal{G} = \{v\} \cup \bigcup_{e \in \mathcal{E}} \big(\{e\} \times (0, \infty)\big)$ be a star graph with star vertex $v \equiv \{(e, 0), e \in \mathcal{E}\}$. As the Walsh Brownian motion is only defined illustratively or with the help of excursion theory in most of the older works, we follow [FK14, Definition 2.1] for a rigorous context:

(19.1) Definition. *A strong Markov process $W = (W^1, W^2)$ on $\mathcal{G}$ is a Walsh Brownian motion (or Walsh process) on $\mathcal{G}$ with weights $(p^e_2, e \in \mathcal{E})$, if $p^e_2 \geq 0$ for all $e \in \mathcal{E}$ and $\sum_{e \in \mathcal{E}} p^e_2 = 1$, and with $\mu := \sum_{e \in \mathcal{E}} p^e_2 \, \varepsilon_e$, the process W satisfies:*

(i) W^2 is a reflecting Brownian motion on $\mathbb{R}_+$;

(ii) if $W_0 = v$, then for $t > 0$, the distribution of W^1_t is given by μ;

(iii) if $W_0 = (e, x)$ with $x > 0$, then $W^1_t = e$ holds for all $t < H_v$, and on $t > H_v$, the distribution of W^1_t is equal to μ and independent of $(W^2_t, t \geq 0)$.

For a Walsh process $W_t = (W^1_t, W^2_t)$, $t \geq 0$, we will denote the "radial process" by

$$|W_t| := W^2_t, \quad t \geq 0.$$

[BPY89] contains a list of various existence proofs. In this paper, the authors first gain insight into the structure of the semigroup of a Walsh Brownian motion (see lemma (19.2) below), which they then use to derive a Feller process W satisfying the conditions of definition (19.1). A more natural approach in view of the above process description is the construction via the application of Itô excursion theory [Itô72], generalized from the skew Brownian motion [Sal86, Example 5.7] to the star graph case. [Lej06] gives a comprehensive survey on construction methods for skew Brownian motions. Details on the construction in the context of star graphs can also be found in [FK14, Section 2].

19.2. Basic Results

The semigroup of the Walsh Brownian motion can be obtained using its strong Markov property at the first hitting time of the vertex. The process then decomposes into a one-dimensional Brownian motion on the starting vertex killed on hitting the origin, followed by a reflecting Brownian motion on the edges chosen by the weight distribution $\mu = \sum_{e\in\mathcal{E}} p_2^e \, \varepsilon_e$. The closed form of the semigroup is given in [Wal78, Equations (2.1)–(2.2)] in a more general context. By inserting the discrete distribution μ, we get:

(19.2) Lemma. *The semigroup $(T_t^W, t \geq 0)$ of the Walsh process reads for all $f \in b\mathscr{B}(\mathcal{G})$, $t \geq 0$, $(l, x) \in \mathcal{G}$:*

$$T_t^W f(l,x) = \sum_{e\in\mathcal{E}} p_2^e \Big(T_t^{|B|} f(e,\,\cdot\,) + T_t^{[0,\infty)} \big(f(l,\,\cdot\,) - f(e,\,\cdot\,) \big) \Big)(x),$$

with $(T_t^{|B|}, t \geq 0)$, $(T_t^{[0,\infty)}, t \geq 0)$ being the semigroups of the reflecting Brownian motion, the standard Brownian motion killed when hitting the origin respectively, as introduced in examples (16.2) *and* (16.3).

In particular, we have $T_t^W f(v) = \sum_{e\in\mathcal{E}} p_2^e \, T_t^{|B|} f(e,\,\cdot\,)(0)$, so the resolvent of the Walsh process at the star vertex v is obtained with the help of example (16.2):

$$\begin{aligned} U_\alpha^W f(v) &= \sum_{e\in\mathcal{E}} p_2^e \, U_\alpha^{|B|} f(e,\,\cdot\,)(0) \\ &= \sum_{e\in\mathcal{E}} p_2^e \, \frac{2}{\sqrt{2\alpha}} \int_0^\infty e^{-\sqrt{2\alpha}x} f(e,x)\,dx. \end{aligned} \tag{19.3}$$

As the semigroups of reflected and killed Brownian motion are Feller semigroups, the Feller property of the Walsh Brownian motion is immediate (cf. [BPY89, Theorem 2.1]):

(19.4) Theorem. *$(T_t^W, t \geq 0)$ is a Feller semigroup on $\mathcal{G}$.*

Furthermore, the closed form (19.3) of the resolvent yields:

(19.5) Theorem. *The generator of W reads $A = \frac{\Delta}{2}$, with domain*

$$\mathscr{D}(A) = \{ f \in \mathcal{C}_0^2(\mathcal{G}) : \sum_{e\in\mathcal{E}} p_2^e f_e'(v) = 0 \}.$$

We will always work with a continuous version of the Walsh Brownian motion, whose existence is obvious when constructed via Itô excursion theory, but which can also be obtained from the semigroup considerations of [BPY89, Lemmas 2.2, 2.3, Theorem 2.4]:

(19.6) Theorem. *There exists a version $(W_t, t \geq 0)$ of the Walsh Brownian motion on the star graph $\mathcal{G}$ which is continuous, and for which $(|W_t|, t \geq 0)$ is a reflecting Brownian motion on $\mathbb{R}_+$.*

Therefore, properties which only depend on $|W|$ or on the behavior of W on one edge can be derived from the respective properties of a Brownian motion on $\mathbb{R}$ or on $\mathbb{R}_+$. For instance, the passage time formulas of subsection 14.3 can be used in appropriate cases for the Walsh Brownian motion W as well.

As the edge process $(W_t^1, t \geq 0)$ is independent of the radial process $(W_t^2 = |W_t|, t \geq 0)$ (and thus of its local time), the following result is a direct consequence of theorem (15.4):

(19.7) Lemma. *The joint distribution of (W_t, L_t), $t \geq 0$, at the star vertex v is given by*

$$\mathbb{E}_v^W\big(f(W_t, L_t)\big) = \sum_{e \in \mathcal{E}} p_2^e \int_0^\infty \int_0^\infty f\big((e,x),y\big)\, \frac{2(x+y)}{\sqrt{2\pi t^3}}\, e^{-\frac{(x+y)^2}{2t}}\, dx\, dy, \quad f \in \mathscr{B}(\mathcal{G}).$$

(19.8) Example. Consider the “Dirichlet Walsh process” W^D, that is the Walsh process W killed at the first hitting time of the star vertex v: With $H_v := \inf\{t \geq 0 : W_t = v\}$, it is defined by

$$W_t^D := \begin{cases} W_t, & t < H_v, \\ \Delta, & t \geq H_v. \end{cases}$$

By theorem (19.6), the Walsh process W just behaves like a standard (reflecting) Brownian motion on the starting edge until hitting the star vertex. So the Dirichlet Walsh process W^D, with fixed starting edge, equals the Dirichlet process $B^{[0,\infty)}$ on the half line (see example (16.3)). Therefore, when identifying $\{(e,\Delta), e \in \mathcal{E}\} \equiv \Delta$, we get $\mathbb{P}_{(e,x)}$-a.s. for any $(e,x) \in \mathcal{G}$:

$$\forall t \geq 0 : W_t^D = \big(e, B_t^{[0,\infty)}\big).$$

Thus, the resolvent of W^D reads, for $\alpha > 0$, $f \in b\mathscr{B}(\mathcal{G})$, $(e,x) \in \mathcal{G}$,

$$U_\alpha^{W,D} f(e,x) = U_\alpha^{[0,\infty)}\big(f(e,\,\cdot\,)\big)(x).$$

Our findings of example (16.3) imply that $(U_\alpha^{W,D}, \alpha > 0)$ preserves $\mathcal{C}_0(\mathcal{G})$. Furthermore, they give

$$\begin{aligned} U_\alpha^{W,D} f'(e,0+) &= 2 \int_0^\infty e^{-\sqrt{2\alpha}x}\, f(e,x)\, dx, \\ U_\alpha^{W,D} f''(v) &= -2 f(v). \end{aligned}$$

The domain of the generator then reads

$$\mathscr{D}(A^D) = \{f \in \mathcal{C}_0^2(\mathcal{G}) : f(v) = 0\}.$$

For later use, we also remark that for all $(e,x) \in \mathcal{G}$,

$$U_\alpha^{W,D} 1(e,x) = \mathbb{E}_x^B\Big(\int_0^{H_0} e^{-\alpha t}\, dt\Big) = \frac{1}{\alpha}\, \mathbb{E}_x^B\big(1 - e^{-\alpha H_0}\big) = \frac{1}{\alpha}\, \big(1 - e^{-\sqrt{2\alpha}x}\big). \qquad \blacksquare$$

20. Brownian Motions on Metric Graphs

We are ready to introduce and study the main class of stochastic processes of this thesis. As already explained in the introduction, it is suitable to characterize Brownian motions on metric graphs by their generators, which will be the goal of this section.

After finally giving the rigorous definition of a "Brownian motion on a metric graph", we collect some basic properties of such a process by utilizing its locally "one-dimensional Brownian behavior" on the edges and by applying our previous findings for the half-line and interval cases. We are then able to analyze the resolvents—yielding their Feller property—and the generators of Brownian motions on metric graphs, giving explicit formulas for the computation of their "Feller–Wentzell" boundary conditions.

These results constitute the fundamental basis for the pathwise constructions given in the upcoming sections 21 and 22.

20.1. Definition

Following the definitions of the half-line and interval cases (16.1), (17.1), and thus extending the definition of [KPS12a] to the discontinuous setting, we define a Brownian motion on a metric graph $\mathcal{G}$ to be a right continuous, strong Markov process on $\mathcal{G}$ which behaves on every edge like the standard one-dimensional Brownian motion. That is, the local coordinate of such a process, if stopped once it leaves its starting edge, needs to be equivalent to the Brownian motion on $\mathbb{R}$, stopped when leaving the corresponding interval of the process' initial edge:

(20.1) Definition. *Let $X = (\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ be a right continuous, strong Markov process on a metric graph $\mathcal{G}$. X is a Brownian motion on $\mathcal{G}$, if for all $g = (l,x) \in \mathcal{G}$, the random time*

$$H_X := \inf\big\{t \geq 0 : X_t \notin l^0\big\}, \quad \text{with } l^0 = \{l\} \times (0, \rho_l),$$

is a stopping time over $(\mathscr{G}_t, t \geq 0)$, and for all $n \in \mathbb{N}$, $f_1, \dots, f_n \in b\mathscr{B}(\mathcal{G})$, $t_1, \dots, t_n \in \mathbb{R}_+$,

$$\mathbb{E}_{(l,x)}\big(f_1(X_{t_1 \wedge H_X}) \cdots f_n(X_{t_n \wedge H_X})\big) = \mathbb{E}_x^B\big(f_1(l, B_{t_1\wedge H_B}) \cdots f_n(l, B_{t_n \wedge H_B})\big)$$

holds, with B being the Brownian motion on $\mathbb{R}$ and $H_B := \inf\big\{t \geq 0 : B_t \notin (0, \rho_l)\big\}$.

The technical requirement of the first hitting time H_X of the closed set $\complement l^0$ being a stopping time is always satisfied if we are working in the context of usual hypotheses (cf. theorem (3.8)). It can also be achieved if we ensure the *continuity* of the process X until H_X (see lemma (3.7)), that is, continuity while the process runs inside any edge. While the latter condition is not implied by the above definition, it is a desirable property which may be implemented by constructing a Brownian motion on a metric graph with the help of continuous excursions of a “standard” one-dimensional Brownian motion, as done in sections 21 and 22.

20.2. Basic Properties

We first need to collect some basic properties of Brownian motions on metric graphs. Most of them are implicitly used without proof in earlier works, such as in [IM63], [Kni81], or [KPS12a], and may be attained quite easily in the continuous setting. However, it seems to us that a little bit more care is needed for discontinuous Brownian motions. For instance, it is not evident from its very definition that a Brownian motion on a metric graph will (a.s.) behave continuously during an excursion on some edge.

For all that follows, let X be a Brownian motion on a metric graph $\mathcal{G}$, H_X be the first exit time from $l^0 = \{l\} \times (0, \rho_l)$ for a given initial point $g = (l, x) \in \mathcal{G}$, as well as B be the one-dimensional Brownian motion with the first exit time H_B from the corresponding edge interval $(0, \rho_l)$, as specified in definition (20.1). As usual, we identify any edge $l \in \mathcal{L}$ with its geometric representation $\{l\} \times [0, \rho_l]$, where we set $[0, \rho_l] := [0, +\infty)$ if $\rho_l = +\infty$.

We start with some basic results on H_X:

(20.2) Lemma. *For all $t \geq 0$,*

$$\{H_X \leq t\} = \{X_{t \wedge H_X} \in \complement l^0\} \quad \text{and} \quad \{H_B \leq t\} = \{B_{t \wedge H_B} \in \complement (0, \rho_l)\}.$$

Proof. For any right continuous process Y on $(E, \mathscr{E})$ and every debut H_A of a closed set $A \in \mathscr{E}$, it is $Y_{H_A} \in A$. Thus, if $H_A \leq t$, then

$$Y_{t \wedge H_A} = Y_{H_A} \in A.$$

On the other hand, if $Y_{t \wedge H_A} \in A$, then $H_A \leq (t \wedge H_A) \leq t$.

Now apply this general result to $Y := X$, $A := \complement l^0$, and to $Y := B$, $A := \complement (0, \rho_l)$. □

(20.3) Corollary. *For all $g = (l, x) \in \mathcal{G}$,*

$$\mathbb{P}_{(l,x)} \circ H_X^{-1} = \mathbb{P}_x^B \circ H_B^{-1},$$

especially

$$\mathbb{P}_{(l,x)}(H_X < +\infty) = \mathbb{P}_x^B(H_B < +\infty) = 1.$$

These results will be considerately improved in theorem (20.8) below. For the time being, they are sufficient to deduce a slightly more general property of the distributions of the stopped Brownian motion:

(20.4) Lemma. *For all $g=(l,x)\in\mathcal{G}$, $n\in\mathbb{N}$, $f_1,\dots,f_n,h\in b\mathscr{B}(\mathcal{G})$, $0\le t_1\le\dots\le t_n$,*

$$\begin{aligned}&\mathbb{E}_{(l,x)}\big(f_1(X_{t_1})\cdots f_n(X_{t_n})\,h(X_{H_X});t_n<H_X\big)\\&=\mathbb{E}^B_x\big(f_1(l,B_{t_1})\cdots f_n(l,B_{t_n})\,h(l,B_{H_B});t_n<H_B\big).\end{aligned}$$

Proof. Observe that, because $H_X<+\infty$ a.s. (see corollary (20.3)) and $X_{s\wedge H_X}=X_{H_X}$ holds for all $s\ge H_X$, we have

$$\lim_{s\to\infty}h(X_{s\wedge H_X})=h(X_{H_X})\quad\text{a.s.},$$

and analogously,

$$\lim_{s\to\infty}h(l,B_{s\wedge H_B})=h(l,B_{H_B})\quad\text{a.s.}.$$

Thus, by using LDCT and the definition of a Brownian motion on a metric graph, we conclude that

$$\begin{aligned}&\mathbb{E}_{(l,x)}\big(f_1(X_{t_1})\cdots f_n(X_{t_n})\,h(X_{H_X});t_n<H_X\big)\\&=\lim_{s\to\infty}\mathbb{E}_{(l,x)}\big(f_1(X_{t_1\wedge H_X})\cdots f_n(X_{t_n\wedge H_X})\,h(X_{s\wedge H_X})\,\mathbb{1}_{l^0}(X_{t\wedge H_X})\big)\\&=\lim_{s\to\infty}\mathbb{E}^B_x\big(f_1(l,B_{t_1\wedge H_B})\cdots f_n(B_{t_n\wedge H_B})\,h(l,B_{s\wedge H_B})\,\mathbb{1}_{l^0}(l,B_{t\wedge H_B})\big)\\&=\mathbb{E}^B_x\big(f_1(l,B_{t_1})\cdots f_n(l,B_{t_n})\,h(l,B_{H_B});t_n<H_B\big).\end{aligned}$$ □

This lemma allows us to achieve equivalent defining properties for Brownian motions on metric graphs. They will turn out to be more suitable for our work, as they are based on the (partial) resolvent and the exit behavior of the process rather than on its stopped distributions:

(20.5) Theorem. *Let X be a right continuous, strong Markov process on $\mathcal{G}$. X is a Brownian motion on $\mathcal{G}$, if and only if for all $g=(l,x)\in\mathcal{G}$, the following assertions hold:*

(i) *for all $\alpha>0$, $f\in b\mathscr{B}(\mathcal{G})$,*

$$\mathbb{E}_{(l,x)}\Big(\int_0^{H_X}e^{-\alpha t}f(X_t)\,dt\Big)=\mathbb{E}^B_x\Big(\int_0^{H_B}e^{-\alpha t}f(l,B_t)\,dt\Big);$$

(ii) $\mathbb{P}_{(l,x)}\circ\big(H_X,X_{H_X}\big)^{-1}=\mathbb{P}^B_x\circ\big(H_B,(l,B_{H_B})\big)^{-1}$.

Proof. Necessity follows directly from lemma (20.4).

Now let (i) and (ii) hold true. As X and B are right continuous, strong Markov processes and H_X, H_B are debuts of closed sets, the stopped processes $X_{\cdot\wedge H_X}$, $B_{\cdot\wedge H_B}$ are indeed right continuous, strong Markov processes (see section 8). Let $(\widetilde{T}_t,t\ge 0)$ and $(\widetilde{T}^B_t,t\ge 0)$ be their respective semigroups, that is, consider for $f\in b\mathscr{B}(\mathcal{G})$ and $f_l:=f(l,\cdot)\in b\mathscr{B}([0,\rho_l])$:

$$\begin{aligned}\widetilde{T}_tf(l,x)&=\mathbb{E}_{(l,x)}\big(f(X_{t\wedge H_X})\big),\\\widetilde{T}^B_tf_l(x)&=\mathbb{E}^B_x\big(f_l(B_{t\wedge H_X})\big).\end{aligned}$$

As the stopped process $X_{\cdot \wedge H_X}$ is strongly Markovian, Dynkin's formula (3.16) for decomposition of its resolvent at H_X gives for all $\alpha > 0$:

$$\begin{aligned}\int_0^\infty e^{-\alpha t}\, \widetilde{T}_t f(l,x)\, dt &= \mathbb{E}_{(l,x)}\Big(\int_0^{H_X} e^{-\alpha t}\, f(X_t)\, dt\Big) \\ &\quad + \mathbb{E}_{(l,x)}\Big(e^{-\alpha H_X}\, \mathbb{E}_{X_{H_X}}\Big(\int_0^\infty e^{-\alpha t}\, f(X_{t\wedge H_X})\, dt\Big)\Big).\end{aligned}$$

With $X_{H_X} \in \complement l^0$, we have $H_X = 0$ $\mathbb{P}_{X_{H_X}}$-a.s., thus the above decomposition becomes

$$\int_0^\infty e^{-\alpha t}\, \widetilde{T}_t f(l,x)\, dt = \mathbb{E}_{(l,x)}\Big(\int_0^{H_X} e^{-\alpha t}\, f(X_t)\, dt\Big) + \frac{1}{\alpha}\, \mathbb{E}_{(l,x)}\Big(e^{-\alpha H_X}\, f(X_{H_X})\Big).$$

Analogously, we get by decomposing the resolvent of $B_{\cdot \wedge H_B}$:

$$\int_0^\infty e^{-\alpha t}\, \widetilde{T}^B_t f_l(x)\, dt = \mathbb{E}^B_x\Big(\int_0^{H_B} e^{-\alpha t}\, f(l, B_t)\, dt\Big) + \frac{1}{\alpha}\, \mathbb{E}^B_x\Big(e^{-\alpha H_B}\, f(l, B_{H_B})\Big).$$

Using (i) and (ii) immediately yields

$$\int_0^\infty e^{-\alpha t}\, \widetilde{T}_t f(l,x)\, dt = \int_0^\infty e^{-\alpha t}\, \widetilde{T}^B_t f_l(x)\, dt,$$

holding true for all $\alpha > 0$ and all $f \in b\mathcal{C}(\mathcal{G})$, $(l,x) \in \mathcal{G}$. As the mappings $t \mapsto \widetilde{T}_t f(l,x)$ and $t \mapsto \widetilde{T}^B_t f_l(x)$ are right continuous, the uniqueness theorem for Laplace transforms (cf. [Dyn65, Lemma 1.1]) asserts that

$$\forall t \geq 0: \quad \widetilde{T}_t f(l,x) = \widetilde{T}^B_t f_l(x).$$

As $X_{\cdot \wedge H_X}$, $B_{\cdot \wedge H_B}$ are Markov processes with the "same" semigroup, we are able to show inductively that for all $(l,x) \in \mathcal{G}$, $f_1, \ldots, f_n \in b\mathcal{C}(\mathcal{G})$, $0 \leq t_1 \leq \cdots \leq t_n$,

$$\begin{aligned}&\mathbb{E}_{(l,x)}\big(f_1(X_{t_1\wedge H_X}) \cdots f_n(X_{t_n \wedge H_X})\big) \\ &= \mathbb{E}_{(l,x)}\big(f_1(X_{t_1\wedge H_X}) \cdots f_{n-1}(X_{t_{n-1}\wedge H_X})\, \mathbb{E}_{X_{t_{n-1}\wedge H_X}}\big(f_n(X_{(t_n - t_{n-1})\wedge H_X})\big)\big) \\ &= \mathbb{E}_{(l,x)}\big(f_1(X_{t_1\wedge H_X}) \cdots f_{n-1}(X_{t_{n-1}\wedge H_X})\, \widetilde{T}_{t_n - t_{n-1}} f_n(X_{t_{n-1}\wedge H_X})\big) \\ &= \cdots \\ &= \mathbb{E}^B_x\big(f_1(l, B_{t_1\wedge H_B}) \cdots f_{n-1}(l, B_{t_{n-1}\wedge H_B})\, \widetilde{T}^B_{t_n - t_{n-1}}\big(f_n(l,\,\cdot\,)\big)(B_{t_{n-1}\wedge H_B})\big) \\ &= \mathbb{E}^B_x\big(f_1(l, B_{t_1\wedge H_B}) \cdots f_n(l, B_{t_n\wedge H_B})\big),\end{aligned}$$

which is easily extended to $f_1, \ldots, f_n \in b\mathscr{B}(\mathcal{G})$ by using the MCT. $\square$

With the help of this theorem, we can further refine the properties of the first exit time H_X. Indeed, despite of its potential discontinuities, the Brownian motion can only exit its initial edge by hitting vertices incident with it:

(20.6) Corollary. *For all $g = (l,x) \in \mathcal{G}$,*

$$H_X = H_{\partial(l)} \quad \mathbb{P}_{(l,x)}\text{-a.s.}.$$

Proof. As $\partial(l) \subseteq \complement l^0$, we always have $H_X = H_{\complement l^0} \le H_{\partial(l)}$.
Using (ii) of theorem (20.5) gives

$$\mathbb{P}_{(l,x)}\big(X_{H_X} \in l\big) = \mathbb{P}_x^B\big(B_{H_B} \in [0, \rho_l]\big) = 1.$$

On the other hand, $X_{H_X} \in \complement l^0$ holds, as $\complement l^0$ is closed and X is right continuous. So we conclude that $X_{H_X} \in l \cap \complement l^0 = \partial(l)$ a.s., which results in $H_{\partial(l)} \le H_X$ a.s.. □

It immediately follows that

$$\forall t \ge 0: \quad X_{t \wedge H_X} \in l \quad \mathbb{P}_{(l,x)}\text{-a.s.},$$

because if otherwise $X_{t \wedge H_X} \in \complement l \subseteq \complement l^0$, then $H_X \le t \wedge H_X$ and so $X_{H_X} = X_{t \wedge H_X} \notin l$, contradicting to $X_{H_X} = X_{H_{\partial(l)}} \in \partial(l) \subseteq l$.

This seemingly small result implies that any Brownian motion, stopped on leaving the open interior of its starting edge, remains on this edge (especially at the exit time):

(20.7) Theorem. *For all $g = (l, x) \in \mathcal{G}$,*

$$\mathbb{P}_{(l,x)}\big(\forall t \ge 0 : X_{t \wedge H_X} \in l\big) = 1.$$

Proof. We are going to use the section theorem, cf. [DM78, IV-83, p. 137f]. Assume the contrary, that is,

$$\mathbb{P}_{(l,x)}\big(\exists t \ge 0 : X_{t \wedge H_X} \notin l\big) > 0.$$

Consider the optional set

$$A := \big\{(t, x) \in \mathbb{R}_+ \times \Omega : X_{t \wedge H_X}(\omega) \notin l\big\},$$

and the projection $\pi\colon \mathbb{R}_+ \times \Omega \to \Omega$ onto the second coordinate. Then, by the assumption, there exists $\varepsilon > 0$ such that

$$\mathbb{P}\big(\pi(A)\big) > \varepsilon.$$

Now, the section theorem asserts that there exists a stopping time R with

(i) for all $\omega \in \Omega$ with $R(\omega) < +\infty$: $\big(R(\omega), \omega\big) \in A$, that is, $X_{R \wedge H_X}(\omega) \notin l$, and

(ii) $\mathbb{P}_{(l,x)}(R < +\infty) \ge \mathbb{P}\big(\pi(A)\big) - \varepsilon > 0$.

Especially, we have $\mathbb{P}_{(l,x)}(X_{R \wedge H_X} \notin l) \ge \mathbb{P}_{(l,x)}(R < +\infty) > 0$.
However, we are going to show that for every stopping time R,

$$\mathbb{P}_{(l,x)}\big(X_{R \wedge H_X} \notin l\big) = 0$$

holds true, which yields a contradiction to the above: Because $X_{H_X} = X_{H_{\partial(l)}} \in \partial(l) \subseteq l$,

$$\mathbb{P}_{(l,x)}\big(X_{R \wedge H_X} \notin l; R \ge H_X\big) = \mathbb{P}_{(l,x)}\big(X_{H_X} \notin l; R \ge H_X\big) = 0,$$

so it remains to compute

$$\begin{aligned}\mathbb{P}_{(l,x)}\big(X_{R\wedge H_X}\notin l\big)&=\mathbb{P}_{(l,x)}\big(X_R\notin l, R<H_X\big)\\&\le\mathbb{P}_{(l,x)}\big(X_R\in\complement l^0, R<H_X\big)\\&=\mathbb{E}_{(l,x)}\big(\mathbb{P}_{X_R}(H_X=0);R<H_X\big),\end{aligned}$$

where in the last step we used the fact that for all $g\in\mathcal{G}$,

$$\mathbb{P}_g(H_X=0)=\left\{\begin{matrix}1, & g\in\complement l^0\\ 0, & g\in l^0\end{matrix}\right\}=\mathbb{1}_{\complement l^0}(g),$$

which is an immediate consequence of H_X being the debut of the closed set $\complement l^0$ for right right continuous, normal process X. Next, the strong Markov property of X implies $\mathbb{P}_{X_R}(H_X=0)=\mathbb{P}_{(l,x)}(H_X\circ R=0\,|\,\mathscr{F}_{R+})$, so by using this together with the terminal time property of H_X and $\{R<H_X\}\in\mathscr{F}_R$ (see, e.g., [BG69, Proposition I.6.8]), we get

$$\mathbb{P}_{(l,x)}\big(X_{R\wedge H_X}\notin l\big)=\mathbb{P}_{(l,x)}\big(H_X=R,R<H_X\big)=0.\qquad\square$$

We are now able to restrict our attention to the initial edge (and thus to its local coordinate) of the Brownian motion when considering the process stopped on leaving this edge, allowing us to gain full insight into its exit distributions.

In the following results, we set as usual $[0,\rho_l]:=[0,+\infty)$ if $\rho_l=+\infty$.

(20.8) Theorem. *Let X be a Brownian motion on $\mathcal{G}$, B be the standard one-dimensional Brownian motion, as well as $\pi^2\colon\mathcal{G}\to\mathbb{R}_+$ be the projection onto the local coordinate. Then for every $g=(l,x)\in\mathcal{G}$, and for $A\in\mathscr{B}(\mathcal{G})$ with $A':=\pi^2(A\cap l)\subseteq[0,\rho_l]$ being open (in the topology of $[0,\rho_l]$), the following holds true:*

$$\mathbb{P}_{(l,x)}\circ\big(H_A^{X'},X'_{H_A^{X'}}\big)^{-1}=\mathbb{P}^B_x\circ\big(H^{B'}_{A'},(l,B'_{H^{B'}_{A'}})\big)^{-1},$$

where $X':=X_{\cdot\wedge H_X}$, $B':=B_{\cdot\wedge H_B}$, and $H_A^{X'}$, $H_{A'}^{B'}$ are the first hitting times of A, A' for X', B' respectively.

Proof. It follows from theorem (20.7) that $\widetilde{X}':=\pi^2(X_{\cdot\wedge H_X})$ is a right continuous process with values in $[0,\rho_l]$, having the same finite dimensional distributions as $B'=B_{\cdot\wedge H_B}$.

Let Y be the canonical right continuous coordinate process on $[0,\rho_l]$, and define the path mappings $\Phi^{\widetilde{X}'}$ and $\Phi^{B'}$ from $\widetilde{X}'$ and B' to the space of all right continuous maps $\mathbb{R}_+\to[0,\rho_l]$, as given in subsection 7.2. Especially, we have

$$\forall t\ge0:\quad Y_t\circ\Phi^{\widetilde{X}'}=\widetilde{X}'_t\quad\text{and}\quad Y_t\circ\Phi^{B'}=B'_t.$$

Consider the debut of $A'\in\mathscr{B}\big([0,\rho_l]\big)$ for Y:

$$H_{A'}^Y:=\inf\{t\ge0:Y_t\in A'\}.\tag{20.9}$$

$H^Y_{A'}$ and $Y_{H^Y_{A'}}$ are $\mathscr{F}^Y_\infty$-measurable (as the hitting time of any open set is a stopping time over $(\mathscr{F}^Y_{t+}, t \geq 0)$, cf. section 3). If $A \in \mathscr{B}(\mathcal{G})$ with $\pi^2(A \cap l) = A'$, then we have

$$\begin{aligned} H^Y_{A'} \circ \Phi^{\widetilde{X}'} &= \inf\{t \geq 0 : Y_t \circ \Phi^{\widetilde{X}'} \in A'\} \\ &= \inf\{t \geq 0 : \pi^2(X_{t \wedge H_X}) \in \pi^2(A \cap l)\} \\ &= \inf\{t \geq 0 : X_{t \wedge H_X} \in A\} \\ &= H^{X'}_A, \end{aligned}$$

where we used theorem (20.7) for the third identity. This gives for any $\omega^X \in \Omega^X$:

$$\begin{aligned} Y_{H^Y_{A'}} \circ \Phi^{\widetilde{X}'}(\omega^X) &= Y_{H^Y_{A'}(\Phi^{\widetilde{X}'}(\omega^X))}(\Phi^{\widetilde{X}'}(\omega^X)) \\ &= \widetilde{X}'_{H^{X'}_A}(\omega^X) \\ &= \pi^2(X'_{H^{X'}_A})(\omega^X). \end{aligned}$$

Analogously, we get

$$H^Y_{A'} \circ \Phi^{B'} = H^{B'}_{A'} \quad \text{and} \quad Y_{H^Y_{A'}} \circ \Phi^{B'} = B'_{H^{B'}_{A'}}.$$

Thus, for any $f \in \mathscr{B}([0,+\infty]) \otimes \mathscr{B}([0,\rho_l])$, setting $G := f(H^Y_{A'}, Y_{H^Y_{A'}}) \in \mathscr{F}^Y_\infty$ gives

$$\begin{aligned} \mathbb{E}_{(l,x)}\big(f(H^{X'}_A, \pi^2(X'_{H^{X'}_A}))\big) &= \mathbb{E}_{(l,x)}(G \circ \Phi^{X'}) \\ &= \mathbb{E}^B_x(G \circ \Phi^{B'}) \\ &= \mathbb{E}^B_x\big(f(H^{B'}_{A'}, B'_{H^{B'}_{A'}})\big), \end{aligned}$$

which together with theorem (20.7) concludes the proof. □

(20.10) Remark. As easily observed in its proof, the above theorem can also be stated for any $A \in \mathscr{B}(\mathcal{G})$ with $A' := \pi^2(A \cap l) \in \mathscr{B}([0,\rho_l])$, as long as the first hitting time $H^Y_{A'}$ of A', as defined in (20.9), attains $\mathscr{F}^Y_\infty$-measurability, with $\mathscr{F}^Y_\infty = \sigma(Y_t, t \geq 0)$ being the σ-algebra generated by a suitable coordinate process Y on $[0,\rho_l]$.

For instance, this is the case if A is a closed set and the Brownian motion X is known to be continuous up to the hit of A, cf. lemma (3.7), as we can then consider the continuous canonical coordinate process Y in the proof instead. ■

We are usually interested in the exit distributions of the "original" Brownian motion X on a metric graph instead of the stopped process X', so we lift the results of theorem (20.8) from X' to X (the same remark on the limitation to open subsets A' also applies here):

(20.11) Corollary. *Let $g = (l,x) \in \mathcal{G}$, $A \in \mathscr{B}(\mathcal{G})$.*

(i) If $A \subseteq l$ and $A' := \pi^2(A) \subseteq [0,\rho_l]$ is open, then theorem (20.8) *holds true.*

(ii) If $A \subseteq l^0$ and $A' := \pi^2(A^\complement \cap l) \subseteq [0, \rho_l]$ is open, then

$$\mathbb{P}_{(l,x)} \circ \left(H^X_{\complement A}, X_{H^X_{\complement A}}\right)^{-1} = \mathbb{P}^B_x \circ \left(H^B_{A'}, (l, B_{H^B_{A'}})\right)^{-1}.$$

Proof. (i) The requirements of theorem (20.8) are fulfilled, as $A' = \pi^2(A) = \pi^2(A \cap l)$.

(ii) Theorem (20.8) gives

$$\mathbb{P}_{(l,x)} \circ \left(H^{X'}_{\complement A}, X'_{H^{X'}_{\complement A}}\right)^{-1} = \mathbb{P}^B_x \circ \left(H^{B'}_{A'}, (l, B'_{H^{B'}_{A'}})\right)^{-1}.$$

We will consider both distributions separately.
As $\complement A \supseteq \complement l^0$, it is $H^X_{\complement A} \leq H_X$ and therefore

$$X'_{H^X_{\complement A}} = X_{H^X_{\complement A} \wedge H_X} = X_{H^X_{\complement A}}.$$

Furthermore, we observe that

$$\begin{aligned} H^{X'}_{\complement A} &= \inf\{t \geq 0 : X_{t \wedge H_X} \in \complement A\} \\ &= \inf\{t \in [0, H_X] : X_t \in \complement A\} \\ &= \inf\{t \geq 0 : X_t \in \complement A\} \\ &= H^X_{\complement A}, \end{aligned}$$

where the third identity follows again from $H^X_{\complement A} \leq H_X$: If $H^X_{\complement A} < H_X$, the identity is clear. If $H^X_{\complement A} = H_X$, then as $\complement l^0$ is closed, we have $X_{H_X} \in \complement l^0 \subseteq \complement A$, so H_X lies in both sets, thus concluding that both infima are equal. In summary, this gives

$$\mathbb{P}_{(l,x)} \circ \left(H^{X'}_{\complement A}, X'_{H^{X'}_{\complement A}}\right)^{-1} = \mathbb{P}_{(l,x)} \circ \left(H_{\complement A}, X_{H^X_{\complement A}}\right)^{-1}.$$

Turning to the part for the Brownian motion B, observe that $A \subseteq \{l\} \times (0, \rho_l)$. This means that $A' = \pi^2(\complement A \cap l)$ contains the points 0 and (if l is an internal edge) ρ_l. Thus, we have

$$H^{B'}_{A'} = H^B_{A'} \leq H_B,$$

which shows

$$B'_{H^{B'}_{A'}} = B_{H^B_{A'} \wedge H_B} = B_{H^B_{A'}},$$

resulting in

$$\mathbb{P}^B_x \circ \left(H^{B'}_{A'}, (l, B'_{H^{B'}_{A'}})\right)^{-1} = \mathbb{P}^B_x \circ \left(H^B_{A'}, (l, B_{H^B_{A'}})\right)^{-1}. \qquad \square$$

(20.12) Lemma. *Let X be a Brownian motion on $\mathcal{G}$. Then, for any $f \in b\mathscr{B}(\mathcal{G})$, $\alpha > 0$ and $g = (l, x) \in \mathcal{G}$, the resolvent of X reads, if $l = e \in \mathcal{E}$,*

$$U_\alpha f(g) = U_\alpha^{D,e} f(g) + e^{-\sqrt{2\alpha}\, d(\partial(e),g)}\, U_\alpha f\big(\partial(e)\big),$$

and if $l = i \in \mathcal{I}$,

$$\begin{aligned} U_\alpha f(g) = U_\alpha^{D,i} f(g) &+ \frac{\sinh\big(\sqrt{2\alpha}\, d(\partial_+(i), g)\big)}{\sinh(\sqrt{2\alpha}\, \rho_i)}\, U_\alpha f\big(\partial_-(i)\big) \\ &+ \frac{\sinh\big(\sqrt{2\alpha}\, d(\partial_-(i), g)\big)}{\sinh(\sqrt{2\alpha}\, \rho_i)}\, U_\alpha f\big(\partial_+(i)\big), \end{aligned}$$

with

$$\begin{aligned} U_\alpha^{D,e} f(g) &:= U_\alpha^{[0,\infty)} f_l\big(d(\partial(e), g)\big), \quad g \in e, \\ U_\alpha^{D,i} f(g) &:= U_\alpha^{[0,\rho_i]} f_l\big(d(\partial_-(i), g)\big), \quad g \in i, \end{aligned} \tag{20.13}$$

where $\big(U_\alpha^{[0,\infty)}, \alpha > 0\big)$ and $\big(U_\alpha^{[0,\rho_i]}, \alpha > 0\big)$ are the resolvents of the one-dimensional Brownian motion killed on leaving $[0, \infty)$, $[0, \rho_i]$ respectively, which are given in examples (16.3) and (17.3).

Proof. The decomposition of the resolvent at the stopping time H_X with the help of Dynkin's formula (3.16) yields for $g = (l, x) \in \mathcal{G}$, $f \in b\mathscr{B}(\mathcal{G})$:

$$U_\alpha f(g) = \mathbb{E}_g\Big(\int_0^{H_X} e^{-\alpha t}\, f(X_t)\, dt\Big) + \mathbb{E}_g\big(e^{-\alpha H_X}\, U_\alpha f(X_{H_X})\big).$$

Thus, by theorem (20.5), we have

$$U_\alpha f(g) = \mathbb{E}_x^B\Big(\int_0^{H_B} e^{-\alpha t}\, f(l, B_t)\, dt\Big) + \mathbb{E}_x^B\big(e^{-\alpha H_B}\, U_\alpha f(l, B_{H_B})\big).$$

With $H_B = \inf\{t \geq 0 : B_t = 0\}$ or $H_B = \inf\big\{t \geq 0 : B_t \in \{0, \rho_l\}\big\}$ depending on whether $l \in \mathcal{E}$ or $l \in \mathcal{I}$, the passage time formulas of the one-dimensional Brownian motion (cf. subsection 14.3) conclude the proof: We only need to note that for any $g = (l, x) \in \mathcal{G}$, we have $\partial_-(l) = (l, 0)$, $x = d\big(\partial_-(l), g\big)$ and $\partial_+(l) = (l, \rho_l)$, $\rho_l - x = d\big(\partial_+(l), g\big)$ in case $l \in \mathcal{I}$, whereas $\partial(l) = (l, 0)$, $x = d\big(\partial(l), g\big)$ in case $l \in \mathcal{E}$. $\square$

As seen in the examinations for the resolvents $\big(U_\alpha^{[0,\infty)}, \alpha > 0\big)$ and $\big(U_\alpha^{[a,b]}, \alpha > 0\big)$ of the "Dirichlet" Brownian motions on $[0, \infty)$ and $[a, b]$ (cf. examples (16.3) and (17.3)),

- $\big(U_\alpha^{[0,\infty)}, \alpha > 0\big)$ maps $b\mathscr{B}([0, \infty))$ on $b\mathcal{C}([0, \infty))$ and $\mathcal{C}_0([0, \infty))$ on $\mathcal{C}_0^2([0, \infty))$, and assumes the boundary values $U^{[0,\infty)} f(0) = 0$, $U^{[0,\infty)} f''(0) = -2f(0)$,
- $\big(U_\alpha^{[0,\rho_i]}, \alpha > 0\big)$ maps $b\mathscr{B}([0, \rho_i])$ on $b\mathcal{C}([0, \rho_i])$ and $\mathcal{C}([0, \rho_i])$ on $\mathcal{C}^2([0, \rho_i])$, and assumes the boundary values $U^{[0,\rho_i]} f(x) = 0$, $U^{[0,\rho_i]} f''(x) = -2f(x)$, for $x \in \{0, \rho_i\}$.

Thus, the resolvents defined in equation (20.13) are continuous functions, twice continuously differentiable inside their respective edge for any $f \in \mathcal{C}_0(\mathcal{G})$, and assume the values

$$\begin{aligned} U_\alpha^{D,e} f(\partial(e)) &= 0, \quad U_\alpha^{D,e} f''(\partial(e)) = -2f(\partial(e)), \\ U_\alpha^{D,i} f(\partial_-(i)) &= 0, \quad U_\alpha^{D,i} f''(\partial_-(i)) = -2f(\partial_-(i)), \\ U_\alpha^{D,i} f(\partial_+(i)) &= 0, \quad U_\alpha^{D,i} f''(\partial_+(i)) = -2f(\partial_+(i)). \end{aligned}$$

Therefore, these boundary values for resolvents $U^{D,e}$, $U^{D,i}$ of various edges e, i, incident with the same vertex, coincide on their mutual vertex. Then, by the decompositions given in lemma (20.12) for the resolvent $(U_\alpha, \alpha > 0)$ of a Brownian motion on a metric graph, $U_\alpha f$ extends to a twice continuously differentiable function on $\mathcal{G}$, yielding:

(20.14) Corollary. *The resolvent $(U_\alpha, \alpha > 0)$ of a Brownian motion on a metric graph maps $b\mathscr{B}(\mathcal{G})$ on $b\mathcal{C}(\mathcal{G})$ and $\mathcal{C}_0(\mathcal{G})$ on $\mathcal{C}_0^2(\mathcal{G})$.*

(20.15) Theorem. *Let X be a Brownian motion on $\mathcal{G}$ with generator A. Then X is a Feller process, uniquely determined by its generator $A = \frac{1}{2}\Delta$, with $\mathscr{D}(A) \subseteq \mathcal{C}_0^2(\mathcal{G})$.*

Proof. Theorem (5.13) together with corollary (20.14) and the right continuity of X immediately show the Feller property of X. Uniqueness is guaranteed by theorem (5.9).

Let $f \in \mathscr{D}(A)$. Then by (5.10), there exist $h \in \mathcal{C}_0(\mathcal{G})$ and $\alpha > 0$ with $f = U_\alpha h$, and $U_\alpha h \in \mathcal{C}_0^2(\mathcal{G})$ holds by corollary (20.14). Differentiating the decomposition given in lemma (20.12) twice yields for $g = (l, x) \in \mathcal{G}$, in case $l = i \in \mathcal{I}$:

$$\begin{aligned} \frac{1}{2} f''(g) &= \frac{1}{2} U_\alpha^{D,i} h''(g) + \alpha \frac{\sinh\big(\sqrt{2\alpha}\, d(\partial_-(i), g)\big)}{\sinh(\sqrt{2\alpha})} U_\alpha h(\partial_-(i)) \\ &\qquad + \alpha \frac{\sinh\big(\sqrt{2\alpha}\, d(\partial_+(i), g)\big)}{\sinh(\sqrt{2\alpha})} U_\alpha h(\partial_+(i)), \\ &= \alpha\, U_\alpha^{D,i} h(g) - h(g) + \alpha \frac{\sinh\big(\sqrt{2\alpha}\, d(\partial_-(i), g)\big)}{\sinh(\sqrt{2\alpha})} U_\alpha h(\partial_-(i)) \\ &\qquad + \alpha \frac{\sinh\big(\sqrt{2\alpha}\, d(\partial_+(i), g)\big)}{\sinh(\sqrt{2\alpha})} U_\alpha h(\partial_+(i)), \\ &= \alpha\, U_\alpha h(g) - h(g), \end{aligned}$$

and in case $l = e \in \mathcal{E}$:

$$\begin{aligned} \frac{1}{2} f''(g) &= \frac{1}{2} U_\alpha^{D,e} h''(g) + \alpha\, e^{-\sqrt{2\alpha}\, d(\partial_-(e), g)}\, U_\alpha h(\partial_-(e)) \\ &= \alpha\, U_\alpha^{D,e} h(g) - h(g) + \alpha\, e^{-\sqrt{2\alpha}\, d(\partial_-(e), g)}\, U_\alpha h(\partial_-(e)) \\ &= \alpha\, U_\alpha h(g) - h(g). \end{aligned}$$

Thus, for any $f \in \mathscr{D}(A)$, we have $f \in \mathcal{C}_0^2(\mathcal{G})$ and $Af = \frac{\Delta}{2} f$ on $\mathcal{G}$. $\square$

20.3. Computing the Generator: Feller's Theorem

As any Brownian motion on a metric graph is a Feller process with generator $A = \frac{1}{2}\Delta$, it is uniquely characterized by its generator domain, more accurately: by the generator's boundary conditions. We are going to extend the classical results of the half-line and interval cases by generalizing the approach of [Kni81, Lemma 6.2] and [IM63, Section 8]:

(20.16) Theorem. *Let X be a Brownian motion on $\mathcal{G}$ with generator A. Then, for every $v \in \mathcal{V}$, there exist $c_1^v \geq 0$, $c_2^{v,l} \geq 0$ for each $l \in \mathcal{L}(v)$, $c_3^v \geq 0$ and a measure c_4^v on $\mathcal{G}\backslash\{v\}$, satisfying*

$$c_1^v + \sum_{l \in \mathcal{L}(v)} c_2^{v,l} + c_3^v + \int_{\mathcal{G}\backslash\{v\}} \big(1 - e^{-d(v,g)}\big)\, c_4^v(dg) = 1,$$

such that for every $f \in \mathscr{D}(A)$, the relation

$$c_1^v\, f(v) - \sum_{l \in \mathcal{L}(v)} c_2^{v,l}\, f_l'(v) + c_3^v\, Af(v) - \int_{\mathcal{G}\backslash\{v\}} \big(f(g) - f(v)\big)\, c_4^v(dg) = 0$$

holds. The constants and the measure only depend on the process' exit behavior from any arbitrarily small neighborhood of v. They are given by

$$\begin{aligned} c_1^v &= c_1^{v,\Delta} + c_1^{v,\infty}, \\ \textit{with} \quad c_1^{v,\Delta} &= \lim_{n\to\infty} \frac{\mathbb{P}_v(X_{\tau_{\varepsilon_n}} = \Delta)}{\mathbb{E}_v(\tau_{\varepsilon_n})\, K_{\varepsilon_n}^v}, \quad c_1^{v,\infty} = \sum_{e \in \mathcal{E}} \bar{\mu}^v\big(\{(e,+\infty)\}\big), \\ c_2^{v,l} &= \begin{cases} \bar{\mu}^v\big(\{(l,0+)\}\big), & l \in \mathcal{E}(v), \\ \bar{\mu}^v\big(\{(l,0+)\}\big), & l \in \mathcal{I}(v), v = \partial_-(l), \\ \bar{\mu}^v\big(\{(l,\rho_l-)\}\big), & l \in \mathcal{I}(v), v = \partial_+(l), \end{cases} \\ c_3^v &= \lim_{n\to\infty} \frac{1}{K_{\varepsilon_n}^v}, \\ c_4^v(dg) &= \frac{1}{1 - e^{-d(v,g)}}\, \bar{\mu}^v(dg), \end{aligned}$$

where for every $\varepsilon > 0$, $\tau_\varepsilon = \inf\{t \geq 0 : X_t \in \complement\overline{B_\varepsilon(v)}\}$,

$$K_\varepsilon^v = 1 + \frac{\mathbb{P}_v(X_{\tau_\varepsilon} = \Delta)}{\mathbb{E}_v(\tau_\varepsilon)} + \int_{\mathcal{G}\backslash\{v\}} \big(1 - e^{-d(v,g)}\big)\, \nu_\varepsilon^v(dg),$$

ν_ε^v and μ_ε^v are measures on $\mathcal{G}\backslash\{v\}$ defined by

$$\begin{aligned} \nu_\varepsilon^v(dg) &= \frac{\mathbb{P}_v(X_{\tau_\varepsilon} \in dg)}{\mathbb{E}_v(\tau_\varepsilon)}, \\ \mu_\varepsilon^v(dg) &= \big(1 - e^{-d(v,g)}\big)\, \frac{\nu_\varepsilon^v(dg)}{K_\varepsilon^v}, \end{aligned}$$

as well as $\bar\mu^v_\varepsilon$, $\bar\mu^v$ are measures on $\overline{\mathcal{G}\backslash\{v\}}$ with

$$\begin{aligned}\bar\mu^v_\varepsilon(dg) &= \mu^v_\varepsilon(dg\cap(\mathcal{G}\backslash\{v\})),\\ \bar\mu^v &= \lim_{n\to\infty}\bar\mu^v_{\varepsilon_n}\end{aligned}$$

(in the sense of weak convergence), and $(\varepsilon_n, n\in\mathbb{N})$ is a sequence of positive numbers converging to zero such that all of the above limits exist.

Proof. Let $v\in\mathcal{V}$. For all $\varepsilon>0$, define the first exit time of X from $\overline{B_\varepsilon(v)}$ by

$$\tau_\varepsilon := \inf\big\{t\geq 0 : X_t\in \complement\overline{B_\varepsilon(v)}\big\}.$$

In case v is a trap, we can compute the generator directly: Then

$$Af(v) = \lim_{t\downarrow 0}\frac{\mathbb{E}_v\big(f(X_t)\big)-f(v)}{t} = 0$$

holds true, thus choosing $c^v_3 = 1$ and $c^v_1 = c^{v,l}_2 = c^v_4 = 0$ for all $l\in\mathcal{L}(v)$ gives

$$c^v_1 f(v) - \sum_{l\in\mathcal{L}(v)} c^{v,l}_2 f'_l(v) + c^v_3 Af(v) - \int \big(f(g)-f(v)\big)\, c^v_4(dg) = 0.$$

This choice coincides with the definition of the parameters in the theorem, because in the case of a trap v, we have $\mathbb{E}_v(\tau_\varepsilon) = +\infty$ for all $\varepsilon>0$, all (scaled) exit distributions read $\mathbb{P}_v(X_{\tau_\varepsilon}=\Delta) = \nu^v_\varepsilon = \mu^v_\varepsilon = 0$, and thus $K^v_\delta = 1$ holds for all $\varepsilon>0$ as well as $\bar\mu^v = 0$.

If v is not a trap, then due to X being Feller (see theorem (20.15)), $\mathbb{E}_v(\tau_\varepsilon)<+\infty$ holds true for all $\varepsilon>0$ sufficiently small by theorem (5.16), and thus Dynkin's formula (3.18) is applicable for every $f\in\mathscr{D}(A)$ (cf. remark (5.11)). It yields

$$\begin{aligned}Af(v) &= \lim_{\varepsilon\downarrow 0}\frac{\mathbb{E}_v\big(f(X_{\tau_\varepsilon})\big)-f(v)}{\mathbb{E}_v(\tau_\varepsilon)}\\ &= \lim_{\varepsilon\downarrow 0}\Big(-f(v)\,\frac{\mathbb{P}_v(X_{\tau_\varepsilon}=\Delta)}{\mathbb{E}_v(\tau_\varepsilon)} + \int_{\mathcal{G}\backslash\{v\}}\big(f(g)-f(v)\big)\,\nu^v_\varepsilon(dg)\Big),\end{aligned}\tag{20.17}$$

with ν^v_ε being measures on $\mathcal{G}\backslash\{v\}$, defined by

$$\nu^v_\varepsilon(dg) := \frac{\mathbb{P}_v(X_{\tau_\varepsilon}\in dg)}{\mathbb{E}_v(\tau_\varepsilon)},\quad \varepsilon>0,$$

as the support of X_{τ_ε} is the completion of $\complement\overline{B_\varepsilon(v)}$ in $\mathcal{G}$ and therefore is a subset of $\mathcal{G}\backslash\{v\}$.

Introducing the normalizing constants

$$K^v_\varepsilon := 1 + \frac{\mathbb{P}_v(X_{\tau_\varepsilon}=\Delta)}{\mathbb{E}_v(\tau_\varepsilon)} + \int_{\mathcal{G}\backslash\{v\}}\big(1-e^{-d(v,g)}\big)\,\nu^v_\varepsilon(dg),\quad \varepsilon>0,$$

equation (20.17) implies (as $\frac{1}{K^v_\varepsilon}\in[0,1]$ for all $\varepsilon>0$) that

$$0 = \lim_{\varepsilon\downarrow 0}\Big(f(v)\,\frac{\mathbb{P}_v(X_{\tau_\varepsilon}=\Delta)}{\mathbb{E}_v(\tau_\varepsilon)\,K^v_\varepsilon} + Af(v)\,\frac{1}{K^v_\varepsilon} - \int_{\mathcal{G}\backslash\{v\}}\big(f(g)-f(v)\big)\,\frac{\nu^v_\varepsilon(dg)}{K^v_\varepsilon}\Big).\tag{20.18}$$

We rescale the measures ν_ε^v by introducing measures

$$\mu_\varepsilon^v(dg) := \big(1 - e^{-d(v,g)}\big)\,\frac{\nu_\varepsilon^v(dg)}{K_\varepsilon^v}, \quad \varepsilon > 0,$$

on $\mathcal{G}\backslash\{v\}$. It is immediate that equation (20.18) then is equivalent to

$$0 = \lim_{\varepsilon \downarrow 0} \Big(f(v)\,\frac{\mathbb{P}_v(X_{\tau_\varepsilon} = \Delta)}{\mathbb{E}_v(\tau_\varepsilon)\,K_\varepsilon^v} + Af(v)\,\frac{1}{K_\varepsilon^v} - \int_{\mathcal{G}\backslash\{v\}} \frac{f(g) - f(v)}{1 - e^{-d(v,g)}}\,\mu_\varepsilon^v(dg)\Big). \tag{20.19}$$

Let $\bar{\mu}_\varepsilon^v$ be the extensions of the measures μ_ε^v to the compactification $\overline{\mathcal{G}\backslash\{v\}}$ of $\mathcal{G}\backslash\{v\}$ (see subsection 18.4 for details on the compactification of a subspace of a metric graph), that is, we define the measures $\bar{\mu}_\varepsilon^v$ on $\overline{\mathcal{G}\backslash\{v\}}$ by

$$\bar{\mu}_\varepsilon^v(dg) := \mu_\varepsilon^v\big(dg \cap (\mathcal{G}\backslash\{v\})\big), \quad \varepsilon > 0.$$

Then the above identity (20.19) remains valid for $\bar{\mu}_\varepsilon^v$ instead of μ_ε^v, where

$$g \mapsto \frac{f(g) - f(v)}{1 - e^{-d(v,g)}}$$

is continuously extended from $\mathcal{G}\backslash\{v\}$ to $\overline{\mathcal{G}\backslash\{v\}}$ by

$$\begin{aligned} \forall l \in \mathcal{L}(v): \quad & \lim_{g \to v,\, g \in l^0} \frac{f(g) - f(v)}{1 - e^{-d(v,g)}} = f_l'(v), \\ \forall e \in \mathcal{E}: \quad & \lim_{g \to \infty,\, g \in e^0} \frac{f(g) - f(v)}{1 - e^{-d(v,g)}} = -f(v), \end{aligned}$$

because $f \in \mathscr{D}(A) \subseteq \mathcal{C}_0^2(\mathcal{G}) \subseteq \mathcal{C}_0^{0,2}(\mathcal{G})$ and $f \in \mathscr{D}(A) \subseteq \mathcal{C}_0^2(\mathcal{G}) \subseteq \mathcal{C}_0(\mathcal{G})$.

As $\mathcal{C}(\overline{\mathcal{G}\backslash\{v\}})$ is separable (see theorem (18.17)) and all measures $\bar{\mu}_\varepsilon^v$, $\varepsilon > 0$, are bounded by 1, there exists a sequence $(\varepsilon_n, n \in \mathbb{N})$ of strictly positive numbers, converging to zero, such that $(\bar{\mu}_{\varepsilon_n}^v, n \in \mathbb{N})$ converges weakly to a measure $\bar{\mu}^v$ on $\overline{\mathcal{G}\backslash\{v\}}$.[2]

The sequences $\big(\frac{1}{K_\varepsilon^v}, \varepsilon > 0\big)$ and $\big(\frac{\mathbb{P}_v(X_{\tau_\varepsilon} = \Delta)}{\mathbb{E}_v(\tau_\varepsilon)\,K_\varepsilon^v}, \varepsilon > 0\big)$ are bounded by 1 as well, thus by choosing appropriate subsequences of $(\varepsilon_n, n \in \mathbb{N})$ and naming them $(\varepsilon_n, n \in \mathbb{N})$ again if necessary, we also obtain the existence of

$$\begin{aligned} c_1^{v,\Delta} &:= \lim_{n \to \infty} \frac{\mathbb{P}_v(X_{\tau_{\varepsilon_n}} = \Delta)}{\mathbb{E}_v(\tau_{\varepsilon_n})\,K_{\varepsilon_n}^v}, \\ c_3^v &:= \lim_{n \to \infty} \frac{1}{K_{\varepsilon_n}^v}. \end{aligned}$$

[2]This can be shown by employing the standard argument used in Helly's selection theorem: Let $\mathscr{S} := \{h_m, m \in \mathbb{N}\}$ be a countable, dense subset of $\mathcal{C}(\overline{\mathcal{G}\backslash\{v\}})$, and $(\tilde{\varepsilon}_n, n \in \mathbb{N})$ a sequence of strictly positive numbers, converging to zero. As all measures are bounded by 1, the "array" $\big(\int \frac{h_m}{\|h_m\|}\,d\bar{\mu}_{\tilde{\varepsilon}_n}^v, m, n \in \mathbb{N}\big)$ is bounded by 1. By the diagonal method (see, e.g., [Bil79, Theorem 25.13]), it is possible to choose a subsequence $(\varepsilon_n, n \in \mathbb{N})$ of $(\tilde{\varepsilon}_n, n \in \mathbb{N})$ such that $\lim_n \int h_m\,d\bar{\mu}_{\varepsilon_n}^v$ exists for all $m \in \mathbb{N}$, that is for all functions in a dense subset of $\mathcal{C}(\overline{\mathcal{G}\backslash\{v\}})$. Thus, $\lim_n \int f\,d\bar{\mu}_{\varepsilon_n}^v$ exists for all $f \in \mathcal{C}(\overline{\mathcal{G}\backslash\{v\}})$ and defines a positive linear functional on $\mathcal{C}(\overline{\mathcal{G}\backslash\{v\}})$. Therefore, by the Riesz–Markov–Kakutani representation theorem, there exists a measure on $\bar{\mu}^v$ on $\overline{\mathcal{G}\backslash\{v\}}$ which satisfies $\lim_n \int f\,d\bar{\mu}_{\varepsilon_n}^v = \int f\,d\bar{\mu}^v$.

Inserting everything in equation (20.19) shows

$$0 = c_1^{v,\Delta}\, f(v) + c_3^v\, Af(v) - \int_{\overline{\mathcal{G}\backslash\{v\}}} \frac{f(g)-f(v)}{1-e^{-d(v,g)}}\, \bar{\mu}^v(dg),$$

or equivalently

$$\begin{aligned} 0 = \Big(c_1^{v,\Delta} + \sum_{e\in\mathcal{E}} \bar{\mu}^v\big(\{(e,+\infty)\}\big)\Big)\, f(v) - \sum_{\substack{l\in\mathcal{L}(v),\\ v=\partial_-(l)}} \bar{\mu}^v\big(\{(l,0+)\}\big)\, f_l'(v) \\ - \sum_{\substack{l\in\mathcal{I}(v),\\ v=\partial_+(l)}} \bar{\mu}^v\big(\{(l,\rho_l-)\}\big)\, f_l'(v) + \ c_3^v\, Af(v) - \int_{\mathcal{G}\backslash\{v\}} \frac{f(g)-f(v)}{1-e^{-d(v,g)}}\, \bar{\mu}^v(dg), \end{aligned}$$

so setting $c_1^v := c_1^{v,\Delta} + \sum_{e\in\mathcal{E}} \bar{\mu}^v\big(\{(e,+\infty)\}\big)$, $c_2^{v,l} := \bar{\mu}^v\big(\{(l,0+)\}\big)$ or $c_2^{v,l} := \bar{\mu}^v\big(\{(l,\rho_l-)\}\big)$ for each $l\in\mathcal{L}(v)$ depending on whether $v\in\partial_-(l)$ or $v\in\partial_+(l)$, and defining the measure c_4^v on $\mathcal{G}\backslash\{v\}$ by $c_4^v(dg) := \frac{1}{1-e^{-d(v,g)}}\, \bar{\mu}^v(dg)$ yields the result

$$0 = c_1^v\, f(v) - \sum_{l\in\mathcal{L}(v)} c_2^{v,l}\, f_l'(v) + c_3^v\, Af(v) - \int_{\mathcal{G}\backslash\{v\}} \big(f(g)-f(v)\big)\, c_4^v(dg).$$

This completes the proof, as insertion of the definitions offers the normalization

$$\begin{aligned} & c_1^v + \sum_{l\in\mathcal{L}(v)} c_2^{v,l} + c_3^v + \int_{\mathcal{G}\backslash\{v\}} \big(1-e^{-d(v,g)}\big)\, c_4^v(dg) \\ &= c_1^{v,\Delta} + c_3^v + \int_{\overline{\mathcal{G}\backslash\{v\}}} \bar{\mu}^v(dg) \\ &= \lim_{n\to\infty} \frac{1}{K_{\varepsilon_n}^v} \Big(\frac{\mathbb{P}_v(X_{\tau_{\varepsilon_n}} = \Delta)}{\mathbb{E}_v(\tau_{\varepsilon_n})} + 1 + \int_{\mathcal{G}\backslash\{v\}} \big(1-e^{-d(v,g)}\big)\, \nu_{\varepsilon_n}^v(dg)\Big) \\ &= 1. \end{aligned}$$

□

By examining the proof, the reader may observe that the "Brownian" property of X was not used anywhere. Indeed, the above result holds true for any Feller process (we will not need this fact).

Theorem (20.16) gives explicit (albeit rather unwieldy) expressions for the boundary condition of a Brownian motion. As we will need to utilize them quite frequently, we assign the following, supposably appropriate name:

(20.20) Definition. *For any Brownian motion X on a metric graph $\mathcal{G}$, the collection*

$$\big(c_1^{v,\Delta}, c_1^{v,\infty}, (c_2^{v,l})_{l\in\mathcal{L}(v)}, c_3^v, c_4^v\big)_{v\in\mathcal{V}}$$

as defined in theorem (20.16) is called Feller–Wentzell data of X.

If no distinction is necessary, $c_1^{v,\Delta}$ and $c_1^{v,\infty}$ are combined, denoted by $c_1^v = c_1^{v,\Delta} + c_1^{v,\infty}$.

(20.21) Theorem. *Let X be a Brownian motion on a metric graph $\mathcal{G}$. Then X is a Feller process with generator $A = \frac{1}{2}\Delta$, and for every vertex $v \in \mathcal{V}$ there exist constants $p_1^v \geq 0$, $p_2^{v,l} \geq 0$ for each $l \in \mathcal{L}(v)$, $p_3^v \geq 0$ and a measure p_4^v on $\mathcal{G}\backslash\{v\}$ with*

$$p_1^v + \sum_{l \in \mathcal{L}(v)} p_2^{v,l} + p_3^v + \int \left(1 - e^{-d(v,g)}\right) p_4^v(dg) = 1,$$

and

$$p_4^v(\mathcal{G}\backslash\{v\}) = +\infty, \quad \text{if} \quad \sum_{l \in \mathcal{L}(v)} p_2^{v,l} + p_3^v = 0, \tag{20.22}$$

such that the domain of A reads

$$\begin{aligned} \mathscr{D}(A) \subseteq \Big\{ & f \in \mathcal{C}_0^2(\mathcal{G}) : \forall v \in \mathcal{V} : \\ & p_1^v f(v) - \sum_{l \in \mathcal{L}(v)} p_2^{v,l} f_l'(v) + \frac{p_3^v}{2} f''(v) - \int \left(f(g) - f(v)\right) p_4^v(dg) = 0 \Big\}. \end{aligned} \tag{20.23}$$

Proof. It only remains to show condition (20.22): Assume that there is a $v \in \mathcal{V}$ such that $p_2^{v,l} = 0$ for all $l \in \mathcal{L}(v)$, $p_3^v = 0$ and p_4^v is a finite measure on $\mathcal{G}\backslash\{v\}$. For any $f \in \mathcal{C}_0(\mathcal{G})$, $\alpha > 0$, $u = \alpha U_\alpha f$ is an element of $\mathscr{D}(A)$, so by (20.23) it especially fulfills

$$\alpha U_\alpha f(v) \left(p_1^v + p_4^v(\mathcal{G}\backslash\{v\})\right) - \int \alpha U_\alpha f(g)\, p_4^v(dg) = 0.$$

Letting $\alpha \to +\infty$ yields with theorem (5.13) and LDCT (as $\|\alpha U_\alpha f\| \leq \|f\|$) that

$$f(v) \left(p_1^v + p_4^v(\mathcal{G}\backslash\{v\})\right) = \int f(g)\, p_4^v(dg),$$

for all $f \in \mathcal{C}_0(\mathcal{G})$. But then p_4^v must the Dirac measure in v, scaled by $p_1^v + p_4^v(\mathcal{G}\backslash\{v\}) > 0$, which is impossible. □

(20.24) Remark. On any non-vertex point $g = (l, x) \in \mathcal{G}^0$ of the graph $\mathcal{G}$, the generator A of any Brownian motion X on $\mathcal{G}$ reads

$$Af(g) = \frac{1}{2} \frac{\partial^2}{\partial x^2} f(l, x), \quad f \in \mathscr{D}(A),$$

being the usual differentiation of a function defined on some open subset of $\mathbb{R}$. It is therefore necessary for the first derivate f' of f to exist and be continuous at g, that is,

$$\lim_{\xi \downarrow x} f_l'(\xi) = \lim_{\xi \uparrow x} f_l'(\xi).$$

Therefore, if we introduce a new vertex v' at $g = (l, x) \in \mathcal{G}^0$, splitting the original edge l into two new edges l_1', l_2' (as done in subsection 18.2 in order to eliminate tadpoles), the original Brownian motion X will satisfy the boundary condition

$$\frac{1}{2} f_{l_1'}'(v') + \frac{1}{2} f_{l_2'}'(v') = 0, \quad f \in \mathscr{D}(A),$$

at the new vertex v'. Thus, we can always assume that we are able to introduce "trivial" vertices inside of existing edges which do not change the generator or the Feller–Wentzell data of the underlying Brownian motion, in case the "non-skew" boundary condition above is chosen at the new vertices. ■

As already mentioned in the introduction, it is not trivial to show that the boundary conditions, as given in theorem (20.21), are also sufficient for the generator domain. This means that, in general, we do not gain the complete description of the generator. [KPS12a, Section 3] shows that equality in equation (20.23) is attained if the Brownian motion is continuous up to its lifetime. In the discontinuous setting, we were able to prove the corresponding result for the interval case in subsection 17.2, demonstrating the technical difficulties that already arise for metric graphs with two vertex points. For metric graphs with only one vertex, the proof is fortunately much simpler:

(20.25) Lemma. *Let X be a Brownian motion on a star graph $\mathcal{G}$ with star point v, and let $p_1 \geq 0$, $p_2^e \geq 0$ for each $e \in \mathcal{E}$, $p_3 \geq 0$ and a measure p_4 on $\mathcal{G}\backslash\{v\}$ be given with*

$$p_1 + \sum_{e\in\mathcal{E}} p_2^e + p_3 + \int_{\mathcal{G}\backslash\{v\}} (1 - e^{-x})\, p_4\big(d(e,x)\big) = 1,$$

such that the generator of X is $A = \frac{\Delta}{2}$ and its domain satisfies $\mathscr{D}(A) \subseteq \mathscr{D}$, with

$$\mathscr{D} := \Big\{ f \in \mathcal{C}_0^2(\mathcal{G}) : \\ p_1 f(v) - \sum_{e\in\mathcal{E}} p_2^e f_e'(v) + \frac{p_3}{2} f''(v) - \int_{\mathcal{G}\backslash\{v\}} \big(f(g) - f(v)\big)\, p_4(dg) = 0 \Big\}.$$

Then $\mathscr{D}(A) = \mathscr{D}$.

Proof. As in the proof for the interval case, we will employ lemma (5.12): For $\alpha > 0$, let $f \in \mathscr{D}$ with $\frac{\Delta}{2} f = \alpha f$. As f solves this differential equation on every edge, f must be of the form

$$f(e,x) = c_1^e\, e^{-\sqrt{2\alpha}x} + c_2^e\, e^{\sqrt{2\alpha}x}, \quad e \in \mathcal{E},\ x \geq 0,$$

for some $c_1^e, c_2^e \in \mathbb{R}$, for each $e \in \mathcal{E}$. However, $f \in \mathcal{C}_0(\mathcal{G})$ holds true, so as f needs to vanish at infinity, it is $c_2^e = 0$ for all $e \in \mathcal{E}$. But then, in order to be continuous at the star vertex, all the c_1^e need to coincide. Therefore, setting $c_1^e = c$ for all $e \in \mathcal{E}$ results in

$$f(e,x) = c\, e^{-\sqrt{2\alpha}x}, \quad e \in \mathcal{E},\ x \geq 0.$$

As $f \in \mathscr{D}(A) \subseteq \mathscr{D}$, the boundary condition for f now yields

$$c \Big(p_1 + \sqrt{2\alpha} \sum_{e\in\mathcal{E}} p_2^e + \alpha\, p_3 + \int_{\mathcal{G}\backslash\{v\}} (1 - e^{-\sqrt{2\alpha}x})\, p_4^v\big(d(e,x)\big) \Big) = 0,$$

which is only possible for $c = 0$, as all of the summands in the parentheses are non-negative, but must add up to a positive number due to the provided normalization $p_1 + \sum_{e\in\mathcal{E}} p_2^e + p_3 + \int_{\mathcal{G}\backslash\{v\}} (1 - e^{-x})\, p_4\big(d(e,x)\big) = 1$.

Thus $\frac{\Delta}{2} f = \alpha f$, $f \in \mathscr{D}$, is only solved by $f = 0$, completing the proof. □

(20.26) Theorem. *Let X be a Brownian motion on star graph $\mathcal{G}$ with star point v. Then X is a Feller process with generator $A = \frac{1}{2}\Delta$, and there exist constants $p_1 \geq 0$, $p_2^e \geq 0$ for each $e \in \mathcal{E}$, $p_3 \geq 0$ and a measure p_4 on $\mathcal{G} \backslash \{v\}$ with*

$$p_1 + \sum_{e \in \mathcal{E}} p_2^e + p_3 + \int_{\mathcal{G} \backslash \{v\}} \left(1 - e^{-d(v,g)}\right) p_4^v(dg) = 1,$$

and

$$p_4(\mathcal{G} \backslash \{v\}) = +\infty, \quad \textit{if} \quad \sum_{e \in \mathcal{E}} p_2^e + p_3 = 0,$$

such that the domain of A reads

$$\mathscr{D}(A) = \Big\{ f \in \mathcal{C}_0^2(\mathcal{G}) : \\ p_1 f(v) - \sum_{e \in \mathcal{E}} p_2^e f_e'(v) + \frac{p_3}{2} f''(v) - \int_{\mathcal{G} \backslash \{v\}} \big(f(g) - f(v)\big)\, p_4(dg) = 0 \Big\}.$$

Furthermore, X is uniquely characterized by this set of normalized constants.

21. Construction of all Brownian Motions on a Star Graph

We are going to construct all Brownian motions on a star graph by extending the ingenious approach of [IM63] for the half-line case, which was explained in section 16. Afterwards, in subsection 21.12, we will use our construction results to gain further insight into the properties of these processes. This will be necessary for the treatment in section 22, where Brownian motions on star graphs serve as basic "building blocks" for Brownian motions on general metric graphs.

In all that follows, let $\mathcal{G}$ be a fixed star graph with star vertex v and set of external edges $\mathcal{E}$. For keeping notations readable in the following construction, we will assume that $\mathcal{E} = \{1, \ldots, n\}$ holds with $n = |\mathcal{E}|$. As usual, we consider the geometrical representation

$$\mathcal{G} = \{v\} \cup \bigcup_{e=1}^{n} \big(\{e\} \times [0, \infty)\big)$$

of the graph $\mathcal{G}$, with all initial points $(e, 0)$, $e \in \mathcal{E}$, being identified with the vertex v.

Furthermore, with regard to the assertions of Feller's theorem (20.21), we assume that we are given a fixed set of boundary weights

$$p_1 \geq 0, \ p_2^e \geq 0 \text{ for each } e \in \mathcal{E}, \ p_3 \geq 0, \ p_4 \text{ measure on } \mathcal{G} \backslash \{v\},$$

satisfying $\int_{\mathcal{G} \backslash \{v\}} \left(1 - e^{-d(v,g)}\right) p_4(dg) < +\infty$ and

$$p_4(\mathcal{G} \backslash \{v\}) = +\infty, \quad \text{if} \quad p_3 = 0 \text{ and } p_2^e = 0 \text{ for all } e \in \mathcal{E}.$$

We define the partial and total reflection weights by (with $\frac{0}{0} := 0$)

$$q_2^e := \frac{p_2^e}{p_2} \quad \text{for } e \in \mathcal{E}, \quad \text{with } p_2 := \sum_{e \in \mathcal{E}} p_2^e,$$

and decompose the jump measure on the separate edges by introducing for each $e \in \mathcal{E}$ a measure p_4^e on $(0, +\infty)$ by

$$p_4^e(A) := p_4(\{e\} \times A), \quad A \in \mathscr{B}\big((0, +\infty)\big).$$

Then, as $d\big(v, (e,x)\big) = x$ on any star graph, the measures p_4^e, $e \in \mathcal{E}$, also satisfy

$$\int_{(0,\infty)} (1 - e^{-x})\, p_4^e(dx) < +\infty,$$

and $p_4^e\big((0,\infty)\big) = +\infty$ holds for some $e \in \mathcal{E}$, if $p_3 = 0$ and $p_2^e = 0$ for all $e \in \mathcal{E}$.

(21.1) Remark. Notice that we do not require the parameters $\big(p_1, (p_2^e)_{e\in\mathcal{E}}, p_3, p_4\big)$ to be normalized. This will turn out to be helpful in subsection 21.12. ■

(21.2) Remark. The upcoming, extensive construction in this section is only necessary for measures p_4 which admit $p_4\big(\mathcal{G}\backslash\{v\}\big) = +\infty$. If $p_1 \geq 0$, $p_2^e \geq 0$ for each $e \in \mathcal{E}$, $p_3 \geq 0$, and p_4 is a *finite* measure on $\mathcal{G}\backslash\{v\}$, normalized by

$$p_1 + \sum_{e\in\mathcal{E}} p_2^e + p_3 + \int_{\mathcal{G}\backslash\{v\}} (1 - e^{-x})\, p_4(d(e,x)) = 1,$$

there is a much simpler way to construct a Brownian motion X on the star graph $\mathcal{G}$ with generator domain

$$\mathscr{D}(A) = \Big\{ f \in \mathcal{C}_0^2(\mathcal{G}) : \\ p_1 f(v) - \sum_{e\in\mathcal{E}} p_2^e f_e'(v) + \frac{p_3}{2} f''(v) - \int_{\mathcal{G}\backslash\{v\}} \big(f(g) - f(v)\big)\, p_4(dg) = 0 \Big\},$$

which we briefly cover now:

If $p_2 > 0$, following the construction of [KPS12b] and [KPS12c], start with the Walsh process W on $\mathcal{G}$ with reflection weights $(q_2^e = p_2^e/p_2, e \in \mathcal{E})$ and local time $(L_t, t \geq 0)$ at the star vertex v. Then implement the stickiness parameter p_3 by "slowing down" W at the vertex via the canonical approach of time change, as given in [KPS12c, Section 2]: For $\gamma := p_3/p_2$, introduce the new time scale τ by defining its inverse by

$$\tau^{-1} : \mathbb{R}_+ \to \mathbb{R}_+, \quad t \mapsto t + \gamma\, L_t,$$

and consider the sticky Walsh process

$$W_t^s := W_{\tau(t)}, \quad t \geq 0,$$

with its new local time $L^s_t = L_{\tau(t)}$, $t \geq 0$, as seen in [KPS12c, Equation (2.22)]. Next, following [KPS12c, Section 3], introduce an exponentially distributed random variable S with rate $\beta := \frac{p_1 + p_4(\mathcal{G}\backslash\{v\})}{p_2}$, independent of W^s, and kill W^s when its local time exceeds S, that is, at the random time

$$\zeta_{\beta,\gamma} := \inf\{t \geq 0 : L^s_t > S\},$$

to obtain the process

$$W^g_t := \begin{cases} W^s_t, & t < \zeta_{\beta,\gamma}, \\ \Delta, & t \geq \zeta_{\beta,\gamma}. \end{cases}$$

[KPS12c, Theorem 3.7] shows that W^g is a Brownian motion on $\mathcal{G}$ with generator

$$\mathscr{D}(A^g) = \Big\{ f \in \mathcal{C}^2_0(\mathcal{G}) : \\ (p_1 + p_4(\mathcal{G}\backslash\{v\}))f(v) - \sum_{e\in\mathcal{E}} p^e_2 f'_e(v) + \frac{p_3}{2} f''(v) = 0 \Big\}.$$

Now adjoin an absorbing, isolated point $\square$ to the state space $\mathcal{G}$ and let X^q be the independent copies process resulting from W^g, as explained in subsection 13.1, where W^g is revived whenever it dies with the transfer measure

$$\forall g \in \mathcal{G} : \quad k^0(g, \cdot\,) := q, \quad \text{with} \quad q := \frac{p_1\,\varepsilon_\square + p_4}{p_1 + p_4(\mathcal{G}\backslash\{v\})}.$$

Then, by following exactly the proof of lemma (21.67), using [KPS12b, Lemma 1.12] and [KPS12c, Corollary 3.5] for the results needed on ψ_α, we see that X^q is a Brownian motion on $\mathcal{G} \cup \{\square\}$ with generator

$$\mathscr{D}(A^q) \subseteq \Big\{ f \in \mathcal{C}^2_0(\mathcal{G}) : \\ -\sum_{e\in\mathcal{E}} p^e_2 f'_e(v) + \frac{p_3}{2} f''(v) - \int_{(\mathcal{G}\backslash\{v\})\cup\{\square\}} \big(f(g) - f(v)\big)\,(p_1\,\varepsilon_\square + p_4)(dg) = 0 \Big\}.$$

Finally, map the absorbing set $\{\square\}$ to Δ as explained in subsection 12.2 to form the Brownian motion $X := \psi(X^q)$ on $\mathcal{G}$. Then lemma (22.2) shows that the generator of X satisfies

$$\mathscr{D}(A) \subseteq \Big\{ f \in \mathcal{C}^2_0(\mathcal{G}) : \\ p_1 f(v) - \sum_{e\in\mathcal{E}} p^e_2 f'_e(v) + \frac{p_3}{2} f''(v) - \int_{\mathcal{G}\backslash\{v\}} \big(f(g) - f(v)\big)\,p_4(dg) = 0 \Big\}.$$

However, X is a Brownian motion on a star graph, so lemma (20.25) asserts that $\mathscr{D}(A)$ indeed equals the right-hand set.

If $p_2 = 0$ and $p_3 > 0$, the resulting process is simpler. In this case, the construction follows exactly the same lines as above, except that instead of considering a standard

Walsh process W, we start with a Walsh process W^a absorbed at the star vertex v, which is then killed when W^a has stopped at v for an independent, exponentially distributed time with rate $\beta := \frac{p_1+p_4(\mathcal{G}\backslash\{v\})}{p_3}$. As seen in [KPS12b, Subsection 1.4], the domain of the resulting Brownian motion W^e reads

$$\begin{aligned}\mathscr{D}(A^e) &= \Big\{ f \in \mathcal{C}_0^2(\mathcal{G}) : \beta f(v) + \frac{1}{2} f''(v) = 0 \Big\} \\ &= \Big\{ f \in \mathcal{C}_0^2(\mathcal{G}) : (p_1 + p_4(\mathcal{G}\backslash\{v\})) f(v) + \frac{p_3}{2} f''(v) = 0 \Big\}.\end{aligned}$$

The remaining construction for the implementation of the jumps then proceeds as above.

The case $p_2 = 0$ and $p_3 = 0$ is impossible if p_4 is finite, as seen in (20.22). ■

In the following, we will always assume that $p_2 > 0$ or p_4 is infinite.

21.1. Definitions

The main ingredients for our construction will be a Walsh process W on $\mathcal{G}$ and a family of subordinators $(Q^e, e \in \mathcal{E})$, which are used to control the jumps to the respective external edges. We are introducing them on an appropriate, mutual space now:

Let $\hat{W} = \big(\Omega^W, \mathscr{F}^W, (\mathscr{F}_t^W)_{t\geq 0}, (\hat{W}_t)_{t\geq 0}, (\hat{\Theta}_t^W)_{t\geq 0}, (\mathbb{P}_{(e,x)}^W)_{(e,x)\in\mathcal{G}}\big)$ be a Walsh process on $\mathcal{G}$ with edge weights $q_2^e = p_2^e/p_2$, $e \in \mathcal{E}$,[3] and $(\hat{L}_t, t \geq 0)$ be the local time of $\hat{W}$ at v. We have for all $s, t \geq 0$ (as the local time is an additive functional by lemma (15.2)):

$$\hat{W}_s \circ \hat{\Theta}_t^W = \hat{W}_{s+t}, \qquad \hat{L}_s \circ \hat{\Theta}_t^W = \hat{L}_{s+t} - \hat{L}_t.$$

For each $e \in \mathcal{E}$, let $\hat{Q}^e = \big(\Omega^{Q,e}, \mathscr{F}^{Q,e}, (\mathscr{F}_t^{Q,e})_{t\geq 0}, (\hat{Q}_t^e)_{t\geq 0}, (\hat{\Theta}_t^{Q,e})_{t\geq 0}, (\mathbb{P}_q^{Q,e})_{q\in\mathbb{R}}\big)$ be a subordinator with Lévy measure p_4^e and drift 0 realized as canonical coordinate process on the space $\Omega^{Q,e}$ of all càdlàg functions. By example (6.32), we then have natural translation and centering operators $(\hat{\gamma}_q^{Q,e}, q \in \mathbb{R})$ and $\hat{\Gamma}^{Q,e}$ at our disposal.

Let $\hat{Q} := (\hat{Q}^1, \ldots, \hat{Q}^n)$ be the Cartesian product of the processes $(\hat{Q}^e, e \in \mathscr{E})$, that is,

$$\hat{Q} = \big(\Omega^Q, \mathscr{F}^Q, (\mathscr{F}_t^Q)_{t\geq 0}, (\hat{Q}_t)_{t\geq 0}, (\hat{\Theta}_t^Q)_{t\geq 0}, (\mathbb{P}_{(q^1,\ldots,q^n)}^Q)_{(q^1,\ldots,q^n)\in\mathbb{R}^n}\big)$$

with sample space $\Omega^Q := \prod_{e\in\mathscr{E}} \Omega^{Q,e}$, σ-algebra $\mathscr{F}^Q := \bigotimes_{e\in\mathscr{E}} \mathscr{F}^{Q,e}$, the process being $\hat{Q}_t := (\hat{Q}_t^1, \ldots, \hat{Q}_t^n)$ for any $t \geq 0$, equipped with its natural filtration $(\mathscr{F}_t^Q, t \geq 0)$, shift operators $\hat{\Theta}_t^Q := \hat{\Theta}_t^{Q,1} \times \cdots \times \hat{\Theta}_t^{Q,n}$, $t \geq 0$, translation operators $\hat{\gamma}_{(q^1,\ldots,q^n)}^Q := \hat{\gamma}_{q^1}^{Q,1} \times \cdots \times \hat{\gamma}_{q^n}^{Q,n}$, $q^1, \ldots, q^n \in \mathbb{R}$, centering operator $\hat{\Gamma}^Q := \hat{\Gamma}^{Q,1} \times \cdots \times \hat{\Gamma}^{Q,n}$, as well as initial measures $\mathbb{P}_{(q^1,\ldots,q^n)}^Q := \mathbb{P}_{q^1}^{Q,1} \otimes \cdots \otimes \mathbb{P}_{q^n}^{Q,n}$ for all $q^1, \ldots, q^n \in \mathbb{R}$.

By construction, the processes $\hat{Q}^1, \ldots, \hat{Q}^n$ are independent, so by lemma (6.25), the set N of simultaneous jumps of $\hat{Q}^1, \ldots, \hat{Q}^n$ is a measurable null set. As the natural shift,

[3] If $p_2 = 0$ (this requires $p_4 = +\infty$), then consider a Walsh process with arbitrary weight distribution, for instance use $q_2^e = 1/n$ for all $e \in \mathcal{E}$. Any choice leads to the correct boundary condition, as will be seen in subsection 21.11.

translation and centering operators do not change or introduce new discontinuities, they map $\Omega^Q \backslash N$ into itself. Therefore, we are able to restrict the process $\hat{Q}$ together with all its operators to $\Omega^Q \backslash N$, naming this new sample space again Ω^Q. Thus, at most one of the processes $\hat{Q}^1, \ldots, \hat{Q}^n$ has a jump at any given time $t > 0$.

Now combine the Walsh process and the subordinators independently in one space by defining the product space $\Omega := \Omega^W \times \Omega^Q$ with σ-algebra $\mathscr{F} := \mathscr{F}^W \otimes \mathscr{F}^Q$ and product measures $\mathbb{P}_{(e,x),(q^1,\ldots,q^n)} := \mathbb{P}^W_{(e,x)} \otimes \mathbb{P}^Q_{(q^1,\ldots,q^n)}$ for $(e,x) \in \mathcal{G}$, $(q^1, \ldots, q^n) \in \mathbb{R}^n$. As we will typically want to start the subordinators at the origin, we furthermore set

$$\mathbb{P}_g := \mathbb{P}_{g,(0,\ldots,0)}, \quad g \in \mathcal{G}.$$

With the help of the canonical projections $\pi^W \colon \Omega^W \times \Omega^Q \to \Omega^W$, $\pi^Q \colon \Omega^W \times \Omega^Q \to \Omega^Q$, and $\pi^{Q,e} \colon \Omega^{Q,1} \times \cdots \times \Omega^{Q,n} \to \Omega^{Q,e}$, $e \in \mathcal{E}$, we set for any $t \geq 0$, $q \in \mathbb{R}^n$:

$$\begin{aligned}
&W_t := \hat{W}_t \circ \pi^W, \quad L_t := \hat{L}_t \circ \pi^W, \quad \Theta^W_t := (\hat{\Theta}^W_t \circ \pi^W) \times \pi^Q, \\
&Q_t := \hat{Q}_t \circ \pi^Q, \quad Q^e_t := \hat{Q}_t \circ \pi^{Q,e} \circ \pi^Q, \ e \in \mathcal{E}, \\
&\Theta^Q_t := \pi^W \times (\hat{\Theta}^Q_t \circ \pi^Q), \quad \gamma^Q_q := \pi^W \times (\hat{\gamma}^Q_q \circ \pi^Q), \quad \Gamma^Q := \pi^W \times (\hat{\Gamma}^Q \circ \pi^Q).
\end{aligned}$$

Define the processes $(P_t, t \geq 0)$ and $(P^e_t, t \geq 0)$, $e \in \mathcal{E}$, by

$$\begin{aligned}
P_e(t) &:= p_2\, t + Q^e(t) + \sum_{f \in \mathcal{E}, f \neq e} Q^f(t-), \quad e \in \mathcal{E}, \\
P(t) := P_0(t) &:= p_2\, t + \sum_{e \in \mathcal{E}} Q^e(t),
\end{aligned}$$

where, as usual, we set $Q^e(t-) := \lim_{s \uparrow\uparrow t} Q^e(s)$ for $t > 0$, and $Q^e(0-) := Q^e(0)$. Furthermore, for any vector $\eta = (\eta^e, e \in \mathcal{E})$ of real numbers with $\eta^e \leq 0$ for all except at most one $e \in \mathcal{E}$, we construct a function $E(\eta) \colon \mathcal{G} \to \mathcal{E}$ by setting

$$E(\eta)(l,x) := \begin{cases} e, & \exists e \in \mathcal{E} : \eta^e > 0, \\ l, & \forall e \in \mathcal{E} : \eta^e \leq 0. \end{cases}$$

For all $e \in \mathcal{E}$, define the processes $(\eta^e_t, t \geq 0)$ by

$$\eta^e_t := \big(P_e P^{-1} - \mathrm{id}\big)(L_t), \quad t \geq 0.$$

Finally, we define the stochastic process $(X_t, t \geq 0)$ on $\mathcal{G}$, by setting

$$X_t := \big(E(\eta^e_t, e \in \mathcal{E}) \circ W_t, \eta_t + |W_t|\big), \quad t \geq 0.$$

For later use, we also set

$$\varrho_t := P^{-1}(L_t) = \inf\{s \geq 0 : P_s > L_t\}, \quad t \geq 0.$$

21.2. Remarks on the Definition

The process $(X_t, t \geq 0)$ will turn out to be a Brownian motion on $\mathcal{G}$ which realizes the reflection parameters $(p_2^e, e \in \mathcal{E})$ and the jump measure p_4 in the boundary condition of the generator. It is a generalization of Itô–McKean's construction on the half line, which was explained in section 16.

Indeed, the local coordinate of X_t is, by definition, just Itô–McKean's "basic" Brownian motion on the half line, namely $PP^{-1}(L_t) - L_t + |W_t|$. However, we need to adjust their construction by a process which controls the edges of the Brownian motion: We cannot use the edge process of $(W_t, t \geq 0)$, as this would change the edge whenever $|W_t|$ is at v, even if the translated excursion $PP^{-1}(L_t) - L_t + |W_t|$ is not finished yet, see figure 21.1.

Therefore, we need to "overwrite" the edge process of $(W_t, t \geq 0)$ to being constant on some edge $e \in \mathcal{E}$, as long as there is a "jump excursion" on this edge. There does not seem to be a straight-forward way to define such an "overwriting process". Our solution is the introduction of the auxiliary processes P_e, $e \in \mathcal{E}$, which are modifications of the process P, namely, being right continuous at the jump times of their own edge e, and left continuous at jump times of the other edges. Therefore, on jump excursions on their own edge, $P_e P^{-1}$ will have "upper triangles" (which is equivalent to $\eta_t^e > 0$) just as PP^{-1}, but "lower triangles" (which is equivalent to $\eta_t^e < 0$) on jump excursions to other edges, see figure 21.2 and remark (21.7). Thus, it is possible to derive the current edge of a jump excursion from the paths of $t \mapsto P_e P^{-1}(L_t)$, $e \in \mathcal{E}$, or equivalently from the processes $(\eta_t^e, t \geq 0)$, $e \in \mathcal{E}$.

The process $E(\eta_t^e, e \in \mathcal{E})$ thus chooses which (if any) of the jump excursion times $\eta_t^e, e \in \mathcal{E}$, is currently greater than zero (that is, which "triangle" is the "upper triangle"), and holds the motion on this edge $e \in \mathcal{E}$ for the remaining length $\eta_t^e > 0$ of this excursion; during this time Itô–McKean's Brownian motion $\eta_t + |W_t|$ on the local coordinate behaves like a standard Brownian motion. On the other hand, if all jump excursion times are zero, then $E(\eta_t^e, e \in \mathcal{E})$ just uses the original edge of the Walsh process $(W_t, t \geq 0)$ and $\eta_t = 0$ holds true, so both coordinates of X_t coincide with both original coordinates of W_t. This means that, as long as there is no jump excursion, X_t is just W_t. We will make these explanations rigorous now.

In order to ensure that the process $(X_t, t \geq 0)$ is well-defined, it is necessary that there is at most one $e \in \mathcal{E}$ with $\eta_t^e > 0$ at any time $t \geq 0$. This will be shown below in lemma (21.10). To this end, we need to analyze the defining functions $P_e P^{-1}$, $e \in \mathcal{E} \cup \{0\}$. The difference between the functions P_e, $e \in \mathcal{E}$, are rather subtle: If we define the set of all jumps of the subordinator Q^e by $J_e := \{t > 0 : \Delta Q^e(t) \neq 0\}$, $e \in \mathcal{E}$, then the set of all jumps reads $J := \biguplus_{e \in \mathcal{E}} J_e$, as there are no simultaneous jumps. By definition, $P_e(t) = P(t)$ holds true for all $e \in \mathcal{E}$ if $t \in \complement J$, whereas for $t \in J$, we have

$$P_e(t) = \begin{cases} P(t), & t \in J_e, \\ P(t-), & t \notin J_e, \end{cases}$$

that is, the function P_e is right continuous at the jumps of Q^e, and left continuous with a positive jump discontinuity at the jumps of all other subordinators Q^f, $f \neq e$. We collect these first findings:

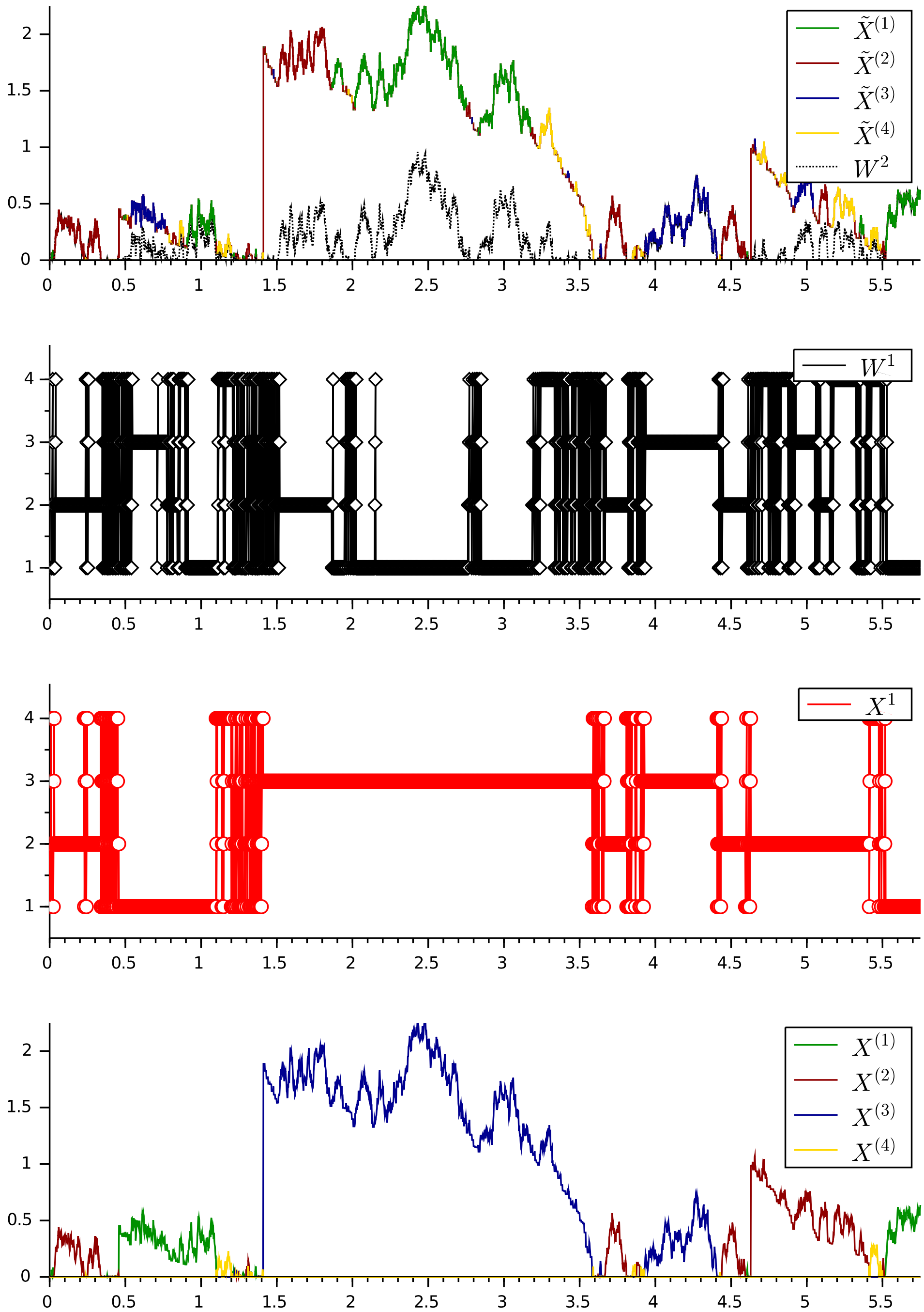


Figure 21.1: Construction approach for Brownian motions on a star graph: Illustration of the Walsh process $W_t = (W_t^1, W_t^2)$ and the resulting Brownian motion with jumps $X_t = (X_t^1, X_t^2) = \big(E(\eta_t^e, e \in \mathcal{E}) \circ W_t, PP^{-1}(L_t) - L_t + |W_t|\big)$. The incorrect process $\widetilde{X}_t = (W_t^1, PP^{-1}(L_t) - L_t + |W_t|)$, pictured in the first graph, already implements the desired radial process, however switches edges during jump excursions whenever the original process W hits the vertex. Thus, the edge process W^1 must be transformed to X^1 in order to "hold" the current edge during jump excursions. $X^{(e)}$, $\widetilde{X}^{(e)}$ represent the process parts of X, $\widetilde{X}$ on the corresponding edges $e \in \mathcal{E} = \{1, 2, 3, 4\}$.

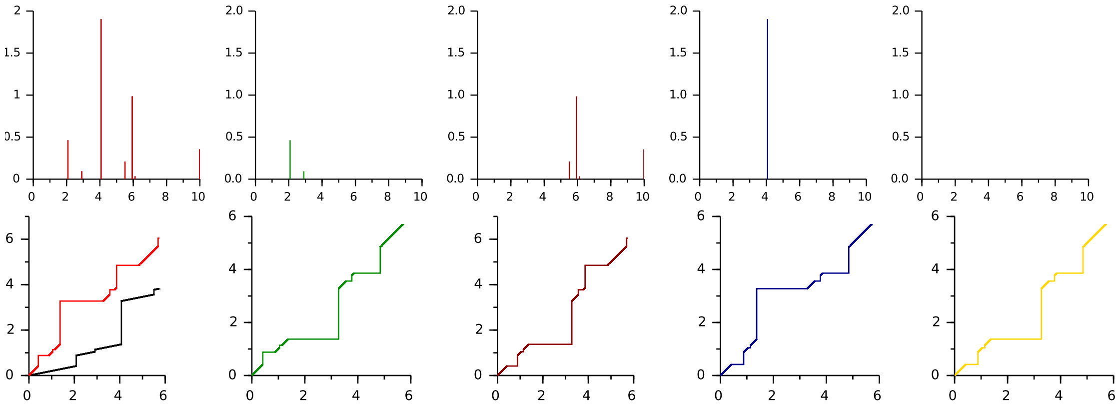


Figure 21.2: Illustration of the extension of Itô–McKean's approach to the star graph: The left-hand graphs show the jumps of the complete subordinator Q and the resulting processes P (black) and PP^{-1} (red) used in Itô–McKean's construction in the half-line case. The other graphs show (in the first row) the decomposition into the subordinator parts Q^e of the corresponding edges $e \in \mathcal{E}$ (with color assignment like in figure 21.1), as well as (in the second row) the resulting processes $P_e P^{-1}$, which feature the needed "upper triangles" for "own jumps" and "lower triangles" for "other jumps".

(21.3) Lemma. *For every $e \in \mathcal{E}$, let $J_e = \{t > 0 : \Delta Q^e(t) \neq 0\}$ be the set of all jumps of the subordinator Q^e, and set $J = \biguplus_{e \in \mathcal{E}} J_e$. Then, for all $e \in \mathcal{E}$, $t \geq 0$,*

$$P_e(t) = \begin{cases} P(t), & t \in J_e \cup \complement J, \\ P(t-), & t \in J \cap \complement J_e. \end{cases}$$

Before we are able to proceed with the analysis of $P_e P^{-1}$, we need to collect some properties of *pseudo-inverses* (or *generalized inverses*). These results can mostly be found scattered in the literature, for instance in [EH13]. There, the authors consider the left continuous pseudo-inverse instead (see [FWTK12] for the differences in the definitions), so we need to reiterate some proofs.

(21.4) Definition. *An increasing function $f\colon [0,\infty) \to [0,\infty]$ has a level of constancy at $t_0 \geq 0$ of length $h > 0$, if $f(t) = f(t_0+)$ for all $t \in (t_0, t_0+h)$, $f(t) < f(t_0)$ for all $t < t_0$, and $f(t) > f(t_0+)$ for all $t > t_0 + h$.*

(21.5) Lemma. *Let $P\colon [0,\infty) \to [0,\infty)$ be a right continuous, strictly increasing function. Then, the generalized inverse*

$$P^{-1}\colon [0,\infty) \to [0,\infty], \quad t \mapsto P^{-1}(t) := \inf\{s \geq 0 : P(s) > t\}$$

admits:

(i) P^{-1} is right continuous and increasing;

(ii) *for all* $t \geq 0$*:* $P^{-1}P(t) = P^{-1}(P(t-)) = t$*;*

(iii) *for all* $t \geq 0$ *with* $P^{-1}(t) < +\infty$*:* $PP^{-1}(t) \geq t$*;*

(iv) *for all* $t \in \operatorname{ran}(P)$*:* $PP^{-1}(t) = t$*;*

(v) *for all* $t, u \geq 0$*:* $P^{-1}(t) \leq u$*, if and only if* $t \leq P(u)$*;*

(vi) P^{-1} *is continuous,*

(vii) P *has a jump at* $t > 0$ *of height* h*, if and only if* P^{-1} *has a level of constancy at* $P(t-)$ *of length* h*.*

Proof. (i) Right continuity has been proved in [FWTK12, Theorem 2]. The definition of P^{-1} directly implies that it is increasing.

(ii) Let $t \geq 0$. As P is strictly increasing, it is $\{s \geq 0 : P(s) > P(t)\} = (t, \infty)$, and $\{s \geq 0 : P(s) > P(t-)\}$ equals either (t, ∞) or $[t, \infty)$, depending on whether t is a point of continuity or not.

(iii) Let $t \geq 0$ with $P^{-1}(t) < +\infty$. Then there exists a sequence $(s_n, n \in \mathbb{N})$ in $\{s \geq 0 : P(s) > t\}$ which strictly decreases to $P^{-1}(t)$. But then $P(s_n) > t$ for all $n \in \mathbb{N}$, and as P is right continuous, $PP^{-1}(t) = \lim_n P(s_n) \geq t$.

(iv) Let $t \in \operatorname{ran}(P)$. Then, as P is injective, there exists one and only one $\tilde{s} \in [0, \infty)$ with $P(\tilde{s}) = t$. But then $P(s) > t$ for all $s > \tilde{s}$, therefore $P^{-1}(t) = \tilde{s}$ and $PP^{-1}(t) = P(\tilde{s}) = t$.

(v) If $P^{-1}(t) \leq u$, then $P(u) \geq PP^{-1}(t) \geq t$ by (iii). If $P(u) \geq t$, then, as P is strictly increasing, $\{s \geq 0 : P(s) > t\} \subseteq (u, \infty)$, so $P^{-1}(t) \leq u$.

(vi) Assume P^{-1} is discontinuous. As it is right continuous, there exist $t > 0$, $a < b$, such that

$$P^{-1}(t-) \leq a < b = P^{-1}(t).$$

Let $u \in [a, b)$. Then, for all $s < t$, we have $P^{-1}(s) \leq P^{-1}(t-) \leq a \leq u$, which by (v) implies that $s \leq P(u)$. Thus, $P(u) \geq t$ holds true. But $P(u) \ngtr t$, because $u < b = \inf\{s \geq 0 : P(s) > t\}$. Therefore, $P(u) = t$ for all $u \in [a, b)$, which contradicts the strict increase of P.

(vii) Let $t_0 > 0$ be a point of discontinuity of P with jump height $h > 0$, that is,

$$P(t_0) - P(t_0-) = h.$$

Let $y_0 := P(t_0-)$. Then, as $P(s) \leq P(t_0-) = y_0$ for all $s \leq t_0$ and $P(t_0) = y_0 + h$,

$$P^{-1}(y_0) = t_0 = P^{-1}(y_0 + h).$$

As P^{-1} is increasing, it must be constant on (y_0, y_0+h). It remains to show that this set is maximal: As P is strictly increasing, for any $y < y_0$ there exists $s < t_0$ such that $P(s) < y < y_0 = P(t_0-)$, so $P^{-1}(y) \le s < t_0$. Furthermore, as P is right continuous and $P(t_0) = y_0 + h$, it is $P^{-1}(y) > t_0$ for any $y > y_0 + h$.

Now, let (y_0, y_0+h) be a constancy set of length h for P^{-1}, with

$$P^{-1}(y_0) = P^{-1}(y_0+h) = t_0.$$

Then $P(t_0) \ge y_0 + h$ by (v). But indeed $P(t_0) = y_0 + h$, as otherwise there exists $y > y_0 + h$ with $P(t_0) > y$, which would imply $P^{-1}(y) = P^{-1}(y_0+h) = t_0$. On the other hand, $P(t) \le y_0$ for all $t < t_0$, so $P(t_0-) \le y_0$. But $P(t_0-) = y_0$, as otherwise there exists $y < y_0$ with $P(t_0-) < y$, which would imply $P^{-1}(y) = P^{-1}(y_0) = t_0$. Therefore, there is a jump at $t_0 = P^{-1}(y_0)$ with height $P(t_0) - P(t_0-) = h$.

□

(21.6) Lemma. *Let $L\colon [0,\infty) \to [0,\infty)$ be a continuous, increasing function. Then, the generalized inverses*

$$L^{-1}\colon [0,\infty) \to [0,\infty], \quad t \mapsto L^{-1}(t) := \inf\{s \ge 0 : L(s) > t\}$$

and

$$L_-^{-1}\colon [0,\infty) \to [0,\infty], \quad t \mapsto L_-^{-1}(t) := \inf\{s \ge 0 : L(s) \ge t\}$$

admit:

(i) L^{-1} is right continuous and increasing;

(ii) for all $t, u \ge 0$: $L^{-1}(t) < u$, if and only if $t < L(u)$;

(iii) L_-^{-1} is left continuous and increasing;

(iv) for all $t, u \ge 0$: $L_-^{-1}(t) \le u$, if and only if $t \le L(u)$.

Proof. (i) is covered by [FWTK12, Theorem 2], (iii) by [EH13, Proposition 1].

Turning to (ii), let $L^{-1}(t) < u$. Then there exists a sequence $(s_n, n \in \mathbb{N})$ decreasing to $L^{-1}(t)$ with $L(s_n) > t$ for all $n \in \mathbb{N}$. But then $s_n < u$ for almost all $n \in \mathbb{N}$, and as L is increasing, it follows that $L(u) \ge L(s_n) > t$. On the other hand, if $t < L(u)$, then, as L is continuous, $L(u') > t$ for some $u' < u$, and therefore $L^{-1}(t) \le u' < u$.

It remains to prove (iv): If $L_-^{-1}(t) \le u$, then by monotonicity and continuity of L, $L(u) \ge L\big(L_-^{-1}(t)\big) \ge t$. Conversely, if $t \le L(u)$, then $u \in \{s \ge 0 : L(s) \ge t\}$ and thus $L_-^{-1}(t) \le u$. □

We are ready to analyze the functions $t \mapsto P_e P^{-1}(t)$, $e \in \mathcal{E} \cup \{0\}$:

(21.7) Remark. The function P is strictly increasing, as $p_2 > 0$ or $p_4^e\big((0,\infty)\big) = +\infty$ for at least one $e \in \mathcal{E}$ (cf. theorem (6.19)). Thus, PP^{-1} has a level of constancy at some time $P(t-)$ of length h, if and only if P^{-1} has, so by (vii) of lemma (21.5), if and only if there exists a jump of P at time $t \in J$ of height h. Therefore, we can decompose $\mathbb{R}_+$ into $\mathscr{D} := \{t \geq 0 : PP^{-1}(t) = t\}$ and

$$\mathscr{D}^\complement = \bigcup_{n\in\mathbb{N}} [l_n^-, l_n^+),$$

where each interval (l_n^-, l_n^+) corresponds to some jump of P at time t_n of height l_n via

$$l_n^- = P(t_n-), \quad l_n^+ - l_n^- = l_n, \quad n \in \mathbb{N}.$$

Then, by definition of P, it is

$$l_n^+ = l_n^- + l_n = P(t_n-) + \big(P(t_n) - P(t_n-)\big) = P(t_n), \quad n \in \mathbb{N},$$

and by (ii) of lemma (21.5), we also have

$$P^{-1}(l_n^-) = P^{-1}\big(P(t_n-)\big) = t_n, \quad n \in \mathbb{N}.$$

This gives for each $n \in \mathbb{N}$,

$$\forall t \in \big[l_n^-, l_n^+\big) = \big[P(t_n-), P(t_n)\big) : \quad PP^{-1}(t) = PP^{-1}(l_n^-) = P(t_n). \tag{21.8}$$

For every P_eP^{-1}, $e \in \mathcal{E}$, the same decomposition holds true, that is, we have $P_eP^{-1}(t) = P_eP_e^{-1}(t) = t$ for all $t \in \mathscr{D}$. However, observe that by lemma (21.3),

$$\forall t \in \big[l_n^-, l_n^+\big) = \big[P(t_n-), P(t_n)\big) : \quad P_eP^{-1}(t) = P_e(t_n) = \begin{cases} P(t_n), & t_n \in J_e, \\ P(t_n-), & t_n \notin J_e. \end{cases} \tag{21.9}$$

■

(21.10) Lemma. *For all $t \geq 0$, the following holds true:*

(i) There is at most one $e \in \mathcal{E}$ with $\eta_t^e > 0$.

(ii) $\eta_t > 0$, if and only if $\eta_t^e > 0$ for exactly one $e \in \mathcal{E}$.

Proof. As there are no simultaneous jumps by construction, all jump times t_n, $n \in \mathbb{N}$, are pairwise distinct, so the intervals $\big(P(t_n-), P(t_n)\big)$, $n \in \mathbb{N}$, are pairwise disjoint and there is exactly one $e \in \mathcal{E}$ with $t_n \in J_e$. Thus, we have for all $e \in \mathcal{E}$, $n \in \mathbb{N}$, $t \in \big[l_n^-, l_n^+\big) = \big[P(t_n-), P(t_n)\big)$,

$$P_eP^{-1}(t) - t = \begin{cases} P(t_n) - t > 0, & t_n \in J_e, \\ P(t_n-) - t \leq 0, & t_n \notin J_e, \end{cases}$$

and $P_eP^{-1}(t) - t = 0$ for all $t \in \mathscr{D}$. Therefore, for any $t \geq 0$, there is at most one $e \in \mathcal{E}$ with $P_eP^{-1}(t) - t > 0$, and in this case $t \in [l_n^-, l_n^+)$ for some $n \in \mathbb{N}$, which is equivalent to $PP^{-1}(t) - t > 0$ by (21.8) and (21.9). □

The path behavior of X is now clear: For $L_t \in \mathscr{D}$, we have $\eta_t = PP^{-1}(L_t) - L_t = 0$, and $\eta_t^e = 0$ for all $e \in \mathcal{E}$ by lemma (21.10), so for these times, it is $X_t = W_t$ by definition. Otherwise, if $L_t \in [l_n^-, l_n^+)$ with $[l_n^-, l_n^+)$ corresponding to a jump $\big(t_n, (e_n, l_n)\big)$, we have $\eta_t^{e_n} = \eta_t > 0$, so $E(\eta_t^e, e \in \mathcal{E}) \circ W_t = e_n$ and $|X_t| = \eta_t + |W_t| = P(t_n) - L_t + |W_t| = l_n^+ - L_t + |W_t|$, which for $t \in L_-^{-1}\big([l_n^-, l_n^+)\big)$ behaves like a Brownian motion started at $l_n^+ - l_n^- = l_n$. In total, we get

$$X_t = \begin{cases} W_t, & L_t \in \mathscr{D}, \\ \big(e, l_n^+ - L_t + |W_t|\,\big), & L_t \in [l_n^-, l_n^+) \text{ with } t_n \in J_e. \end{cases} \tag{21.11}$$

(21.12) Lemma. *The process $(X_t, t \geq 0)$ is right continuous, and it is continuous on any excursion away from v, that is: For any $t \geq 0$ with $X_t \neq v$, $(X_t, t \geq 0)$ is continuous on $[t, t_0]$, with $t_0 := \inf\{s \geq t : X_s = v\}$.*

Proof. As $(W_t, t \geq 0)$ and $(L_t, t \geq 0)$ are continuous, and L_t only grows if W_t is at v, the edge of X_t only changes at some time $t \geq 0$, if either the edge of W_t changes or L_t grows over some l_n^+, in which case $l_n^+ - L_t + |W_t| = 0$ holds true. Thus, as the second coordinate $(\eta_t + |W_t|\,, t \geq 0)$ is right continuous, and the first coordinate only changes if the radial part is at the origin, the resulting process $(X_t, t \geq 0)$ is right continuous.

X is away from v if either W is or if $L \in [l_n^-, l_n^+)$ for some $n \in \mathbb{N}$. In both cases the process behaves continuously in the open interior of these times, which follows from the representation (21.11) and the continuity of W and L. For $t \in L_-^{-1}\big([l_n^-, l_n^+)\big)$, equation (21.11) gives $X_t = \big(e, l_n^+ - L_t + |W_t|\,\big)$, thus we have

$$t_0 := \inf\{s \geq t : X_s = v\} = \inf\{s \geq 0 : L_s \geq l_n^+\},$$

and for every sequence $(t_n, n \in \mathbb{N})$ of times which strictly increases to t_0, $(X_{t_n}, n \in \mathbb{N})$ converges to $(e, 0) = v$. But as X is right continuous and $\{v\}$ is closed, we have $X_{t_0} = v$, so X is also continuous at t_0. $\square$

21.3. Shift and Translation Operators for X

Define the operators $(\gamma_x^P, x \in \mathbb{R})$ on Ω by

$$\gamma_x^P := \gamma_{x/n,\ldots,x/n}^Q, \quad x \in \mathbb{R}.$$

Then $(\gamma_x^P, x \in \mathbb{R})$ and Γ^Q are translation and centering operators for all P_e, $e \in \mathcal{E} \cup \{0\}$, because for $e \in \mathcal{E}$ (for $e = 0$, namely $P_e = P$, the calculation is completely analogous), we obtain by shifting the underlying processes Q^e, $e \in \mathcal{E}$,

$$\begin{aligned} P_e(t) \circ \gamma_x^P &= p_2\, t + Q^e(t) \circ \gamma_{x/n,\ldots,x/n}^Q + \sum_{e \in \mathcal{E}} Q^e(t-) \circ \gamma_{x/n,\ldots,x/n}^Q \\ &= p_2\, t + Q^e(t) + \frac{x}{n} + \sum_{f \neq e} \big(Q^f(t-) + \frac{x}{n}\big) \\ &= P_e(t) + x, \end{aligned} \tag{21.13}$$

and, by using the definition $Q^f(0-) = Q^f(0)$,

$$
\begin{aligned}
P_e(t) \circ \Gamma^Q &= p_2\, t + Q^e(t) \circ \Gamma^Q + \sum_{f \neq e} Q^f(t-) \circ \Gamma^Q \\
&= p_2\, t + Q^e(t) + \sum_{f \neq e} Q^j(t-) - \big(p_2\, 0 + Q^e(0) + \sum_{f \neq e} Q^j(0-)\big) \qquad (21.14)\\
&= P_e(t) - P_e(0).
\end{aligned}
$$

Define the operators Θ_t^X, $t \geq 0$, by

$$\Theta_t^X := \Theta_t^W \otimes (\gamma_{-L_t}^P \circ \Theta_{\varrho_t}^Q),$$

that is, for all $\omega = (\omega^W, \omega^Q) \in \Omega$,

$$\Theta_t^X(\omega) = \big(\Theta_t^W(\omega^W), \gamma_{-L_t(\omega)}^P(\Theta_{\varrho_t(\omega)}^Q(\omega^Q))\big).$$

In order not to complicate the notation even more, we will also write Θ^W, Θ^Q for the lifts of these shift operators from Ω^W, Ω^Q to Ω: For all $\omega = (\omega^W, \omega^Q) \in \Omega$, the formulas $\Theta_t^W(\omega) = \Theta_t^W(\omega^W)$, $\Theta_t^Q(\omega) = \Theta_t^Q(\omega^Q)$ will be used implicitly in this section.

(21.15) Lemma. *$(\Theta_t^X, t \geq 0)$ is a family of shift operators for X.*

Proof. Fix $s, t \geq 0$. It is clear that $\Theta_t^X : \Omega \to \Omega$, as $\Theta_t^W : \Omega^W \to \Omega^W$, $\Theta_t^Q : \Omega^Q \to \Omega^Q$, $\varrho_t(\omega) \geq 0$ for all $\omega \in \Omega$ and $\gamma_x^P : \Omega^Q \to \Omega^Q$ for all $x \in \mathbb{R}$.

We begin by calculating the shift on the subordinator: For all $u \geq 0$, we have

$$
\begin{aligned}
P(\gamma_{-L_t}^P \circ \Theta_{\varrho_t}^Q)^{-1}(u) &= \inf\{s \geq 0 : P(\gamma_{-L_t}^P \circ \Theta_{\varrho_t}^Q)(s) > u\} \\
&= \inf\{s \geq 0 : P(s + \varrho_t) - L_t > u\} \\
&= \big(\inf\{s \geq 0 : P(s) > u + L_t\} - \varrho_t\big) \vee 0 \\
&= \big(P^{-1}(u + L_t) - \varrho_t\big)^+.
\end{aligned}
$$

By noting that $(L_t, t \geq 0)$ is an additive functional and $P^{-1}(L_{s+t}) \geq P^{-1}(L_t)$, we get

$$
\begin{aligned}
P^{-1}(L_s) \circ \Theta_t^X &= P(\gamma_{-L_t} \circ \Theta_{\varrho_t}^Q)^{-1}(L_{s+t} - L_t) \\
&= \big(P^{-1}(L_{s+t}) - \varrho_t\big)^+ \qquad (21.16)\\
&= P^{-1}(L_{s+t}) - \varrho_t.
\end{aligned}
$$

Let $e \in \mathcal{E} \cup \{0\}$. Then, by applying the shift Θ_t^X and the above findings, we obtain

$$
\begin{aligned}
(P_e P^{-1}(L_s) - L_s) \circ \Theta_t^X &= P_e(\gamma_{-L_t}^P \circ \Theta_{\varrho_t}^Q)\big(P^{-1}(L_s) \circ \Theta_t^X\big) - L_s \circ \Theta_t^W \\
&= P_e\big(P^{-1}(L_s) \circ \Theta_t^X + \varrho_t\big) - L_t - (L_{s+t} - L_t) \\
&= P_e P^{-1}(L_{s+t}) - L_{s+t}.
\end{aligned}
$$

By inserting the last two formulas into the definition of X and additionally using

$$W_s \circ \Theta_t^X = W_s \circ \Theta_t^W = W_{s+t},$$

we get $X_s \circ \Theta_t^X = X_{s+t}$.

It remains to prove $\Theta_s^X \circ \Theta_t^X = \Theta_{s+t}^X$. We calculate for $\omega = (\omega^W, \omega^Q)$

$$\begin{aligned}\Theta_s^X(\Theta_t^X(\omega)) &= \big(\Theta_s^W(\Theta_t^W(\omega^W)), \gamma^P_{-L_s(\Theta_t(\omega))}(\Theta^Q_{\varrho_s(\Theta_t(\omega))}(\gamma^P_{-L_t(\omega)}(\Theta^Q_{\varrho_t(\omega)}(\omega^Q))))\big)\\ &= \big(\Theta^W_{s+t}(\omega^W), \gamma^P_{-L_{s+t}(\omega)+L_t(\omega)}(\Theta^Q_{\varrho_{s+t}(\omega)-\varrho_t(\omega)}(\gamma^P_{-L_t(\omega)}(\Theta^Q_{\varrho_t(\omega)}(\omega^Q))))\big),\end{aligned}$$

where we used the shift property of $(\Theta_t^W, t \ge 0)$ on themselves and on the additive functional $(L_t, t \ge 0)$, as well as $\varrho_s \circ \Theta_t^X = P^{-1}(L_{s+t}) - \varrho_t$ by (21.16). Observing that $(\Theta_t^Q, t \ge 0)$ and $(\gamma_x^P, x \in \mathbb{R})$ commute (because the shift operators $(\hat{\Theta}_t^{Q,e}, t \ge 0)$ and translation operators $(\hat{\gamma}_q^{Q,e}, q \in \mathbb{R})$ of the Cartesian parts commute, see (6.32)), we get

$$\begin{aligned}\Theta_s^X(\Theta_t^X(\omega)) &= \big(\Theta^W_{s+t}(\omega^W), \gamma^P_{-L_{s+t}(\omega)+L_t(\omega)} \circ \gamma^P_{-L_t(\omega)} \circ \Theta^Q_{\varrho_{s+t}(\omega)-\varrho_t(\omega)} \circ \Theta^Q_{\varrho_t(\omega)}(\omega^Q)\big)\\ &= \big(\Theta^W_{s+t}(\omega^W), \gamma^P_{-L_{s+t}(\omega)}(\Theta^Q_{\varrho_{s+t}(\omega)}(\omega^Q))\big)\\ &= \Theta^X_{s+t}(\omega).\end{aligned}$$

□

21.4. Suitable Filtration for X

In order to describe for any $t \ge 0$ the mapping

$$X_t = \big(E(P_e(\varrho_t) - L_t, e \in \mathcal{E}) \circ W_t, P(\varrho_t) - L_t + |W_t|\big),$$

the "information" of $\mathscr{F}_t^W$ and "$\mathscr{F}^Q_{\varrho_t}$" is needed. First of all, we must clarify what we mean by the latter σ-algebra, as $\varrho_t = P^{-1}(L_t)$ is certainly not an $(\mathscr{F}_t^Q, t \ge 0)$-stopping time. Following the general definition of a stopped σ-algebra $\mathscr{G}_\tau$, namely

$$\mathscr{G}_\tau = \{A \in \mathscr{G}_\infty \,|\, \forall s \ge 0:\ A \cap \{\tau \le s\} \in \mathscr{G}_s\},$$

we set for each $t \ge 0$

$$\mathscr{F}_t := \{A \in \mathscr{F}_t^W \otimes \mathscr{F}_\infty^Q \,|\, \forall s \ge 0:\ A \cap \{\varrho_t \le s\} \in \mathscr{F}_t^W \otimes \mathscr{F}_s^Q\}.$$

It turns out that $\mathscr{F}_t$ is just the stopped σ-algebra $\widetilde{\mathscr{F}}^t_{\varrho_t}$ as defined in (3.3) for the random time ϱ_t and the filtration $(\widetilde{\mathscr{F}}^t_s, s \ge 0)$ given by

$$\widetilde{\mathscr{F}}^t_s := \mathscr{F}_t^W \otimes \mathscr{F}_s^Q, \quad s \ge 0.$$

For this definition to fit in the context of section 3 and in order to employ the basic results on usual stopped σ-algebras, we immediately show:

(21.17) Lemma. *For every $t \ge 0$, ϱ_t is an $(\widetilde{\mathscr{F}}^t_s, s \ge 0)$-stopping time, that is,*

$$\forall s \ge 0: \quad \{\varrho_t \le s\} \in \mathscr{F}_t^W \otimes \mathscr{F}_s^Q.$$

Proof. By using (v) of lemma (21.5), we see that for any $s, t \geq 0$,

$$\begin{aligned}\{\varrho_t > s\} &= \{P^{-1}(L_t) > s\} = \{L_t > P(s)\} \\ &= \bigcup_{q \in \mathbb{Q}_+} \left(\{L_t > q\} \cap \{q > P(s)\}\right) \ \in \mathscr{F}_t^W \otimes \mathscr{F}_s^Q,\end{aligned}$$

as $(L_t, t \geq 0)$ is adapted to $(\mathscr{F}_t^W, t \geq 0)$ and $(P_s, s \geq 0)$ is adapted to $(\mathscr{F}_s^Q, s \geq 0)$. □

(21.18) Lemma. *The sequence* $(\varrho_t, t \geq 0)$ *is increasing.*

Proof. As $(\hat{Q}_t, t \geq 0)$ is a subordinator, the process $(P_t, t \geq 0)$ and thus $(P_t^{-1}, t \geq 0)$ are increasing. $(L_t, t \geq 0)$ is increasing as well, so is $\left(\varrho_t = P^{-1}(L_t), t \geq 0\right)$. □

By taking the well-known result $\mathscr{G}_{\tau_1} \subseteq \mathscr{G}_{\tau_2}$ for any two stopping times τ_1, τ_2 with $\tau_1 \leq \tau_2$ into account (see, e.g., [CW05, Theorem 1.3.5]), the two lemmas above prove the first part of the following theorem:

(21.19) Theorem. $(\mathscr{F}_t, t \geq 0)$ *is a filtration and* $(X_t, t \geq 0)$ *is adapted to* $(\mathscr{F}_t, t \geq 0)$.

The second part follows from the next set of lemmas:

(21.20) Lemma. $(\varrho_t, t \geq 0)$ *is adapted to* $(\mathscr{F}_t, t \geq 0)$.

Proof. Fix $t \geq 0$. For $\alpha < 0$, it is $\{\varrho_t \leq \alpha\} = \emptyset \in \mathscr{F}_t$. For $\alpha \geq 0$, lemma (21.17) yields

$$\{\varrho_t \leq \alpha\} \in \mathscr{F}_t^W \otimes \mathscr{F}_\alpha^Q \subseteq \mathscr{F}_t^W \otimes \mathscr{F}_\infty^Q,$$

as well as for all $s \geq 0$,

$$\{\varrho_t \leq \alpha\} \cap \{\varrho_t \leq s\} = \{\varrho_t \leq \alpha \wedge s\} \in \mathscr{F}_t^W \otimes \mathscr{F}_{\alpha \wedge s}^Q \subseteq \mathscr{F}_t^W \otimes \mathscr{F}_s^Q.$$

Therefore, we have $\{\varrho_t \leq \alpha\} \in \mathscr{F}_t$ by the definition of $\mathscr{F}_t$. □

For the following results, we define for any collection $\mathscr{A}_1$ of sets and any set A_2 the usual "Cartesian product" of families of sets

$$\mathscr{A}_1 \times A_2 := \{A_1 \times A_2 : A_1 \in \mathscr{A}_1\},$$

and analogously the set $A_2 \times \mathscr{A}_1 = \{A_2 \times A_1 : A_1 \in \mathscr{A}_1\}$.

(21.21) Lemma. *For all* $t \geq 0$, $\mathscr{F}_t^W \times \Omega^Q \subseteq \mathscr{F}_t$.

Proof. Fix $t \geq 0$ and let $\hat{A} \in \mathscr{F}_t^W$. Then $\hat{A} \times \Omega^Q \in \mathscr{F}_t$ holds true, because we have

$$\hat{A} \times \Omega^Q \in \mathscr{F}_t^W \otimes \mathscr{F}_\infty^Q,$$

and for all $s \geq 0$,

$$(\hat{A} \times \Omega^Q) \cap \{\varrho_t \leq s\} \in \mathscr{F}_t^W \otimes \mathscr{F}_s^Q.$$ □

(21.22) Lemma. *For all $e \in \mathcal{E}$, $u \geq 0$, the process $\big(Q^e((\varrho_t - u) \vee 0), t \geq 0\big)$ is adapted to $(\mathscr{F}_t, t \geq 0)$.*

Proof. Fix $e \in \mathcal{E}$, $t \geq 0$ and $\alpha \in \mathbb{R}$. For all $s \geq 0$, we have

$$\begin{aligned}
&\{Q^e((\varrho_t - u) \vee 0) < \alpha\} \cap \{\varrho_t \leq s\} \\
&= \Big(\bigcup_{q \in \mathbb{Q}_+,\ q \leq s} \{Q^e((q - u) \vee 0) < \alpha\} \cap \{\varrho_t < q\}\Big) \\
&\quad \cup \Big(\{Q^e((s - u) \vee 0) < \alpha\} \cap \{\varrho_t = s\}\Big) \\
&\in \mathscr{F}_t^W \otimes \mathscr{F}_s^Q
\end{aligned}$$

and

$$\begin{aligned}
&\{Q^e((\varrho_t - u) \vee 0) < \alpha\} \\
&= \{Q^e((\varrho_t - u) \vee 0) < \alpha\} \cap \bigcup_{s \in \mathbb{N}} \{\varrho_t \leq s\} \\
&= \bigcup_{s \in \mathbb{N}} \Big(\{Q^e((\varrho_t - u) \vee 0) < \alpha\} \cap \{\varrho_t \leq s\}\Big) \\
&\in \mathscr{F}_t^W \otimes \mathscr{F}_\infty^Q,
\end{aligned}$$

so, by definition of $\mathscr{F}_t$, we have shown that $\big\{Q^e((\varrho_t - u) \vee 0) < \alpha\big\} \in \mathscr{F}_t$. $\square$

(21.23) Corollary. *For all $e \in \mathcal{E} \cup \{0\}$, the process $\big(P_e(\varrho_t), t \geq 0\big)$ is $(\mathscr{F}_t, t \geq 0)$-adapted.*

Proof. This is evident for $P = P_0$, as for $t \geq 0$,

$$P(\varrho_t) = p_2\, \varrho_t + \sum_{e \in \mathcal{E}} Q^e(\varrho_t)$$

is $\mathscr{F}_t$-measurable by lemma (21.22) with $u = 0$. For $e \in \mathscr{E}$, $t \geq 0$,

$$Q^e(\varrho_t-) = \lim_{s \uparrow\uparrow \varrho_t} Q^e(s) = \lim_{n \to \infty} Q^e((\varrho_t - \frac{1}{n}) \vee 0)$$

is $\mathscr{F}_t$-measurable, and so is

$$P_e(\varrho_t) = p_2\, \varrho_t + Q^e(\varrho_t) + \sum_{f \neq e} Q^f(\varrho_t-).$$

$\square$

21.5. Strong Markov Property of (W, Q)

In the construction of X, the process P and, thus, the process Q appear shifted by the random time ϱ_t. In order to use Markov arguments when analyzing the process X in the next subsections, we will need to transfer the strong Markov property of $\hat{Q}$ to the part Q of the combined process (W, Q) and then understand how the shifts Θ_t^W of W and $\Theta_{\varrho_t}^Q$ of Q act on this combined process.

The main idea is that, as we only shift the process Q by ϱ_t, we only need to consider this part of the combined process. Therefore, we introduce the filtration $(\overline{\mathscr{F}}^Q_s, s \geq 0)$ by

$$\overline{\mathscr{F}}^Q_s := \mathscr{F}^W_\infty \otimes \mathscr{F}^Q_s, \quad s \geq 0.$$

It is immediate that

$$\overline{\mathscr{F}}^Q_\infty = \mathscr{F}^W_\infty \otimes \mathscr{F}^Q_\infty.$$

Surely, this filtration is large enough for the time shift ϱ_t:

(21.24) Lemma. *For all $t \geq 0$, ϱ_t is an $(\overline{\mathscr{F}}^Q_s, s \geq 0)$-stopping time.*

Proof. This is clear, as for all $s \geq 0$, we have by lemma (21.17):

$$\{\varrho_t \leq s\} \in \mathscr{F}^W_t \otimes \mathscr{F}^Q_s \subseteq \mathscr{F}^W_\infty \otimes \mathscr{F}^Q_s = \overline{\mathscr{F}}^Q_s. \qquad \square$$

As every translated stopping time is again a stopping time, the next result follows:

(21.25) Corollary. *For all $t \geq 0$, $u \geq 0$, $\varrho_t + u$ is an $(\overline{\mathscr{F}}^Q_s, s \geq 0)$-stopping time.*

The next two lemmas show that this new filtration is, of course, larger than the actual filtrations needed, which will be helpful for proving Markov properties later.

(21.26) Lemma. *For any $(\overline{\mathscr{F}}^Q_s, s \geq 0)$-stopping time τ, $\mathscr{F}^W_\infty \times \Omega^Q \subseteq \overline{\mathscr{F}}^Q_\tau$.*

Proof. Let $A \in \mathscr{F}^W_\infty$. Then, for any $s \geq 0$,

$$A \times \Omega^Q \in \mathscr{F}^W_\infty \otimes \mathscr{F}^Q_s \subseteq \mathscr{F}^W_\infty \otimes \mathscr{F}^Q_\infty,$$

and, as τ is an $(\overline{\mathscr{F}}^Q_s, s \geq 0)$-stopping time,

$$(A \times \Omega^Q) \cap \{\tau \leq s\} \in \mathscr{F}^W_\infty \otimes \mathscr{F}^Q_s. \qquad \square$$

(21.27) Lemma. *For all $t \geq 0$, $\mathscr{F}_t \subseteq \overline{\mathscr{F}}^Q_{\varrho_t}$.*

Proof. Using the definitions of both σ-algebras, we get for $t \geq 0$,

$$\begin{aligned}
\mathscr{F}_t &= \{A \in \mathscr{F}^W_t \otimes \mathscr{F}^Q_\infty \,|\, \forall s \geq 0: \ A \cap \{\varrho_t \leq s\} \in \mathscr{F}^W_t \otimes \mathscr{F}^Q_s\} \\
&\subseteq \{A \in \mathscr{F}^W_\infty \otimes \mathscr{F}^Q_\infty \,|\, \forall s \geq 0: \ A \cap \{\varrho_t \leq s\} \in \widetilde{\mathscr{F}}_s\} \\
&= \overline{\mathscr{F}}^Q_{\varrho_t}.
\end{aligned} \qquad \square$$

We are now able to transfer the Markov property and the strong Markov property from $\hat{Q}$ to Q. Observe that the following results are not really representing the Markov properties which were defined and discussed in chapter I, as we will only consider and shift the second part of the combined process (W, Q) here, so everything is still "independent" of the first coordinate. These "partial" time shifts are not commonly treated, because joint Markov processes $\big((X_t, Y_t), t \geq 0\big)$ typically run with a mutual time parameter t and thus are translated collectively by a mutual time shift. Therefore we will need to lift the following "Markov properties" manually.

$(Q_s, s \geq 0)$ is "Markovian" with respect to $(\overline{\mathscr{F}}^Q_s, s \geq 0)$ in the following sense:

(21.28) Lemma. *For all $g \in \mathcal{G}$, $q \in \mathbb{R}^n$, $f \in b\mathscr{B}(\mathbb{R})^{\otimes n}$, $s, t \geq 0$,*

$$\mathbb{E}_{g,q}\big(f(Q_{s+t}) \,|\, \overline{\mathscr{F}}^Q_s\big) = \mathbb{E}_{g,Q_s}\big(f(Q_t)\big).$$

Proof. It is obvious that $(Q_s, s \geq 0)$ is adapted to $(\overline{\mathscr{F}}^Q_s, s \geq 0)$.

The system $\{A \times B : A \in \mathscr{F}^W_\infty, B \in \mathscr{F}^Q_s\}$ is an $\cap$-stable generator of $\overline{\mathscr{F}}^Q_s$, so the claim follows from the definition of the combined process on the product space with the help of Fubini's theorem, as we have

$$\begin{aligned}
\mathbb{E}_{g,q}\big(f(Q_{s+t})\ \mathbb{1}_{A\times B}\big) &= \mathbb{E}^W_g\big(\mathbb{1}_A\big)\ \mathbb{E}^Q_q\big(f(\hat{Q}_{s+t})\,\mathbb{1}_B\big) \\
&= \mathbb{E}^W_g\big(\mathbb{1}_A\big)\ \mathbb{E}^Q_q\big(\mathbb{E}^Q_{\hat{Q}_s}\big(f(\hat{Q}_t)\big)\,\mathbb{1}_B\big) \\
&= \mathbb{E}_{g,q}\big(\mathbb{E}^Q_{Q_s}\big(f(\hat{Q}_t)\big)\,\mathbb{1}_{A\times B}\big) \\
&= \mathbb{E}_{g,q}\big(\mathbb{E}_{g,Q_s}\big(f(Q_t)\big)\,\mathbb{1}_{A\times B}\big)
\end{aligned}$$

for all $A \in \mathscr{F}^W_\infty$, $B \in \mathscr{F}^Q_s$, $g \in \mathscr{G}$, $q \in \mathbb{R}^n$, $f \in b\mathscr{B}(\mathbb{R})^{\otimes n}$, $s, t \geq 0$. □

We are going to reiterate the standard argument which shows that every Feller process is strongly Markovian (see, e.g., [RW00a, Section III.8]), applied to the process Q. This may seem strange, because we already have the strong Markov property of $\hat{Q}$, but it is surprisingly difficult to transfer it directly to Q, as an $(\overline{\mathscr{F}}_s, s \geq 0)$-stopping time also randomizes the first coordinate of (W, Q) and, even if the processes are independent, it does not appear easy to separate both parts in the random time.

$(Q_s, s \geq 0)$ is "strongly Markovian" with respect to $(\overline{\mathscr{F}}^Q_s, s \geq 0)$ in the following sense:

(21.29) Lemma. *For all $g \in \mathcal{G}$, $q \in \mathbb{R}^n$, $f \in b\mathscr{B}(\mathbb{R})^{\otimes n}$, $s \geq 0$ and every stopping time τ over $(\overline{\mathscr{F}}^Q_s, s \geq 0)$,*

$$\mathbb{E}_{g,q}\big(f(Q_{s+\tau}) \,|\, \overline{\mathscr{F}}^Q_\tau\big) = \mathbb{E}_{g,Q_\tau}\big(f(Q_s)\big).$$

Proof. Fix an $(\overline{\mathscr{F}}^Q_s, s \geq 0)$-stopping time τ. Then there exists a decreasing sequence $(\tau_n, n \in \mathbb{N})$ of $(\overline{\mathscr{F}}^Q_s, s \geq 0)$-stopping times with values in $\{\frac{k}{2^n} : n, k \in \mathbb{N}\}$, defined by

$$\tau_n = \tfrac{k}{2^n} \quad \Leftrightarrow \quad \tau \in \big[\tfrac{k-1}{2^n}, \tfrac{k}{2^n}\big), \quad n, k \in \mathbb{N}.$$

In particular, we have $\{\tau_n \leq \frac{k}{2^n}\} = \{\tau < \frac{k}{2^n}\} \in \overline{\mathscr{F}}^Q_{\frac{k}{2^n}}$.

Following the standard argument for deducing the strong Markov property of Feller processes from their Markov property, we compute for all $s \geq 0$, $f \in b\mathcal{C}(\mathbb{R}^n)$, $A \in \overline{\mathscr{F}}^Q_\tau = \{A \in \overline{\mathscr{F}}^Q_\infty \,|\, \forall s \geq 0: \; A \cap \{\tau \leq s\} \in \overline{\mathscr{F}}^Q_s\}$:

$$\begin{aligned}
\mathbb{E}_{g,q}\big(f(Q_{s+\tau})\,\mathbb{1}_A\big) &= \lim_{n\to\infty} \mathbb{E}_{g,q}\big(f(Q_{s+\tau_n})\,\mathbb{1}_A\big)\\
&= \lim_{n\to\infty} \sum_{k\in\mathbb{N}} \mathbb{E}_{g,q}\big(f(Q_{s+\frac{k}{2^n}})\,\mathbb{1}_{A\cap\{\frac{k-1}{2^n}\leq\tau<\frac{k}{2^n}\}}\big)\\
&= \lim_{n\to\infty} \sum_{k\in\mathbb{N}} \mathbb{E}_{g,q}\big(\mathbb{E}_{g,Q_{\frac{k}{2^n}}}(f(Q_s))\,\mathbb{1}_{A\cap\{\frac{k-1}{2^n}\leq\tau<\frac{k}{2^n}\}}\big)\\
&= \lim_{n\to\infty} \mathbb{E}_{g,q}(\mathbb{E}_{g,Q_{\tau_n}}(f(Q_s))\,\mathbb{1}_A)\\
&= \lim_{n\to\infty} \mathbb{E}_{g,q}(\mathbb{E}^Q_{Q_{\tau_n}}(f(\hat{Q}_s))\,\mathbb{1}_A)\\
&= \lim_{n\to\infty} \mathbb{E}_{g,q}(T^Q_s f(Q_{\tau_n})\,\mathbb{1}_A)\\
&= \mathbb{E}_{g,q}(T^Q_s f(Q_\tau)\,\mathbb{1}_A)\\
&= \mathbb{E}_{g,q}(\mathbb{E}_{g,Q_\tau}(f(Q_s))\,\mathbb{1}_A),
\end{aligned}$$

where $(T^Q_s, s \geq 0)$ is the Feller semigroup of $\hat{Q}$. □

We are now ready to infer the Markov property of the combined process (W, Q) with respect to the shifts Θ^W_t and $\Theta^Q_{\varrho t}$ and to the actual filtration $(\mathscr{F}_t, t \geq 0)$: The combined shift operators $\Theta_t := \Theta^W_t \otimes \Theta^Q_{\varrho t}$, $t \geq 0$, on Ω are defined in the intuitive way, that is, for all $\omega = (\omega^W, \omega^Q) \in \Omega$, we consider

$$\Theta_t(\omega) = \Theta^W_t \otimes \Theta^Q_{\varrho t}(\omega) = (\hat{\Theta}^W_t(\omega^W), \hat{\Theta}^Q_{\varrho t((\omega^W,\omega^Q))}(\omega^Q)). \tag{21.30}$$

The basic version of the "Markov property" for (W, Q) with respect to $(\mathscr{F}_t, t \geq 0)$ via the just defined combined shift operators $(\Theta_t, t \geq 0)$ reads:

(21.31) Lemma. *For all $g \in \mathcal{G}$, $q \in \mathbb{R}^n$, $f \in b\mathscr{B}(\mathbb{R})$, $h \in b\mathscr{B}(\mathbb{R})^{\otimes n}$, $r, s, t \geq 0$,*

$$\mathbb{E}_{g,q}(f(W_r)\,h(Q_s) \circ \Theta^W_t \otimes \Theta^Q_{\varrho t} \,|\, \mathscr{F}_t) = \mathbb{E}_{W_t, Q_{\varrho t}}(f(W_r)\,h(Q_s)).$$

Proof. By using $\mathscr{F}_t \subseteq \mathscr{F}^W_t \otimes \mathscr{F}^Q_\infty$ together with the Markov property of W with respect to $(\mathscr{F}^W_t \otimes \mathscr{F}^Q_\infty, t \geq 0)$ (which follows from the product space construction), we obtain

$$\begin{aligned}
&\mathbb{E}_{g,q}(f(W_r) \circ \Theta^W_t\; h(Q_s) \circ \Theta^Q_{\varrho t} \,|\, \mathscr{F}_t)\\
&= \mathbb{E}_{g,q}\big(\mathbb{E}_{g,q}(f(W_r) \circ \Theta^W_t \,|\, \mathscr{F}^W_t \otimes \mathscr{F}^Q_\infty)\; h(Q_s) \circ \Theta^Q_{\varrho t} \,|\, \mathscr{F}_t\big)\\
&= \mathbb{E}_{g,q}\big(\mathbb{E}^W_{W_t}(f(\hat{W}_r))\; h(Q_s) \circ \Theta^Q_{\varrho t} \,|\, \mathscr{F}_t\big).
\end{aligned}$$

Employing $\mathscr{F}_t \subseteq \overline{\mathscr{F}}^Q_{\varrho t}$ (by lemma (21.27)), the adaptedness of W to $(\overline{\mathscr{F}}^Q_{\varrho t}, t \geq 0)$ (by lemmas (21.26) and (21.24)), as well as the "strong Markov property" of Q with respect to ϱ_t (as given in lemma (21.29)), we get

$$\begin{aligned}
&\mathbb{E}_{g,q}\big(f(W_r)\circ\Theta_t^W\, h(Q_s)\circ\Theta_{\varrho t}^Q \mid \mathscr{F}_t\big)\\
&= \mathbb{E}_{g,q}\big(\mathbb{E}_{W_t}^W(f(\hat{W}_r))\ \mathbb{E}_{g,q}(h(Q_s)\circ\Theta_{\varrho t}^Q \mid \overline{\mathscr{F}}{}^Q_{\varrho t}) \mid \mathscr{F}_t\big)\\
&= \mathbb{E}_{g,q}\big(\mathbb{E}_{W_t}^W(f(\hat{W}_r))\ \mathbb{E}_{g,Q_{\varrho t}}^Q(h(Q_s)) \mid \mathscr{F}_t\big)\\
&= \mathbb{E}_{W_t}^W(f(\hat{W}_r))\ \mathbb{E}_{Q_{\varrho t}}^Q(h(\hat{Q}_s))\\
&= \mathbb{E}_{W_t,Q_{\varrho t}}\big(f(W_r)\, h(Q_s)\big),
\end{aligned}$$

where we also used that (W,Q) is adapted to $(\mathscr{F}_t, t\geq 0)$ (see lemmas (21.21) and (21.22)). □

Complying with the usual generalization of Markovian shifts, we lift the above lemma to $\mathscr{F}_\infty^0$-measurable functions by slightly adjusting the routine proof (see, e.g., [BG69, Proposition I.8.4]):

(21.32) Theorem. *For all $g\in\mathcal{G}$, $q\in\mathbb{R}^n$, $Y\in b\mathscr{F}_\infty^0 = b\sigma(W_r, Q_s, r, s\geq 0)$, $t\geq 0$,*

$$\mathbb{E}_{g,q}\big(Y\circ\Theta_t^W\otimes\Theta_{\varrho t}^Q \mid \mathscr{F}_t\big) = \mathbb{E}_{W_t,Q_{\varrho t}}(Y).$$

Proof. Using the MCT, it suffices to show this formula for

$$Y = f_1(W_{r_1})\ \cdots\ f_k(W_{r_k})\, h_1(Q_{s_1})\ \cdots\ h_l(Q_{s_l})$$

with $k,l\in\mathbb{N}$, $f_1,\dots,f_k\in b\mathscr{B}(\mathbb{R})$, $h_1,\dots,h_l\in b\mathscr{B}(\mathbb{R})^{\otimes n}$, as well as $0\leq r_1<\cdots<r_k$, $0\leq s_1<\cdots<s_l$. We can assume that $k=l$ holds, as otherwise we just fill the missing functions with functions identically to 1.

We are going to prove the assertion by an induction over $k\in\mathbb{N}$. The case $k=1$ is already done in lemma (21.31). For the inductive step from $k-1$ to k, we compute

$$\begin{aligned}
&\mathbb{E}_{g,q}\Big(\prod_{i=1}^k f_i(W_{r_i})\circ\Theta_t^W\, h_i(Q_{s_i})\circ\Theta_{\varrho t}^Q \mid \mathscr{F}_t\Big)\\
&= \mathbb{E}_{g,q}\Big(\prod_{i=1}^{k-1} f_i(W_{r_i+t})\, \mathbb{E}_{g,q}\big(f_k(W_{r_k})\circ\Theta_t^W \mid \mathscr{F}^W_{r_{k-1}+t}\otimes\mathscr{F}^Q_\infty\big)\\
&\qquad \prod_{j=1}^k h_j(Q_{s_j})\circ\Theta_{\varrho t}^Q \mid \mathscr{F}_t\Big)\\
&= \mathbb{E}_{g,q}\Big(\prod_{i=1}^{k-1} f_i(W_{r_i+t})\, \mathbb{E}^W_{W_{r_{k-1}+t}}\big(f_k(\hat{W}_{r_k-r_{k-1}})\big)\\
&\qquad \prod_{j=1}^{k-1} h_j(Q_{s_j+\varrho t})\, \mathbb{E}_{g,q}\big(h_k(Q_{s_k})\circ\Theta_{\varrho t}^Q \mid \overline{\mathscr{F}}{}^Q_{s_{k-1}+\varrho t}\big) \mid \mathscr{F}_t\Big),
\end{aligned}$$

where we used $\mathscr{F}_t\subseteq\mathscr{F}^W_{r+t}\otimes\mathscr{F}^Q_\infty$ for all $r\geq 0$ (by the definition of $\mathscr{F}_t$), the measurability of $Q_{s+\varrho t}$ with respect to $\mathscr{F}^W_{r+t}\otimes\mathscr{F}^Q_\infty$ (which follows from the right continuity of Q

and the stopping time property of ϱ_t with respect to $(\widetilde{\mathscr{F}}_s^t = \mathscr{F}_t^W \otimes \mathscr{F}_s^Q, s \geq 0)$, see lemma (21.17)), the Markov property of W with respect to $(\mathscr{F}_t^W \otimes \mathscr{F}_\infty^Q, t \geq 0)$ (by the product construction of (W, Q) with Fubini's theorem), as well as $\mathscr{F}_t \subseteq \overline{\mathscr{F}}^Q_{s+\varrho_t}$ (see lemma (21.27)) and the measurability of W with respect to $\overline{\mathscr{F}}^Q_{s+\varrho_t}$ (see lemma (21.26)).

Next, we again use the "strong Markov property" of Q (see lemma (21.29)) to get

$$\begin{aligned}
&\mathbb{E}_{g,q}\Big(\prod_{i=1}^{k} f_i(W_{r_i})\, h_i(Q_{s_i}) \circ \Theta_t^W \otimes \Theta_{\varrho t}^Q \mid \mathscr{F}_t\Big) \\
&= \mathbb{E}_{g,q}\Big(\prod_{i=1}^{k-1} f_i(W_{r_i+t})\, \mathbb{E}^W_{W_{r_{k-1}+t}}\big(f_k(\hat{W}_{r_k - r_{k-1}})\big) \\
&\qquad \prod_{j=1}^{k-1} h_j(Q_{s_j+\varrho t})\, \mathbb{E}^Q_{Q_{s_{k-1}+\varrho t}}\big(h_k(\hat{Q}_{s_k - s_{k-1}})\big) \mid \mathscr{F}_t\Big) \\
&= \mathbb{E}_{g,q}\Big(\prod_{i=1}^{k-1} \widetilde{f}_i(W_{r_i})\, \widetilde{h}_i(Q_{s_i}) \circ \Theta_t^W \otimes \Theta_{\varrho t}^Q \mid \mathscr{F}_t\Big),
\end{aligned}$$

where we renamed $\widetilde{f}_i = f_i$, $\widetilde{h}_i = h_i$ for $i \in \{1, \ldots, k-1\}$, $\widetilde{f}_{k-1} = f_{k-1}\, \mathbb{E}^W_{\cdot}\big(f_k(\hat{W}_{r_k - r_{k-1}})\big)$, $\widetilde{h}_{k-1} = h_{k-1}\, \mathbb{E}^Q_{\cdot}\big(h_k(\hat{Q}_{s_k - s_{k-1}})\big)$. By using the inductive assumption and performing the above steps in reverse order, we get

$$\begin{aligned}
&\mathbb{E}_{g,q}\Big(\prod_{i=1}^{k} f_i(W_{r_i})\, h_i(Q_{s_i}) \circ \Theta_t^W \otimes \Theta_{\varrho t}^Q \mid \mathscr{F}_t\Big) \\
&= \mathbb{E}_{W_t, Q_{\varrho t}}\Big(\prod_{i=1}^{k-1} \widetilde{f}_i(W_{r_i})\, \widetilde{h}_i(Q_{s_i})\Big) \\
&= \mathbb{E}_{W_t, Q_{\varrho t}}\Big(\prod_{i=1}^{k-1} f_i(W_{r_i})\, h_i(Q_{s_i})\, \widetilde{h}_k(Q_{s_{k-1}})\, \mathbb{E}^W_{W_{r_{k-1}}}\big(f_k(\hat{W}_{r_k - r_{k-1}})\big)\Big) \\
&= \mathbb{E}_{W_t, Q_{\varrho t}}\Big(\prod_{i=1}^{k-1} f_i(W_{r_i})\, h_i(Q_{s_i})\, \widetilde{h}_k(Q_{s_{k-1}})\, \mathbb{E}_{\cdot,\cdot}\big(f_k(W_{r_k}) \mid \mathscr{F}^W_{r_{k-1}} \otimes \mathscr{F}^Q_\infty\big)\Big) \\
&= \mathbb{E}_{W_t, Q_{\varrho t}}\Big(\prod_{i=1}^{k-1} f_i(W_{r_i})\, h_i(Q_{s_i})\, \mathbb{E}_{\cdot,\cdot}\big(h_k(Q_{s_k}) \mid \mathscr{F}^W_\infty \otimes \mathscr{F}^Q_{s_{k-1}}\big)\, f_k(W_{r_k})\Big) \\
&= \mathbb{E}_{W_t, Q_{\varrho t}}\Big(\prod_{i=1}^{k} f_i(W_{r_i})\, h_i(Q_{s_i})\Big),
\end{aligned}$$

where the "$\cdot, \cdot$" in the above expectations serve as place holders for exactly that value which is assumed by the random variable $(W_t, Q_{\varrho t})$ outside of the expectation. $\square$

For later use, we also need to introduce another, coarser filtration:

$$\overline{\overline{\mathscr{F}}}{}^Q_s := \Omega^W \times \mathscr{F}^Q_s, \quad s \geq 0.$$

Then lemma (21.29) has a natural equivalent:

(21.33) Lemma. *For all $g \in \mathcal{G}$, $q \in \mathbb{R}^n$, $f \in b\mathscr{B}(\mathbb{R})$, $h \in b\mathscr{B}(\mathbb{R})^{\otimes n}$, $r, s \geq 0$ and every stopping time τ over $(\overline{\overline{\mathscr{F}}}{}^Q_s, s \geq 0)$,*

$$\mathbb{E}_{g,q}\big(f(W_r)\, h(Q_{s+\tau}) \,\big|\, \overline{\overline{\mathscr{F}}}{}^Q_\tau\big) = \mathbb{E}_{g,Q_\tau}\big(f(W_r)\, h(Q_s)\big).$$

Proof. As $\overline{\overline{\mathscr{F}}}{}^Q_s \subseteq \overline{\mathscr{F}}{}^Q_s$ for all $s \geq 0$, τ is also an $(\overline{\mathscr{F}}{}^Q_s, s \geq 0)$-stopping time with stopped σ-algebras $\overline{\overline{\mathscr{F}}}{}^Q_\tau \subseteq \overline{\mathscr{F}}{}^Q_\tau$, and as W_r is $\overline{\mathscr{F}}{}^Q_\tau$-measurable (see lemma (21.26)), we get

$$\begin{aligned}
\mathbb{E}_{g,q}\big(f(W_r)\, h(Q_{s+\tau}) \,\big|\, \overline{\overline{\mathscr{F}}}{}^Q_\tau\big) &= \mathbb{E}_{g,q}\big(\mathbb{E}_{g,q}\big(f(W_r)\, h(Q_{s+\tau}) \,\big|\, \overline{\mathscr{F}}{}^Q_\tau\big) \,\big|\, \overline{\overline{\mathscr{F}}}{}^Q_\tau\big) \\
&= \mathbb{E}_{g,q}\big(f(W_r)\, \mathbb{E}_{g,q}\big(h(Q_{s+\tau}) \,\big|\, \overline{\mathscr{F}}{}^Q_\tau\big) \,\big|\, \overline{\overline{\mathscr{F}}}{}^Q_\tau\big) \\
&= \mathbb{E}_{g,q}\big(f(W_r)\, \mathbb{E}_{g,Q_\tau}\big(h(Q_s)\big) \,\big|\, \overline{\overline{\mathscr{F}}}{}^Q_\tau\big) \\
&= \mathbb{E}_{g,q}\big(f(W_r) \,\big|\, \overline{\overline{\mathscr{F}}}{}^Q_\tau\big)\, \mathbb{E}_{g,Q_\tau}\big(h(Q_s)\big)
\end{aligned}$$

by lemma (21.29). But as $\overline{\overline{\mathscr{F}}}{}^Q_\tau \subseteq \Omega^W \times \mathscr{F}^Q_\infty$, it follows that

$$\mathbb{E}_{g,q}\big(f(W_r) \,\big|\, \overline{\overline{\mathscr{F}}}{}^Q_\tau\big)\, \mathbb{E}_{g,Q_\tau}\big(h(Q_s)\big) = \mathbb{E}_{g,q}\big(f(W_r)\big)\, \mathbb{E}_{g,Q_\tau}\big(h(Q_s)\big),$$

and taking the product construction of W and Q into account, we conclude that

$$\mathbb{E}_{g,q}\big(f(W_r)\big)\, \mathbb{E}_{g,Q_\tau}\big(h(Q_s)\big) = \mathbb{E}_{g,Q_\tau}\big(f(W_r)\, h(Q_s)\big). \qquad \square$$

(21.34) Theorem. *For all $g \in \mathcal{G}$, $q \in \mathbb{R}^n$, $Y \in b\mathscr{F}^0_\infty = b\sigma(W_r, Q_s, r, s \geq 0)$ and every stopping time τ over $(\overline{\overline{\mathscr{F}}}{}^Q_s, s \geq 0)$,*

$$\mathbb{E}_{g,q}\big(Y \circ \mathrm{id}^W \otimes \Theta^Q_\tau \,\big|\, \overline{\overline{\mathscr{F}}}{}^Q_t\big) = \mathbb{E}_{q,Q_\tau}(Y).$$

Proof. As usual, in regard to the MCT, it suffices to show this formula for

$$Y = f_1(W_{r_1}) \cdots f_k(W_{r_k})\, h_1(Q_{s_1}) \cdots h_l(Q_{s_l})$$

with $k, l \in \mathbb{N}$, $f_1, \ldots, f_k \in b\mathscr{B}(\mathbb{R})$, $h_1, \ldots, h_l \in b\mathscr{B}(\mathbb{R})^{\otimes n}$, as well as $0 \leq r_1 < \cdots < r_k$, $0 \leq s_1 < \cdots < s_l$. It is standard to extend lemma (21.33) to

$$\mathbb{E}_{g,q}\big(h_1(Q_{s_1}) \cdots h_l(Q_{s_l}) \circ \Theta^Q_\tau \,\big|\, \overline{\overline{\mathscr{F}}}{}^Q_\tau\big) = \mathbb{E}_{g,Q_\tau}\big(h_1(Q_{s_1}) \cdots h_l(Q_{s_l})\big).$$

Therefore, by setting $f^W := f_1(W_{r_1}) \cdots f_k(W_{r_k})$, $h^Q := h_1(Q_{s_1}) \cdots h_l(Q_{s_l})$ and using the same techniques as in the proof of lemma (21.33), it remains to compute:

$$\begin{aligned}
\mathbb{E}_{g,q}\big(f^W\, h^Q \circ \mathrm{id}^W \otimes \Theta^Q_\tau \,\big|\, \overline{\overline{\mathscr{F}}}{}^Q_t\big) &= \mathbb{E}_{g,q}\big(f^W\, h^Q \circ \Theta^Q_\tau \,\big|\, \overline{\overline{\mathscr{F}}}{}^Q_t\big) \\
&= \mathbb{E}_{g,q}\big(f^W\, \mathbb{E}_{g,q}\big(h^Q \circ \Theta^Q_\tau \,\big|\, \overline{\mathscr{F}}{}^Q_t\big) \,\big|\, \overline{\overline{\mathscr{F}}}{}^Q_t\big) \\
&= \mathbb{E}_{g,q}\big(f^W \,\big|\, \overline{\overline{\mathscr{F}}}{}^Q_t\big)\, \mathbb{E}_{g,Q_\tau}\big(h^Q\big) \\
&= \mathbb{E}_{g,q}\big(f^W\big)\, \mathbb{E}_{g,Q_\tau}\big(h^Q\big) \\
&= \mathbb{E}_{g,Q_\tau}\big(f^W\, h^Q\big). \qquad \square
\end{aligned}$$

Of course, the roles of W and Q can be interchanged in lemmas (21.28), (21.29), (21.33) and theorem (21.34), giving us for the filtration

$$\overline{\overline{\mathscr{F}}}_t^W := \mathscr{F}_t^W \times \Omega^Q, \quad t \geq 0:$$

(21.35) Theorem. *For all $g \in \mathcal{G}$, $q \in \mathbb{R}^n$, $Y \in b\mathscr{F}_\infty^0 = b\sigma(W_r, Q_s, r, s \geq 0)$ and every stopping time τ over $(\overline{\overline{\mathscr{F}}}_t^W, t \geq 0)$,*

$$\mathbb{E}_{g,q}(Y \circ \Theta_\tau^W \otimes \mathrm{id}^Q \mid \overline{\overline{\mathscr{F}}}_t^W) = \mathbb{E}_{W_\tau,q}(Y).$$

21.6. Strong Markov Property at H_0

Let $H_0 := \inf\{t \geq 0 : X_t = v\}$ be the first entry time of X in the star vertex v.

We are going to show the strong Markov property of X at H_0 next, which will be essential for the proof of both the Markov property and the strong Markov property of X. The Markovian behavior of X at H_0 should appear quite natural, because X is just the underlying Walsh process W until $H_0 = H_0^W$ (with H_0^W being the first entry time of W in v), W is strongly Markovian, and the additional, independent parts of the subordinators only come into play after H_0.

(21.36) Lemma. *It holds $X_t = W_t$ for all $t \leq H_0$ and $H_0 = H_0^W$, $\mathbb{P}_g$-a.s. for all $g \in \mathcal{G}$.*

Proof. Let $g \in \mathcal{G}$. Every identity in this proof will be meant $\mathbb{P}_g = \mathbb{P}_{(g,0)}$-a.s..

Because $(L_t, t \geq 0)$ only grows at $\{t \geq 0 : W_t = 0\}$ and is continuous, we have $L_t = 0$ for all $t \leq H_0^W$. The fact that P starts at 0 and is strictly increasing implies that $P^{-1}(0) = 0$, so we get

$$\forall e \in \mathcal{E} \cup \{0\}, t \leq H_0^W : \; P_e P^{-1}(L_t) = P_e(0) = 0.$$

By checking the definition of X, it is immediate that

$$\forall t \leq H_0^W : \quad X_t = W_t.$$

As $X_t = W_t \neq 0$ for all $t < H_0^W$ and $X_{H_0^W} = W_{H_0^W} = 0$, this also proves $H_0 = H_0^W$. □

(21.37) Corollary. *The processes $(X_{t \wedge H_0}, t \geq 0)$ and $(W_{t \wedge H_0^W}, t \geq 0)$ have the same finite dimensional distributions with respect to $\mathbb{P}_g$ for all $g \in \mathcal{G}$.*

Before we continue with our developments towards the strong Markov property, we remark the following relation for later use:

(21.38) Lemma. *For all $\alpha > 0$, $t \geq 0$, $\omega \in \Omega$,*

$$\mathbb{E}_{X_t(\omega),0}\big(e^{-\alpha H_0}\big) = \mathbb{E}_{W_t(\omega),0}\big(e^{-\alpha L^{-1}(\eta_t(\omega))}\big).$$

Proof. If $\eta_t(\omega) \neq 0$, then there is exactly one $e \in \mathcal{E}$ with $\eta_t^e(\omega) > 0$ by lemma (21.10), so the definition of X and lemma (21.36) yield

$$\mathbb{E}_{X_t(\omega),0}\big(e^{-\alpha H_0}\big) = \mathbb{E}_{(e,\eta_t(\omega)+|W_t(\omega)|),0}\big(e^{-\alpha H_0^W}\big).$$

The first hitting time and the local time of the Walsh process W at the vertex correspond to the respective entities of the underlying (reflecting) Brownian motion B at the origin (see theorem (19.6)), so lemmas (14.5) and (15.9) give

$$\begin{aligned}\mathbb{E}_{(e,\eta_t(\omega)+|W_t(\omega)|),0}\big(e^{-\alpha H_0^W}\big) &= \mathbb{E}^B_{\eta_t(\omega)+|W_t(\omega)|}\big(e^{-\alpha H_0^B}\big)\\ &= e^{-\sqrt{2\alpha}\,(\eta_t(\omega)+|W_t(\omega)|)}\\ &= \mathbb{E}^B_{|W_t(\omega)|}\big(e^{-\alpha L^{-1}(\eta_t(\omega))}\big)\\ &= \mathbb{E}_{W_t(\omega),0}\big(e^{-\alpha L^{-1}(\eta_t(\omega))}\big).\end{aligned}$$

If $\eta_t(\omega) = 0$, then

$$\mathbb{E}_{X_t(\omega),0}\big(e^{-\alpha H_0}\big) = \mathbb{E}_{W_t(\omega),0}\big(e^{-\alpha H_0^W}\big),$$

which completes the proof, as $L^{-1}(0) = H_0^W$. □

We prepare the strong Markov property of X at H_0 with the following result:

(21.39) Lemma. *For all $g \in \mathcal{G}$, $t \geq 0$, $f \in b\mathscr{B}(\mathcal{G})$, $k \in \mathbb{N}$, $f_1, \ldots, f_k \in b\mathscr{B}(\mathcal{G})$ and $0 \leq t_1 < \cdots < t_k$, the following holds true with $J := f_1(X_{t_1\wedge H_0}) \cdots f_k(X_{t_k\wedge H_0})$:*

$$\mathbb{E}_g\big(f(X_{t+H_0})\cdot J\big) = \mathbb{E}_g\big(\mathbb{E}_{X_{H_0}}\big(f(X_t)\big)\cdot J\big).$$

Proof. Consider the process X shifted by H_0, that is

$$X_{t+H_0} = \big(e(L_{t+H_0})\circ W_{t+H_0}, PP^{-1}(L_{t+H_0}) - L_{t+H_0} + |W_{t+H_0}|\big).$$

As $H_0 = H_0^W$ and $L_{H_0} = L_{H_0^W} = 0$, we have

$$L_{t+H_0} = L_{t+H_0} - L_{H_0} = L_t \circ \Theta^W_{H_0^W},$$

and therefore

$$X_{t+H_0} = X_t \circ \big(\Theta^W_{H_0^W} \times \mathrm{id}^Q\big),$$

so shifting by H_0 does not shift Q (this is also clear by the definition of X or by looking at its shift operators). Lemma (21.36) then gives

$$\begin{aligned}&\mathbb{E}_g\big(f(X_{t+H_0})\,f_1(X_{t_1\wedge H_0})\cdots f_k(X_{t_k\wedge H_0})\big)\\ &= \mathbb{E}_g\big(f(X_{t+H_0})\,f_1(W_{t_1\wedge H_0^W})\cdots f_k(W_{t_k\wedge H_0^W})\big)\\ &= \mathbb{E}_g\big(f_1(W_{t_1\wedge H_0^W})\cdots f_k(W_{t_k\wedge H_0^W})\,\mathbb{E}_g\big(f(X_t)\circ(\Theta^W_{H_0^W}\otimes\mathrm{id}^Q)\,\big|\,\overline{\overline{\mathscr{F}}}{}^W_{H_0^W}\big)\big),\end{aligned}$$

with $\overline{\overline{\mathscr{F}}}{}_t^W = \mathscr{F}_t^W \times \Omega^Q$, $t \geq 0$, as defined at the end of subsection 21.5. Using the strong Markov property (21.35) of W with respect to $(\overline{\overline{\mathscr{F}}}{}_t^W, t \geq 0)$ for the stopping time H_0^W, the inner conditional expectation becomes

$$\mathbb{E}_{g,0}\big(f(X_t) \circ (\Theta^W_{H_0^W} \otimes \mathrm{id}^Q) \,\big|\, \overline{\overline{\mathscr{F}}}{}^W_{H_0^W}\big) = \mathbb{E}_{W_{H_0^W},0}\big(f(X_t)\big) = \mathbb{E}_{0,0}\big(f(X_t)\big),$$

which completes the proof by using once again lemma (21.36), yielding

$$\begin{aligned}&\mathbb{E}_g\big(f(X_{t+H_0})\, f_1(X_{t_1 \wedge H_0}) \cdots f_k(X_{t_k \wedge H_0})\big)\\ &= \mathbb{E}_g\big(\mathbb{E}_{X_{H_0}}\big(f(X_t)\big)\, f_1(X_{t_1 \wedge H_0}) \cdots f_k(X_{t_k \wedge H_0})\big).\end{aligned}$$

□

We are almost ready for the first main result, namely the strong Markov property of the process X at H_0, which we would like to prove with the help of Galmarino's theorem (3.22). However, there are no stopping operators for X available on the constructed space Ω, as stopping the process at the vertex v would cause the local time to explode. Therefore, we need to switch to the path space realization $(Y_t, t \geq 0)$ of X. As the process X is right continuous and continuous inside the edges by lemma (21.12), we are able to construct the canonical process $Y_t(\omega) := \omega(t)$, $\omega \in \Omega^Y$, $t \geq 0$, on the path space

$$\begin{aligned}\Omega^Y := \{\omega\colon \mathbb{R}_+ \to \mathcal{G} \,|\, \omega \text{ right continuous} \wedge \forall t \geq 0 \text{ with } \omega(t) \neq v\colon \omega \text{ is}\\ \text{continuous on } [t, t_0], \text{ with } t_0 := \inf\{s \geq t : \omega(s) = v\}\},\end{aligned}$$

equipped with its canonical filtration $\mathscr{F}_t^Y = \sigma(Y_s, s \leq t)$, $t \geq 0$, and mapping operator $\Phi\colon \Omega \to \Omega^Y$, see subsection 7.1.

As $X_t = Y_t \circ \Phi$ holds by equation (7.1), we have for the first entry time H_0^Y of Y in v:

$$H_0^Y \circ \Phi = \inf\{t \geq 0 : Y_t \circ \Phi = v\} = H_0.$$

The space Ω^Y admits the natural shift and stopping operators

$$\Theta_t(\omega) := \omega(\,\cdot\, + t), \quad \alpha_t(\omega) := \omega(\,\cdot\, \wedge t), \quad t \geq 0,\ \omega \in \Omega^Y,$$

as both shifted and stopped paths admit the conditions on Ω^Y. Therefore, we are able to apply Galmarino's theorem (3.22) in the context of Y.

(21.40) Lemma. *H_0^Y is a stopping time over $(\mathscr{F}_t^Y, t \geq 0)$.*

Proof. By definition of Ω^Y, the canonical coordinate process Y is right continuous on $\mathbb{R}_+$ and continuous on $[0, t_0]$, with $t_0 = \inf\{s \geq 0 : Y_s = v\} = H_0^Y$. As $\{v\}$ is a closed subset of the Polish space $\mathcal{G}$, lemma (3.7) yields that H_0^Y is a stopping time over the natural "raw" filtration $(\mathscr{F}_t^Y, t \geq 0)$. □

(21.41) Theorem. $(Y_t, t \geq 0)$ *is strongly Markovian with respect to* $\big((\mathscr{F}_t^Y)_{t \geq 0}, H_0^Y\big)$.

Proof. Galmarino's theorem (3.22) asserts that $\mathscr{F}^Y_{H_0} = \sigma(Y_{t \wedge H_0^Y}, t \geq 0)$. It is therefore sufficient to show that

$$\mathbb{E}_{g,0}\big(f(Y_{t+H_0^Y}) \cdot J\big) = \mathbb{E}_{g,0}\big(\mathbb{E}_{Y_{H_0^Y}}\big(f(Y_t)\big) \cdot J\big)$$

holds for all $g \in \mathcal{G}$, $t \geq 0$, $f \in b\mathscr{B}(\mathcal{G})$, $k \in \mathbb{N}$, $f_1, \dots, f_k \in b\mathscr{B}(\mathcal{G})$, $0 \leq t_1 < \cdots < t_k$, with

$$J := f_1(Y_{t_1 \wedge H_0^Y}) \cdots f_k(Y_{t_k \wedge H_0^Y}).$$

But this is immediately proved by equation (7.2) and lemma (21.39), as

$$\begin{aligned}
&\mathbb{E}^Y_{g,0}\big(f(Y_{t+H_0^Y})\, f_1(Y_{t_1 \wedge H_0^Y}) \cdots f_k(Y_{t_k \wedge H_0^Y})\big) \\
&= \mathbb{E}_{g,0}\big(f(X_{t+H_0})\, f_1(X_{t_1 \wedge H_0}) \cdots f_k(X_{t_k \wedge H_0})\big) \\
&= \mathbb{E}_{g,0}\big(\mathbb{E}_{X_{H_0}}(f(X_t))\, f_1(X_{t_1 \wedge H_0}) \cdots f_k(X_{t_k \wedge H_0})\big) \\
&= \mathbb{E}^Y_{g,0}\big(\mathbb{E}_{Y_{H_0^Y}}(f(Y_t))\, f_1(Y_{t_1 \wedge H_0^Y}) \cdots f_k(Y_{t_k \wedge H_0^Y})\big). \qquad \square
\end{aligned}$$

21.7. Markov Property of X

Next, we need to prepare the proof of the Markov property of X with respect to $(\mathscr{F}_t, t \geq 0)$ by analyzing the action of the time shift $(\Theta_t, t \geq 0)$, as defined in (21.30), on all of the underlying components of X. Let $t \geq 0$ be fixed in this subsection.

For each $e \in \mathscr{E} \cup \{0\}$, we define the increments of the processes P_e and Q^e shifted by $\varrho_t = P^{-1}(L_t)$ for all times $s \geq 0$ by

$$\begin{aligned}
{}^{+}P_e(s) &:= P_e(s + \varrho_t) - P(\varrho_t) = P_e(s + P^{-1}(L_t)) - PP^{-1}(L_t), \\
{}^{+}Q^e(s) &:= Q^e(s + \varrho_t) - Q^e(\varrho_t),
\end{aligned}$$

as well as the centered processes by

$$\begin{aligned}
{}^{0}Q^e(s) &:= Q^e(s) - Q^e(0) = Q \circ \Gamma(s), \\
{}^{0}P(s) &:= P(s) - P(0) = P \circ \Gamma(s).
\end{aligned}$$

We notice (recall equation (21.14)) that

$$(21.42) \qquad {}^{+}P = {}^{0}P \circ \Theta^Q_{\varrho_t} = P \circ \Gamma \circ \Theta^Q_{\varrho_t}, \quad {}^{+}Q^e = {}^{0}Q^e \circ \Theta^Q_{\varrho_t} = Q^e \circ \Gamma \circ \Theta^Q_{\varrho_t},$$

and that the processes ${}^{+}P_e$, $e \in \mathcal{E} \cup \{0\}$, and ${}^{0}P$ are strictly increasing as the underlying processes are (see remark (21.7)).

The main non-trivial parts of X are the excursion times $(\eta^e_t, t \geq 0)$, $e \in \mathcal{E}$. We start by examining how the shift of these components by a time t relates to the basic shifts of the underlying processes Q and W:

(21.43) Lemma. *For all $\omega \in \Omega$, $s \geq 0$,*

$$\eta^e_{t+s}(\omega) = P_e\big(({}^{+}P^{-1}(L_s \circ \Theta^W_t - \eta_t))(\omega)\big) \circ \Theta^Q_{\varrho_t}(\omega) - (L_s \circ \Theta^W_t + L_t)(\omega),$$

and

$$\eta^e_{t+s}(\omega) = P_e\big(\,\cdot\,, {}^{+}P^{-1}(\omega, (L_s \circ \Theta^W_t - \eta_t)(\omega))\big) \circ \Theta^Q_{\varrho_t}(\omega) - (L_s \circ \Theta^W_t + L_t)(\omega).$$

Proof. As P and L are increasing, $P(u) > L_{t+s}$ with $u \geq 0$ implies $u \geq P^{-1}(L_t)$, so

$$\begin{aligned} P^{-1}(L_{t+s}) - P^{-1}(L_t) &= \inf\{u \geq 0 : P(u) > L_{t+s}\} - P^{-1}(L_t) \\ &= \inf\{u \geq P^{-1}(L_t) : P(u) > L_{t+s}\} - P^{-1}(L_t) \\ &= \inf\{u \geq 0 : P(u + P^{-1}(L_t)) > L_{t+s}\} \\ &= \inf\{u \geq 0 : {}^{+}P(u) + PP^{-1}(L_t) > L_s \circ \Theta_t^W + L_t\} \\ &= {}^{+}P^{-1}(L_s \circ \Theta_t^W - \eta_t). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} P_e P^{-1}(L_{t+s})(\omega) &= P_e({}^{+}P^{-1}(L_s \circ \Theta_t^W - \eta_t)(\omega) + P^{-1}(L_t)(\omega))(\omega) \\ &= P_e(({}^{+}P^{-1}(L_s \circ \Theta_t^W - \eta_t))(\omega)) \circ \Theta_{\varrho t}^Q(\omega). \end{aligned}$$ □

(21.44) Lemma. *For all $\omega \in \Omega$, $s \geq 0$,*

$$\eta_{t+s}^e(\omega) = (P_e{}^{0}P^{-1}(L_s - \eta_t(\omega)) - L_s) \circ \Theta_t^W \otimes \Theta_{\varrho t}^Q(\omega) - L_t(\omega).$$

Proof. It is

$$\begin{aligned} &(P_e{}^{0}P^{-1}(L_s - \eta_t(\omega)) - L_s) \circ \Theta_t^W \otimes \Theta_{\varrho t}^Q(\omega) - L_t(\omega) \\ &= P_e{}^{0}P^{-1}((L_s \circ \Theta_t^W - \eta_t)(\omega)) \circ \Theta_{\varrho t}^Q(\omega) - (L_s \circ \Theta_t^W + L_t)(\omega), \end{aligned}$$

so with regard to lemma (21.43), it suffices to show that for all $v \in \mathbb{R}$,

$$P_e({}^{+}P^{-1}(\omega, v)) \circ \Theta_{\varrho t}^Q(\omega) = P_e{}^{0}P^{-1}(v) \circ \Theta_{\varrho t}^Q(\omega)$$

holds true: We have ${}^{+}P = {}^{0}P \circ \Theta_{\varrho t}^Q$ by definition, which results in

$${}^{+}P^{-1} = \left({}^{0}P \circ \Theta_{\varrho t}^Q\right)^{-1} = {}^{0}P^{-1} \circ \Theta_{\varrho t}^Q,$$

because for all $\omega \in \Omega$, $v \in \mathbb{R}$,

$$\begin{aligned} \left({}^{0}P \circ \Theta_{\varrho t}^Q\right)^{-1}(\omega, v) &= \inf\{u \geq 0 : ({}^{0}P \circ \Theta_{\varrho t}^Q)(\omega, u) > v\} \\ &= \inf\{u \geq 0 : {}^{0}P(\Theta_{\varrho t}^Q(\omega), u) > v\} \\ &= {}^{0}P^{-1}(\Theta_{\varrho t}^Q(\omega), v) \\ &= {}^{0}P^{-1}(\,\cdot\,, v) \circ \Theta_{\varrho t}^Q(\omega). \end{aligned}$$

This gives us

$$\begin{aligned} P_e(\,\cdot\,, {}^{+}P^{-1}(\omega, v)) \circ \Theta_{\varrho t}^Q(\omega) &= P_e(\,\cdot\,, {}^{0}P^{-1}(\,\cdot\,, v) \circ \Theta_{\varrho t}^Q(\omega)) \circ \Theta_{\varrho t}^Q(\omega) \\ &= P_e{}^{0}P^{-1}(v) \circ \Theta_{\varrho t}^Q(\omega), \end{aligned}$$

completing the proof. □

(21.45) Theorem. *For all $\omega \in \Omega$, $s \geq 0$,*

$$\begin{aligned}\eta^e_{t+s}(\omega) &= \big({}^0P_e {}^0P^{-1}(L_s - \eta_t(\omega)) - (L_s - \eta_t(\omega))\big) \circ \Theta^W_t \otimes \Theta^Q_{\varrho_t}(\omega)\\ &= \big(P_e P^{-1}(L_s - \eta_t(\omega)) - (L_s - \eta_t(\omega))\big) \circ \big(\operatorname{id}^W \otimes \Gamma^Q\big) \circ \big(\Theta^W_t \otimes \Theta^Q_{\varrho_t}\big)(\omega)\\ &= \eta^e_s \circ \big(\operatorname{id}^W \otimes (\gamma^P_{\eta_t(\omega)} \circ \Gamma^Q)\big) \circ \big(\Theta^W_t \otimes \Theta^Q_{\varrho_t}\big)(\omega).\end{aligned}$$

Proof. The first identity follows directly from the preceding lemma (21.44), as inserting the definitions of ${}^0P_e$ and η_t results in

$$\begin{aligned}\big({}^0P_e(\,\cdot\,) + \eta_t(\omega)\big) \circ \Theta^Q_{\varrho_t}(\omega) &= \big(P_e(\,\cdot\, + \varrho_t) - P(\varrho_t) + \eta_t\big)(\omega)\\ &= P_e \circ \Theta^Q_{\varrho_t}(\omega) - L_t(\omega).\end{aligned}$$

The relation ${}^0P_e = P_e \circ \Gamma^Q$ implies the second identity of the claim, and this expression together with $P^{-1} \circ \gamma^P_{\eta_t(\omega)}(v) = P^{-1}\big(v - \eta_t(\omega)\big)$ for all $v \in \mathbb{R}$ yields the last identity, as

$$\begin{aligned}&\big(P_e P^{-1}(L_s - \eta_t(\omega)) - (L_s - \eta_t(\omega))\big) \circ \big(\operatorname{id}^W \otimes \Gamma^Q\big) \circ \big(\Theta^W_t \otimes \Theta^Q_{\varrho_t}\big)(\omega)\\ &\quad= \big(P_e P^{-1}(L_s) - L_s\big) \circ \big(\operatorname{id}^W \otimes \gamma^P_{\eta_t(\omega)}\big) \circ \big(\operatorname{id}^W \otimes \Gamma^Q\big) \circ \big(\Theta^W_t \otimes \Theta^Q_{\varrho_t}\big)(\omega). \qquad \square\end{aligned}$$

(21.46) Corollary. *For all $\omega \in \Omega$, $s \geq 0$,*

$$X_{t+s}(\omega) = X_s \circ \big(\operatorname{id}^W \otimes (\gamma^P_{\eta_t(\omega)} \circ \Gamma^Q)\big) \circ \big(\Theta^W_t \otimes \Theta^Q_{\varrho_t}\big)(\omega).$$

(21.47) Corollary. *For all $s \geq 0$ with $L_s \circ \Theta^W_t < \eta_t$,*

$$\eta^e_{t+s} = \eta^e_t - L_s \circ \Theta^W_t$$

holds $\mathbb{P}_g$-a.s. for every $g \in \mathcal{G}$.

Proof. As ${}^0P_e$ is strictly increasing and ${}^0P_e(0) = P_e(0) - P(0) = 0$ holds $\mathbb{P}_g = \mathbb{P}_{g,0}$-a.s., we have $P^{-1}(v) = 0$ a.s. for every non-positive number $v \leq 0$. Thus, if $L_s \circ \Theta^W_t < \eta_t$, we get from the first identity of theorem (21.45):

$$\begin{aligned}\eta^e_{t+s}(\omega) &= \big({}^0P_e(0) - (L_s - \eta_t(\omega))\big) \circ \Theta^W_t \otimes \Theta^Q_{\varrho_t}(\omega)\\ &= \big(P_e(\varrho_t) - P(\varrho_t) - (L_s \circ \Theta^W_t - \eta_t)\big)(\omega)\\ &= \big(\eta^e_t - \eta_t - L_s \circ \Theta^W_t + \eta_t\big)(\omega),\end{aligned}$$

where we just inserted the definitions of ${}^0P$ and η_t for the last two identities. $\square$

(21.48) Lemma. *For all $g \in \mathcal{G}$, $f \in b\mathscr{B}(\mathcal{G})$, $t \geq 0$,*

$$\begin{aligned}&\mathbb{E}_{g,0}\Big(\int_0^\infty e^{-\alpha s} f(X_{t+s})\,ds \,\Big|\, \mathscr{F}_t\Big)\\ &= \mathbb{E}_{X_t,0}\Big(\int_0^{H^X_0} e^{-\alpha s} f(X_s)\,ds\Big) + \mathbb{E}_{X_t,0}\Big(e^{-\alpha H^X_0}\, \mathbb{E}_{X_{H^X_0},0}\Big(\int_0^\infty e^{-\alpha s} f(X_s)\,ds\Big)\Big).\end{aligned}$$

Proof. We decompose the integral inside the conditional expectation at the end of the first excursion. For the part of the current excursion (if there is one), we compute

$$\begin{aligned}
&\mathbb{E}_{g,0}\Big(\int_0^\infty \mathbb{1}_{\{L_s\circ\Theta_t^W<\eta_t\}}\, e^{-\alpha s}\, f(X_{t+s})\, ds \,\big|\, \mathscr{F}_t\Big)(\omega)\\
&= \mathbb{E}_{g,0}\Big(\mathbb{E}_{g,0}\Big(\int_0^\infty \mathbb{1}_{\{L_s\circ\Theta_t^W<\eta_t\}}\, e^{-\alpha s}\, f\big(E(\eta_t^e - L_s\circ\Theta_t^W, e\in\mathcal{E})\circ(W_s\circ\Theta_t^W),\\
&\qquad\qquad \eta_t - L_s\circ\Theta_t^W + |W_s|\circ\Theta_t^W\big)\, ds \,\big|\, \mathscr{F}_t^W\otimes\mathscr{F}_\infty^Q\Big) \,\big|\, \mathscr{F}_t\Big)(\omega),
\end{aligned}$$

where we used $\mathscr{F}_t \subseteq \mathscr{F}_t^W \otimes \mathscr{F}_\infty^Q$ (by definition of $\mathscr{F}_t$) and corollary (21.47) for the reduction of the shifted excursion times η_{t+s}^e to η_t^e. The Markov property of W with respect to $(\mathscr{F}_t^W, t\geq 0)$ now gives

$$\begin{aligned}
&\mathbb{E}_{g,0}\Big(\mathbb{E}_{g,0}\Big(\int_0^\infty \mathbb{1}_{\{L_s\circ\Theta_t^W<\eta_t\}}\, e^{-\alpha s}\, f\big(E(\eta_t^e - L_s\circ\Theta_t^W, e\in\mathcal{E})\circ(W_s\circ\Theta_t^W),\\
&\qquad\qquad \eta_t - L_s\circ\Theta_t^W + |W_s|\circ\Theta_t^W\big)\, ds \,\big|\, \mathscr{F}_t^W\otimes\mathscr{F}_\infty^Q\Big) \,\big|\, \mathscr{F}_t\Big)(\omega)\\
&= \mathbb{E}_{g,0}\Big(\mathbb{E}_{W_t(\,\cdot\,),0}\Big(\int_0^\infty \mathbb{1}_{\{L_s<\eta_t(\,\cdot\,)\}}\, e^{-\alpha s}\, f\big(E(\eta_t^e(\,\cdot\,) - L_s, e\in\mathcal{E})\circ W_s,\\
&\qquad\qquad \eta_t(\,\cdot\,) - L_s + |W_s|\big)\, ds\Big) \,\big|\, \mathscr{F}_t\Big)(\omega),
\end{aligned}$$

where the auxiliary arguments "$(\,\cdot\,)$" are meant to be variables of the function inside $\mathbb{E}_{g,0}(\cdots\mid\mathscr{F}_t)$,[4] due to the measurability of η_t^e (see corollary (21.23)) with respect to the σ-algebra $\mathscr{F}_t^W\otimes\mathscr{F}_\infty^Q$ being conditioned on (cf. lemma (3.11), which is analogously provable for Markov processes and deterministic shifts). Adaption of W to $(\mathscr{F}_t, t\geq 0)$ now trivializes the conditional expectation, and the decomposition $\{\eta_t>0\} = \biguplus_{e\in\mathcal{E}}\{\eta_t^e>0\}$ by lemma (21.10) (as the whole integral vanishes for $\eta_t = 0$) together with the relation $\{L_s<\eta_t(\omega)\} = \{s < L_-^{-1}(\eta_t(\omega))\}$ for the left-continuous pseudo-inverse L_-^{-1} of L (see lemma (21.6)) yields

$$\begin{aligned}
&\mathbb{E}_{g,0}\Big(\mathbb{E}_{W_t(\,\cdot\,),0}\Big(\int_0^\infty \mathbb{1}_{\{L_s<\eta_t(\,\cdot\,)\}}\, e^{-\alpha s}\, f\big(E(\eta_t^e(\,\cdot\,) - L_s, e\in\mathcal{E})\circ W_s,\\
&\qquad\qquad \eta_t(\,\cdot\,) - L_s + |W_s|\big)\, ds\Big) \,\big|\, \mathscr{F}_t\Big)(\omega)\\
&= \sum_{e\in\mathcal{E}} \mathbb{1}_{\{\eta_t^e(\omega)>0\}}\, \mathbb{E}_{W_t(\omega),0}\Big(\int_0^{L_-^{-1}(\eta_t(\omega))} e^{-\alpha s}\, f\big(e, \eta_t(\omega) + |W_s| - L_s\big)\, ds\Big).
\end{aligned}$$

[4] That is,

$$\begin{aligned}
&\mathbb{E}_{g,0}\Big(\mathbb{E}_{W_t(\,\cdot\,),0}\Big(\int_0^\infty \mathbb{1}_{\{L_s<\eta_t(\,\cdot\,)\}}\, e^{-\alpha s}\, f\big(E(\eta_t^e(\,\cdot\,) - L_s, e\in\mathcal{E})\circ W_s, \eta_t(\,\cdot\,) - L_s + |W_s|\big)\, ds\Big) \,\big|\, \mathscr{F}_t\Big)\\
&= \mathbb{E}_{g,0}\big(Y \mid \mathscr{F}_t\big)
\end{aligned}$$

with $Y(\omega) := \mathbb{E}_{W_t(\omega),0}\Big(\int_0^\infty \mathbb{1}_{\{L_s<\eta_t(\omega)\}}\, e^{-\alpha s}\, f\big(E(\eta_t^e(\omega) - L_s, e\in\mathcal{E})\circ W_s, \eta_t(\omega) - L_s + |W_s|\big)\, ds\Big)$.

By employing Lévy's characterization of the local time and the distribution of its inverse, as examined in lemmas (15.8) and (15.10), applied to the radial part $|W|$ of Walsh Brownian motion (see lemma (19.6)), and then using lemma (21.36) as well as the definition of X, we conclude that

$$
\begin{aligned}
&\sum_{e\in\mathcal{E}} \mathbb{1}_{\{\eta_t^e(\omega)>0\}} \, \mathbb{E}_{W_t(\omega),0}\Big(\int_0^{L_-^{-1}(\eta_t(\omega))} e^{-\alpha s} \, f\big(e, \eta_t(\omega) + |W_s| - L_s\big) \, ds \Big) \\
&= \sum_{e\in\mathcal{E}} \mathbb{1}_{\{\eta_t^e(\omega)>0\}} \, \mathbb{E}_{(e,\eta_t+|W_t|)(\omega),0}\Big(\int_0^{H_0^W} e^{-\alpha s} \, f(W_s) \, ds \Big) \\
&= \mathbb{1}_{\{\eta_t(\omega)>0\}} \, \mathbb{E}_{X_t(\omega),0}\Big(\int_0^{H_0^X} e^{-\alpha s} \, f(X_s) \, ds \Big).
\end{aligned}
$$

In summary, we have shown that—with the knowledge of the process' history—the part of the shifted first excursion (if there is one currently running) equals the first non-shifted excursion, in case the process is restarted at current state of the process:

$$
\begin{aligned}
&\mathbb{E}_{g,0}\Big(\int_0^\infty \mathbb{1}_{\{L_s\circ\Theta_t^W < \eta_t\}} \, e^{-\alpha s} \, f(X_{t+s}) \, ds \,\Big|\, \mathscr{F}_t \Big) \\
&\quad = \mathbb{1}_{\{\eta_t>0\}} \, \mathbb{E}_{X_t,0}\Big(\int_0^{H_0^X} e^{-\alpha s} \, f(X_s) \, ds \Big).
\end{aligned}
\tag{21.49}
$$

Turning to the part after the first excursion, we get by the definition of X_{t+s}

$$
\begin{aligned}
&\mathbb{E}_{g,0}\Big(\int_0^\infty \mathbb{1}_{\{L_s\circ\Theta_t^W \geq \eta_t\}} \, e^{-\alpha s} \, f(X_{t+s}) \, ds \,\Big|\, \mathscr{F}_t \Big)(\omega) \\
&= \mathbb{E}_{g,0}\Big(\int_0^\infty \mathbb{1}_{\{L_s\circ\Theta_t^W \geq \eta_t\}} \, e^{-\alpha s} \, f\big(E(\eta_{t+s}^e, e\in\mathcal{E}) \circ W_{t+s}, \eta_{t+s} + |W_{t+s}|\big) \, ds \,\Big|\, \mathscr{F}_t \Big)(\omega).
\end{aligned}
$$

Theorem (21.45) reduces the shifted excursion times η_{t+s}^e to η_t^e with the help of shifts and centerings of the underlying processes, thus yielding

$$
\begin{aligned}
&\mathbb{E}_{g,0}\Big(\int_0^\infty \mathbb{1}_{\{L_s\circ\Theta_t^W \geq \eta_t\}} \, e^{-\alpha s} \, f\big(E(\eta_{t+s}^e, e\in\mathcal{E}) \circ W_{t+s}, \eta_{t+s} + |W_{t+s}|\big) \, ds \,\Big|\, \mathscr{F}_t \Big)(\omega) \\
&= \mathbb{E}_{g,0}\Big(\Big(\int_0^\infty \mathbb{1}_{\{L_s \geq \eta_t(\,\cdot\,)\}} \, e^{-\alpha s} \, f\big(E\big(P_e P^{-1}(L_s - \eta_t(\,\cdot\,)) - (L_s - \eta_t(\,\cdot\,)), e\in\mathcal{E}\big) \circ W_s, \\
&\qquad\qquad PP^{-1}(L_s - \eta_t(\,\cdot\,)) - (L_s - \eta_t(\,\cdot\,)) + |W_s|\big) \, ds \Big) \\
&\qquad\qquad\qquad \circ \big(\mathrm{id}^W \otimes \Gamma^Q\big) \circ \big(\Theta_t^W \otimes \Theta_{\varrho_t}^Q\big)(\,\cdot\,) \,\Big|\, \mathscr{F}_t \Big)(\omega),
\end{aligned}
$$

where the auxiliary arguments "$(\,\cdot\,)$" again represent the variable of the function inside $\mathbb{E}_{g,0}(\cdots \mid \mathscr{F}_t)$. Employing the Markov property of (W, Q) with respect to $(\mathscr{F}_t, t \geq 0)$, as shown in theorem (21.32), gives

$$\begin{aligned}
&\mathbb{E}_{g,0}\Big(\Big(\int_0^\infty \mathbb{1}_{\{L_s\geq\eta_t(\,\cdot\,)\}}\, e^{-\alpha s}\, f\big(E\big(P_eP^{-1}(L_s-\eta_t(\,\cdot\,))-(L_s-\eta_t(\,\cdot\,)), e\in\mathcal{E}\big)\circ W_s,\\
&\qquad\qquad PP^{-1}(L_s-\eta_t(\,\cdot\,))-(L_s-\eta_t(\,\cdot\,))+|W_s|\,\big)\, ds\Big)\\
&\qquad\qquad\qquad \circ\big(\mathrm{id}^W\otimes\Gamma^Q\big)\circ\big(\Theta_t^W\otimes\Theta_{\varrho t}^Q\big)(\,\cdot\,)\,\big|\,\mathscr{F}_t\Big)(\omega)\\
&=\mathbb{E}_{(W_t,Q_{\varrho t})(\omega)}\Big(\Big(\int_0^\infty \mathbb{1}_{\{L_s\geq\eta_t(\omega)\}}\, e^{-\alpha s}\, f\big(E\big(P_eP^{-1}(L_s-\eta_t(\omega))-(L_s-\eta_t(\omega)), e\in\mathcal{E}\big)\circ W_s,\\
&\qquad\qquad PP^{-1}(L_s-\eta_t(\omega))-(L_s-\eta_t(\omega))+|W_s|\,\big)\, ds\Big)\\
&\qquad\qquad\qquad \circ\big(\mathrm{id}^W\otimes\Gamma^Q\big)\Big).
\end{aligned}$$

The centering operator can be processed with the help of theorem (6.36) by translating the starting point $Q_{\varrho t}(\omega)$ to 0, and the set $\{L_s\geq\eta_t(\omega)\}$ is decomposed into $\{L_s=\eta_t(\omega)\}$ and $\{s>L^{-1}(\eta_t(\omega))\}$ (see lemma (21.6)), resulting in

$$\begin{aligned}
&\mathbb{E}_{(W_t,Q_{\varrho t})(\omega)}\Big(\Big(\int_0^\infty \mathbb{1}_{\{L_s\geq\eta_t(\omega)\}}\, e^{-\alpha s}\, f\big(E\big(P_eP^{-1}(L_s-\eta_t(\omega))-(L_s-\eta_t(\omega)), e\in\mathcal{E}\big)\circ W_s,\\
&\qquad\qquad PP^{-1}(L_s-\eta_t(\omega))-(L_s-\eta_t(\omega))+|W_s|\,\big)\, ds\Big)\\
&\qquad\qquad\qquad \circ\big(\mathrm{id}^W\otimes\Gamma^Q\big)\Big)\\
&=\mathbb{E}_{W_t(\omega),0}\Big(\int_0^\infty \mathbb{1}_{\{L_s=\eta_t(\omega)\}}\, f(W_s)\, ds\\
&\qquad +\int_{L^{-1}(\eta_t(\omega))}^\infty e^{-\alpha s}\, f\big(E\big(P_eP^{-1}(L_s-\eta_t(\omega))-(L_s-\eta_t(\omega)), e\in\mathcal{E}\big)\circ W_s,\\
&\qquad\qquad PP^{-1}(L_s-\eta_t(\omega))-(L_s-\eta_t(\omega))+|W_s|\,\big)\, ds\Big).
\end{aligned}$$

Now, $\{L_s=\eta_t(\omega)\}$ is a null set for every $\eta_t(\omega)\neq 0$, and as L is an additive functional, $L_s\circ\Theta^W_{L^{-1}(u)}=L_{s+L^{-1}(u)}-L_{L^{-1}(u)}=L_{s+L^{-1}(u)}-u$ holds true, so

$$\begin{aligned}
&\mathbb{E}_{W_t(\omega),0}\Big(\int_0^\infty \mathbb{1}_{\{L_s=\eta_t(\omega)\}}\, f(W_s)\, ds\\
&\qquad +\int_{L^{-1}(\eta_t(\omega))}^\infty e^{-\alpha s}\, f\big(E\big(P_eP^{-1}(L_s-\eta_t(\omega))-(L_s-\eta_t(\omega)), e\in\mathcal{E}\big)\circ W_s,\\
&\qquad\qquad PP^{-1}(L_s-\eta_t(\omega))-(L_s-\eta_t(\omega))+|W_s|\,\big)\, ds\Big),\\
&=\mathbb{1}_{\{\eta_t(\omega)=0\}}\,\mathbb{E}_{W_t(\omega),0}\Big(\int_0^\infty \mathbb{1}_{\{L_s=0\}}\, f(W_s)\, ds\Big)\\
&\quad +\mathbb{E}_{W_t(\omega),0}\Big(e^{-\alpha L^{-1}(\eta_t(\omega))}\,\Big(\int_0^\infty e^{-\alpha s}\, f\big(E\big(P_eP^{-1}(L_s)-L_s, e\in\mathcal{E}\big)\circ W_s,\\
&\qquad\qquad PP^{-1}(L_s)-L_s+|W_s|\,\big)\, ds\Big)\circ\Theta^W_{L^{-1}(\eta_t(\omega))}\Big).
\end{aligned}$$

As the local time vanishes until the first hit of the vertex, $\{L_s = 0\} = \{s \le H_0^W\}$ holds true, and applying the strong Markov property of W with respect to its augmented, right continuous filtration for the stopping time $L^{-1}(\eta_t(\omega))$ (while treating the part of the subordinator Q to be constant, which is possible due to Fubini's theorem), with stopping point $W_{L^{-1}(\eta_t(\omega))} = 0$, yields

$$\begin{aligned}
&\mathbb{1}_{\{\eta_t(\omega)=0\}}\, \mathbb{E}_{W_t(\omega),0}\Big(\int_0^\infty \mathbb{1}_{\{L_s=0\}}\, f(W_s)\, ds\Big) \\
&\quad + \mathbb{E}_{W_t(\omega),0}\Big(e^{-\alpha L^{-1}(\eta_t(\omega))}\, \Big(\int_0^\infty e^{-\alpha s}\, f\big(E(P_e P^{-1}(L_s) - L_s, e \in \mathcal{E}) \circ W_s, \\
&\qquad\qquad\qquad\qquad\qquad\qquad PP^{-1}(L_s) - L_s + |W_s|\,\big)\, ds\Big) \circ \Theta^W_{L^{-1}(\eta_t(\omega))}\Big) \\
&= \mathbb{1}_{\{\eta_t(\omega)=0\}}\, \mathbb{E}_{W_t(\omega),0}\Big(\int_0^{H_0^W} f(W_s)\, ds\Big) \\
&\quad + \mathbb{E}_{W_t(\omega),0}\Big(e^{-\alpha L^{-1}(\eta_t(\omega))}\, \mathbb{E}_{0,0}\Big(\int_0^\infty e^{-\alpha s}\, f(X_s)\, ds\Big)\Big).
\end{aligned}$$

Now, lemma (21.38), the relation $X_{H_0^X} = 0$ on $\{H_0^X < \infty\}$ (by right continuity of X) and the definition of X imply

$$\begin{aligned}
&\mathbb{1}_{\{\eta_t(\omega)=0\}}\, \mathbb{E}_{W_t(\omega),0}\Big(\int_0^{H_0^W} f(W_s)\, ds\Big) \\
&\quad + \mathbb{E}_{W_t(\omega),0}\Big(e^{-\alpha L^{-1}(\eta_t(\omega))}\, \mathbb{E}_{0,0}\Big(\int_0^\infty e^{-\alpha s}\, f(X_s)\, ds\Big)\Big) \\
&= \mathbb{1}_{\{\eta_t(\omega)=0\}}\, \mathbb{E}_{X_t(\omega),0}\Big(\int_0^{H_0^X} f(X_s)\, ds\Big) \\
&\quad + \mathbb{E}_{X_t(\omega),0}\Big(e^{-\alpha H_0^X}\, \mathbb{E}_{X_{H_0^X},0}\Big(\int_0^\infty e^{-\alpha s}\, f(X_s)\, ds\Big)\Big).
\end{aligned}$$

In total, we obtained

$$\begin{aligned}
&\mathbb{E}_{g,0}\Big(\int_0^\infty \mathbb{1}_{\{L_s \circ \Theta_t^W \ge \eta_t\}}\, e^{-\alpha s}\, f(X_{t+s})\, ds \,\big|\, \mathscr{F}_t\Big) \\
&= \mathbb{1}_{\{\eta_t=0\}}\, \mathbb{E}_{X_t,0}\Big(\int_0^{H_0^X} f(X_s)\, ds\Big) + \mathbb{E}_{X_t,0}\Big(e^{-\alpha H_0^X}\, \mathbb{E}_{X_{H_0^X},0}\Big(\int_0^\infty e^{-\alpha s}\, f(X_s)\, ds\Big)\Big),
\end{aligned}$$

which together with above result (21.49) for the first excursion concludes the proof. □

By combining this lemma with the strong Markov property at H_0, we are now able to deduce the Markov property of X. As we only have access to the strong Markov property at H_0 with respect to canonical filtration $(\mathscr{F}_t^Y, t \ge 0)$ of the path space realization Y of X (see the preceding subsection 21.6), we need to restrict our attention to the canonical filtration of X as well:

$$\mathscr{F}_t^X := \sigma(X_s, s \le t), \quad t \ge 0.$$

(21.50) Corollary. *For all $g \in \mathcal{G}$, $f \in b\mathscr{B}(\mathcal{G})$, $t \geq 0$,*

$$\begin{aligned}
&\mathbb{E}_{g,0}\Big(\int_0^\infty e^{-\alpha s} f(X_{t+s})\,ds \,\big|\, \mathscr{F}_t^X\Big) \\
&= \mathbb{E}_{X_t,0}\Big(\int_0^{H_0^X} e^{-\alpha s} f(X_s)\,ds\Big) + \mathbb{E}_{X_t,0}\Big(e^{-\alpha H_0^X}\,\mathbb{E}_{X_{H_0^X},0}\Big(\int_0^\infty e^{-\alpha s} f(X_s)\,ds\Big)\Big).
\end{aligned}$$

Proof. As X is adapted to $(\mathscr{F}_t, t \geq 0)$ by lemma (21.19), we have $\mathscr{F}_t^X \subseteq \mathscr{F}_t$ for all $t \geq 0$. Thus, lemma (21.48) yields

$$\begin{aligned}
&\mathbb{E}_{g,0}\Big(\int_0^\infty e^{-\alpha s} f(X_{t+s})\,ds \,\big|\, \mathscr{F}_t^X\Big) \\
&= \mathbb{E}_{g,0}\Big(\mathbb{E}_{g,0}\Big(\int_0^\infty e^{-\alpha s} f(X_{t+s})\,ds \,\big|\, \mathscr{F}_t\Big) \,\big|\, \mathscr{F}_t^X\Big) \\
&= \mathbb{E}_{g,0}\Big(\mathbb{E}_{X_t,0}\Big(\int_0^{H_0^X} e^{-\alpha s} f(X_s)\,ds\Big) \\
&\qquad + \mathbb{E}_{X_t,0}\Big(e^{-\alpha H_0^X}\,\mathbb{E}_{X_{H_0^X},0}\Big(\int_0^\infty e^{-\alpha s} f(X_s)\,ds\Big)\Big) \,\big|\, \mathscr{F}_t^X\Big).
\end{aligned}$$

The remaining conditional expectation trivializes as X is adapted to $(\mathscr{F}_t^X, t \geq 0)$. □

(21.51) Lemma. *For all $g \in \mathcal{G}$, $f \in b\mathscr{B}(\mathcal{G})$, $t \geq 0$,*

$$\mathbb{E}_g\Big(\int_0^\infty e^{-\alpha s} f(X_{t+s})\,ds \,\big|\, \mathscr{F}_t^X\Big) = \mathbb{E}_{X_t}\Big(\int_0^\infty e^{-\alpha s} f(X_s)\,ds\Big).$$

Proof. Let $g \in \mathcal{G}$, $f \in b\mathscr{B}(\mathcal{G})$ and $t \geq 0$. In theorem (21.41) we showed the strong Markov property of Y at H_0^Y. Therefore, we are able to apply Dynkin's formula (3.17) for the stopping time H_0^Y, resulting in

$$\begin{aligned}
&\mathbb{E}_g^Y\Big(\int_0^{H_0^Y} e^{-\alpha s} f(Y_s)\,ds\Big) + \mathbb{E}_g^Y\Big(e^{-\alpha H_0^Y}\,\mathbb{E}_{Y_{H_0^Y}}^Y\Big(\int_0^\infty e^{-\alpha s} f(Y_s)\,ds\Big)\Big) \\
&= \mathbb{E}_g^Y\Big(\int_0^\infty e^{-\alpha s} f(Y_s)\,ds\Big).
\end{aligned} \tag{21.52}$$

For $n \in \mathbb{N}$, $A_1, \dots, A_n \in \mathscr{B}(\mathcal{G})$, $0 \leq t_1 \leq \cdots \leq t_n \leq t$, set

$$\begin{aligned}
A^X &:= \{X_{t_1} \in A_1, \dots, X_{t_n} \in A_n\} \in \mathscr{F}_t^X, \\
A^Y &:= \{Y_{t_1} \in A_1, \dots, Y_{t_n} \in A_n\} \in \mathscr{F}_t^Y.
\end{aligned}$$

Then, as $\mathbb{E}_{X_t,0}\big(\int_0^\infty e^{-\alpha s} f(X_s)\,ds\big)$ is $\mathscr{F}_t^X$-measurable and the set of all A^X of the above form constitutes an $\cap$-stable generating system of $\mathscr{F}_t^X$, it suffices to prove that

$$\mathbb{E}_g\Big(\int_0^\infty e^{-\alpha s} f(X_{t+s})\,ds\,;\, A^X\Big) = \mathbb{E}_g\Big(\mathbb{E}_{X_t}\Big(\int_0^\infty e^{-\alpha s} f(X_s)\,ds\Big)\,;\, A^X\Big).$$

We start by decomposing the resolvent at H_0^X with the help of lemma (21.48):

$$\begin{aligned}
&\mathbb{E}_g\Big(\int_0^\infty e^{-\alpha s}\, f(X_{t+s})\, ds\,;\, A^X\Big)\\
&= \mathbb{E}_g\Big(\mathbb{E}_{X_t}\Big(\int_0^{H_0^X} e^{-\alpha s}\, f(X_s)\, ds\Big)\\
&\qquad + \mathbb{E}_{X_t}\Big(e^{-\alpha H_0^X}\, \mathbb{E}_{X_{H_0^X}}\Big(\int_0^\infty e^{-\alpha s}\, f(X_s)\, ds\Big)\Big)\,;\, A^X\Big).
\end{aligned}$$

Now we switch to Y via the mapping operator Φ (see equations (7.1) and (7.2)), use the strong Markov property of Y at H_0^Y, embodied in equation (21.52), in order to reunite both resolvent parts, and then switch back to X, yielding

$$\begin{aligned}
&\mathbb{E}_g\Big(\int_0^\infty e^{-\alpha s}\, f(X_{t+s})\, ds\,;\, A^X\Big)\\
&= \mathbb{E}_g\Big(\mathbb{E}_{Y_t\circ\Phi}\Big(\int_0^{H_0^Y\circ\Phi} e^{-\alpha s}\, f(Y_s\circ\Phi)\, ds\Big)\\
&\qquad + \mathbb{E}_{Y_t\circ\Phi}\Big(e^{-\alpha H_0^Y\circ\Phi}\, \mathbb{E}_{Y_{H_0^Y}\circ\Phi}\Big(\int_0^\infty e^{-\alpha s}\, f(Y_s\circ\Phi)\, ds\Big)\Big)\,;\, \Phi^{-1}(A^Y)\Big)\\
&= \mathbb{E}_g^Y\Big(\mathbb{E}_{Y_t}^Y\Big(\int_0^{H_0^Y} e^{-\alpha s}\, f(Y_s)\, ds\Big)\\
&\qquad + \mathbb{E}_{Y_t}^Y\Big(e^{-\alpha H_0^Y}\, \mathbb{E}_{Y_{H_0^Y}}^Y\Big(\int_0^\infty e^{-\alpha s}\, f(Y_s)\, ds\Big)\Big)\,;\, A^Y\Big)\\
&= \mathbb{E}_g^Y\Big(\mathbb{E}_{Y_t}^Y\Big(\int_0^\infty e^{-\alpha s}\, f(Y_s)\, ds\Big)\,;\, A^Y\Big)\\
&= \mathbb{E}_g\Big(\mathbb{E}_{X_t}\Big(\int_0^\infty e^{-\alpha s}\, f(X_s)\, ds\Big)\,;\, A^X\Big).
\end{aligned}$$

□

(21.53) Theorem. $X = (\Omega, \mathscr{F}, (\mathscr{F}_t^X)_{t\geq 0}, (X_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbb{P}_g)_{g\in\mathcal{G}})$ *is a right continuous simple Markov process.*

Proof. The right continuity of $(X_t, t \geq 0)$ has been shown in lemma (21.12). In regard to theorem (2.18), the Markov property has been proved in lemma (21.51). □

21.8. Strong Markov Property of X

(21.54) Theorem. X *is a Feller process.*

Proof. We already know that X is a Markov process, which implies the family $(T_t, t \geq 0)$ defined by

$$T_t f(x) = \mathbb{E}_g\big(f(X_t)\big), \quad t \geq 0, f \in b\mathscr{B}(\mathcal{G}), g \in \mathcal{G},$$

is indeed a Markov semigroup. It is therefore sufficient to show that this semigroup is Feller. We will check property (iv) of theorem (5.13):

By the right continuity and normality of X, LDCT yields

$$\forall g \in \mathcal{G}, f \in \mathcal{C}_0(\mathcal{G}): \quad \lim_{t \downarrow 0} \mathbb{E}_g\big(f(X_t)\big) = \mathbb{E}_g\big(f(X_0)\big) = f(g),$$

so $(T_t, t \geq 0)$ is continuous at 0. It remains to prove that the resolvent $(U_\alpha, \alpha > 0)$ of X preserves $\mathcal{C}_0(\mathcal{G})$, that is, we need to show that

$$\forall \alpha > 0, f \in \mathcal{C}_0(\mathcal{G}): \quad U_\alpha f \in \mathcal{C}_0(\mathcal{G}).$$

To this end, we decompose once again the resolvent of X at H_0 with lemma (21.48) for $t = 0$: Using $X_{H_0} = v$ (by the right continuity of X) and lemma (21.36), we get

$$\begin{aligned}
U_\alpha f(g) &= \mathbb{E}_g\Big(\int_0^\infty e^{-\alpha s} f(X_s)\, ds \Big) \\
&= \mathbb{E}_g\Big(\int_0^{H_0} e^{-\alpha s} f(X_s)\, ds \Big) + \mathbb{E}_g\Big(e^{-\alpha H_0}\, \mathbb{E}_{X_{H_0}}\Big(\int_0^\infty e^{-\alpha s} f(X_s)\, ds \Big)\Big) \\
&= \mathbb{E}_g^W\Big(\int_0^{H_0^W} e^{-\alpha s} f(W_s)\, ds \Big) + \mathbb{E}_g^W\big(e^{-\alpha H_0^W}\big)\, U_\alpha f(v) \\
&= U_\alpha^{W,D} f(g) + e^{-\sqrt{2\alpha}\, d(v,g)}\, U_\alpha f(v),
\end{aligned}$$

with $\big(U_\alpha^{W,D}, \alpha > 0\big)$ being the resolvent of the Walsh process on $\mathcal{G}$ killed at v. It is now immediate that $(U_\alpha, \alpha > 0)$ preserves $\mathcal{C}_0(\mathcal{G})$, because $\big(U_\alpha^{W,D}, \alpha > 0\big)$ preserves $\mathcal{C}_0(\mathcal{G})$ by example (19.8), $g \mapsto \exp\big(- \sqrt{2\alpha}\, d(v,g)\big)$ is continuous and vanishes at infinity, and $\lim_{g \to v} U_\alpha^{W,D} f(g) = 0$ holds true. □

21.9. Local Time of X at the Vertex

As P is strictly increasing, the process $P^{-1}L_t$ grows if and only if $PP^{-1}L_t$ grows, that is, if $L_t \in \mathscr{D}$ (cf. the results of subsection 21.2). But then $X_t = W_t$ must be at v. Furthermore, we showed in equation (21.16) that $t \mapsto P^{-1}L_t$ is an additive functional for X. By this (and also by looking at the path behavior of X), the following result is to be expected (see also [BG69, Section V.3]):

(21.55) Theorem. *The local time $(L_t^X, t \geq 0)$ of X at v is*

$$L_t^X = P^{-1} L_t, \quad t \geq 0.$$

In general, the local time of X at v only depends on the behavior of X at v, and therefore only on the behavior of the second coordinate $(\eta_t + |W_t|, t \geq 0)$ at the origin. This is however exactly the Brownian motion on the half line which was constructed by Itô and McKean, and it was proved in [IM63, Section 14] that $(P^{-1}L_t, t \geq 0)$ is its local time at the origin. So the above theorem is achieved by carrying over their result to our generalization.

21.10. General Brownian Motion $X^\bullet$ on a Star Graph

Up to this point, we only took care of the reflection parameters $(p_2^e, e \in \mathcal{E})$ and the jump distributions $(p_4^e, e \in \mathcal{E})$. We will now implement the stickiness parameter $p_3 \geq 0$ and the killing parameter $p_1 \geq 0$ by using the standard procedures of time change and killing (see sections 9 and 10). To this end, we will now consider the Feller process X as right process in the context of the usual hypotheses (see theorem (5.8)).

For the implementation of stickiness, we define the additive functional $(\tau_t, t \geq 0)$ by

$$\tau_t := t + p_3 L_t^X, \quad t \geq 0,$$

and consider the time-changed process $\big(X_{\tau^{-1}(t)}, t \geq 0\big)$. By theorem (9.3) with analogous considerations as in example (9.2), $\big(X_{\tau^{-1}(t)}, t \geq 0\big)$ is a right process with shift operators $\big(\Theta^X_{\tau^{-1}(t)}, t \geq 0\big)$. Its local time turns out to be $\big(L^X_{\tau^{-1}(t)}, t \geq 0\big)$, which we will only show (and need) partially:

(21.56) Lemma. *$\big(L^X_{\tau^{-1}(t)}, t \geq 0\big)$ is an additive functional for $\big(X_{\tau^{-1}(t)}, t \geq 0\big)$.*

Proof. For any $s, t \geq 0$, we compute

$$\begin{aligned} L^X_{\tau^{-1}(t)} \circ \Theta^X_{\tau^{-1}(s)} &= L^X_{\tau^{-1}(t) \circ \Theta^X_{\tau^{-1}(s)} + \tau^{-1}(s)} - L^X_{\tau^{-1}(s)} \\ &= L^X_{\tau^{-1}(t+s)} - L^X_{\tau^{-1}(s)}, \end{aligned}$$

where we used that $(L_t^X, t \geq 0)$ is an additive functional (as seen in subsection 21.9, cf. equation (21.16)) for the first identity, and for the second identity employed the relation

$$\tau^{-1}(t) \circ \Theta^X_{\tau^{-1}(s)} + \tau^{-1}(s) = \tau^{-1}(t+s), \quad s, t \geq 0,$$

which is a general result for the inverse of any additive functional $(\tau_t, t \geq 0)$ (see, e.g., [Sha88, Proposition 65.8] or the computations in the proof of [Kni81, Theorem 6.4]). □

Now kill this new process $\big(X_{\tau^{-1}(t)}, t \geq 0\big)$ once its local time $\big(L^X_{\tau^{-1}(t)}, t \geq 0\big)$ reaches a certain level: To this end, apply the construction given in subsection 10.2 in order to introduce an exponentially distributed random variable S with mean 1, independent of $\mathscr{F}$, and set

$$\zeta := \inf\{t \geq 0 : p_1 \, L^X_{\tau^{-1}(t)} > S\}.$$

Establish the definitive process $X^\bullet$ resulting from killing $\big(X_{\tau^{-1}(t)}, t \geq 0\big)$ at ζ by

$$\forall t \geq 0: \quad X_t^\bullet := \begin{cases} X_{\tau^{-1}(t)}, & t < \zeta, \\ \Delta, & t \geq \zeta. \end{cases}$$

Lemma (21.56) and theorem (10.2) yield the following:

(21.57) Theorem. *$X^\bullet$ is a right process.*

21.11. Resolvent and Generator of $X^\bullet$

We will conclude our construction by showing that $X^\bullet$ is indeed the process which implements the correct boundary conditions into the generator. Let $(U^\bullet_\alpha, \alpha > 0)$ be the resolvent and $A^\bullet$ be the generator of $X^\bullet$.

We first trace the resolvent $U^\bullet$ of $X^\bullet$ back to the components of X:

(21.58) Lemma. *For $\alpha > 0$, $f \in b\mathcal{C}(\mathcal{G})$, $g \in \mathcal{G}$,*

$$U^\bullet_\alpha f(g) = \mathbb{E}_g\Big(\int_0^\infty e^{-\alpha t}\, e^{-(p_1+\alpha p_3)L^X_t}\, f(X_t)\, d\tau(t)\Big).$$

Proof. The definition of $X^\bullet$ and the independence of S from everything else yield

$$\begin{aligned}
U^\bullet_\alpha f(g) &= \mathbb{E}_g\Big(\int_0^\infty e^{-\alpha t}\, f(X^\bullet_t)\, dt\Big) \\
&= \mathbb{E}_g\Big(\int_0^\infty e^{-\alpha t}\, \mathbb{1}_{\{p_1 L^X_{\tau^{-1}(t)} < S\}}\, f(X_{\tau^{-1}(t)})\, dt\Big) \\
&= \mathbb{E}_g\Big(\int_0^\infty e^{-\alpha t}\, \Big(\int_{p_1 L^X_{\tau^{-1}(t)}}^\infty e^{-s}\, ds\Big)\, f(X_{\tau^{-1}(t)})\, dt\Big) \\
&= \mathbb{E}_g\Big(\int_0^\infty e^{-\alpha t}\, e^{-p_1 L^X_{\tau^{-1}(t)}}\, f(X_{\tau^{-1}(t)})\, dt\Big).
\end{aligned}$$

As $(\tau_t, t \geq 0)$ is increasing and bijective, the substitution rule for Stieltjes integrals (see, e.g., [FT12]) gives

$$\begin{aligned}
U^\bullet_\alpha f(g) &= \mathbb{E}_g\Big(\int_0^\infty e^{-\alpha\tau(\tau^{-1}(t))}\, e^{-p_1 L^X_{\tau^{-1}(t)}}\, f(X_{\tau^{-1}(t)})\, dt\Big) \\
&= \mathbb{E}_g\Big(\int_0^\infty e^{-\alpha\tau(t)}\, e^{-p_1 L^X_t}\, f(X_t)\, d\tau(t)\Big),
\end{aligned}$$

and inserting the definition $\tau_t = t + p_3 L^X_t$ completes the proof. □

We are now ready to completely calculate the resolvent of $X^\bullet$. The form of the resolvent is well-known for the case of the half line, see [IM63, Section 15], or [Rog83, Theorem 3] for a different approach via excursion theory. As we constructed $X^\bullet$ pathwise, we will follow the computational techniques of [IM63] in order to prove the following theorem:

(21.59) Theorem. *For $\alpha > 0$, $f \in b\mathcal{C}(\mathcal{G})$, $g \in \mathcal{G}$,*

$$U^\bullet_\alpha f(g) = U^{W,D}_\alpha f(g) + e^{-\sqrt{2\alpha}\, d(v,g)}\, U^\bullet_\alpha f(v)$$

holds, with $(U^{W,D}_\alpha, \alpha > 0)$ being the resolvent of the Walsh process on $\mathcal{G}$ killed at v (as given in example (19.8)), and

$$U^\bullet_\alpha f(v) = \frac{\sum_{e\in\mathcal{E}} p_2^e\, 2\int_0^\infty e^{-\sqrt{2\alpha}x} f(e,x)\, dx + p_3\, f(v) + \int U^{W,D}_\alpha f(g)\, p_4(dg)}{p_1 + \sqrt{2\alpha}\, p_2 + \alpha\, p_3 + \int_0^\infty (1 - e^{-\sqrt{2\alpha} l})\, p_4^\Sigma(dl)}$$

holds with $p_4^\Sigma = \sum_{e\in\mathcal{E}} p_4^e$.

Proof. Consider the first hitting time of the vertex v for $X^\bullet$, that is,

$$H_0^\bullet := \inf\{t \geq 0 : X_t^\bullet = v\}.$$

We observe that the transformation effects from X to $X^\bullet$ only take effect after the first hitting of v, so $X_t^\bullet = X_t = W_t$ for all $t \leq H_0^\bullet = H_0^X = H_0^W$ (see also lemma (21.36)). In addition, $X^\bullet_{H_0^\bullet} = v$ holds by right continuity of $X^\bullet$. The application of Dynkin's formula (3.16) for the decomposition of the Feller (thus strongly Markovian) process $X^\bullet$ at the stopping time $H_0^\bullet$ therefore yields

$$\begin{aligned} U_\alpha^\bullet f(g) &= \mathbb{E}_g\Big(\int_0^{H_0^\bullet} e^{-\alpha t}\, f(X_t)\, dt\Big) + \mathbb{E}_g\big(e^{-\alpha H_0^\bullet}\, U_\alpha^\bullet f(X^\bullet_{H_0^\bullet})\big) \\ &= \mathbb{E}_g^W\Big(\int_0^{H_0^W} e^{-\alpha t}\, f(W_t)\, dt\Big) + \mathbb{E}_g^W\big(e^{-\alpha H_0^W}\big)\, U_\alpha^\bullet f(v). \end{aligned}$$

The Laplace transform of the first hitting time of the vertex reads $\mathbb{E}_g^W\big(e^{-\alpha H_0^W}\big) = e^{-\sqrt{2\alpha}\, d(v,g)}$ by lemma (14.5), as the Walsh process W behaves on any edge like a reflecting Brownian motion (see theorem (19.6)).

It remains to analyze the resolvent at the vertex v: Continuing the computations of lemma (21.58), we obtain by inserting the definition of τ_t and using that L_t^X only grows at $X_t = v$, that

$$\begin{aligned} U_\alpha^\bullet f(v) &= \mathbb{E}_v\Big(\int_0^\infty e^{-\alpha t}\, e^{-(p_1+\alpha p_3)L_t^X}\, f(X_t)\, d\tau(t)\Big) \\ &= \mathbb{E}_v\Big(\int_0^\infty e^{-\alpha t}\, e^{-(p_1+\alpha p_3)L_t^X}\, f(X_t)\, dt\Big) \\ &\quad + p_3\, f(v)\, \mathbb{E}_v\Big(\int_0^\infty e^{-\alpha t}\, e^{-(p_1+\alpha p_3)L_t^X}\, dL^X(t)\Big). \end{aligned}$$

Decomposing $\mathbb{R}_+$ into $\bigcup_{n\in\mathbb{N}} \big[L_-^{-1}(l_n^-), L_-^{-1}(l_n^+)\big)$ and its complement, and using that $X_t = \big(e_n, l_n^+ + |W_t| - L_t\big)$ holds for $t \in \big[L_-^{-1}(l_n^-), L_-^{-1}(l_n^+)\big)$, $n \in \mathbb{N}$, and $X_t = W_t$ otherwise (see subsection 21.2, especially equation (21.11)) results in

$$\begin{aligned} U_\alpha^\bullet f(v) &= \sum_{n\in\mathbb{N}} \mathbb{E}_v\Big(\int_{L_-^{-1}(l_n^-)}^{L_-^{-1}(l_n^+)} e^{-\alpha t}\, e^{-(p_1+\alpha p_3)P^{-1}L_t}\, f\big(e_n, l_n^+ + |W_t| - L_t\big)\, dt\Big) \\ &\quad + \mathbb{E}_v\Big(\int_0^\infty e^{-\alpha t}\, e^{-(p_1+\alpha p_3)P^{-1}L_t}\, f(W_t)\, dt\Big) \\ &\quad - \sum_{n\in\mathbb{N}} \mathbb{E}_v\Big(\int_{L_-^{-1}(l_n^-)}^{L_-^{-1}(l_n^+)} e^{-\alpha t}\, e^{-(p_1+\alpha p_3)P^{-1}L_t}\, f(W_t)\, dt\Big) \\ &\quad + p_3\, f(v)\, \mathbb{E}_v\Big(\int_0^\infty e^{-\alpha t}\, e^{-(p_1+\alpha p_3)P^{-1}L_t}\, dP^{-1}L_t\Big) \\ &=: u_1 + u_2 - u_3 + u_4. \end{aligned}$$

We are going to compute these four expressions one after the other:

We start with u_1: The functions l_n^-, l_n^+, $n \in \mathbb{N}$, only depend on Ω^Q. We begin by computing the (conditional) expectation with respect to the space Ω^W. Fubini's theorem asserts that while integrating on Ω^W, we can treat l_n^-, l_n^+, $n \in \mathbb{N}$, as constants (this will not be annotated in the formulas below to keep them reasonably readable), therefore

$$u_1 = \sum_{n\in\mathbb{N}} \mathbb{E}_0^Q\Big(\mathbb{E}_v^W\Big(\int_0^{L_-^{-1}(l_n^+)-L_-^{-1}(l_n^-)} e^{-\alpha(t+L_-^{-1}(l_n^-))}\, e^{-(p_1+\alpha p_3)P^{-1}(l_n^-)}$$
$$f\big(e_n, l_n^+ + |W_{t+L_-^{-1}(l_n^-)}| - L_{t+L_-^{-1}(l_n^-)}\big)\, dt\Big)\Big).$$

Using $L_-^{-1}(l_n^+) - L_-^{-1}(l_n^-) = L_-^{-1}(l_n) \circ \Theta^W_{L_-^{-1}(l_n^-)}$ by lemma (15.7), the additive functional property $L_{t+L_-^{-1}(l_n^-)} = L_t \circ \Theta^W_{L_-^{-1}(l_n^-)} - L_{L_-^{-1}(l_n^-)}$, with $L_{L_-^{-1}(l_n^-)} = l_n^-$ by continuity of L, as well as $l_n^+ - l_n^- = l_n$ by remark (21.7), then yields for u_1

$$\sum_{n\in\mathbb{N}} \mathbb{E}_0^Q\Big(\mathbb{E}_v^W\Big(e^{-\alpha L_-^{-1}(l_n^-)}\, e^{-(p_1+\alpha p_3)P^{-1}(l_n^-)}$$
$$\mathbb{E}_0^W\Big(\Big(\int_0^{L_-^{-1}(l_n)} e^{-\alpha t}\, f(e_n, l_n + |W_t| - L_t)\, dt\Big) \circ \Theta^W_{L_-^{-1}(l_n^-)} \,\Big|\, \mathscr{F}^W_{\Theta^W_{L_-^{-1}(l_n^-)}}\Big)\Big)\Big).$$

W is strongly Markovian with respect to the stopping time $L_-^{-1}(l_n^-)$, with the stopping point being given by $W_{L_-^{-1}(l_n^-)} = v$ (as L only grows at v), so by also using that $L_-^{-1}(l_n^-) = L^{-1}(l_n^-)$ holds a.s. by lemma (15.8), it follows that

$$u_1 = \sum_{n\in\mathbb{N}} \mathbb{E}_0^Q\Big(e^{-(p_1+\alpha p_3)P^{-1}(l_n^-)}\, \mathbb{E}_0^W\Big(e^{-\alpha L^{-1}(l_n^-)}$$
$$\mathbb{E}_0^W\Big(\int_0^{L^{-1}(l_n)} e^{-\alpha t}\, f(e_n, l_n + |W_t| - L_t)\, dt\Big)\Big)\Big).$$

Now the process $\big(l_n + |W_t| - L_t, t \le L^{-1}(l_n)\big)$ started at 0 behaves just like the standard Brownian motion $(B_t, t \le H_0^B)$ started at l_n (cf. lemma (15.10)). By using lemma (15.9) for the characteristic function of L^{-1}, we thus get

$$\begin{aligned}
u_1 &= \sum_{n\in\mathbb{N}} \mathbb{E}_0^Q\Big(e^{-(p_1+\alpha p_3)P^{-1}(l_n^-)}\, \mathbb{E}_v^W\Big(e^{-\alpha L^{-1}(l_n^-)}\, \mathbb{E}_{l_n}^B\Big(\int_0^{H_0^B} e^{-\alpha t}\, f(e_n, B_t)\, dt\Big)\Big)\Big) \\
&= \sum_{n\in\mathbb{N}} \mathbb{E}_0^Q\Big(e^{-(p_1+\alpha p_3)P^{-1}(l_n^-)}\, \mathbb{E}_v^W\Big(e^{-\alpha L^{-1}(l_n^-)}\Big)\, U_\alpha^{[0,\infty)}(f(e_n,\cdot))(l_n)\Big) \\
&= \sum_{n\in\mathbb{N}} \mathbb{E}_0^Q\Big(e^{-\alpha l_n^-}\, e^{-(p_1+\alpha p_3)P^{-1}(l_n^-)}\, U_\alpha^{W,D} f(e_n, l_n)\Big).
\end{aligned}$$

Representing P by its random measure N, with jump times $(t_n, n \in \mathbb{N})$ and jump marks $\big((e_n, l_n), n \in \mathbb{N}\big)$ as discussed in remark (21.7), results in

$$\begin{aligned}
u_1 &= \sum_{n\in\mathbb{N}} \mathbb{E}_0^Q\Big(e^{-\alpha P(t_n-)}\, e^{-(p_1+\alpha p_3)t_n}\, U_\alpha^{W,D} f(e_n, l_n)\Big) \\
&= \mathbb{E}_0^Q\Big(\int e^{-\alpha P(t-)}\, e^{-(p_1+\alpha p_3)t}\, U_\alpha^{W,D} f(g)\, N(dt \times dg)\Big).
\end{aligned}$$

Employing theorem (6.29) for the last integral, we conclude that

$$\begin{aligned} u_1 &= \int_0^\infty e^{-t(\sqrt{2\alpha}p_2+\int_0^\infty (1-e^{-\sqrt{2\alpha}l})\, p_4^\Sigma(dl))}\, e^{-(p_1+\alpha p_3)t} dt \cdot \int U_\alpha^{W,D} f(g)\, p_4(dg) \\ &= \frac{\int U_\alpha^{W,D} f(g)\, p_4(dg)}{p_1+\sqrt{2\alpha}\, p_2+\alpha\, p_3+\int_0^\infty \big(1-e^{-\sqrt{2\alpha}l}\big)\, p_4^\Sigma(dl)}. \end{aligned}$$

The computations for u_3 follow the same path, but are easier. By using the same techniques as for u_1, we get

$$\begin{aligned} u_3 &= \sum_{n\in\mathbb{N}} \mathbb{E}_0^Q\Big(e^{-(p_1+\alpha p_3)P^{-1}(l_n^-)} \\ &\qquad\qquad \mathbb{E}_v^W\Big(\int_0^{L_-^{-1}(l_n)\circ\Theta^W_{L_-^{-1}(l_n^-)}} e^{-\alpha(t+L_-^{-1}(l_n^-))}\, f\big(W_t\circ\Theta^W_{L_-^{-1}(l_n^-)}\big)\, dt\Big)\Big) \\ &= \sum_{n\in\mathbb{N}} \mathbb{E}_0^Q\Big(e^{-(p_1+\alpha p_3)P^{-1}(l_n^-)}\, e^{-\alpha L^{-1}(l_n^-)}\, \mathbb{E}_v^W\Big(\int_0^{L^{-1}(l_n)} f(W_t)\, dt\Big)\Big). \end{aligned}$$

Applying Dynkin's formula (3.16) for the decomposition at the stopping time $L^{-1}(l_n)$ (see lemma (15.6)) yields

$$\begin{aligned} U_\alpha^W f(v) &= \mathbb{E}_v^W\Big(\int_0^{L^{-1}(l_n)} e^{-\alpha t}\, f(W_t)\, dt\Big) + \mathbb{E}_v^W\Big(e^{-\alpha L^{-1}(l_n)}\, U_\alpha^W f\big(W_{L^{-1}(l_n)}\big)\Big) \\ &= \mathbb{E}_v^W\Big(\int_0^{L^{-1}(l_n)} e^{-\alpha t}\, f(W_t)\, dt\Big) + e^{-\sqrt{2\alpha}l_n}\, U_\alpha^W f(v), \end{aligned}$$

thus resulting in

$$\begin{aligned} u_3 &= \sum_{n\in\mathbb{N}} \mathbb{E}_0^Q\Big(e^{-(p_1+\alpha p_3)P^{-1}(l_n^-)}\, e^{-\sqrt{2\alpha}l_n^-}\, \big(1-e^{-\sqrt{2\alpha}l_n}\big)\, U_\alpha^W f(v)\Big) \\ &= \sum_{n\in\mathbb{N}} \mathbb{E}_0^Q\Big(e^{-(p_1+\alpha p_3)t_n}\, e^{-\sqrt{2\alpha}P(t_n-)}\, \big(1-e^{-\sqrt{2\alpha}l_n}\big)\Big)\cdot U_\alpha^W f(v) \\ &= \frac{\int_0^\infty \big(1-e^{-\sqrt{2\alpha}l}\big)\, p_4^\Sigma(dl)}{p_1+\sqrt{2\alpha}\, p_2+\alpha\, p_3+\int_0^\infty \big(1-e^{-\sqrt{2\alpha}l}\big)\, p_4^\Sigma(dl)}\cdot U_\alpha^W f(v), \end{aligned}$$

where the last identity follows again from theorem (6.29) together with

$$\int \big(1-e^{-\sqrt{2\alpha}\pi^2(e,x)}\big)\, p_4\big(d(e,x)\big) = \int_0^\infty \big(1-e^{-\sqrt{2\alpha}x}\big)\, p_4^\Sigma(dx).$$

Coming to u_2, we first recall the joint distribution of (W_t, L_t) (see lemma (19.7)):

$$\mathbb{E}_v^W\big(f(W_t,L_t)\big) = \sum_{e\in\mathcal{E}} q_2^e \int_0^\infty \int_0^\infty f\big((e,x),y\big)\, \frac{2(x+y)}{\sqrt{2\pi t^3}}\, e^{-\frac{(x+y)^2}{2t}}\, dx\, dy.$$

Using the independence of W and Q, as well as the distribution for (W_t, L_t), gives

$$\begin{aligned}
u_2 &= \mathbb{E}_0^Q\Big(\sum_{e\in\mathcal{E}} q_2^e \int_0^\infty \int_0^\infty \int_0^\infty e^{-\alpha t}\, e^{-(p_1+\alpha p_3)P^{-1}(y)}\, f\big((e,x)\big)\, \frac{2(x+y)}{\sqrt{2\pi t^3}}\, e^{-\frac{(x+y)^2}{2t}}\, dt\, dx\, dy\Big) \\
&= \sqrt{2\alpha}\, U_\alpha^W f(v)\, \mathbb{E}_0^Q\Big(\int_0^\infty e^{-\sqrt{2\alpha} y}\, e^{-(p_1+\alpha p_3)P^{-1}(y)}\, dy\Big),
\end{aligned}$$

where we used, with $z = x + y > 0$, that

$$\begin{aligned}
\int_0^\infty e^{-\alpha t}\, \frac{z}{\sqrt{2\pi t^3}}\, e^{-\frac{z^2}{2t}}\, dt &= \int_0^\infty e^{-\alpha t}\, \frac{\partial}{\partial z}\big(\frac{1}{\sqrt{2\pi t}}\, e^{-\frac{z^2}{2t}}\big)\, dt \\
&= \frac{\partial}{\partial z}\, \frac{1}{\sqrt{2\alpha}}\, e^{-\sqrt{2\alpha} z} \\
&= e^{-\sqrt{2\alpha} z},
\end{aligned}$$

and, by the closed form (19.3) of the resolvent of W, that

$$\sum_{e\in\mathcal{E}} q_2^e\, 2 \int_0^\infty e^{-\sqrt{2\alpha} x}\, f\big((e,x)\big)\, dx = \sqrt{2\alpha}\, U_\alpha^W f(v).$$

We compute the remaining expectation separately, for $\lambda := \sqrt{2\alpha}$, $\beta := p_1 + \alpha p_3$:

$$\begin{aligned}
\lambda \int_0^\infty e^{-\lambda t}\, e^{-\beta P^{-1}(t)}\, dt &= -\int_0^\infty e^{-\beta P^{-1}(t)}\, de^{-\lambda t} \\
&= \lim_{t\to\infty} e^{-\beta P^{-1}(t)}\, e^{-\lambda t} - e^{-\beta P^{-1}(0)}\, e^{-\lambda 0} - \int_0^\infty e^{-\lambda t}\, de^{-\beta P^{-1}(t)} \\
&= 1 - \beta \int_0^\infty e^{-\lambda t}\, e^{-\beta P^{-1}(t)}\, dP^{-1}(t) \\
&= 1 - \beta \int_0^\infty e^{-\lambda P(t)}\, e^{-\beta t}\, dt.
\end{aligned}$$

As $P(t-) = P(t)$ a.s., we conclude by using equation (6.21) that

$$\begin{aligned}
u_2 &= U_\alpha^W f(v)\, \Big(1 - (p_1 + \alpha p_3) \int_0^\infty \mathbb{E}_0^Q\big(e^{-\sqrt{2\alpha} P(y)}\big)\, e^{-(p_1+\alpha p_3) y}\, dy\Big) \\
&= U_\alpha^W f(v)\, \Big(1 - (p_1 + \alpha p_3) \int_0^\infty e^{-y(\sqrt{2\alpha} p_2 + \int_0^\infty (1-e^{-\sqrt{2\alpha} l})\, p_4^\Sigma(dl))}\, e^{-(p_1+\alpha p_3) y}\, dy\Big) \\
&= U_\alpha^W f(v)\, \Big(1 - \frac{p_1 + \alpha p_3}{p_1 + \sqrt{2\alpha} p_2 + \alpha p_3 + \int_0^\infty \big(1 - e^{-\sqrt{2\alpha} l}\big)\, p_4^\Sigma(dl)}\Big) \\
&= \frac{\sqrt{2\alpha}\, p_2 + \int_0^\infty \big(1 - e^{-\sqrt{2\alpha} l}\big)\, p_4^\Sigma(dl)}{p_1 + \sqrt{2\alpha}\, p_2 + \alpha\, p_3 + \int_0^\infty \big(1 - e^{-\sqrt{2\alpha} l}\big)\, p_4^\Sigma(dl)} \cdot U_\alpha^W f(v).
\end{aligned}$$

It remains to compute u_4: If $p_1 + \alpha p_3 \neq 0$, then

$$\begin{aligned}
u_4 &= -\frac{p_3\, f(v)}{p_1 + \alpha p_3} \mathbb{E}_v\Big(\int_0^\infty e^{-\alpha t}\, de^{-(p_1+\alpha p_3)P^{-1} L_t} \Big) \\
&= -\frac{p_3\, f(v)}{p_1 + \alpha p_3} \mathbb{E}_v\Big(\lim_{t\to\infty} e^{-\alpha t}\, e^{-(p_1+\alpha p_3)P^{-1} L_t} - e^{-\alpha 0}\, e^{-(p_1+\alpha p_3)P^{-1} L_0} \\
&\qquad\qquad - \int_0^\infty e^{-(p_1+\alpha p_3)P^{-1} L_t}\, de^{-\alpha t} \Big) \\
&= \frac{p_3\, f(v)}{p_1 + \alpha p_3} \Big(1 - \alpha\, \mathbb{E}_v\Big(\int_0^\infty e^{-\alpha t}\, e^{-(p_1+\alpha p_3)P^{-1} L_t}\, dt \Big)\Big),
\end{aligned}$$

and observing that the last expectation is just u_2 with $f = 1$, we get with $U_\alpha^W 1 = \frac{1}{\alpha}$:

$$\begin{aligned}
u_4 &= \frac{p_3\, f(v)}{p_1 + \alpha p_3} \Big(1 - \frac{\sqrt{2\alpha}\, p_2 + \int_0^\infty (1 - e^{-\sqrt{2\alpha} l})\, p_4^\Sigma(dl)}{p_1 + \sqrt{2\alpha}\, p_2 + \alpha\, p_3 + \int_0^\infty (1 - e^{-\sqrt{2\alpha} l})\, p_4^\Sigma(dl)} \Big) \\
&= \frac{p_3\, f(v)}{p_1 + \sqrt{2\alpha}\, p_2 + \alpha\, p_3 + \int_0^\infty (1 - e^{-\sqrt{2\alpha} l})\, p_4^\Sigma(dl)}.
\end{aligned}$$

If $p_1 + \alpha p_3 = 0$, that is, if $p_1 = p_3 = 0$, then $u_4 = 0$ holds by its definition, which is in accord with the above formula for u_4.

Adding everything up, we get

$$U_\alpha^\bullet f(v) = \frac{\int U_\alpha^{W,D} f(g)\, p_4(dg) + \sqrt{2\alpha}\, p_2\, U_\alpha^W f(v) + p_3\, f(v)}{p_1 + \sqrt{2\alpha}\, p_2 + \alpha\, p_3 + \int_0^\infty (1 - e^{-\sqrt{2\alpha} l})\, p_4^\Sigma(dl)},$$

and insertion of the closed form for $U_\alpha^W f(v)$ (see equation (19.3)) completes the proof. □

It was already shown in theorem (21.57) that $X^\bullet$ is a right process. By checking the resolvent condition (iv) of theorem (5.13) with the help of the decomposition given in the above theorem (21.59) (the resolvent $(U_\alpha^{W,D}, \alpha > 0)$ of the killed Walsh process preserves $\mathcal{C}_0(\mathcal{G})$ by example (19.8)), we obtain the next result:

(21.60) Corollary. *$X^\bullet$ is a Feller process.*

We finish the construction on the star graph by showing that the process $X^\bullet$ implements the desired boundary conditions:

(21.61) Theorem. *$X^\bullet$ is a Brownian motion on $\mathcal{G}$. Its generator reads $A^\bullet = \frac{1}{2}\Delta$ with*

$$\begin{aligned}
\mathscr{D}(A^\bullet) = \Big\{ f \in \mathcal{C}_0^2(\mathcal{G}) : \\
p_1\, f(v) - \sum_{e\in\mathcal{E}} p_2^e\, f_e'(v) + \frac{p_3}{2} f''(v) - \int \big(f(g) - f(v)\big)\, p_4(dg) = 0 \Big\}.
\end{aligned}$$

Proof. Let $H_0^\bullet$ be the first entry time of $X^\bullet$ in v. As the transformation effects of subsection 21.10 only take effect after the first hitting of v, we have by lemma (21.36)

$$\forall t \le H_0^\bullet = H_0 = H_0^W : \quad X_t^\bullet = X_t = W_t.$$

Thus the stopped process $(X^\bullet_{t\wedge H_0^\bullet}, t \ge 0)$ behaves identically to a stopped Walsh process $(W_{t\wedge H_0^W}, t \ge 0)$, which by theorem (19.6) fulfills the defining conditions (20.1) of a Brownian motion on the metric graph $\mathcal{G}$. In addition, $X^\bullet$ is right continuous and strongly Markovian by theorem (21.57), therefore it is a Brownian motion on the star graph $\mathcal{G}$.

In view of lemma (20.25), we only need to show that the domain of the generator lies inside the right-hand set. Because $X^\bullet$ is Feller, $\mathscr{D}(A^\bullet) = U_\alpha^\bullet(\mathcal{C}_0(\mathcal{G}))$ holds true for any $\alpha > 0$ (see (5.10)), therefore it is enough to prove that every potential $U_\alpha^\bullet f$, $f \in \mathcal{C}_0(\mathcal{G})$, satisfies the above-stated boundary condition: The derivatives of $U_\alpha^{W,D} f$ were already computed in example (19.8) (it is $f_e'(v) = f'(e,0+)$ there), so the first formula of theorem (21.59) gives for $g = (e,x) \in \mathcal{G}$, by setting $\psi_\alpha(g) := e^{-\sqrt{2\alpha}\, d(v,g)} = e^{-\sqrt{2\alpha} x}$:

$$\begin{aligned}
U_\alpha^\bullet f_e'(v) &= U_\alpha^{W,D} f_e'(v) + \psi_\alpha'(v)\, U_\alpha^\bullet f(v) \\
&= 2\int_0^\infty e^{-\sqrt{2\alpha}x} f(e,x)\,dx - \sqrt{2\alpha}\, U_\alpha^\bullet f(v), \\
U_\alpha^\bullet f''(v) &= U_\alpha^{W,D} f''(v) + \psi_\alpha''(v)\, U_\alpha^\bullet f(v) \\
&= -2f(v) + 2\alpha\, U_\alpha^\bullet f(v), \\
U_\alpha^\bullet f(g) - U_\alpha^\bullet f(v) &= U_\alpha^{W,D} f(g) - (1 - e^{-\sqrt{2\alpha}x})\, U_\alpha^\bullet f(v).
\end{aligned}$$

By using these relations and then inserting the closed form of $U_\alpha^\bullet f(v)$ as given in theorem (21.59), we get

$$\begin{aligned}
&p_1\, U_\alpha^\bullet f(v) - \sum_{e\in\mathcal{E}} p_2^e\, U_\alpha^\bullet f_e'(v) + \frac{p_3}{2}\, U_\alpha^\bullet f''(v) - \int \big(U_\alpha^\bullet f(g) - U_\alpha^\bullet f(v)\big)\, p_4(dg) \\
&= \Big(p_1 + \sqrt{2\alpha}\, p_2 + \alpha\, p_3 + \int (1 - e^{-\sqrt{2\alpha}x})\, p_4^\Sigma(dx)\Big) \cdot U_\alpha^\bullet f(v) \\
&\quad - \Big(2\sum_{e\in\mathcal{E}} p_2^e \int_0^\infty e^{-\sqrt{2\alpha}x} f(e,x)\,dx + \frac{p_3}{2}\, 2\, f(v) + \int U_\alpha^{W,D} f(g)\, p_4(dg)\Big) \\
&= 0.
\end{aligned}$$

□

21.12. Further Results on the Generator of a Star Graph

We are going to gain further insight into the star-graph case and derive results which will be necessary for our upcoming developments on the general case.

We first turn to the question on whether the generator of a Brownian motion on a star graph is uniquely characterized by the Feller–Wentzell data arising from Feller's theorem (20.26). Of course, the generator domain $\mathscr{D}(A)$ determines any Brownian motion by theorem (5.9). Therefore, we need to ensure that no two different sets of boundary data give rise to the same set $\mathscr{D}(A)$, which does not seem obvious in the presence of non-local boundary conditions.

(21.62) Lemma. *For a star graph $\mathcal{G}$ with star point v, let $c_1 \geq 0$, $c_2^e \geq 0$ for each $e \in \mathcal{E}$, $c_3 \geq 0$, c_4 a measure on $\mathcal{G}\backslash\{v\}$ as well as $p_1 \geq 0$, $p_2^e \geq 0$ for each $e \in \mathcal{E}$, $p_3 \geq 0$, p_4 a measure on $\mathcal{G}\backslash\{v\}$ be given, which satisfy*

$$c_1 + \sum_{e\in\mathcal{E}} c_2^e + c_3 + \int_{\mathcal{G}\backslash\{v\}} \big(1 - e^{-d(v,g)}\big)\, c_4^v(dg) = 1,$$
$$p_1 + \sum_{e\in\mathcal{E}} p_2^e + p_3 + \int_{\mathcal{G}\backslash\{v\}} \big(1 - e^{-d(v,g)}\big)\, p_4^v(dg) = 1.$$

If

$$\begin{aligned}&\Big\{f \in \mathcal{C}_0^2(\mathcal{G}) : c_1 f(v) - \sum_{e\in\mathcal{E}} c_2^e f_e'(v) + \frac{c_3}{2} f''(v) - \int_{\mathcal{G}\backslash\{v\}} \big(f(g) - f(v)\big)\, c_4(dg) = 0\Big\}\\ =&\Big\{f \in \mathcal{C}_0^2(\mathcal{G}) : p_1 f(v) - \sum_{e\in\mathcal{E}} p_2^e f_e'(v) + \frac{p_3}{2} f''(v) - \int_{\mathcal{G}\backslash\{v\}} \big(f(g) - f(v)\big)\, p_4(dg) = 0\Big\},\end{aligned}$$

then

$$c_1 = p_1, \quad \forall e \in \mathcal{E}: \ c_2^e = p_2^e, \quad c_3 = p_3, \quad c_4 = p_4.$$

Proof. Let X^p, X^c be Brownian motions on the star graph $\mathcal{G}$, constructed with the techniques of subsections 21.1–21.11, which implement the boundary condition at v given by the p's, c's. With A^p, U^p and A^c, U^c being the generators and resolvents of X^p, X^c respectively, theorem (21.61) asserts that

$$\mathscr{D}(A^c) = \Big\{f \in \mathcal{C}_0^2(\mathcal{G}) : c_1 f(v) - \sum_{e\in\mathcal{E}} c_2^e f_e'(v) + \frac{c_3}{2} f''(v) - \int \big(f(g) - f(v)\big)\, c_4(dg) = 0\Big\},$$
$$\mathscr{D}(A^p) = \Big\{f \in \mathcal{C}_0^2(\mathcal{G}) : p_1 f(v) - \sum_{e\in\mathcal{E}} p_2^e f_e'(v) + \frac{p_3}{2} f''(v) - \int \big(f(g) - f(v)\big)\, p_4(dg) = 0\Big\}.$$

Then, by assumption, the generators and thus the resolvents of X^p and X^c coincide, so especially we have $U_\alpha^p f(v) = U_\alpha^c f(v)$ for all $\alpha > 0$, $f \in b\mathcal{C}(\mathcal{G})$. Theorem (21.59) then yields

$$\begin{aligned}&\frac{\sum_{e\in\mathcal{E}} p_2^e\, 2\int_0^\infty e^{-\sqrt{2\alpha}x} f(e,x)\,dx + p_3 f(0) + \int U_\alpha^{W,D} f(g)\, p_4(dg)}{p_1 + \sqrt{2\alpha}p_2 + \alpha p_3 + \int_0^\infty (1 - e^{-\sqrt{2\alpha}l})\, p_4^\Sigma(dl)}\\ =&\frac{\sum_{e\in\mathcal{E}} c_2^e\, 2\int_0^\infty e^{-\sqrt{2\alpha}x} f(e,x)\,dx + c_3 f(0) + \int U_\alpha^{W,D} f(g)\, c_4(dg)}{c_1 + \sqrt{2\alpha}c_2 + \alpha c_3 + \int_0^\infty (1 - e^{-\sqrt{2\alpha}l})\, c_4^\Sigma(dl)}.\end{aligned} \tag{21.63}$$

By inserting $f = 1$, we get

$$\begin{aligned}&\frac{1}{\alpha}\,\frac{\sqrt{2\alpha}p_2 + \alpha p_3 + \int \big(1 - e^{-\sqrt{2\alpha}x}\big)\, p_4^\Sigma(dx)}{p_1 + \sqrt{2\alpha}p_2 + \alpha p_3 + \int_0^\infty (1 - e^{-\sqrt{2\alpha}l})\, p_4^\Sigma(dl)}\\ =&\frac{1}{\alpha}\,\frac{\sqrt{2\alpha}c_2 + \alpha c_3 + \int \big(1 - e^{-\sqrt{2\alpha}x}\big)\, c_4^\Sigma(dx)}{c_1 + \sqrt{2\alpha}c_2 + \alpha c_3 + \int_0^\infty (1 - e^{-\sqrt{2\alpha}l})\, c_4^\Sigma(dl)},\end{aligned}$$

so when introducing

$$\tilde{p}_\alpha := \sqrt{2\alpha} p_2 + \alpha p_3 + \int \left(1 - e^{-\sqrt{2\alpha} x}\right) p_4^\Sigma(dx),$$
$$\tilde{c}_\alpha := \sqrt{2\alpha} c_2 + \alpha c_3 + \int \left(1 - e^{-\sqrt{2\alpha} x}\right) c_4^\Sigma(dx),$$

it follows that

$$\frac{\tilde{p}_\alpha}{p_1 + \tilde{p}_\alpha} = \frac{\tilde{c}_\alpha}{c_1 + \tilde{c}_\alpha}. \tag{21.64}$$

If $p_1 \neq 0$, consider $D := \frac{c_1}{p_1}$. Then $c_1 = D\, p_1$ holds, and the equation above implies $\tilde{c}_\alpha = D\, \tilde{p}_\alpha$, that is

$$\sqrt{2\alpha} c_2 + \alpha c_3 + \int \left(1 - e^{-\sqrt{2\alpha} x}\right) c_4^\Sigma(dx) = D \left(\sqrt{2\alpha} p_2 + \alpha p_3 + \int \left(1 - e^{-\sqrt{2\alpha} x}\right) p_4^\Sigma(dx) \right).$$

Dividing both sides by α and letting $\alpha \to \infty$ yields $c_3 = D\, p_3$ by lemma (1.18), so

$$\sqrt{2\alpha} c_2 + \int \left(1 - e^{-\sqrt{2\alpha} x}\right) c_4^\Sigma(dx) = D \left(\sqrt{2\alpha} p_2 + \int \left(1 - e^{-\sqrt{2\alpha} x}\right) p_4^\Sigma(dx) \right). \tag{21.65}$$

Now dividing by $\sqrt{2\alpha}$ and letting $\alpha \to \infty$ again yields $c_2 = D p_2$, thus as well

$$\int \left(1 - e^{-\sqrt{2\alpha} x}\right) c_4^\Sigma(dx) = D \int \left(1 - e^{-\sqrt{2\alpha} x}\right) p_4^\Sigma(dx).$$

But then

$$c_1 + c_2 + c_3 + \int \left(1 - e^{-\sqrt{2\alpha} x}\right) c_4^\Sigma(dx) = D \Big(p_1 + p_2 + p_3 + \int \left(1 - e^{-\sqrt{2\alpha} x}\right) p_4^\Sigma(dx) \Big),$$

and by inserting $\alpha = \frac{1}{2}$, the normalizations of the c's and p's imply $D = 1$.

Thus, we have $c_1 = p_1$ and $c_3 = p_3$. Coming back to equation (21.63), we obtain

$$\begin{aligned} &\sum_{e \in \mathcal{E}} p_2^e \, 2 \int_0^\infty e^{-\sqrt{2\alpha} x} f(e, x)\, dx + \int U_\alpha^{W,D} f(g)\, p_4(dg) \\ = &\sum_{e \in \mathcal{E}} c_2^e \, 2 \int_0^\infty e^{-\sqrt{2\alpha} x} f(e, x)\, dx + \int U_\alpha^{W,D} f(g)\, c_4(dg) \end{aligned}$$

for all $\alpha > 0$, $f \in b\mathcal{C}(\mathcal{G})$. Fix $e \in \mathcal{E}$. By approximation with the help of LDCT, we can insert the function f with $f = 1$ on $e^0 = \{e\} \times (0, \infty)$ and $f = 0$ otherwise in the above equation, yielding

$$\frac{1}{\sqrt{2\alpha}} p_2^e + \frac{1}{\alpha} \int \left(1 - e^{-\sqrt{2\alpha} x}\right) p_4^e(dx) = \frac{1}{\sqrt{2\alpha}} c_2^e + \frac{1}{\alpha} \int \left(1 - e^{-\sqrt{2\alpha} x}\right) c_4^e(dx).$$

Multiplying by $\sqrt{2\alpha}$ and letting $\alpha \to \infty$ gives $p_2^e = c_2^e$. Therefore,

$$\forall \alpha > 0: \quad \int \big(1 - e^{-\sqrt{2\alpha}x}\big)\, p_4^e(dx) = \int \big(1 - e^{-\sqrt{2\alpha}x}\big)\, c_4^e(dx),$$

which by theorem (1.20) is only possible if $p_4^e = c_4^e$. This completes the proof for $p_1 \neq 0$.

If $p_1 = 0$, equation (21.64) implies that $c_1 = 0$ or $\tilde{p}_\alpha = 0$ for all $\alpha > 0$. The latter is impossible, so $c_1 = p_1 = 0$. Now using equation (21.63) again with $\alpha = \frac{1}{2}$ and $f = \mathbb{1}_{\{v\}}$ (by approximating f with $\mathcal{C}_0(\mathcal{G})$-functions, using LDCT), and utilizing the normalizations of the c's and p's, we get

$$\frac{p_3}{1 - \frac{p_3}{2}} = \frac{c_3}{1 - \frac{c_3}{2}},$$

so $c_3 = p_3$.

First assume $p_3 \neq 0$. Inserting $c_3 = p_3$ in equation (21.63) with $f = \mathbb{1}_{\{v\}}$ gives

$$\frac{p_3}{\sqrt{2\alpha}p_2 + \alpha p_3 + \int \big(1 - e^{-\sqrt{2\alpha}x}\big)\, p_4^\Sigma(dx)} = \frac{p_3}{\sqrt{2\alpha}c_2 + \alpha p_3 + \int \big(1 - e^{-\sqrt{2\alpha}x}\big)\, c_4^\Sigma(dx)},$$

which is equivalent to

$$\sqrt{2\alpha}p_2 + \int \big(1 - e^{-\sqrt{2\alpha}x}\big)\, p_4^\Sigma(dx) = \sqrt{2\alpha}c_2 + \int \big(1 - e^{-\sqrt{2\alpha}x}\big)\, c_4^\Sigma(dx),$$

yielding equation (21.65) for $D = 1$. Thus, the rest of the proof then proceeds exactly as in the case $p_1 \neq 0$.

If $p_1 = 0$ and $p_3 = 0$, we have already seen that $c_1 = 0$ and $c_3 = 0$ as well. Using equation (21.63) with $\alpha = \frac{1}{2}$ and $f(e,x) = e^{-\beta x}$ for any $\beta > 0$, it follows, as the the c's and p's are normalized, with remark (16.5) that

$$\begin{aligned} &\frac{2}{1+\beta}\, p_2 + \frac{2}{1-\beta^2} \int_0^\infty \big(e^{-\beta x} - e^{-x}\big)\, p_4^\Sigma(dx) \\ &= \frac{2}{1+\beta}\, c_2 + \frac{2}{1-\beta^2} \int_0^\infty \big(e^{-\beta x} - e^{-x}\big)\, c_4^\Sigma(dx), \end{aligned}$$

with the integrals being finite, because $e^{-\beta x} - e^{-x} = -e^{-x}\big(1 - e^{-(\beta-1)x}\big)$ for $\beta > 1$ and $0 \leq e^{-\beta x} - e^{-x} \leq 1 - e^{-x}$ for $0 < \beta \leq 1$. Multiplying both sides by β and letting $\beta \to +\infty$ yields $p_2 = c_2$, because $\int_0^\infty \frac{e^{-\beta x} - e^{-x}}{\beta - 1}\, p_4^\Sigma(dx) \to 0$ for $\beta \to +\infty$. But then

$$\int \big(e^{-\beta x} - e^{-x}\big)\, p_4^\Sigma(dx) = \int \big(e^{-\beta x} - e^{-x}\big)\, c_4^\Sigma(dx)$$

holds for all $\beta > 0$, and by adding $\int (1 - e^{-x}) p_4^\Sigma(dx) = 1 - p_2 = 1 - c_2 = \int (1 - e^{-x}) c_4^\Sigma(dx)$ to both sides and setting $\beta := \sqrt{2\alpha}$, we get for all $\alpha > 0$

$$\int \big(1 - e^{-\sqrt{2\alpha}x}\big)\, p_4^\Sigma(dx) = \int \big(1 - e^{-\sqrt{2\alpha}x}\big)\, c_4^\Sigma(dx).$$

The rest of the proof then proceeds as above. □

We are just now in a position to state that our construction of the general Brownian motion on a star graph (as done in subsections 21.1–21.11) with the boundary conditions given in the beginning of this section indeed implements the corresponding Feller–Wentzell data:

(21.66) Theorem. *Let $X^\bullet$ be constructed as above for given data $p_1 \geq 0$, $p_2^e \geq 0$, $e \in \mathcal{E}$, $p_3 \geq 0$, p_4 measure on $\mathcal{G}\backslash\{v\}$, such that $p_1 + \sum_{e\in\mathcal{E}} p_2^e + p_3 + \int \left(1 - e^{-x}\right) p_4^\Sigma(dx) = 1$. Then with $c_1^v =: c_1$, $c_2^{v,e} =: c_2^e$ for $e \in \mathcal{E}$, $c_3^v =: c_3$, $c_4^v =: c_4$ as given in Feller's theorem* (20.16) *for the process $X^\bullet$, the Feller–Wentzell data of $X^\bullet$ reads*

$$c_1 = p_1, \quad c_2^e = p_2^e,\ e \in \mathcal{E}, \quad c_3 = p_3, \quad c_4 = p_4.$$

Proof. By applying Feller's theorem (20.16) for the star graph (20.26), we have

$$\begin{aligned}\mathscr{D}(A^\bullet) = \Big\{ & f \in \mathcal{C}_0^2(\mathcal{G}) : \\ & c_1 f(v) - \sum_{e\in\mathcal{E}} c_2^e f_e'(v) + \frac{c_3}{2} f''(v) - \int_{\mathcal{G}\backslash\{v\}} \big(f(g) - f(v)\big)\, c_4(dg) = 0 \Big\},\end{aligned}$$

while theorem (21.61) gives

$$\begin{aligned}\mathscr{D}(A^\bullet) = \Big\{ & f \in \mathcal{C}_0^2(\mathcal{G}) : \\ & p_1 f(v) - \sum_{e\in\mathcal{E}} p_2^e f_e'(v) + \frac{p_3}{2} f''(v) - \int_{\mathcal{G}\backslash\{v\}} \big(f(g) - f(v)\big)\, p_4(dg) = 0 \Big\}.\end{aligned}$$

Applying lemma (21.62) yields the result. □

We are going to employ the above result in order to show that the rather artificial part c_1^∞ of the killing weight $c_1 = c_1^\Delta + c_1^\infty$ in Feller's theorem (20.16) indeed vanishes in the star-graph case (here, the star vertex v is left out in the notation of the Feller–Wentzell data). This will be essential for our process construction on general metric graphs in section 22, see remark (22.7).

We achieve this as follows: Starting with the Brownian motion $X^\bullet$ which implements the killing parameter $c_1 = c_1^\Delta + c_1^\infty$, we revive this process at its killing times with the identical copies method established in subsection 13.1 via some revival distribution k. As killing can be interpreted as a jump to Δ, which is now transformed to a jump to a revival point chosen by k, we expect the killing weight c_1 to be transformed into a jump part $c_1 k$, which is then added to the original jump weight c_4. However, an analysis of the boundary conditions for the revived process via two different methods shows a discrepancy: The resolvent of the revived process can be decomposed with Dynkin's formula at the revival time, showing that the "full" killing parameter $c_1 = c_1^\Delta + c_1^\infty$ is shifted to the jump measure. But when tracing back the explicit formulas of Feller's theorem for the Feller–Wentzell data of the revived process to the original process $X^\bullet$, it is seen that only the "natural" killing weight c_1^Δ is transformed, while leaving the "artificial" killing portion c_1^∞ unaltered. As the Feller–Wentzell data uniquely characterizes the process, this is only possible if c_1^∞ already vanishes for the original process $X^\bullet$.

We are carrying out this program, starting with the analysis of the resolvent of the revived Brownian motion:

(21.67) Lemma. *Let $X^\bullet$ be a Brownian motion on the star graph $\mathcal{G}$ with generator*

$$\mathscr{D}(A^\bullet) = \Big\{ f \in \mathcal{C}_0^2(\mathcal{G}) : \\ c_1 f(v) - \sum_{e\in\mathcal{E}} c_2^e f_e'(v) + \frac{c_3}{2} f''(v) - \int_{\mathcal{G}\setminus\{v\}} \big(f(g) - f(v)\big)\, c_4(dg) = 0 \Big\}.$$

Let q be a probability measure on $\mathcal{G}$, and X^q be the identical copies process, as constructed in subsection 13.1, resulting from successive revivals of $X^0 := X^\bullet$ with the revival kernel K^0, which is defined by the transfer measure (see lemma (11.5)*)*

$$k^0(g, \cdot\,) := q, \quad g \in \mathcal{G}.$$

Then X^q is a Brownian motion on $\mathcal{G}$ with generator

$$\mathscr{D}(A^q) = \Big\{ f \in \mathcal{C}_0^2(\mathcal{G}) : \\ - \sum_{e\in\mathcal{E}} p_2^e f_e'(v) + \frac{p_3}{2} f''(v) - \int_{\mathcal{G}\setminus\{v\}} \big(f(g) - f(v)\big)\, (p_4 + p_1\, q)(dg) = 0 \Big\}.$$

Proof. We decompose the resolvent at the first revival time R^1 with the help of Dynkin's formula (3.16): As the process X^q up to the time R^1 equals the original process $X^\bullet$ up to its lifetime ζ, we have by theorem (11.19), for any $f \in \mathcal{C}_0(\mathcal{G})$, $g \in \mathcal{G}$:

$$\begin{aligned} U_\alpha^q f(g) &= \mathbb{E}_g\Big(\int_0^{R^1} e^{-\alpha t}\, f(X_t^q)\, dt\Big) + \mathbb{E}_g\big(e^{-\alpha R^1}\, U_\alpha f(X_{R^1}^q)\big) \\ &= \mathbb{E}_g\Big(\int_0^{\zeta} e^{-\alpha t}\, f(X_t^\bullet)\, dt\Big) + \mathbb{E}_g\big(e^{-\alpha R^1}\, K(U_\alpha^q f)\big) \\ &= U_\alpha^\bullet f(g) + \psi_\alpha(g)\, q(U_\alpha^q f), \end{aligned}$$

with $(U_\alpha^\bullet, \alpha > 0)$ being the resolvent of $X^\bullet$, and

$$\begin{aligned} \psi_\alpha(g) &= \mathbb{E}_g\big(e^{-\alpha\zeta}\big) \\ &= 1 - \alpha\, \mathbb{E}_g\Big(\int_0^\infty e^{-\alpha t}\, \mathbb{1}_{\mathcal{G}}(X_t^\bullet)\, dt\Big) \\ &= 1 - \alpha\, U_\alpha^\bullet \mathbb{1}_{\mathcal{G}}(g). \end{aligned}$$

As $X^\bullet$ is Feller and $\psi_\alpha \in \mathcal{C}_0(\mathcal{G})$ by theorem (21.59) and example (19.8), $(U_\alpha^q, \alpha > 0)$ preserves $\mathcal{C}_0(\mathcal{G})$ as well. Furthermore, X^q is right continuous and normal by definition, so X^q is Feller by theorem (5.13). As $X_t^q = X_t^\bullet$ holds for all $t \le H_0$ and $X^\bullet$ is a Brownian motion on $\mathcal{G}$, X^q is also a Brownian motion on $\mathcal{G}$.

Let $h \in \mathscr{D}(A^q)$. Then by (5.10), there exists an $f \in \mathcal{C}_0(\mathcal{G})$ with $h = U_\alpha^q f$. As $\Delta \notin \mathcal{G}$ is isolated, we have $\mathbb{1}_{\mathcal{G}} \in b\mathcal{C}(\mathcal{G})$, so both

$$1 - \psi_\alpha = \alpha\, U_\alpha^\bullet \mathbb{1}_{\mathcal{G}}$$

and $U^\bullet f$ fulfill the boundary conditions for $X^\bullet$, as the proof of theorem (21.61) is also applicable to functions in $b\mathcal{C}(\mathcal{G})$. We are ready to compute the boundary conditions for X^q: Using our findings, we get

$$\begin{aligned}
&\frac{p_3}{2}\, U^q_\alpha f''(v) \\
&= \frac{p_3}{2} \big(U^\bullet_\alpha f + \psi_\alpha\, q(U^q_\alpha f)\big)''(v) \\
&= \frac{p_3}{2}\, U^\bullet_\alpha f''(v) - \frac{p_3}{2}\,(1-\psi_\alpha)''(v)\, q(U^q_\alpha f) \\
&= -p_1\, U^\bullet_\alpha f(v) + \sum_{e\in\mathcal{E}} p_2^e\, U^\bullet_\alpha f'_e(v) + \int \big(U^\bullet_\alpha f(g) - U^\bullet_\alpha f(v)\big)\, p_4(dg) \\
&\quad - \Big(- p_1 (1-\psi_\alpha(v)) - \sum_{e\in\mathcal{E}} p_2^e\, (\psi_\alpha)'_e(v) - \int \big(\psi_\alpha(g) - \psi_\alpha(v)\big)\, p_4(dg)\Big) q(U^q_\alpha f) \\
&= -p_1\, U^q_\alpha f(v) + \sum_{e\in\mathcal{E}} p_2^e\, U^q_\alpha f'_e(v) + \int \big(U^q_\alpha f(g) - U^q_\alpha f(v)\big)\, p_4(dg) \\
&\quad + p_1\, q(U^q_\alpha f),
\end{aligned}$$

and as q is a probability measure, we have

$$q(U^q_\alpha f) = \int \big(U^q_\alpha f(g) - U^q_\alpha f(v)\big)\, q(dg) + U_\alpha f(v),$$

so it follows that

$$\frac{p_3}{2}\, U^q_\alpha f''(v) = \sum_{e\in\mathcal{E}} p_2^e\, U^q_\alpha f'_e(v) + \int \big(U^q_\alpha f(g) - U^q_\alpha f(v)\big)\, (p_4 + p_1\, q)(dg).$$

Lemma (20.25) completes the proof. □

Next, we deduce the Feller–Wentzell data of the revived process from the respective Feller–Wentzell data of the original process by explicitly computing the formulas given in Feller's theorem (20.16):

(21.68) Lemma. *Let X be a Brownian motion on the star graph $\mathcal{G}$ with generator*

$$\begin{aligned}
\mathscr{D}(A^X) = \Big\{ & f \in \mathcal{C}_0^2(\mathcal{G}) : \\
& (c_1^\Delta + c_1^\infty)\, f(v) - \sum_{e\in\mathcal{E}} c_2^e f'_e(v) + \frac{c_3}{2} f''(v) - \int_{\mathcal{G}\setminus\{v\}} \big(f(g) - f(v)\big)\, c_4(dg) = 0 \Big\}
\end{aligned}$$

for the Feller–Wentzell data $\big(c_1^\Delta, c_1^\infty, (c_2^e)_{e\in\mathcal{E}}, c_3, c_4\big)$ satisfying the usual normalization

$$c_1^\Delta + c_1^\infty + \sum_{e\in\mathcal{E}} c_2^e + c_3 + \int_{\mathcal{G}\setminus\{v\}} \big(1 - e^{-d(v,g)}\big)\, c_4^v(dg) = 1.$$

If $c_1 = c_1^\Delta + c_1^\infty > 0$, construct Y as the instant return process of X, that is, as the identical copies process of X (as constructed in subsection 13.1) with the revival kernel K^0 being defined by the transfer measure (see lemma (11.5))

$$k^0(g,\,\cdot\,) = \varepsilon_v.$$

Then Y is a Brownian motion on $\mathcal{G}$ with generator

$$\mathscr{D}(A^Y) = \Big\{ f \in \mathcal{C}_0^2(\mathcal{G}) : \\ c_1^\infty f(v) - \sum_{e\in\mathcal{E}} c_2^e f_e'(v) + \frac{c_3}{2} f''(v) - \int_{\mathcal{G}\backslash\{v\}} \big(f(g) - f(v)\big)\, c_4(dg) = 0 \Big\}.$$

Proof. It has already been argued in the proof of lemma (21.67) that the revived process Y is a Brownian motion on $\mathcal{G}$. As we will need to compare the formulas given in Feller's theorem (20.16) for the Feller–Wentzell data of the processes X and Y, we indicate the defining entities for X, Y by the corresponding superscript, that is, for instance

$$\tau_\varepsilon^X = \inf\{t \ge 0 : X_t \in \complement\overline{B_e(v)}\}, \quad \tau_\varepsilon^Y = \inf\{t \ge 0 : Y_t \in \complement\overline{B_e(v)}\}, \quad \varepsilon > 0.$$

If $\mathbb{P}_v(\tau_\varepsilon^X < \zeta^X) = 0$ for all $\varepsilon > 0$, that is, if $\mathbb{P}_v(\tau_\varepsilon^X = \zeta^X) = 1$ holds for all $\varepsilon > 0$, then (depending on whether $\mathbb{E}_v(\tau_\varepsilon^X)$ is infinite or finite) v is an absorbing point or a holding point for X, and in the latter case X must jump directly from v to Δ after an exponential holding time. The generators for these two cases read

$$\mathscr{D}(A^X) = \{f \in \mathcal{C}_0^2(\mathcal{G}) : \frac{1}{2} f''(v) = 0\},$$

and

$$\mathscr{D}(A^X) = \{f \in \mathcal{C}_0^2(\mathcal{G}) : c_1 f(v) + \frac{c_3}{2} f''(v) = 0\}$$

with $c_1^\infty = 0$, as $\nu_\varepsilon^X = 0$ holds for all $\varepsilon > 0$ in Feller's theorem (20.16) by definition. But in both cases, the revived process Y is just the Brownian motion absorbed in v, so

$$\mathscr{D}(A^Y) = \{f \in \mathcal{C}_0^2(\mathcal{G}) : \frac{1}{2} f''(v) = 0\},$$

conforming to the claim of the lemma.

Otherwise, there is some $\varepsilon > 0$ with $\mathbb{P}_v(\tau_\varepsilon^X < \zeta^X) > 0$. As $\tau_{\varepsilon'}^X \le \tau_\varepsilon^X$ holds for all $\varepsilon' < \varepsilon$, we then have for all $\varepsilon > 0$ sufficiently small

$$\mathbb{P}_v(\tau_\varepsilon^X < \zeta^X) > 0.$$

We need to compare τ_ε^Y with τ_ε^X: While τ_ε^X can be realized by X jumping to Δ or entering $\mathcal{G}\backslash\overline{B_\varepsilon(v)}$, τ_ε^Y is only realized if X enters $\mathcal{G}\backslash\overline{B_\varepsilon(v)}$. If τ_ε^X is realized by X jumping to Δ, then Y restarts at v and $\tau_\varepsilon^Y = R^1 + \tau_\varepsilon^Y \circ \Theta_{R^1}$ holds true, with the first revival time R^1 being equal to the death time ζ^X of X. Due to the strong Markov property, the number of revivals of Y before leaving $B_\varepsilon(v)$ is geometrically distributed, so

$$\begin{aligned}\mathbb{E}_v(\tau_\varepsilon^Y) = \sum_{n\in\mathbb{N}_0} \big(n\,\mathbb{E}_v(\tau_\varepsilon^X \,|\, \tau_\varepsilon^X = \zeta^X) + \mathbb{E}_v(\tau_\varepsilon^X \,|\, \tau_\varepsilon^X < \zeta^X)\big)\\ \cdot \mathbb{P}_v(\tau_\varepsilon^X = \zeta^X)^n\,\mathbb{P}_v(\tau_\varepsilon^X < \zeta^X),\end{aligned} \tag{21.69}$$

which gives

$$\begin{aligned}\mathbb{E}_v(\tau_\varepsilon^Y) &= \frac{1-\mathbb{P}_v(\tau_\varepsilon^X < \zeta^X)}{\mathbb{P}_v(\tau_\varepsilon^X < \zeta^X)}\,\mathbb{E}_v(\tau_\varepsilon^X \,|\, \tau_\varepsilon^X = \zeta^X) + \mathbb{E}_v(\tau_\varepsilon^X \,|\, \tau_\varepsilon^X < \zeta^X)\\ &= \frac{1}{\mathbb{P}_v(\tau_\varepsilon^X < \zeta^X)}\,\mathbb{E}_v(\tau_\varepsilon^X).\end{aligned} \tag{21.70}$$

Before continuing, we prove equation (21.69) rigorously: We start by decomposing τ_ε^Y with respect to the revival times $(R^n, n\in\mathbb{N})$ of the concatenated process Y, that is

$$\mathbb{E}_v(\tau_\varepsilon^Y) = \sum_{n\in\mathbb{N}_0} \mathbb{E}_v(\tau_\varepsilon^Y\,;\, R^n \le \tau_\varepsilon^Y < R^{n+1}).$$

Before the first revival time, Y behaves just like X, so

$$\begin{aligned}\mathbb{E}_v(\tau_\varepsilon^Y\,;\, R^0 \le \tau_\varepsilon^Y < R^1) &= \mathbb{E}_v(\tau_\varepsilon^X\,;\, \tau_\varepsilon^X < \zeta^X)\\ &= \mathbb{E}_v(\tau_\varepsilon^X \,|\, \tau_\varepsilon^X < \zeta^X)\,\mathbb{P}_v(\tau_\varepsilon^X < \zeta^X).\end{aligned} \tag{21.71}$$

After the n-th revival, we are using the strong Markov property of Y together with

$$\mathbb{E}_x\big(f(Y_{R^n}) \,|\, \mathscr{F}_{R^n-}\big) = \int f\, d\varepsilon_v = f(v) \quad \text{on } \{R^n < \infty\}$$

by theorem (11.19) and the definition of the revival kernel in order to compute

$$\begin{aligned}&\mathbb{E}_v\big(\tau_\varepsilon^Y\,;\, R^n \le \tau_\varepsilon^Y < R^{n+1}\big)\\ &= \mathbb{E}_v\big(\tau_\varepsilon^Y \circ \Theta_{R^n} + R^n\,;\, R^n \le \tau_\varepsilon^Y, \tau_\varepsilon^Y \circ \Theta_{R^n} < R^{n+1}\circ\Theta_{R^n}\big)\\ &= \mathbb{E}_v\big(\mathbb{E}_{Y_{R^n}}(\tau_\varepsilon^Y\,;\, \tau_\varepsilon^Y < R^{n+1}) + R^n\,\mathbb{P}_{Y_{R^n}}(\tau_\varepsilon^Y < R^{n+1})\,;\, R^n \le \tau_\varepsilon^Y\big)\\ &= \mathbb{E}_v\big(\mathbb{E}_v(\tau_\varepsilon^Y\,;\, \tau_\varepsilon^Y < R^1) + R^n\,\mathbb{P}_v(\tau_\varepsilon^Y < R^1)\,;\, R^n \le \tau_\varepsilon^Y\big)\\ &= \mathbb{E}_v(\tau_\varepsilon^X \,|\, \tau_\varepsilon^X < \zeta^X)\,\mathbb{P}_v(\tau_\varepsilon^X < \zeta^X)\,\mathbb{P}_v(R^n \le \tau_\varepsilon^Y)\\ &\quad + \mathbb{E}_v(R^n\,;\, R^n \le \tau_\varepsilon^Y)\,\mathbb{P}_v(\tau_\varepsilon^X < \zeta^X),\end{aligned}$$

where we used equation (21.71) as well as the relation $\mathbb{P}_v(\tau_\varepsilon^Y < R^1) = \mathbb{P}_v(\tau_\varepsilon^X < \zeta^X)$ for the last identity. It remains to show

$$\mathbb{P}_v(R^n \le \tau_\varepsilon^Y) = \mathbb{P}_v(\tau_\varepsilon^X = \zeta^X)^n \tag{21.72}$$

and

$$\mathbb{E}_v(R^n\,;\, R^n \le \tau_\varepsilon^Y) = n\,\mathbb{E}_v(\tau_\varepsilon^X \,|\, \tau_\varepsilon^X = \zeta^X)\,\mathbb{P}_v(\tau_\varepsilon^X = \zeta^X)^n \tag{21.73}$$

for all $n \in \mathbb{N}_0$, which will be done inductively: For equation (21.72), the cases $n = 0$ and $n = 1$ are clear, and employing the same techniques as above, we conclude that

$$\begin{aligned}\mathbb{P}_v\big(R^{n+1} \leq \tau_\varepsilon^Y\big) &= \mathbb{P}_v\big(R^{n+1} \circ \Theta_{R^n} \leq \tau_\varepsilon^Y \circ \Theta_{R^n}, R^n \leq \tau_\varepsilon^Y\big)\\ &= \mathbb{E}_v\big(\mathbb{P}_v(R^1 \leq \tau_\varepsilon^Y)\,;\, R^n \leq \tau_\varepsilon^Y\big)\\ &= \mathbb{P}_v(R^1 \leq \tau_\varepsilon^Y)\,\mathbb{P}_v(R^n \leq \tau_\varepsilon^Y)\\ &= \mathbb{P}_v(\tau_\varepsilon^X = \zeta^X)^{n+1}.\end{aligned}$$

For equation (21.73), the case $n = 0$ is again clear, and $n = 1$ is straight forward, as

$$\begin{aligned}\mathbb{E}_v(R^1\,;\, R^1 \leq \tau_\varepsilon^Y) &= \mathbb{E}_v(\zeta^X\,;\, \zeta^X \leq \tau_\varepsilon^X)\\ &= \mathbb{E}_v(\tau_\varepsilon^X\,;\, \tau_\varepsilon^X = \zeta^X)\\ &= \mathbb{E}_v(\tau_\varepsilon^X \,|\, \tau_\varepsilon^X = \zeta^X)\,\mathbb{P}_v(\tau_\varepsilon^X = \zeta^X).\end{aligned}$$

The general case requires the same course of actions as used for equation (21.72): It is

$$\begin{aligned}\mathbb{E}_v\big(R^{n+1}\,;\, R^{n+1} \leq \tau_\varepsilon^Y\big) &= \mathbb{E}_v\big(R^{n+1} \circ \Theta_{R^n} + R^n\,;\, R^{n+1} \circ \Theta_{R^n} \leq \tau_\varepsilon^Y \circ \Theta_{R^n}, R^n \leq \tau_\varepsilon^Y\big)\\ &= \mathbb{E}_v\big(\mathbb{E}_v(R^1\,;\, R^1 \leq \tau_\varepsilon^Y) + R^n\,\mathbb{P}_v(R^1 \leq \tau_\varepsilon^Y)\,;\, R^n \leq \tau_\varepsilon^Y\big),\end{aligned}$$

and using the inductive assumption for $\mathbb{E}_v(R^n\,;\, R^n \leq \tau_\varepsilon^Y)$ and the closed form (21.72) for $\mathbb{P}_v(R^n \leq \tau_\varepsilon^Y)$ yields

$$\begin{aligned}\mathbb{E}_v(R^{n+1}\,;\, R^{n+1} \leq \tau_\varepsilon^Y) &= \mathbb{E}_v(\tau_\varepsilon^X \,|\, \tau_\varepsilon^X = \zeta^X)\,\mathbb{P}_v(\tau_\varepsilon^X = \zeta^X)\,\mathbb{P}_v(\tau_\varepsilon^X = \zeta^X)^n\\ &\quad + \mathbb{P}_v(\tau_\varepsilon^X = \zeta^X)\,n\,\mathbb{E}_v(\tau_\varepsilon^X \,|\, \tau_\varepsilon^X = \zeta^X)\,\mathbb{P}_v(\tau_\varepsilon^X = \zeta^X)^n.\end{aligned}$$

This finishes the proof of equations (21.69) and (21.70).

Next, we need to compare the exit distributions from $\overline{B_\varepsilon(v)}$ of Y with the ones of X: It seems obvious that Y exits exactly like X, if X does not exit by jumping to Δ, so

$$\mathbb{P}_v(Y_{\tau_\varepsilon^Y} \in B) = \mathbb{P}_v(X_{\tau_\varepsilon^X} \in B \,|\, \tau_\varepsilon^X < \zeta^X), \quad B \in \mathscr{B}(\mathcal{G}). \tag{21.74}$$

The rigorous proof of this claim is not very complicated: Decomposing the probability on the left-hand side via the revival times gives

$$\mathbb{P}_v(Y_{\tau_\varepsilon^Y} \in B\,;\, \tau_\varepsilon^Y < R^1) = \mathbb{P}_v(X_{\tau_\varepsilon^X} \in B\,;\, \tau_\varepsilon^X < \zeta^X).$$

As $\tau_\varepsilon^Y \circ R^n = \tau_\varepsilon^Y - R^n$ on $\{\tau_\varepsilon^Y > R^n\}$, it follows that

$$\begin{aligned}&\mathbb{P}_v\big(Y_{\tau_\varepsilon^Y} \in B\,;\, R^n < \tau_\varepsilon^Y < R^{n+1}\big)\\ &= \mathbb{P}_v\big(Y_{\tau_\varepsilon^Y} \circ \Theta_{R^n} \in B\,;\, R^n < \tau_\varepsilon^Y, \tau_\varepsilon^Y \circ \Theta_{R^n} < R^{n+1} \circ \Theta_{R^n}\big)\\ &= \mathbb{E}_v\big(\mathbb{P}_v(Y_{\tau_\varepsilon^Y} \in B\,;\, \tau_\varepsilon^Y < R^1)\,;\, R^n < \tau_\varepsilon^Y\big)\\ &= \mathbb{P}_v\big(X_{\tau_\varepsilon^X} \in B\,;\, \tau_\varepsilon^X < \zeta^X\big)\,\mathbb{P}_v(\tau_\varepsilon^X = \zeta^X)^n,\end{aligned}$$

where we used $Y_t = X_t$ for all $t < R^1$ as well as equation (21.72) for the last identity. As $\tau_\varepsilon^Y \neq R^n$ for all $n \in \mathbb{N}_0$, this proves equation (21.74), because

$$\begin{aligned}\mathbb{P}_v(Y_{\tau_\varepsilon^Y} \in B) &= \mathbb{P}_v(X_{\tau_\varepsilon^X} \in B\,;\, \tau_\varepsilon^X < \zeta^X) \sum_{n\in\mathbb{N}_0} \mathbb{P}_v(\tau_\varepsilon^X = \zeta^X)^n \\ &= \mathbb{P}_v(X_{\tau_\varepsilon^X} \in B\,;\, \tau_\varepsilon^X < \zeta^X)\, \frac{1}{\mathbb{P}_v(\tau_\varepsilon^X < \zeta^X)}.\end{aligned}$$

In order to calculate the domain of the generator A^Y of Y, we need to reiterate the proof of Feller's theorem (20.16): Because v is not a trap, theorem (5.16) shows that $\mathbb{E}_v(\tau_\varepsilon^X) < +\infty$ for all sufficiently small $\varepsilon > 0$, so equation (21.70) yields $\mathbb{E}_v(\tau_\varepsilon^Y) < +\infty$. Furthermore, as seen in the proof of lemma (21.67), Y is Feller, so Dynkin's formula is applicable for any $f \in \mathscr{D}(A^Y)$. Then, as Y cannot jump to Δ at all, we have

$$\begin{aligned}A^Y f(v) &= \lim_{\varepsilon\downarrow 0} \frac{\mathbb{E}_v\big(f(Y_{\tau_\varepsilon^Y})\big) - f(v)}{\mathbb{E}_v(\tau_\varepsilon^Y)} \\ &= \lim_{\varepsilon\downarrow 0} \int_{\mathcal{G}\setminus\{v\}} \big(f(g) - f(v)\big) \frac{\mathbb{P}_v(Y_{\tau_\varepsilon^Y} \in dg)}{\mathbb{E}_v(\tau_\varepsilon^Y)},\end{aligned}$$

and inserting equations (21.74), (21.70) and the measure $\nu_\varepsilon^X = \nu_\varepsilon^v$, as defined in Feller's theorem (20.16) for the process X, yields

$$\begin{aligned}A^Y f(v) &= \lim_{\varepsilon\downarrow 0} \int_{\mathcal{G}\setminus\{v\}} \big(f(g) - f(v)\big) \frac{\mathbb{P}_v(X_{\tau_\varepsilon^X} \in dg)/\mathbb{P}_v(\tau_\varepsilon^X < \zeta^X)}{\mathbb{E}_v(\tau_\varepsilon^X)/\mathbb{P}_v(\tau_\varepsilon^X < \zeta^X)} \\ &= \lim_{\varepsilon\downarrow 0} \int_{\mathcal{G}\setminus\{v\}} \big(f(g) - f(v)\big)\, \nu_\varepsilon^X(dg).\end{aligned}$$

By now exactly following the proof of Feller's theorem for the process Y, but using

$$K_\varepsilon^X = 1 + \frac{\mathbb{P}_v(X_{\tau_\varepsilon^X} = \Delta)}{\mathbb{E}_v(\tau_\varepsilon^X)} + \int_{\mathcal{G}\setminus\{v\}} \big(1 - e^{-d(v,g)}\big)\, \nu_\varepsilon^X(dg)$$

instead of the normalization K_ε^Y (where the second summand would be missing), we get

$$c_1^\infty f(v) - \sum_{e\in\mathcal{E}} c_2^e f_e'(v) + \frac{c_3}{2} f''(v) - \int_{\mathcal{G}\setminus\{v\}} \big(f(g) - f(v)\big)\, c_4(dg) = 0.$$

In comparison to the boundary condition of A^X, the only term missing is c_1^Δ, which is due to $\mathbb{P}_v(Y_{\tau_\varepsilon^Y} = \Delta) = 0$. □

We quickly remark that, if $c_1^\Delta \neq 0$, the boundary conditions in lemmas (21.67) and (21.68) are not normalized anymore, but we can always renormalize them if needed.

We have thus shown that, when reviving the process $X^\bullet$, the killing parameter c_1 or c_1^Δ transforms into a jump part: The resolvent calculation (21.67) proves that c_1 is completely transformed, while the approach via Feller's theorem (21.68) only transforms c_1^Δ and leaves c_1^∞ as "killing portion" intact. This discrepancy will be employed now:

(21.75) Lemma. *Let X be a Brownian motion on a star graph $\mathcal{G}$ with star vertex v. Then $c_1^{v,\infty} = 0$ holds true in Feller's theorem* (20.16).

Proof. Let $c_1 = c_1^{\Delta} + c_1^{\infty}$, $(c_2^e, e \in \mathcal{E})$, c_3, c_4 be given as in Feller's theorem (20.16) for a Brownian motion X on a star graph $\mathcal{G}$ with vertex v. By lemma (20.25), the generator of X reads

$$\mathscr{D}(A^X) = \Big\{ f \in \mathcal{C}_0^2(\mathcal{G}) : \\ c_1 f(v) - \sum_{e \in \mathcal{E}} c_2^e f_e'(v) + \frac{c_3}{2} f''(v) - \int_{\mathcal{G}\setminus\{v\}} \big(f(g) - f(v)\big)\, c_4(dg) = 0 \Big\}.$$

Define

$$\tilde{s} := \sum_{e \in \mathcal{E}} c_2^e + \frac{c_3}{2} + \int_{\mathcal{G}\setminus\{v\}} \big(1 - e^{-d(v,g)}\big)\, c_4(dg).$$

By recalling equation (20.22), we see that $\tilde{s} > 0$.

Assume $c_1^{\infty} \neq 0$. Consider the instant return process Y of X, that is the identical copies process, as constructed in subsection 13.1, resulting from successive revivals of X at the killing point v. Lemma (21.67) applied with the revival distribution $q = \varepsilon_v$ gives

$$\mathscr{D}(A^Y) = \Big\{ f \in \mathcal{C}_0^2(\mathcal{G}) : \\ - \sum_{e \in \mathcal{E}} \tilde{c}_2^e f_e'(v) + \frac{\tilde{c}_3}{2} f''(v) - \int_{\mathcal{G}\setminus\{v\}} \big(f(g) - f(v)\big)\, \tilde{c}_4(dg) = 0 \Big\},$$

with renormalized boundary weights

$$\forall e \in \mathcal{E} : \ \tilde{c}_2^e := \tilde{s}^{-1}\, c_2^e, \quad \tilde{c}_3 := \tilde{s}^{-1}\, c_3, \quad \tilde{c}_4 := \tilde{s}^{-1}\, c_4.$$

On the other hand, it is $c_1 \geq c_1^{\infty} > 0$. So lemma (21.68) is applicable and shows that

$$\mathscr{D}(A^Y) = \Big\{ f \in \mathcal{C}_0^2(\mathcal{G}) : \\ \tilde{\tilde{\mathrm{c}}}_1^{\infty} f(v) - \sum_{e \in \mathcal{E}} \tilde{\tilde{\mathrm{c}}}_2^e f_e'(v) + \frac{\tilde{\tilde{\mathrm{c}}}_3}{2} f''(v) - \int_{\mathcal{G}\setminus\{v\}} \big(f(g) - f(v)\big)\, \tilde{\tilde{\mathrm{c}}}_4(dg) = 0 \Big\},$$

with renormalized boundary weights

$$\tilde{\tilde{\mathrm{c}}}_1^{\infty} := \tilde{\tilde{\mathrm{s}}}^{-1}\, c_1^{\infty}, \quad \forall e \in \mathcal{E} : \ \tilde{\tilde{\mathrm{c}}}_2^e := \tilde{\tilde{\mathrm{s}}}^{-1}\, c_2^e, \quad \tilde{\tilde{\mathrm{c}}}_3 := \tilde{\tilde{\mathrm{s}}}^{-1}\, c_3, \quad \tilde{\tilde{\mathrm{c}}}_4 := \tilde{\tilde{\mathrm{s}}}^{-1}\, c_4,$$

for $\tilde{\tilde{\mathrm{s}}} := c_1^{\infty} + \tilde{s}$.

As both of the above sets of $\mathscr{D}(A^Y)$ are equal, lemma (21.62) yields $\tilde{\tilde{\mathrm{c}}}_1^{\infty} = 0$, which is impossible, as $\tilde{\tilde{\mathrm{s}}}\, \tilde{\tilde{\mathrm{c}}}_1^{\infty} = c_1^{\infty} > 0$ by assumption. $\square$

22. Construction of all Brownian Motions on a Metric Graph

After having prepared the necessary process transformations in chapter II, collected the characteristics of Brownian motions on a metric graph in section 20 and built up all Brownian motions on star graphs in section 21, we are now in a position to give a complete pathwise construction of Brownian motions on a given metric graph for any admissible set of Feller–Wentzell data. We already announced the mode of construction: We will begin with Brownian motions on star graphs which implement the corresponding "local" boundary conditions (including "small jumps") at their respective vertices. When the process is started on one of these star graphs and approaches (or jumps to) the vicinity of another vertex, it is killed and revived on the relevant subgraph with the help of the concatenation techniques developed in sections 11 and 13. That way, we achieve a Brownian motion on a general metric graph by successive pastings of partial Brownian motions on star graphs. The accurate construction approach will be laid out in the following.

As usual, we will assume any metric graph discussed here to have no tadpoles, as such edges can always be "broken up" into two non-tadpoles, cf. subsection 18.2 and remark (20.24) for the discussion concerning tadpoles.

22.1. Our Agenda

Technically, we will not start with star graphs, but with the "target" metric graph which we then decompose into subgraphs. This is necessary, as the subgraphs (that is, at some level, star graphs) must be chosen appropriately in order to construct the correct complete graph at the end, and the topology of the "target" graph is required for the pathwise construction and the specification of the Feller–Wentzell data.

To this end, let $\mathcal{G} = (\mathcal{V}, \mathcal{I}, \mathcal{E}, \partial, \rho)$ be a metric graph having at least two vertices. We will break $\mathcal{G}$ up by decomposing the set of vertices into $\mathcal{V} = \mathcal{V}^{-1} \uplus \mathcal{V}^{+1}$ and defining two "subgraphs" $\widetilde{\widetilde{\mathcal{G}}}^j$, $j \in \{-1, +1\}$, which possess the respective vertices $\mathcal{V}^j$ as well as all of the original edges (with their combinatorial structure) not incident with the other vertices $\mathcal{V}^{-j}$. As internal edges i which are incident with vertices of both subgraphs are lost, we need to replace them by new external "shadow" edges e_i^{-1}, e_i^{+1} on the respective subgraphs, see the upper graph of figure 22.1.

By iteratively decomposing the subgraphs further up to the level of star graphs, we are able to introduce Brownian motions on $\widetilde{\widetilde{\mathcal{G}}}^{-1}$ and $\widetilde{\widetilde{\mathcal{G}}}^{+1}$ with the desired boundary behavior at their vertices. In order to paste the two processes—and thus the two graphs—together, we need to "cut out" the excrescent parts of the external "shadow" edges by removing them from the subgraphs and killing the partial Brownian motions whenever they hit the removed locations. The remaining parts of these external edges need to be reorientated where necessary (as vertices are always initial points of external edges) and then are mapped to the original internal edges in order to achieve "real" subgraphs $\mathcal{G}^{-1}$ and $\mathcal{G}^{+1}$ of the original graph $\mathcal{G}$, see the lower graph of figure 22.1.

The resulting Brownian motions on $\mathcal{G}^{-1}$ and $\mathcal{G}^{+1}$ can now be pasted together with the help of the alternating copies technique established in subsection 13.2, namely by

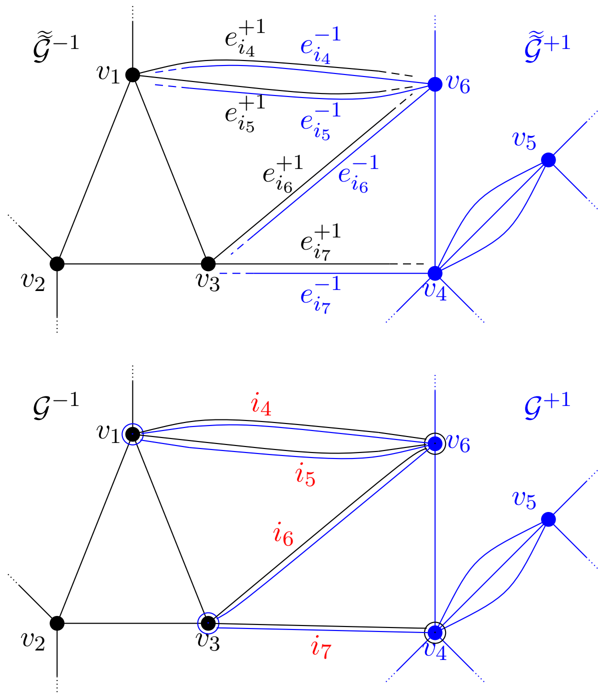
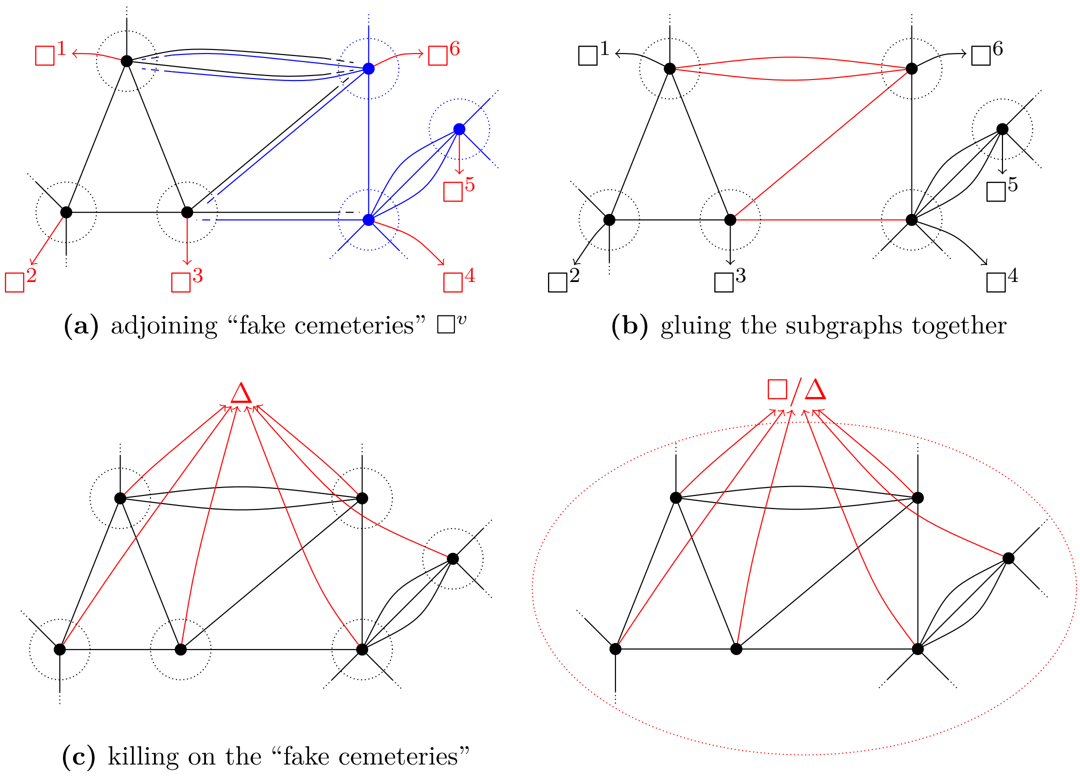


Figure 22.1: Decomposition and gluing of metric graphs: The metric graph $\mathcal{G}$ of figure 18.1 is decomposed into two "subgraphs" $\widetilde{\widetilde{\mathcal{G}}}^{-1}$ and $\widetilde{\widetilde{\mathcal{G}}}^{+1}$ with vertices $\mathcal{V}^{-1} = \{v_1, v_2, v_3\}$ and $\mathcal{V}^{+1} = \{v_4, v_5, v_6\}$, where the internal edges i, which are incident with vertices of both subgraphs, are replaced by new external edges e_i^{-1}, e_i^{+1} on the respective subgraphs. By performing the transformations explained in subsection 22.1, subsets of these "subgraphs" are mapped to the subsets $\mathcal{G}^{-1}$, $\mathcal{G}^{+1}$ of the graph $\mathcal{G}$.

reviving the subprocesses at the other subgraph whenever they leave the remaining part of one of their shadow vertices (and thus are killed).

This construction approach will cause two main technical difficulties, which will prescribe the order of applied transformations: Firstly, the "global" jumps, that is jumps to other vertices or subgraphs, can only be implemented once the gluing is complete, as their jump destinations do not exist for the original Brownian motions on the subgraphs. They will be implemented by an instant return process with an appropriate revival measure. Moreover, the implementation of the killing portions p_1^v, $v \in \mathcal{V}$, via jumps to the cemetery must be postponed until the gluing procedure and the introduction of the global jumps is complete. The reason is that, as just mentioned, both procedures will apply the technique of identical/alternating copies, which is based on reviving the process and would therefore cancel any killing effect beforehand.

The above-mentioned restrictions and interactions of the applied techniques result in some rather unwieldy "workarounds" in the upcoming complete construction. We are giving an overview of the construction steps now, the mathematical justifications will follow in subsections 22.2–22.5.



(a) adjoining "fake cemeteries" $\square^v$

(b) gluing the subgraphs together

(c) killing on the "fake cemeteries"

(d) adjoining $\square$, reviving with an appropriate measure, killing on $\square$ again

Figure 22.2: Completing the construction of Brownian motions on a metric graph: Illustrated are the steps that are performed in the construction of the target Brownian motion on the complete graph, when starting with Brownian motions on the subgraphs which already implement the correct reflection, stickiness and "local" jump parameters. The dotted lines indicate the range of the implemented jump measures.

Assume that we are given a metric graph $\mathcal{G} = (\mathcal{V}, \mathcal{I}, \mathcal{E}, \partial, \rho)$ and boundary weights

$$(p_1^v, (p_2^{v,l})_{l \in \mathcal{L}(v)}, p_3^v, p_4^v)_{v \in \mathcal{V}},$$

which satisfy the conditions of Feller's theorem (20.21).

As we cannot introduce the "global" jumps yet, we choose, for each $v \in \mathcal{V}$, a distance $\delta^v > 0$ such that δ^v is smaller than the lengths of all edges emanating from v, and define the restricted jump measure and "extended" killing parameter by

$$\begin{aligned} q_4^v &:= p_4^v(\,\cdot\, \cap B_{\delta^v}(v)), \\ q_1^v &:= p_1^v + p_4^v(\complement B_{\delta^v}(v)). \end{aligned}$$

We are going to construct the complete Brownian motion with the just given boundary weights iteratively. That is, we decompose the metric graph into two subgraphs $\widetilde{\widetilde{\mathcal{G}}}^{-1}$

and $\widetilde{\widetilde{\mathcal{G}}}^{+1}$ as explained above, and assume that there exist two Brownian motions $\widetilde{\widetilde{X}}^{-1}$, $\widetilde{\widetilde{X}}^{+1}$ thereon which implement the boundary conditions

$$\big(q_1^v, (p_2^{v,l})_{l\in\widetilde{\widetilde{\mathcal{L}}}^j(v)}, p_3^v, q_4^v\big)_{v\in\mathcal{V}^j}, \quad j\in\{-1,+1\},$$

where we set the reflection parameters for the adjoined "shadow" edges to $p_2^{v,e_i^j} = p_2^{v,i}$.

As the gluing procedure only works for processes with infinite lifetime, we further adjoin for every vertex $v\in\mathcal{V}$ an absorbing "fake" cemetery point $\square^v$ to the respective subgraph $\widetilde{\widetilde{\mathcal{G}}}^j$, and assimilate the killing parameter into the jump measure by reviving the subprocesses at $\square^v$ whenever they die at v, see figure 22.2a. Then the new processes possess the boundary conditions

$$\big(0, (p_2^{v,l})_{l\in\widetilde{\widetilde{\mathcal{L}}}^j(v)}, p_3^v, q_1^v\,\varepsilon_{\square^v} + q_4^v\big)_{v\in\mathcal{V}^j}, \quad j\in\{-1,+1\}.$$

Next, we glue both processes together and obtain a process on the complete graph $\mathcal{G}$, as illustrated in figure 22.2b, with boundary conditions

$$\big(0, (p_2^{v,l})_{l\in\mathcal{L}(v)}, p_3^v, q_1^v\,\varepsilon_{\square^v} + q_4^v\big)_{v\in\mathcal{V}}.$$

In order to introduce the global jumps, we split the jump to $\square^v$, with original weight $q_1^v = p_1^v + p_4^v\big(\complement B_{\delta^v}(v)\big)$, into real killing with weight p_1^v and non-local jumps relative to the measure $p_4^v\big(\,\cdot\,\cap\complement B_{\delta^v}(v)\big)$. To this end, we need to kill the process again: By mapping the absorbing points $\{\square^v, v\in\mathcal{V}\}$ to the "real" cemetery Δ, see figure 22.2c, we obtain a newly killed process with boundary conditions

$$\big(q_1^v, (p_2^{v,l})_{l\in\mathcal{L}(v)}, p_3^v, q_4^v\big)_{v\in\mathcal{V}}.$$

We adjoin another absorbing "fake" cemetery point $\square$ and construct the next process as instant revival process with revival distribution $\big(p_1^v\,\varepsilon_\square + p_4^v\big(\,\cdot\,\cap\complement B_{\delta^v}(v)\big)\big)/q_1^v$. This process now implements jumps relative to the measure $p_1^v\,\varepsilon_{\square^v} + p_4^v\big(\,\cdot\,\cap\complement B_{\delta^v}(v)\big)$, which adds to the already existing jump measure $q_4^v = p_4^v\big(\,\cdot\,\cap B_{\delta^v}(v)\big)$, thus satisfying the boundary conditions

$$\big(0, (p_2^{v,l})_{l\in\mathcal{L}(v)}, p_3^v, p_1^v\,\varepsilon_\square + p_4^v\big)_{v\in\mathcal{V}}.$$

Finally, we transform the jumps to $\square$ into killing by mapping $\square$ to Δ, and obtain the complete boundary condition

$$\big(p_1^v, (p_2^{v,l})_{l\in\mathcal{L}(v)}, p_3^v, p_4^v\big)_{v\in\mathcal{V}}.$$

As just explained, we need to perform many process transformations in the complete construction, while keeping track of the resulting boundary conditions. In order to keep our results comprehensible, we first analyze the two main components—killing on an absorbing set and introduction of jumps via the instant revival process—and their effects on the generator separately in the next two subsections.

22.2. Killing a Brownian Motion on an Absorbing Set

In this subsection, we examine how killing a Brownian motion on an absorbing set F affects the boundary conditions of its generator. It will turn out that the jump portion which originally led to F is just transformed into the killing portion, as any jump to F is now immediately triggering the killing.

We implement the killing transformation by mapping the absorbing set F to Δ, using the techniques of subsection 12.2, that is, we consider the process $\psi(X)$ for the map

$$\psi\colon \mathcal{G} \to \mathcal{G}\backslash F,\ x \mapsto \psi(x) := \begin{cases} x, & x \in \mathcal{G}\backslash F, \\ \Delta, & x \in F. \end{cases} \tag{22.1}$$

It has been shown there that the transformed process $\psi(X)$ is a right process in case X is a right process and F is an isolated and absorbing set for X.

We are able to obtain the following set of necessary boundary conditions by directly computing the generator of the transformed process:

(22.2) Lemma. *Let X be a Brownian motion on $\mathcal{G}$ with generator*

$$\mathscr{D}(A^X) \subseteq \Big\{ f \in \mathcal{C}_0^2(\mathcal{G}) : \forall v \in \mathcal{V} : \\ c_1^v f(v) - \sum_{l \in \mathcal{L}(v)} c_2^{v,l} f_l'(v) + \frac{c_3}{2} f''(v) - \int_{\mathcal{G}\backslash\{v\}} \big(f(g) - f(v)\big)\, c_4^v(dg) = 0 \Big\},$$

and $F \subsetneq \mathcal{G}$ be an isolated, absorbing set for X. Let $Y := \psi(X)$ be the process on $\mathcal{G}\backslash F$ resulting from killing X on F, with ψ as given in equation (22.1)*. Then the domain of the generator of Y satisfies*

$$\mathscr{D}(A^Y) \subseteq \Big\{ f \in \mathcal{C}_0^2(\mathcal{G}\backslash F) : \forall v \in \mathcal{V}\backslash F : \\ \big(c_1^v + c_4^v(F)\big) f(v) - \sum_{l \in \mathcal{L}(v)} c_2^{v,l} f_l'(v) + \frac{c_3}{2} f''(v) - \int_{\mathcal{G}\backslash(F \cup \{v\})} \big(f(g) - f(v)\big)\, c_4^v(dg) = 0 \Big\}.$$

Proof. For all $f \in \mathscr{D}(A^Y)$, we have for $g \in \mathcal{G}\backslash F$

$$\begin{aligned} A^X(f \circ \psi)(g) &= \lim_{t \downarrow 0} \frac{\mathbb{E}_g\big(f \circ \psi(X_t)\big) - f \circ \psi(g)}{t} \\ &= \lim_{t \downarrow 0} \frac{\mathbb{E}_g\big(f(Y_t)\big) - f(g)}{t}, \end{aligned}$$

which exists and is equal to $A^Y f(g)$. On the other hand, if $g \in F$, then $X_t \in F$ holds for all $t \geq 0$, $\mathbb{P}_g$-a.s., because F is absorbing for X, and it follows that

$$A^X(f \circ \psi)(g) = \lim_{t \downarrow 0} \frac{\mathbb{E}_g\big(f \circ \psi(X_t)\big) - f \circ \psi(g)}{t} = \lim_{t \downarrow 0} \frac{\mathbb{E}_g\big(f(\Delta)\big) - f(\Delta)}{t} = 0.$$

Thus, we have $f \circ \psi \in \mathscr{D}(A^X)$ for all $f \in \mathscr{D}(A^Y)$, and $A^X(f \circ \psi) = A^Y f\, \mathbb{1}_{\complement F}$ in this case.

So, if $f \in \mathscr{D}(A^Y)$, then $f \circ \psi$ fulfills the boundary condition for X, that is

$$\begin{aligned} 0 &= c_1^v f(\psi(v)) - \sum_{l \in \mathcal{L}(v)} c_2^{v,l} f_l'(\psi(v)) + \frac{c_3}{2} f''(\psi(v)) - \int_{\mathcal{G}\backslash\{v\}} \big(f(\psi(g)) - f(\psi(v))\big)\, c_4^v(dg) \\ &= c_1^v f(v) - \sum_{l \in \mathcal{L}(v)} c_2^{v,l} f_l'(v) + \frac{c_3}{2} f''(v) - \int_{\mathcal{G}\backslash(F \cup \{v\})} \big(f(g) - f(v)\big)\, c_4^v(dg) + f(v)\, c_4^v(F), \end{aligned}$$

for all $v \in \mathcal{V}\backslash F$, where we used $f(\psi(g)) = f(\Delta) = 0$ for all $g \in F$. □

However, this proof generally does not provide us with the Feller–Wentzell data of the killed process (as we are only able to compare Feller–Wentzell data with the boundary data of the generator in the star graph case, cf. theorem (21.66)). Therefore, we need to derive it manually by checking its definitions given in Feller's theorem (20.16):

(22.3) Lemma. *Let X be a Brownian motion on $\mathcal{G}$ with Feller–Wentzell data*

$$\big(c_1^{v,\Delta}, c_1^{v,\infty}, (c_2^{v,l})_{l \in \mathcal{L}(v)}, c_3^v, c_4^v\big)_{v \in \mathcal{V}},$$

and $F \subsetneq \mathcal{G}$ be an isolated, absorbing set for X. Let $Y := \psi(X)$ be the process on $\mathcal{G}\backslash F$ resulting from killing X on F, with ψ as given in equation (22.1). If $\mathcal{G}\backslash F$ is a metric graph and Y is a Brownian motion on $\mathcal{G}\backslash F$, then the Feller–Wentzell data of Y reads

$$\big(c_1^{v,\Delta} + c_4^v(F), c_1^{v,\infty}, (c_2^{v,l})_{l \in \mathcal{L}(v)}, c_3^v, c_4^v(\,\cdot\, \cap F^{\complement})\big)_{v \in \mathcal{V}\backslash F}.$$

Proof. Fix $v \in \mathcal{V}\backslash F$. The processes' exit behaviors totally coincide, except if X exits from a small neighborhood of v by jumping into F (then Y jumps to Δ). Thus, we have with the notations of Feller's theorem (20.16) (where we indicate the corresponding process in the superscript), that for all $\varepsilon > 0$ small enough, $\mathbb{E}_v(\tau_\varepsilon^X) = \mathbb{E}_v(\tau_\varepsilon^Y)$ holds, and

$$\begin{aligned} \mathbb{P}_v\big(Y_{\tau_\varepsilon^Y} \in dg \cap (\mathcal{G}\backslash F)\big) &= \mathbb{P}_v\big(X_{\tau_\varepsilon^X} \in dg \cap (\mathcal{G}\backslash F)\big), \\ \mathbb{P}_v\big(Y_{\tau_\varepsilon^Y} = \Delta\big) &= \mathbb{P}_v\big(X_{\tau_\varepsilon^X} \in \{\Delta\} \cup F\big). \end{aligned}$$

Therefore, we have $\nu_\varepsilon^{Y,v} = \nu_\varepsilon^{X,v}(\,\cdot\, \cap (\mathcal{G}\backslash F))$ and, as $d(v,f) = +\infty$ for all $f \in F$,

$$\int_F \big(1 - e^{-d(v,g)}\big)\, \nu_\varepsilon^{X,v}(dg) = \nu_\varepsilon^{X,v}(F) = \frac{\mathbb{P}_v(X_{\tau_\varepsilon^Y} \in F)}{\mathbb{E}_v(\tau_\varepsilon^X)}.$$

It follows that

$$\begin{aligned} K_\varepsilon^{Y,v} &= 1 + \frac{\mathbb{P}_v(Y_{\tau_\varepsilon^Y} = \Delta)}{\mathbb{E}_v(\tau_\varepsilon^Y)} + \int_{\mathcal{G}\backslash\{v\}} \big(1 - e^{-d(v,g)}\big)\, \nu_\varepsilon^{Y,v}(dg) \\ &= 1 + \frac{\mathbb{P}_v(Y_{\tau_\varepsilon^X} = \Delta)}{\mathbb{E}_v(\tau_\varepsilon^X)} + \int_{\mathcal{G}\backslash\{v\} \cup F} \big(1 - e^{-d(v,g)}\big)\, \nu_\varepsilon^{X,v}(dg) \\ &= K_\varepsilon^{X,v}. \end{aligned}$$

As F is isolated, we get $\bar{\mu}^{Y,v} = \bar{\mu}^{X,v}(\,\cdot\,\cap(\overline{\mathcal{G}\backslash\{v\}}\backslash F))$, and conclude that

$$\begin{aligned}
c_1^{Y,v,\Delta} &= \lim_{\varepsilon\downarrow 0}\Big(\frac{\mathbb{P}_v(X_{\tau_\varepsilon^Y} = \Delta)}{\mathbb{E}_v(\tau_\varepsilon^X)\,K_\varepsilon^{X,v}} + \frac{\mathbb{P}_v(X_{\tau_\varepsilon^Y} \in F)}{\mathbb{E}_v(\tau_\varepsilon^X)\,K_\varepsilon^{X,v}}\Big) \\
&= c_1^{X,v,\Delta} + \bar{\mu}^{X,v}(F) \\
&= c_1^{X,v,\Delta} + c_4^{X,v}(F), \\
c_1^{Y,v,\infty} &= c_1^{X,v,\infty}, \\
c_2^{Y,v,l} &= c_2^{X,v,l}, \quad l \in \mathcal{L}(v), \\
c_3^{Y,v} &= c_3^{X,v}, \\
c_4^{Y,v} &= c_4^{X,v}(\,\cdot\,\cap(\mathcal{G}\backslash F)). \qquad \square
\end{aligned}$$

(22.4) Remark. We will apply lemma (22.3) in the following context: Let X be a Brownian motion on $\mathcal{G}$ and F be an isolated and absorbing set for X, such that for its first entry time $H_F := \inf\{t \geq 0 : X_t \in F\}$ and H_X as given in definition (20.1),

$$H_X < H_F \quad \mathbb{P}_g\text{-a.s.}$$

holds true for all $g \in \complement F$.

It then follows from theorem (12.5) that the killed process $Y = \psi(X)$ is a right process, and therefore strongly Markovian. If $\mathcal{G}\backslash F$ is a metric graph, then, as $H_Y = H_X$ and $Y_t = X_t$ for all $t \leq H_X < H_F$, the properties of theorem (20.5) follow for Y from the respective ones of X. Thus, Y is a Brownian motion on $\mathcal{G}\backslash F$, and lemma (22.3) can be applied in order to deduce the Feller–Wentzell data of Y.

The condition above is especially satisfied if F can only be reached from $\complement F$ via jumps from vertices, which, as F is isolated and thus has positive distance from any vertex $v \in \mathcal{V}\backslash F$, cannot happen immediately due to the normality of the process. ■

22.3. Introduction of Non-Local Jumps

We will introduce the "non-local" jumps, namely jumps to other subgraphs, with the help of the technique of instant revivals as established in subsection 13.1. In order to prepare this approach, we examine the effect of this method on the Feller–Wentzell data. Similar results were already attained in the examinations concerning Brownian motions on star graphs in subsection 21.12, cf. especially lemmas (21.67) and (21.68). The next lemma shows that, as expected, the killing weight will be transformed to an additional jump portion with distribution given by the revival kernel. It also clarifies that this technique can only be used for the implementation of finite jump measures.

(22.5) Lemma. *Let X be a Brownian motion on $\mathcal{G}$ with Feller–Wentzell data*

$$(c_1^{v,\Delta}, c_1^{v,\infty}, (c_2^{v,l})_{l\in\mathcal{L}(v)}, c_3^v, c_4^v)_{v\in\mathcal{V}},$$

and $c_1^{v,\Delta} > 0$. Let Y be the instant revival process, constructed of X with revival kernel

$$k(v,\,\cdot\,) = \kappa^v, \quad v \in \mathcal{V},$$

for some probability measure κ^v on $\mathcal{G}$, and $k(g, \cdot\,) = \varepsilon_g$ for all $g \notin \mathcal{V}$. Suppose that for every $v \in \mathcal{V}$ there exists $\delta > 0$ such that

(i) $\kappa^v(B_\delta(v)) = 0$, *and*

(ii) for all $\varepsilon < \delta$, $X_{\tau_\varepsilon^X} \in B_\delta(v)$ *holds* $\mathbb{P}_v^X$*-a.s. on* $\{\tau_\varepsilon^X < \zeta^X\}$.

Then, Y is a Brownian motion on $\mathcal{G}$. For all $v \in \mathcal{V}$, the generator A^Y of Y satisfies for every $f \in \mathscr{D}(A^Y)$

$$c_1^{v,\infty} f(v) - \sum_{l \in \mathcal{L}(v)} c_2^{v,l} f_l'(v) + c_3^v\, Af(v) - \int_{\mathcal{G}\setminus\{v\}} \big(f(g) - f(v)\big)\, \big(c_4^v + c_1^{v,\Delta}\, \kappa^v\big)(dg) = 0.$$

If additionally $d(v,x) = +\infty$ holds for every $x \in \operatorname{supp} \kappa^v$, then the Feller–Wentzell data of Y at v reads

$$\big(0, c_1^{v,\infty}, (c_2^{v,l})_{l \in \mathcal{L}(v)}, c_3^v, c_4^v + c_1^{v,\Delta}\, \kappa^v\big).$$

Proof. By theorem (13.1), Y is a right process and thus strongly Markovian. As $Y_t = X_t$ holds for all $t \leq H_Y$, definition (20.1) or its equivalent characterization (20.5) imply that Y is a Brownian motion on $\mathcal{G}$.

Fix $v \in \mathcal{V}$. We are going to reiterate the proof of Feller's theorem (20.16) for the process Y and compare the components evolving in the generators of Y and X. As usual, components of X of Feller's theorem at vertex v will be named c_1^X, ν_ε^X, K_ε^X, etc., instead of c_1^v, ν_ε^v, K_ε^v. This proof will be based on the following two main principles:

(i) Due to assumption (i), the processes Y and X are equivalent in a neighborhood of v, more precisely: There exists $\delta > 0$ (e.g. being the minimum of δ in assumption (i) and the minimal length of all edges incident with v) such that

$$\forall \varepsilon \leq \delta: \quad \mathbb{E}_v^Y(\tau_\varepsilon^Y) = \mathbb{E}_v^X(\tau_\varepsilon^X),$$

and for all $n \in \mathbb{N}$, $f_1, \dots, f_n \in b\mathscr{B}(\mathcal{G})$, $0 \leq t_1 < \ldots < t_n$,

$$\mathbb{P}_v^Y\big(f_1(Y_{t_1}) \cdots f_1(Y_{t_n}); t_n < \tau_\delta^Y\big) = \mathbb{P}_v^X\big(f_1(X_{t_1}) \cdots f_1(X_{t_n}); t_n < \tau_\delta^X\big).$$

In particular, we have for all $\varepsilon < \delta$

$$\mathbb{P}_v^Y\big(Y_{\tau_\varepsilon^Y} \in A \,|\, Y_{\tau_\varepsilon^Y} \in B_\delta(v)\big) = \mathbb{P}_v^X\big(X_{\tau_\varepsilon^X} \in A \,|\, X_{\tau_\varepsilon^X} \in B_\delta(v)\big), \quad A \in \mathscr{B}(\mathcal{G}).$$

(ii) Due to assumption (ii), the process X only has jumps from v into $B_\delta(v)$ or to Δ, that is,

$$\forall \varepsilon < \delta: \quad \mathbb{P}_v^X\big(X_{\tau_\varepsilon^X} \in B_\delta(v) \cup \{\Delta\}\big) = 1.$$

Therefore, Y only can jump into $\complement B_\delta(v)$ if the underlying process X is killed and revived again, which yields

$$\mathbb{P}_v^Y\big(Y_{\tau_\varepsilon^Y} \in \complement B_\delta(v)\big) = \mathbb{P}_v^X\big(X_{\tau_\varepsilon^X} = \Delta\big),$$

and the jump distribution is given by the reviving kernel

$$\mathbb{P}_v^Y\big(Y_{\tau_\varepsilon^Y} \in A \,|\, Y_{\tau_\varepsilon^Y} \in \complement B_\delta(v)\big) = \kappa^v(A), \quad A \in \mathscr{B}(\mathcal{G}).$$

Furthermore, the revived process Y is not able to die at all, yielding

$$\mathbb{P}_v^Y\big(Y_{\tau_\varepsilon^Y} = \Delta\big) = 0.$$

Let $f \in \mathscr{D}(A^Y)$ and fix $v \in \mathcal{V}$. The vertex v cannot be a trap for Y, as otherwise v would either be a trap for X, which is impossible by $c_1^{v,\Delta} > 0$, or Y would be revived at v when X dies there, which contradicts assumption (i). Thus, Dynkin's formula (3.17) yields

$$Af(v) = \lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}_v^Y\big(f(Y_{\tau_\varepsilon^Y})\big) - f(v)}{\mathbb{E}_v^Y(\tau_\varepsilon^Y)}.$$

We are going to reiterate the steps in the proof of Feller's theorem (20.16) for the process Y, but will be using the normalization factor K_ε^X of X instead of K_ε^Y. This will not pose any problems because $K_\varepsilon^X \geq K_\varepsilon^Y$ holds true, which is seen as follows: With the scaled exit distributions from $\complement B_\varepsilon(v)$

$$\nu_\varepsilon^Y(A) = \frac{\mathbb{P}_v^Y\big(Y_{\tau_\varepsilon^Y} \in A\big)}{\mathbb{E}_v^Y(\tau_\varepsilon^Y)}, \quad \nu_\varepsilon^X(A) = \frac{\mathbb{P}_v^X\big(X_{\tau_\varepsilon^X} \in A\big)}{\mathbb{E}_v^X(\tau_\varepsilon^X)}, \quad A \in \mathscr{B}(\mathcal{G}\backslash\{v\}),$$

for Y and X, we have for all $\varepsilon > 0$,

$$\begin{aligned}
K_\varepsilon^X &= 1 + \frac{\mathbb{P}_v^X\big(X_{\tau_\varepsilon} = \Delta\big)}{\mathbb{E}_v^X(\tau_\varepsilon^X)} + \int_{\mathcal{G}\backslash\{v\}} \big(1 - e^{-d(v,g)}\big)\, \nu_\varepsilon^X(dg) \\
&= 1 + \frac{\mathbb{P}_v^Y\big(Y_{\tau_\varepsilon} \in \complement B_\delta(v)\big)}{\mathbb{E}_v^Y(\tau_\varepsilon^Y)} + \int_{B_\delta(v)} \big(1 - e^{-d(v,g)}\big)\, \nu_\varepsilon^Y(dg) \\
&\geq 1 + \frac{\mathbb{P}_v^Y\big(Y_{\tau_\varepsilon} = \Delta\big)}{\mathbb{E}_v^Y(\tau_\varepsilon^Y)} + \int_{\mathcal{G}\backslash\{v\}} \big(1 - e^{-d(v,g)}\big)\, \nu_\varepsilon^Y(dg) \\
&= K_\varepsilon^Y,
\end{aligned}$$

because $\mathbb{P}_v^Y\big(Y_{\tau_\varepsilon} = \Delta\big) = 0$ and

$$\frac{\mathbb{P}_v^Y\big(Y_{\tau_\varepsilon} \in \complement B_\delta(v)\big)}{\mathbb{E}_v^Y(\tau_\varepsilon^Y)} = \int_{\complement B_\delta(v)} 1\, \nu_\varepsilon^Y(dg) \geq \int_{\complement B_\delta(v)} \big(1 - e^{-d(v,g)}\big)\, \nu_\varepsilon^Y(dg). \tag{22.6}$$

Thus, by exactly following the proof of Feller's theorem (20.16), we get

$$\lim_{\varepsilon \downarrow 0} \Big(f(v)\, \frac{\mathbb{P}_v^Y\big(Y_{\tau_\varepsilon} = \Delta\big)}{\mathbb{E}_v^Y(\tau_\varepsilon)\, K_\varepsilon^X} + Af(v)\, \frac{1}{K_\varepsilon^X} - \int_{\mathcal{G}\backslash\{v\}} \big(f(g) - f(v)\big)\, \frac{\nu_\varepsilon^Y(dg)}{K_\varepsilon^X} \Big) = 0.$$

However, it is $\mathbb{P}_v^Y(Y_{\tau_\varepsilon} = \Delta) = 0$, and the exit distributions of Y decompose into

$$\nu_\varepsilon^Y(A) = \frac{\mathbb{P}_v^Y\big(Y_{\tau_\varepsilon^Y} \in A \cap B_\delta(v)\big)}{\mathbb{E}_v^Y(\tau_\varepsilon^Y)} + \frac{\mathbb{P}_v^Y\big(Y_{\tau_\varepsilon^Y} \in A \cap B_\delta(v)^\complement\big)}{\mathbb{E}_v^Y(\tau_\varepsilon^Y)},$$

with

$$\begin{aligned}
\frac{\mathbb{P}^Y_v\big(Y_{\tau^Y_\varepsilon} \in A \cap B_\delta(v)\big)}{\mathbb{E}^Y_v(\tau^Y_\varepsilon)} &= \frac{\mathbb{P}^Y_v\big(Y_{\tau^Y_\varepsilon} \in A \,|\, Y_{\tau^Y_\varepsilon} \in B_\delta(v)\big)}{\mathbb{P}^Y_v\big(Y_{\tau^Y_\varepsilon} \in B_\delta(v)\big)} \, \frac{1}{\mathbb{E}^Y_v(\tau^Y_\varepsilon)} \\
&= \frac{\mathbb{P}^X_v\big(X_{\tau^X_\varepsilon} \in A \,|\, X_{\tau^X_\varepsilon} \in B_\delta(v)\big)}{\mathbb{P}^X_v\big(Y_{\tau^X_\varepsilon} \in B_\delta(v)\big)} \, \frac{1}{\mathbb{E}^X_v(\tau^X_\varepsilon)} \\
&= \frac{\mathbb{P}^X_v\big(X_{\tau^X_\varepsilon} \in A \cap B_\delta(v)\big)}{\mathbb{E}^X_v(\tau^X_\varepsilon)} \\
&= \nu^X_\varepsilon(A),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\mathbb{P}^Y_v\big(Y_{\tau^Y_\varepsilon} \in A \cap B_\delta(v)^\complement\big)}{\mathbb{E}^Y_v(\tau^Y_\varepsilon)} &= \mathbb{P}^Y_v\big(Y_{\tau^Y_\varepsilon} \in A \,|\, Y_{\tau^Y_\varepsilon} \in B_\delta(v)^\complement\big) \, \frac{\mathbb{P}^Y_v\big(Y_{\tau^Y_\varepsilon} \in B_\delta(v)^\complement\big)}{\mathbb{E}^X_v(\tau^X_\varepsilon)} \\
&= \kappa^v(A) \, \frac{\mathbb{P}^X_v(X_{\tau^X_\varepsilon} = \Delta)}{\mathbb{E}^X_v(\tau^X_\varepsilon)}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\lim_{\varepsilon \downarrow 0} \Big(A f(v) \, \frac{1}{K^X_\varepsilon} - \int_{\mathcal{G}\setminus\{v\}} \big(f(g) - f(v)\big) \, \frac{\nu^X_\varepsilon(dg)}{K^X_\varepsilon} \\
- \frac{\mathbb{P}^X_v(X_{\tau^X_\varepsilon} = \Delta)}{\mathbb{E}^X_v(\tau^X_\varepsilon) \, K^X_\varepsilon} \int_{\complement B_\delta(v)} \big(f(g) - f(v)\big) \, \kappa^v(dg) \Big) = 0,
\end{aligned}$$

and knowing that $\frac{1}{K^X_{\varepsilon_n}}$, $\frac{\nu^X_{\varepsilon_n}(dg)}{K^X_{\varepsilon_n}}$, $\frac{\mathbb{P}^X_v(X_{\tau^X_{\varepsilon_n}} = \Delta)}{\mathbb{E}^X_v(\tau^X_{\varepsilon_n}) \, K^X_{\varepsilon_n}}$ converge along the same sequence $(\varepsilon_n, n \in \mathbb{N})$ given by Feller's theorem (20.16) for X, we conclude that

$$\begin{aligned}
c_1^{v,\infty} f(v) - \sum_{l \in \mathcal{L}(v)} c_2^{v,l} f_l'(v) + c_3^v \, A f(v) - \int_{\mathcal{G}\setminus\{v\}} \big(f(g) - f(v)\big) \, c_4^v(dg) \\
- c_1^{v,\Delta} \int_{\complement B_\delta(v)} \big(f(g) - f(v)\big) \, \kappa^v(dg) = 0.
\end{aligned}$$

In case every point in the support of κ^v has distance $+\infty$ from v, equation (22.6) shows that $K^X_\varepsilon = K^Y_\varepsilon$ holds true, so by the definition of the Feller–Wentzell data in Feller's theorem (20.16), the above set of boundary conditions at v for Y coincides with the Feller–Wentzell data of Y at v. □

The reader may notice that the resulting boundary data for Y given in lemma (22.5) might not satisfy the normalization condition of the Feller–Wentzell data of Feller's theorem (20.16), in case the support of κ^v does not have infinite distance from v.

(22.7) Remark. Observe in above lemma (22.5) that the revival of a process upon its death with a revival distribution κ only transforms the "real" killing parameter c_1^Δ into an additional jump part $c_1^\Delta\,\kappa$, while leaving the artificial killing portion c_1^∞ intact. The main explanation is that c_1^∞ does not represent the effect of "killing" in the sense of jumps to the cemetery point Δ. It is rather caused by an explosion of the process, triggered by ever-growing jumps when the process approaches a vertex point (which can be seen by surveying the proof of Feller's theorem (20.16)), and this effect is not transformed by the revival technique.

In the Brownian context, we do not expect any effects which would contribute to c_1^∞, and we indeed showed in lemma (21.75) that c_1^∞ vanishes for all Brownian motions on star graphs which were constructed in section 21. As they will form the building blocks of the Brownian motions on a general metric graph, the Feller–Wentzell data of all processes considered here will satisfy

$$\forall v \in \mathcal{V} : c_1^{v,\infty} = 0. \qquad \blacksquare$$

22.4. Gluing the Graphs Together

We are going to discuss the main construction method, namely the pasting of the subgraphs and their Brownian motions thereon. As already disclosed in subsection 22.1, this technique will compromise several steps, so we will further divide this subsection in order to keep ourselves oriented.

22.4.1. Decomposition of the Graph $\mathcal{G}$ into $\widetilde{\widetilde{\mathcal{G}}}^{-1}$, $\widetilde{\widetilde{\mathcal{G}}}^{+1}$

Let $\mathcal{G} = \big(\mathcal{V}, \mathcal{E}, \mathcal{I}, \partial, \rho\big)$ be a metric graph. We partition $\mathcal{G}$ into two graphs by choosing disjoint, non-empty sets $\mathcal{V}^{-1}$, $\mathcal{V}^{+1}$, such that $\mathcal{V} = \mathcal{V}^{-1} \uplus \mathcal{V}^{+1}$ holds, and decompose the set of edges into

$$\begin{aligned}
\mathcal{E} &= \mathcal{E}^{-1} \uplus \mathcal{E}^{+1}, && \text{with } \mathcal{E}^j := \{e \in \mathcal{E} : \partial(e) \in \mathcal{V}^j\}, \\
\mathcal{I} &= \mathcal{I}^{-1} \uplus \mathcal{I}^{+1} \uplus \mathcal{I}_s, && \text{with } \mathcal{I}^j := \{i \in \mathcal{I} : \partial_-(i) \in \mathcal{V}^j, \partial_+(i) \in \mathcal{V}^j\}, \\
\mathcal{I}_s &:= \mathcal{I}_s^{-1} \uplus \mathcal{I}_s^{+1}, && \text{with } \mathcal{I}_s^j := \{i \in \mathcal{I} : \partial_-(i) \in \mathcal{V}^j, \partial_+(i) \notin \mathcal{V}^j\}.
\end{aligned}$$

As most of the following construction will be performed for both partial graphs in parallel, we will always assume that $j \in \{-1, +1\}$ when nothing else is said.

We define the metric graphs $\widetilde{\widetilde{\mathcal{G}}}^{-1}$, $\widetilde{\widetilde{\mathcal{G}}}^{+1}$ by

$$\widetilde{\widetilde{\mathcal{G}}}^j := \big(\mathcal{V}^j, \mathcal{E}^j \cup \mathcal{E}_s^j, \mathcal{I}^j, \partial^j, \rho^j\big),$$

equipped with additional external "shadow" edges

$$\mathcal{E}_s^j := \{e_i^j, i \in \mathcal{I}_s\}, \quad \text{with } \forall i \in \mathcal{I}_s : \quad e_i^j \notin \mathcal{E} \cup \mathcal{E}_s^{-j} \cup \mathcal{I},$$

where the original graph's combinatorial structure and edge lengths are naturally transfered to $\widetilde{\widetilde{\mathcal{G}}}^{-1}$, $\widetilde{\widetilde{\mathcal{G}}}^{+1}$ by setting

$$\begin{aligned}
&\partial^j\Big|_{\mathcal{E}^j \cup (I^j \times I^j)} := \partial\big|_{\mathcal{E}^j \cup (I^j \times I^j)}, \quad \partial^j(e_i^j) := \begin{cases} \partial_-(i), & i \in \mathcal{I}_s^j, \\ \partial_+(i), & i \in \mathcal{I}_s^{-j}, \end{cases} \\
&\rho^j\Big|_{\mathcal{E}^j \cup I^j} := \rho\big|_{\mathcal{E}^j \cup I^j}, \quad \rho^j\Big|_{\mathcal{E}_s^j} := +\infty.
\end{aligned}$$

For later use, we also define the "shadow length" of an external "shadow" edge by

$$\rho_s(e_i^j) := \rho(i), \quad e_i^j \in \mathcal{E}_s^{-1} \cup \mathcal{E}_s^{+1}.$$

The excrescent parts of the shadow edges, which will be removed in the following development before gluing both subgraphs together, are named

$$\widetilde{\mathcal{G}}_s^j := \bigcup_{e \in \mathcal{E}_s^j} (\{e\} \times [\rho_s(e), +\infty)).$$

22.4.2. Introducing the Brownian Motion $\widetilde{\widetilde{X}}^j$ on $\widetilde{\widetilde{\mathcal{G}}}^j$

Let $\widetilde{\widetilde{X}}^{-1}$, $\widetilde{\widetilde{X}}^{+1}$ be Brownian motions on $\widetilde{\widetilde{\mathcal{G}}}^{-1}$, $\widetilde{\widetilde{\mathcal{G}}}^{+1}$ respectively, which admit the hypotheses of right processes, feature infinite lifetimes, have the Feller–Wentzell data

$$\left(0, 0, (p_2^{v,l})_{l \in \widetilde{\widetilde{\mathcal{L}}}^j(v)}, p_3^v, p_4^v\right)_{v \in \mathcal{V}^j},$$

are continuous inside every edge (cf. lemma (21.12)), and satisfy for all $v \in \mathcal{V}^j$

$$\forall \varepsilon < \delta: \quad \mathbb{P}_v^j\big(\widetilde{\widetilde{X}}^j_{\widetilde{\widetilde{\tau}}^j_\varepsilon} \in \widetilde{\mathcal{G}}_s^j\big) = 0, \tag{22.8}$$

with $\delta := \min\{\rho_i, i \in \mathcal{I}_s\}$ and $\widetilde{\widetilde{\tau}}^j_\varepsilon := \inf\{t \geq 0 : d(\widetilde{\widetilde{X}}^j_t, \widetilde{\widetilde{X}}^j_0) > \varepsilon\}$.

By gluing the graphs $\widetilde{\widetilde{\mathcal{G}}}^{-1}$ and $\widetilde{\widetilde{\mathcal{G}}}^{+1}$ (and thus the Brownian motions $\widetilde{\widetilde{X}}^{-1}$ and $\widetilde{\widetilde{X}}^{+1}$ thereon) together, we are going to show the following main result:

(22.9) Theorem. *There exists a Brownian motion X on $\mathcal{G}$ with Feller–Wentzell data*

$$\left(c_1^v, (c_2^{v,l})_{l \in \mathcal{L}(v)}, c_3^v, c_4^v\right)_{v \in \mathcal{V}},$$

such that for each $v \in \mathcal{V}$, it holds that $c_1^v = 0$, $c_3^v = p_3^v$, $c_4^v = p_4^v \circ (\psi^j)^{-1}$, with ψ^j being defined by equation (22.11), *and*

$$\begin{aligned} i \in \mathcal{I}(v): \quad & c_2^{v,i} = \begin{cases} p_2^{v,i}, & i \in \mathcal{I}^{-1}(v) \cup \mathcal{I}^{+1}(v), \\ p_2^{v,e_i^j}, & i \in \mathcal{I}_s(v), \text{ with } j \in \{-1,+1\} \text{ such that } v \in \mathcal{V}^j, \end{cases} \\ e \in \mathcal{E}(v): \quad & c_2^{v,e} = p_2^{v,e}. \end{aligned}$$

We are going to construct the process X of the theorem above explicitly via alternating copies of transformed processes X^{-1}, X^{+1} of $\widetilde{\widetilde{X}}^{-1}$, $\widetilde{\widetilde{X}}^{+1}$. Before that, we need to kill the original processes $\widetilde{\widetilde{X}}^{-1}$ and $\widetilde{\widetilde{X}}^{+1}$ on the excrescent shadow edges and reorientate the remaining parts in order to comply to the direction of the original internal edges of $\mathcal{G}$.

22.4.3. Defining $\widetilde{X}^j$ by Killing $\widetilde{\widetilde{X}}^j$ on $\widetilde{\mathcal{G}}^j_s$

Consider the first entry time into $\widetilde{\mathcal{G}}^j_s$ of the prototype Brownian motion $\widetilde{\widetilde{X}}^j$ on $\widetilde{\widetilde{\mathcal{G}}}^j$:

$$\widetilde{\widetilde{\tau}}^j := \inf\{t \geq 0 : \widetilde{\widetilde{X}}^j_t \in \widetilde{\mathcal{G}}^j_s\}.$$

We define $\widetilde{X}^j$ to be the process obtained by killing $\widetilde{\widetilde{X}}^j$ at the terminal time $\widetilde{\widetilde{\tau}}^j$, that is

$$\widetilde{X}^j_t := \begin{cases} \widetilde{\widetilde{X}}^j_t, & t < \widetilde{\widetilde{\tau}}^j, \\ \Delta, & t \geq \widetilde{\widetilde{\tau}}^j, \end{cases}$$

on the topological subspace $\widetilde{\mathcal{G}}^j$ of $\widetilde{\widetilde{\mathcal{G}}}^j$ given by

$$\widetilde{\mathcal{G}}^j := \widetilde{\widetilde{\mathcal{G}}}^j \backslash \widetilde{\mathcal{G}}^j_s = \mathcal{V}^j \cup \bigcup_{l \in \mathcal{E}^j \cup \mathcal{I}^j} \big(\{l\} \times (0, \rho_l)\big) \cup \bigcup_{e \in \mathcal{E}^j_s} \big(\{e\} \times (0, \rho_s(e))\big).$$

(22.10) Lemma. *$\widetilde{X}^j$ is a right process on $\widetilde{\mathcal{G}}^j$ with lifetime $\widetilde{\widetilde{\tau}}^j$.*

Proof. $\widetilde{\widetilde{X}}^j$ is a right process with infinite lifetime. By employing theorem (10.1), it suffices to observe that $\widetilde{\widetilde{\tau}}^j$ is the debut of the closed, thus nearly optional set $\widetilde{\mathcal{G}}^j_s$, and the regular set of the killing time $\widetilde{\widetilde{\tau}}^j$ reads

$$F := \big\{g \in \widetilde{\widetilde{\mathcal{G}}}^j : \mathbb{P}^j_g(\widetilde{\widetilde{\tau}}^j = 0) = 1\big\} = \widetilde{\mathcal{G}}^j_s,$$

as $\widetilde{\widetilde{X}}^j$ is a right continuous, normal process and $\widetilde{\mathcal{G}}^j_s$ is closed. □

We would like to point out that the just introduced processes $\widetilde{X}^j$ are not Brownian motions on a metric graph anymore, as $\widetilde{\mathcal{G}}^j$ is not a metric graph. Thus, we will not be able to apply any results on Brownian motions for $\widetilde{X}^j$ in the upcoming development.

22.4.4. Letting X^j be the Mapping of $\widetilde{X}^j$ to the Subspace $\mathcal{G}^j \subseteq \mathcal{G}$

We need to fit the subspaces $\widetilde{\mathcal{G}}^j$ of $\widetilde{\widetilde{\mathcal{G}}}^j$ to the corresponding subspaces of $\mathcal{G}$. To this end, introduce the topological subspaces $\mathcal{G}^{-1}$, $\mathcal{G}^{+1}$ of $\mathcal{G}$ by

$$\mathcal{G}^j := \mathcal{V}^j \cup \bigcup_{l \in \mathcal{E}^j \cup \mathcal{I}^j \cup \mathcal{I}_s} \big(\{l\} \times (0, \rho_l)\big),$$

and consider the mapping $\psi^j : \widetilde{\mathcal{G}}^j \to \mathcal{G}^j$ defined by

$$\begin{aligned} &\forall i \in \mathcal{I}_s, x \in (0, \rho_i) : \quad \psi^j\big((e^j_i, x)\big) := \begin{cases} (i, x), & i \in \mathcal{I}^j_s, \\ (i, \rho_i - x), & i \in \mathcal{I}^{-j}_s, \end{cases} \\ &\psi^j = \mathrm{id} \ \text{ otherwise.} \end{aligned} \tag{22.11}$$

Clearly, ψ^j is a bijective mapping with its inverse $(\psi^j)^{-1} =: \varphi^j : \mathcal{G}^j \to \widetilde{\mathcal{G}}^j$ being given by

$$\begin{aligned} &\forall i \in \mathcal{I}_s, x \in (0, \rho_i) : \quad \varphi^j\big((i, x)\big) := \begin{cases} (e^j_i, x), & i \in \mathcal{I}^j_s, \\ (e^j_i, \rho_i - x), & i \in \mathcal{I}^{-j}_s, \end{cases} \\ &\varphi^j = \mathrm{id} \ \text{ otherwise.} \end{aligned}$$

Furthermore, ψ^j is a continuous mapping, as it is continuous inside every edge and its preimages of balls with radius $\varepsilon > 0$ around vertices $v \in \mathcal{V}^j$ read

$$\begin{aligned}
(\psi^j)^{-1}(B_\varepsilon(v)) &= (\psi^j)^{-1}\Big(\{v\} \cup \bigcup_{e \in \mathcal{E}^j(v)} (\{e\} \times (0,\varepsilon)) \cup \bigcup_{\substack{i \in \mathcal{I}^j(v) \cup \mathcal{I}^j_s(v) \\ \partial_-(i)=v}} (\{i\} \times (0,\varepsilon)) \\
&\qquad \cup \bigcup_{\substack{i \in \mathcal{I}^j(v) \cup \mathcal{I}^j_s(v) \\ \partial_+(i)=v}} (\{i\} \times (\rho_i - \varepsilon, \rho_i))\Big), \\
&= \{v\} \cup \bigcup_{e \in \mathcal{E}^j(v)} (\{e\} \times (0,\varepsilon)) \cup \bigcup_{\substack{i \in \mathcal{I}^j(v) \\ \partial_-(i)=v}} (\{i\} \times (0,\varepsilon)) \\
&\qquad \cup \bigcup_{\substack{i \in \mathcal{I}^j(v) \\ \partial_+(i)=v}} (\{i\} \times (\rho_i - \varepsilon, \rho_i)) \cup \bigcup_{e \in \mathcal{E}^j_s(v)} (\{e\} \times (0,\varepsilon)) \\
&= \widetilde{B}_\varepsilon(v).
\end{aligned}$$

$\widetilde{X}^j$ is a right process on $\widetilde{\mathcal{G}}^j$, ψ^j is a bijective and measurable map from $\widetilde{\mathcal{G}}^j$ onto $\mathcal{G}^j$, and $t \mapsto \psi^j(\widetilde{X}^j_t)$ is right continuous (as ψ^j is continuous and $t \mapsto \widetilde{X}^j_t$ is right continuous). Thus, the following result is a direct consequence of theorem (12.3):

(22.12) Lemma. *The process $X^j := \psi^j(\widetilde{X}^j)$, resulting from the state space mapping of $\widetilde{X}^j$ by ψ^j (cf. section 12), is a right process on $\psi^j(\widetilde{\mathcal{G}}^j) = \mathcal{G}^j$ with lifetime $\zeta^j = \widetilde{\zeta}^j = \widetilde{\widetilde{\tau}}^j$.*

22.4.5. Constructing X as Alternating Copies of X^{-1}, X^{+1}

Apply the technique of subsection 13.2 to define the process X obtained from forming alternating copies of X^{-1}, X^{+1} via the transfer kernels K^{-1}, K^{+1}, which are defined by

$$\begin{aligned}
K^{-1} &= \sum_{i \in I_s^{-1}} \varepsilon_{\partial_+(i)} \, \mathbb{1}_{\{i\}}\big(\pi^1(X^{-1}_{\zeta^{-1}-})\big) + \sum_{i \in I_s^{+1}} \varepsilon_{\partial_-(i)} \, \mathbb{1}_{\{i\}}\big(\pi^1(X^{-1}_{\zeta^{-1}-})\big), \\
K^{+1} &= \sum_{i \in I_s^{-1}} \varepsilon_{\partial_-(i)} \, \mathbb{1}_{\{i\}}\big(\pi^1(X^{+1}_{\zeta^{+1}-})\big) + \sum_{i \in I_s^{+1}} \varepsilon_{\partial_+(i)} \, \mathbb{1}_{\{i\}}\big(\pi^1(X^{+1}_{\zeta^{+1}-})\big).
\end{aligned} \tag{22.13}$$

That is, the transfer kernels implement the following rules for $j \in \{-1,+1\}$:

(i) X is revived as X^{+1} at $v = \partial_{-j}(i)$, if X^{-1} dies on $i \in \mathcal{I}^j_s$;

(ii) X is revived as X^{-1} at $v = \partial_j(i)$, if X^{+1} dies on $i \in \mathcal{I}^j_s$.

For later use, we already remark the following combined formula of the above definitions for the transfer kernels K^j, $j \in \{-1,+1\}$:

$$K^j = k^j(i) := \begin{cases} \varepsilon_{\partial_+}(i), & i \in \mathcal{I}^j_s, \\ \varepsilon_{\partial_-}(i), & i \in \mathcal{I}^{-j}_s, \end{cases} \quad \text{for} \quad i := \pi^1(X^j_{\zeta^j-}). \tag{22.14}$$

(22.15) Lemma. *K^j is a transfer kernel from X^j to E^{-j}.*

Proof. With probability 1, the process $\widetilde{\widetilde{X}}^j$ cannot realize $\widetilde{\widetilde{\tau}}^j$ through a direct jump from any vertex $v \in \mathcal{V}^j$: Otherwise, this would imply $\mathbb{P}^j_v(\widetilde{\widetilde{X}}^j_{\widetilde{\widetilde{\tau}}^j_\varepsilon} \in \widetilde{\mathcal{G}}^j_s) > 0$, as $\widetilde{\widetilde{\tau}}^j \geq \widetilde{\widetilde{\tau}}^j_\varepsilon$ holds for $\varepsilon < \delta$, contradicting our basic assumption (22.8). Furthermore, $\widetilde{\widetilde{X}}^j$ is continuous on every edge, so $\widetilde{\widetilde{X}}^j_{\widetilde{\widetilde{\tau}}^j-}$ exists and is equal to $\widetilde{\widetilde{X}}^j_{\widetilde{\widetilde{\tau}}^j}$. Thus,

$$\widetilde{X}^j_{\zeta^j-} = \lim_{t \uparrow\uparrow \zeta^j} \widetilde{X}^j_t = \lim_{t \uparrow\uparrow \widetilde{\widetilde{\tau}}^j} \widetilde{\widetilde{X}}^j_t = \widetilde{\widetilde{X}}^j_{\widetilde{\widetilde{\tau}}^j}$$

exists in $\{(e, \rho_s(e)), e \in \mathcal{E}^j_s\}$, and

$$\pi^1(X^j_{\zeta^j-}) = \pi^1(\psi^j(\widetilde{X}^j_{\zeta^j-})) = \pi^1(\psi^j(\widetilde{\widetilde{X}}^j_{\widetilde{\widetilde{\tau}}^j})) \tag{22.16}$$

exists in $\mathcal{I}_s$. Therefore, example (11.3) yields $\pi^1(X^j_{\zeta^j-}) \in \mathscr{F}^j_{[\zeta^j-]}$, so K^j is indeed a probability kernel K from $(\Omega^j, \mathscr{F}^j_{[\zeta^j-]})$ to $(E^{-j}, \mathscr{E}^{-j})$, that is, a transfer kernel. □

Let

- τ^{-1}_{-1} be the first entry time of X^{-1} into $\mathcal{G}^{-1}\backslash\mathcal{G}^{+1}$,
- τ^{+1}_{+1} be the first entry time of X^{+1} into $\mathcal{G}^{+1}\backslash\mathcal{G}^{-1}$.

Then, according to theorem (13.2), X is a right process on $\mathcal{G} = \mathcal{G}^{-1} \cup \mathcal{G}^{+1}$, in case the following conditions hold true for all $g \in \mathcal{G}^{-1} \cap \mathcal{G}^{+1}$, $h^{-1}, h^{+1} \in b\mathscr{B}(\mathcal{G})$:

(i) $\mathbb{E}^{-1}_g\Big(\int_0^{\tau^{-1}_{-1}} e^{-\alpha t} f(X^{-1}_t)\, dt\Big) = \mathbb{E}^{+1}_g\Big(\int_0^{\tau^{+1}_{+1}} e^{-\alpha t} f(X^{+1}_t)\, dt\Big)$;

(ii) $\mathbb{E}^{-1}_g(e^{-\alpha\tau^{-1}_{-1}}\, h^{-1}(X^{-1}_{\tau^{-1}_{-1}});\, \tau^{-1}_{-1} < \zeta^{-1}) = \mathbb{E}^{+1}_g(e^{-\alpha\zeta^{+1}}\, K^{+1}h^{-1};\, \zeta^{+1} < \tau^{+1}_{+1})$,
$\mathbb{E}^{+1}_g(e^{-\alpha\tau^{+1}_{+1}}\, h^{+1}(X^{+1}_{\tau^{+1}_{+1}});\, \tau^{+1}_{+1} < \zeta^{+1}) = \mathbb{E}^{-1}_g(e^{-\alpha\zeta^{-1}}\, K^{-1}h^{+1};\, \zeta^{-1} < \tau^{-1}_{-1})$.

We are preparing the proof of these equalities. By construction, we have

$$\begin{aligned} \mathcal{G}^{-1} \cap \mathcal{G}^{+1} &= \bigcup_{i \in \mathcal{I}_s} (\{i\} \times (0, \rho_i)), \\ \mathcal{G}^j\backslash\mathcal{G}^{-j} &= \mathcal{V}^j \cup \bigcup_{l \in \mathcal{E}^j \cup \mathcal{I}^j} (\{l\} \times (0, \rho_l)). \end{aligned}$$

By using the definition of X^j and observing that $\varphi^j(\mathcal{G}^j\backslash\mathcal{G}^{-j}) = \mathcal{G}^j\backslash\mathcal{G}^{-j}$, we get

$$\begin{aligned} \tau^j_j &= \inf\{t \geq 0 : X^j_t \in \mathcal{G}^j\backslash\mathcal{G}^{-j}\} \\ &= \inf\{t \geq 0 : \psi^j(\widetilde{X}^j_t) \in \mathcal{G}^j\backslash\mathcal{G}^{-j}\} \\ &= \inf\{t \geq 0 : \widetilde{X}^j_t \in \mathcal{G}^j\backslash\mathcal{G}^{-j}\}. \end{aligned}$$

The process $\widetilde{X}^j$ was constructed by killing $\widetilde{\widetilde{X}}^j$ at $\widetilde{\widetilde{\tau}}^j$, thus, by introducing the first exit times of $\widetilde{\widetilde{X}}^j$ from the shadow edges

$$\widetilde{\widetilde{\tau}}^j_j := \inf\big\{t \geq 0 : \widetilde{\widetilde{X}}^j_t \in \mathcal{G}^j \backslash \mathcal{G}^{-j}\big\} = \inf\big\{t \geq 0 : \widetilde{\widetilde{X}}^j_t \notin \bigcup_{i \in \mathcal{I}_s} (\{e^j_i\} \times (0, \infty))\big\},$$

we obtain the relation

$$\tau^j_j \wedge \zeta^j = \widetilde{\widetilde{\tau}}^j_j \wedge \widetilde{\widetilde{\tau}}^j. \tag{22.17}$$

Turning to the actual proof of (i) and (ii), let $g \in \mathcal{G}^{-1} \cap \mathcal{G}^{+1}$, that is, $g = (i, x)$ for some $i \in \mathcal{I}_s$, $x \in (0, \rho_i)$. Choose $j \in \{-1, +1\}$ such that $i \in \mathcal{I}^j_s$. By tracing X^j back to $\widetilde{\widetilde{X}}^j$ and employing that the latter is a Brownian motion on $\widetilde{\widetilde{\mathcal{G}}}^j$, lemma (20.4) and corollary (20.11) yield

$$\begin{aligned}
\mathbb{E}^{-j}_g\Big(\int_0^{\tau^{-j}_{-j}} e^{-\alpha t}\, f\big(X^{-j}_t\big)\, dt\Big) &= \mathbb{E}^{-j}_{(\psi^{-j})^{-1}(g)}\Big(\int_0^{\tau^{-j}_{-j}} e^{-\alpha t}\, f\big(\psi^{-j}(\widetilde{\widetilde{X}}^{-j}_t, \widetilde{\widetilde{\tau}}^{-j})\big)\, dt\Big) \\
&= \mathbb{E}^{-j}_{(e^{-j}_i, \rho(i)-x)}\Big(\int_0^{\widetilde{\widetilde{\tau}}^{-j}_{-j} \wedge \widetilde{\widetilde{\tau}}^{-j}} e^{-\alpha t}\, f\big(\psi^{-j}(\widetilde{\widetilde{X}}^{-j}_t)\big)\, dt\Big) \\
&= \mathbb{E}^B_{\rho(i)-x}\Big(\int_0^{\tau_0 \wedge \tau_{\rho(i)}} e^{-\alpha t}\, f\big(\psi^{-j}(e^{-j}_i, B_t)\big)\, dt\Big) \\
&= \mathbb{E}^B_{\rho(i)-x}\Big(\int_0^{\tau_0 \wedge \tau_{\rho(i)}} e^{-\alpha t}\, f\big(i, \rho(i) - B_t\big)\, dt\Big),
\end{aligned} \tag{22.18}$$

and analogously

$$\begin{aligned}
\mathbb{E}^j_g\Big(\int_0^{\tau^j_j} e^{-\alpha t}\, f\big(X^j_t\big)\, dt\Big) &= \mathbb{E}^j_{(e^j_i, x)}\Big(\int_0^{\widetilde{\widetilde{\tau}}^j_j \wedge \widetilde{\widetilde{\tau}}^j} e^{-\alpha t}\, f\big(\psi^j(\widetilde{\widetilde{X}}^j_t)\big)\, dt\Big) \\
&= \mathbb{E}^B_x\Big(\int_0^{\tau_0 \wedge \tau_{\rho(i)}} e^{-\alpha t}\, f\big(i, B_t\big)\, dt\Big).
\end{aligned} \tag{22.19}$$

Let $(\gamma_y, y \in \mathbb{R})$ be the translation operators and ι be the reflection operator (see subsection 6.5) for the one-dimensional Brownian motion B. They give

$$\begin{aligned}
\mathbb{E}^B_x\big(Y \circ \gamma_y\big) &= \mathbb{E}^B_{x+y}\big(Y\big), \\
\mathbb{E}^B_x\big(Y \circ \iota\big) &= \mathbb{E}^B_{-x}\big(Y\big)
\end{aligned}$$

for all $x, y \in \mathbb{R}$, $Y \in b\mathscr{F}^B_\infty$, as well as

$$\begin{aligned}
\tau_x \circ \gamma_y &= \inf\{t \geq 0 : B_t \circ \gamma_y = x\} = \inf\{t \geq 0 : B_t + y = x\} = \tau_{x-y}, \\
\tau_x \circ \iota &= \inf\{t \geq 0 : B_t \circ \iota = x\} = \inf\{t \geq 0 : -B_t = x\} = \tau_{-x}.
\end{aligned}$$

Then, an application of the theorems (6.35) and (6.42) results in

$$\begin{aligned}
&\mathbb{E}^B_{\rho(i)-x}\Big(\int_0^{\tau_0\wedge\tau_{\rho(i)}} e^{-\alpha t}\, f\big(i,\rho(i)-B_t\big)\,dt\Big)\\
&= \mathbb{E}^B_{-x}\Big(\Big(\int_0^{\tau_0\wedge\tau_{\rho(i)}} e^{-\alpha t}\, f\big(i,\rho(i)-B_t\big)\,dt\Big)\circ\gamma_{\rho(i)}\Big)\\
&= \mathbb{E}^B_{-x}\Big(\int_0^{\tau_{-\rho(i)}\wedge\tau_0} e^{-\alpha t}\, f\big(i,-B_t\big)\,dt\Big)\\
&= \mathbb{E}^B_{x}\Big(\Big(\int_0^{\tau_{-\rho(i)}\wedge\tau_0} e^{-\alpha t}\, f\big(i,-B_t\big)\,dt\Big)\circ\iota\Big)\\
&= \mathbb{E}^B_{x}\Big(\int_0^{\tau_0\wedge\tau_{\rho(i)}} e^{-\alpha t}\, f\big(i,B_t\big)\,dt\Big),
\end{aligned}$$

which proves the equality of (22.18) and (22.19), and thus concludes (i).

Coming to (ii), we will prove both assertions simultaneously, as they only differ in the initial process. Let $j\in\{-1,+1\}$. We start by reducing the first integral to $\widetilde{\widetilde{X}}{}^j$, and obtain with the help of equation (22.17) the identity

$$\begin{aligned}
&\mathbb{E}^{-j}_g\big(e^{-\alpha\tau^{-j}_{-j}}\,h^{-j}\big(X^{-j}_{\tau^{-j}_{-j}}\big);\ \tau^{-j}_{-j}<\zeta^{-j}\big)\\
&= \mathbb{E}^{-j}_{(\psi^{-j})^{-1}(g)}\Big(e^{-\alpha\widetilde{\widetilde{\tau}}{}^{-j}_{-j}}\,h^{-j}\big(\psi^{-j}\big(\widetilde{\widetilde{X}}{}^{-j}_{\widetilde{\widetilde{\tau}}{}^{-j}_{-j}}\big)\big);\ \widetilde{\widetilde{\tau}}{}^{-j}_{-j}<\widetilde{\widetilde{\tau}}{}^{-j}\Big),
\end{aligned}$$

where $(\psi^{-j})^{-1}(g)=(e^{-j}_i,\rho_i-x)$ or $(\psi^{-j})^{-1}(g)=(e^{-j}_i,x)$ depending on whether $i\in\mathcal{I}^j_s$ or $i\in\mathcal{I}^{-j}_s$. For all that follows, we define for any $g\in\widetilde{\widetilde{\mathcal{G}}}{}^j$ the first hitting time $\widetilde{\widetilde{H}}{}^j_g$ of $\{g\}$ by the process $\widetilde{\widetilde{X}}{}^j$. By the continuity of $\widetilde{\widetilde{X}}{}^j$ inside the edges, we see that $\mathbb{P}^j_{(e^j_i,y)}$-a.s. for any $i\in\mathcal{I}_s$, the relation

$$\widetilde{\widetilde{\tau}}{}^j_j=\widetilde{\widetilde{H}}{}^j_v\quad\text{on }\{\widetilde{\widetilde{\tau}}{}^j_j<\widetilde{\widetilde{\tau}}{}^j\}=\{\widetilde{\widetilde{H}}{}^j_v<\widetilde{\widetilde{H}}{}^j_{(e^j_i,\rho_s(e^j_i))}\}$$

holds true with $v:=\partial(e^j_i)$, so we have

$$\psi^{-j}\big(\widetilde{\widetilde{X}}{}^{-j}_{\widetilde{\widetilde{\tau}}{}^{-j}_{-j}}\big)=\partial(e^{-j}_i)=\begin{cases}\partial_+(i), & i\in\mathcal{I}^j_s,\\ \partial_-(i), & i\in\mathcal{I}^{-j}_s.\end{cases}$$

Thus, we get

$$\begin{aligned}
&\mathbb{E}^{-j}_g\big(e^{-\alpha\tau^{-j}_{-j}}\,h^{-j}\big(X^{-j}_{\tau^{-j}_{-j}}\big);\ \tau^{-j}_{-j}<\zeta^{-j}\big)\\
&= \mathbb{E}^{-j}_{(\psi^{-j})^{-1}(g)}\Big(e^{-\alpha\widetilde{\widetilde{\tau}}{}^{-j}_{-j}}\,h^{-j}\big(\psi^{-j}\big(\widetilde{\widetilde{X}}{}^{-j}_{\widetilde{\widetilde{\tau}}{}^{-j}_{-j}}\big)\big);\ \widetilde{\widetilde{\tau}}{}^{-j}_{-j}<\widetilde{\widetilde{\tau}}{}^{-j}\Big)\\
&= \begin{cases}\mathbb{E}^{-j}_{(e^{-j}_i,\rho(i)-x)}\Big(e^{-\alpha\widetilde{\widetilde{H}}{}^j_v}\,h^{-j}\big(\partial_+(i)\big);\ \widetilde{\widetilde{H}}{}^{-j}_v<\widetilde{\widetilde{H}}{}^{-j}_{(e^{-j}_i,\rho_s(e^{-j}_i))}\Big), & i\in\mathcal{I}^j_s,\\
\mathbb{E}^{-j}_{(e^{-j}_i,x)}\Big(e^{-\alpha\widetilde{\widetilde{H}}{}^j_v}\,h^{-j}\big(\partial_-(i)\big);\ \widetilde{\widetilde{H}}{}^{-j}_v<\widetilde{\widetilde{H}}{}^{-j}_{(e^{-j}_i,\rho_s(e^{-j}_i))}\Big), & i\in\mathcal{I}^{-j}_s.\end{cases}
\end{aligned}$$

But $\widetilde{\widetilde{X}}^{-j}$ is a Brownian motion on $\widetilde{\widetilde{\mathcal{G}}}^{-j}$, so using corollary (20.11) together with remark (20.10) and recalling $\rho_s(e_i^{-j}) = \rho(i)$ yield

$$\begin{aligned} & \mathbb{E}_g^{-j}\big(e^{-\alpha\tau_{-j}^{-j}}\, h^{-j}\big(X^{-j}_{\tau^{-j}_{-j}}\big);\, \tau^{-j}_{-j} < \zeta^{-j}\big) \\ & = \begin{cases} h^{-j}\big(\partial_+(i)\big)\, \mathbb{E}^B_{\rho(i)-x}\big(e^{-\alpha\tau_0};\, \tau_0 < \tau_{\rho(i)}\big), & i \in \mathcal{I}^j_s, \\ h^{-j}\big(\partial_-(i)\big)\, \mathbb{E}^B_x\big(e^{-\alpha\tau_0};\, \tau_0 < \tau_{\rho(i)}\big), & i \in \mathcal{I}^{-j}_s. \end{cases} \end{aligned} \tag{22.20}$$

Next, we employ the same techniques as above in order to compute the right-hand sides of (ii). Equations (22.16) and (22.17) give

$$\begin{aligned} & \mathbb{E}^j_g\big(e^{-\alpha\zeta^j}\, K^j h^{-j};\, \zeta^j < \tau^j_j\big) \\ & = \mathbb{E}^j_{(\psi^j)^{-1}(g)}\Big(e^{-\alpha\widetilde{\widetilde{\tau}}^j}\, k^j\big(\pi^1\big(\psi^j\big(\widetilde{\widetilde{X}}^j_{\widetilde{\widetilde{\tau}}^j}\big)\big)\big)\, h^{-j};\, \widetilde{\widetilde{\tau}}^j < \widetilde{\widetilde{\tau}}^j_j\Big). \end{aligned}$$

We observe that

$$\widetilde{\widetilde{\tau}}^j = \widetilde{\widetilde{H}}^j_{(e^j_i, \rho_s(e^j_i))} \quad \text{on } \{\widetilde{\widetilde{\tau}}^j < \widetilde{\widetilde{\tau}}^j_j\} = \{\widetilde{\widetilde{H}}^j_{(e^j_i, \rho_s(e^j_i))} < \widetilde{\widetilde{H}}^j_v\}$$

holds, as $\widetilde{\widetilde{\tau}}^j_j \leq \widetilde{\widetilde{\tau}}^j$ in case $\widetilde{\widetilde{\tau}}^j = \widetilde{\widetilde{H}}^j_{(e^j_k, \rho_s(e^j_k))}$ for some other $k \neq i$. Thus, we have

$$\pi^1\big(\psi^j\big(\widetilde{\widetilde{X}}^j_{\widetilde{\widetilde{\tau}}^j}\big)\big) = \pi^1\big(\psi^j\big((e^j_i, \rho_s(e^j_i))\big)\big) \quad \mathbb{P}_{(e^j_i, x)}\text{-a.s. on } \{\widetilde{\widetilde{\tau}}^j < \widetilde{\widetilde{\tau}}^j_j\},$$

and because ψ^j maps e^j_i to i, the definition of the transfer kernel K^j, which was summarized in equation (22.14), gives

$$K^j = k^j\big(\pi^1\big(\psi^j\big((e^j_i, \rho_s(e^j_i))\big)\big)\big) = \begin{cases} \varepsilon_{\partial_+(i)}, & i \in \mathcal{I}^j_s, \\ \varepsilon_{\partial_-(i)}, & i \in \mathcal{I}^{-j}_s. \end{cases}$$

This results in

$$\begin{aligned} & \mathbb{E}^j_g\big(e^{-\alpha\zeta^j}\, K^j h^{-j};\, \zeta^j < \tau^j_j\big) \\ & = \mathbb{E}^j_{(\psi^j)^{-1}(g)}\Big(e^{-\alpha\widetilde{\widetilde{\tau}}^j}\, k^j\big(\pi^1\big(\psi^j\big(\widetilde{\widetilde{X}}^j_{\widetilde{\widetilde{\tau}}^j}\big)\big)\big)\, h^{-j};\, \widetilde{\widetilde{\tau}}^j < \widetilde{\widetilde{\tau}}^j_j\Big) \\ & = \begin{cases} \mathbb{E}^j_{(e^j_i, x)}\Big(e^{-\alpha\widetilde{\widetilde{H}}^j_{(e^j_i, \rho_s(e^j_i))}}\, h^{-j}\big(\partial_+(i)\big);\, \widetilde{\widetilde{H}}^j_{(e^j_i, \rho_s(e^j_i))} < \widetilde{\widetilde{H}}^j_v\Big), & i \in \mathcal{I}^j_s, \\ \mathbb{E}^j_{(e^j_i, \rho(i)-x)}\Big(e^{-\alpha\widetilde{\widetilde{H}}^j_{(e^j_i, \rho_s(e^j_i))}}\, h^{-j}\big(\partial_-(i)\big);\, \widetilde{\widetilde{H}}^j_{(e^j_i, \rho_s(e^j_i))} < \widetilde{\widetilde{H}}^j_v\Big), & i \in \mathcal{I}^{-j}_s \end{cases} \\ & = \begin{cases} h^{-j}\big(\partial_+(i)\big)\, \mathbb{E}^B_x\big(e^{-\alpha\tau_{\rho(i)}};\, \tau_{\rho(i)} < \tau_0\big), & i \in \mathcal{I}^j_s, \\ h^{-j}\big(\partial_-(i)\big)\, \mathbb{E}^B_{\rho(i)-x}\big(e^{-\alpha\tau_{\rho(i)}};\, \tau_{\rho(i)} < \tau_0\big), & i \in \mathcal{I}^{-j}_s. \end{cases} \end{aligned} \tag{22.21}$$

Now, the first passage time formulas of lemma (14.5) yield

$$\mathbb{E}^B_{\rho(i)-x}\big(e^{-\alpha\tau_0}; \tau_0 < \tau_{\rho(i)}\big) = \frac{\sinh\big(\sqrt{2\alpha}\, x\big)}{\sinh\big(\sqrt{2\alpha}\, \rho(i)\big)} = \mathbb{E}^B_x\big(e^{-\alpha\tau_{\rho(i)}}; \tau_{\rho(i)} < \tau_0\big),$$

$$\mathbb{E}^B_x\big(e^{-\alpha\tau_0}; \tau_0 < \tau_{\rho(i)}\big) = \frac{\sinh\big(\sqrt{2\alpha}\, (\rho(i) - x)\big)}{\sinh\big(\sqrt{2\alpha}\, \rho(i)\big)} = \mathbb{E}^B_{\rho(i)-x}\big(e^{-\alpha\tau_{\rho(i)}}; \tau_{\rho(i)} < \tau_0\big).$$

A comparison of equations (22.20) and (22.21) then proves the equalities in (ii).

We have shown that the conditions of theorem (13.2) are fulfilled and thus have proved:

(22.22) Lemma. *The process X, obtained by forming alternating copies of X^{-1} and X^{+1} via the transfer kernels K^{-1} and K^{+1}, as defined by equation* (22.13), *is a right process on $\mathcal{G}^{-1} \cup \mathcal{G}^{+1} = \mathcal{G}$.*

22.4.6. Proving that X is a Brownian Motion on $\mathcal{G}$

As just seen, X is a right process and therefore a strong Markov process on $\mathcal{G}$. In regard to theorem (20.5), it suffices to analyze the stopped resolvent and the exit behavior from any edge in order to show that X is indeed a Brownian motion on $\mathcal{G}$:

(22.23) Lemma. *X is a Brownian motion on $\mathcal{G}$.*

Proof. For mutual edges $i \in \mathcal{I}_s$, we choose $j \in \{-1,+1\}$ such that $i \in \mathcal{I}_s^j$. Then we have $X_t = X_t^j$ for all $t < \tau_j^j$ and $X_{R^1} \in \partial(i)$, $\mathbb{P}_{(i,x)}$-a.s., so $H_X = \tau_j^j \wedge R^1$ holds true, which together with equation (22.19) yield

$$\begin{aligned}
\mathbb{E}_{(i,x)}\Big(\int_0^{H_X} e^{-\alpha t}\, f(X_t)\, dt\Big) &= \mathbb{E}^j_{(i,x)}\Big(\int_0^{\tau_j^j\wedge\zeta^j} e^{-\alpha t}\, f(X_t^j)\, dt\Big) \\
&= \mathbb{E}^B_x\Big(\int_0^{\tau_0\wedge\tau_{\rho(i)}} e^{-\alpha t}\, f\big(i, B_t\big)\, dt\Big) \\
&= \mathbb{E}^B_x\Big(\int_0^{H_B} e^{-\alpha t}\, f(i, B_t)\, dt\Big).
\end{aligned}$$

For non-mutual edges $l \notin \mathcal{I}_s$, on the other hand, choose $j \in \{-1,+1\}$ with $(l,x) \in \widetilde{\widetilde{\mathcal{G}}}{}^j$. Then $X_t^j = \widetilde{\widetilde{X}}{}_t^j$ holds for all $t < \tau_j^j$, $\mathbb{P}_{(l,x)}$-a.s., and as $\widetilde{\widetilde{X}}{}^j$ is itself Brownian motion on $\widetilde{\widetilde{\mathcal{G}}}{}^j$, the above identity follows immediately.

Coming to the exit distribution from an edge, the identity

$$\mathbb{P}_{(l,x)} \circ \big(H_X, X_{H_X}\big)^{-1} = \mathbb{P}^B_x \circ \big(H_B, (l, B_{H_B})\big)^{-1}$$

follows for edges $l \notin \mathcal{I}_s$ from the corresponding property of $\widetilde{\widetilde{X}}{}^{-1}$ or $\widetilde{\widetilde{X}}{}^{+1}$ by theorem (20.5). In case $i \in \mathcal{I}_s$, we choose again $j \in \{-1,+1\}$ with $i \in \mathcal{I}_s^j$. Then, by employing equations (22.20), (22.21) and $H_X = \tau_j^j \wedge R^1$ $\mathbb{P}_{(i,x)}$-a.s., we get for all $\alpha > 0$, $h \in b\mathscr{B}(\mathcal{G})$

$$\begin{aligned}
&\mathbb{E}_{(i,x)}\big(e^{-\alpha H_X} h(X_{H_X})\big) \\
&= \mathbb{E}^j_{(i,x)}\big(e^{-\alpha\tau_j^j} h(X^j_{\tau_j^j});\, \tau_j^j < \zeta^j\big) + \mathbb{E}^j_{(i,x)}\big(e^{-\alpha\zeta^j} K^j g;\, \zeta^j < \tau_j^j\big) \\
&= \mathbb{E}^B_x\big(e^{-\alpha\tau_0}\, h\big(\partial_-(i)\big);\, \tau_0 < \tau_{\rho(i)}\big) + \mathbb{E}^B_x\big(e^{-\alpha\tau_{\rho(i)}}\, h\big(\partial_+(i)\big);\, \tau_{\rho(i)} < \tau_0\big) \\
&= \mathbb{E}^B_x\big(e^{-\alpha H_B} h(i, B_{H_B})\big),
\end{aligned}$$

which results in

$$\mathbb{P}_{(i,x)} \circ \big(H_X, X_{H_X}\big)^{-1} = \mathbb{P}^B_x \circ \big(H_B, (i, B_{H_B})\big)^{-1}. \qquad \square$$

22.4.7. Computing the Feller–Wentzell Data of X

The Feller–Wentzell data of X, as given in Feller's theorem (20.16), is derived from its exit distributions from any arbitrarily small neighborhood of each vertex. X is constructed via alternating copies of X^{-1} and X^{+1}, so we first need to analyze their respective exit behavior. To this end, we consider the exit times of X^j

$$\tau^j_\varepsilon := \inf\{t \geq 0 : d(X^j_t, X^j_0) > \varepsilon\}$$

together with the exit distributions $X^j_{\tau^j_\varepsilon}$ for all small $\varepsilon > 0$. As we only have information on $\widetilde{\widetilde{X}}{}^j$, we need to trace back the required data to these original processes. Fix $v \in \mathcal{V}$ and choose $j \in \{-1, +1\}$ such that $v \in \mathcal{V}^j$.

Using the definition of X^j and the fact that ψ^j is an isometry, we get for all $\varepsilon > 0$

$$\begin{aligned}\tau^j_\varepsilon &= \inf\{t \geq 0 : d(\psi^j(\widetilde{X}^j_t), \psi^j(\widetilde{X}^j_0)) > \varepsilon\}\\ &= \inf\{t \geq 0 : d(\widetilde{X}^j_t, \widetilde{X}^j_0) > \varepsilon\}\\ &=: \widetilde{\tau}^j_\varepsilon.\end{aligned}$$

By its definition, $\widetilde{X}^j_t = \widetilde{\widetilde{X}}{}^j_t$ holds for all $t < \widetilde{\widetilde{\tau}}{}^j$, and as $\complement B_\varepsilon(v) \supseteq \widetilde{\mathcal{G}}^j_s$, we obtain

$$\forall \varepsilon < \delta: \quad \widetilde{\widetilde{\tau}}{}^j_\varepsilon \leq \widetilde{\widetilde{\tau}}{}^j \quad \mathbb{P}^j_v\text{-a.s.}.$$

More precisely, we even get

$$\forall \varepsilon < \delta: \quad \widetilde{\widetilde{\tau}}{}^j_\varepsilon < \widetilde{\widetilde{\tau}}{}^j, \quad \text{if} \quad \widetilde{\widetilde{\tau}}{}^j_\varepsilon \neq +\infty, \quad \mathbb{P}^j_v\text{-a.s.},$$

because

$$\begin{aligned}\mathbb{P}^j_v(\widetilde{\widetilde{\tau}}{}^j_\varepsilon = \widetilde{\widetilde{\tau}}{}^j, \widetilde{\widetilde{\tau}}{}^j_\varepsilon < +\infty) &= \mathbb{P}^j_v(\widetilde{\widetilde{\tau}}{}^j_\varepsilon = \widetilde{\widetilde{\tau}}{}^j, \widetilde{\widetilde{X}}_{\widetilde{\widetilde{\tau}}{}^j} \in \widetilde{\mathcal{G}}^j_s, \widetilde{\widetilde{\tau}}{}^j_\varepsilon < +\infty)\\ &= \mathbb{P}^j_v(\widetilde{\widetilde{\tau}}{}^j_\varepsilon = \widetilde{\widetilde{\tau}}{}^j, \widetilde{\widetilde{X}}_{\widetilde{\widetilde{\tau}}{}^j_\varepsilon} \in \widetilde{\mathcal{G}}^j_s, \widetilde{\widetilde{\tau}}{}^j_\varepsilon < +\infty)\\ &\leq \mathbb{P}^j_v(\widetilde{\widetilde{X}}_{\widetilde{\widetilde{\tau}}{}^j_\varepsilon} \in \widetilde{\mathcal{G}}^j_s) = 0.\end{aligned}$$

Therefore, we see that for all $\varepsilon < \delta$,

$$\begin{aligned}\widetilde{\widetilde{\tau}}{}^j_\varepsilon &= \inf\{t \in [0, \widetilde{\widetilde{\tau}}{}^j) : d(\widetilde{\widetilde{X}}{}^j_t, \widetilde{\widetilde{X}}{}^j_0) > \varepsilon\} \wedge \widetilde{\widetilde{\tau}}{}^j\\ &= \inf\{t \in [0, \widetilde{\widetilde{\tau}}{}^j) : d(\widetilde{X}^j_t, \widetilde{X}^j_0) > \varepsilon\} \wedge \widetilde{\widetilde{\tau}}{}^j\\ &= \inf\{t \geq 0 : d(\widetilde{X}^j_t, \widetilde{X}^j_0) > \varepsilon\}\\ &=: \widetilde{\tau}^j_\varepsilon,\end{aligned}$$

where we used that $\widetilde{X}^j$ is a subprocess of $\widetilde{\widetilde{X}}{}^j$ with lifetime $\widetilde{\widetilde{\tau}}{}^j$, that is

$$d(\widetilde{X}^j_{\widetilde{\widetilde{\tau}}{}^j}, \widetilde{X}^j_0) = d(\Delta, \widetilde{X}^j_0) = +\infty > \varepsilon.$$

We have thus shown:

(22.24) Lemma. *Let $v \in \mathcal{V}^j$. For all $\varepsilon < \delta$, it holds $\mathbb{P}_v^j$-a.s. that*

$$\tau_\varepsilon^j = \widetilde{\tau}_\varepsilon^j = \widetilde{\widetilde{\tau}}_\varepsilon^j,$$

and

$$\widetilde{\widetilde{\tau}}_\varepsilon^j < \widetilde{\widetilde{\tau}}^j, \quad \text{if} \quad \widetilde{\widetilde{\tau}}_\varepsilon^j < +\infty.$$

(22.25) Corollary. *For all $v \in \mathcal{V}^j$, $\varepsilon < \delta$, the exit distribution of X^j is given by*

$$X^j_{\tau_\varepsilon^j} = \begin{cases} \psi^j(\widetilde{\widetilde{X}}{}^j_{\widetilde{\widetilde{\tau}}_\varepsilon^j}), & \widetilde{\widetilde{\tau}}_\varepsilon^j < +\infty, \\ \Delta, & \widetilde{\widetilde{\tau}}_\varepsilon^j = +\infty. \end{cases}$$

We are ready to compute the Feller–Wentzell data of X. By lemmas (22.24) and (22.12), we have for all $\varepsilon < \delta$

$$\tau_\varepsilon^j = \widetilde{\widetilde{\tau}}_\varepsilon^j < \widetilde{\widetilde{\tau}}^j = \zeta^j \qquad \text{on } \{\zeta^j < +\infty\},$$

so $\tau_\varepsilon^j < \zeta^j$ a.s. holds. On the other hand, $X_t = X_t^j$ holds for all $t < R^1 = \zeta^j$ (more formally, $X_t^j(\omega_i) = X_t\big((\omega_1, \omega_2, \ldots)\big)$ with $i = 1$ if $j = -1$, and $i = 2$ if $j = +1$) by the construction of X, yielding

$$\mathbb{P}_v \circ \big(\tau_\varepsilon, X_{\tau_\varepsilon}\big)^{-1} = \mathbb{P}_v^j \circ \big(\tau_\varepsilon^j, X^j_{\tau_\varepsilon^j}\big)^{-1}.$$

Thus, if v is not a trap, then $\widetilde{\widetilde{\tau}}_\varepsilon^j < +\infty$ holds $\mathbb{P}_v^j$-a.s. for all sufficiently small $\varepsilon > 0$ (see theorem (5.16)), and therefore $\tau_\varepsilon < +\infty$ holds $\mathbb{P}_v$-a.s. as well. By using the notations of Feller's theorem (20.16) and backtracking X to $\widetilde{\widetilde{X}}{}^j$, we compute for $\varepsilon < \delta$

$$\begin{aligned} \forall A \in \mathscr{B}(\mathcal{G}\backslash\{v\}): \quad \nu_\varepsilon^v(A) &= \frac{\mathbb{P}_v(X_{\tau_\varepsilon} \in A)}{\mathbb{E}_v(\tau_\varepsilon)} \\ &= \frac{\mathbb{P}_v^j(X^j_{\tau_\varepsilon^j} \in A)}{\mathbb{E}_v^j(\tau_\varepsilon^j)} \\ &= \frac{\mathbb{P}_v^j\big(\psi^j(\widetilde{\widetilde{X}}{}^j_{\widetilde{\widetilde{\tau}}_\varepsilon^j}) \in A\big)}{\mathbb{E}_v^j(\widetilde{\widetilde{\tau}}_\varepsilon^j)} \\ &= \widetilde{\widetilde{\nu}}_\varepsilon^{j,v}\big((\psi^j)^{-1}(A)\big), \end{aligned}$$

where we naturally extend, here and in all that follows, the mapping $\psi^j : \widetilde{\mathcal{G}}^j \to \mathcal{G}^j$ to $\psi^j : \widetilde{\mathcal{G}}^j \to \mathcal{G}$. This gives

$$\begin{aligned} K_\varepsilon^v &= 1 + \frac{\mathbb{P}_v(X_{\tau_\varepsilon} = \Delta)}{\mathbb{E}_v(\tau_\varepsilon)} + \int_{\mathcal{G}\backslash\{v\}} \big(1 - e^{-d(v,g)}\big)\, \nu_\varepsilon^v(dg) \\ &= 1 + \frac{\mathbb{P}_v^j(\widetilde{\widetilde{X}}{}^j_{\widetilde{\widetilde{\tau}}_\varepsilon^j} = \Delta)}{\mathbb{E}_v^j(\widetilde{\widetilde{\tau}}_\varepsilon^j)} + \int_{\widetilde{\mathcal{G}}^j\backslash\{v\}} \big(1 - e^{-d(v,\psi^j(g))}\big)\, \widetilde{\widetilde{\nu}}_\varepsilon^{j,v}(dg) \\ &= 1 + \frac{\mathbb{P}_v^j(\widetilde{\widetilde{X}}{}^j_{\widetilde{\widetilde{\tau}}_\varepsilon^j} = \Delta)}{\mathbb{E}_v^j(\widetilde{\widetilde{\tau}}_\varepsilon^j)} + \int_{\widetilde{\widetilde{\mathcal{G}}}^j\backslash\{v\}} \big(1 - e^{-d(v,g)}\big)\, \widetilde{\widetilde{\nu}}_\varepsilon^{j,v}(dg) \\ &= \widetilde{\widetilde{K}}_\varepsilon^{j,v}, \end{aligned}$$

because ψ^j is an isometry with $\psi^j(v) = v$, and as $\widetilde{\nu}^v_\varepsilon(\widetilde{\widetilde{\mathcal{G}}}^j \backslash \widetilde{\mathcal{G}}^j) = 0$ holds due to the basic assumption (22.8). Renormalization yields, again because ψ is an isometry,

$$\begin{aligned}
\forall A \in \mathscr{B}(\mathcal{G}\backslash\{v\}): \quad \mu^v_\varepsilon(A) &= \int_A \big(1 - e^{-d(v,g)}\big) \frac{\nu^v_\varepsilon(dg)}{K^v_\varepsilon} \\
&= \int_{\psi^{-1}(A)} \big(1 - e^{-d(v,\psi(g))}\big) \frac{\widetilde{\nu}^{j,v}_\varepsilon(dg)}{\widetilde{\widetilde{K}}^v_\varepsilon} \\
&= \widetilde{\mu}^{j,v}_\varepsilon\big((\psi^j)^{-1}(A)\big).
\end{aligned}$$

Next, introduce the topological subspaces $\overline{\widetilde{\mathcal{G}}^j\backslash\{v\}}$ of $\overline{\widetilde{\widetilde{\mathcal{G}}}^j\backslash\{v\}}$ and $\overline{\mathcal{G}^j\backslash\{v\}}$ of $\overline{\mathcal{G}\backslash\{v\}}$, and consider the continuous extension from $\psi^j : \widetilde{\mathcal{G}}^j \to \mathcal{G}^j$ to $\overline{\psi}^j : \overline{\widetilde{\mathcal{G}}^j\backslash\{v\}} \to \overline{\mathcal{G}\backslash\{v\}}$. Continuity of $\overline{\psi}^j$ dictates that the new points $\overline{\widetilde{\mathcal{G}}^j\backslash\{v\}}\backslash\widetilde{\mathcal{G}}^j$ are mapped to

$$\begin{aligned}
i \in \mathcal{I}^j_s(v): \quad & \overline{\psi}^j\big((e^j_i, 0+)\big) = \lim_{x \downarrow\downarrow 0} \psi^j\big((e^j_i, x)\big) = \lim_{x \downarrow\downarrow 0} (i, x) = (i, 0+), \\
& \overline{\psi}^j\big((e^j_i, \rho_i-)\big) = \lim_{x \uparrow\uparrow \rho(i)} \psi^j\big((e^j_i, x)\big) = \lim_{x \uparrow\uparrow \rho(i)} (i, x) = (i, \rho_i), \\
i \in \mathcal{I}^{-j}_s(v): \quad & \overline{\psi}^j\big((e^j_i, 0+)\big) = \lim_{x \downarrow\downarrow 0} \psi^j\big((e^j_i, x)\big) = \lim_{x \downarrow\downarrow 0} (i, \rho_i - x) = (i, \rho_i), \\
& \overline{\psi}^j\big((e^j_i, \rho_i-)\big) = \lim_{x \uparrow\uparrow \rho(i)} \psi^j\big((e^j_i, x)\big) = \lim_{x \uparrow\uparrow \rho(i)} (i, \rho_i - x) = (i, 0+),
\end{aligned} \tag{22.26}$$

and analogously

$$\begin{aligned}
i \in \mathcal{I}^j(v): \quad & \overline{\psi}^j\big((i, 0+)\big) = (i, 0+), \quad \text{if } v = \partial_-(i), \\
& \overline{\psi}^j\big((i, \rho_i-)\big) = (i, \rho_i-), \ \text{if } v = \partial_+(i), \\
e \in \mathcal{E}^j(v): \quad & \overline{\psi}^j\big((e, 0+)\big) = (e, 0+), \\
e \in \mathcal{E}^j: \quad & \overline{\psi}^j\big((e, +\infty)\big) = (e, +\infty).
\end{aligned} \tag{22.27}$$

Proceeding in the course of the proof of Feller's theorem (20.16) for $\widetilde{\widetilde{X}}^j$, we extend the measures $\widetilde{\mu}^{j,v}_\varepsilon$ to measures $\overline{\widetilde{\widetilde{\mu}}}^{j,v}_\varepsilon$ on $\overline{\widetilde{\widetilde{\mathcal{G}}}^j\backslash\{v\}}$ by

$$\overline{\widetilde{\widetilde{\mu}}}^{j,v}_\varepsilon(A) := \widetilde{\mu}^v_\varepsilon\big(A \cap (\widetilde{\widetilde{\mathcal{G}}}^j\backslash\{v\})\big), \quad A \in \mathscr{B}(\overline{\widetilde{\widetilde{\mathcal{G}}}^j\backslash\{v\}}),$$

and choose a sequence of positive numbers $(\varepsilon_n, n \in \mathbb{N})$ converging to zero, such that $(\overline{\widetilde{\widetilde{\mu}}}^{j,v}_{\varepsilon_n}, n \in \mathbb{N})$ converges weakly to a measure $\overline{\widetilde{\widetilde{\mu}}}^{j,v}$. When also extending the measures μ^v_ε to measures $\overline{\mu}^v_\varepsilon$ on $\overline{\mathcal{G}\backslash\{v\}}$, we obtain

$$\begin{aligned}
\forall A \in \mathscr{B}(\overline{\mathcal{G}\backslash\{v\}}): \quad \overline{\mu}^v_\varepsilon(A) &:= \mu^v_\varepsilon\big(A \cap (\mathcal{G}\backslash\{v\})\big) \\
&= \widetilde{\mu}^{j,v}_\varepsilon\big((\psi^j)^{-1}(A \cap (\mathcal{G}\backslash\{v\})\big) \\
&= \widetilde{\mu}^{j,v}_\varepsilon\big((\overline{\psi}^j)^{-1}(A) \cap (\widetilde{\mathcal{G}}^j\backslash\{v\})\big) \\
&= \widetilde{\mu}^{j,v}_\varepsilon\big((\overline{\psi}^j)^{-1}(A) \cap (\widetilde{\widetilde{\mathcal{G}}}^j\backslash\{v\})\big) \\
&= \overline{\widetilde{\widetilde{\mu}}}^{j,v}_\varepsilon \circ (\overline{\psi}^j)^{-1}(A).
\end{aligned}$$

Then, by the continuous mapping theorem, $(\bar{\mu}^v_{\varepsilon_n}, n \in \mathbb{N})$ converges weakly to the measure

$$\bar{\mu}^v = \bar{\tilde{\tilde{\mu}}}^{j,v} \circ (\bar{\psi}^j)^{-1}$$

on $\overline{\mathcal{G}\backslash\{v\}}$. We summarize all our results up to this point:

(22.28) Lemma. *Let $v \in \mathcal{V}^j$, and K^v_ε, μ^v_ε, $\bar{\mu}^v$ and $\tilde{\tilde{K}}^{j,v}_\varepsilon$, $\tilde{\tilde{\mu}}^{j,v}$, $\bar{\tilde{\tilde{\mu}}}^{j,v}$ be defined as in Feller's theorem* (20.16) *for the Brownian motions X, $\tilde{\tilde{X}}^j$ respectively. Then,*

(i) $K^v_\varepsilon = \tilde{\tilde{K}}^{j,v}_\varepsilon$ *for all $\varepsilon < \delta$,*

(ii) $\mu^v_\varepsilon = \tilde{\tilde{\mu}}^{j,v}_\varepsilon \circ (\psi^j)^{-1}$ *for all $\varepsilon < \delta$,*

(iii) $(\mu^v_{\varepsilon_n}, n \in \mathbb{N})$ *converges weakly along the same sequence $(\varepsilon_n, n \in \mathbb{N})$ of that positive numbers for which $(\tilde{\tilde{\mu}}^{j,v}_{\varepsilon_n}, n \in \mathbb{N})$ converges weakly to $\bar{\tilde{\tilde{\mu}}}^{j,v}$, and its limit reads*

$$\bar{\mu}^v = \bar{\tilde{\tilde{\mu}}}^{j,v} \circ (\bar{\psi}^j)^{-1}.$$

We are now ready to compute the Feller–Wentzell data of the glued process X, thus completing the proof of theorem (22.9):

Proof of theorem (22.9). We have already proved in lemma (22.23) that X is a Brownian motion on $\mathcal{G}$. It only remains to compute the Feller–Wentzell data of X by employing lemma (22.28). To this end, let $v \in \mathcal{V}$ and choose $j \in \{-1, +1\}$ such that $v \in \mathcal{V}^j$.

The killing parameters are given by

$$\begin{aligned}
c_1^{v,\Delta} &= \lim_{n\to\infty} \frac{\mathbb{P}_v(X_{\tau_{\varepsilon_n}} = \Delta)}{\mathbb{E}_v(\tau_{\varepsilon_n}) K^v_{\varepsilon_n}} = \lim_{n\to\infty} \frac{\mathbb{P}^j_v(\tilde{\tilde{X}}^j_{\tilde{\tilde{\tau}}^j_{\varepsilon_n}} = \Delta)}{\mathbb{E}^j_v(\tilde{\tilde{\tau}}^j_{\varepsilon_n}) \tilde{\tilde{K}}^{j,v}_{\varepsilon_n}} = p_1^{v,\Delta}, \\
c_1^{v,\infty} &= \sum_{e\in\mathcal{E}} \bar{\mu}^v\big(\{(e,+\infty)\}\big) = \sum_{e\in\mathcal{E}^j\cup\mathcal{E}^j_s} \bar{\tilde{\tilde{\mu}}}^{j,v}\big(\{(e,+\infty)\}\big) = p_1^{v,\infty},
\end{aligned}$$

and thus vanish, as $p_1^v = p_1^{v,\Delta} + p_1^{v,\infty} = 0$ holds by assumption.

The reflection parameters are defined as

$$c_2^{v,l} = \begin{cases} \bar{\mu}^v\big(\{(l,0+)\}\big), & l \in \mathcal{E}(v), \\ \bar{\mu}^v\big(\{(l,0+)\}\big), & l \in \mathcal{I}(v), v = \partial_-(l), \\ \bar{\mu}^v\big(\{(l,\rho_l-)\}\big), & l \in \mathcal{I}(v), v = \partial_+(l). \end{cases}$$

For $e \in \mathcal{E}(v)$, the relation $(\bar{\psi}^j)^{-1}\big((e,0+)\big) = (e,0+)$ immediately yields $c_2^{v,e} = p_2^{v,e}$. For $i \in \mathcal{I}(v)$, we need to distinguish some cases, using equations (22.26) and (22.27): For $i \in \mathcal{I}(v)$ with $v = \partial_-(i)$, that is if $i \in \mathcal{I}^j(v) \cup \mathcal{I}^j_s(v)$, we have

$$c_2^{v,i} = \bar{\mu}^v\big(\{(i,0+)\}\big) = \begin{cases} \bar{\tilde{\tilde{\mu}}}^{j,v}\big(\{(i,0+)\}\big) = p_2^{v,i}, & i \in \mathcal{I}^j(v), \\ \bar{\tilde{\tilde{\mu}}}^{j,v}\big(\{(e^j_i,0+)\}\big) = p_2^{v,e^j_i}, & i \in \mathcal{I}^j_s(v), \end{cases}$$

while for $i \in \mathcal{I}(v)$ with $v = \partial_+(i)$, that is if $i \in \mathcal{I}^{-j}(v) \cup \mathcal{I}_s^{-j}(v)$, we have

$$c_2^{v,i} = \bar{\mu}^v(\{(i, \rho_i-)\}) = \begin{cases} \widetilde{\widetilde{\mu}}^{j,v}(\{(i, \rho_i-)\}) = p_2^{v,i}, & i \in \mathcal{I}^{-j}(v), \\ \widetilde{\widetilde{\mu}}^{j,v}(\{(e_i^j, 0+)\}) = p_2^{v,e_i^j}, & i \in \mathcal{I}_s^{-j}(v). \end{cases}$$

The diffusion parameter is given by

$$c_3^v = \lim_{n\to\infty} \frac{1}{K_{e_n}^v} = \lim_{n\to\infty} \frac{1}{\widetilde{\widetilde{K}}_{e_n}^{j,v}} = p_3^v.$$

For all $A \in \mathscr{B}(\mathcal{G}\backslash\{v\})$, the jump distribution is computed by

$$\begin{aligned} c_4^v(A) &= \int_A \frac{1}{1 - e^{-d(v,g)}} \bar{\mu}^v(dg) \\ &= \int_{(\psi^j)^{-1}(A)} \frac{1}{1 - e^{-d(v,\psi^j(g))}} \widetilde{\widetilde{\mu}}^{j,v}(dg) \\ &= p_4^v \circ (\psi^j)^{-1}(A), \end{aligned}$$

as $\overline{\psi^j}$ is an extension from $\psi^j : \widetilde{\mathcal{G}}^j \to \mathcal{G}$ and an isometry. □

22.5. Completing the Construction

We are ready to carry out the program that was laid out in subsection 22.1.

(22.29) Theorem. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{I}, \partial, \rho)$ be a metric graph, and for every $v \in \mathcal{V}$ let constants $p_2^{v,l} \geq 0$ for $l \in \mathcal{L}(v)$, $p_3^v \geq 0$ and a measure p_4^v on $\mathcal{G}\backslash\{v\}$ be given, satisfying*

$$\sum_{l\in\mathcal{L}(v)} p_2^{v,l} + p_3^v + \int_{\mathcal{G}\backslash\{v\}} (1 - e^{-d(v,g)})\, p_4^v(dg) = 1,$$

and

$$p_4^v(\mathcal{G}\backslash\{v\}) = +\infty, \quad \text{if} \quad \sum_{l\in\mathcal{L}(v)} p_2^{v,l} + p_3^v = 0,$$

as well as $p_4^v(\complement\overline{B_\delta(v)}) = 0$ for some $\delta \in (0, \min_{l\in\mathcal{L}} \rho_l)$. Then there exists a Brownian motion X on $\mathcal{G}$ which has infinite lifetime, is continuous inside all edges, satisfies $X_{\tau_\varepsilon} \in B_\delta(v)$ $\mathbb{P}_v$-a.s. for all $\varepsilon < \delta$, $v \in \mathcal{V}$, and admits the Feller–Wentzell data

$$\big(0, (p_2^{v,l})_{l\in\mathcal{L}(v)}, p_3^v, p_4^v\big)_{v\in\mathcal{V}}.$$

Proof. We proceed via an induction over the count $n := |\mathcal{V}|$ of vertices. If $n = 1$, then $\mathcal{G}$ is a star graph, and the construction of section 21 together with theorems (21.61), (21.66) and lemma (21.12) yields the result.

Assume now that such Brownian motions exist for all metric graphs with less than n vertices. Let $\mathcal{G}$ be a metric graph with n vertices $\mathcal{V} = \{v_1, \ldots, v_n\}$ and boundary data as given in the theorem. Decompose the graph into $\widetilde{\widetilde{\mathcal{G}}}^{-1}$ and $\widetilde{\widetilde{\mathcal{G}}}^{+1}$, as done in subsection 22.4, for $\mathcal{V}^{-1} = \{v_1, \ldots, v_{n-1}\}$ and $\mathcal{V}^{+1} = \{v_n\}$. Then the conditions of the theorem are satisfied for these graphs $\widetilde{\widetilde{\mathcal{G}}}^{-1}$, $\widetilde{\widetilde{\mathcal{G}}}^{+1}$ with $n-1$ vertices, one vertex respectively, and corresponding boundary data $(p_2^{v,l} \geq 0, l \in \widetilde{\widetilde{\mathcal{L}}}^j(v))$, $p_3^v \geq 0$ and $p_4^v \circ \psi^j$ (as ψ^j is an isometry, this data satisfies the normalization requirements). Therefore, there exist Brownian motions $\widetilde{\widetilde{X}}^j$ on $\widetilde{\widetilde{\mathcal{G}}}^j$ with infinite lifetime which are continuous inside all edges, satisfy $\widetilde{\widetilde{X}}^j_{\widetilde{\widetilde{\tau}}^j_\varepsilon} \in B_\delta(v)$ $\mathbb{P}_v^j$-a.s. for all $v \in \mathcal{V}^j$ and admit the Feller–Wentzell data

$$\big(0, (p_2^{v,l})_{l \in \widetilde{\widetilde{\mathcal{L}}}^j(v)}, p_3^v, p_4^v \circ \psi^j\big)_{v \in \mathcal{V}^j}$$

with $p^{v,e_i^j} := p_2^{v,i}$ for $i \in \mathcal{I}_s(v)$, $v \in \mathcal{V}^j$. We then follow the construction of subsection 22.4 in order to glue $\widetilde{\widetilde{X}}^{-1}$ and $\widetilde{\widetilde{X}}^{+1}$ together, and theorem (22.9) concludes the proof. $\square$

In order to implement the killing parameter and the "global" jumps, we first need to adjoin the "fake cemeteries" $\square^v$, $v \in \mathcal{V}$:

(22.30) Theorem. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{I}, \partial, \rho)$ be a metric graph, and for every $v \in \mathcal{V}$ let constants $p_1^v \geq 0$, $p_2^{v,l} \geq 0$ for $l \in \mathcal{L}(v)$, $p_3^v \geq 0$ and a measure p_4^v on $\mathcal{G} \backslash \{v\}$ be given with*

$$p_1^v + \sum_{l \in \mathcal{L}(v)} p_2^{v,l} + p_3^v + \int_{\mathcal{G} \backslash \{v\}} \big(1 - e^{-d(v,g)}\big)\, p_4^v(dg) = 1,$$

and

$$p_4^v(\mathcal{G} \backslash \{v\}) = +\infty, \quad \text{if} \quad \sum_{l \in \mathcal{L}(v)} p_2^{v,l} + p_3^v = 0,$$

as well as $p_4^v\big(\complement \overline{B_\delta(v)}\big) = 0$ for some $\delta \in \big(0, \min_{l \in \mathcal{L}} \rho_l\big)$. Then there exists a Brownian motion X on $\mathcal{G} \cup \{\square^v, v \in \mathcal{V}\}$ with $\{\square^v, v \in \mathcal{V}\}$ being an isolated, absorbing set for X, such that X has infinite lifetime, is continuous inside all edges, satisfies $X_{\tau_\varepsilon} \in B_\delta(v) \cup \{\square^v\}$ $\mathbb{P}_v$-a.s. for all $\varepsilon < \delta$, $v \in \mathcal{V}$, and admits the Feller–Wentzell data

$$\big(0, (p_2^{v,l})_{l \in \mathcal{L}(v)}, p_3^v, p_4^v + p_1^v\, \varepsilon_{\square^v}\big)_{v \in \mathcal{V}}.$$

Proof. This proof proceeds analogously to the proof of theorem (22.29), except that we need to adjoin the isolated points $\square^v$, $v \in \mathcal{V}$, to the partial processes and revive them there before gluing the partial graphs together.

If $|\mathcal{V}| = 1$, then $\mathcal{G}$ is a star graph, and the construction of section 21 (again with theorems (21.61), (21.66), and lemma (21.12)) gives a Brownian motion on $\mathcal{G}$ with the needed properties and Feller–Wentzell data

$$\big(p_1^v, (p_2^{v,l})_{l \in \mathcal{L}(v)}, p_3^v, p_4^v\big).$$

By concatenating it with the constant process on $\{\square^v\}$ with the technique of subsection 11.1, revive this Brownian motion on a new, isolated, absorbing point $\square^v$. Then a computation which exactly follows the proof of lemma (22.5) yields that the revived process is a Brownian motion on $\mathcal{G} \cup \{\square^v\}$ and its Feller–Wentzell data at v reads

$$\big(0, (p_2^{v,l})_{l \in \mathcal{L}(v)}, p_3^v, p_4^v + p_1^v \, \varepsilon_{\square^v}\big).$$

Now let $\mathcal{V} = \{v_1, \dots, v_n\}$, and assume that the assertion of the theorem holds for any graph with less than n vertices. Then decompose the graph $\mathcal{G}$ into $\widetilde{\widetilde{\mathcal{G}}}^{-1}$ and $\widetilde{\widetilde{\mathcal{G}}}^{+1}$, as done in subsection 22.4, for $\mathcal{V}^{-1} = \{v_1, \dots, v_{n-1}\}$ and $\mathcal{V}^{+1} = \{v_n\}$. By assumption, there exist Brownian motions $\widetilde{\widetilde{X}}^j$ on $\widetilde{\widetilde{\mathcal{G}}}^j \cup \{\square^v, v \in \mathcal{V}^j\}$ with the needed path properties and Feller–Wentzell data

$$\big(0, (p_2^{v,l})_{l \in \widetilde{\widetilde{\mathcal{L}}}^j(v)}, p_3^v, p_4^v \circ \psi^j + p_1^v \, \varepsilon_{\square^v}\big)_{v \in \mathcal{V}^j}$$

with $p^{v,e_i^j} := p_2^{v,i}$ for $i \in \mathcal{I}_s(v)$, $v \in \mathcal{V}^j$. We then again follow the construction of subsection 22.4 to glue $\widetilde{\widetilde{X}}^{-1}$ and $\widetilde{\widetilde{X}}^{+1}$ together, and theorem (22.9) yields the result. $\square$

It remains to implement the "global" jumps:

(22.31) Theorem. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{I}, \partial, \rho)$ be a metric graph, and for every $v \in \mathcal{V}$ let constants $p_1^v \geq 0$, $p_2^{v,l} \geq 0$ for $l \in \mathcal{L}(v)$, $p_3^v \geq 0$ and a measure p_4^v on $\mathcal{G} \backslash \{v\}$ be given with*

$$p_1^v + \sum_{l \in \mathcal{L}(v)} p_2^{v,l} + p_3^v + \int_{\mathcal{G} \backslash \{v\}} \big(1 - e^{-d(v,g)}\big) \, p_4^v(dg) = 1,$$

and

$$p_4^v(\mathcal{G} \backslash \{v\}) = +\infty, \quad \textit{if} \quad \sum_{l \in \mathcal{L}(v)} p_2^{v,l} + p_3^v = 0.$$

Then there exists a Brownian motion X on $\mathcal{G}$ which is continuous inside all edges, such that its generator satisfies

$$\begin{aligned} \mathscr{D}(A) \subseteq \Big\{ & f \in \mathcal{C}_0^2(\mathcal{G}) : \forall v \in \mathcal{V} : \\ & p_1^v f(v) - \sum_{l \in \mathcal{L}(v)} p_2^{v,l} \, f_l'(v) + \frac{p_3^v}{2} f''(v) - \int_{\mathcal{G} \backslash \{v\}} \big(f(g) - f(v)\big) \, p_4^v(dg) = 0 \Big\}. \end{aligned}$$

Proof. Let $\delta > 0$ with $\delta < \min_{l \in \mathcal{L}} \rho_l$, and define for every $v \in \mathcal{V}$

$$q_1^v := p_1^v + p_4^v(\complement B_v(\delta)), \quad q_4^v := p_4^v|_{B_v(\delta)}.$$

Then, introducing the normalizing factor

$$c_0^v := \Big(q_1^v + \sum_{l \in \mathcal{L}(v)} p_2^{v,l} + p_3^v + \int_{\mathcal{G} \backslash \{v\}} \big(1 - e^{-d(v,g)}\big) \, q_4^v(dg) \Big)^{-1}$$

enables us to employ theorem (22.30) in order to construct a Brownian motion X^1 on $\mathcal{G} \cup \{\Box^v, v \in \mathcal{V}\}$ which has infinite lifetime, is continuous inside all edges, satisfies $X_{\tau_\varepsilon} \in B_\delta(v) \cup \{\Box^v\}$ $\mathbb{P}_v$-a.s. for all $\varepsilon < \delta$, $v \in \mathcal{V}$, and has the Feller–Wentzell data

$$\big(c_0^v\,(0, (p_2^{v,l})_{l \in \mathcal{L}(v)}, p_3^v, q_4^v + q_1^v\,\varepsilon_{\Box^v})\big)_{v \in \mathcal{V}}.$$

As X^1 has infinite lifetime, we can use the construction of subsection 11.4 to adjoin a new, isolated, absorbing point $\Box$ to X^1, resulting in a Brownian motion X^2 on the metric graph $\mathcal{G} \cup \{\Box^v, v \in \mathcal{V}\} \cup \{\Box\}$ with the same Feller–Wentzell data as X^1 for all $v \in \mathcal{V}$, and additional Feller–Wentzell data $(0,0,1,0)$ at the new vertex $\Box$.

Let X^3 be the right process on $\mathcal{G} \cup \{\Box\}$ which results from killing X^2 on the absorbing set $\{\Box^v, v \in \mathcal{V}\}$ (see subsection 12.2). As X^3 is strong Markov and $X^3_t = X^2_t$ for all $t \leq H_{\mathcal{V}}$, X^3 is a Brownian motion on $\mathcal{G} \cup \{\Box\}$, and lemma (22.3) asserts that the Feller–Wentzell data of X^3 reads

$$\big(c_0^v\,(q_1^v, (p_2^{v,l})_{l \in \mathcal{L}(v)}, p_3^v, q_4^v)\big)_{v \in \mathcal{V}}.$$

Now construct X^4 as the revived process obtained from X^3 by the identical copies method of subsection 13.1 with revival distributions

$$\kappa^v := (q_1^v)^{-1}\,\big(p_1^v\,\varepsilon_\Box + p_4^v|_{\complement B_\delta(v)}\big), \quad v \in \mathcal{V}.$$

Then by lemma (22.5), X^4 is a Brownian motion on $\mathcal{G} \cup \{\Box\}$, and its generator satisfies

$$\begin{aligned}
\mathscr{D}(A) \subseteq \Big\{ f \in \mathcal{C}_0^2(\mathcal{G} \cup \{\Box\}) : \forall v \in \mathcal{V} :& \\
&- \sum_{l \in \mathcal{L}(v)} c_0^v\, p_2^{v,l}\, f_l'(v) + \frac{c_0^v\, p_3^v}{2} f''(v) \\
&- \int_{(\mathcal{G} \setminus \{v\}) \cup \{\Box\}} (f(g) - f(v))\, c_0^v (p_4^v|_{B_\delta(v)} + p_1^v\,\varepsilon_\Box + p_4^v|_{B_\delta(v)^\complement})(dg) = 0 \Big\} \\
= \Big\{ f \in \mathcal{C}_0^2(\mathcal{G} \cup \{\Box\}) : \forall v \in \mathcal{V} :& \\
&- \sum_{l \in \mathcal{L}(v)} p_2^{v,l}\, f_l'(v) + \frac{p_3^v}{2} f''(v) \\
&- \int_{(\mathcal{G} \setminus \{v\}) \cup \{\Box\}} (f(g) - f(v))\, (p_4^v + p_1^v\,\varepsilon_\Box)(dg) = 0 \Big\}.
\end{aligned}$$

Finally, employ again the construction of subsection 12.2 in order to kill X^4 on the isolated, absorbing set $\{\Box\}$ to obtain the Brownian motion X^5 on $\mathcal{G}$. Lemma (22.2) asserts that the domain of its generator satisfies

$$\begin{aligned}
\mathscr{D}(A) \subseteq \Big\{ f \in \mathcal{C}_0^2(\mathcal{G}) : \forall v \in \mathcal{V} :& \\
&p_1^v f(v) - \sum_{l \in \mathcal{L}(v)} p_2^{v,l}\, f_l'(v) + \frac{p_3^v}{2} f''(v) - \int_{\mathcal{G} \setminus \{v\}} (f(g) - f(v))\, p_4^v(dg) = 0 \Big\}. \quad \Box
\end{aligned}$$

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