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Asset Prices and Alternative Characterizations of the Pricing Kernel

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Non-Technical Summary

There is empirical evidence that stock prices are not governed by a geometric Brownian motion. For example, empirical studies report time-varying expected returns and time-varying volatilities which contradicts the assumption that asset prices are governed by a geometric Brownian motion.

In spite of this empirical evidence we still lack a model for stock prices which is consistent with empirical findings and has a sound economic foundation. In this paper we propose the displaced diffusion process as an alternative to the geometric Brownian motion. The advantage of this model is that it is consistent with a representative investor economy and it is flexible enough to be consistent with mean reversion, respectively mean aversion, and time-varying volatilities.

In addition, this paper illustrates how investors preferences influence asset price processes.

Asset Prices and Alternative Characterizations of the Pricing Kernel[1]

Erik Lüders[2]

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Abstract

In a continuous-time representative investor economy with an exogenously given information process, asset prices are derived for alternative characterizations of the pricing kernel. In addition to the characterization of forward prices in a general representative investor economy a detailed analysis of forward prices for the HARA-class is given. In particular, analytical and numerical solutions of forward prices are derived for a representative investor with non-constant relative risk aversion. The derived asset prices are consistent with empirically well documented characteristics as mean reversion and random volatility. Hence, they are viable alternatives to the geometric Brownian motion.

Keywords: equilibrium price processes, displaced diffusion process, random volatility, mean-reversion

JEL Classification:G12

1 Introduction

This paper analyzes asset price processes for alternative characterizations of the pricing kernel. Analytical solutions for the asset price are derived in a representative investor economy with the representative investor's utility function being from the hyperbolic absolute risk aversion (HARA) class. The HARA-class is chosen for two reasons. First, these utility functions are widely used in financial economics and secondly, they include the case of constant and non-constant relative risk aversion. Thus, this paper contributes to the literature on the economic foundation of stochastic processes for asset price processes. Especially the impact on the price process of different assumptions on the representative investor's utility function is analyzed.

Recent articles have analyzed the relationship between preferences and equilibrium asset prices. The papers of Bick [2] and [3], He and Leland [24], Hodges and Carverhill [27], Hodges and Selby [26] and Pham and Touzi [47] start from exogenously given asset price processes and investigate whether these price processes can be supported by a representative investor economy with a state-independent utility function. In addition they provide results to the question which utility functions are consistent with which kind of asset price processes. These papers show that asset price processes imply strong assumptions about preferences.[1] The papers of Franke, Stapleton and Subrahmanyam [19] and Lüders and Peisl [40] differ from the aforementioned mainly because they introduce an exogenous information process as a starting point. This information process models the process of investors' expectations about the final value of the asset and, thus, determines the distribution of the terminal value of the asset. Hence, these papers start from an exogenously given distribution of the final value and derive the forward price of this asset from the assumption on the pricing kernel, respectively the representative investor's utility function. The results in Franke, Stapleton and Subrahmanyam [19] and Lüders and Peisl [40] are more constructive than previous results because they show how expectations and preferences yield certain asset price processes. For example, the geometric Brownian motion as assumed in Black and Scholes [4] is discussed and it is shown that an information process of the geometric Brownian motion type and a representative investor with constant relative risk aversion yield such an asset price process. In particular, the necessary and sufficient conditions for a geometric

[1] See also Cuoco and Zapatero [12] and Wang [52].

Brownian motion are derived.

This paper is closely related to Franke, Stapleton and Subrahmanyam [19] and Lüders and Peisl [40]. Again, the exogenous information process is the starting point. However, in contrast to Lüders and Peisl [40] the main focus is on the impact of the representative investor's utility function on asset prices. Furthermore, while Franke, Stapleton and Subrahmanyam [19] provide strong results showing the relationship between asset prices and preferences, in this paper analytical solutions for the asset price are derived. Assuming a displaced diffusion process for the information process we get closed form solutions for the equilibrium asset price when the representative investor is not constant relative risk averse. Moreover, in these models asset returns depend on the level of the asset price. Hence, these models may provide some explanation for the predictability of asset returns, a stylized fact which is in contrast to the geometric Brownian motion, as well as the empirically well documented fact that asset prices are not two-parameter lognormally distributed.[2] The paper is also closely related to Camara [7] and Rubinstein [50]. In Camara [7] risk neutral valuation relationships are derived for three-parameter lognormally distributed variables in a discrete time model. The displaced diffusion process assumed in this paper implies that the final value is three-parameter lognormally distributed. Rubinstein [50] proposes the displaced diffusion process to model the value of firms. His main argument in favor of the displaced diffusion is that the assets of a firm may be decomposed into those that are relatively risky and those that are relatively riskless. Hence, Rubinstein [50] provides a justification for the information process assumed in parts of this paper. An additional aspect of our paper is that it gives a utility theoretic foundation for the assumption that an asset price is governed by a displaced diffusion process since it is shown that these processes are consistent with non-constant relative risk aversion.

This paper is organized as follows. In section 2 asset prices are discussed in a continuous-time arbitrage-free economy. Then, the problem is analyzed under the additional assumption that a representative investor with a state-independent utility function of terminal wealth exists.[3] Finally, the main

[2] These facts are documented empirically either in studies of option prices or directly in studies of asset price processes. See for example Fama and French [16], Poterba and Summers [48], Kothari and Shanken [38],Jackwerth [29], Jackwerth and Rubinstein [30], Rubinstein [51], Canina and Figlewski [8] or Kim and Kon [37].

[3] For a discussion of the difference between the arbitrage-free economy and the economy

purpose of this paper is to provide alternative characterizations of viable forward price processes, i.e. to derive price processes which are consistent with a representative investor economy. Therefore, section 4 provides a detailed discussion of forward prices and forward price processes when the representative investor's utility function is of the HARA-class. Also closed form and numerical solutions are derived. More precisely, we characterize 3 classes of market models. The first class are the standard Black-Scholes economies characterized by an information process of the geometric Brownian motion type and constant relative risk aversion. The second class are models consistent with non-constant relative risk aversion which can be interpreted as displaced Black Scholes economies. Analytical solutions for the forward price process are derived for this class also. Finally, for the third and most general class numerical solutions for the forward price are derived. Section 5 summarizes the main results.

2 The general characterization

In this paper we consider a market with a given time horizon $T > 0$ and the one-dimensional standard Brownian motion W on a given filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ where $(\mathcal{F}_t)_{t\ \in[0,T]}$ is the filtration generated by W augmented by all the $\mathcal{F}$-null sets, with $\mathcal{F} = \mathcal{F}_T$. It is assumed that at least one risky asset is traded and hence the market is complete. If the market is arbitrage free, then the forward price of any non-dividend paying asset is given by[4]

$$F_{t,T} = E\left(F_{\tau,T}\Phi_{t,\tau} \middle| \mathcal{F}_t \right), \qquad \text{for all } t, \tau \in [0,T] \text{ and } t \leq \tau, \tag{1}$$

where $\Phi_{t,\tau}$ is the pricing kernel for the time-period $[t, \tau]$. Since the market is complete, the prcing kernel is unique. $F_{s,T}$ is the forward price of the asset at time $s \in [0,T]$ with delivery date T. Since throughout this work we will only discuss forward prices with delivery date T, we simplify notation and write F_t. Also, since the information structure discussed here is always complete and therefore the relevant filtration will always be obvious, for

with a representative investor with state-independent utility function see also Decamps and Lazrak [13].

[4]For a detailed discussion of arbitrage-free markets see for example the seminal paper of Harrison and Kreps [23] or the textbooks of Musiela and Rutkowski [45] and Karatzas and Shreve [35], [36].

notational convenience we will write $E_t(\bullet)$ instead of $E(\bullet|\mathcal{F}_t)$. Furthermore, unless stated differently expectations are always taken under the objective probability measure P. Finally, throughout the paper we will assume that the asset pays no dividend.

Equation (1), in the "pricing kernel notation", basically states that a probability measure $\widetilde{P}$ equivalent to the objective probability measure P exists under which the forward price process is a martingale. The existence of such an equivalent martingale measure is equivalent to the absence of arbitrage possibilities. In addition, it is well known from Girsanov's theorem that the equivalent martingale measure (which is uniquely determined in complete markets) is defined by a density process. This density process is governed by a stochastic differential equation

$$\begin{aligned} d\Phi_t^{(\lambda)} &= -\Phi_t^{(\lambda)}\lambda_t dW_t\ , \quad 0 \leqslant t \leqslant T, \\ \Phi_0^{(\lambda)} &= 1, \end{aligned} \tag{2}$$

with $P\left(\int_0^T \lambda_t^2 dt < \infty\right) = 1$ and λ being the ratio of the instantaneous drift of the asset and the instantaneous volatility.[5] Furthermore, it is clear that $\Phi^{(\lambda)}$ is a P-martingale and the unique solution to equation (2) is

$$\Phi_t^{(\lambda)} = \exp\left(-\int_0^t \lambda_u dW_u - \frac{1}{2}\int_0^t |\lambda_u|^2 du\right)$$

for $0 \leqslant t \leqslant T$.

In addition, we will define the final value of the asset by introducing an information process, i.e. a stochastic process which characterizes the representative investor's expectations about the final value of the asset. Hence, intuitively speaking, we consider the process

$$I_t = E_t(X_T)\ , \quad 0 \leqslant t \leqslant T,$$

where X_T is the exogenous final value and $X_T \equiv F_T$. Since the information process characterizes a process of conditional expectations, this process is a martingale. However, following Franke, Stapleton and Subrahmanyam [19] and Lüders and Peisl [40], we define the final value X_T by postulating some information process. Hence, the expected distribution of X_T is derived from

[5] Note that we write Φ_t for $\Phi_{0,t}$. $\Phi_{t,T}$ is defined by $\frac{\Phi_T}{\Phi_t}$.

the exogenously given information process and since at the terminal date T the following identity holds $I_T \equiv F_T \equiv X_T$, equation (1) can be rewritten as

$$F_t = E_t\left(I_T \Phi_{t,T}\right) \ , \quad 0 \leqslant t \leqslant T,$$

for some information process I and some pricing kernel Φ with

$$\Phi_{t,T} \equiv \frac{\Phi_T}{\Phi_t} \ , \quad 0 \leqslant t \leqslant T.$$[6]

Thus, with the pricing kernel and the information process given, the asset price is completely characterized.[7] Therefore, in the following sections we will analyze this relationship in a representative investor economy, i.e. an economy where the pricing kernel is determined by the utility function of the representative investor.

3 Asset prices in a representative investor economy

In this section we will now analyze asset price processes under the assumption that a representative investor with a state-independent utility function U of wealth F_T exists and that the utility function U belongs to the set of twice continuously differentiable, strictly increasing and strictly concave functions defined on $(0, \infty)$. It is well known that in the equilibrium of such a representative investor economy the following equality must hold

$$\Phi_{0,T} = \frac{\frac{\partial}{\partial x} U\left(F_T\right)}{a}, \tag{3}$$

for some scalar $a > 0$.[8] The pricing kernel $\Phi_{0,T}$ is a probability density by definition. Therefore, the expected value of the pricing kernel $\Phi_{0,T}$ must be equal to one, i.e. $E\left(\Phi_{0,T}\right) = 1$. This yields that the scalar a is equal to the expected marginal utility, i.e. $a = E\left(\frac{\partial}{\partial x} U\left(F_T\right)\right)$.

[6] See for example Lüders and Peisl [40].

[7] For a discussion see also Franke, Stapleton and Subrahmanyam [19] and Lüders and Peisl [40]. For a derivation of information processes from observations see Lüders and Peisl [41].

[8] See Cox and Huang [9] and [10] and Karatzas, Lehoczky and Shreve [34].

From the preceding section we know that the asset price is given by

$$F_t = E_t \left(F_T \Phi_{t,T} \right) \ , \quad 0 \leqslant t \leqslant T,$$

with

$$\Phi_{t,T} = \frac{\Phi_{0,T}}{E_t \left(\Phi_{0,T} \right)}$$

and since the pricing kernel is a martingale, i.e. $\Phi_{0,t} = E_t \left(\Phi_{0,T} \right)$,

$$F_t = \frac{E_t \left(F_T \Phi_{0,T} \right)}{\Phi_{0,t}} \ , \quad 0 \leqslant t \leqslant T.$$

Let us replace F_T by the exogenous final value I_T of the information process, this yields

$$F_t = \frac{E_t \left(I_T \Phi_{0,T} \right)}{\Phi_{0,t}} \ , \quad 0 \leqslant t \leqslant T. \tag{4}$$

Since, in this section we are interested in the asset price under alternative characterizations of the pricing kernel we may assume a simple one-dimensional information process I which is governed by the following stochastic differential equation

$$\begin{aligned} dI_t &= I_t \sigma dW_t \ , \quad 0 \leqslant t \leqslant T, \\ I_0 &> 0 \end{aligned} \tag{5}$$

with σ being a constant coefficient. It follows from equation (3) by $\Phi_{0,t} = E_t \left(\Phi_{0,T} \right)$ and the theorem of Feynman-Kac that the pricing kernel $\Phi_{0,t}$ can be characterized by a continuous deterministic function $\alpha \left(I_t, t \right)$ of I_t and time t.[9] If $\Phi_{0,T}$ is a deterministic function of I_T, then obviously $\Phi_{0,T} I_T$ is also a deterministic function of I_T. It follows that $E_t \left(I_T \Phi_{0,T} \right)$ can be characterized by a function $\beta \left(I_t, t \right)$. Since both functions $\alpha \left(I_t, t \right)$ and $\beta \left(I_t, t \right)$ characterize martingales, they have to satisfy the following deterministic partial differential equations

$$\begin{aligned} \frac{\partial \alpha \left(x, t \right)}{\partial t} + \frac{1}{2} \frac{\partial^2 \alpha \left(x, t \right)}{\partial x^2} \sigma^2 x^2 &= 0, \\ \alpha \left(x, T \right) &= \frac{\frac{\partial}{\partial x} U \left(x \right)}{a}, \end{aligned}$$

[9] The theorem of Feynman-Kac is discussed for example in Karatzas and Shreve [35] or Oksendal [46].

and

$$
\begin{aligned}
\frac{\partial \beta(x,t)}{\partial t}+\frac{1}{2}\frac{\partial^2 \beta(x,t)}{\partial x^2}\sigma^2 x^2 &= 0, \\
\beta(x,T) &= x\frac{\frac{\partial}{\partial x}U(x)}{a},
\end{aligned}
$$

where the difference between $\alpha(I_t,t)$ and $\beta(I_t,t)$ is due to the boundary conditions. Moreover, application of Itô's lemma yields that $\alpha(I_t,t)=\Phi_{0,t}$ and $\beta(I_t,t)$ are governed by the following stochastic differential equations

$$
\begin{aligned}
d\alpha(I_t,t) &= \frac{\partial \alpha(I_t,t)}{\partial x}\sigma I_t dW_t\ , \quad 0\leqslant t\leqslant T, \qquad (6)\\
\alpha(I_T,T) &= \frac{\frac{\partial}{\partial x}U(I_T)}{a}
\end{aligned}
$$

$$
\begin{aligned}
d\beta(I_t,t) &= \frac{\partial \beta(I_t,t)}{\partial x}\sigma I_t dW_t\ , \quad 0\leqslant t\leqslant T, \qquad (7)\\
\beta(I_T,T) &= I_T\frac{\frac{\partial}{\partial x}U(I_T)}{a}
\end{aligned}
$$

Note that equation (6) and equation (7) are backward stochastic differential equations since the final values are given. Furthermore, it follows immediately from equation (4) that the forward price F_t can be characterized by

$$
F_t=\digamma(t,\alpha_t,\beta_t)=\frac{\beta(I_t,t)}{\alpha(I_t,t)}\ , \quad 0\leqslant t\leqslant T,
$$

with

$$
\begin{aligned}
\alpha_t &= \alpha(I_t,t)\ , \quad 0\leqslant t\leqslant T,\\
\beta_t &= \beta(I_t,t)
\end{aligned}
$$

Applying Itô's lemma yields the following backward stochastic differential equation for the forward price $F_t=\digamma(t,\alpha_t,\beta_t)$: [10]

$$
d\digamma = \left\{\frac{\beta(I_t,t)}{(\alpha(I_t,t))^3}\left(\frac{\partial \alpha(I_t,t)}{\partial x}\sigma I_t\right)^2\right\}dt
$$

[10] See Appendix $A.1$

$$\begin{aligned}
& -\left\{\frac{1}{\left(\alpha\left(I_t,t\right)\right)^2}\frac{\partial\alpha\left(I_t,t\right)}{\partial x}\frac{\partial\beta\left(I_t,t\right)}{\partial x}\left(\sigma I_t\right)^2\right\}dt \\
& +\sigma I_t\left\{\frac{1}{\alpha\left(I_t,t\right)}\frac{\partial\beta\left(I_t,t\right)}{\partial x}-\frac{\beta\left(I_t,t\right)}{\left(\alpha\left(I_t,t\right)\right)^2}\frac{\partial\alpha\left(I_t,t\right)}{\partial x}\right\}dW_t \ , \\
0 &\leqslant t\leqslant T, \\
F\left(T,\alpha_T,\beta_T\right) &= I_T
\end{aligned}$$

4 The representative investor economy with HARA-utility

We will now analyze the forward price process for a representative investor with a HARA-utility function with constant relative risk aversion and with non-constant relative risk aversion. We start by analyzing the case, when the representative investor has constant relative risk aversion and the information process is a geometric Brownian motion without drift and constant volatility. We call this the standard Black-Scholes economy since this leads to the asset price process assumed by Black and Scholes. After discussing the standard Black-Scholes economy we consider a representative investor with non-constant relative risk aversion but HARA-class. We will analyze decreasing and increasing relative risk aversion. First, we present analytical formulas for the forward price. Then we also give some numerical solutions. Before analyzing the asset prices we briefly discuss HARA-utility functions which seem to be the most popular class of utility functions in modern finance theory.[11]

4.1 Discussion of HARA-utility functions

All members of the HARA-class can be written as a function $U(x)$ of wealth (resp. consumption) x with[12]

[11] See for example Ingersoll[28], Gollier [21] or Merton[44] - especially Merton [42]. For an application in a non-expected utility framework see Franke and Weber [20]. They use a negative HARA-function to model the risk measure in a risk-value framework.

[12] For a discussion of utility functions including the HARA class see for example Duffie [14], Ingersoll [28], Gollier [21], Merton [44] or [42].

$$
\begin{aligned}
U(x) &= \frac{1-\gamma}{\gamma}\left(\frac{x}{1-\gamma}+\theta\right)^{\gamma}, \qquad (8)\\
\gamma &\neq 1,\\
\theta &\in \mathbb{R}.
\end{aligned}
$$

The utility function is defined over the domain $\frac{x}{1-\gamma}+\theta > 0$. Thus, we have the restriction that wealth must satisfy

$$
\begin{aligned}
x &> -\theta(1-\gamma), \quad \text{for } \gamma < 1\\
x &< -\theta(1-\gamma), \quad \text{for } \gamma > 1
\end{aligned}
$$

Note that for $\gamma < 1$ we get a lower bound for wealth x which is equal to $-\theta(1-\gamma)$. This lower bound is negative for positive θ and positive for negative θ. $\gamma > 1$ implies an upper bound which is negative for negative θ and positive for positive θ. Thus, in an asset pricing context $\gamma > 1$ implies very strong restrictions on the distribution of final wealth. Hence, in the following analysis we will assume $\gamma < 1$. A lower bound is less restrictive, especially when the lower bound imposed by the utility function is negative since such a utility function is consistent for example with two-parameter lognormally distributed final wealth.

4.2 Constant relative risk aversion: a foundation for the standard Black-Scholes economy

As aforementioned and already shown in many articles, Black-Scholes option prices are consistent with constant relative risk aversion.[13] Since the Black-Scholes option pricing formula is one of the cornerstones of modern finance theory and since constant relative risk aversion is a natural benchmark for asset price characteristics we will now give a detailed analysis of asset prices in a representative investor economy when the representative investor's utility function is of the constant relative risk aversion-type. More precisely, in this section we assume that the coefficient θ in equation (8) is zero. In addition, we assume that the representative investor's expectations about the final

[13] See for example Merton [43], Rubinstein [49], Brennan [6], Franke, Stapleton and Subrahmanyam [19] or Lüders and Peisl [40].

value are characterized by the following stochastic differential equation:

$$\begin{aligned} dI_t &= I_t \sigma dW_t \ , \quad 0 \leqslant t \leqslant T, \\ I_0 &> 0 \end{aligned} \tag{9}$$

with σ being constant. Thus, the final value X_T of the asset is implicitly given by the final value of the information process I. While an information process characterized by the stochastic differential equation (9) implies very restrictive assumptions on the exogenous final value X_T as well as the information flow in the economy, we assume such an information process here in order to concentrate on the effect of the pricing kernel on the forward price (resp. the forward price process).[14]

In such a representative investor economy the following equality must hold

$$F_t = E_t\left(I_T \frac{\frac{\partial U(I_T)}{\partial x}}{E_t\left(\frac{\partial U(I_T)}{\partial x}\right)}\right) = E_t\left(I_T \frac{\left(\frac{I_T}{1-\gamma}\right)^{\gamma-1}}{E_t\left(\left(\frac{I_T}{1-\gamma}\right)^{\gamma-1}\right)}\right) \ , \quad 0 \leqslant t \leqslant T,$$

where the final value X_T has been substituted by I_T. Since γ is a constant parameter, this simplifies to

$$F_t = E_t\left(\frac{\left(\frac{1}{1-\gamma}\right)^{\gamma-1} I_T^{\gamma}}{E_t\left(\left(\frac{1}{1-\gamma}\right)^{\gamma-1} I_T^{\gamma-1}\right)}\right) = \frac{E_t\left(I_T^{\gamma}\right)}{E_t\left(I_T^{\gamma-1}\right)} \ , \quad 0 \leqslant t \leqslant T, \tag{10}$$

Thus, to get the forward price we have to calculate the conditional γ^{th}, respectively the $(\gamma-1)^{th}$ non-central moments. Since with the given information process I the final value $I_T = I_t \exp\left(-\frac{1}{2}\sigma^2 (T-t) + \int_t^T \sigma dW_s\right)$ is lognormally distributed with

$$E_t(I_T) = I_t \ , \quad 0 \leqslant t \leqslant T,$$

$$Var_t(I_T) = I_t^2 \left(\exp\left(\sigma^2 (T-t)\right) - 1\right) \ , \quad 0 \leqslant t \leqslant T,$$

[14] For a discussion of different characterizations of information processes see Lüders and Peisl [40] and [39].

an analytical solution for the asset price F_t is easily derived: First calculate the conditional mean and the conditional variance of $\ln I_T$

$$Var_t(\ln I_T) = \sigma^2 (T-t) \ , \quad 0 \leqslant t \leqslant T,$$

$$E_t(\ln I_T) = \ln I_t - \frac{1}{2}\sigma^2 (T-t) \ , \quad 0 \leqslant t \leqslant T,$$

then using the fact that

$$E_t(I_T^n) = \exp\left(nE_t(\ln I_T) + n^2 \frac{Var_t(\ln I_T)}{2}\right)$$

yields

$$\begin{aligned} E_t(I_T^n) &= \exp\left(n \ln I_t - \frac{1}{2}n\sigma^2 (T-t) + \frac{1}{2}n^2\sigma^2 (T-t)\right) \quad (11) \\ &= \exp\left(n \ln I_t + \frac{1}{2}n\sigma^2 (T-t)(n-1)\right) \ , \quad 0 \leqslant t \leqslant T. \end{aligned}$$

Inserting equation (11) into equation (10) yields the following analytical solution for the asset price:

$$\begin{aligned} F_t &= A(t) I_t \ , \quad 0 \leqslant t \leqslant T, \quad (12) \\ A(t) &= \exp\left((\gamma - 1)\sigma^2 (T-t)\right). \end{aligned}$$

With equation (12) it is easily seen that neither the expected gross return $E_t(R_{t,T}) = E_t\left(\frac{F_T}{F_t}\right) = \frac{I_t}{F_t}$ nor the expected log-return $E_t(r_{t,T}) = E_t(\ln(R_{t,T}))$ depend on F_t or I_t

$$\begin{aligned} E_t(R_{t,T}) &= \exp\left((1-\gamma)\sigma^2 (T-t)\right), \\ E_t(r_{t,T}) &= (1-\gamma)\sigma^2 (T-t) \ , \quad 0 \leqslant t \leqslant T, \end{aligned}$$

moreover, the log-return r_t per unit of time is constant, i.e.

$$\frac{E_t(r_{t,T})}{(T-t)} = (1-\gamma)\sigma^2 = \text{ constant, } 0 \leqslant t < T.$$

Hence, given that the representative investor expects that the final value is two-parameter lognomally distributed and the representative investor is constant relative risk averse we get an asset pricing model that cannot explain empirical findings as mean-reversion. Using also the fact that the information process I is an Itô process, we get the following stochastic differential equation for the forward price process F by applying Itô's Lemma: [15]

$$\begin{aligned} dF_t &= \underbrace{(1-\gamma)\,\sigma^2}_{=\frac{E_t\left(r_{t,T}\right)}{(T-t)}} F_t \; dt + \sigma F_t \; dW_t \; , \quad 0 \leqslant t \leqslant T, \\ F_T &= I_T. \end{aligned} \tag{13}$$

From equation (12) and equation (13) we can deduce some additional properties of the asset price process in the standard Black-Scholes economy. First, from equation (13) it is obvious, why we call this setting the standard Black-Scholes economy. It is exactly this setting which yields the geometric Brownian motion of the forward price with constant instantaneous drift μ and constant instantaneous volatility σ, i.e.

$$dF_t = \mu F_t \; dt + \sigma F_t dW_t \; , \quad 0 \leqslant t \leqslant T, \tag{14}$$

where $\mu = \frac{E_t\left(r_{t,T}\right)}{(T-t)} = (1-\gamma)\,\sigma^2$. It is well known (see for example Karatzas and Shreve [35] and [36], Musiela Rutkowski [45] or Lüders and Peisl [39] in a similar setting) and has been briefly discussed in section 2 that the instantaneous drift μ in an arbitrage-free market is determined by $\mu = \lambda\sigma$ where λ is called the market price of risk. Thus, in this model the market price of risk λ is equal to $(1-\gamma)\,\sigma$ and depends on the level of relative risk aversion $(1-\gamma)$ as well as the instantaneous volatility σ of the forward price process. Note also that the instantaneous volatility of the forward price process is equal to the instantaneous volatility of the information process. Furthermore, we see that the elasticity of the pricing kernel with respect to the asset price $\frac{\mu}{\sigma^2} = \eta_t^F$ is constant and equal to the level of relative risk aversion $(1-\gamma)$.[16] In addition, the asset price given by equation (12) is linear with respect to the level of the information process I. This property also implies that the information process I and the forward price process F have the same instantaneous volatility σ. This result can be made even stronger:

[15] For an alternative derivation see Appendix A.2

[16] For a discussion of the elasticity of the pricing kernel see Appendix A.3, Franke, Stapleton and Subrahmanyam [19] or Lüders and Peisl [40].

Proposition 1 *Suppose that the information process I is a one-dimensional Itô process*

$$\begin{aligned} dI_t &= I_t \zeta dW_t \ , \quad 0 \leqslant t \leqslant T, \\ I_0 &> 0 \end{aligned}$$

with ζ some deterministic function of t and I_t and that the forward price represents an equilibrium in the representative investor economy such that the forward price is a deterministic function of t and I_t. Then, the instantaneous volatility of the asset price process is equal to the instantaneous volatility of the information process if and only if the asset price F_t is a linear function of I_t, i.e. $F_t = aI_t$.

Proof. First we have to prove the necessity of the linear pricing relationship, i.e. that equal instantaneous volatilities of the information process I and the forward price process F imply a linear pricing rule. Since

$$F_t = E_t \left(F_T \Phi_{t,T} \right) \ , \quad 0 \leqslant t \leqslant T,$$

where because of the assumed economy the pricing kernel $\Phi_{t,T}$ is a deterministic function of t and I_t, it follows from the Theorem of Feynman-Kac that the forward price F_t is given by a deterministic function $v : [0, T] \times \mathbb{R} \to \mathbb{R}$ by

$$F_t = v(t, I_t) \ , \quad 0 \leqslant t \leqslant T.$$

Applying Itô's Lemma yields

$$dF_t = \left\{ \frac{\partial}{\partial t} v(t, I_t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} v(t, I_t) \zeta^2 I_t^2 \right\} dt + \frac{\partial}{\partial x} v(t, I_t) \zeta I_t dW_t \ , \quad 0 \leqslant t \leqslant T, \tag{15}$$

while the usual representation of the forward price process F is given by

$$dF_t = \mu F_t \, dt + \Sigma F_t dW_t \ , \quad 0 \leqslant t \leqslant T, \tag{16}$$

with μ and Σ some arbitrary processes. Thus, we have

$$\frac{\partial}{\partial x} v(t, I_t) \zeta I_t = \Sigma v(t, I_t) \ , \quad 0 \leqslant t \leqslant T,$$

and requiring equal instantaneous volatilities ($\zeta = \Sigma$) leads to the following deterministic differential equation

$$\begin{aligned} \frac{\partial}{\partial x} v(t, x) x &= v(t, x) \ , \quad 0 \leqslant t \leqslant T, \\ \text{with } v(T, x) &= x \end{aligned}$$

for the function $v(t, I_t)$, which states that the elasticity of the forward price F_t with respect to I_t is 1. However, this differential equation is of the linear type and therefore the solution $v(t, x) = \Gamma(t) x$ is unique. This proves that equal instantaneous volatilities imply a linear pricing rule. Now let us turn to the sufficiency of a linear pricing rule for equal instantaneous volatilities. Since the forward price F_t is given by a deterministic function $v(t, I_t)$, linearity in I_t implies $v(t, I_t) = \Gamma(t) I_t$ and therefore also equal instantaneous volatilities $(\zeta = \Sigma)$. ■

With this result we can derive implications for the instantaneous drift.

Proposition 2 *Assume a repesentative investor economy with the information process I governed by the one-dimensional Itô process*

$$\begin{aligned} dI_t &= I_t \zeta dW_t \ , \quad 0 \leqslant t \leqslant T, \\ I_0 &> 0 \end{aligned}$$

with ζ some deterministic function of t and I_t. In addition assume that the instantaneous volatility of the asset price process is equal to the instantaneous volatility of the information process or equivalently the asset price F_t is a linear function of I_t, then the instantaneous drift of the asset price is constant or a function of time t only.

Proof. Linearity implies that the forward price F_t is given by a function $v(t, I_t) = \Gamma(t) I_t$. Using the fact that

$$\frac{\partial}{\partial t} v(t, x) + \frac{1}{2} \frac{\partial^2}{\partial x^2} v(t, x) \zeta^2 x^2 = v(t, x) \mu$$

and $\frac{\partial^2}{\partial x^2} v(t, x) = 0$, it is obvious that

$$\frac{\partial}{\partial t} \Gamma(t) = \Gamma(t) \mu$$

and thus, μ is independent of x. ■

Thus, we have seen that a linear pricing rule and equality of the instantaneous volatilities of the information process and of the forward price process are equivalent. Furthermore, a linear pricing rule implies that the instantaneous drift of the information process is constant or a function of time t, only. However, in general the inverse is not true, i.e. an only time-dependent instantaneous drift does not necessarily imply a linear pricing rule. But, if we assume that the volatility ζ of the information process depends on time t only, then an only time-dependent drift of the forward price process is equivalent to a linear pricing rule.[17]

Consider again the expected gross return $E_t(R_{t,T}) = \frac{I_t}{F_t}$ and the expected log-return $E_t(r_{t,T}) = E_t(\ln(R_{t,T}))$. It is clear that mean-reversion in returns for example implies that the return depends on the level of I_t (either directly or through the forward price F_t). Thus, to get a better understanding of asset returns, let us analyze the question under which conditions $E_t(R_{t,T})$ (resp. $E_t(r_{t,T})$) are independent of I_t. For this analysis we do not require some special information process. First, note that $E_t(R_{t,T})$ is characterized by a function $R(t,x) = \frac{x}{F(t,x)}$ with $F_t = F(t, I_t)$.[18] Hence, independence of $E_t(R_{t,T})$ from I_t implies that $\frac{\partial R(t,x)}{\partial x} = 0$ which yields the following condition

$$\frac{F(t,x) - x\frac{\partial F(t,x)}{\partial x}}{(F(t,x))^2} = 0 \qquad \text{, for all } t \in [0,T] \text{ and for all } x \in \mathbb{R}^+ \tag{17}$$

which implies that $F(t,x)$ is a solution to the following deterministic differential equation

$$F(t,x) - x\frac{\partial F(t,x)}{\partial x} = 0 \qquad \text{, for all } t \in [0,T] \text{ and for all } x \in \mathbb{R}^+. \tag{18}$$

The unique solution is $F(t,x) = A(t)\,x$ for some function $A(t)$. Hence, any linear pricing rule leads to a gross return which is independent of I_t while non-linear pricing rules lead to a dependence of the expected gross return $E_t(R_{t,T})$ on I_t. Obviously, the same holds for the log-return r_t. Finally, let us also analyze subperiod-returns:

$$\begin{aligned} R_{t,t+\tau} &= \frac{F_{t+\tau}}{F_t}, \\ \text{resp. } r_{t,t+\tau} &= \ln\left(\frac{F_{t+\tau}}{F_t}\right). \end{aligned}$$

[17]This follows because the function $v(t,x) = \Gamma(t)\,x$ is the unique solution to the deterministic partial differential equation

$$\begin{aligned} \frac{\partial}{\partial t} v(t,x) + \frac{\partial^2}{\partial x^2} v(t,x)\,\zeta(t)^2 x^2 &= v(t,x)\,\mu(t) \\ v(T,x) &= x \end{aligned}$$

where the instantaneous drift $\mu(t)$ is a function of time t only.

[18]Note that for convenience, the function characterizing the forward price F_t is also denoted by F.

Empirical studies report autocorrelation in subperiod returns. However, analyzing expected subperiod returns

$$\begin{aligned} E_t\left(R^{\mathrm{CRRA}}_{t,t+\tau}\right) &= \frac{\exp\left(\left(\gamma-1\right)\sigma^2\left(T-\left(t+\tau\right)\right)\right)}{\exp\left(\left(\gamma-1\right)\sigma^2\left(T-t\right)\right)} E_t\left(\frac{I_{t+\tau}}{I_t}\right) \\ &= \exp\left(\left(\gamma-1\right)\sigma^2\left(-\tau\right)\right) \\ &= \exp\left(\left(1-\gamma\right)\sigma^2\left(\tau\right)\right) \end{aligned}$$

or the corresponding expected log-returns

$$E_t\left(r^{\mathrm{CRRA}}_{t,t+\tau}\right) = \left(1-\gamma\right)\sigma^2\left(\tau\right).$$

in the standard Black-Scholes economy shows that these returns do not depend on the processes I or F. They do only depend on the relative risk aversion and the conditional variance of $\ln F_{t+\tau}$ which is equal for every subperiod with equal lengt τ and it grows linearly with τ. Moreover, conditional and unconditional expected returns are the same, i.e.

$$\begin{aligned} E_t\left(R^{\mathrm{CRRA}}_{t,t+\tau}\right) &= E\left(R^{\mathrm{CRRA}}_{t,t+\tau}\right) \\ E_t\left(r^{\mathrm{CRRA}}_{t,t+\tau}\right) &= E\left(r^{\mathrm{CRRA}}_{t,t+\tau}\right) \end{aligned}$$

Thus, linear pricing rules are not consistent with recent empirical studies which, for example, document mean-reversion.

While we have discussed intensively the case of constant relative risk aversion and an information process governed by a geometric Brownian motion with constant volatility, we will now turn to the case when the representative investor does not have a utility function with constant relative risk aversion. While non-constant relative risk aversion -especially decreasing relative risk aversion- seems to be more realistic than constant relative risk aversion we will see in the following section that the pricing rules become much more inconvenient to handle.[19]

Two different classes of models will be discussed. The first class is closely related to the standard Black-Scholes economy, although it is consistent with increasing and decreasing relative risk aversion. Technically, the models of this class are displaced Black-Scholes economies since terminal wealth is supposed to be three-parameter lognormally distributed in contrast to the two-parameter lognormally distributed terminal wealth in the standard Black-Scholes economy. The second class is technically not that closed to the

[19] However, note that empirical evidence is mixed. See for example Gollier [21], Elton and Gruber [15], Blume and Friend [5] and Zhou [53].

standard Black-Scholes economy. Numerical solutions for forward prices in this class of models are presented, too. However, the numerical solutions are restricted to HARA-utility functions exhibiting increasing relative risk aversion. In contrast to HARA-utility functions exhibiting decreasing relative risk aversion, assuming increasing relative risk aversion is consistent with an information process which is governed by a geometric Brownian motion with constant volatility and drift zero. This follows from the discussion of HARA-utility functions in section 4.1 since in contrast to decreasing relative risk aversion, increasing relative risk aversion does not imply a strictly positive lower bound for terminal wealth. Thus, the advantage of this class of models is that final wealth can still be assumed to be two-parameter lognormally distributed which allows an immediate comparison with the standard Black-Scholes economy.

4.3 The displaced Black-Scholes economy

In this section we will now analyze asset price processes in the economy we introduced before assuming that the representative investor's utility function is characterized by

$$U\left(X_T\right) = \frac{1-\gamma}{\gamma}\left(\frac{X_T}{1-\gamma}+\theta\right)^{\gamma}, \tag{19}$$

with $\gamma < 1$ and $\theta \neq 0$. It is clear from the above discussion of HARA-utility functions that in order to get viable forward prices, terminal wealth X_T has to admit the following condition

$$X_T \geq \underbrace{-\theta\left(1-\gamma\right)}_{\text{lower bound}}, \tag{20}$$

which implies for decreasing relative risk aversion ($\theta < 0$), for example, a stirctly positive lower bound and hence it implies that the information process cannot be governed by the stochastic differential equation (9) since this would imply that

$$\inf_{\omega\in\Omega} X_T\left(\omega\right) = 0$$

which contradicts the strict positive lower bound. To avoid such inconsistencies between the distribution of terminal wealth and the utility function let us assume that $Y_T = X_T + \theta\left(1-\gamma\right)$ is two-parameter lognormally distributed which implies that terminal wealth is three-parameter lognormally distributed.[20] Hence, compared to the standard Black-Scholes economy terminal wealth is displaced by $\theta\ (1-\gamma)$. This has also an impact on the information process process I with

$$I_t = E_t\left(X_T\right) \ , \quad 0 \leqslant t \leqslant T.$$

As usual in this class of models, the final wealth X_T is defined implicitly by the information process characterizing investors' expectations, therefore $X_T = I_T$. Since Y_T is two-parameter lognormally distributed we assume that the information process $\widehat{I}$ with

$$\widehat{I}_t = E_t\left(Y_T\right) \ , \quad 0 \leqslant t \leqslant T,$$

is governed by

$$\begin{aligned} d\widehat{I}_t &= \widehat{I}_t \sigma dW_t \ , \quad 0 \leqslant t \leqslant T, \\ \widehat{I}_0 &> 0, \end{aligned}$$

with σ constant. This information process $\widehat{I}$ is the same as for example in Franke, Stapleton and Subrahmanyam [19] and equals the information in the Black-Scholes economy. Hence, if terminal wealth was defined by Y_T this would be the Black-Scholes economy. However, the relevant information process describing the representative investor's expectations about terminal wealth X_T in this displaced Black-Scholes economy is defined by

$$I_t = \widehat{I}_t - \theta\left(1-\gamma\right) \ , \quad 0 \leqslant t \leqslant T,$$

which yields the following stochastic differential equation for the information process I : [21]

$$\begin{aligned} dI_t &= \sigma I_t \left(1 + \frac{\theta\left(1-\gamma\right)}{I_t}\right) dW_t \ , \quad 0 \leqslant t \leqslant T, \\ I_0 &> -\theta\left(1-\gamma\right). \end{aligned}$$

[20] Such a transformation of variable is also used in Franke [17].

[21] Note that this information process is still a martingale. The difference between this information process and the information process assumed for example in Franke, Stapleton and Subrahmanyam [19], i.e. a positive martingale with constant instantaneous volatility is as follows: while the information process in Franke, Stapleton and Subrahmanyam [19] yields a two-parameter lognormal distribution the information process assumed here yields a three-parameter log-normal distribution (for a discussion of the two- and three-parameter lognormal distribution see for example Aitchison and Brown [1],Crow and Shimizu [11] or Johnson and Kotz [33]).

This stochastic process is a displaced diffusion process in the spirit of Rubinstein [50].

With the new information process and the utility function given by equation (19) all information needed to determine the equilibrium asset price is given. The following proposition establishes an analytical solution for the equilibrium forward price (resp. the equilibrium forward price process) in this economy.

Proposition 3 *Suppose that the information process I is governed by the stochastic differential equation*

$$\begin{aligned} dI_t &= I_t \sigma \left(1 + \frac{\theta (1-\gamma)}{I_t}\right) dW_t \ , \quad 0 \leqslant t \leqslant T, \\ I_0 &> -\theta (1-\gamma) , \end{aligned} \tag{21}$$

and that the representative investor maximizes expected utility over terminal wealth and his utility function is given by

$$U(x) = \frac{1-\gamma}{\gamma} \left(\frac{x}{1-\gamma} + \theta\right)^{\gamma} .$$

Then, the forward price is given by the following formula

$$\begin{aligned} F_t &= A(t) I_t \\ &\quad + \theta (1-\gamma)(A(t) - 1) \ , \\ 0 &\leqslant t \leqslant T, \end{aligned} \tag{22}$$

with

$$A(t) = \exp\left((\gamma - 1) \sigma^2 (T-t)\right) ,$$

and the asset price process F is governed by the following stochastic differential equation

$$\begin{aligned} dF_t &= F_t \underbrace{\lambda \Sigma_t}_{\mu^{NCRRA}} dt + F_t \Sigma dW_t \ , \quad 0 \leqslant t \leqslant T, \\ F_T &= I_T, \end{aligned} \tag{23}$$

with

$$\begin{aligned} \lambda &= (1-\gamma) \sigma, \\ \Sigma_t &= \sigma \left(1 + \frac{\theta (1-\gamma)}{F_t}\right) . \end{aligned}$$

Proof. See Appendix A.4.

It is easily seen that the standard Black-Scholes economy is a special case of the model discussed here. It follows from equation (22) that the forward price in the displaced economy is given by

$$F_t = A(t)\left(\widehat{I}_t\right) - \theta(1-\gamma), \quad 0 \leqslant t \leqslant T, \tag{24}$$

respectively

$$F_t + \theta(1-\gamma) = A(t)\left(I_t + \theta(1-\gamma)\right), \quad 0 \leqslant t \leqslant T, \tag{25}$$

hence for for $\theta = 0$ we have the Black-Scholes world. Moreover, note that $F_t + \theta(1-\gamma)$ is a linear function of $I_t + \theta(1-\gamma)$ but in contrast to the Black-Scholes economy F_t is now an affine-linear function of I_t, i.e. $F_t = c + a\ I_t$. Equation (22) and equation (23) reveal that the equilibrium forward price is also governed by a displaced diffusion process as proposed by Rubinstein [50]

In addition, since we are now in a world with non-constant relative risk aversion it does not come as a surprise that the elasticity of the pricing kernel

$$\frac{\mu^{\text{NCRRA}}}{\Sigma_t^2} = \frac{(1-\gamma)}{\left(1 + \frac{\theta(1-\gamma)}{F_t}\right)} \tag{26}$$

is not constant. Furthermore, in contrast to the geometric Brownian motion of the standard Black-Scholes economy, the instantaneous drift μ^{NCRRA} depends on the level of the forward price

$$\mu^{\text{NCRRA}} = (1-\gamma)\,\sigma^2\left(1 + \frac{\theta(1-\gamma)}{F_t}\right). \tag{27}$$

Also interesting is the fact that the instantaneous volatility of the forward price process in the displaced Black-Scholes economy depends on the level of the forward price as well

$$\Sigma_t = \sigma\left(1 + \frac{\theta(1-\gamma)}{F_t}\right).$$

and hence the instantaneous volatility is random. The results for the instantaneous return $\frac{dF_t}{F_t}$ or more precisely the instantaneous logreturn $d\ln(F_t)$

confirm these findings:[22]

$$\begin{aligned} d\ln(F_t) &= \left\{(1-\gamma)\,\sigma\Sigma_t - \frac{1}{2}\Sigma_t^2\right\} dt + \Sigma_t dW_t, \\ 0 &\leqslant t \leqslant T. \end{aligned} \tag{28}$$

with the instantaneous drift $\mu^{\text{logreturn}}$ of $d\ln(F_t)$given by

$$\mu^{\text{logreturn}} = \Sigma_t^2 \left(\frac{(1-\gamma)\,\sigma}{\Sigma_t} - \frac{1}{2}\right), \quad 0 \leqslant t \leqslant T. \tag{29}$$

Hence, equation (28) and equation (29) show the characteristics of the return of the forward price process. Moreover, the expected subperiod-returns

$$\begin{aligned} E_t\left(R_{t,t+\tau}\right) &= E_t\left(\frac{F_{t+\tau}}{F_t}\right), \\ \text{resp. } E_t\left(r_{t,t+\tau}\right) &= E_t\left(\ln\left(\frac{F_{t+\tau}}{F_t}\right)\right) \end{aligned}$$

do also depend on the level of the processes F resp. I. More precisely, they admit the following representation

$$E_t\left(R_{t,t+\tau}\right) = \frac{A(t+\tau)\left(I_t + \theta(1-\gamma)\right) - \theta(1-\gamma)}{A(t)\left(I_t + \theta(1-\gamma)\right) - \theta(1-\gamma)}.$$

Hence, the derived forward price processes in the displaced Black-Scholes economy are very flexible and have many interesting characteristics which are in line with empirical findings.

The equilibrium forward price processes derived here are also similar to the classical Ornstein-Uhlenbeck process and other classical mean-reversion (mean-aversion) processes. However, note that Bick [3] has shown that independently of the information process, the Ornstein-Uhlenbeck is not consistent with a representative investor economy. Hence, the stochastic differential equation (23) is an interesting alternative for example to the Ornstein-Uhlenbeck process which is the continuous-time equivalent of an autoregressive process of order one.[23]

Let us now discuss the case of decreasing relative risk aversion ($\theta < 0$) and the case of increasing relative risk aversion ($\theta > 0$).

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[22] While the instantaneous return is commonly defined as $\frac{dF_t}{F_t}$ it is somewhat informal. However, defining the instantaneous return as $\ln(F_t)$ and applying Itô's Lemma yields almost the same stochastic process and it is formally correct.

[23] See for example Gourieroux and Jasiak [22].

4.3.1 Decreasing relative risk aversion

With decreasing relative risk averion ($\theta < 0$) we find that the elasticity of the pricing kernel is decreasing. This is obvious from equation (26). It follows immediately from equation (27) that for decreasing relative risk aversion, i.e. negative θ, μ^{NCRRA} increases with the level of the forward price. Thus, decreasing relative risk aversion leads to a forward price process which is mean-averting, that is the return of the forward price process moves in the same direction as the level of the forward price.[24] Let us also analyze the expected gross return $E_t\left(R_{t,T}\right)$ and the expected logreturn $E_t\left(r_{t,T}\right)$. In this economy the expected gross return can be written as

$$E_t\left(R_{t,T}\right) = \frac{I_t}{F_t} = \frac{1}{\exp\left(\left(\gamma-1\right)\sigma^2\left(T-t\right)\right) + \frac{\theta(1-\gamma)(\exp((\gamma-1)\sigma^2(T-t))-1)}{I_t}}. \quad (30)$$

Differentiation of the expected gross return with respect to the information yields

$$\frac{\partial R_t\left(t,I_t\right)}{\partial x} = \frac{\theta\left(1-\gamma\right)\left(\exp\left(\left(\gamma-1\right)\sigma^2\left(T-t\right)\right)-1\right)}{\left(F_t\right)^2} > 0.$$

Thus, the expected gross return (resp. the expected logreturn) depends positively on F_t, implying that the higher the realized forward price the higher will be the expected gross return (resp. the expected logreturn). Similar results hold also for the expected sub-period returns. Differentiation of the conditionally expected subperiod return $\mu^{\mathrm{DRRA}}\left(t,\tau,I_t\right) = E_t\left(R^{\mathrm{DRRA}}_{t,t+\tau}\right)$ with respect to the state variable

$$\frac{\partial \mu^{\mathrm{DRRA}}\left(t,\tau,I_t\right)}{\partial x} = \theta\left(1-\gamma\right)\frac{A\left(t\right)-A\left(t+\tau\right)}{\left(F_t\right)^2} > 0$$

shows the positive relationship between the level and the expected return. Thus, we have the result that expected returns depend positively on I_t and F_t.

[24] This result is related to He and Leland [24], p. 614. However, He and Leland derive only qualitative results and the volatility of the asset price is exogenously given in their setting. Furthermore, in contrast to our results He and Leland derive "Proposition 2 demonstrates that mean reversion is naturally associated with preferences that exhibit decreasing relative risk aversion, when the volatility of stock return is constant. Similarly mean aversion processes are naturally associated with preferences that exhibit increasing relative risk aversion". However, as will be discussed at the end of this section, their results rely on the assumption that the volatility of stock returns is constant.

To sum up, in this model the representative investor is decreasing relative risk averse and hence the pricing kernel has decreasing elasticity, but contrary to intuition returns are not mean-reverting but depend positively on F_t. How can we explain this result? The crucial point in this model is that because the HARA-utility function is defined only for $X_T \geq -\theta(1-\gamma)$ for decreasing relative risk aversion ($\theta < 0$) we cannot assume that wealth is two-parameter lognormally distributed and therefore the information process governed by the stochastic differential equation (9) is not appropriate. In order to get a viable model with decreasing relative risk aversion, we defined the distribution of final wealth X_T by

$$X_T = \widehat{I}_T - \theta(1-\gamma)$$

where $\widehat{I}_T$ is two-parameter lognormally distributed which yields a lower bound for final wealth X_T equal to

$$\inf_{\omega\in\Omega} X_T(\omega) = -\theta(1-\gamma).$$

However, this changes the information process: The instantaneous volatility of the information process $I_t = E_t(X_T)$ is increasing with I_t. This implies that the asset becomes more risky the higher I_t resp. F_t. Thus, the fact that returns are increasing with I_t resp. F_t is due to the fact that risk is higher for high I_t (F_t) and thus the investor requires a higher return although he is decreasing relative risk averse. Moreover, the results are not directly comparable with for example He and Leland [24] or Bick [3]. The important difference to these papers is that with our approach (see also Franke, Stapleton and Subrahmanyam [19] or Lüders and Peisl [40]) the distribution of terminal wealth is exogenously given. Those papers analyze price processes which are consistent with certain utility functions without considering the distirbution of terminal wealth.

To conclude, while the market price of risk λ is the same in this economy with decreasing relative risk aversion as in the standard Black-Scholes economy the elasticity of the pricing kernel changes from constant to decreasing. Also the instantaneous volatility is different from that in the standard Black-Scholes economy. First, by definition of the information process the instantaneous volatility of the information process which equals

$$\sigma\left(1+\frac{\theta(1-\gamma)}{I_t}\right)$$

does now depend positively on the level of I_t. But in contrast to the standard Black-Scholes economy the instantaneous volatility of the information process and the instantaneous volatility of the forward price process are not equal. While the functional form of both is the same, the instantaneous volatility Σ of the forward price is a function of the forward price and since $F_t \leq I_t$ it follows that the instantaneous volatility of the forward price is lower than the instantaneous volatility of the information process. Finally, the model in this section provides analytical solutions for forward prices when the representative investor is decreasing relative risk averse. The analysis of the forward prices and returns showed that this model is consistent for example with non-constant volatilities and expected returns. However, in contrast to most empirical studies with this model we get mean-aversion.

4.3.2 Increasing relative risk aversion

It remains to discuss asset prices resp. asset price processes for increasing relative risk aversion, that is $\theta > 0$. From the analysis in the preceding section the results are obvious. For example, the expected gross return $E_t(R_{t,T})$, the expected subperiod returns $E_t(R_{t,\tau})$ and the instantaneous drift of the forward price depend negatively on I_t. The other results can be transferred from the case of decreasing relative risk aversion to the case of increasing relative risk aversion as well. For example, the volatility of the forward price process is now given by

$$\sigma\left(1+\frac{\theta(1-\gamma)}{F_t}\right), \text{ with } \theta > 0,$$

which decreases with the level of F_t. Hence, the volatility of the forward price in this model is consistent with the so called leverage effect which says that the volatility of assets is higher in bear-markets than in bull-markets.

Thus, although the assumption that the representative investor is increasing relative risk averse might be in conflict with usual assumptions on preferences, the model in this section is consistent with stylized facts on volatility and returns. Moreover, it shows that mean-reversion does not necessarily imply decreasing relative risk aversion.

4.3.3 A comment on the three-parameter lognormal distribution

The information process assumed in the preceding sections has already been briefly discussed and the equilibrium forward price process has been analyzed in detail. However, to get a better intuition let us add some remarks on these processes as well as on the implied distribution of final wealth.[25] First it should be emphasized that the implied three-parameter lognormal distribution is closely related to the two-parameter lognormal distribution of wealth, which is implied by the standard Black-Scholes dynamics. To be precise, the difference between the two distributions is that the three-parameter lognormal distribution has an additional threshold parameter which displaces the distribution. We denote the lognormal distribution by $\Lambda\left(\widehat{\tau},\widehat{\mu},\widehat{\sigma}^2\right)$. $\widehat{\tau}$ is the threshold parameter and for $\widehat{\tau}=0$ we have the two-parameter lognormal distribution which is denoted by $\Lambda\left(\widehat{\mu},\widehat{\sigma}^2\right)$. $\widehat{\mu}$ is the mean of the corresponding normally distributed variable and $\widehat{\sigma}^2$ the variance, i.e. if $X\sim\Lambda\left(\widehat{\tau},\widehat{\mu},\widehat{\sigma}^2\right)$ then $Z=\ln\left(X-\widehat{\tau}\right)$ is normally distributed with $N\left(\widehat{\mu},\widehat{\sigma}^2\right)$. Finally, the density function $f\left(x\right)$ of X having $\Lambda\left(\widehat{\tau},\widehat{\mu},\widehat{\sigma}^2\right)$ is

$$f\left(x\right)=\begin{cases}\frac{1}{\sqrt{2\pi}\widehat{\sigma}(x-\widehat{\tau})}\exp\left(-\frac{(\ln(x-\widehat{\tau})-\widehat{\mu})}{2\widehat{\sigma}^2}\right), & x>\widehat{\tau},\\ 0\,, & x\leq\widehat{\tau}.\end{cases}$$

What makes the three-parameter lognormal distribution less convenient in the asset pricing context than the two-parameter lognormal distribution is especially that the calculation of non-central moments becomes much less elegant.[26] While the non-central moments of the two-parameter lognormal distribution are simply

$$E\left(Z^{\gamma}\right)=\exp\left(\gamma\widehat{\mu}+\gamma^2\frac{\widehat{\sigma}^2}{2}\right)$$

the moments of the three-parameter lognormal distribution are given by:[27]

$$E\left(Z^{\gamma}\right)=\sum_{\alpha=0}^{\gamma}\left(\binom{\gamma}{\alpha}\widehat{\tau}^{\gamma-\alpha}\exp\left(\alpha\widehat{\mu}+\frac{1}{2}\alpha^2\widehat{\sigma}^2\right)\right),\qquad\gamma=0,1,2,....$$

However, as argued by Rubinstein [50] the displaced diffusion process or equivalently the three-parameter lognormal distribution is a reasonable alternative to the popular geometric Brownian motion. His argument in favor

[25] A detailed discussion of the lognormal distribution is given in Aitchison and Brown [1],Crow and Shimizu [11] or Johnson and Kotz [33]. Camara [7] discusses the two- and three-parameter lognormal distribution in the context of risk neutral valuation relationships. The corresponding stochatic process, i.e. the displaced diffusion process, is discussed for example in Rubinstein [50].

[26] Note that central moments are the same for both distributions.

[27] See Heyde [25].

of the displaced diffusion process can be summarized as follows. Suppose, the company invests in some risky asset A with its price governed by a geometric Brownian motion and in addition in some riskless asset B, then as is also clear from our analysis, the value of the firm follows a displaced diffusion process with terminal value being three-parameter lognormally distributed. In addition, as shown in Rubinstein [50] and Camara [7] options written on a tree-parameter lognormally distributed asset may explain the observed deviations from Black-Scholes prices. However, the studies of Jackwerth and Rubinstein (see for example [31] and [32]) indicate that the displaced diffusion model does not explain deviations from Black-Scholes prices satisfactorily for the post-1987 crash market.

4.4 Non-constant relative risk aversion and a non-displaced information process

The results of the preceding section are based on the assumption that terminal wealth is three-parameter lognormally distributed. This has the unpleasant implication that it is not clear whether the asset price characteristics which differ from those in the Black-Scholes economy are merely driven by the assumption on the information process or the fact that the representative investor is not assumed to be constant relative risk averse. While the HARA-utility function implies a three-parameter lognormal distribution for the case of decreasing relative risk aversion, for increasing relative risk averse investors we may assume that terminal wealth is two-parameter lognormally distributed without violating the assumption of non-satiation. Therefore, we will now assume that the representative investor is increasing relative risk averse and the information process is governed by a geometric Brownian motion with constant instantaneous volatility and drift zero, i.e. we assume that the information process I is characterized by the stochastic differential equation (9). The assumption that the information process follows such a geometric Brownian motion implies that I_T is conditionally two-parameter lognormal.

To analyze the asset price process under these assumptions, we start again from the basic valuation equation in a representative investor economy:

$$F_t = E_t \left(I_T \frac{\frac{\partial U(I_T)}{\partial x}}{E_t\left(\frac{\partial U(I_T)}{\partial x}\right)} \right). \tag{31}$$

The utility function $U(I_T)$ is given by equation (19). To simplify the resulting pricing equation we define a new variable Y_T :

$$Y_T = I_T + \theta(1-\gamma).$$

Inserting into equation (31) and simplifying yields:

$$F_t = \frac{E_t(Y_T^{\gamma})}{E_t\left(Y_T^{\gamma-1}\right)} - \theta(1-\gamma). \tag{32}$$

Obviously, the pricing equation for the non-constant relative risk aversion case and the pricing equation for the constant relative risk aversion case (equation 10) are very similar. Again the asset price depends on the ratio of the conditional γ^{th} and $(\gamma-1)^{th}$ non-central moments. However, in the case of constant relative risk aversion the moments have to be calculated for final wealth while with non-constant relative risk aversion we have to calculate the moments of a transformed variable. In addition, since in this section the information process I is assumed to be governed by the stochastic differential equation (9) and hence terminal wealth is two-parameter lognormally distributed, the variable Y_T is three-parameter lognormally distributed with the threshold parameter equal to $\theta(1-\gamma)$.

The following figures show the numerical evaluation of equation (32). Figure 1 displays the forward price F_t in terms of the conditionally expected final value (I_t) and the conditional variance of $\ln I_T$. The threshold parameter $\theta(1-\gamma)$ is assumed to be 100 and $\gamma = -1$. Note that the forward price is displayed for values of the variance of $\ln I_T$ between 0.1 and 1.9. It is probably easier to interpret after transforming the conditional variance of $\ln I_T$ $(var_t(\ln I_T))$ into the instantaneous variance σ^2 of the information process and the time to maturity $(T-t)$ according to

$$\sigma^2 = \frac{var_t(\ln I_T)}{T-t}$$

Thus, if for example we assume that the instantaneous variance σ^2 is 0.1 then the figure displays time to maturity from 1 to 19. To plot the graphs 2527 pricing rules (equation 32) are evaluated numerically.

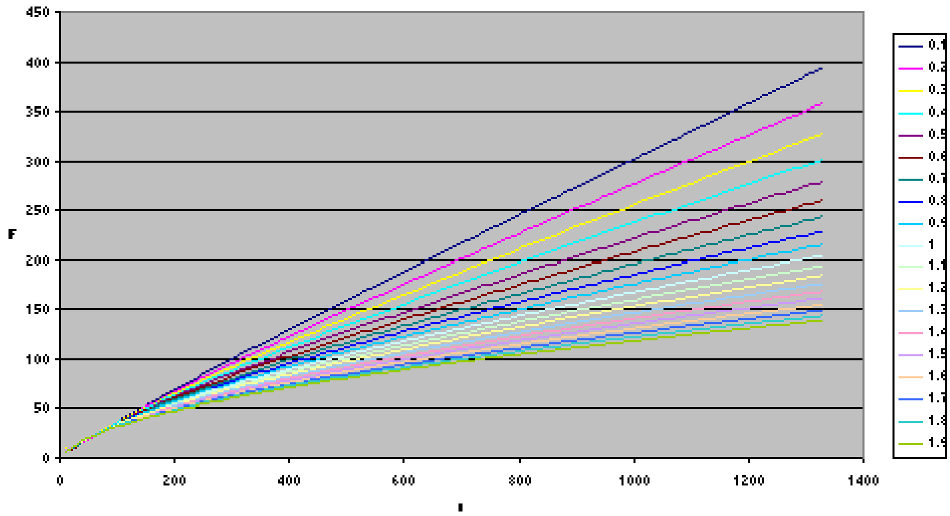

Figure 1: Forward prices F_t in terms of I_t for different levels of the conditional variance of $\ln I_T$

Figure 1 shows that in contrast to the Black-Scholes economy the forward price F_t is not linear in I_t. For the economy analyzed here, i.e. the information process governed by a geometric Brownian motion with constant instantaneous volatility and drift zero and an increasing relative risk averse investor the forward price F_t is concave in I_t. For smaller values of the conditional variance of $\ln I_T$ the forward price becomes more and more linear and since by definition $F_T = I_T$ it is clear that at time T the forward price is linear in I_T. Figure 2 illustrates the relationship between the forward price F_t, the conditional expectation of the final value (I_t) and the conditional variance of the logarithm of the final value $(var_t(\ln I_T))$.

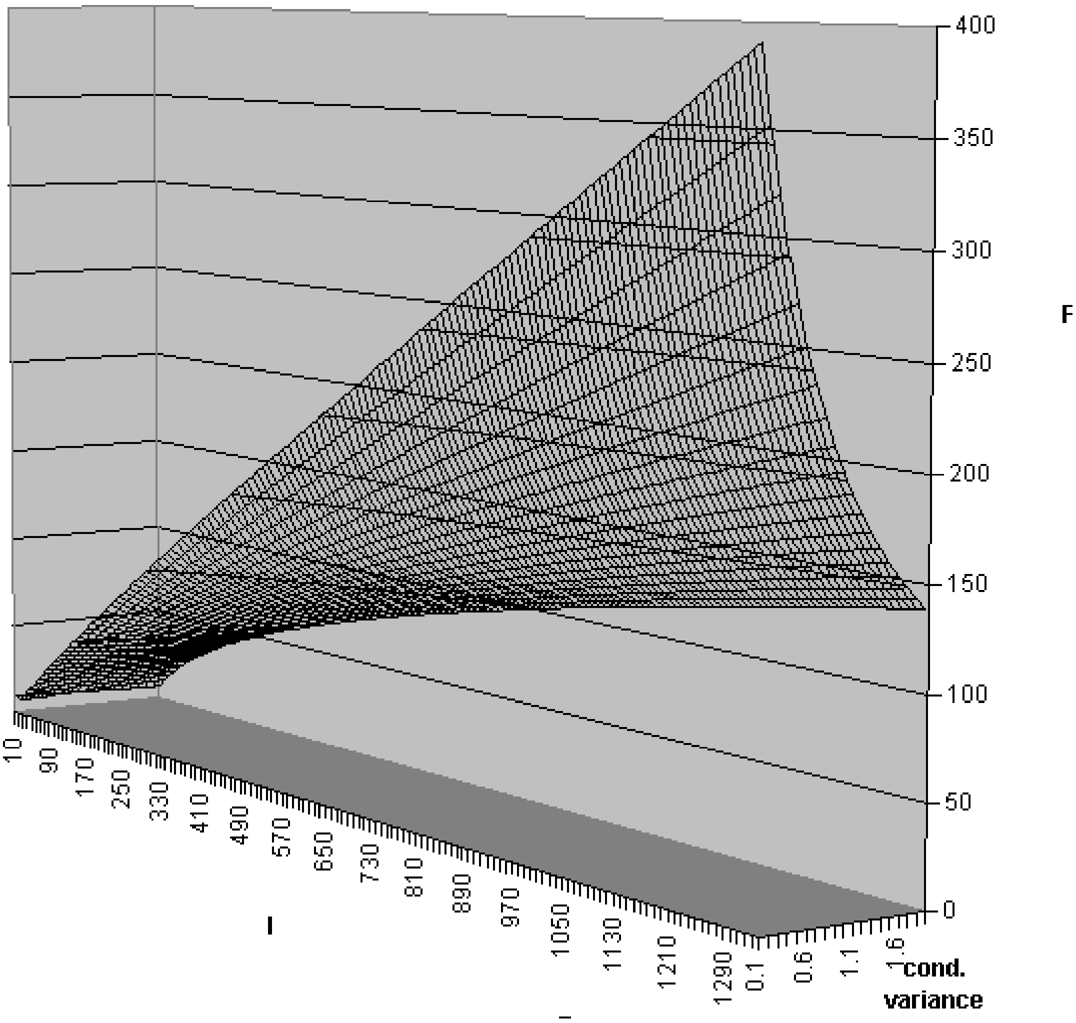

Figure 2: The forward price in terms of I_t and the conditional variance of $\ln I_T$.

The concavity of F_t with respect to I_t does not come as a surprise. With increasing relative risk aversion the investor demands a higher risk premium the wealthier he is. With constant relative risk aversion the forward price is a linear function of I_t representing the fact that the investor requires the same expected gross return independent of his wealth. Therefore, with increasing relative risk aversion, asset prices increase less with increasing I_t the higher I_t reflecting the fact that the investor requires a higher risk premium with increasing wealth.

In order to understand the implications of the functional form of the forward price for the forward price process we need the following proposition.

Proposition 4 *Assume that $I_t > 0$.*[28] *Then, the expected return depends positively (resp. negatively) on I_t if and only if*

$$\frac{F(t, I_t)}{I_t} > \frac{\partial F(t, I_t)}{\partial x}$$

resp.

$$\frac{F(t, I_t)}{I_t} < \frac{\partial F(t, I_t)}{\partial x}$$

and the instantaneous volatility Σ of the forward price F with

$$\begin{aligned} dF_t &= \mu F_t dt + \Sigma F_t dW_t \ , \quad 0 \leqslant t \leqslant T, \\ & \mu, \Sigma \text{ some deterministic functions of } t \text{ and } I_t \end{aligned}$$

is lower (resp. higher) than the instantaneous volatility σ of the information process.

Proof. It follows from the definition of the forward price F_t and application of Itô's Lemma that

$$\begin{aligned} \Sigma F(t, I_t) &= \frac{\partial F}{\partial x}(t, I_t)\, \sigma I \\ \Sigma &= \frac{\partial F}{\partial x}(t, I_t) \frac{I}{F} \sigma \end{aligned}$$

Positive dependence implies

$$\frac{\partial F}{\partial x}(t, I_t) \frac{I}{F} < 1$$

and thus

$$\Sigma < \sigma.$$

On the other hand

$$\Sigma < \sigma$$

[28] The assumption that x resp. I_t is strictly positive is almost equivalent to limited liability. With this assumption and ruling out arbitrage possibilities it follows that the forward price $F(t, x)$ must be strictly positive, too.

implies

$$\frac{\partial F}{\partial x}(t, I_t)\frac{I}{F} < 1$$

which implies positive dependence. The proof for negative dependence is analogous.

■

Proposition 4 shows the properties of asset prices and asset returns under non-linear pricing rules. From figure 1 and figure 2 we see that at least for some values of $var_t(\ln I_T)$ and I_t

$$\frac{F_t}{I_t} > \frac{\partial F(t, I_t)}{\partial x}$$

which according to proposition 4 implies a positive dependence of the expected gross return R_t on I_t. Furthermore, the instantaneous volatility of the forward price will be smaller than the instantaneous volatility of the information process.

5 Conclusion

There is empirical evidence that asset prices are not governed by geometric Brownian motions. First, there is a substantial body of evidence which documents that financial prices are not two-parameter lognormally distributed. In addition many papers show that asset returns are mean-reverting and finally empirical studies on option prices report significant deviations from Black-Scholes prices which is also an empirical fact against the geometric Brownian motion. Moreover, from a theoretical point of view the geometric Brownian motion is not convincing as a characterization of asset price processes since it relies on the following two assumptions (i) the information process is governed by a geometric Brownian motion with constant instantaneous volatility and drift zero and (ii) the representative investor is constant relative risk averse.

In this paper equilibrium forward price processes are analyzed for alternative characterizations of the pricing kernel. Moreover, analytical and numerical solutions of equilibrium asset prices are derived for HARA-utility functions including decreasing and increasing relative risk aversion. In particular we propose the displaced diffusion process as an alternative model for asset price processes. This process has many desirable properties: it is a viable price process, i.e. it is consistent with a representative investor economy. Moreover it is a generalization of the standard geometric Brownian motion, since it is consistent with increasing, decreasing and constant relative risk averion. Finally, the displaced diffusion process is consistent with many empirically well documented facts as random volatility and mean reversion.

To sum up, in this paper equilibrium forward prices are derived. In contrast to the Black-Scholes world the equilibrium forward prices are usually not linear in I_t (the information process) and thus returns may depend on the level of the forward price. Especially, the equilibrium forward prices for non-constant relative risk averse investors are not linear in I_t. Hence, these price processes are consistent with recent empirical findings providing evidence that asset prices are not governed by geometric Brownian motions with constant parameters.

A Appendix

A.1 Derivation of the general characterization of the forward price process

The forward price is characterized by

$$F_t = \frac{\beta(I_t, t)}{\alpha(I_t, t)}, \quad 0 \leqslant t \leqslant T, \tag{33}$$

with

$$\begin{aligned} d\alpha(I_t, t) &= \frac{\partial \alpha(I_t, t)}{\partial x} \sigma I_t dW_t, \quad 0 \leqslant t \leqslant T, \\ \alpha(I_T, T) &= \frac{\frac{\partial}{\partial x} U(I_T)}{a} \end{aligned} \tag{34}$$

$$\begin{aligned} d\beta(I_t, t) &= \frac{\partial \beta(I_t, t)}{\partial x} \sigma I_t dW_t, \quad 0 \leqslant t \leqslant T, \\ \beta(I_T, T) &= I_T \frac{\frac{\partial}{\partial x} U(I_T)}{a} \end{aligned} \tag{35}$$

Applying Itô's lemma yields the following characterization for the forward price $F_t = \digamma(t, \alpha_t, \beta_t)$ (for notational simplicity we write simply $\digamma$ instead of $\digamma(t, \alpha_t, \beta_t)$)

$$d\digamma = \frac{\partial \digamma}{\partial \alpha} d\alpha + \frac{\partial \digamma}{\partial \beta} d\beta + \frac{1}{2}\left(\frac{\partial^2 \digamma}{\partial \alpha^2} d\langle \alpha \rangle + \frac{\partial^2 \digamma}{\partial \beta^2} d\langle \beta \rangle + 2\frac{\partial^2 \digamma}{\partial \alpha \partial \beta} d\langle \alpha, \beta \rangle\right)$$

substituting equations (33), (34), (35) and simplifying yields

$$\begin{aligned}
d\digamma &= \left\{\frac{\beta(I_t,t)}{(\alpha(I_t,t))^3}\left(\frac{\partial \alpha(I_t,t)}{\partial x}\sigma I_t\right)^2\right\} dt \\
&\quad - \left\{\frac{1}{(\alpha(I_t,t))^2}\frac{\partial \alpha(I_t,t)}{\partial x}\frac{\partial \beta(I_t,t)}{\partial x}(\sigma I_t)^2\right\} dt \\
&\quad + \sigma I_t \left\{\frac{1}{\alpha(I_t,t)}\frac{\partial \beta(I_t,t)}{\partial x} - \frac{\beta(I_t,t)}{(\alpha(I_t,t))^2}\frac{\partial \alpha(I_t,t)}{\partial x}\right\} dW_t\ , \\
0 &\leqslant t \leqslant T, \\
\digamma(T, \alpha_T, \beta_T) &= I_T
\end{aligned}$$

A.2 An alternative derivation of Black Scholes asset price processes

In this appendix we use a different approach to derive the forward price process. This alternative derivation gives some nice insights how to look differently at this pricing problem.

Assume that the information process is governed by the stochastic differential equation

$$\begin{aligned} dI_t &= I_t \sigma dW_t, \\ I_0 &> 0, \end{aligned} \tag{36}$$

with σ constant and assume that the utility function of the representative investor $U(x)$ is a member of the Hyperbolic Absolute Risk Aversion (HARA) family with constant relative risk aversion ($\theta = 0$), i.e.

$$U(x) = \frac{1-\gamma}{\gamma}\left(\frac{x}{1-\gamma}\right)^{\gamma}. \tag{37}$$

It is well known, that in equilibrium the following condition must hold

$$\Phi_{0,T} = \frac{\frac{\partial}{\partial x} U\left(F_T\right)}{a},$$

for some scalar $a > 0$ and $F_T = I_T$. Thus, inserting equation (37) yields the following condition for an equilibrium in the economy under consideration

$$a\ \Phi_{0,T} = \left(\frac{I_T}{1-\gamma}\right)^{\gamma-1}$$

Since the pricing kernel $\Phi_{0,t}$ is a martingale, i.e. $\Phi_{0,t} = E_t\left(\Phi_{0,T}\right)$ we can apply Feynman-Kac which yields a function $g : [0,T] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying the deterministic partial differential equation

$$\frac{\partial g\left(t,x\right)}{\partial t} + \frac{1}{2}\frac{\partial^2 g\left(t,x\right)}{\partial x^2}\sigma^2 x^2 = 0 \tag{38}$$

with the boundary condition

$$g\left(T,x\right) = \frac{1}{a}\left(\frac{x}{1-\gamma}\right)^{\gamma-1}$$

and $\Phi_{0,\cdot}$ can be characterized by $\Phi_{0,t} = g\left(t, I_t\right),\ t \in [0,T]$. Solving equation (38) yields the following analytical solution for the pricing kernel when the information process is governed by the stochastic differential equation (36) and the utility function of the representative investor is given by equation (37)

$$\Phi_{0,t} = K\ \exp\left(\frac{1}{2}\left(1-\gamma\right)\left(\gamma-2\right)\sigma^2\left(t-T\right)\right)\ \left(\frac{I_t}{1-\gamma}\right)^{\gamma-1} \tag{39}$$

with K some constant to scale $\Phi_{0,0} = 1$. Rewriting the pricing kernel in differential form yields

$$\begin{aligned} d\Phi_{0,t} &= -\underbrace{\left(1-\gamma\right)\sigma}_{\text{market price of risk}}\Phi_{0,t}dW_t, \\ \Phi_{0,0} &= 1. \end{aligned}$$

Since we know that the equilibrium forward price in the economy under consideration is per definition a martingale under the equivalent martingale measure defined by the pricing kernel given in equation (39), we know that there is a process Γ such that the process F admits

$$F_t = I_T - \int_t^T (1-\gamma)\,\sigma\Gamma_s \; ds - \int_t^T \Gamma_s dW_t \quad 0 \leqslant t \leqslant T. \tag{40}$$

Equation (40) is true for the forward price of any cash flow in the economy under consideration, except that because of the included condition $F_T = I_T$ in equation (40) it characterizes the forward price of the cash flow I_T. To get an analytical solution for the forward price F_t especially to get a characterization for the process Γ we make use of the fact that because $F_t = E_t\left(I_T \Phi_{t,T}\right)$ with the pricing kernel being a deterministic function of I_t and t the forward price is a deterministic function of I_t and t, also. Thus, we know that the forward price is characterized by a function $\widehat{\chi}$ with $F_t = \widehat{\chi}\left(t, I_t\right)$ which solves the following deterministic partial differential equation:

$$\begin{aligned} \frac{\partial \widehat{\chi}\left(t,x\right)}{\partial t} + \frac{1}{2}\frac{\partial^2 \widehat{\chi}\left(t,x\right)}{\partial^2 x} x^2 \sigma^2 - \left(1-\gamma\right)\sigma \frac{\partial \widehat{\chi}\left(t,x\right)}{\partial x} x\sigma &= 0, \\ \widehat{\chi}\left(T,x\right) &= x. \end{aligned}$$

Trying $\widehat{\chi}\left(t, I_t\right) = A\left(t\right) I_t$ with $A\left(t\right)$ some deterministic function of time t we get the following closed form solution for the forward price

$$F_t = \exp\left(\left(1-\gamma\right)\sigma^2\left(t-T\right)\right) I_t.$$

A.3 Characterization of the elasticity of the pricing kernel

We start from the fact that the pricing kernel $\Phi_{0,t}$ can be characterized by a function h satisfying the Feynman-Kac partial differential equation

$$\begin{aligned} 0 &= \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}\mu x + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}\Sigma^2 x^2 \\ h(T,x) &= \frac{\frac{\partial}{\partial x}U(x)}{a} \end{aligned}$$

by $\Phi_{0,t} = h(t, F_t)$ if the forward price is governed by

$$\begin{aligned} dF_t &= \mu F_t + \Sigma F_t dW_t \ , \quad 0 \le t \le T, \\ F_0 &> 0 \end{aligned}$$

with μ and Σ arbitrary borel functions of t and F_t. Furthermore, it follows from Itô's Lemma that the pricing kernel is governed by the following stochastic differential equation

$$\begin{aligned} d\Phi_{0,t} &= \frac{\partial}{\partial x}h(t,F_t)\ \Sigma_t\ F_t\, dW_t \ , \quad 0 \le t \le T, \\ \Phi_{0,0} &= 1. \end{aligned}$$

To derive the elasticity of the pricing kernel we compare this to the usual characterization of the pricing kernel

$$\begin{aligned} d\Phi_{0,t} &= -\Phi_{0,t}\,\kappa_t\, dW_t \ , \quad 0 \le t \le T, \\ \Phi_{0,0} &= 1. \end{aligned}$$

Hence, we get

$$\frac{\kappa_t}{\Sigma_t} = -\frac{\frac{\partial}{\partial x}h(t,F_t)}{h(t,F_t)}\ F_t \ , \quad 0 \le t \le T. \tag{41}$$

Equation (41) shows that in the economy under consideration the ratio of the instantaneous volatility of the Girsanov-process, i.e. the market price of risk, and the instantaneous volatility of the asset are equal to the elasticity of the pricing kernel. In T the following relation holds

$$\left.\frac{\kappa_t}{\Sigma_t}\right|_{t=T} = -\frac{\frac{\partial}{\partial x} h\left(T, F_T\right)}{h\left(T, F_T\right)} F_T = -\frac{\frac{\partial^2}{\partial x^2} U\left(F_T\right)}{\frac{\partial}{\partial x} U\left(F_T\right)} F_T\ ,$$

thus the relative risk aversion of the representative agent equals the elasticity of the pricing kernel.[29]

[29] For a more detailed derivation and discussion see for example Decamps and Lazrak [13], Franke, Stapleton, Subrahmanyam [19] and Lüders and Peisl [40]. For a derivation and discussion of the pricing kernel in an equilibrium model not relying on the representative agent assumption see for example Franke, Stapleton, Subrahmanyam [18].

A.4 Proof of Proposition 3

Given the information process (21) note that the process $\widehat{I}_t = I_t + \theta(1-\gamma)$ is governed by a geometric Brownian motion with constant volatility σ and drift zero

$$\begin{aligned} d\widehat{I}_t &= \sigma \widehat{I}_t dW_t\ , \quad 0 \leqslant t \leqslant T, \\ \widehat{I}_0 &> 0. \end{aligned}$$

Now, let us derive the corresponding pricing kernel in terms of $\widehat{I}$. Using the fact that in equilibrium the following condition must hold

$$\Phi_{0,T} = \frac{\frac{\partial}{\partial x} U(F_T)}{a},$$

for some scalar $a > 0$ and $F_T = I_T$. Inserting the utility function and simplifying yields

$$a\ \Phi_{0,T} = \left(\frac{1}{1-\gamma}\right)^{\gamma-1} \left(\widehat{I}_T\right)^{\gamma-1}.$$

Since the pricing kernel $\Phi_{0,t}$ is a martingale, i.e. $\Phi_{0,t} = E_t(\Phi_{0,T})$ we can apply Feynman-Kac which yields a function $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfying the deterministic partial differential equation

$$\frac{\partial g(t,x)}{\partial t} + \frac{1}{2}\frac{\partial^2 g(t,x)}{\partial x^2}\sigma^2 x^2 = 0 \tag{42}$$

with the boundary condition

$$g(T,x) = \underbrace{\frac{1}{a}\left(\frac{1}{1-\gamma}\right)^{\gamma-1}}_{=K} (x)^{\gamma-1}$$

and $\Phi_{0,t} = g(t, I_t)$. Solving this deterministic partial differential equation yields the following formula for the unique pricing kernel in the economy

$$\begin{aligned} \Phi_{0,t} &= H\ \exp\left(\left(-\gamma\frac{1}{2}+1\right)(1-\gamma)\sigma^2(T-t)\right) \cdot \qquad (43) \\ &\quad \cdot K \underbrace{(I_t + \theta(1-\gamma))}_{=\widehat{I}_t}{}^{\gamma-1}\ , \\ 0 &\leqslant t \leqslant T, \end{aligned}$$

with H some constant to scale $\Phi_{0,0} = 1$. Rewriting the pricing kernel in differential form yields

$$\begin{aligned} d\Phi_{0,t} &= -\underbrace{(1-\gamma)\,\sigma}_{\text{market price of risk}} \Phi_{0,t} dW_t, \ , \quad 0 \leqslant t \leqslant T, \\ \Phi_{0,0} &= 1. \end{aligned}$$

Since we know that the equilibrium forward price in the economy under consideration is per definition a martingale under the equivalent martingale measure defined by the pricing kernel given in equation (43), we know that there is a process Γ such that the process F admits

$$F_t = I_T - \int_t^T (1-\gamma)\,\sigma\Gamma_s \; ds - \int_t^T \Gamma_s dW_s \ , \quad 0 \leqslant t \leqslant T. \tag{44}$$

To prove the proposition consider an asset in the economy with forward price P_t and final value $P_T = \widehat{I}_T$. Note that the introduction of P is purely for technical reasons. It facilitates the derivation of an analytical solution for the forward price F. The pricing kernel and therefore the market price of risk are uniquely determined in the economy and since P_t can be characterized by a deterministic function of t and $\widehat{I}_t$ applying Itô's Lemma yields the following stochastic differential equation for the forward price

$$\begin{aligned} dP_t &= P_t\,(1-\gamma)\,\sigma dt + P_t \sigma dW_t \ , \quad 0 \leqslant t \leqslant T, \\ P_T &= \widehat{I}_T. \end{aligned}$$

Thus, P_t follows a geometric Brownian motion as the equilibrium asset price in a Black-Scholes economy. In addition, we know the unique market price of risk in the economy and by definition $F_T = I_T = \widehat{I}_T - \theta\,(1-\gamma)$. Since $\theta\,(1-\gamma)$ is constant, $F_t = P_t - \theta\,(1-\gamma)$. Applying Itô's Lemma yields

$$\begin{aligned} dF_t &= F_t\,(1-\gamma)\,\sigma^2 \left(1 + \frac{\theta\,(1-\gamma)}{F_t}\right) dt + F_t \sigma \left(1 + \frac{\theta\,(1-\gamma)}{F_t}\right) dW_t, \\ 0 &\leqslant t \leqslant T, \\ F_T &= I_T. \end{aligned}$$

Indeed, this stochastic differential equation characterizes the equilibrium forward price process of I_T since it is consistent with the unique pricing kernel in the economy and it satisfies $F_T = I_T$. Furthermore, we can give an analytical formula for the forward price $F_t = P_t - \theta(1-\gamma)$ in terms of the information process and the pricing kernel since $P_t = \exp((\gamma-1)\sigma^2(T-t))\widehat{I_t}$:

$$F_t = \exp\big((\gamma-1)\sigma^2(T-t)\big)\widehat{I_t} - \theta(1-\gamma) \ , \quad 0 \leqslant t \leqslant T,$$

using

$$\widehat{I_t} = I_t + \theta(1-\gamma) \ , \quad 0 \leqslant t \leqslant T,$$

yields

$$\begin{aligned} F_t &= \exp\big((\gamma-1)\sigma^2(T-t)\big) \\ &\quad (I_t + \theta(1-\gamma)) - \theta(1-\gamma) \ , \\ 0 &\leqslant t \leqslant T, \end{aligned}$$

which can be written as

$$\begin{aligned} F_t &= \exp\big((\gamma-1)\sigma^2(T-t)\big) I_t \\ &\quad +\theta(1-\gamma)\big(\exp\big((\gamma-1)\sigma^2(T-t)\big) - 1\big), \\ 0 &\leqslant t \leqslant T, \end{aligned} \tag{45}$$

■

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