The direct and the inverse problem of finite type Fermi curves of two-dimensional double-periodic Schrödinger operators

Inauguraldissertation
zur Erlangung des akademischen Grades
eines Doktors der Naturwissenschaften
der Universität Mannheim

vorgelegt von

Eva Lübcke
aus Siegen

Mannheim, 2017

Dekan: Dr. Bernd Lübcke, Universität Mannheim
Referent: Professor Dr. Martin Ulrich Schmidt, Universität Mannheim
Korreferent: Professor Dr. Horst Knörrer, ETH Zürich

Tag der mündlichen Prüfung: 11. Januar 2018

Abstract. The aim of this thesis is to give answers to the questions raised by the direct and the inverse problem of Fermi curves of the time-independent two-dimensional double-periodic Schrödinger operator with finite type potential. Generally, the potential can be associated with a complex curve – the Fermi curve – and a line bundle corresponding to the eigenfunctions on this curve. Throughout this work, it is assumed that the potential is of finite type. In this scenario, the associated curve is a Riemann surface and the line bundle corresponds to a divisor on this curve. The advantage of Fermi curves of finite type is that their properties are easier to analyze than the general kind. Usually, the set of finite type potentials is dense in the set of all potentials that lead to a Schrödinger operator. Therefore, it can be expected that the properties of Fermi curves of finite type transfer to the general case.

The direct problem is concerned with the properties of a Fermi curve with given finite type potential. The inverse problem comprises three aspects: the reconstruction of a unique potential from some given data, the determination of the isospectral set for a given potential and the construction of the moduli space, i.e. the space which contains all possible potentials which belong to a Schrödinger operator.

In the investigation of the direct problem, it is first shown that the Fermi curve is a one-dimensional variety in $\mathbb{C}^2$ and the asymptotic behavior of this curve is studied. Moreover, an important symmetry of the Fermi curve is analyzed which is expressed in terms of a holomorphic involution on this curve. It will become visible that only non-special divisors on the Fermi curve which have a certain behavior under this involution are divisors corresponding to a double-periodic finite type potential. During this procedure, a regular operator-valued 1-form is constructed which is necessary to show that the divisor has the mentioned behavior under the involution. This behavior can be correlated with the geometry of the Fermi curve by showing that the divisor can obey the restrictions induced by the involution if and only if the latter has exactly two fixed points.

In the second part of this work, the inverse problem is analyzed. Hereby, we first illustrate which kind of data defines a finite type potential uniquely. One datum of particular interest is – among others – a compact Riemann surface with two marked points at infinity and another such datum is a divisor. Both, the Riemann surface and the divisor, must be equipped with the symmetry induced by the holomorphic involution. Only then can the given data yield the normalization of a Fermi curve corresponding to the two-dimensional double-periodic Schrödinger operator with unique finite type potential. Later in this thesis, it is illustrated that the isospectral set of a given Fermi curve is parametrized by the Prym variety. This is the subset of the Jacobian variety whose elements are antilinear under a holomorphic involution which is induced by the symmetry of the Fermi curve. The main emphasis here is on real-valued potentials. When focusing on this case, the Fermi curve is additionally equipped with a second symmetry in terms of an antiholomorphic involution. The results from [Natanzon, 2004] are used to explain the structure of the real Prym variety. This structure is analyzed with help of both types of involutions. The thesis concludes with a discussion of whether the connected components of the real Prym variety contain special divisors.

Zusammenfassung Ziel dieser Arbeit ist es, die Fragen, welche durch das direkte und das inverse Problem von Fermikurven endlichen Typs des zeitunabhängingen, zweidimensionalen, doppelperiodischen Schrödingeroperators aufgeworfen werden, zu beantworten. Im Allgemeinen kann ein Potential einer komplexen Kurve – der Fermikurve – und einem Linienbündel, welches den Eigenfunktionen auf dieser Kurve entspricht, zugeordnet werden. In dieser Arbeit wird angenommen, dass das betrachtet Potential von endlichem Typ ist. In diesem Fall ist die zu dem Potential gehörige Kurve eine Riemannsche Fläche und das Eigenbündel kann durch einen Divisor auf dieser Kurve charakterisiert werden. Der Vorteil von Fermikurven endlichen Typs ist, dass deren Eigenschaften einfacher zu analysieren sind als die Eigenschaften im Allgemeinen. Normalerweise liegt die Menge der Potentiale endlichen Typs dicht in der Menge aller Potentiale, die zu einem Schrödingeroperator gehören. Daher ist zu erwarten, dass sich Eigenschaften der Fermikurven endlichen Typs auf den allgemeinen Fall übertragen. Das direkte Problem befasst sich mit den Eigenschaften einer Fermikurve zu gegebem Potential endlichen Typs. Das inverse Problem umfasst drei Aspekte: die Rekonstruktion eines eindeutigen Potentials aus gegebenen Daten, der Bestimmung der Isospektralmenge für ein gegebenes Potential und der Konstruktion des Modulraums. In der Untersuchung des direkten Problems wird zunächst gezeigt, dass die Fermikurve eine eindimensionale Varietät in $\mathbb{C}^2$ ist und das asymptotische Verhalten dieser Kurve wird untersucht. Außerdem wird eine Symmetrie der Fermikurve gezeigt, welche durch eine holomorphe Involution ausgedrückt wird. Es wird gezeigt, dass nur nicht-spezielle Divisoren, welche eine bestimmtes Verhalten unter dieser Involution haben, zu einem doppelperiodischen Potential endlichen Typs gehören. In Zuge dieser Betrachtungen wird eine reguläre, operatorwertige 1-Form konsturiert. Mit Hilfe dieser kann gezeigt wrden, dass der Divisor das erwähnte Verhalten unter der Involution hat. Dieses Verhalten wird zu der Geometrie der Fermikurve in Bezug gesetzt: es wirdindem gezeigt, dass der Divisor nur genau dann die durch die Involution vorgegebene Symmetrie erfüllt, wenn die Involution genau zwei Fixpunkte hat. Im zweiten Teil dieser Arbeit wird das inverse Problem betrachtet. Hierbei leiten wir zunächst die Daten her, welche eine Fermikurve endlichen Typs eindeutig charakterisieren, Ein Datum von besonderem Interesse ist eine kompakte Riemannsche Fläche mit zwei ausgezeichneten Punkten bei Unendlich und ein anderes Datum ist ein Divisor. Sowohl die Riemannsche Fläche als auch der Divisor müssen einer Symmetrie genügen, welche durch die holomorphe Involution induziert ist. Nur dann können die gegebenen Daten die Normalisierung einer Fermikurve eines zweidimensionalen, doppelperiodischen Schrödingeroperators mit einem Potential endlichen Typs sein. Es wird gezeigt, dass die Prymvarietät die Isospektralmenge einer gegebenen Fermikurve parametrisiert. Dies ist die Teilmenge der Jacobivarietät, deren Elemente antilinear unter einer holomorphen Involution sind, welche der Symmetrie der Fermikurve entspringt. Der Schwerpunkt liegt hierbei auf reellwertigen Potentialen. In diesem Fall genügt die Fermikurve zusätzlich einer zweiten Symmetrie, welche durch eine antiholomorphe Involution ausgedrückt wird. Ergebnisse aus [Natanzon, 2004] werden verwendet, um die Struktur der reellen Prymvarietät zu erläutern.
Diese Struktur wird mit Hilfe beider Involutionen untersucht. Die Arbeit endet mit einer Diskussion, ob die Zusammenhangskomponenten der reellen Prymvarietät spezielle Divisoren enthalten.

Contents

Introduction

The Schrödinger equation is one of the fundamental equations in quantum mechanics. It was postulated for the first time by Erwin Schrödinger in 1926. This thesis analyzes the direct and the inverse problem of the two-dimensional time-independent Schrödinger operator $-\triangle + u$ with double-periodic finite type potential u for a fixed energy level. A more elaborate explanation of what is meant by this will be given later in the introduction.
The inverse problem is of particular interest in physics which, as an empirical science, verifies theoretical statements with help of data obtained from measurements. In order to describe the inverse problem, it is necessary to know sufficient information about the structure of this data. This can be done by considering the direct problem. In other words, considering the direct problem can be seen as dealing with the question: *"What shall I observe or measure to be able to determine what phenomenom it is?"*
In this work, the eigenvalue problem of the time-independent two-dimensional Schrödinger operator $-\triangle + u$ with finite potential $u \in C(\mathbb{R}^2/\Gamma)$ is taken into account, where Γ is a two-dimensional, positively orientated geometric lattice in $\mathbb{R}^2$. For $(x, y) \in \mathbb{R}^2$, the corresponding equation reads as

$$(-\triangle + u(x, y))\psi(x, y) = \lambda\psi(x, y), \tag{0.1}$$

where $\triangle$ is the Laplace differential operator. Furthermore, the eigenfunctions ψ are restricted to functions that are quasiperiodic with respect to the same lattice Γ. That means for all $(x, y) \in \mathbb{R}^2$ and all $\gamma \in \Gamma$, there holds

$$\psi((x, y) + \gamma) = e^{2\pi\iota(k_1 x + k_2 y)}\psi(x, y). \tag{0.2}$$

In the phase $e^{2\pi\iota(k_1 x + k_2 y)}$, one can – encoded in the domain of the operator – note the parameter $k = (k_1, k_2) \in \mathbb{C}^2$. Then the set of all $k \in \mathbb{C}^2$ for which a non-trivial ψ that obeys (0.1) with $\lambda = 0$ as well as (0.2) exists is called the Fermi curve $F(u)$ of the potential u. $F(u)$ is a variety in $\mathbb{C}^2$ which is invariant under translation by the dual lattice Γ^* of Γ. Due to this invariance, the actual Fermi curve $F(u)/\Gamma^* \subset \mathbb{C}^2/\Gamma^*$ can be found.
This thesis gives answers to questions which are raised by the so-called direct problem of the Fermi curves for Schrödinger operators with finite type potential as well as some aspects of the inverse problem.
The question that arises is: *"What defines the property of a potential to be of finite type?"*
The free Fermi curve $F(0)/\Gamma^*$, i.e. the Fermi curve with zero-potential, consists of two complex planes which intersect each other in infinitely many discrete double points $k_\nu^\pm$ indexed by $\nu \in \Gamma^*$.

All of these double points resolve in the normalization. This normalization – consisting of two copies of $\mathbb{C}$ – can be compactified by adding two points $Q^{\pm}$ at infinity. Additionally, we will see that the Fermi curve $F(u)/\Gamma^*$ converges to $F(0)/\Gamma^*$ asymptotically. We want to determine in which cases the normalization of $F(u)/\Gamma^*$ has two open ends as the normalization of $F(0)/\Gamma^*$ so that we can also compactify it by adding two points $Q^{\pm}$ at infinity. Herefore, we consider the asymptotics of $F(u)/\Gamma^* \subset \mathbb{C}^2/\Gamma^*$. Because Γ^* is a real two-dimensional lattice, it suffices to consider the asymptotics for large imaginary part of $[k] \in F(u)/\Gamma^*$. This is what we mean if we say 'far outside'. It will turn out that one has to distinguish between two parts 'far outside': small open neighborhoods of the double points $k_\nu^{\pm}$ and a remainder which is bounded away from these open neighborhoods. The Fermi curve contained in this remainder is a one-dimensional manifold which means that this part is topologically the same as the free Fermi curve. We also want to know how $F(u)/\Gamma^*$ looks like inside of these open neighborhoods 'far outside'. There are two different cases: either $F(u)/\Gamma^*$ has a double point which is close to $k_\nu^{\pm}$ and the normalization of $F(u)/\Gamma^*$ locally consists of two sheets or $F(u)/\Gamma^*$ has a handle and the normalization locally consists of only one sheet. In order to compactify the normalization of $F(u)/\Gamma^*$ – as it is possible in the free case – $F(u)/\Gamma^*$ may only have finitely many handles. In this case, it is biholomorphic to the normalization of $F(0)/\Gamma^*$ in all parts 'far outside'. Therefore, one property of a Fermi curve of finite type is that in all small open neighborhoods of $k_\nu^{\pm}$, the Fermi curve has only double points which means the corresponding normalization has two open ends, and therefore can be compactified. A second important property is that the lift of the eigenfunction to this compactified normalization is a meromorphic function.

The general philosophy behind the restriction to finite type potentials is twofold: On the one hand, one assumes that the class of finite type potentials is dense in the set of all potentials. On the other hand, the class of finite type potentials is easier to handle than the general case because one needs much less analysis. This allows to study some phenomenons regarding the Fermi curves of finite type and the associated line bundle which should also hold in the general case, yet are easier to analyze.

We want to point out that we only consider finite type potentials for one fixed energy level $\lambda = 0$. In [Feldman et al., 1992], it is shown that besides trivial examples, there exists no real-valued potential such that the curve corresponding to all energy levels, i.e. the Bloch variety, can be compactified.

By the direct problem, we mean that we assume that a certain potential u is known, and therefore we can employ the following approach: We study the asymptotic behavior of this curve for large $\|\operatorname{Im}(k)\|$ as explained above and deduce some more properties concerning the geometric structure of Fermi curves of finite type. Most important, we show the classical trisection of the Fermi curve which was first shown in [Krichever, 1995] and can – for real valued potentials – also be found in [Feldman et al., 2000, Sections 2.5 and 3.18]: A compact part which may contain arbitrary singularities, and a part 'far outside' that decays again into two distinct parts: separated from the small open neighborhoods of the double points $k_\nu^\pm$, the Fermi curve is a one-dimensional manifold and in the neighborhoods of these double points, the two different shapes of the Fermi curve as described above can occur.

The Fermi curve contains all parameters k which allow us to find a non-trivial solution of (0.1) with $\lambda = 0$ obeing (0.2). Hence, it can be seen as kind of a 'spectrum' of fixed energy levels. Accordingly, it is a natural question to ask: *"How does the eigenspace behave at particular values of this 'spectrum'?"* This eigenspace defines a line bundle on the Fermi curve. On Riemann surfaces, each line bundle uniquely defines a divisor. However, the Fermi curve is a singular curve: In this case, the line bundle can be associated with a so-called generalized divisor on the Fermi curve. Generalized divisors are finitely generated subsheaves of the germs of the meromorphic functions on $F(u)/\Gamma^*$. A generalized divisor on a Riemann surface is always associated with a classical divisor, so this is one appropriate possibility to define an analogon to divisors for singular curves. The generalized divisor used in this thesis can be seen as the pole divisor of the normalized eigenfunctions in the regular part of the Fermi curve.

We then define regular finite type potentials. These are finite type potentials for which we can use a unique classical divisor instead of a generalized divisor. For these potentials, we also deduce several properties which have to holds for the divisor of a normalized eigenfunction. Among others, we explore a connection between a symmetry of the Fermi curve and the divisor which will be useful when regarding the questions of the inverse problem. This symmetry of $F(u)/\Gamma^*$ is given in terms of a holomorphic involution σ acting on the Fermi curve. The connection is expressed in terms of a linear equivalence $D + \sigma(D) \simeq K + Q^+ + Q^-$, where K is the canonical divisor on the compactified normalization of $F(u)/\Gamma^*$ and $Q^\pm$ are the two points which are added to the normalization such that the latter is compact. These points are fixed points of the holomorphic involution. One central part of this work is to show that the above linear equivalence can hold if and only if σ has exactly the two fixed points Q^+ and Q^-. Besides that we show some additional properties of D and analyze which extra conditions come into play if we assume that the given potential u is real-valued. We also show that there are two meromorphic differentials with prescribed poles at $Q^\pm$ which take the periodicity conditions with respect to Γ into account. All this will be explained in more detail during the outline of this work hereinafter. After we have gathered enough information in the direct problem, we address ourselves to the inverse problem.

In the inverse problem, the approach works just the other way around. Here, we assume that so-called spectral data – a compact Riemann surface X with two marked points together with a divisor D, a holomorphic involution σ and two meromorphic differentials with prescribed poles and periods – is given and show under which conditions this data describes the normalization of a Fermi curve corresponding to a unique potential u. We will see that we need to assume that the divisor D obeys the above linear equivalence to be able to reconstruct a unique potential and a unique eigenfunction of the Schrödinger operator corresponding to the given data. The inverse problem classically decays into two distinct questions:

1. **The isospectral problem:** A fixed potential u_0 is given and one asks: *"How to determine all potentials u such that $F(u) = F(u_0)$?"* This question can be answered by finding a description of all possible ways to deform the divisor corresponding to u_0 so that the deformed divisor on the same Fermi curve is still a divisor corresponding to an eigenfunction of the Schrödinger operator.

2. **The moduli problem:** A Fermi curve $F(u)$ is given and one asks: *"How to describe all possible deformations of this curve such that the resulting curve is still a Fermi curve to a possibly different potential u than the given one?"*

In this work, we give a first hint on how to answer the questions raised by the isospectral problem for regular finite type potentials in the sense that we show that the so-called isospectral set – mapped to the Jacobian of a given Fermi curve – is parametrized by the so-called Prym variety. This is the subset of the Jacobian which is antisymmetric under σ. We focus on real-valued potentials and construct the real part of the Prym variety in analogy to [Natanzon, 2004].
For our setting, the answer to the questions raised by the moduli problem is an application of the deformation theory from the so-far unpublished paper [Carberry and Schmidt, 2017]. We have explained the structure of this deformation theory in Appendix C to give a more complete picture of the inverse problem, even though these are not results we worked out. As soon as [Carberry and Schmidt, 2017] is published, we refer the reader to this paper for the corresponding proofs. Additionally to the results from [Carberry and Schmidt, 2017], we explain one convenient possibility how to apply this theory on finite type Fermi curves and add some minor details which are important for the scenario considered in this work.

The main focus of this work is divided up into two parts: Part I comprises Chapters 2 to 4 and answers the question raised by the direct problem as far as it is necessary for this work. Part II envolves with the inverse problem and consists of Chapters 5 and 6.
Chapter 1 gives a brief introduction of the two-dimensional Schrödinger equation with periodic potential as it is used in this work. We introduce two equivalent formulations of the quasiperiodicity condition of the Schrödinger operator: the formulation mentioned above, in which the eigenfunctions have to be quasiperiodic with respect to Γ as in (0.2) and another formulation, where the eigenfunctions are double-periodic with respect to Γ. To obtain double-periodicity of the eigenfunctions, one has to modify the Laplacian as $\triangle_k := \triangle + 4\pi\iota\langle k, \nabla\rangle - 4\pi^2 k^2$. Then the Schrödinger operator reads as $-\triangle_k + u$. Both formulations are used in this work: The advantage of the first formulation is that it can also be formulated for parameters $[k] \in \mathbb{C}^2/\Gamma^*$ so that we can consider the eigenfunctions as objects defined on all of $F(u)/\Gamma^*$. This can be exploited later in this work to define finite potentials u in such a way that the Fermi curve $F(u)/\Gamma^*$ can uniquely be associated to another curve which can be compactified by adding two points Q^+ and Q^- at infinity – remember that the Fermi curve itself is generically not compactifiable.

The second formulation is useful for the determination of the asymptotic behavior of the Schrödinger equation: The periodicity of the eigenfunctions with respect to Γ enables us to apply Fourier transformation to the Schrödinger operator $-\triangle_k +u$. The Fourier transform of the modified Laplacian depends on elements of the dual lattice Γ^*. Translating k in some direction in which the dual lattice is involved yields a way to gain insight into the behavior of the operator if the imaginary part of k becomes infinitely large. Because of the translation invariance of $F(u)$ under the real two-dimensional lattice Γ^*, it suffices to consider the asymptotics only for large imaginary values of k. Afterwards, we define the Fermi curve of a Schrödinger operator as the set of points $[k] \in \mathbb{C}^2/\Gamma^*$ for which $[k]$ is a possible parameter for the Schrödinger operator with fixed potential u such that the kernel of this operator is non-empty. In the same breath, we also introduce the Bloch variety of a Schrödinger operator. This is the set of points $([k], \lambda) \in \mathbb{C}^2/\Gamma^* \times \mathbb{C}$ for which λ is an eigenvalue for the Schrödinger operator and k is a possible parameter for this eigenvalue with $\psi \not\equiv 0$. It will become obvious at the end of Chapter 2 and the beginning of Section 3.3 why we also consider this curve. We then show with help of the spectral projection that the Fermi curve is a variety in $\mathbb{C}^2/\Gamma^*$ and that the Bloch variety is a variety in $\mathbb{C}^2/\Gamma^* \times \mathbb{C}$. After this first glance at which kind of object the Fermi curve is, we show the well-known fact that it has some symmetries in terms of involutions. The first involution, which we call σ, holds for all Fermi curves and can be seen as mirroring the Fermi curve at the origin of $\mathbb{C}^2$. The second involution, named τ_1, only exists on Fermi curves associated to real-valued potentials. It acts on $F(u)$ as $k \mapsto -\bar{k}$. Both of these symmetries turn out to be important in the remainder of this work. This is followed by a brief discussion of two concrete examples of Fermi curves which we can determine explicitly: Fermi curves with zero potential and Fermi curves with constant potential. Hereby, we orientate us on [Feldman et al., 2000, Chapter 3.16]. The Fermi curve with zero potential becomes crucial when considering the asymptotics of an arbitrary Fermi curve $F(u)$ because it turns out that all Fermi curves are converging to the free Fermi curve for large imaginary value of k. Moreover, we see in this discussion that the free Fermi curve has the form of two complex planes which intersect each other at infinitely many discrete double points. This already gives a broad hint that we have to consider two different cases for the asymptotics: the part of the Fermi curve which is contained in small open neighborhoods of these double points as well as the rest of the Fermi curve. Equipped with the knowledge about the scenario we are in, we finish the first chapter by reformulating the two questions raised by the inverse problem to illuminate them a bit more with respect to the additional knowledge we have achieved in the first chapter.

In Chapter 2, we give an explicit analysis of the asymptotics for large imaginary values of k. We first show some basic results concerning the regularity properties of the resolvent and consider the behavior of the resolvent with help of Fourier transformation for large imaginary values of k. The key to more insight into the asymptotic behaviour of the Fermi curve is a decomposition of the Schrödinger operator in terms of its Fourier series for the asymptotic case. Away from the Fermi curve, the resolvent is a regular operator which is not defined on the Fermi curve itself.

However, the so-called reduced resolvent is also defined on the Fermi curve. This is the part of the resolvent which describes the complement of the eigenvalues of the Fermi curve. We will see that asymptotically, all but maximal two small eigenvalues are bounded away from zero. The part of the resolvent corresponding to these two small eigenvalues is spanned by either one or two Fourier modi of the corresponding Fourier series. It turns out that it suffices to consider the one- respectively two-dimensional eigenspaces of these small eigenvalues – parametrized by k – to describe the Fermi curve. We will see that in the generic case – i.e. away from the double points $k_\nu^\pm$ – the eigenspace on the free Fermi curve is one-dimensional and the eigenspace at $k_\nu^\pm$ is spanned by the two eigenfunctions correponding to the sheets of the free Fermi curve which intersect in $k_\nu^\pm$. For large $\| \mathrm{Im}(k)\|$, the eigenspace of an arbitrary Fermi curve can be decomposed in the eigenspace of the free Schrödinger equation and an error which describes the deviation from a Fermi curve with continuous potential to the free Fermi curve. We are be able to show that this error tends to zero in the limit of $\| \mathrm{Im}(k)\| \to \infty$. This way to describe the asymptotics is orientated on the methods used in [Klauer, 2011, Section 4.5] to describe the zero set which locally and asymptotically describes the Fermi curve inside of small open neighborhoods of $k_\nu^\pm \in F(0)/\varGamma^*$. A similar way to show the asymptotic behavior of the Fermi curve can also be found in [Feldman et al., 2000, Sections 3.16 and 3.17]. Transferring these asymptotics to the Fermi curve in $\mathbb{C}^2/\varGamma^*$ finally enables us to describe the partition of the Fermi curve into three parts: The first part is compact and an one-dimensional variety in $\mathbb{C}^2/\varGamma^*$. The second part 'far outside' looks like two complex planes, where out of each plane a huge hole and infinitely many small holes are cut out. The third part is contained in the infinitely many small holes that are cut out of the second part. It comprises the parts of $F(u)/\varGamma^*$ which are contained in small open neighborhoods of $k_\nu^\pm$ 'far outside'. We show that inside each of these small open neighborhoods, the Fermi curve might either have an ordinary double point or a handle if $\| \mathrm{Im}\, k\|$ is sufficiently large. In the wake of this, we also make some statements about the number of connected components of the regular parts of the Fermi curve and the Bloch variety as well as about the set of points contained in these varieties which are branch points with respect to specific coverings or which correspond to eigenvalues of higher order. These are necessary tools hereinafter to define finite type potentials and to describe the line bundle on Fermi curves of finite type.

Chapter 3 is concerned with the eigenfunctions ψ of the Schrödinger operator with eigenvalue $\lambda = 0$. In Section 3.1 we take into account that the Fermi curve is a singular curve and introduce so-called generalized divisors, see [Hartshorne, 1986]. These are subsheaves of the sheaf of meromorphic functions on the Fermi curve which are finitely generated submodules of the holomorphic functions on the Fermi curve. This definition constitutes one possibility to generalize the concept of divisors which is well-known on Riemann surfaces. We oriented this part of the thesis on the results in [Klein et al., 2016]. Hereby, we also introduce regular differential forms on a singular curve X – defined as in [Serre, 1988, Chapter IV §3] or [Rosenlicht, 1952] – as those meromorphic differential forms on X whose products with all functions which are holomorphic in some neighborhood of any point of

X have no residues at this point. These are the natural generalization of holomorphic differential forms on Riemann surfaces for complex curves. Afterwards, a more tangible characterization for such n-forms is given for hypersurfaces of dimension n. This is used frequently in the remainder of this work and is taken from [Schmidt, 2002].

Equipped with the concept of generalized divisors on singular curves, we are able to define a generalized divisor corresponding to a unique, normalized eigenfunction of the Schrödinger equation in Section 3.2. At regular points of the Fermi curve, this corresponds to the pole divisor of the unique eigenfunction which is normalized to 1 at the origin $(x, y) = (0, 0)$.

In Section 3.3, we take a close look at the spectral projection on the Bloch variety. It is possible to construct another projection from the spectral projection which can be restricted to the Fermi curve. The latter is kind of a 'modified spectral projection' that enables us to show that there exists a regular projection-valued 1-form ω on the Fermi curve. The regularity of this 1-form is the main ingredient to show in the next chapter that the linear equivalence $D + \sigma(D) \simeq K + Q^+ + Q^-$ holds for a divisor D which can be associated to the generalized divisor if u is a regular finite-type potential. The existence of a similar 1-form is also shown in [Krichever, 1989, Chapter III] for orthogonal lattices under the assumption that the Fermi curve is a Riemann surface.

The first task in Chapter 4 is to define finite type potentials. We motivate this by showing that the normalization of the free Fermi curve $F(0)/\Gamma^*$ can be compactified by adding two points at infinity. Afterwards we define finite type potentials as potentials of such type that the normalization of $F(u)/\Gamma^*$ can also be compactified by adding two points $Q^\pm$ at infinity and that the lift of the normalized eigenfunction to the normalization is a meromorphic function with finitely many poles. In the course of this, we take another uniquely defined one-sheeted covering of the Fermi curve into account. This is defined in [Klein et al., 2016] and denoted as the middleding of a singular curve. It is the most desingularized one-sheeted covering of $F(u)/\Gamma^*$ such that the lift of the generalized divisor on the Fermi curve to this covering is still a generalized divisor. It turns out that the property of a potential to be of finite type can be characterized with help of this one-sheeted covering: finite type potentials are just the potentials for which the middleding can be compactified. To show that the latter characterization is a feasible definition of finite type potentials, the regular 1-form defined from the modified spectral projection comes into play for the first time. Moreover, we show that the regular 1-form on $F(u)/\Gamma^*$ can be lifted to a regular 1-form ω on the middleding. After that, regular finite type potentials are defined as those potentials u for which the middleding and the normalization of $F(u)/\Gamma^*$ coincide. This allows to exploit the advantages of both of these one-sheeted coverings of the Fermi curve: The normalization is a Riemann surface, whereas the middleding is in general a singular curve. However, the sheaf generated by the normalized eigenfunction does generically not define a generalized divisor on the normalization, because it is not a finitely generated submodule of the holomorphic functions of the normalization. It can be shown that the lift to the middleding of the generalized divisor on $F(u)/\Gamma^*$ corresponding to the normalized eigenfunction is again a generalized divisor. Therefore, if the middleding and the normalization are the same one-sheeted covering, this lift can uniquely be associated to a classical divisor. Moreover, we are able to exploit the regularity properties we have shown before for the lift of the differential ω to the middleding. Defining the generalized divisor and deducing the definition for finite type potentials is one of the main tasks of this work. We explore the properties of the eigendivisor D on the normalization of a Fermi curve which corresponds to the Schrödinger operator with regular finite type potential. The first step in Section 4.2.1 is to show a relation already known, compare for example [Novikov and Veselov, 1984, Veselov, 1984, Novikov and Veselov, 1986], for which we could not find an explicit proof in the literature. This property correlates the involution σ, i.e. the geometric structure of the Fermi curve, with the eigendivisor: the linear equivalence $D + \sigma(D) \simeq K + Q^+ + Q^-$ which we have already mentioned above. In [Novikov and Veselov, 1984], it is remarked that I. R. Shafarevich and V. V. Shokurov indicated that the linear equivalence shown in Section 4.2.1 can hold if and only if the involution σ has exactly two fixed points. This is shown in Section 4.2.2 which is another central aspekt of this work. To do so, one first has to take the quotient X_σ of X by the holomorphic involution σ into account. This quotient space is a compact Riemann surface and $\pi_\sigma : X \to X_\sigma$ is a two-sheeted covering. We utilize this covering to construct a cycle basis of $H_1(X, \mathbb{Z})$ with respect to the symmetries induced by σ out of a given cycle basis of $H_1(X_\sigma, \mathbb{Z})$. Secondly, we consider the images of divisors D obeying $D + \sigma(D) \simeq K + Q^+ + Q^-$ under the Abel map into the Jacobi variety of X. In Appendix A, we give some more information relevant to the theory in the background of the results presented here and also provide a brief introduction how our results can be seen in the more general context which was given in [Mumford, 1971] about the Prym variety of a Riemann surface which is equipped with a holomorphic involution. After that, we show in Section 4.2.3 that another symmetry for the divisors holds for real-valued potentials. In Section 4.2.4, we show that the divisor of a finite type potential is non-special. Non-speciality is a property that assures that the preimage in the set of positive divisors of degree g of a point $x \in \mathrm{Jac}(X)$ under the Abel map is unique. Hereby, we use a slightly modified version in the definition of non-speciality which also takes the marked points $Q^\pm$ into account. The reason to use this modified version becomes obvious in Chapter 5. The last step in the observations on the eigendivisor in Section 4.2.5 then shows that the properties of the eigendivisor we have mentioned above still hold if we normalize the eigenfunctions with respect to an arbitrary $(x, y) \in \mathbb{R}^2$. This is done with help of the Krichever construction [Krichever, 1977], a method to define a double-periodic flow in $\mathbb{R}^2$ on the normalization from a given Mittag Leffler distribution which appears again in Chapter 5. Last but not least, we show in Section 4.3 that for every potential u, we can define two meromorphic differentials on X from the parameter k and the lattice Γ as $d(\hat{\gamma}_1 k_1 + \hat{\gamma}_2 k_2)$ and $d(\check{\gamma}_1 k_1 + \check{\gamma}_2 k_2)$. These differentials are uniquely defined by their poles on the compactified normalization X of $F(u)/\Gamma^*$ and their periods with respect to $H_1(X, \mathbb{Z})$. We will see that these differential forms are holomorphic on $X \setminus \{Q^+, Q^-\}$ and have poles of second order at $Q^\pm$. Moreover, it will become visible that the integrals of these 1-forms over elements of $H_1(X, \mathbb{Z})$ are integer. These differentials

enable us in the inverse problem to define a map $X \to \mathbb{C}^2$ so that the image of this map just equals the Fermi curve.
The second part of the work is about the inverse problem. In Chapter 5, we assume that a rather abstract set of data is given. From this, we reconstruct a unique potential of the Schrödinger operator and its eigenfunctions in terms of Baker-Akhiezer functions [Krichever, 1977]. For the more general case of a two-dimensional periodic Schrödinger operator with magnetic field, the existence of a unique Baker Akhiezer function is shown in terms of θ-functions in [Dubrovin, 1981]. The set of data comprises a compact Riemann surface with two marked points Q^+ and Q^- together with two meromorphic differentials with prescribed pole behavior at Q^+ and Q^- and prescribed periods with respect to the first homology group of X, a holomorphic involution σ and a divisor D which obeys $D + \sigma(D) \simeq K + Q^+ + Q^-$. Getting insight into which role this linear equivalence plays in the reconstruction of the eigenfunction and the potential is another important part of this work. We show that it is this linear equivalence which allows the reconstruction of a unique Baker-Akhiezer function ψ and a unique potential u so that ψ is an eigenfunction of $-\triangle + u$ with double-periodic u with respect to some 2-dimensional lattice $\mathit{\Gamma} \subset \mathbb{R}^2$ which is also encoded in the given spectral data. Without this restriction on the divisor, the reconstructed operator would additionally contain a magnet field. Moreover, the complex analytic properties of the given differentials make it possible to tinker a unique map $k : X \to \mathbb{C}^2$ such that the image of this map equals the Fermi curve. Afterwards, we show that it is possible to reconstruct these functions uniquely from the given data. We also state a restriction on the divisors in the spectral data such that the reconstructed potential is real-valued.
The second part of the inverse problem in Chapter 6 answers some of the questions raised by the isospectral problem. We show that the isospectral set is parametrized by the so-called Prym variety corresponding to the normalization of a given Fermi curve with regular finite type potential. Hereby, the focus is on real-valued potentials. One of the main parts of this work is to explain the structure of the Prym variety for real-valued potentials. This work was done in order to be able to use statements from [Natanzon, 2004], which were not proven with the necessary rigor in the original manuscript. Our results of this can be found in Section 6.2 and the basis used for the results shown there can be found in Appendix B. Since we consider these rather as ‘tools’ in the background to fully understand the structure of the real Prym variety and because large parts of the corresponding results in [Natanzon, 2004] were more or less correct – despite not being worked out entirely – we attached these in the appendix.
Section 6.2 starts with the definition of so-called real curves (X, τ) as compact Riemann surfaces which are endowed with an antiholomorphic involution τ. After characterizing the different types of such real curves, we take a close look at the properties of holomorphic differential forms which obey a certain transformation behavior with respect to the additional real structure. Hereby, we make some existence statements which need the results from Appendix B: There, a 1-1-connection between real Fuchsian groups and the existence of real spinors is shown. A real Fuchsian group is

a Fuchsian group with an additional structure corresponding to the antiholomorphic involution of X and real spinors are spinors which obey certain properties on the fixed point sets of τ on X. The existence statements on holomorphic differential forms with certain properties on these fixed point sets shown in Section 6.2.2 then derive from the existence theorems on the real spinors introduced in Appendix B. Section 6.2.3 is concerned with giving complete proofs of the assertions on M-curves from [Natanzon, 2004] because there large parts were missing. M-curves are real curves on which the fixed point set of the antiholomorphic involution has the maximal number of connected components.
In Section 6.2.4, we introduce the real part of the Jacobian variety as it was proposed in [Natanzon, 2004], characterize the connected components of the real part of the Prym variety and determine which connected component contains only non-singular divisors. Hereby, we noted that in [Natanzon, 2004] an additional transformation behavior of the lattice to define the Jacobian as $\mathbb{C}^g$ modulo this lattice is overlooked. Moreover, it is the main effort in this Section to understand the characterization of the connected components of the real Jacobian since the corresponding arguments are missing in [Natanzon, 2004].
Finally, we describe in Section 6.2.5 the real parts of the Prym variety. The results on their structure can be deduced from the foregoing section by considering the quotient X_σ of a real curve (X, τ) with holomorphic involution σ as a real curve. To this we can apply the results from the previous parts of Section 6.2. Lifting the objects on X_σ obtained from these results to X makes it possible to describe the real part of the Prym variety of X. To describe whether a connected component of the real Prym variety contains images of non-special divisors obeying the linear equivalence, a property called positive or negative definiteness is defined in [Natanzon, 2004]. We modified it in a way that it fits to the setup and explain in detail why this modified definition is necessary. This definition is then used to gain clarity about the existence of non-singular connected components of the real Prym variety.
Concluding, we can say that this thesis considers many different aspects of the questions raised by both, the direct and the inverse problem of finite type Fermi curves of the Schrödinger operator. In the part about the direct problem all questions could be answered. Even though many results of this part are considered as known in the common literature, we could not find any sources which gather all this knowledge. For example in the second part of Chapter 2, beginning with Section 2.3, some properties of the Fermi curve are presented rather detailed, which are necessary for the rest of this work but for which we could not find the corresponding proofs. Moreover, we also could not find a motivation why finite type potentials are assumed to exist and hope that the discussion about the line bundle on the singular Fermi curve in Chapter 5 answers this question. Also the way the Baker Akhiezer functions are determined in the literature we know uses more complicated tools which are avoided in this work. In Chapter 5, we only use basic knowledge on Riemann surface theory together with the Krichever construction [Krichever, 1977] to show that a unique potential can be constructed from some given spectral data. Moreover, we could

not find other sources than [Natanzon, 2004] to describe the real part of the Prym variety which parametrizes the isospectral set. We hope that the way we presented this topic provides all the necessary details to bring clarity in the properties of the real Prym variety of a finite type Fermi curve and enables further research on the isospectral set for real-valued finite type potentials.

1. The Schrödinger equation and Fermi curves

1.1. The operators

The initial point of this work is the eigenvalue problem of the time-independent, two dimensional Schrödinger operator $-\triangle + u$. This reads as

$$(-\triangle + u(x,y))\psi(x,y) = \lambda\psi(x,y), \tag{1.1}$$

where $(x,y) \in \mathbb{R}^2$ and $\triangle = \partial_x^2 + \partial_y^2$ is the usual Laplace differential operator. Furthermore, we consider the potential u to be double-periodic and continuous and the solution ψ to be quasiperiodic with respect to the same non-degenerate, positively orientated geometric lattice $\varGamma \subseteq \mathbb{R}^2$. This lattice shall be generated by $\hat{\gamma}$ and $\check{\gamma}$, where $\det(\hat{\gamma}, \check{\gamma}) \neq 0$. Then $\varGamma$ is the discrete set

$$\varGamma := \{\gamma = m\hat{\gamma} + n\check{\gamma} \mid m, n \in \mathbb{Z}\}.$$

Without loss of generality, we assume that $\varGamma$ is spanned by $\hat{\gamma} := \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $\check{\gamma}$ into an arbitrary direction linearly independent from $\hat{\gamma}$. All other choices of generators $(\hat{\gamma}, \check{\gamma})$ of $\varGamma$ only differ from this choice by a linear transformation. So for all $(x,y) \in \mathbb{R}^2$ and all $\gamma \in \varGamma$ one has

$$u((x,y) + \gamma) = u(x,y) \tag{1.2}$$

and

$$\psi((x,y) + \gamma) = e^{2\pi\iota\langle k,\gamma\rangle}\psi(x,y), \tag{1.3}$$

where $k \in \mathbb{C}^2$ is some boundary condition and $\langle\cdot,\cdot\rangle$ is the complex bilinear dot product which is defined as

$$\langle \left(\begin{smallmatrix}x_1\\y_1\end{smallmatrix}\right), \left(\begin{smallmatrix}x_2\\y_2\end{smallmatrix}\right)\rangle = (x_1y_1 + x_2y_2) \quad \text{with} \quad \left(\begin{smallmatrix}x_1\\y_1\end{smallmatrix}\right), \left(\begin{smallmatrix}x_2\\y_2\end{smallmatrix}\right) \in \mathbb{C}^2.$$

As usual, we will denote equation (1.1) together with (1.2) and (1.3) as the *two-dimensional Schrödinger equation*. The corresponding operator is the same as the one considered in [Klauer, 2011]. To keep this work self-contained, we will now introduce the used notation and objects.

Due to the quasiperiodicity, we take $(x,y) \in \mathbb{R}^2$ out of a fundamental domain of $\mathbb{R}^2/\varGamma$ which is defined as follows.

Definition 1.1 (Fundamental Domain)**.** Let $\hat{\gamma}, \check{\gamma}$ be a basis of $\mathit{\Gamma}$ and p be an arbitrary point in $\mathbb{R}^2$. Then the *fundamental domain* is the cell representing the action of $\mathit{\Gamma}$ on $\mathbb{R}^2$ by

$$\mathit{\Delta} := \{(x,y) \in \mathbb{R}^2 \mid (x,y) = p + t\hat{\gamma} + s\check{\gamma} \text{ with } (t,s) \in [0,1]^2\}. \tag{1.4}$$

If we pick out an element of $\mathit{\Delta}$ to investigate the behavior of some functions nearby or at this element, we always assume that the base point p is chosen in such a way that the picked out element is in the interior of $\mathit{\Delta}$.

In this work, we consider continuous potentials $u \in C(\mathbb{R}^2/\mathit{\Gamma}) := C(\mathbb{R}^2/\mathit{\Gamma}, \mathbb{C})$ and assume that the eigenfunctions ψ are contained in $L^2(\mathit{\Delta}, \mathbb{C})$. Hereby, $L^2(\mathit{\Delta}, \mathbb{C})$ is the Banach space of L^2-functions which map from $\mathit{\Delta}$ to $\mathbb{C}$ with

$$\|f\|_2 := \left(\int_{\mathit{\Delta}} |f|^2 \, \mathrm{d}\mu\right)^{1/2} < \infty,$$

where $d\mu$ denotes the Lebesgue measure. Likewise $L^2(\mathbb{R}^2/\mathit{\Gamma}, \mathbb{C})$ is defined, where the integral in the norm is also taken over some fundamental domain $\mathit{\Delta}$. We do not distinguish between measurable functions and equivalence classes with respect to the kernel of $\|\cdot\|_2$. Furthermore, we write $L^2(\mathit{\Delta}) := L^2(\mathit{\Delta}, \mathbb{C})$ as well as $L^2(\mathbb{R}^2/\mathit{\Gamma}) := L^2(\mathbb{R}^2/\mathit{\Gamma}, \mathbb{C})$. We assume that the potential u is either non-constant in both directions of $\mathit{\Gamma}$ or constant. The case of a potential which is non-constant in one direction can be reduced to the 1-dimensional Schrödinger operator.
The scalar $\lambda \in \mathbb{C}$ in (1.1) is the eigenvalue of $-\triangle + u$ with respect to an eigenfunction $\psi(\lambda) \not\equiv 0$. For fixed boundary condition $k \in \mathbb{C}^2$, the set of all possible eigenvalues λ in equation (1.1) is called the spectrum of $-\triangle + u$. Obviously, this set depends on k.
We denote by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ the bilinear form which is for $f, g \in L^2(\mathit{\Delta})$ respectively $f, g \in L^2(\mathbb{R}^2/\mathit{\Gamma})$ defined as

$$\langle\!\langle f, g \rangle\!\rangle = \int_{\mathit{\Delta}} fg \, d\mu. \tag{1.5}$$

To obtain that $-\triangle + u$ is a closed, unbounded operator, one has to consider $-\triangle + u$ as an operator which depends on k since the domain on which this operator is closed depends on k. In other words: by varying k one obtains a family of differential operators with different domains. In [Klauer, 2011, Section 3.1] is shown a way how to obtain a family of differential operators which is equivalent to the two-dimensional Schrödinger equation and which also varies for different k, but for which the domain remains the same, i.e. a way to modify the Schrödinger operator such that an eigenfunction ψ_k of the modified operator is also periodic with respect to $\mathit{\Gamma}$. More precisely, one can formulate the whole problem described above by (1.1), (1.2) and (1.3) equivalently on $\mathbb{R}^2/\mathit{\Gamma}$ by considering

$$(-\triangle_k + u)\psi_k = \lambda\psi_k, \tag{1.6}$$

where

$$\triangle_k := \nabla_k^2 = \triangle + 4\pi\iota\langle\nabla, k\rangle - 4\pi^2 k^2 \tag{1.7}$$

with $\nabla_k := \nabla + 2\pi\iota k$. The last equality sign in (1.7) follows since k does not depend on $(x, y) \in \mathbb{R}^2$, and therefore $\langle k, \nabla\rangle = \langle\nabla, k\rangle$. It is furthermore shown in [Klauer, 2011, Propositions 3.1.6 and 3.17] that for all $k \in \mathbb{C}^2$, the operator $\triangle_k$ is formally a normal operator which maps the space of $\mathit{\Gamma}$-periodic functions to itself. The last statement holds since differential operators are local and $\triangle_k$ does not depend on $(x, y) \in \mathbb{R}^2$.

Note that the eigenfunction $\psi(k, \lambda, (x, y))$ depends on $k \in \mathbb{C}^2$, $\lambda \in \mathbb{C}$ as well as on $(x, y) \in \mathit{\Delta}$. For brevity, we omit the dependencies on the parameters which are either not important for the respective considerations or for which it is clear from the context which values are taken into account. We write for example $\psi(k)$ if we are only interested in the dependence of this function on k for arbitrary (x, y) and λ or if it is clear which values (x, y) and λ are taken into account. Likewise, we write $\psi(x, y)$, $\psi(\lambda)$, $\psi(k, \lambda)$ or $\psi(k, (x, y))$. In case that we are not interested in any of the dependencies on k, (x, y) and λ we just write ψ. Analogous notation is used for the eigenfunction ψ_k in the formulation of (1.6).

The next Lemma shows that the two mentioned formulations are indeed equivalent. It is partly shown in [Klauer, 2011, Theorem 3.1.10, Lemma 4.2.2 and Proposition 4.2.4]:

Lemma 1.2. *Equations* (1.1), (1.2) *and* (1.3) *are fulfilled by* k, λ, ψ *and* u *if and only if for* $(x, y) \in \mathbb{R}^2$ *holds*

$$(-\triangle_k + u(x, y))\psi_k(x, y) = \lambda\psi_k(x, y),$$

where $\psi_k(x, y) = e^{-2\pi\iota\langle k, \binom{x}{y}\rangle}\psi(k, (x, y))$. *Furthermore,* $\psi_k((x, y) + \gamma) = \psi_k(x, y)$ *for all* $\gamma \in \mathit{\Gamma}$ *and* $\psi_{k+\kappa}(x, y) = e^{-2\pi\iota\langle\kappa, \binom{x}{y}\rangle}\psi_k(x, y)$ *for all* $\kappa \in \mathit{\Gamma}^*$.

Proof. Basically, the proof uses that $\triangle e^{2\pi\iota\langle k, \binom{x}{y}\rangle}\psi(x, y) = e^{2\pi\iota\langle k, \binom{x}{y}\rangle} \triangle_k \psi(x, y)$ and that the multiplication with $e^{2\pi\iota\langle k, \cdot\rangle}$ is bijective. The first equality follows by direct calculations. Therefore,

$$\begin{aligned}
\lambda\psi(k, (x, y)) &= (-\triangle + u(x, y))\psi(k, (x, y)) \\
&= (-\triangle + u(x, y))e^{2\pi\iota\langle k, \binom{x}{y}\rangle} \underbrace{e^{-2\pi\iota\langle k, \binom{x}{y}\rangle}\psi(k, (x, y))}_{=\psi_k(x,y)} \\
&= e^{2\pi\iota\langle k, \binom{x}{y}\rangle}(\triangle_k + u(x, y))\psi_k(x, y) \\
\Leftrightarrow \lambda\psi_k(x, y) &= (\triangle_k + u(x, y))\psi_k(x, y).
\end{aligned}$$

The periodicity of ψ_k with respect to $\mathit{\Gamma}$ follows immediately from the quasiperiodicity (1.3) of ψ with $k \in \mathbb{C}^2$:

$$\begin{aligned}
\psi_k((x, y) + \gamma) &= e^{-2\pi\iota\langle k, \binom{x}{y} + \gamma\rangle}\psi((x, y) + \gamma) = e^{-2\pi\iota\langle k, \binom{x}{y}\rangle}e^{-2\pi\iota\langle k, \gamma\rangle} \underbrace{\psi(k, ((x, y) + \gamma))}_{=e^{2\pi\iota\langle k, \gamma\rangle}\psi(k,(x,y))} \\
&= e^{-2\pi\iota\langle k, \binom{x}{y}\rangle}\psi(k, (x, y)) = \psi_k(x, y).
\end{aligned}$$

To see the quasiperiodicity with respect to Γ^*, we use modified Wirtinger operators. In analogy to (1.7), we modify the Wirtinger operators $\partial := \frac{1}{2}(\partial_x + \iota\partial_y)$ and $\bar\partial := \frac{1}{2}(\partial_x - \iota\partial_y)$ by adding boundary conditions $k \in \mathbb{C}^2$ as

$$\partial_k := \partial + \pi(\iota k_1 + k_2) \quad \text{and} \quad \bar\partial_k := \bar\partial + \pi(\iota k_1 - k_2). \tag{1.8}$$

Then for all $k \in \mathbb{C}^2$, there holds

$$\begin{aligned}\bar\partial_k \partial_k &= \frac{1}{4}\left((\partial_x + \iota\partial_y + 2\pi(\iota k_1 + k_2))(\partial_x - \iota\partial_y) + 2\pi(\iota k_1 - k_2)\right)\\ &= \frac{1}{4}\left(\partial_x^2 + \partial_y^2 - 4\pi^2(k_1^2 + k_2^2) + 4\pi\iota(k_1\partial_x + k_2\partial_y)\right)\\ &= \frac{1}{4}(\triangle - 4\pi^2 k^2 + 4\pi\iota\langle k, \nabla\rangle) = \frac{1}{4}\triangle_k\,.\end{aligned} \tag{1.9}$$

For $k \in \mathbb{C}^2$ and $\kappa \in \Gamma^*$ it is

$$\begin{aligned}e^{-2\pi\iota\langle\kappa,\binom{x}{y}\rangle}\partial_k e^{2\pi\iota\langle\kappa,\binom{x}{y}\rangle} &= e^{-2\pi\iota\langle\kappa,\binom{x}{y}\rangle}\partial e^{2\pi\iota\langle\kappa,\binom{x}{y}\rangle} + \pi(\iota k_1 + k_2)\\ &= \partial\pi(\iota\kappa_1 + \kappa_2) + \pi(\iota k_1 + k_2) = \partial_{k+\kappa}\end{aligned}$$

and likewise $\bar\partial_{k+\kappa} = e^{-2\pi\iota\langle\kappa,\binom{x}{y}\rangle}\bar\partial_k e^{2\pi\iota\langle\kappa,\binom{x}{y}\rangle}$. Taking these two results together yields that

$$\triangle_{k+\kappa} = e^{-2\pi\iota\langle\kappa,\binom{x}{y}\rangle}\triangle_k e^{2\pi\iota\langle\kappa,\binom{x}{y}\rangle}$$

and thus $\psi_k(k+\kappa,(x,y)) = e^{-2\pi\iota\langle\kappa,\binom{x}{y}\rangle}\psi_k(k,(x,y))$. □

Corollary 1.3. *For $u \in C(\mathbb{R}^2/\Gamma)$, let $k \in \mathbb{C}^2$ be such that there exists a non-trivial $\psi \in L^2(\Delta)$ which solves $(-\triangle + u)\psi(k) = \lambda\psi(k)$ and obeys* (1.3)*. Then for every $\kappa \in \Gamma^*$, there exists a non-trivial $\psi(k+\kappa) \in L^2(\Delta)$ which solves $(-\triangle + u)\psi(k+\kappa) = \lambda\psi(k+\kappa)$ and it is $\psi(k+\kappa) = \psi(k)$.*

Proof. The operator $-\triangle + u$ with boundary value k equals the operator $-\triangle + u$ with boundary value $k+\kappa$ and $\kappa \in \Gamma^*$ since $\langle\kappa,\gamma\rangle \in \mathbb{Z}$ for all $\gamma \in \Gamma$ and $\kappa \in \Gamma^*$. Therefore, the domains of these two operators coincide. More precisely, Lemma 1.2 yields that

$$\psi(k+\kappa,(x,y)) = e^{2\pi\iota\langle k+\kappa,\binom{x}{y}\rangle}\psi_{k+\kappa}(x,y) = e^{2\pi\iota\langle\kappa,\binom{x}{y}\rangle}e^{2\pi\iota\langle k,\binom{x}{y}\rangle}\psi_k(x,y) = e^{2\pi\iota\langle\kappa,\binom{x}{y}\rangle}\psi(k,(x,y)).$$

□

From now on, we denote both formulations described above as the Schrödinger equation. We term these viewpoints in the same way as it is done in the unpublished paper [Schmidt, 2002, Chapter 2] for the Dirac operator with periodic potentials. Corresponding to the two formulations of the Schrödinger equation introduced above, the eigenfunctions can be characterized in two different ways. For this we first note that we can also find a one-dimensional representation of Γ with help of the dual lattice Γ^* of Γ. This is defined by the scalar product representation of the linear forms

$$\Gamma^* := \{\kappa \in \mathbb{R}^2 \mid \langle \kappa, \gamma \rangle \in \mathbb{Z} \text{ for all } \gamma \in \Gamma\}.$$

The generators of the dual lattice Γ^* are denoted by $\hat{\kappa}$ and $\check{\kappa}$. With the generators $\hat{\gamma} = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $\check{\gamma}$ of Γ, we define them as

$$\hat{\kappa} := \frac{1}{\hat{\gamma}_1\check{\gamma}_2 - \hat{\gamma}_2\check{\gamma}_1}\begin{pmatrix}\check{\gamma}_2\\-\check{\gamma}_1\end{pmatrix} = \begin{pmatrix}1\\-\frac{\check{\gamma}_1}{\check{\gamma}_2}\end{pmatrix} \quad \text{and} \quad \check{\kappa} := \frac{1}{\hat{\gamma}_1\check{\gamma}_2 - \hat{\gamma}_2\check{\gamma}_1}\begin{pmatrix}-\hat{\gamma}_2\\\hat{\gamma}_1\end{pmatrix} = \begin{pmatrix}0\\\frac{1}{\check{\gamma}_2}\end{pmatrix} \tag{1.10}$$

since the choice $\langle \hat{\kappa}, \hat{\gamma} \rangle = \langle \check{\kappa}, \check{\gamma} \rangle = 1$ and $\langle \check{\kappa}, \hat{\gamma} \rangle = \langle \check{\kappa}, \hat{\gamma} \rangle = 0$ is natural for the generators of Γ^* in our eyes. So Γ^* is also a two-dimensional real lattice. Hereby, $\hat{\gamma}_1\check{\gamma}_2 - \hat{\gamma}_2\check{\gamma}_1 = \check{\gamma}_2 \neq 0$ holds because Γ is a two-dimensional lattice. For an arbitrary $k \in \mathbb{C}^2$, a one-dimensional representation of Γ is given by the map $\gamma \mapsto e^{2\pi i \langle k, \gamma \rangle} \in \mathbb{C} \setminus \{0\}$ in $GL(\mathbb{C})$ with $k \in \mathbb{C}^2$. The representations for $k \neq k'$ are equal if $k - k' \in \Gamma^*$ since then one has

$$e^{2\pi\iota\langle k,\gamma\rangle} = e^{2\pi\iota\langle k-(k-k'),\gamma\rangle} = e^{2\pi\iota\langle k',\gamma\rangle}.$$

So the set of all one-dimensional representations of Γ can be identified with $\mathbb{C}^2/\Gamma^*$. Let $[k] \in \mathbb{C}^2/\Gamma^*$ be the equivalence class defined by $\{\tilde{k} \in \mathbb{C}^2 \mid \exists\, \kappa \in \Gamma^* : \tilde{k} = k + \kappa\}$. Then every $[k] \in \mathbb{C}^2/\Gamma^*$ induces a line bundle on the torus $\mathbb{R}^2/\Gamma$ with cocycles $e^{2\pi i \langle k, \left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\rangle}$ whose sections are functions ψ on $\mathbb{R}^2$ which fulfill the same quasiperiodicity condition (1.3) on the boundary of Δ as the eigenfunctions of the Schrödinger operator (1.1) with Γ-periodic potential u.

Definition 1.4. The formulation of Schrödinger equation (1.1) with $u \in C(\mathbb{R}^2/\Gamma)$ and quasiperiodicity of the eigenfunctions (1.3) we denote as the *fundamental domain* formulation.

Note that this formulation of the Schrödinger equation (1.1) is invariant under translations of $k \mapsto k + \kappa$, where $\kappa \in \Gamma^*$: For $\kappa \in \Gamma^*$ and $\gamma \in \Gamma$ one has $e^{2\pi\iota\langle k+\kappa,\gamma\rangle} = e^{2\pi\iota\langle k,\gamma\rangle}$, so the quasiperiodicity condition (1.3) does not change under this shift and also neither the Schrödinger equation (1.1) nor the periodicity of the potential (1.2) are effected by it. The periodicity of the eigenfunctions with respect to Γ^* is shown in Corollary 1.3. Ergo, one can consider equivalence classes $[k] \in \mathbb{C}^2/\Gamma^*$ where $[k] = [k']$ if and only if $k - k' \in \Gamma^*$. As before, we omit the dependency on $[k]$ of the eigenfunctions and write ψ instead of $\psi([k])$ if it is not necessary to take this dependency into account.

The other viewpoint, i.e. considering $-\triangle_k + u$ with eigenfunctions that are periodic with respect to Γ, has the advantage that the eigenfunctions ψ_k are in some sense 'global' since they exist on the whole torus $\mathbb{R}^2/\Gamma$. In this formulation, Lemma 1.2 shows that the eigenfunctions are quasiperiodic with respect to Γ^* and periodic with respect to Γ.

Definition 1.5. The functions $e^{-2\pi\iota\langle k,\binom{x}{y}\rangle}$ are global, non-vanishing sections on the torus $\mathbb{R}^2/\Gamma$ of the line bundle at $[k] \in \mathbb{C}^2/\Gamma^*$. Since a line bundle is trivial if and only if it has a nowhere vanishing section, these functions are called *trivializations.* One can describe the sections ψ_k by $\psi_k = e^{-2\pi\iota\langle k,\binom{x}{y}\rangle}\psi$ which are usual functions multiplied with these trivializations. However, in general $e^{2\pi\iota\langle k,\binom{x}{y}\rangle} \neq e^{2\pi\iota\langle k+\kappa,\binom{x}{y}\rangle}$ for $\kappa \in \Gamma^*$, so the trivializations are different for each $k \in \mathbb{C}^2$ and the sections have to obey the following quasiperiodicity condition with respect to Γ^*:

$$\psi_k(k+\kappa,(x,y)) = e^{-2\pi\iota\langle \kappa,\binom{x}{y}\rangle}\psi_k(k,(x,y)) \text{ for } \kappa \in \Gamma^*.$$

It is shown in Lemma 1.2 that $\psi_k(k,(x,y))$ obeys these two conditions. We will use the trivialization formulation when the asymptotic behavior of the Fermi curve is considered, i.e. for $k \in \mathbb{C}^2$ with large imaginary value. In the other parts of this work the fundamental domain formulation is used. This is because then it is possible to consider the Fermi curve as a subset of $\mathbb{C}^2/\Gamma^*$.
Later on, we will also make use of the transpose of the Schrödinger operators of $-\triangle_k + u$ and $-\triangle + u$. Therefore, this operator is introduced here. Obviously, $-\triangle_k + u : L^2(\mathbb{R}^2/\Gamma) \to L^2(\mathbb{R}^2/\Gamma)$ as well as $-\triangle + u : L^2(\varDelta) \to L^2(\varDelta)$ are linear and bounded operators for $u \in C(\mathbb{R}^2/\Gamma, \mathbb{C})$. The dual space of $L^2(\mathbb{R}^2/\Gamma)$ is again $L^2(\mathbb{R}^2/\Gamma)$ and the dual space of $L^2(\varDelta)$ is $L^2(\varDelta)$. Then the adjoint or transposed operator $(-\triangle_k + u)^T : L^2(\mathbb{R}^2/\Gamma) \to L^2(\mathbb{R}^2/\Gamma)$ of $-\triangle_k + u$ is defined as follows: We consider for every $g \in L^2(\mathbb{R}^2/\Gamma, \mathbb{C})$ the map

$$L^2(\mathbb{R}^2/\Gamma) \to \mathbb{C}, \quad f \mapsto \langle\!\langle g, (-\triangle_k + u)f \rangle\!\rangle.$$

This map defines a continuous, linear functional on $L^2(\mathbb{R}^2/\Gamma)$. So by the Riesz-Fischer Theorem [Reed and Simon, 1980, Theorem II.4], for every $g \in L^2(\mathbb{R}^2/\Gamma)$, there exists a unique element $h \in L^2(\mathbb{R}^2/\Gamma)$ such that for all $f \in L^2(\mathbb{R}^2/\Gamma)$

$$\langle\!\langle g, (-\triangle_k + u)f \rangle\!\rangle = \langle\!\langle h, f \rangle\!\rangle.$$

Then $(-\triangle_k + u)^T g = h$ defines the transposed Schrödinger operator. Likewise the transposed operator $(-\triangle + u)^T$ of $-\triangle + u$ is defined with the only difference that for fixed k, the dual space of $\{\psi(k,\cdot) \in L^2(\varDelta) \mid \psi(k,(x+y)+\gamma) = e^{2\pi\iota\langle k,\binom{x}{y}\rangle}\psi(k,(x,y))\}$ is given by $\{\varphi \in L^2(\varDelta) \mid \varphi_k(k,(x+y)+\gamma) = e^{-2\pi\iota\langle k,\binom{x}{y}\rangle}\varphi(k,(x,y))\}$ since then the product $\varphi\psi$ is periodic with respect to Γ, and so one can apply partial integration to obtain $\langle\!\langle \nabla\varphi(k,\cdot), \psi(k,\cdot) \rangle\!\rangle = -\langle\!\langle \varphi(k,\cdot), \nabla\psi(k,\cdot) \rangle\!\rangle$. Next, the exact form of $(-\triangle_k + u)^T$ is determined.

Lemma 1.6. *The transpose of the Schrödinger operator* (1.6) *equals* $-\triangle_{-k} + u$ *with eigenfunction* φ_k. *If* ψ_k *is in the kernel of* $-\triangle_k + u$, *then* ψ_{-k} *is in the kernel of* $(\triangle_k + u)^T$.

Proof. At first, we note that $u(x,y) \in \mathbb{C}$, so $u^t(x,y) = u(x,y)$. The same holds for $\lambda \in \mathbb{C}$. To determine the transposed operator of $-\triangle_k + u$ consider first the transposed operator of

$\nabla_k = \nabla + 2\pi\iota k$. For $\psi_k \in L^2(\mathbb{R}^2/\Gamma)$ and $\varphi_k \in L^2(\mathbb{R}^2/\Gamma)$ it is

$$\langle\!\langle \nabla_k \varphi_k, \psi_k \rangle\!\rangle = \int_\Delta (\nabla + 2\pi\iota k)\varphi_k \cdot \psi_k \, d\mu = \int_\Delta \varphi_k \cdot (-\nabla + 2\pi\iota k)\psi_k \, d\mu.$$

The second equality sign follows by partial integration, where the term evaluated at the boundary of Δ is zero since ψ_k and φ_k are assumed to be periodic on $\mathbb{R}^2$ with respect to the lattice Γ. So $\nabla_k^T := -\nabla + 2\pi ik$. This implies for the transposed Laplacian with boundary conditions

$$\begin{aligned} \triangle_k^T &= (\nabla_k^T)^2 = \nabla_k^T \cdot \nabla_k^T = (-\nabla + 2\pi ik)(-\nabla + 2\pi ik) \\ &= \triangle - 4\pi ik \cdot \nabla - 4\pi k^2 = \triangle_{-k}. \end{aligned}$$

Let $(-\triangle_k + u)\psi_k(k, \cdot) = 0$. Then $(-\triangle_{-k} + u)\psi_k(-k, \cdot) = 0$. □

1.2. The Bloch variety and the Fermi curve

Definition 1.7. For the Schrödinger equation (1.6), the *Bloch variety* of the Schrödinger operator $-\triangle_k + u$ is defined as

$$B(u)/\Gamma^* := \{([k], \lambda) \in \mathbb{C}^2/\Gamma^* \times \mathbb{C} \mid (k, \lambda) \in B(u)\}, \tag{1.11}$$

where

$$B(u) := \{(k, \lambda) \in \mathbb{C}^2 \times \mathbb{C} \mid \text{There is a } \psi_k \in L^2(\mathbb{R}^2/\Gamma) \setminus \{0\} \text{ such that } (-\triangle_k + u)\psi_k = \lambda\psi_k\}.$$

$B(u)$ is invariant under translations by $\kappa \in \Gamma^*$ as we will see below in Lemma 1.8. So strictly speaking, the Bloch variety is defined as in (1.11). However, both $B(u)$ and $B(u)/\Gamma^*$ are denoted as the Bloch variety. If it is not clear from the context which version is used, this will be mentioned explicitly. The subset of one eigenvalue λ in (1.11) is

$$F_\lambda(u) := \{k \in \mathbb{C}^2 \mid \text{There is a } \psi_k \in L^2(\mathbb{R}^2/\Gamma) \setminus \{0\} \text{ such that } (-\triangle_k + u)\psi_k = \lambda\psi_k\}.$$

$F_\lambda(u)$ is called the *Fermi curve* to the eigenvalue λ. Since changing the fixed value λ by adding a constant only changes the potential u by subtracting a constant, we consider the Fermi curve belonging to $\lambda = 0$ in the sequel and call it $F(u)$. In case we are only interested in $[k] \in \mathbb{C}^2/\Gamma^*$, $F(u)/\Gamma^*$ is considered which is the actual Fermi curve. Both, $F(u)$ and $F(u)/\Gamma^*$, are denoted as the Fermi curve. We point out which one is meant if necessary.

The following Lemma is taken from [Klauer, 2011, Lemma 4.3.1]. It justifies why it is possible to consider $B(u)/\Gamma^*$ respectively $F(u)/\Gamma^*$. Since the proof is so short and essential, the basic steps are repeated here.

Lemma 1.8. *For all $u \in L^2(\mathbb{R}/\varGamma)$ and for all $\kappa \in \varGamma^*$, the Bloch variety $B(u)$ and the Fermi curve $F(u)$ are invariant under transformations $k \mapsto k + \kappa$.*

Sketch of the proof. Let $k \in B(u)$. Then there exists an eigenfunction $\psi_k \in L^2(\mathbb{R}^2/\varGamma)$ with $\psi_k \not\equiv 0$ and an eigenvalue $\lambda \in \mathbb{C}$ such that (1.6) holds. Multiplying (1.6) from the left side by $e^{-2\pi\iota\langle\kappa,\binom{x}{y}\rangle}$ yields

$$-e^{-2\pi\iota\langle\kappa,\binom{x}{y}\rangle} \triangle_k e^{2\pi\iota\langle\kappa,\binom{x}{y}\rangle} e^{-2\pi\iota\langle\kappa,\binom{x}{y}\rangle}\psi_k + u(x,y)e^{-2\pi\iota\langle\kappa,\binom{x}{y}\rangle}\psi_k = \lambda e^{-2\pi\iota\langle\kappa,\binom{x}{y}\rangle}\psi_k.$$

Due to (1.9), this yields that the transformation $k \mapsto k + \kappa$ induces nothing but a transformation of the eigenfunction and does not influence the Bloch variety $B(u)$. Obviously, the same arguments apply for $F(u)$. □

Next, we show that for $u \in C(\mathbb{R}^2/\varGamma, \mathbb{C})$, the resolvent

$$\operatorname{Res}(\lambda, k, u) := (-\triangle_k + u - \lambda)^{-1} \tag{1.12}$$

of the Schrödinger operator $-\triangle_k + u$ is a compact operator from $L^2(\mathbb{R}^2/\varGamma)$ to $L^2(\mathbb{R}^2/\varGamma)$. The set of compact operators on a Banach space V to a Banach space W are denote as $\mathcal{K}(V, W)$ and for $V = W$ as $\mathcal{K}(V)$. To show this, some basics about Fourier transformation is used. Hereby, let $\ell^2(\varGamma^*) := \ell^2(\varGamma^*, \mathbb{C})$ be the space of all square summable sequences with values in $\mathbb{C}$. More precisely, let ω be the space of all sequences in $\mathbb{C}$. Then

$$\ell^2(\varGamma^*) := \{(x_\kappa)_{\kappa\in\varGamma^*} \in \omega \mid \sum_{\kappa\in\varGamma^*} |x_\kappa|^2 < \infty\}$$

which is a Banach space when it is equipped with $\|(x_n)_{n\in\mathbb{N}}\|_{\ell^2(\varGamma)} := \sum_{n=1}^\infty |x_n|^2$. The two spaces $L^2(\mathbb{R}^2/\varGamma)$ and $\ell^2(\varGamma^*)$ can be related. An examples which enlightens this connection is given by the map

$$\mathbb{R}^2 \to \mathbb{C}, \quad x \mapsto e^{2\pi\iota\langle\kappa,x\rangle}$$

for $\kappa \in \mathbb{C}^2$. If $\kappa \in \varGamma^*$, then this function is periodic with respect to $\varGamma$ and one can consider it as a function on $\mathbb{R}^2/\varGamma$. With the fundamental domain $\varDelta$ of $\mathbb{R}^2/\varGamma$, we define the Fourier transform as

$$\mathcal{F} : L^2(\mathbb{R}^2/\varGamma) \to \ell^2(\varGamma^*), \quad f \mapsto \left(\kappa \mapsto \frac{1}{\operatorname{Vol}(\varDelta)} \int_\varDelta e^{-2\pi\iota\langle\kappa,x\rangle} f(x)\, dx\right).$$

We call $\hat{f} := \mathcal{F}(f)$. Vice versa, the Fourier Inversion Theorem [Reed and Simon, 1975, Theorem IX.1] yields that the inverse Fourier transform for sequences in $\ell^2(\varGamma^*)$ is defined as

$$\mathcal{F}^{-1} : \ell^2(\varGamma^*) \to L^2(\mathbb{R}^2/\varGamma), \quad \hat{f} \mapsto \left(x \mapsto \sum_{\kappa\in\varGamma^*} e^{2\pi\iota\langle\kappa,x\rangle} \hat{f}(\kappa)\right),$$

where $\mathcal{F}^{-1}(\hat{f})(x)$ is called the Fourier series of $\hat{f}$. Furthermore, deriving a function in $L^2(\mathbb{R}^2/\Gamma)$ is a multiplication operator in $\ell^2(\Gamma^*)$, i.e. for a multi-index α one has $\widehat{\partial^\alpha f} = (2\pi\iota\kappa)^\alpha \hat{f}$.

Definition 1.9. We denote by $\mathcal{F} : L^2(\mathbb{R}^2/\Gamma) \to \ell^2(\Gamma^*)$ the *Fourier transform* and by $\mathcal{F}^{-1} : \ell^2(\Gamma^*) \to L^2(\mathbb{R}^2/\Gamma)$ the *inverse Fourier transform.* Let $O : L^2(\mathbb{R}^2/\Gamma) \to L^2(\mathbb{R}^2/\Gamma)$ be an operator. We then define the *Fourier transform* $\hat{O}$ of O as

$$\hat{O} : \ell^2(\Gamma^*) \to \ell^2(\Gamma^*), \quad \hat{\psi}_k \mapsto \mathcal{F}(O\mathcal{F}^{-1}(\hat{\psi}_k)).$$

Vice versa, for $\hat{O} : \ell^2(\Gamma^*) \to \ell^2(\Gamma^*)$, the *inverse Fourier transform* O is defined as

$$O : L^2(\mathbb{R}^2/\Gamma) \to L^2(\mathbb{R}^2/\Gamma), \quad \psi_k \mapsto \mathcal{F}^{-1}(\hat{O}\mathcal{F}(\psi_k)).$$

The Fourier transform will be very useful in the first two chapters of this work. We first use it here to show the following theorem.

Theorem 1.10. *Let $u_0 \in C(\mathbb{R}^2/\Gamma, \mathbb{C})$. For every $c > 0$ and every open bounded subset $K \in \mathbb{C}^2$, there exists a $\lambda > 0$ such that $(k, u) \mapsto (-\triangle_k + u + \lambda)^{-1}$ defines a map $K \times B_c(u_0, \|\cdot\|_\infty) \to \mathcal{K}(L^2(\mathbb{R}^2/\Gamma))$ which is holomorphic in k and u. Furthermore, there also exists an open bounded subset W containing λ_0 such that the map*

$$\lambda \mapsto (\lambda - \triangle_k + u)^{-1}$$

is a boundedly invertible and holomorphic for $(\lambda, u, k) \in W \times U \times K$.

Proof. It is

$$(-\triangle_k + u + \lambda)^{-1} = (-\triangle_k + \lambda)^{-1}(\mathbb{1} + u(-\triangle_k + \lambda)^{-1})^{-1}.$$

For $f \in L^2(\mathbb{R}^2/\Gamma)$, one has due to Parseval's identity [Reed and Simon, 1980, Theorem II.6]

$$\|(-\triangle_k + \lambda)^{-1} f\|_{L^2(\mathbb{R}^2/\Gamma)} = \|\widehat{(-\triangle_k + \lambda)^{-1} f}\|_{\ell^2(\Gamma^*)} = \left(\sum_{\kappa\in\Gamma^*} \left| \frac{\hat{f}(\kappa)}{\lambda + 4\pi(k+\kappa)^2} \right|^2 \right)^{1/2}. \tag{1.13}$$

Since $\|(k+\kappa)^2\| \to \infty$ for $\|\kappa\| \to \infty$ and $k \in K$, the infimum of $\mathrm{Re}(k + \kappa^2)$ for $\kappa \in \Gamma^*$ and $k \in K$ exists. We choose $\lambda \in \mathbb{R}$ such that

$$\lambda > -\inf_{\substack{\kappa\in\Gamma^* \\ k\in K}} 4\pi \,\mathrm{Re}(k+\kappa)^2.$$

Then $\lambda + 4\pi \,\mathrm{Re}(k+\kappa)^2 > 0$ for all $\kappa \in \Gamma^*$. Using again Parseval's identity yields

$$\|(-\triangle_k + \lambda)^{-1} f\|_{L^2(\mathbb{R}^2/\Gamma)} \leq \underbrace{\frac{1}{\lambda + \inf_{\kappa\in\Gamma^*}(4\pi(k+\kappa)^2)}}_{<\infty} \|f\|_{L^2(\mathbb{R}^2/\Gamma)}.$$

Choosing λ additionally in such a way that $\lambda > -\inf_{\substack{\kappa\in\Gamma^*\\ k\in K}} 4\pi\,\mathrm{Re}(k+\kappa)^2 + c$ gives then

$$\|(-\triangle_k+\lambda)^{-1}f\|_{L^2(\mathbb{R}^2/\Gamma)} < \frac{1}{c+4\pi\|\,\mathrm{Im}(k+\kappa)^2\|}\|f\|_{L^2(\mathbb{R}^2/\Gamma)} \leq c^{-1}\|f\|_{L^2(\mathbb{R}^2/\Gamma)}.$$

Together with Hölder's inequality we obtain from this that

$$\begin{aligned}\|u(-\triangle_k+\lambda)^{-1}f\|_{L^2(\mathbb{R}^2/\Gamma)} &= \|u\|_{L^\infty(\mathbb{R}^2/\Gamma,\mathbb{C})}\|(-\triangle_k+\lambda)^{-1}f\|_{L^2(\mathbb{R}^2/\Gamma)}\\ &< c\|(-\triangle_k+\lambda)^{-1}f\|_{L^2(\mathbb{R}^2/\Gamma)} < \|f\|_{L^2(\mathbb{R}^2/\Gamma)}.\end{aligned}$$

Accordingly, by Neumann's Theorem $(\mathbb{1}+u(-\triangle_k+\lambda)^{-1})^{-1}$ converges and thus

$$\|(-\triangle_k+u+\lambda)^{-1}f\|_{L^2(\mathbb{R}^2/\Gamma)} \leq C\|f\|_{L^2(\mathbb{R}^2/\Gamma)}.$$

So for λ sufficiently large, $(-\triangle_k+u+\lambda)^{-1}$ is a bounded operator from $L^2(\mathbb{R}^2/\Gamma)$ to $L^2(\mathbb{R}^2/\Gamma)$. The compactness of this operator can be shown as follows: Fourier transforming $(-\triangle_k+\lambda)^{-1}$, compare Definition 1.9, yields that

$$\widehat{(-\triangle_k+\lambda)^{-1}} = \sum_{\kappa\in\Gamma^*}\frac{1}{\lambda+4\pi(k+\kappa)^2}$$

is a multiplication operator which can be written as the limit of finite range operators which is compact, compare [Reed and Simon, 1980, Theorem VI.13]. Since $\mathcal{F}^{-1}$ is bounded and the composition of a compact operator with a bounded operator is again compact, $(-\triangle_k+\lambda)^{-1}$ is a compact operator. Because the operator $(\mathbb{1}+u(-\triangle_k+\lambda)^{-1})^{-1}$ is bounded, the same arguments apply to $(-\triangle_k+u+\lambda)^{-1}$ and show that this is also a compact operator. To see that the map

$$K\times B_c(u_0)\to\mathcal{K}(L^2(\mathbb{R}^2/\Gamma)),\quad (k,u)\mapsto(-\triangle_k+u+\lambda)^{-1}$$

is holomorphic in $u\in B_c(u_0)$, the Neumann series is used again. We have seen before that $(\mathbb{1}+u(-\triangle_k+\lambda)^{-1})^{-1}=\sum_{n=0}^\infty(u(-\triangle_k+\lambda)^{-1})^n$ converges for our choice of λ. So

$$(-\triangle_k+u+\lambda)^{-1} = (-\triangle_k+\lambda)^{-1}(\mathbb{1}+u(-\triangle_k+\lambda)^{-1})^{-1} = \sum_{n=0}^\infty(-\triangle_k+\lambda)^{-1}(-u(-\triangle_k+\lambda)^{-1})^n. \tag{1.14}$$

The last term can be read as the Taylor series of $(-\triangle_k+\lambda+u)^{-1}$ with respect to u at $u_0=0$ and thus this operator is holomorphic in u. Holomorphy in k can also be seen in equation (1.14): By definition of $\triangle_k$ in (1.7), this is a holomorphic operator in k. So it follows from (1.13) that $(-\triangle_k+\lambda)^{-1}$ is for $\lambda>0$ sufficiently large holomorphic in k since $\widehat{-\triangle_k+\lambda}$ is the uniform limit of functions holomorphic in k. Because the inverse Fourier transform is complex linear, also

$-\triangle_k + u = \mathcal{F}^{-1}(\widehat{-\triangle_k + \lambda})$ is holomorphic. So $(-\triangle_k + \lambda)^{-1}(u(-\triangle_k + \lambda)^{-1})^n$ is holomorphic in k for each $n \in \mathbb{N}_0$, and therefore the right hand side of (1.14) is the uniform limit of functions holomorphic in k which is holomorphic in k. Due to the Riesz-Schauder Theorem [Reed and Simon, 1980, Theorem VI.15], the spectrum of $(\lambda_0 + \triangle_k - u)^{-1}$ contains at most the accumulation point zero. The other points are eigenvalues of finite order. Therefore, the set

$$S(k,u) := \{\lambda \in \mathbb{C} \setminus \{\lambda_0\} \mid (\lambda_0 - \lambda)^{-1} \text{ is in the spectrum of } (\lambda_0 - \triangle_k + u)^{-1}\}$$

is discrete. For $\lambda \in \mathbb{C} \setminus (S(k,u) \cup \{\lambda_0\})$, the value $(\lambda_0 - \lambda)^{-1}$ is in the resolvent set of $(\lambda_0 - \triangle_k + u)^{-1}$. So $(\lambda_0 - \lambda)^{-1} - (\lambda_0 - \triangle_k + u)^{-1}$ is boundedly invertible. Moreover,

$$\begin{aligned}
&(\lambda_0 - \lambda)^{-1}(\lambda_0 - \triangle_k + u)^{-1}\left((\lambda_0 - \lambda)^{-1} - (\lambda_0 - \triangle_k + u)^{-1}\right)^{-1} = \\
&\qquad = \left(\left((\lambda_0 - \lambda)^{-1} - (\lambda_0 - \triangle_k + u)^{-1}\right)(\lambda_0 - \lambda)(\lambda_0 - \triangle_k + u)\right)^{-1} = \\
&\qquad\qquad = (\lambda_0 - \triangle_k + u - (\lambda_0 - \lambda)^{-1}) = (\lambda - \triangle_k + u)^{-1}.
\end{aligned}$$

For this reason, the map $(u,k,\lambda) \to (\lambda - \triangle_k + u)^{-1}$ is defined on $\{(u,k,\lambda) \in U \times K \times B_\varepsilon(\lambda_0) \mid \lambda \notin S(k,u) \cup \{\lambda_0\}\}$ and the eigenvalue λ_0 is an isolated pole of the resolvent map (1.12). Consequently, for λ_0 sufficiently large, there exists an open neighborhood W of λ_0 such that for all $\lambda \in W$ the above considerations we have made for λ_0 also hold for λ. Holomorphy in λ one sees with help of the resolvent equation for fixed (k_0, u_0) and by using the continuity of the resolvent Res in λ on $B_\varepsilon(\lambda_0)$, i.e.

$$\mathrm{Res}(\lambda, k_0, u_0) - \mathrm{Res}(\mu, k_0, u_0) = (\lambda - \mu)\mathrm{Res}(\lambda, k_0, u_0)\mathrm{Res}(\mu, k_0, u_0). \qquad \square$$

Next, we want to show that the Bloch variety can be considered as a two-dimensional subvariety of $\mathbb{C}^3$ and the Fermi curve as a subvariety of $\mathbb{C}^2$. The following definitions and statements are taken from [Gunning and Rossi, 1965, Chapter II].

Definition 1.11 ([Gunning and Rossi, 1965, Definition II.1])**.** Let U be a domain in $\mathbb{C}^2$. $V \subset U$ is a *holomorphic subvariety* if for every $z \in U$, there is a neighborhood U_z of z and finitely many functions $f_1, \ldots, f_m$ which are holomorphic on U_z such that

$$V \cap U_z = \{x \in U_z \mid f_1(x) = \cdots = f_m(x) = 0\}.$$

Theorem 1.12 ([Gunning and Rossi, 1965, Theorem II.3])**.** *Let F be a collection of functions which are holomorphic on an open set $U \subset \mathbb{C}^n$. Then $V(F) := \{x \in U \mid f(x) = 0 \text{ for all } f \in F\}$ is a subvariety of U.*

Definition 1.13. A *holomorphic variety* is a second-countable Hausdorff topological space V for which there exists a covering by open subsets V_α and homeomorphisms $F_\alpha : V_\alpha \to W_\alpha$ between the subsets $V_\alpha \subset V$ and holomorphic subvarieties W_α of open sets $U_\alpha \subseteq \mathbb{C}^{n_\alpha}$ such that for each nonempty intersection $V_\alpha \cap V_\beta$, the decomposition

$$F_\alpha \circ F_\beta^{-1} : F_\beta(V_\alpha \cap V_\beta) \to F_\alpha(V_\alpha \cap V_\beta)$$

is a biholomorphic map.

Since these are the only kind of varieties considered in this work, the add-on holomorphic is left away in the sequel. To show that $B(u)$ is a variety consisting of subvarieties in $\mathbb{C}^3$ and that $F(u)$ is a variety consisting of subvarieties in $\mathbb{C}^2$, the fact that the resolvent is a compact operator on $L^2(\mathbb{R}^2/\mathit{\Gamma})$ is used. From this we deduce a representation for the Bloch variety $B(u)$ as well as for the Fermi curve $F(u)$ as zero sets of functions which are holomorphic in (k, λ) respectively holomorphic in k. The base to show this is the next theorem which can also be found in [Klauer, 2011, Theorem 4.1.3]. However, the fact that $F(u)$ is a variety in $\mathbb{C}^2$ is essential for the considerations in this work. Accordingly, the proof given in [Klauer, 2011, Theorem 4.1.3] is repeated here, where some details are emphasized differently since a different viewpoint than the one in [Klauer, 2011] is necessary in the sequel.

Theorem 1.14. *[[Klauer, 2011], Theorem. 4.1.3] Let $u_0 \in C(\mathbb{R}^2/\mathit{\Gamma}, \mathbb{C})$, $k_0 \in \mathbb{C}^2$ and $\lambda_0 \in \mathbb{C}^2$ such that $(u_0, k_0, \lambda_0) \in (u_0, B(u_0))$, i.e. there exists a $\psi_k \in L^2(\mathbb{R}^2/\mathit{\Gamma})$ which is not identically zero such that*

$$(-\triangle_{k_0} + u_0)\psi_k = \lambda_0 \psi_k.$$

Then there are open neighborhoods $U \subset C(\mathbb{R}^2/\mathit{\Gamma})$, $K \subset \mathbb{C}^2$ and $B_\varepsilon(\lambda_0) \subset \mathbb{C}$ with $\varepsilon > 0$ of u_0, k_0 and λ_0 such that for all $(u, k, \lambda) \in U \times K \times B_\varepsilon(\lambda_0)$, there exists a finite-dimensional subspace $\mathit{\Sigma}(u, k)$ of $L^2(\mathbb{R}^2/\mathit{\Gamma})$ independent of λ which is invariant under the Schrödinger operator $(-\triangle_k + u)$ with $\dim \mathit{\Sigma}(u, k) = \dim \mathit{\Sigma}(u_0, k_0)$. Furthermore, the intersection of the graph of the map $u \mapsto B(u)$ with $U \times K \times B_\varepsilon(\lambda_0)$ is the zero locus of the determinant

$$(u, k, \lambda) \mapsto \det(\lambda + \triangle_k - u)|_{\mathit{\Sigma}(u,k)}.$$

This map is holomorphic in k, λ and u.

Proof. It follows from Theorem 1.10 that there are bounded open neighborhoods $U \subset C(\mathbb{R}^2/\mathit{\Gamma})$ of u_0 and $K \subset \mathbb{C}^2$ of k_0 such that there exists a fixed $\lambda_1 > 0$ for which the resolvent $(\lambda_1 + \triangle_k - u)^{-1}$ is contained in $\mathcal{K}(L^2(\mathbb{R}^2/\mathit{\Gamma}))$ for all $(u, k) \in U \times K$. Moreover, the proof of this theorem gives that λ_0 is an isolated pole of the resolvent $(\lambda - \triangle_{k_0} + u_0)^{-1}$. Let $\varepsilon > 0$ be so small that $B_\varepsilon(\lambda_0)$ contains no other eigenvalue of $-\triangle_{k_0} + u_0$. The Nagumo Theorem [Nagumo and Yamaguchi, 1993, Article 22] yields that

$$P_{u_0}(k_0) := \frac{1}{2\pi\iota} \oint_{|\xi|=\varepsilon} (\lambda_0 + \xi + \triangle_{k_0} - u_0)^{-1}\, d\xi$$

is a projector on the generalized eigenspaces corresponding to λ_0 of the Schrödinger operator $-\triangle_{k_0} + u_0$ which is invariant under this operator. The derivation of this generalized eigenprojection can for example be found in [Kato, 1980, Section II.1.4]. It projects on the sum of the eigenspaces belonging to the sheets meeting in (k_0, λ_0). This situation is roughly depicted in Figure 1.1 as a projection from $\mathbb{C}^3$ to $\mathbb{R}^2$ for two sheets meeting at (k_0, λ_0) with fixed potential u_0. Moreover, it is

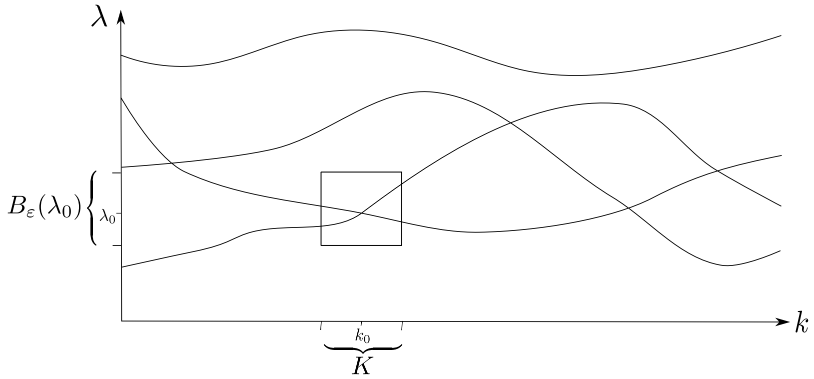


Figure 1.1.: Sketch of an example: two different sheets around (k_0, λ_0, u_0) running together, where the picture in $\mathbb{C}^3$ is projected to $\mathbb{R}^2$ and neglected the open neighborhood of u_0.

known from Theorem 1.10 that the resolvent is compact if λ is not an eigenvalue of $-\triangle_{k_0} + u_0$ and that the eigenvalues of $-\triangle_{k_0} + u_0$ are discrete. So for a closed curve around $\lambda_0 \in \operatorname{Spec}(-\triangle_{k_0} + u_0)$ with sufficiently small diameter such that this curve contains no eigenvalue of $-\triangle_{k_0} + u_0$, also $P_{u_0}(k_0)$ is compact and the image $\Sigma(u_0, k_0)$ of $P_{u_0}(k_0)$ is finite-dimensional. Therefore, one can choose a basis $f_1, \ldots, f_m$ of $\Sigma(u_0, k_0) \subset L^2(\mathbb{R}^2/\Gamma)$. Likewise one can define the spectral projection $P^T_{u_0}(k_0)$ for the transposed operator $(-\triangle + u_0)^T$ for λ_0, where the image of this projection is called $\Sigma^T(u_0, k_0) \subset L^2(\mathbb{R}^2/\Gamma)$. By the same arguments as for $-\triangle_{-k} + u$, one can choose a basis $g_1, \ldots, g_m$ of the corresponding space $\Sigma^T(u_0, k_0)$. We choose $g_1, \ldots, g_m$ such that they yield a dual basis of $f_1, \ldots, f_m$, i.e. such that $\langle\langle g_i, f_j \rangle\rangle = \delta_{ij}$. $P^T_{u_0}(k_0)$ is the transposed operator of $P_{u_0}(k_0)$. Since $P_{u_0}(k_0)$ is a projection on $\Sigma(u_0, k_0)$, one has $P_{u_0}(k_0)|_{\Sigma(u_0,k_0)} = \mathbb{1}_{\Sigma(u_0,k_0)}$. So

$$\langle\langle g_i, f_j \rangle\rangle = \langle\langle g_i, P_{u_0}(k_0) f_j \rangle\rangle = \langle\langle g_i, P_{u_0}(k_0)^2 f_j \rangle\rangle = \langle\langle P^T_{u_0}(k_0, \lambda_0) g_i, P_{u_0}(k_0) f_j \rangle\rangle$$

and the matrix $\left(\langle\langle P_{u_0}(k_0)^T g_i, P_{u_0}(k_0) f_j \rangle\rangle\right)_{i,j=1}^m$ is regular. Moreover, the map $(-\triangle_{k_0} + u_0) P_{u_0}(k_0) : \Sigma(u_0, k_0) \to \Sigma(u_0, k_0)$ is linear and $\Sigma(u_0, k_0)$ is finite-dimensional. Accordingly, the invariance of $P_{u_0}(k_0)$ under $-\triangle_{k_0} + u_0$ yields that there exists a matrix $(A_{ij}(u_0, k_0, \lambda_0)^m_{i,j=1}$ of $-\triangle_{k_0} + u_0$ such that for all $l, i = 1, \ldots, m$

$$\langle\langle g_l, (\lambda_0 - \triangle_{k_0} + u_0) f_i \rangle\rangle = \sum_{j=1}^m A_{ij} \langle\langle g_l, f_i \rangle\rangle.$$

However, this matrix is build with respect to fixed u_0, k_0 and λ_0, so it does not depend on k, u or λ. In the next step, this will be changed by constructing a matrix from this which depends holomorphically on k and λ and continuously on u. To do so, remember first that by Theorem 1.10 one can choose a bounded, open neighborhood K of k_0 and a bounded open neighborhood U of u_0 such that the resolvent is holomorphic with respect to k and u on these neighborhoods for all λ_0 which are in the resolvent set of $-\triangle_{k_0}+u_0$. Let moreover $\varepsilon > 0$ be so small that $B_\varepsilon(\lambda_0)$ contains no other eigenvalue of $-\triangle_k + u$. Then for all $(u,k) \in U \times K$, the same arguments as above apply to the operators

$$P_u(k) := \frac{1}{2\pi\iota}\oint_{|\xi|=\varepsilon}(\lambda_0+\xi-\triangle_k+u)^{-1}\,\mathrm{d}\xi.$$

These $P_u(k,\lambda_0)$ are also projectors which are invariant under $(-\triangle_k + u)$ and of constant rank and they map onto finite-dimensional subspaces $\Sigma(u,k)$.
Let us choose K and $\varepsilon > 0$ such that $(K\times\partial B_\varepsilon(\lambda_0))\cap B(u) = \emptyset$ for all $u \in U$. Additionally, we choose $K \times B_\varepsilon(\lambda_0)$ sufficiently small such that also the matrix defined by $\big((\langle\!\langle P_u(k)^T g_i, P_u(k)f_j\rangle\!\rangle)_{i,j=1\ldots,m}\big)$ is regular, i.e. the determinant, which is a continuous map in λ, does not equal zero. We denote the finite-dimensional subspace $\Sigma(u,k)$ in the neighborhood $K \times B_\varepsilon(\lambda_0)$ of (k_0,λ_0) as the space which is spanned by $P_u(k)f_i$ for $i = 1,\ldots,m$. We have chosen K and $\varepsilon > 0$ such that $K\times\partial B_\varepsilon(\lambda_0)$, i.e. the paths of integration over the resolvent for $k \in K$ and $u \in U$, does not intersect any singularity of the resolvent. Thus, by Theorem 1.10, $P_u(k)$ is holomorphic in $k \in K$ and $u \in U$ since the resolvent is holomorphic. Because the finite-dimensional subspaces $\Sigma(u,k)$ are invariant under $\lambda+\triangle_k-u$, one can represent $\lambda+\triangle_k-u$ in the basis of $\Sigma(u,k)$ given by $(P_u(k)f_i)_{i=1}^m$ and the corresponding dual basis is given by $\big(P_u(k)^T g_i\big)_{i=1}^m$. This yields a matrix representation $A(u,k,\lambda)$ of $\lambda+\triangle_k-u$ which is holomorphic in k and u. For all $l,i = 1,\ldots,m$, there holds

$$\langle\!\langle P_u^T(k)g_l,(\lambda-\triangle_k+u)P_u(k)f_i\rangle\!\rangle = \sum_{j=1}^m A_{ij}(u,k,\lambda)\langle\!\langle P_u^T(k)g_l,P_u(k)f_i\rangle\!\rangle,$$

where A_{ij} are the entries of the matrix $A(u,k,\lambda)$. The matrix $A(u,k,\lambda)$ defined like this is singular if and only if the triple (u,k,λ) is in the graph of $u \mapsto B(u)$ since then $\lambda-\triangle_k+u = 0$. Because $(\langle\!\langle P_u^T(k,\lambda)g_i, P_u(k)f_j\rangle\!\rangle)_{i,j=1}^m$ is regular, this implies that $\det(A(u,k,\lambda)) = 0$ if and only if the triple (u,k,λ) is in the graph of $u \mapsto B(u)$. Furthermore, $\det(A(u,k,\lambda))$ is obviously holomorphic in $\lambda \in B_\varepsilon(\lambda_0)$ and also holomorphic in $k \in K$ and $u \in U$ since $A(u,k,\lambda)$ is . □

So it is possible to find local matrix representations for the operator $\lambda+\triangle_k-u$ on the Bloch variety as well as for the Schrödinger operator $-\triangle_k+u$ on the Fermi curve. From now on, saying “the Bloch variety can locally be considered as the zero set of a holomorphic function” shall mean that for fixed $u \in C(\mathbb{R}^2/\Gamma)$, there exists a $(k_0,\lambda_0) \in B(u)$ and a small open neighborhood $U_{k_0} \times B_\varepsilon(\lambda_0) \subset \mathbb{C}^2 \times \mathbb{C}$ containing (k_0,λ_0) so that on this open neighborhood, the representation of the Bloch variety as in the the above proof holds. Likewise we also speak of local considerations

on $B(u)/\Gamma^*$ or on the Fermi curve $F(u)/\Gamma^*$ respectively $F(u)$.
The next two Corollaries follows immediately from Theorem 1.12.

Corollary 1.15. *$B(u)$ and $F(u)$ are holomorphic varieties in the sense of Definition 1.13, where $B(u)$ is locally defined as a holomorphic subvariety of $\mathbb{C}^3$ and $F(u)$ is locally defined as a holomorphic subvariety of $\mathbb{C}^2$ in the sense of Definition 1.11.*

Proof. That $B(u)$ is a holomorphic subvariety of $\mathbb{C}^3$ and $F(u)$ a holomorphic subvariety of $\mathbb{C}^2$ follows directly from the proof of Theorem 1.14 since both can be represented locally as the zero set of a holomorphic function in $(k,\lambda) \in \mathbb{C}^3$ respectively in $k \in \mathbb{C}^2$. We show that $B(u)$ is also a holomorphic variety. The proof for $F(u)$ then follows by setting $\lambda = 0$. Let $V, W \subset \mathbb{C}^3$ be open subsets such that $B(u) \cap V \cap W \neq \emptyset$. Let (k_0, λ_0) be a point in this intersection set. Then the spectral projection at this point again yields a matrix $A(u,k,\lambda)$ such that $\det(A(u,k,\lambda)) = 0$ describes $B(u)$ on this intersection set. If $\dim \Sigma(u_0, k_0)$ is smaller than the corresponding dimensions on U and V, the spectral projection only projects on the sheets of $B(u)$ contained in a small open neighborhood of (k_0, λ_0) contained in $U \cap V$. Restricting the definition of the spectral projection on U as well as of the spectral projection V on $\Sigma(u_0, k_0)$ yields that these spectral projections, via which we have defined $A(u,k,\lambda)$, coincide on this eventually smaller subspace since it is in both case defined through the basis elements which span $\Sigma(u_0, k_0)$. For this reason, the zero sets describing $B(u)$ on U respectively V coincide on $U \cap V$. □

Corollary 1.16. *Under the assumptions in Theorem 1.14, the operator $\lambda + \triangle_k - u$ can locally around (k_0, λ_0), i.e. for $(k,\lambda) \in K \times B_\varepsilon(\lambda_0)$, be represented as an $m \times m$-matrix $A(u,k,\lambda)$ depending holomorphically on k and λ, where m is the dimension of the generalized eigenspace at (k_0, u_0). The transposed Schrödinger operator $(\lambda + \triangle_{-k} - u)^T$ introduced in Lemma 1.6, can locally around $(-k_0, \lambda_0)$ be represented as the $m \times m$-matrix $A^T(u,k,\lambda)$.*

Proof. The first part of the corollary is clear from the proof of Theorem 1.14. The second part is also easy to see by simple linear algebra. We use the notation from the proof of Theorem 1.14. By the same arguments as in the foregoing proof, one has

$$(\lambda + \triangle_{-k} + u)^T g_l = \sum_{j=1}^{m} B_{lj} g_j,$$

where $g_1, \ldots, g_m$ is the basis of $\Sigma^T(u,k)$. This leads to a matrix $B(u,k,\lambda)$ with coefficients B_{ij}. However, we also know that

$$\begin{aligned}\langle\!\langle (\lambda + \triangle_{-k} - u)^T P_u^T(k) g_l, P_u(k) f_i \rangle\!\rangle &= \langle\!\langle P_u^T(k) g_\ell, (\lambda + \triangle_k - u) P_u(k) f_i \rangle\!\rangle \\ &= \sum_{j=1}^{m} A_{ji} \langle\!\langle P_u^T(k) g_l, P_u(k) f_i \rangle\!\rangle\end{aligned}$$

for $i, l = 1, \ldots, m$. Therefore, $B(u,k,\lambda) = A^T(u,k,\lambda)$. □

1.3. Involutions

In this section, some symmetry of the Fermi curve are shown which will be crucial in this work. More precisely, it is shown that there is a holomorphic involution σ on $F(u)$. This involution will turn out to be a very important property of the Fermi curve for the remainder of this work. Furthermore, in case that the potential u is real-valued, the reality condition can be expressed in terms of an antiholomorphic involution τ_1 on $F(u)$. That means a Fermi curve with complex-valued potential has one involution and a Fermi curve with real-valued potential three since the composition of the τ_1 and σ is also an involution. This will be important in Section 6.2.

Lemma 1.17. *(a) For $u \in C(\mathbb{R}^2/\Gamma, \mathbb{C})$, one has $(-\triangle_k + u)^T = -\triangle_{-k} + u$ which induces a holomorphic involution $\sigma : \mathbb{C}^2 \to \mathbb{C}^2$, $k \mapsto -k$ such that $F(u)$ is invariant under σ. If ψ_k is in the kernel of $-\triangle_k + u$, then $\sigma^*\psi_k$ is an element in the kernel of $(-\triangle_{-k} + u)^T$.*

(b) For $u \in C(\mathbb{R}^2/\Gamma, \mathbb{R})$, there are two antiholomorphic involutions $\tau_1 : \mathbb{C}^2 \to \mathbb{C}^2$, $k \mapsto -\bar{k}$ and $\tau_2 : \mathbb{C}^2 \to \mathbb{C}^2$, $k \mapsto \bar{k}$ such that the Fermi curve is invariant under these involutions.

(i) If ψ_k is an element in the kernel of $-\triangle_k + u$, then $\tau_1^*\bar{\psi}_k$ is an element in the kernel of $-\triangle_{-\bar{k}} + u$. This involution is indicated by $\overline{(-\triangle_k + u)} = -\triangle_{-\bar{k}} + u$.

(ii) The hermitian Schrödinger operator $(-\triangle_k + u)^* = -\triangle_{\bar{k}} + u$ induces the involution $\tau_2 := \sigma \circ \tau_1 = \tau_1 \circ \sigma$. If ψ_k is an element in the kernel of $-\triangle_k + u$, then $\varphi = \tau_2^*\bar{\psi}_k$ is an element in the kernel of $-\triangle_{\bar{k}} + u$.

(iii) If the antiholomorphic involution $\tau_1 : \mathbb{C}^2 \to \mathbb{C}^2$ leaves $F(u)$ invariant and additionally $\tau_1^*\bar{\psi}_k = c(k)\psi_k$ for all ψ_k which are in the kernel of $-\triangle_k + u$ with $k \in F(u)$ and $c(k) \neq 0$, then u is real-valued.

Proof. (a) At first, we note that $u(x,y) \in \mathbb{C}$, so $u^T(x,y) = u(x,y)$, and therefore also $F(u) = F(u^T)$. It is shown in Lemma 1.6 that $\triangle_k^T = \triangle_{-k}$, so $(-\triangle_k + u)^T$ leads to the same Fermi curve as $-\triangle_k + u$ only with different boundary values for the eigenfunctions. Applying σ to the Schrödinger equation yields that

$$0 = \sigma^*((\triangle_k + u)\psi_k) = (\triangle_k + u)^T \sigma^*\psi_k = (\triangle_{-k} + u)c(k)\psi_k$$

with $c(k) \neq 0$. So $\sigma^*\psi_k(k, \cdot) = c(k)\psi_k(-k, \cdot)$.

(b) For determining the complex conjugated operator, consider

$$\overline{\nabla_k} = \overline{\nabla + 2\pi\iota k} = \nabla + \overline{2\pi\iota k} = \nabla - 2\pi\iota\bar{k} = \nabla_{-\bar{k}}$$

Therefore, the complex conjugate of the Laplacian with boundary condition is

$$\overline{\triangle_k} = \overline{\nabla_k} \cdot \overline{\nabla_k} = \nabla_{-\bar{k}} \cdot \nabla_{-\bar{k}} = (\nabla - 2\pi i\bar{k}) \cdot (\nabla - 2\pi i\bar{k}) = \triangle - 4\pi i\bar{k}\nabla - 4\pi\bar{k}^2 = \triangle_{-\bar{k}}.$$

This implies for the Schrödinger operator that $\overline{\triangle_k + u} = \triangle_{-\bar{k}} + \bar{u}$. So for real-valued u, the Fermi curve is invariant under the antiholomorphic involution $\tau_1 : k \mapsto -\bar{k}$. Combining both involutions σ and τ_1 yields a third involution $\tau_2 = \sigma \circ \tau_1 = \tau_1 \circ \sigma$ with $\tau_2(k) = \bar{k}$. This corresponds to the hermitian operator $(-\triangle + u)^*$ which is determined by

$$\overline{\triangle_k}^T = (-\nabla - 2\pi i\bar{k})(-\nabla - 2\pi i\bar{k}) = \triangle + 4\pi i\bar{k} \cdot \nabla - 4\pi \bar{k}^2 = \triangle_{\bar{k}}.$$

Since $F(u)$ is invariant under τ_1 and σ, it is also invariant under τ_2.

(i) Let ψ_k be an element in the kernel of $-\triangle_k + u$. Then

$$\tau_1^*(\overline{(-\triangle_k + u)\psi_k}) = (-\triangle_{-\bar{k}} + u)\tau_1^*\bar{\psi}_k.$$

(ii) Let ψ_k be an element in the kernel of $-\triangle_k + u$. Then

$$\tau_2^*(\overline{(-\triangle_k + u)^T\psi_k}) = (-\triangle_{\bar{k}} + u)\tau_2^*\bar{\psi}_k.$$

(iii) Let now $F(u)$ be invariant under $\tau_1 : k \mapsto -\bar{k}$ and assume that for $\psi_k \in \ker(-\triangle_k + u)$ with $k \in F(u)$ holds $\tau_1^*\bar{\psi}_k = \psi_k$. Then $(-\triangle_k + u)\psi_k = 0$ yields

$$0 = \tau_1^*(\overline{(-\triangle_k + u)\psi_k}) = \overline{(-\triangle_{-\bar{k}} + u)}\tau_1^*\bar{\psi}_k = c(k)\overline{(-\triangle_{-\bar{k}} + u)}\psi_k = c(k)(-\triangle_k + \bar{u})\psi_k.$$

Thus ψ_k is an eigenfunction of $-\triangle_k + u$ and $-\triangle_k + \bar{u}$ for all $k \in F(u)$. So $\bar{u} = u$.

□

1.4. Two examples

We give two explicit examples of Fermi curves: the Fermi curve with zero potential and Fermi curves with constant potential. The first one will be very important in the sequel since it turns out in Chapter 2 that all Fermi curves are 'asymptotically free', i.e. converge to $F(0)$ for $\|\operatorname{Im}(k)\| \to \infty$.

1.4.1. The Free Fermi Curve

In [Klauer, 2011, Theorem 4.2.5], an explicit representation of the free Fermi curve $F(0)/\Gamma^*$ in terms of $(k_1, k_2) \in \mathbb{C}$ is given.

Lemma 1.18 ([Klauer, 2011, Theorem 4.2.5]). *The free Fermi curve can be written as* $F(0) = \mathcal{R} + \Gamma^*$, *where*

$$\mathcal{R} := \{k \in \mathbb{C}^2 \mid k^2 = 0\} = \{k \in \mathbb{C}^2 \mid (k_1 + \imath k_2)(k_1 - \imath k_2) = 0\}, \tag{1.15}$$

provided that the pairs of distinct points (k_ν^-, k_ν^+) *with*

$$k_\nu^\mp := \frac{1}{2}(\pm\nu_1 + \iota\nu_2, -\iota\nu_1 \pm \nu_2) \tag{1.16}$$

are identified to double points for all $\nu \in \Gamma^* \setminus \{0\}$*.* $\mathcal{R}$ *is a system of representatives for the quotient* $F(0)/\Gamma^*$*.*

Sketch of the proof. The equality $F(0) = \mathcal{R} + \Gamma^*$ follows by Fourier transforming $\triangle_k \psi_k = 0$ with $\triangle_k = 4\partial_k \overline{\partial}_k$ as in (1.8), i.e. for all $\nu \in \Gamma^*$, there has to hold

$$(\iota(\nu_1 + k_1) - (\nu_2 + k_2))\,(\iota(\nu_1 + k_1) + (\nu_2 + k_2))\,\hat{\psi}_k(\nu) = 0.$$

Since $\psi_k \not\equiv 0$, there always exists a $\nu \in \Gamma^*$ such that $\hat{\psi}_k(\nu) \neq 0$. So $F(0) \subseteq \mathcal{R} + \Gamma^*$. Conversely, let $k \in \mathcal{R} + \Gamma^*$. Then there exists a $\nu \in \Gamma^*$ such that

$$(k_1 + \nu_1) - \iota(k_2 + \nu_2) = 0 \;\text{ or }\; (k_1 + \nu_1) + \iota(k_2 + \nu_2) = 0.$$

Let ψ_k be a function whose Fourier transform is given by $\hat{\psi}(\kappa) = \delta_{\nu\kappa}$. Then ψ_k solves $-\triangle_k \psi_k = 0$ and thus also $\mathcal{R} + \Gamma^* \subseteq F(0)$.
The fact that k_ν^+ and k_ν^- need to be identified for each $\nu \in \Gamma^*$ can be seen by answering the question when the difference of two elements $k, \tilde{k} \in F(0)$ is contained in the dual lattice Γ^*. This is also done in [Klauer, 2011, Proof of Theorem 4.2.5], but since this fact is essential for the picture of $F(0)/\Gamma^*$, this step is outlined here as well. So let $k, \tilde{k} \in F(0)$ such that $k - \tilde{k} \in \Gamma^*$. Then there are the two possibilities $k = k_1 \pm \iota k_2$ and $\tilde{k} = \tilde{k}_1 \pm \iota \tilde{k}_2$ or $k = k_1 \pm \iota k_2$ and $\tilde{k} = \tilde{k}_1 \mp \iota \tilde{k}_2$. In the first case it follows from $k - \tilde{k} = \nu \in \Gamma^*$ that $k = \tilde{k}$ since $\nu = \nu_1 \pm \iota\nu_2 = 0$ with $\nu_1, \nu_2 \in \mathbb{R}$ can only hold for $\nu_1 = \nu_2 = 0$. In the second case it is $(k_1 \pm \tilde{k}_1) - \iota(k_2 \mp \tilde{k}_2) = 0$. Using $\nu = k - \tilde{k}$ leads to

$$k_1 + \tilde{k}_1 = 2k_1 - k_1 + \tilde{k}_1 = 2k_1 - \nu_1 \;\text{ and }\; k_2 + \tilde{k}_2 = 2k_2 - k_2 + \tilde{k}_2 = 2k_2 - \nu_2 = 0$$

and hence

$$2k_1 - \nu_1 - \iota\nu_2 = 0 \;\text{ and }\; \nu_1 - \iota(2k_2 - \nu_2) = 0.$$

These lines intersect only in the double points as in (1.16) with $k = k_\nu^+$ and $\tilde{k} = k_\nu^-$ for $\nu \in \Gamma^*$. □

We use the notation for $F(0)$ as it is also done in [Feldman et al., 2000, §16], i.e. $F(0)$ is represented as the union

$$F(0) = \bigcup_{\nu \in \Gamma^*} (\mathcal{R}_+(\nu) \cup \mathcal{R}_-(\nu)) \tag{1.17}$$

of infinitely many straight lines

$$\mathcal{R}_\pm(\nu) = \{(k_1, k_2) \in \mathbb{C}^2 \;\mid\; (k_1 + \nu_1) \pm \iota(k_2 + \nu_2) = 0\} \tag{1.18}$$

with $\nu \in \Gamma^*$. Then $\mathcal{R} = \mathcal{R}_+(0) \cup \mathcal{R}_-(0)$. For brevity, we set $\mathcal{R}_\pm := \mathcal{R}_\pm(0)$. As we have seen in the foregoing prove, it is $\mathcal{R}_\pm(\nu) \cap \mathcal{R}_\pm(\widetilde{\nu}) = \emptyset$ for $\nu, \widetilde{\nu} \in \Gamma^*$ with $\nu \neq 0$ and $\nu \neq \widetilde{\nu}$ and

$$\mathcal{R}_+ \cap \mathcal{R}_-(\nu) = \{k_\nu^+\},\ \mathcal{R}_- \cap \mathcal{R}_+(\nu) = \{k_\nu^-\},\ \mathcal{R}_+ \cap \mathcal{R}_-(-\nu) = \{k_{-\nu}^+\},\ \mathcal{R}_+(-\nu) \cap \mathcal{R}_- = \{k_{-\nu}^-\}.$$

and the map $k \mapsto k + \nu$ maps $\mathcal{R}_+(-\nu) \cap \mathcal{R}_-$ to $\mathcal{R}_+ \cap \mathcal{R}_-(\nu)$. Since only a finite number of the line pairs $\mathcal{R}_+(\nu)$ and $\mathcal{R}_-(\nu)$ can intersect any bounded subset of $\mathbb{C}^2$, the union in (1.17) is locally finite.

Lemma 1.19. *The eigenspaces of the free Schrödinger operator $-\triangle$ respectively its transpose $-\triangle^T$ are 1-dimensional on*

$$\mathcal{R} \setminus \left\{k_\nu^\pm \mid \nu \in \Gamma^* \setminus \{0\}\right\}.$$

There, the eigenfunction of $-\triangle$ normalized such that $\psi_N^0(0,0) = 1$ is given by $\psi_N^0(k,(x,y)) = e^{2\pi\iota\langle k, \binom{x}{y}\rangle}$ and the eigenfunction of $-\triangle^T$ normalized as $\varphi_N^0(0,0) = 1$ is given by $\varphi_N^0(k,(x,y)) = e^{-2\pi\iota\langle k, \binom{x}{y}\rangle}$. In all double points $k_\nu^\pm$, the eigenspaces of $\triangle$ respectively $\triangle^T$ are 2-dimensional with

$$\psi^0(k,(x,y)) \in \operatorname{span}\left(e^{2\pi\iota\langle k, \binom{x}{y}\rangle}, e^{2\pi\iota\langle k\pm\nu, \binom{x}{y}\rangle}\right)$$

respectively

$$\varphi^0(k,(x,y)) \in \operatorname{span}\left(e^{-2\pi\iota\langle k, \binom{x}{y}\rangle}, e^{-2\pi\iota\langle k\pm\nu, \binom{x}{y}\rangle}\right).$$

Proof. Let $\psi^0(x,y) \not\equiv 0$ be the eigenfunction of the free Schrödinger operator (1.1) with quasiperiodicity (1.3). This can be calculated with help of the Fourier series of the eigenfunction $\psi_k^0(x,y)$ which is periodic in $\mathbb{R}^2$ with respect to Γ:

$$\psi^0(x,y) = e^{2\pi\iota\langle k, \binom{x}{y}\rangle} \psi_k^0(x,y) = \sum_{\kappa \in \Gamma^*} e^{2\pi\iota\langle k+\kappa, x\rangle} \widehat{\psi}_k^0(\kappa). \tag{1.19}$$

Let now $k \in \mathbb{C}^2$ be fixed and let $\kappa \in \Gamma^*$. Then $- \triangle\, \psi^0(k,(x,y)) = 0$ with $\psi^0(k) \not\equiv 0$ reads in the Fourier space as

$$\sum_{\kappa \in \Gamma^*} 4\pi^2 (k+\kappa)^2 e^{2\pi\iota\langle k+\kappa, \binom{x}{y}\rangle} \hat{\psi}_k^0(\kappa) = 0.$$

This equation can only hold if $(k+\kappa)^2 \widehat{\psi}_k^0(\kappa) = 0$ for all $\kappa \in \Gamma^*$. Since the eigenfunction is not identically zero, $(k+\kappa)^2$ has to vanish for at least one $\kappa \in \Gamma^*$. Hence, for an arbitrary $\kappa \in \Gamma^*$, we want to find a $k \in \mathbb{C}^2$ such that $(k+\kappa)^2 = 0$. Without loss of generality, let $\kappa = 0$. Then for all other $\kappa' \in \Gamma^* \setminus \{0\}$ also $k + \kappa'$ is a solution since the Fermi curve $F(0)$ is invariant under translation by $\kappa \in \Gamma^*$, compare Lemma 1.8. An element k is contained in the zero set of k^2 if and only if $k_1^2 + k_2^2 = 0$. This yields exactly two solutions of k_1, where $k_1 = -\iota k_2$ correspond to $\mathcal{R}_+$ and $k_1 = \iota k_2$ to $\mathcal{R}_-$. In order to determine the solution $\psi^0(k)$ from this, it is necessary to know whether other coefficients $\hat{\psi}_k^0(\kappa)$ for $\kappa \in \Gamma^* \setminus \{0\}$ in (1.19) are not necessarily equal to zero. That means for k chosen in such a way that $k^2 = 0$, we have to answer the question if there is another $\kappa \neq 0$ such that $(k+\kappa)^2 = 0$ holds with the same k and $\kappa \in \Gamma^* \setminus \{0\}$. By Lemma 1.18 it is clear that such a second solution can only exist in the double points $k_\kappa^\pm$ at which $\mathcal{R}_\pm$ intersects $\mathcal{R}_\mp(\kappa)$. At those points,

$$\psi^0(x,y) \in \operatorname{span}\left(\widehat{\psi}_k^0(0)e^{2\pi\iota\langle k,\binom{x}{y}\rangle}, \widehat{\psi}_k^0(\nu)e^{2\pi\iota\langle k\pm\nu,x\rangle}\right)$$

and at all non-double points on $\mathcal{R}$, the eigenfunction is given by $\psi^0 = \widehat{\psi}_k^0(0)e^{2\pi\iota\langle k,\cdot\rangle}$. For the normalized eigenfunction $\psi_N(k)$ the zeroth Fourier coefficient is $\widehat{\psi}_k^0 = 1$. Also at $k = 0$, where $\mathcal{R}_+$ and $\mathcal{R}_-$ intersect, the eigenspace is one-dimensional since there the eigenfunctions belonging to k such that $k_1 = \iota k_2$ and the eigenfunction belonging to k such that $k_1 = -\iota k_2$ both become 1. The same argumentation yields the assertions on the dual eigenfunction φ_k^0 on $\mathcal{R}$ as mentioned above. □

It is clear from the foregoing that the free Fermi curve $F(0)$ can locally be represented as a unique one- or two-sheeted Weierstrass covering, compare [de Jong and Pfister, 2012, Weierstaß Preparation Theorem 2.3.4]. Weierstraß coverings are local coverings $F(u) \to \mathbb{C}, (k_1, k_2) \mapsto k_1$ such that locally around $(k_{0,1}, k_{0,2}) \in F(u)$ the Fermi curve can be represented as the zero set of a polynomial in k_2 with coefficients that are holomorphic in k_1, highest coefficient equal to one and all lower coefficients vanishing at $k_1 = k_{0,1}$. The degree of this polynomial equals the number of sheets which meet in $(k_{0,1}, k_{0,2})$. Even though this covering is obvious for $F(0)$, it is given here explicitly since it will used in the sequel to compare $F(u)$ with $F(0)$ asymptotically by considering the local Weierstraß coverings of $F(u)$. Lemma 1.18 yields that for $k_0 \in F(0)$ and $U \subset \mathbb{C}^2$ an open subset around k_0, $F(0) \cap U$ can be parametrized by setting $z_1 = k_{0,1} - k_1$ and $z_2 = k_{0,2} - k_2$. Then $F(0)$ can locally on an open subset $U = U_1 \times U_2 \subset \mathbb{C}^2$ containing exactly one double point $k_\nu^\pm$ be represented as $f_0(z_1, z_2) := (k_{0,1} - z_1)^2 + (k_{0,2} - z_2)^2 = 0$. If U does not contain a double point, it can either be represented as $f_0^+(z_1, z_2) := (k_{0,1} - z_1) - \iota(k_{0,2} - z_2) = 0$ or as $f_0^-(z_1, z_2) := (k_{0,1} - z_1) + \iota(k_{0,2} - z_2) = 0$. All of these local descriptions of course coincide with the local representations of $F(0)$ as the zero set of $\det A_0(z_1, z_2)$, where A_0 is the 1×1-respectively 2×2-matrix as introduced in the proof of Theorem 1.14 since this representation is unique by the Weierstraß Preparation Theorem [de Jong and Pfister, 2012, Theorem 3.2.4].

1.4.2. Fermi curves with constant potential

In [Klauer, 2011, Theorem 4.4.1] it is furthermore shown that also for constant potentials there is an explicit formula to determine $F(4\pi^2 u_0)$. Note that for $x \in \mathbb{C}^2$, the norm $\|x\| := \sqrt{x_1\bar{x}_1 + x_2\bar{x_2}}$ is used.

Lemma 1.20 ([Klauer, 2011, Theorem 4.4.1])**.** *Let $4\pi^2 u_0$ be a constant potential and*

$$\mathcal{R}(u_0) := \{k \in \mathbb{C}^2 \mid (k_2 - \iota k_1)(k_2 + \iota k_1) + u_0 = 0\}.$$

Then $F(4\pi^2 u_0) = \mathcal{R}(u_0) + \mathit{\Gamma}^*$ *and the set* $\mathcal{R}(u_0)$ *is a system of representatives for* $F(4\pi^2 u_0)/\mathit{\Gamma}^*$ *provided that the pairs of distinct points* $(k_\nu^-(u_0), k_\nu^+(u_0))$ *given by*

$$k_\nu^\mp(u_0) := \frac{1}{2}\left(\pm\nu_1 + \iota\nu_2\sqrt{1+\frac{4u_0}{\|\nu\|^2}}, -\iota\nu_1\sqrt{1+\frac{4u_0}{\|\nu\|^2}} \pm \nu_2\right) \tag{1.20}$$

are identified to double points for all $\nu \in \mathit{\Gamma}^* \setminus \{0\}$ *and provided that the constant potential does not cause any double points to coincide. In particular,* $\mathcal{R}(u_0)$ *is a continuous deformation of* $\mathcal{R}$ *such that the deformation of a double points* $k_\nu^\pm$ *of* $\mathcal{R}$ *is still a double point* $k_\nu^\pm(u_0)$*. Furthermore, for every* $\varepsilon > 0$*, there exists a* $\delta > 0$ *such that for* $\nu \in \mathit{\Gamma}_\delta^*$ *one has* $\|k_\nu^\pm - k_\nu^\pm(u_0)\| < \varepsilon$*.*

Proof. Fourier transforming the Schrödinger operator with constant potential leads to

$$F(4\pi^2 u_0) = \{k \in \mathbb{C}^2 \mid \exists \kappa \in \mathit{\Gamma}^* : ((k_2+\kappa_2) - \iota(k_1+\kappa_1))((k_2+\kappa_2) + \iota(k_1+\kappa_1)) + u_0 = 0\}.$$

Therefore, as in the proof of Lemma 1.18, $F(4\pi^2 u_0) = \mathcal{R}(u_0) + \mathit{\Gamma}^*$. Also as in the case with zero potential, the double points of $F(4\pi^2 u_0)$ must lie on $F(4\pi^2 u_0)$ and differ only by the corresponding element of $\mathit{\Gamma}^* \setminus \{0\}$, i.e.

$$(k_\nu^+(u_0))^2 + u_0 = 0, \quad (k_\nu^-(u_0))^2 + u_0 = 0, \quad k_\nu^-(u_0) - k_\nu^+(u_0) = \nu. \tag{1.21}$$

Inserting the last equation into the second one to eliminate k_ν^- and then subtracting the first equation from this yields $\|\nu\|^2 + 2(\nu, k_\nu^+) = 0$. Assuming $\nu_1 \neq 0$ and denoting $k_\nu^+(u_0) = (k_1, k_2)$ leads to $k_1 = -\frac{1}{2\nu_1}\|\nu\|^2 - \frac{\nu_2}{\nu_1}k_2$. By inserting this into the first equation in (1.21) one gets

$$\begin{aligned}\left(\frac{\|\nu\|^2}{2\nu_1} + \frac{\nu_2}{\nu_1}k_2\right)^2 + k_2^2 + u_0 = 0 \Leftrightarrow \ & \frac{\|\nu\|^4}{4\nu_1^2} + \frac{\|\nu\|^2\nu_2}{\nu_1^2}k_2 + \frac{\nu_2^2}{\nu_1^2}k_2^2 + k_2^2 + u_0 = 0\\ \Leftrightarrow \ & k_2^2 + \nu_2 k_2 + \frac{\nu_1^2 u_0}{\|\nu\|^2} + \frac{\|\nu\|^2}{4} = 0,\end{aligned}$$

and therefore

$$k_2^\pm = -\frac{\nu_2}{2} \pm \sqrt{\frac{\nu_2^2}{4} - \frac{\|\nu\|^2}{4} - \frac{\nu_1^2 u_0}{\|\nu\|^2}} = \frac{1}{2}\left(-\nu_2 \pm \iota\nu_1\sqrt{1+\frac{4u_0}{\|\nu\|^2}}\right).$$

So

$$\begin{aligned}k_1 &= -\frac{1}{2\nu_1}\|\nu\|^2 - \frac{\nu_2}{2\nu_1}\left(-\nu_2 \pm \iota\nu_1\sqrt{1+\frac{4u_0}{\|\nu\|^2}}\right)\\ &= -\frac{\|\nu\|^2}{2\nu_1} + \frac{\nu_2^2}{2\nu_1} \mp \frac{\iota\nu_2}{2}\sqrt{1+\frac{4u_0}{\|\nu\|^2}}\\ &= -\frac{1}{2}\left(\nu_1 \pm \iota\nu_2\sqrt{1+\frac{4u_0}{\|\nu\|^2}}\right).\end{aligned}$$

Then

$$\begin{aligned} k_\nu^-(u_0) = k_\nu^+(u_0) + \nu &= \frac{1}{2}\left(-\nu_1 \pm \iota\nu_2\sqrt{1+\frac{4u_0}{\|\nu\|^2}}, -\nu_2 \pm \iota\nu_1\sqrt{1+\frac{4u_0}{\|\nu\|^2}}\right) + (\nu_1,\nu_2) \\ &= \frac{1}{2}\left(\nu_1 \pm \iota\nu_2\sqrt{1+\frac{4u_0}{\|\nu\|^2}}, \nu_2 \pm \iota\nu_1\sqrt{1+\frac{4u_0}{\|\nu\|^2}}\right). \end{aligned}$$

If $\nu_1 = 0$, we may assume $\nu_2 \neq 0$ since $\nu \neq 0$ and analogous calculations give another set of double points. Still, it suffices to consider only one of these two sets of solutions: Denote these four points as $k_\nu^\pm(u_0)_\pm$, where the second index $\pm$ is representing the sign in front of the square root. The dual lattice Γ^* is invariant under transforming $\nu \mapsto -\nu$ and this transformation maps $k_\nu^\pm(u_0)_\pm \mapsto k_{-\nu}^\pm(u_0)_\mp$. So the entire set of solutions is contained in (1.20) and the choice of the sign in front of the terms of the square roots is arbitrary. Since u_0 is constant, $\sqrt{1+\frac{4u_0}{\|\nu\|}} \to 1$ for $\nu \to \infty$ and thus also $\|k_\nu^\pm - k_\nu^\pm(u_0)\| \to 0$. □

Note that the eigenfunctions of $-\triangle_k + 4\pi^2 u_0$ are the same as the eigenfunctions of $-\triangle_k$ with the only difference that the eigenspace now becomes two-dimensional at the double points $k_\nu^\pm(4\pi^2 u_0)$.

1.5. The direct and the inverse problem

Before we go on with the main parts of this work, we want to recall the following: In the title of this work we promised to consider parts of the so-called inverse problem of the Schrödinger equation (1.1). Taking a look at the map $F : u \mapsto F(u)$, the inverse problem wants to answer the following two questions:

1. *The isospectral problem:* Considering the set of potentials belonging to a Fermi curve $F(u)$ with fixed potential u one asks how the fiber $F^{-1}(F(u))$ can be parametrized. In other words, the question is how to change certain properties of the eigenfunctions on $F(u)/\Gamma^*$ such that the transformed eigenfunctions are still belonging to $F(u)/\Gamma^*$.

2. *The moduli problem:* Here is asked what values of $F(u)$ are possible and how these values are parametrized when the potential u changes. In other words, one question is which complex curves obeyed with what kind of 'data' describe Fermi curves for some potential u. And another question is how these curves can be deformed such that they still are Fermi curves for some other potential $\widetilde{u}$.

In order to answer what kind of information we have about the Fermi curve such that we can assume the right properties of the given data later on, the so-called *direct problem* is considered first in this work. Hereby, we assume that a potential u is given and deduce as many properties of $F(u)/\Gamma^*$ and the corresponding divisor from this as will be necessary to find answers for some questions raised by the inverse problem. Therefore, the direct problem is considered in Part I and then partly answers to the question raised by the inverse problem are given in Part II.

Part I.

The direct problem for finite type potentials

2. Asymptotic freeness

Our aim in this chapter is to understand the asymptotic behavior of the Fermi curve $F(u)$ for $\|\operatorname{Im}(k)\| \to \infty$. We will deduce that the Fermi curve $F(u)/\Gamma^*$ can be divided in three different parts: one compact part and one part 'far outside', where the part 'far outside' is divided into so-called excluded domains, i.e. the parts of the Fermi curve which are contained in small open neighborhoods of the double points of the free Fermi curve, and a rest which can be shown to be a one-dimensional manifold. This was first observed in [Krichever, 1995, Theorem 3.3]. For real-valued potentials $u \in L^2(\mathbb{R}^2/\Gamma, \mathbb{R})$, this is also explained very detailed in [Feldman et al., 2000, Chapter 3, §17 and §18]. In [Feldman et al., 2000, Chapter 2, §5 and Chapter 3, §16 – §18], one can also find the general procedure and an analogous analysis for Heat curves, i.e. curves corresponding to the heat equation with periodic potential.
For the Fermi curves of the two-dimensional periodic Schrödinger equation, only the asymptotics for $\|\operatorname{Im}(k)\| \to \infty$ are of interest since Γ^* is a real two-dimensional lattice and $F(u)$ is periodic with respect to Γ^*, see Lemma 1.8. To do so, k is taken from a compact subset in $\mathbb{C}^2$ and translate it into the direction of the double points $k_\nu^\pm$ of $F(0)$ in (1.16) for $\nu \in \Gamma^*$ and $\|\nu\| \to \infty$. We will see that the approach $k \mapsto k + k_\nu^\pm$ makes sense for two reasons: First of all, it follows from Lemma 1.18 that $k \in F(0)$ implies $k + k_\nu^\pm \in F(0)$ for all $\nu \in \Gamma^*$. Secondly, $\|\nu\| = 2\|\operatorname{Im}(k_\nu^-)\| = 2\|\operatorname{Im}(k_\nu^- + \kappa)\| = 2\|\operatorname{Im}(k_\nu^+)\|$ for all $\nu, \kappa \in \Gamma^*$ and we know from Lemma 1.8 that $F(u)$ is invariant under translations of $\kappa \in \Gamma^*$. Ergo, it suffices to consider the translations for ν with $\kappa = 0$ and the asymptotic behavior of the Fermi curve for translations of k into the direction of k_ν^- or k_ν^+ with $\|\nu\| \to \infty$ yields the full asymptotics of the Fermi curve. We now define the compact subsets of $\mathbb{C}^2$ containing k and which will be translated in the direction of the double points. It would be convenient to include the invariance of $F(u)$ under translations of dual lattice vectors $\nu \in \Gamma^*$. However, this alone is not sufficient because $\mathbb{C}^2/\Gamma^*$ is not compact. Nevertheless, the proof of Lemma 1.18 shows that at least $F(0)$ is also invariant under translations by the double points $k_\nu^\pm$. This motivates the next definition.

Definition 2.1. We denote the lattice generated by ν and k_ν^- with $\nu \in \Gamma^*$ as $\Gamma_{\mathbb{C}}^*$. We define $\Delta_{\mathbb{C}}$ as a fixed fundamental domain of $\mathbb{C}^2/\Gamma_{\mathbb{C}}^*$ such that $(0,0)$ is in the interior of $\Delta_{\mathbb{C}}$.

The lattice $\Gamma_{\mathbb{C}}^*$ has four real directions, so $\mathbb{C}^2/\Gamma_{\mathbb{C}}^*$ is compact and hence also $\Delta_{\mathbb{C}}$ is. Since $k_\nu^+ - k_\nu^- = \nu$, it does not matter whether k_ν^+ or k_ν^- is used in this definition and the above definition yields the lattice generated by all dual lattice vectors ν and all double points $k_\nu^\pm$. Moreover,

$$\mathbb{C}^2 = \bigcup_{\kappa,\nu\in\Gamma^*} (\kappa + k_\nu^- + \Delta_\mathbb{C}).$$

For brevity, when considering $F(u)$ 'far outside', we define

$$\mathbb{C}^2_\delta := \left\{ k \in \mathbb{C}^2 \;\middle|\; \|\operatorname{Im}(k)\| > \frac{1}{\sqrt{2}}\delta^{-1} \right\} \tag{2.1}$$

for some small $\delta > 0$. This set contains translations of $k \in \Delta_\mathbb{C}$ by κ and $k_\nu^\pm$ with $\nu, \kappa \in \Gamma^*$ and $\|\nu\| > \delta^{-1}$ since $\|k_\nu^\pm\| = \frac{1}{\sqrt{2}}\|\nu\|$. To gather the double points 'far outside', we define for all $\delta > 0$

$$\Gamma^*_\delta := \{\nu \in \Gamma^* \mid \|\nu\| > \delta^{-1}\}.$$

Furthermore, we set for $\varepsilon > 0$

$$\Delta^\varepsilon_\mathbb{C} := \{k \in \Delta_\mathbb{C}, \mid \operatorname{dist}(k, F(0)) \geq \varepsilon\}.$$

Then $\Delta^\varepsilon_\mathbb{C}$ is a closed subset of a compact set, hence also compact. To make notation easier in the sequel, we define for $k_0 \in \mathbb{C}^2$ and some subset $M \subset \mathbb{C}^2$

$$k_0 + M := \{k_0 + k \mid k \in M\}.$$

To show that the Fermi curve $F(u)$ with $u \in C(\mathbb{R}^2/\Gamma)$ is 'asymptotically free' in the sense indicated above, an approach which is proposed in [Klauer, 2011] for the excluded domains is used. We transfer these results also on the remaining part of $F(u) \cap \mathbb{C}^2_\delta$ for $\delta > 0$ sufficiently small. In [Klauer, 2011], another way to see the asymptotic freeness on this part of the Fermi curve is indicated: it is suggested implicitly that the asymptotic freeness of the resolvent of the Schrödinger operator yields the full asymptotics for $F(u)$ bounded away from small open balls around the double points $k_\nu^\pm$ with $\nu \in \Gamma^*$. This argument we do not understand. The problem which might occur in our eyes is that one would have to use the asymptotic freeness of $(-\triangle_k + u)^{-1}$ to estimate the spectral projection of $-\triangle_k + u$ for $\lambda = 0$ and compare it to the spectral projection of $-\triangle_k$ for $\lambda = 0$. More precisely, to define the spectral projection and estimate it with help of Theorem 2.5 ([Klauer, 2011, Theorem 4.3.8]) as suggested in [Klauer, 2011], we think that one has to use the formulation of the spectral projection of both operators as in the proof of Theorem 1.14. This is defined by integrating the resolvent $\triangle^{-1}$ respectively $(\triangle - u)^{-1}$ over a small circle around $\lambda = 0$ such that no other eigenvalues of $\triangle^{-1}$ respectively $(\triangle - u)^{-1}$ are contained in the interior of this circle. This 'separation' of the eigenvalues is necessary since otherwise, the projection is not unique. However, the eigenvalues for the free Schrödinger operator can be described as a covering over

k and then read as $\lambda(k + k_\nu^\pm) = (k + \kappa + k_\nu^\pm)^{-2}$ with $\kappa \in \Gamma^*$ and $k \in \Delta_\mathbb{C}$. So for $\|\nu\| \to \infty$, all eigenvalues of $\triangle^{-1}_{(k+\kappa+k_\nu^\pm)}$ converge to zero since they depend on $k + k_\nu^\pm$. We think that all these eigenvalues asymptotically accumulate around zero and hence estimating the eigenprojections by estimating the integrands under the integral of the spectral projections is not an option. So in a certain sense, the Schrödinger operators considered here have 'too much' of this property 'to separate the eigenvalues' since they become distant from each other so fast that all eigenvalues of the resolvent converge too fast to zero. Therefore, we do not see how the technique proposed in [Klauer, 2011] can be applied to show the asymptotic freeness of $F(u)$ away from the excluded domains.
To come over these difficulties, another method will be used which is proposed in [Schmidt, 2002] for the Dirac operator and transferred to the asymptotics of the Fermi curve of the Schrödinger operator in the excluded domains in [Klauer, 2011, Section 4]. Hereby, the excluded domains denote some small open neighborhoods of $k_\nu^\pm$ with $\nu \in \Gamma_\delta^*$ and $\delta > 0$ sufficiently small. We use this approach to show also away from the excluded domains, $F(u)$ behaves asymptotically free. For the excluded domains the results from [Klauer, 2011] are directly transferred. For the remaining part, we make small adjustments. Therefore, some results from [Klauer, 2011] are necessary. These are summarized in the following section to keep this work self-contained and if necessary extend in the appropriate way such that it can be use for our purposes later on.

2.1. Preliminaries for the asymptotics

For Banach spaces X and Y, $\mathcal{B}(X,Y)$ denotes the space of all bounded operators from X to Y. Furthermore, the Sobolev space of L^2-integrable functions which are once weakly differentiable is considered. This is defined as

$$W^{1,2}(\mathbb{R}^2/\Gamma) := \Big\{ f \in L^2(\mathbb{R}^2/\Gamma, \mathbb{C}) \ \mid \ \forall\, |\alpha| \le 1\, \exists\, f_\alpha \in L^2(\mathbb{R}^2/\Gamma, \mathbb{C}) \\ \forall\, \varphi \in C_0^\infty(\mathbb{R}^2/\Gamma) : \int f_\alpha \varphi \, d\mu = (-1)^{|\alpha|} \int f \partial^\alpha \varphi \, d\mu \Big\},$$

where α is some multi-index and $\partial^\alpha f := f_\alpha$ is the weak derivative of f. $W^{1,2}(\mathbb{R}^2/\Gamma)$ is equipped with the norm

$$\|f\|_{W^{1,2}(\mathbb{R}^2/\Gamma)} := \|f\|_{L^2(\mathbb{R}^2/\Gamma)} + \|\partial_x f\|_{L^2(\mathbb{R}^2/\Gamma)} + \|\partial_y f\|_{L^2(\mathbb{R}^2/\Gamma)}.$$

Lemma 2.2. *Let $k \in \mathbb{C}^2 \setminus F(0)$. Then the map defined by*

$$K \to \mathcal{B}(L^2(\mathbb{R}^2/\Gamma), W^{1,2}(\mathbb{R}^2/\Gamma)), \quad k \mapsto -\triangle_k^{-1}$$

is holomorphic.

Proof. We denote the Fourier transform of the $W^{1,2}$-norm as $\|\cdot\|_{\widehat{W^{1,2}}(\mathbb{R}^2/\Gamma)} := \widehat{\|\cdot\|}_{W^{1,2}(\Gamma^*)}$. Using Parseval's identity [Reed and Simon, 1980, Theorem II.6] and partial integration, where we take into account that $f \in L^2(\mathbb{R}^2/\Gamma)$ is periodic with respect to Γ, we obtain

$$\begin{aligned}\|f\|^2_{W^{1,2}(\mathbb{R}/\Gamma,\mathbb{C})} &= \int_\Delta |\partial_x f|^2 + |\partial_y f|^2 + |f|^2 \\ &= \frac{1}{2}\int_\Delta |\bar{f}(\triangle f)|^2 + |(\triangle \bar{f})f|^2 + |f|^2 \\ &= \sum_{\kappa\in\Gamma^*} |\hat{\psi}_k(\kappa)|^2(1+4\pi^2\kappa) = \|f\|^2_{\widehat{W^{1,2}}(\Gamma^*)}.\end{aligned}$$

Let K be any compact subset of $\mathbb{C}^2 \setminus F(0)$ and $k \in K$. Then Fourier-transforming yields

$$\begin{aligned}\|-\triangle_k^{-1} f\|_{W^{1,2}(\mathbb{R}^2/\Gamma)} &= \|\widehat{-\triangle_k^{-1} f}\|_{\widehat{W^{1,2}}(\mathbb{R}^2/\Gamma,\mathbb{C})} \\ &= \frac{1}{\mathrm{Vol}(\Delta)}\left(\sum_{\kappa\in\Gamma^*}\left|\frac{\hat{f}(\kappa)}{-4\pi^2(k+\kappa)^2}\right|^2(1-4\pi^2\|\kappa\|^2)\right)^{1/2} \\ &\leq \left(\frac{c}{\inf\limits_{\substack{\kappa\in\Gamma^*\\k\in K}}\|k+\kappa\|^2} + \frac{\tilde{c}}{\inf\limits_{\substack{\kappa\in\Gamma^*\\k\in K}}\|k+\kappa\|}\right)\|f\|_{L^2(\mathbb{R}^2/\Gamma)}\end{aligned}$$

with some constants $c, \tilde{c} \in \mathbb{R}^+$. Since we can find a compact subset K for any $k \in \mathbb{C}^2 \setminus F(0)$, the image $-\triangle_k^{-1}$ is a bounded operator from $L^2(\mathbb{R}^2/\Gamma)$ to $W^{1,2}(\mathbb{R}^2/\Gamma)$. Holomorphy of the map $k \mapsto -\triangle_k^{-1}$ follows as in the proof of Theorem 1.10. □

The next Lemma will be used frequently to show the asymptotic freeness of $F(u)$. It can also be found in [Klauer, 2011, Lemma 4.3.3]. However, the value of the infimum given there is not determined correctly in [Klauer, 2011] and the corresponding calculations are missing. Therefore, now a more precise proof how to determine this infimum is given.

Lemma 2.3. *Let $K \subset \mathbb{C}^2 \setminus F(0)$ be a compact set. Then for $\kappa, \nu \in \Gamma^*$, one has*

$$\lim_{\|\nu\|\to\infty} \inf_{\substack{\kappa\in\Gamma^*\\k\in K}} |(\kappa + k + k_\nu^\pm)^2| = \infty.$$

Proof. Let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence in Γ^* such that $\lim_{n\to\infty}\|\nu_n\| = \infty$. Consider for fixed ν_n

$$f(\kappa_n, k_n) = \left|\left(\kappa_n + k_n + k_{\nu_n}^\pm\right)^2\right| = \left\|\left(\kappa_n + k_n + k_{\nu_n}^\pm\right)\right\|^2$$

and choose $\kappa_n \in \Gamma^*$ and $k_n \in K$ as vectors such that f attains its infimum. The set K is compact, Γ^* is discrete and f is continuous with $\lim_{\|\kappa\|\to\infty} f(\kappa, k_n) = \infty$. So these vectors exist. Since the infimum of f over $\kappa \in \Gamma^*$ is bigger or equal than the infimum of f over $\kappa \in \mathbb{R}^2$, we extend the domain of $\kappa \in \Gamma^*$ to $\kappa \in \mathbb{R}^2$. Due to $(k_\nu^\pm)^2 = 0$ it is with $k_{n,1} = a_n + \iota b_n$ and $k_{n,2} = c_n + \iota d_n$, where $a_n, b_n, c_n, d_n \in \mathbb{R}$ for all $n \in \mathbb{N}$,

$$\begin{aligned}
\left\|\left(\kappa_n + k_n + k_{\nu_n}^\pm\right)\right\|^2 &= \left\|\begin{pmatrix}\kappa_{n,1}\\ \kappa_{n,2}\end{pmatrix} + \begin{pmatrix}a_n + \iota b_n\\ c_n + \iota d_n\end{pmatrix} + \frac{1}{2}\begin{pmatrix}\pm\nu_{n,1} + \iota\nu_{n,2}\\ -\iota\nu_{n,1} \pm \nu_{n,2}\end{pmatrix}\right\|^2\\
&= \left(\kappa_{n,1} + a_n \pm \frac{1}{2}\nu_{n,1}\right)^2 + \left(b_n + \frac{1}{2}\nu_{n,2}\right)^2 + \left(\kappa_{n,2} + c_n \pm \frac{1}{2}\nu_{n,2}\right)^2 + \left(d_n - \frac{1}{2}\nu_{n,1}\right)^2\\
&= \kappa_{n,1}^2 + a_n^2 + \frac{1}{4}\nu_{n,1}^2 + 2a_n\kappa_{n,1} \pm \kappa_{n,1}\nu_{n,1} \pm a_n\nu_{n,1} + b_n^2 + b_n\nu_{n,2} + \frac{1}{4}\nu_{n,2}^2 +\\
&\quad + \kappa_{n,2}^2 + c_n^2 + \frac{1}{4}\nu_{n,2}^2 + 2c_n\kappa_{n,2} \pm \nu_{n,2}\kappa_{n,2} \pm \nu_{n,2}c_n + d_n^2 - d_n\nu_{n,1} + \frac{1}{4}\nu_{n,1}^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\partial_{\kappa_1} f(\kappa_n, k_n) = 2(\kappa_{n,1} + a_n) \pm \nu_{n,1} = 0 &\Leftrightarrow \kappa_{n,1} = -a_n \mp \frac{1}{2}\nu_{n,1},\\
\partial_{\kappa_2} f(\kappa_n, k_n) = 2(\kappa_{n,2} + c_n) \pm \nu_{n,2} = 0 &\Leftrightarrow \kappa_{n,2} = -c_n \mp \frac{1}{2}\nu_{n,2}
\end{aligned}$$

and $\operatorname{Hess}(f)(\kappa, k_n) = \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$ for all $\kappa \in \mathbb{R}^2$. So the infimum is attained for $\kappa_n = -\operatorname{Re}(k_n + k_{\nu_n}^\pm)$ and for $n \to \infty$ we get

$$f(\kappa_n, k_n) = \|\operatorname{Im}(k_n + k_{\nu_n}^\pm)\|^2 = \|\operatorname{Im}(k_n)\|^2 + 2\langle\operatorname{Im}(k_n), \operatorname{Im}(k_{\nu_n}^\pm)\rangle + \|\nu_n\|^2 \to \infty.$$

The boundedness of the imaginary part of $k_n \in K$ assures that the behavior of the infimum over $\kappa_n \in \Gamma^*$ and $k_n \in K$ also goes to infinity for $n \to \infty$. □

Note that the operator $-\triangle_{k+k_\nu^\pm}$ is depending on $\nu \in \Gamma^*$. So we consider this as a sequence of operators parametrized by $\nu \in \Gamma^*$. Thereby, we want to show that certain estimates hold for such sequences of operators for all $\nu \in \Gamma_\delta^*$ with $\delta > 0$ sufficiently small, i.e. for large imaginary part of the boundary value.

Lemma 2.4. *[Klauer, 2011, Lemma 4.3.7] Let $u \in C(\mathbb{R}^2/\Gamma)$. Then for all sufficiently small $\varepsilon, \tilde{\varepsilon} > 0$ there is a $\delta(\varepsilon, \tilde{\varepsilon}) > 0$ such that for all $k \in \Delta_{\mathbb{C}}^\varepsilon$ and $\nu \in \Gamma_\delta^*$ the operator $u\triangle_{k+k_\nu^\pm}$ exists and is bounded from $L^2(\mathbb{R}^2/\Gamma)$ to $L^2(\mathbb{R}^2/\Gamma)$ with*

$$\|u\,\triangle_{k+k_\nu^-}^{-1}\| < \tilde{\varepsilon}.$$

Proof. Let $k_0 \in \Delta_{\mathbb{C}}^\varepsilon$. Due to Theorem 1.10, there exists a neighborhood K of k_0 such that the resolvent $\triangle_k^{-1}$ exists for all $k \in K$ and is bounded on $L^2(\mathbb{R}^2/\Gamma^*, \mathbb{C})$. We choose K in such a way that $\operatorname{dist}(K, F(0)) > \frac{\varepsilon}{2}$. Since $F(0) \subset \mathbb{C}^2$ is invariant under translations by double points $k_\kappa^\pm$ for all $\kappa \in \Gamma^*$, this also holds for $k \in \{k_\kappa^\pm + \widetilde{k} \mid \widetilde{k} \in K\}$.

For $f \in L^2(\mathbb{R}^2/\Gamma)$ and $\kappa \in \Gamma^*$, one has due to Parseval's identity [Reed and Simon, 1980, Theorem II.6], Hölder's inequality [Reed and Simon, 1975, Proposition IX.4.2] and $\|u\| < c(u)$

$$\begin{aligned} \|u \, \triangle_{k+k_\nu^-}^{-1} f\|_{L^2(\mathbb{R}^2/\Gamma)} &\leq c(u) \| \triangle_{k+k_\nu^-}^{-1} f\|_{L^2(\mathbb{R}^2/\Gamma)} = c(u) \|\widehat{u \, \triangle_{k+k_\nu^-}^{-1} f}\|_{\ell^2(\Gamma^*)} \\ &= c(u) \left(\sum_{\kappa \in \Gamma^*} \left| \frac{\hat{f}(\kappa)}{-4\pi^2 (k + k_\kappa^\pm + \kappa)^2} \right|^2 \right)^{1/2} \\ &\leq \frac{c(u,K)}{\inf\limits_{\substack{\kappa \in \Gamma^* \\ k \in K}} |(k + k_\nu^- + \kappa)^2|} \left(\sum_{\nu \in \Gamma^*} |\hat{f}(\kappa)|^2 \right)^{1/2} \\ &= \frac{c(u,K)}{\inf\limits_{\substack{\kappa \in \Gamma^* \\ k \in K}} |(k + k_\nu^- + \kappa)^2|} \|f\|_{L^2(\mathbb{R}^2/\Gamma)}, \end{aligned}$$

where $c(u,K) \in \mathbb{R}^+$ is some constant depending on u and the neighborhood K of k.
We know from Lemma 2.3 that for all $\tilde{\varepsilon} > 0$, there exists a $\delta_K > 0$ depending on K and $\tilde{\varepsilon}$ such that for all $\nu \in \Gamma_\delta^*$ one has

$$c(u,K) \Big(\inf_{\substack{\kappa \in \Gamma^* \\ k \in K}} |(\kappa + k + k_\nu^\pm)^2| \Big)^{-1} < \tilde{\varepsilon}.$$

So the operator norm of $u \triangle_{k+k_\nu^{-1}}^{-1}$ on K is smaller than $\tilde{\varepsilon}$. One can repeat this procedure for all $k \in \Delta_{\mathbb{C}}^{\varepsilon}$ to obtain the same results for different sets K corresponding to the different k. Since $\Delta_{\mathbb{C}}^{\varepsilon}$ is compact, it can be covered by finitely many of such K. Choosing δ as the minimum of all δ_K from a finite covering of $\Delta_{\mathbb{C}}^{\varepsilon}$ yields the desired result. □

The proof of the following Theorem is completely analogous to the proof of [Klauer, 2011, Theorem 4.3.8], where it is shown for a wider class of potentials. Nevertheless, we repeat the proof here since we believe it is crucial for the steps hereinafter.

Theorem 2.5. *Let $u \in C(\mathbb{R}^2/\Gamma)$. Then for all sufficiently small $\varepsilon, \tilde{\varepsilon} > 0$, there exists a $\delta(\varepsilon, \tilde{\varepsilon}) > 0$ such that for all $k \in \Delta_{\mathbb{C}}^{\varepsilon}$ and all $\nu \in \Gamma_\delta^*$, the operators $\triangle_{k+k_\nu^-}^{-1}$ and $(\triangle_{k+k_\nu^-} - u)^{-1}$ exist and are bounded from $L^2(\mathbb{R}^2/\Gamma)$ to $L^2(\mathbb{R}^2/\Gamma)$ with*

$$\|(\triangle_{k+k_\nu^-} - u)^{-1} - \triangle_{k+k_\nu^-}^{-1}\| < \tilde{\varepsilon}. \tag{2.2}$$

Proof. It is shown in Lemma 2.2 that the resolvent $\triangle_{k+k_\nu^-}^{-1}$ exists and is bounded from $L^2(\mathbb{R}^2/\Gamma)$ to $L^2(\mathbb{R}^2/\Gamma)$ for $k + k_\nu^- \in \mathbb{C}^2 \setminus F(0)$. By Lemma 2.4 we know that for sufficient small $0 < \varepsilon' \leq \varepsilon$, there is a $\delta' > 0$ such that for all $k \in \Delta_{\mathbb{C}}^{\varepsilon}$ and all $\nu \in \Gamma_{\delta'}^*$, the operator $u \triangle_{k+k_\nu^-}^{-1}$ exists and is bounded on $L^2(\mathbb{R}^2/\Gamma)$ with

$$\|u \triangle_{k+k_\nu^-}^{-1}\| < \varepsilon'.$$

For a fixed and sufficiently small $\varepsilon' > 0$, $(\mathbb{1} - u\triangle_{k+k_\nu^-})^{-1}$ can be expressed by its Neumann series, compare [Reed and Simon, 1980, Corollary in VI.3, formula (VI.2)] and thus

$$\|(\triangle_{k+k_\nu^-} - u)^{-1} - \triangle_{k+k_\nu^-}^{-1}\| \leq \| \triangle_{k+k_\nu^-}^{-1} \| \left\| \sum_{n=0}^{\infty} (u\triangle_{k+k_\nu^-}^{-1})^n - \mathbb{1} \right\| \leq \| \triangle_{k+k_\nu^-}^{-1} \| \frac{\varepsilon'}{1-\varepsilon'}.$$

Lemma 2.2 yields that $\triangle_{k+k_\nu^-}^{-1}$ is uniformly bounded in k for all $\nu \in \Gamma^*$ since $\Delta_{\mathbb{C}}^{\varepsilon}$ is compact. Furthermore, $F(0)$ is invariant under translations by the double points k_κ^-. So by Lemma 2.3, the right hand side of the above inequality is smaller than ε for all $k \in \Delta_{\mathbb{C}}^{\varepsilon}$ and all $\nu \in \Gamma_{\delta'}^*$ for sufficiently small ε' such that $\frac{\varepsilon'}{1-\varepsilon'} < \tilde{\varepsilon}$. □

The foregoing considerations yield immediately that $F(u) \cap \mathbb{C}_\delta^2$ has to be contained in a small ε-tube around $F(0) \cap \mathbb{C}_\delta^2$ for $\delta > 0$ sufficiently small. More precisely:

Corollary 2.6.

Let $u \in C(\mathbb{R}^2/\Gamma)$. Then for all $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that for all $k + k_\nu^\pm \in F(u)$ with $k \in \Delta_{\mathbb{C}}$ and $\nu \in \Gamma_\delta^*$ one has

$$\operatorname{dist}(k + k_\nu^\pm, F(0)) = \operatorname{dist}(k, F(0)) = \min_{\tilde{k} \in F(0)} \|k - \tilde{k}\| < \varepsilon.$$

Proof. It is shown in Lemma 2.2 that for $k \in \Delta_{\mathbb{C}}^{\varepsilon}$ and for all $\nu \in \Gamma^*$, the free resolvent $-\triangle_{k+k_\nu^-}^{-1}$ is a regular operator. Due to Theorem 2.5, for all $\varepsilon > 0$, there exists a $\delta' > 0$ such that (2.2) holds for $k \in \Delta_{\mathbb{C}}^{\varepsilon}$ and $\nu \in \Gamma_{\delta'}^*$. So for $\delta' > 0$ small enough, also $(-\triangle_{k+k_\nu^-} - u)^{-1}$ is a regular operator for $k \in \Delta_{\mathbb{C}}^{\varepsilon}$. Hence, the set of singularities of $(-\triangle_{k+k_\nu^-} + u)^{-1}$ is contained in $\{k \in \mathbb{C}^2 \mid k = k_\nu^- + k' \text{ and } k' \in \Delta_{\mathbb{C}} \setminus \Delta_{\mathbb{C}}^{\varepsilon}\}$. This yields that for $\nu \in \Gamma_{\delta'}^*$ and $k \in \Delta_{\mathbb{C}}$, all values of $k + k_\nu^- \in \mathbb{C}^2$ that are describing the Fermi curve $F(u)$ are contained in an ε-tube around $F(0)$. Let $(\delta')^{-1} \leq \|\nu\| = 2\|\operatorname{Im}(k_\nu^-)\|$ for all $\nu \in \Gamma_{\delta'}^*$. Choosing $\delta := \frac{1}{2}\delta'$ yields $\|\operatorname{Im}(k_\nu^-)\| \geq \delta^{-1}$ for all $\nu \in \Gamma_{\delta'}^*$, and therefore the assertion. □

2.2. Matrix decomposition of the Schrödinger operator.

To see the asymptotic freeness of $F(u)$, we will adapt the technique used in [Klauer, 2011, Section 4.5] – which is used there to describe $F(u)$ in the excluded domains – for all parts of $F(u) \cap \mathbb{C}_\delta^2$ with $\delta > 0$ sufficiently small. Therefore, we repeat the whole theory for both cases and cite in which part of [Klauer, 2011] the appropriate results for the excluded domains are shown. The following definitions correspond to the decomposition of $L^2(\mathbb{R}^2/\Gamma)$ as given in [Klauer, 2011, Definition 4.5.1] for the excluded domains.

Definition 2.7 ([Klauer, 2011, Definition 4.5.1])**.** Let E_0 be the one-dimensional complex Banach space generated by $\psi^0 := 1$ and for all $\nu \in \Gamma^*$, let $E_{\pm\nu}$ be the two-dimensional complex Banach space generated by the Fourier modi $\psi^{\pm\nu} := e^{2\pi\iota\langle k,\pm\nu\rangle}$ and ψ^0. Then the canonical projections

$$\pi_0 : L^2(\mathbb{R}^2/\Gamma) \to E_0, \qquad \psi \mapsto \hat{c}(0)\psi^0$$
$$\pi_\nu : L^2(\mathbb{R}^2/\Gamma) \to E_{\pm\nu}, \qquad \psi \mapsto \hat{c}(0)\psi^0 + \hat{c}(\pm\nu)\psi^{\pm\nu}$$

are both bounded linear operators, and therefore also $\pi_0^\perp = (\mathbb{1} - \pi_0)$ and $\pi_{\pm\nu}^\perp := (\mathbb{1} - \pi_{\pm\nu})$ are bounded and linear. We define $E_0^\perp := \mathrm{Im}(\pi_0^\perp)$ and $E_{\pm\nu}^\perp := \mathrm{Im}(\pi_{\pm\nu}^\perp)$. If it makes no or only minor differences whether we consider E_0 or $E_{\pm\nu}$, we write E respectively $E^\perp$ and π respectively $\pi^\perp$.

With this definitions, $L^2(\mathbb{R}^2/\Gamma) = E_0 \oplus E_0^\perp$ as well as $L^2(\mathbb{R}^2/\Gamma) = E_{\pm\nu} \oplus E_{\pm\nu}^\perp$. Since we are going to consider some operators on these subspaces or images under these operators restricted to these subspaces, the following decomposition of linear operators on $L^2(\mathbb{R}^2/\Gamma)$ is very useful.

Definition 2.8 ([Klauer, 2011, Definition 4.5.3]). Let $T : L^2(\mathbb{R}^2/\Gamma) \to L^2(\mathbb{R}^2/\Gamma)$ be a linear operator. We define the linear operators

$$A := \pi T|_E : E \to E, \qquad B := \pi T|_{E^\perp} : E^\perp \to E,$$
$$C := \pi^\perp T|_E : E \to E^\perp, \qquad D := \pi^\perp T|_{E^\perp} : E^\perp \to E^\perp.$$

We call the operator A the restriction of T to E and the operator D the restriction of T to $E^\perp$, even though it is actually the restriction projected to the respective space.

With respect to the decomposition $L^2(\mathbb{R}^2/\Gamma) = E \oplus E^\perp$, the operator T can be represented as

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

We have seen in Lemma 1.19 that for $k + k_\nu^\pm \in F(0) \setminus \{k_\nu^\pm \mid \nu \in \Gamma^*\}$ with $k \in \Delta_\mathbb{C}$, the kernel of $\triangle_{k_\nu^\pm + k}$ consists of E_0 and that the kernel of $\triangle_{k_\nu^\pm}$ consists of $E_{\pm\nu}$, i.e. the singular support of the resolvent $\triangle_{k_\nu^\pm + k}$ is contained in E_0 and the singular support of $\triangle_{k_\nu^\pm}^{-1}$ is contained in $E_{\pm\nu}$. Accordingly, the free resolvent restricted to $E_{\pm\nu}^\perp$ respectively $E_0^\perp$ is a regular operator which we denote as the *reduced resolvent*. We will concretize this in Proposition 2.11 which is for $E_{\pm\nu}$ also shown in [Klauer, 2011, Proposition 4.5.4].

Before we start with this, we introduce the way how we choose k in the asymptotics hereinafter. We want to consider the asymptotics of $F(u)$ in small, neighborhoods of $F(0)$ and distinguish between the parts of the Fermi curve 'far outside' which are close to the double points $k_\nu^\pm \in F(0)$ and the part 'far outside' which is bounded away from these double points. Therefore, it is necessary to define open sets around the free Fermi curve such that for $\delta > 0$ sufficiently small $F(u) \cap \mathbb{C}_\delta^2$ is contained in them and such that the local considerations hold for all ν with $\nu \in \Gamma^*_\delta$. We define

$$B_\varepsilon(k^\pm_{\Gamma^*})^c := \mathbb{C}^2 \setminus \bigcup_{\kappa,\nu\in\Gamma^*} B_\varepsilon(k^\pm_\kappa + \nu).$$

Then the set $\{k \in \mathbb{C}^2 \mid k \in F(u) \cap B_\varepsilon(k^\pm_{\Gamma^*})^c\}$ is the set of all points on $F(u)$ which are ε-far away from the double points $k^\pm_\nu$ of $F(0)$. We again consider translations of small open neighborhoods of $k_0 \in \mathcal{R}$ by $k^\pm_\nu$ for $\|\nu\| \to \infty$. To apply the decomposition of the resolvent of the Schrödinger operator into a singular and a regular part away from the excluded domains, it is necessary to ensure that we consider only neighborhoods of such $k_0 \in \mathcal{R} \setminus \{k^\pm_\nu \mid \nu \in \Gamma^*\}$ which contain no parts of $\mathcal{R}(\kappa)$ for $\kappa \in \Gamma^* \setminus \{0\}$. In this case, the kernel of the Schrödinger operator in this neighborhood consist only of E_0.
Using again the norm $\|\cdot\|$ which is indicated by the hermitian scalar product for elements of $\mathbb{C}^2$, the following proposition holds.

Proposition 2.9. *Let $k \in \mathcal{R}_\pm \cap B_\varepsilon(k^\pm_{\Gamma^*})^c$. Then for all $\tilde{k} \in \mathcal{R}_\mp(\nu)$, there holds*

$$\|k - \tilde{k}\| \geq \|k - k^\pm_\nu\| \geq \min_{\nu\in\Gamma^*}\{\|k - k^-_\nu\|, \|k - k^+_\nu\|\}$$

and for all $\tilde{k} \in \mathcal{R}_\pm(\nu)$, there holds

$$\|k - \tilde{k}\| \geq \|\nu\| \geq \min\{\|\kappa\| \mid \kappa \in \Gamma^* \setminus \{0\}\}.$$

Proof. A normal vector of $\mathcal{R}_+(\nu)$ is given by $\left(\begin{smallmatrix}1\\ \iota\end{smallmatrix}\right)$ and a normal vector of $\mathcal{R}_-(\nu)$ is given by $\left(\begin{smallmatrix}1\\ -\iota\end{smallmatrix}\right)$ and with respect to the hermitian scalar product $\cdot$ it is

$$\begin{pmatrix}1\\ \iota\end{pmatrix} \cdot \begin{pmatrix}1\\ -\iota\end{pmatrix} = 1 \cdot 1 + \iota \cdot \overline{(-\iota)} = 0.$$

So $\mathcal{R}_+(\nu)$ and $\mathcal{R}_-(\kappa)$ are both affine hyperplanes of $\mathbb{C}^2$ which are perpendicular to each other. So $\mathcal{R}_\pm$ and $\mathcal{R}_\pm(\nu)$ are parallel to each other and the minimal distance between two affine hyperplanes which are parallel is the distance measured orthogonally, it is $\operatorname{dist}(\mathcal{R}_\pm, \mathcal{R}_\pm(\nu)) = \|\nu\|$ for all $\nu \in \Gamma^*$ and for every $k \in \mathcal{R}_\pm \cap B_\varepsilon(k^\pm_{\Gamma^*})^c$, there exists a $\tilde{k} \in \mathcal{R}_\pm(\nu)$ such that

$$\|k - \tilde{k}\| = \operatorname{dist}(\mathcal{R}_\pm, \mathcal{R}_\pm(\nu)) = \|\nu\|.$$

For the second case, let $k \in \mathcal{R}_+ \cap B_\varepsilon(k^\pm_{\Gamma^*})^c$ and $\tilde{k} \in \mathcal{R}_-(\nu)$ with $\nu \in \Gamma^* \setminus \{0\}$. Then

$$k_1 + \iota k_2 = 0 \text{ and } (\tilde{k}_1 + \nu_1) - \iota(\tilde{k}_2 + \nu_2) = 0$$

and hence $k_2 = \iota k_1$ and $\tilde{k}_2 = -\iota(\tilde{k}_1 + \nu_1 - \iota\nu_2)$. So the triangle inequality yields

$$\begin{aligned}\|k - \tilde{k}\| = |k_1 - \tilde{k}_1| + |k_2 - \tilde{k}_2| &= |k_1 - \tilde{k}_1| + |\iota k_1 + \iota(\tilde{k}_1 + \nu_1 - \iota\nu_2)| \\ &= |k_1 - \tilde{k}_1| + |k_1 + \tilde{k}_1 + \nu_1 - \iota\nu_2| \\ &\geq |2k_1 + \nu_1 - \iota\nu_2| = 2\left|k_1 - \frac{1}{2}(-\nu_1 + \iota\nu_2)\right| = 2|k_1 - k^+_{\nu,1}|.\end{aligned}$$

Together with $|k_1 - k^+_{\nu,1}| = |k_2 - k^+_{\nu,2}|$ and $\mathcal{R}_+ \cap \mathcal{R}_-(\nu) = \{k^+_\nu\}$, we obtain

$$\min_{\tilde{k}\in\mathcal{R}_-(\nu)} \|k - \tilde{k}\| = \|k - k^+_\nu\|.$$

Analogous calculations yield for $k \in \mathcal{R}_- \cap B_\varepsilon(k^*_\Gamma)^c$ and $\tilde{k} \in \mathcal{R}_+(\nu)$ that $\min_{\tilde{k}\in\mathcal{R}_+(\nu)}(\|k - \tilde{k}\|) = \|k - k^-_\nu\|$ and taking these two equalities together yields the assertion. □

Remark 2.10. This Lemma together with Corollary 2.6 motivates the following setting which we will use to formulate the asymptotics. Let $\varepsilon > 0$ be chosen such that 4ε is smaller than one quarter of the minimal distance of the generators of the lattice $\Gamma^*_\mathbb{C}$ from Definition 2.1 and such that the Fermi curve $F(u)$ is contained in an ε-tube around $F(0)$. The last choice is possible due to Corollary 2.6. We then consider $B_{4\varepsilon}(0) = \{k \in \mathbb{C}^2 \mid \|k\| < 4\varepsilon\}$. For $\delta > 0$ sufficiently small and $\nu \in \Gamma^*_\delta$, we denote the sets $k^\pm_\nu + B_{4\varepsilon}(0)$ as excluded domains.
To describe the asymptotics bounded away from the double points $k^\pm_\nu$, we choose $k_0 \in \mathcal{R}\cap B_{4\varepsilon}(k^\pm_{\Gamma^*})^c$. Then due to Proposition 2.9, k_0 is bounded away by 4ε from $k^\pm_\nu$, thus $B_{2\varepsilon}(k_0)$ does not intersect an ε-tube around $\mathcal{R}(\nu)$ for $\nu \in \Gamma^*_\delta$. That means by considering $k + k^\pm_\nu$ with $k \in B_{2\varepsilon}(k_0)$, one considers indeed only the part of the Fermi curve which is ε-close to $F(0)$ and bounded away by 2ε from $k^\pm_\nu$. Why we are choosing 4ε and 2ε instead of the seemingly more natural choice ε and 2ε will be explained later in Remark 2.30.

Part (b) of the following proposition is shown in [Klauer, 2011, Proposition 4.5.4].

Proposition 2.11. *(a) Let $\varepsilon > 0$ and $k_0 \in \mathcal{R}\cap B_{4\varepsilon}(k^\pm_{\Gamma^*})^c$. Then for all $k \in B_{2\varepsilon}(k_0)$, the resolvent of the free Schrödinger operator as an operator on $L^2(\mathbb{R}^2/\Gamma)$ has the $E_0 \oplus E_0^\perp$-decomposition as in Definition 2.8*

$$\triangle^{-1}_{k+k^\pm_\nu} = \begin{pmatrix} S_0(k + k^\pm_\nu) & 0 \\ 0 & R_0(k + k^\pm_\nu) \end{pmatrix},$$

where R_0 is the reduced resolvent of the free Schrödinger operator. The reduced resolvent $R_0(k + k^\pm_\nu)$ is holomorphic and bounded in $k \in B_{2\varepsilon}(k_0)$, i.e. for the operator norm, there holds

$$\sup_{k\in B_{2\varepsilon}(k_0)} \sup_{\nu\in\Gamma^*\setminus\{0\}} \|R_0(k + k^\pm_\nu)\| < \infty.$$

With respect to the basis ψ^0 *of* E_0*, the singular part of the free Schrördinger operator reads as*

$$S_0(k) = -\frac{1}{4\pi^2}k^{-2}.$$

(b) Let $\tilde{\varepsilon} > 0$ *be smaller than half of the distance between the generators of* $\Gamma^*_{\mathbb{C}}$*. Then the resolvent of the free Schrödinger operator as an operator on* $L^2(\mathbb{R}^2/\Gamma)$ *has for all* $\nu \in \Gamma^* \setminus \{0\}$ *and* $k \in B_{\tilde{\varepsilon}}(0)$ *the* $E_{\pm\nu} \oplus E^{\perp}_{\pm\nu}$*-decomposition as in Definition 2.8*

$$\triangle^{-1}_{k+k^{\pm}_{\nu}} = \begin{pmatrix} S_{\pm\nu}(k+k^{\pm}_{\nu}) & 0 \\ 0 & R_{\pm\nu}(k+k^{\pm}_{\nu}) \end{pmatrix},$$

where $R_{\pm\nu}$ *is the reduced resolvent of the free Schrödinger operator. For* $k \in B_{\tilde{\varepsilon}}(0)$ *and* $\nu \in \Gamma^*$*,* $R_{\pm\nu}(k+k^{\pm}_{\nu})$ *is holomorphic and bounded, i.e. for the operator norm, there holds*

$$\sup_{k\in B_{\varepsilon}(0)} \sup_{\nu\in\Gamma^*\setminus\{0\}} \|R_{\pm\nu}(k+k^{\pm}_{\nu})\| < \infty.$$

With respect to the basis $\psi^0, \psi^{\pm\nu}$ *of* $E_{\pm\nu}$*, the singular part of the free Schrödinger operator reads as*

$$S_{\pm\nu}(k) = \begin{pmatrix} -4\pi^2k^2 & 0 \\ 0 & -4\pi^2(k+\nu)^2 \end{pmatrix}^{-1}.$$

Note that $\tilde{\varepsilon}$ is chosen in such a way that it holds for 8ε and thus also for 4ε with ε as in Remark 2.10. These two cases will be necessary hereinafter.

Proof. Let $\tilde{k} \in \mathbb{C}^2$ be a pole of $k \mapsto \triangle^{-1}_k$, i.e. there exists a $\kappa \in \Gamma^*$ such that $\tilde{k} \in \mathcal{R}(\kappa)$. Then due to Lemma 1.18,

$$\tilde{k}_2 + \kappa_2 = \iota(\tilde{k}_1 + \kappa_1) \ \text{ or } \tilde{k}_2 + \kappa_2 = -\iota(\tilde{k}_1 + \kappa_1). \tag{2.3}$$

Furthermore, we already know that the Fourier transform of $\triangle^{-1}_k$ is a diagonal operator with $-\frac{1}{4\pi^2(k+\kappa)^2}$ on the diagonals with $\kappa \in \Gamma^*$ and with some bijection between Γ^* and $\mathbb{N}$ to sort the diagonal entries.

(a) Let $\tilde{k} = k_0 \in B_{4\varepsilon}(k^{\pm}_{\Gamma^*})^c$. Then also $k_0 + k^{\pm}_{\nu} \in B_{4\varepsilon}(k^{\pm}_{\Gamma^*})^c \cap$ for all $\nu \in \Gamma^*$. Without loss of generality, let $k_0 \in \mathcal{R}^{\pm}$ and $k_0 \notin \mathcal{R}^{\mp}$. Then k_0 fulfills exactly one of the conditions in (2.3) with $\kappa = 0$, since Γ^* is discrete. Thus, for $k \in B_{2\varepsilon}(k_0) \subset B_{2\varepsilon}(k_{\Gamma^*})^c$, also only the equation in (2.3) with $\kappa = 0$ which holds for k_0 can equal zero. Therefore, $\triangle^{-1}_{k+k^{\pm}_{\nu}}$ becomes only singular on E_0 and is regular when it is restricted to $E^{\perp}_0$. The form of S_0 is given by the Fourier transform of the resolvent restricted to E_0 and maps $\hat{\psi}^0$ to $-\frac{1}{4\pi^2(k+k^{\pm}_{\nu})^2}\hat{\psi}^0$.

(b) Let now $\tilde{k} = k^{\pm}_{\nu}$ for some $\nu \in \Gamma^*$. Then one of the equalities in (2.3) holds for $\kappa = 0$ and the other one for $\kappa = \pm\nu$ as can be seen in the proof of Lemma 1.18. So for any other pole

$\tilde{k} = k_\nu^\pm + k$ of the resolvent with $k \in B_\varepsilon(0)$, the corresponding equality in (2.3) can also only hold for either the same κ or 0 since Γ^* is discrete. The rest of the argumentation is analogous to (i) with the only difference that in the neighborhood of $k_\nu^\pm$, the free Schrödinger operator is singular on $E_{\pm\nu}$.

The estimate of the norm follows in both cases as in Lemma 2.2. The only difference is that now also $k \in F(0)$ is allowed since only the regular part of the free Schrödinger operator is considered. □

Remark 2.12. For abbreviation, one can also write the above statements as

$$\triangle_k^{-1} = S(k) + R(k),$$

where $S(k) := \pi \triangle_k^{-1} |_E$ and $R(k) := \pi^\perp \triangle_k^{-1} |_E^\perp$. If we want to point out whether we consider these operators in a neighborhood of $k_\nu^\pm$ or in the neighborhood of a regular point of $F(0)$, we put an index $\pm\nu$ respectively 0 to the corresponding operators S and R. No index shall imply that the assertions hold for both variants. We also neglect the dependence of S and R on k as long as it is not necessary for the asymptotics.

For the reduced resolvent, the same statements as we have seen in Section 2.1 for $k \in \Delta_{\mathbb{C}}^\varepsilon$ can be shown for all $k \in \Delta_{\mathbb{C}}$, so especially also for $k \in F(0)$ since the reduced resolvent is regular on $F(0)$. Therefore, the following Lemma is obvious to expect. Part (b) is shown in [Klauer, 2011, Lemma 4.5.9].

Lemma 2.13. *(a) For all $\varepsilon > 0$ as in Remark 2.10, there is a $\delta(\varepsilon) > 0$ such that for all $k \in B_{2\varepsilon}(k_0)$, all $k_0 \in \mathcal{R} \cap B_{4\varepsilon}(k_{\Gamma^*}^\pm)^c$ and all $\nu \in \Gamma_\delta^*$*

$$\inf_{\substack{k \in B_{2\varepsilon}(k_0) \\ \kappa \in \Gamma^* \setminus \{0\}}} |(\kappa + k + k_\nu^\pm)^2| > \varepsilon^{-1}.$$

(b) For all $\tilde{\varepsilon} > 0$ which are smaller than half of the distance between the generators of $\Gamma_{\mathbb{C}}^*$, there is a $\delta(\tilde{\varepsilon}) > 0$ such that for all $\nu \in \Gamma_\delta^*$

$$\inf_{\substack{k \in B_{\tilde{\varepsilon}(0)} \\ \kappa \in \Gamma^* \setminus \{0, \pm\nu\}}} |(\kappa + k + k_\nu^\pm)^2| > \varepsilon^{-1}.$$

Proof. The proof of both statements is more or less analogous to the proof of Lemma 2.3. In case (a), there is only exactly one zero of $\kappa \mapsto |(\kappa + k + k_\nu^\pm)^2|$. Without loss of generality, we assume again that $\kappa = 0$ and due to the choice of ε, this zero does not coincide with any double point $k_\nu^\pm$. However, $\kappa = 0$ is excluded from the infimum. The same holds in a neighborhood of the double points $k_\nu^\pm$ with the only difference that in this case, the only zeros of $\kappa \mapsto |(\kappa + k + k_\nu^\pm)^2|$ are given by $\kappa \in \{0, \pm\nu\}$ which are also excluded from the infimum, and therefore (b) holds. □

We also need an analogon to Lemma 2.4 for the reduced resolvent, but in stronger norms and also for reversed order of potential and Laplacian. First of all, note that $W^{1,2}(\mathbb{R}^2/\Gamma) \subset L^2(\mathbb{R}^2/\Gamma)$. Thus, the direct composition of $L^2(\mathbb{R}^2/\Gamma)$ in Definition 2.8 transfers also on $W^{1,2}(\mathbb{R}^2/\Gamma)$ and we also denote this composition as $W^{1,2}(\mathbb{R}^2/\Gamma, \mathbb{C}) = E + E^\perp$.

Lemma 2.14. *(a) For all $\varepsilon > 0$ as in Remark 2.10 and all $u \in C(\mathbb{R}^2/\Gamma)$, there is a $\delta(\varepsilon, u) > 0$ such that for all $k \in B_{2\varepsilon}(k_0)$ with $k_0 \in \mathcal{R} \cap B_{4\varepsilon}(k^\pm_{\Gamma^*})^c$ and all $\nu \in \Gamma^*_\delta$, the operator $\pi_0^\perp(\triangle^{-1}_{k+k^\pm_\nu} u|_{E^\perp_{\pm\nu}})$ is in $\mathcal{B}(L^2(\mathbb{R}^2/\Gamma), W^{1,2}(\mathbb{R}^2/\Gamma))$ with*

$$\|\pi_0^\perp(\triangle^{-1}_{k+k^\pm_\nu} u|_{E^\perp_0})\| < \varepsilon.$$

(b) For all $\varepsilon > 0$ as in Remark 2.10 and all $u \in C(\mathbb{R}^2/\Gamma)$, there exists a $\delta(\varepsilon, u) > 0$ such that for all $k \in B_{4\varepsilon}(0)$ and all $\nu \in \Gamma^*_\delta$, the operator $\pi^\perp_{\pm\nu}(\triangle^{-1}_{k+k^\pm_\nu} u|_{E^\perp_{\pm\nu}})$ is in $\mathcal{B}(L^2(\mathbb{R}^2/\Gamma), W^{1,2}(\mathbb{R}^2/\Gamma))$ with

$$\|\pi^\perp_{\pm\nu}(\triangle^{-1}_{k+k^\pm_\nu} u|_{E^\perp_{\pm\nu}})\| < \varepsilon.$$

Proof. Proposition 2.11 yields that

$$\pi^\perp \triangle^{-1}_{k+k^\pm_\nu} u|_E = \triangle^{-1}_{k+k^\pm_\nu} \pi^\perp u|_E.$$

The multiplication operator $u : L^2(\mathbb{R}^2/\Gamma) \to L^2(\mathbb{R}^2/\Gamma)$ is bounded since $u \in C(\mathbb{R}^2/\Gamma)$. Moreover, it follows as in Lemma 2.2 that $- \triangle_{k+k^\pm_\nu} |_{E^\perp} \in \mathcal{B}(L^2(\mathbb{R}^2/\Gamma), W^{1,2}(\mathbb{R}^2/\Gamma))$. Hence, for $\delta > 0$ sufficiently small, the restriction of $\triangle_{k+k^\pm_\nu} u$ to $E^\perp_0$ is bounded for $k \in B_{2\varepsilon}(k_0)$ respectively the restriction to $E^\perp_{\pm\nu}$ is bounded for $k \in B_{4\varepsilon}(0)$. As in the proof of Lemma 2.2, there holds in the case of (a) for the operator norm

$$\| - \pi_0^\perp \triangle^{-1}_{k+k^\pm_\nu} |_{E^\perp}\| \leq \frac{c}{\inf\limits_{\substack{\kappa \in \Gamma^* \setminus \{0\} \\ k \in K}} \|k + k^\pm_\nu + \kappa\|^2} + \frac{\tilde{c}}{\inf\limits_{\substack{\kappa \in \Gamma^* \setminus \{0\} \\ k \in K}} \|k + k^\pm_\nu + \kappa\|}$$

with some constants $c, \tilde{c} \in \mathbb{R}^+$. For $\delta > 0$ sufficiently small and $\nu \in \Gamma^*_\delta$, the term on the right hand side is smaller than ε. Analogously the case (b) is shown. The only difference is that the infimum in κ is taken over $\Gamma \setminus \{0, \pm\nu\}$.
The estimate of the norm is also shown as the estimate in Lemma 2.14 since for $\psi \in L^2(\mathbb{R}^2/\Gamma)$, there holds

$$\|\pi^\perp \triangle^{-1}_{k+k^\pm_\nu} u\psi\|_{W^{1,2}(\mathbb{R}^2/\Gamma)} \leq \| \triangle^{-1}_{k+k^\pm_\nu} \|_{W^{1,2}(\mathbb{R}^2/\Gamma)} \|\pi^\perp u\psi\|_{L^2(\mathbb{R}^2/\Gamma)}.$$

So by Lemma 2.13, there exists a $\delta > 0$ depending only on u and ε such that $\| - \triangle^{-1}_{k+k^\pm_\nu} u\| < \varepsilon$. □

Lemma 2.15 ([Schmidt, 2002, Lemma 3.21])**.** *Let E be a Banach space, F a closed subspace of E and $T : E \to F$ a bounded linear operator. Then the operator $\mathbb{1}_E - T$ is boundedly invertible if* *and only if* $\mathbb{1}_F - T|_F$ *is boundedly invertible, where* $\mathbb{1}_E$ *and* $\mathbb{1}_F$ *denote the identity in* E *and* F*, respectively. Furthermore, if* $(\mathbb{1}_F - T|_F)^{-1}$ *exists, then*

$$(\mathbb{1}_E - T)^{-1} = \mathbb{1}_E + (\mathbb{1}_F - T|_F)^{-1}T.$$

Proof. Let $(\mathbb{1}_E-T)^{-1}$ exist and be bounded. Since the codomain of T is F, one has $(\mathbb{1}_E-T)F \subseteq F$. Let $x \in E \setminus F$ and assume that $(\mathbb{1}_E - T)x \in F$. Then $x - Tx \in F$ and $Tx \in F$, so also $x \in F$. Thus, one also has $(\mathbb{1}_E - T)(E \setminus F) \subseteq (E \setminus F)$. Next, ee want to deduce $F \subseteq (\mathbb{1}_E - T)F$. Suppose that there exists a $g \in F$ such that there is no $f \in F$ with $(\mathbb{1}_E - T)f = g$. Due to the surjectivity of $(\mathbb{1}_E - T)$, it follows that there exits an $f \in E \setminus F$ such that $(\mathbb{1}_E - T)f = g$ which shows that $g \in E \setminus F$. By the same means, also $E \setminus F \subseteq (\mathbb{1}_E - T)(E \setminus F)$, and therefore $(\mathbb{1}_E - T)F = F$ and $(\mathbb{1}_E - T)(E \setminus F) = E \setminus F$. So the inverse of $\mathbb{1}_F - T|_F$ exists and is given by $(\mathbb{1}_E - T)^{-1}|_F$. It is bounded since $(\mathbb{1}_E - T)^{-1}$ is bounded.
Conversely, let $(\mathbb{1}_F - T|_F)^{-1}$ exist and be bounded on F. Since $(\mathbb{1}_E)|_F = \mathbb{1}_F$ and $T : E \to F$, one has $(\mathbb{1}_E - T)|_F = \mathbb{1}_F - T|_F$ and accordingly $(\mathbb{1}_E - T)(\mathbb{1}_F - T|_F)^{-1} = \mathbb{1}_F$. Since $T : E \to F$, this equality yields

$$(\mathbb{1}_E - T)(\mathbb{1}_E + (\mathbb{1}_F - T|_F)^{-1}T) = \mathbb{1}_E - T + (\mathbb{1}_E - T)(\mathbb{1}_F - T|_F)^{-1}T = \mathbb{1}_E,$$

and so $\mathbb{1}_E + (\mathbb{1}_F - T|_F)^{-1}T$ is a right inverse of $\mathbb{1}_E - T$. $T : E \to F$ implies

$$T(\mathbb{1}_E - T) = T\mathbb{1}_E - TT = \mathbb{1}_F T - T|_F T = (\mathbb{1}_F - T|_F)T,$$

so $\mathbb{1}_E + (\mathbb{1}_F - T|_F)^{-1}T$ is also a left inverse of $\mathbb{1}_E - T$. Hence,

$$\begin{aligned}(\mathbb{1}_E + (\mathbb{1}_F - T|_F)^{-1}T)(\mathbb{1}_E - T) &= \mathbb{1}_E - T + (\mathbb{1}_F - T|_F)^{-1}T(\mathbb{1}_E - T)\\ &= \mathbb{1}_E - T + (\mathbb{1}_F - T|_F)^{-1}(\mathbb{1}_F - T|_F)T\\ &= \mathbb{1}_E - T + T = \mathbb{1}_E.\end{aligned}$$

□

We show now that formally, one has a decomposition of $(\triangle + u)^{-1}$ similar to the decomposition of $\triangle^{-1}$ given in Remark 2.12.

Proposition 2.16. *Formally, it is*

$$(\triangle_k - u)^{-1} = (\mathbb{1} - Ru)^{-1} R + (\mathbb{1} - Ru)^{-1} S(u) (\mathbb{1} - uR)^{-1}, \tag{2.4}$$

where

$$S(u) := \left(\mathbb{1} - Su(\mathbb{1} - Ru)^{-1}\right)^{-1} S.$$

R and S are to be understood as operators acting on $L^2(\mathbb{R}^2/\Gamma)$ and not as operators acting on $E^\perp$ respectively on E. Formally shall indicate two things: First of all, that the above equality holds whenever the operators on both sides of the equality exist and are bounded. Secondly, that the sets of functions in $L^2(\mathbb{R}^2/\Gamma)$ for which they are unbounded coincide.

Proof. It is $\triangle_k^{-1} = S + R$, see Proposition 2.11. Moreover,

$$R(\mathbb{1} - uR)^{-1} = \left((\mathbb{1} - uR)R^{-1}\right)^{-1} = (R^{-1} - u)^{-1} = \left(R^{-1}(\mathbb{1} - Ru)\right)^{-1} = (\mathbb{1} - Ru)^{-1}R.$$

So

$$\begin{aligned} \triangle_k^{-1} &= R + S - Su(\mathbb{1} - uR)^{-1}R + Su(\mathbb{1} - uR)^{-1}R \\ &= \left(\mathbb{1} - Su(\mathbb{1} - Ru)^{-1}\right) R + S\left(\mathbb{1} + uR(\mathbb{1} - uR)^{-1}\right). \end{aligned}$$

Using Neumann series one gets that

$$\mathbb{1} + uR(\mathbb{1} - uR)^{-1} = \mathbb{1} + uR\sum_{n=0}^{\infty}(uR)^n = \sum_{n=0}^{\infty}(uR)^n = (\mathbb{1} - uR)^{-1}$$

and hence $\triangle_k^{-1} = \left(\mathbb{1} - Su(\mathbb{1} - Ru)^{-1}\right) R + S(\mathbb{1} - uR)^{-1}$. So formally, it is

$$\left(\mathbb{1} - Su(\mathbb{1} - Ru)^{-1}\right)^{-1} \triangle_k^{-1} = R + \left(\mathbb{1} - Su(\mathbb{1} - Ru)^{-1}\right)^{-1} S(\mathbb{1} - uR)^{-1}.$$

The left hand side of this equation equals $(\mathbb{1} - Ru)(\triangle_k - u)^{-1}$ since

$$\begin{aligned} \left(\mathbb{1} - Su(\mathbb{1} - Ru)^{-1}\right)^{-1} &= (\mathbb{1} - Ru)(\mathbb{1} - Ru)^{-1}\left(\mathbb{1} - Su(\mathbb{1} - Ru)^{-1}\right)^{-1} \\ &= (\mathbb{1} - Ru)\left((\mathbb{1} - Su(\mathbb{1} - Ru)^{-1})(\mathbb{1} - Ru)\right)^{-1} \\ &= (\mathbb{1} - Ru)(\mathbb{1} - Su - Ru)^{-1} \\ &= (\mathbb{1} - Ru)\left(\mathbb{1} - \triangle_k^{-1}u)\right)^{-1} \end{aligned} \tag{2.5}$$

and $(\triangle_k - u)^{-1} = (\mathbb{1} - \triangle_k^{-1}u)^{-1}\triangle_k^{-1}$. Taking all of this together finally yields the desired decomposition of $(\triangle_k - u)^{-1}$. □

The first part of (b) in the next proposition is also shown in [Klauer, 2011, Proposition 4.5.15].

Proposition 2.17. *(a) For all $\varepsilon > 0$ as in Remark 2.10 and all $u \in C(\mathbb{R}^2/\Gamma)$, there is a $\delta(\varepsilon, u) > 0$ such that for all $k \in B_{2\varepsilon}(k_0)$ with $k_0 \in \mathcal{R} \cap B_\varepsilon(k_{\Gamma^*}^\pm)^c$ and all $\nu \in \Gamma_\delta^*$, the operators*

$$\pi_0 u\left(\mathbb{1} - R(k + k_\nu^\pm)u\right)^{-1}\Big|_{E_0} \quad \text{and} \quad (\mathbb{1} - R(k + k_\nu^\pm)u)^{-1}$$

exist and are bounded.

(b) For all ε as in Remark 2.10 and all $u \in C(\mathbb{R}^2/\Gamma)$, there is a $\delta(\varepsilon, u) > 0$ such that for all $k \in B_{4\varepsilon}(0)$ and all $\nu \in \Gamma^*_\delta$, the operators

$$\pi_{\pm\nu} u \left(\mathbb{1} - R(k + k^{\pm}_{\nu})u\right)^{-1}\Big|_{E_{\pm\nu}} \text{ and } (\mathbb{1} - R(k + k^{\pm}_{\nu})u)^{-1}$$

exist and are bounded.

Proof. By Lemma 2.14, we know that for every $\varepsilon > 0$ under the given conditions in (a) respectively (b), there exists a $\delta(\varepsilon, u) > 0$ such that for all $k \in B_{2\varepsilon}(k_0)$ respectively $k \in B_{4\varepsilon}(0)$ and $\nu \in \Gamma^*_\delta$ the operator

$$\pi^{\perp}(R_{k+k^{\pm}_{\nu}} u)|_{E^{\perp}}$$

is bounded by $\varepsilon > 0$. For sufficiently small $\varepsilon > 0$, the Neumann Theorem yields that

$$\mathbb{1}_{E^{\perp}} - \pi^{\perp} R(k + k^{\pm}_{\nu})u|_{E^{\perp}}$$

is invertible. By Lemma 2.15, this implies that in this case $\mathbb{1} - R(k+k^{\pm}_{\nu})u$ is invertible on $L^2(\mathbb{R}^2/\Gamma)$. So the operator $u(\mathbb{1} - R(k + k^{\pm}_{\nu})u)^{-1}$ exists and is bounded from $L^2(\mathbb{R}^2/\Gamma)$ to $L^2(\mathbb{R}^2/\Gamma)$ and the same is true for the restriction $\pi_E(u(\mathbb{1} - Ru)^{-1})|_E$ of this operator to E. □

We want to get the explicit form of a matrix such that the zero set of the determinant of this matrix locally equals the values of $F(u)$ for some $u \in C(\mathbb{R}^2/\Gamma)$. Again, part (b) of the next theorem can be found in [Klauer, 2011, Theorem 4.5.19].

Theorem 2.18. *(a) For all $\varepsilon > 0$ as in Remark 2.10 and all $u \in C(\mathbb{R}^2/\Gamma)$, there is a $\delta(\varepsilon, u) > 0$ such that for all $k \in B_{2\varepsilon}(k_0)$ with $k_0 \in \mathcal{R} \cap B_\varepsilon(k^{\pm}_{\Gamma^*})^c$ and all $\nu \in \Gamma^*_\delta$, the Fermi curve can locally on $k^{\pm}_{\nu} + B_{2\varepsilon}(k_0)$ be described as the zero set of the in k holomorphic mapping*

$$k \mapsto \det\left(-4\pi^2(k + k^{\pm}_{\nu})^2 + \mathcal{A}_0(k + k^{\pm}_{\nu}, u)\right),$$

where

$$\mathcal{A}_0(k + k^{\pm}_{\nu}, u) := \pi_0 u \left(\mathbb{1} - Ru\right)^{-1}|_{E_0}.$$

(b) For all $\varepsilon > 0$ as in Remark 2.10 and all $u \in C(\mathbb{R}^2/\Gamma)$, there is a $\delta(\varepsilon, u) > 0$ such that for all $k \in B_{4\varepsilon}(0)$ and all $\nu \in \Gamma^*_\delta$, the Fermi curve can locally on $k^{\pm}_{\nu} + B_{4\varepsilon}(0)$ be described as the zero set of the in k holomorphic mapping

$$k \mapsto \det\left(\begin{pmatrix} -4\pi^2(k + k^{\pm}_{\nu})^2 & 0 \\ 0 & -4\pi^2(k + k^{\mp}_{\nu})^2 \end{pmatrix} + \mathcal{A}_{\pm\nu}(k + k^{\pm}_{\nu}, u)\right),$$

where

$$\mathcal{A}_{\pm\nu}(k + k^{\pm}_{\nu}, u) := \pi_{\pm\nu} u \left(\mathbb{1} - Ru\right)^{-1}|_{E_{\pm\nu}}.$$

Proof. In the setting of (a), it is for $k \in B_{2\varepsilon}(k_0)$ and $\nu \in \Gamma_\delta^*$

$$S_0(k + k_\nu^\pm) = \frac{1}{-4\pi^2(k + k_\nu^\pm)^2}$$

and in the setting of (b), since $k_\nu^\pm \pm \nu = k_\nu^\mp$, one has for $k \in B_{4\varepsilon}(0)$ that

$$S_{\pm\nu}(k + k_\nu^\pm) = \begin{pmatrix} -4\pi^2(k + k_\nu^\pm)^2 & 0 \\ 0 & -4\pi^2(k + k_\nu^\mp)^2 \end{pmatrix}^{-1}.$$

Combining the results of Proposition 2.17 and Proposition 2.16 yields that $k \in F(u)$ for generic $u \in C(\mathbb{R}^2/\Gamma)$ if and only if the operator in the decomposition (2.4) has a pole. The reduced resolvent R is regular and unequal to zero and it is shown in Proposition 2.17 that the same holds for $(\mathbb{1} - Ru)^{-1}$. Analogous argumentation as in the proof of this Proposition yields that also $(\mathbb{1} - uR)^{-1}$ is regular and unequal to zero. More precisely, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $(1 - R(k + k_\nu^\pm)u)^{-1}$ and $(1 - uR(k + k_\nu^\pm))^{-1}$ are regular for $\nu \in \Gamma_\delta^*$ and the respective choices of k. So all terms but $S(u)$ in the decomposition (2.4) are regular and unequal to zero for $\delta > 0$ sufficiently small. This implies together with Lemma 2.15 that $k \in F(u) \setminus F(4\pi^2\hat{u}_0)$ if and only if $S(u)$ restricted to E

$$\left(\mathbb{1}_E - S(k + k_\nu^\pm)\mathcal{A}(k + k_\nu(\hat{u}_0), u)\right)^{-1} S(k + k_\nu^\pm)\Big|_E$$

has a pole or equivalently if

$$\det(S(k + k_\nu^\pm)^{-1} - \mathcal{A}(k + k_\nu^\pm, u)) = 0.$$

To see that the same holds for $k \in F(0)$, note that as in the proof of Proposition 2.16 it is due to equation (2.5)

$$\begin{aligned}(\triangle_{k+k_\nu^\pm} - u)^{-1} &= (\mathbb{1} - \triangle_{k+k_\nu^\pm}^{-1} u)^{-1} \triangle_{k+k_\nu^\pm}^{-1} \\ &= (\mathbb{1} - Ru)^{-1} \left(\mathbb{1} - Su(\mathbb{1} - Ru)^{-1}\right)^{-1} (R + S)\end{aligned}$$

which is a meromorphic operator in k. It is shown in Proposition 2.17 that for $\delta > 0$ sufficiently small $(\mathbb{1} - R(k + k_\nu^\pm)u)^{-1}$ is regular and bounded for all $\nu \in \Gamma_\delta^*$. $R + S$ is an operator of block diagonal form with zeros on the off diagonals, so $(R+S)^{-1} = R^{-1} + S^{-1}$. Hence, for $k \in F(u) \setminus F(0)$ and for $k \in F(u) \cap F(0)$, all poles of $(\triangle_{k+k_\nu^\pm} - u)^{-1}$ are contained in

$$\begin{aligned}&\left(\mathbb{1} - Su(\mathbb{1} - Ru)^{-1}\right)^{-1} (R + S) = \left((R + S)^{-1}(\mathbb{1} - Su(\mathbb{1} - Ru)^{-1})\right)^{-1} \\ &= \left((R + S)^{-1} - (R + S)^{-1} Su\,(\mathbb{1} - Ru)^{-1}\right)^{-1} = \left(R^{-1} + S^{-1} - \pi u(\mathbb{1} - Ru)^{-1}\right)^{-1}.\end{aligned}$$

The last equality holds due to $R^{-1}S = 0$ and $S^{-1}S = \mathbb{1}_E$, so $(R + S)^{-1}(u) = \pi u$. The last expression of these equalities can only have a pole if $(R+S)^{-1} - \pi u(\mathbb{1} - Ru)^{-1} = 0$. Because $\operatorname{Im}(R^{-1}) \subset E^{\perp}$ and $\operatorname{Im}\left(S^{-1} - \pi u(\mathbb{1} - Ru)\right) \subset E$, this can only hold for functions contained in the kernel of R. So for $k \in F(0)$ and $\nu \in \Gamma_\delta^*$, all poles of $(\triangle_{k+k_\nu^\pm} - u)^{-1}$ are contained in the set of poles of $\left(S^{-1} - \pi u(\mathbb{1} - Ru)^{-1}\right)^{-1}$ or equivalently, are contained in

$$\left\{k \in \mathbb{C}^2 \mid \det(S^{-1} - \mathcal{A}(u, k + k_\nu^\pm)) = 0\right\}.$$

□

Note that an arbitrary potential $u \in C(\mathbb{R}^2/\Gamma)$ can be represented as

$$u(x,y) = \frac{1}{\mu(\Delta)} \sum_{\kappa \in \Gamma^*} \hat{u}(\kappa) e^{2\pi\iota \langle \kappa, \binom{x}{y} \rangle} = 4\pi^2 \hat{u}_0 + \frac{1}{\mu(\Delta)} \sum_{\kappa \in \Gamma^* \setminus \{0\}} \hat{u}(\kappa) e^{2\pi\iota \langle \kappa, \binom{x}{y} \rangle},$$

where $\hat{u}_0 := \frac{\hat{u}(0)}{4\pi^2 \mu(\Delta)}$. This is also done in [Klauer, 2011, Section 4.4] to obtain stronger asymptotics for $F(u)$ which are not necessary for this work. However, this decomposition is necessary for the next Lemma. Part (b) of it is a modified version from [Klauer, 2011][Lemma 4.5.21].

Lemma 2.19. *(a) For all $\varepsilon > 0$ as in Remark 2.10 and all $u \in C(\mathbb{R}^2/\Gamma)$, there is a $\delta(\varepsilon, u) > 0$ such that for all $k \in B_{2\varepsilon}(k_0)$ with $k_0 \in \mathcal{R} \cap B_\varepsilon(k_{\Gamma^*}^\pm)^c$ and all $\nu \in \Gamma_\delta^*$, there holds*

$$|\hat{u}_0| - \varepsilon < \|\mathcal{A}_0(k + k_\nu^\pm, u)\| < |\hat{u}_0| + \varepsilon$$

in the usual operator norm.

(b) Let $\mathcal{A}_{\pm\nu}(k + k_\nu^\pm, u) = \begin{pmatrix} a(k+k_\nu^\pm,u) & b(k+k_\nu^\pm,u) \\ c(k+k_\nu^\pm,u) & d(k+k_\nu^\pm,u) \end{pmatrix}$. For all $\varepsilon > 0$ as in Remark 2.10 and all $u \in C(\mathbb{R}^2/\Gamma)$, there is a $\delta(\varepsilon, u) > 0$ such that for all $k \in B_{4\varepsilon}(0)$ and all $\nu \in \Gamma_\delta^*$, there holds

$$\begin{aligned} |\hat{u}_0| - \varepsilon <& \|a(k + k_\nu^\pm, u)\|, \|d(k + k_\nu^\pm, u)\| < |\hat{u}_0| + \varepsilon \\ & \|b(k + k_\nu^\pm, u)\|, \|c(k + k_\nu^\pm, u)\| < \varepsilon. \end{aligned}$$

in the usual operator norm.

Proof. Again, the proof of (a) and (b) is analogous up to different open subsets $B_{2\varepsilon}(k_0)$ respectively $B_{4\varepsilon}(0)$. Therefore, let $k \in B_{2\varepsilon}(k_0)$ for (a) and $k \in B_{4\varepsilon}(0)$ in the setting of (b). One has

$$\mathcal{A}(k + k_\nu^\pm, u) = \pi u(\mathbb{1} - R(k + k_\nu^\pm)u)^{-1}|_E,$$

where the second part converges to the identity due to Lemma 2.14. So the norm of the whole operator converges to $\|\pi u\|$. In case (a), this implies

$$\lim_{\|\nu\| \to \infty} \mathcal{A}(k + k_\nu^\pm, u) = \lim_{\|\nu\| \to \infty} \pi_0 u(\mathbb{1} - R(k + k_\nu^\pm)u)^{-1} = \pi_0 u = \hat{u}_0.$$

In the case of (b), one also has that for $\|\nu\| \to \infty$, $\mathcal{A}(k + k_\nu^\pm, u)$ converges to $\pi_{\pm\nu} u$, where

$$\pi_{\pm\nu} u = \begin{pmatrix} \hat{u}_0 & \hat{u}(\nu) e^{2\pi\iota\langle \nu, \binom{x}{y} \rangle} \\ \hat{u}(-\nu) e^{2\pi\iota\langle -\nu, \binom{x}{y} \rangle} & \hat{u}_0 \end{pmatrix} \to \begin{pmatrix} \hat{u}_0 & 0 \\ 0 & \hat{u}_0 \end{pmatrix} \text{ for } \|\nu\| \to \infty$$

because $\frac{1}{|T|} \left(\int_T (\hat{u}(\pm\nu) e^{2\pi\iota\langle \pm\nu, \binom{x}{y} \rangle})^2 dA \right)^{1/2} \to 0$ for $\|\nu\| \to \infty$. □

More precisely, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $(1 - R(k + k_\nu^\pm) u)^{-1}$ and $(1 - u R(k + k_\nu^\pm))^{-1}$ are regular for $\nu \in \Gamma_\delta^*$ and the respective choices of k.

Lemma 2.20. *(a) For all $\varepsilon > 0$ as in Remark 2.10 and all $u \in C(\mathbb{R}^2/\Gamma)$, there is a $\delta(\varepsilon, u) > 0$ such that for all $k \in B_{2\varepsilon}(k_0)$ with $k_0 \in \mathcal{R} \cap B_\varepsilon(k_{\Gamma^*}^\pm)^c$ and all $\nu \in \Gamma_\delta^*$, the matrix $\mathcal{A}_{\pm\nu}(k + k_\nu^\pm, u)$ is continuously differentiable in k and*

$$\lim_{\|\nu\| \to \infty} \left\| \frac{\partial}{\partial k} \mathcal{A}_0(k + k_\nu^\pm, u) \right\| = 0$$

uniformly in k.

(b) For all $\varepsilon > 0$ as in Remark 2.10 and all $u \in C(\mathbb{R}^2/\Gamma)$, there is a $\delta(\varepsilon, u) > 0$ such that for all $k \in B_{4\varepsilon}(0)$ and all $\nu \in \Gamma_\delta^*$, the matrix $\mathcal{A}_{\pm\nu}(k + k_\nu^\pm, u)$ is continuously differentiable in k and

$$\lim_{\|\nu\| \to \infty} \left\| \frac{\partial}{\partial k} \mathcal{A}_{\pm\nu}(k + k_\nu^\pm, u) \right\| = 0$$

uniformly in k.

Proof. By Proposition 2.17, $\mathcal{A}(k + k_\nu^\pm, u)$ exists and is holomorphic in k for $k \in B_{2\varepsilon}(k_0)$ in the case of (a) and $k \in B_{4\varepsilon}(0)$ in the case of (b) and $\nu \in \Gamma_\delta^*$ with $\delta > 0$ sufficiently small. Then

$$\frac{\partial}{\partial k} u = 0 \text{ and } \frac{\partial}{\partial k} R = R \frac{\partial}{\partial k} \triangle_{k + k_\nu^\pm} |_{E^\perp} R.$$

We abbreviate $C := \frac{\partial}{\partial k} \triangle_{k + k_\nu^\pm} |_{E^\perp}$ and denote the derivative with respect to k with a prime. The components of the Fourier transform of C are given by $-8\pi^2 (k_i + k_{\nu,i}^\pm + \kappa_i)$ for $i \in \{1, 2\}$ and hence the Fourier transform of RC is uniformly bounded. Then

$$\begin{aligned} (u(\mathbb{1} - Ru)^{-1})' &= u((\mathbb{1} - Ru)^{-1})' = -u(\mathbb{1} - Ru)^{-1} (\mathbb{1} - Ru)' (\mathbb{1} - Ru)^{-1} \\ &= u(\mathbb{1} - Ru)^{-1} R' u (\mathbb{1} - Ru)^{-1} = u(\mathbb{1} - Ru)^{-1} RCRu(\mathbb{1} - Ru)^{-1} \end{aligned}$$

and since $\pi^\perp(\triangle_{k + k_\nu^\pm}) = \triangle_{k + k_\nu^\pm} \pi^\perp$

$$\frac{\partial}{\partial k} \mathcal{A}(k + k_\nu^\pm, u) = -\pi u (\mathbb{1} - Ru)^{-1} RCRu (\mathbb{1} - Ru)^{-1} |_E.$$

By Lemmata 2.14 and 2.15, the norm of Ru vanishes uniformly for $\|\nu\| \to \infty$ and the norms of the remaining operators are uniformly bounded. So the assertion follows. □

Corollary 2.21. *(a) For all $\varepsilon > 0$ as in Remark 2.10, all $\tilde{\varepsilon} > 0$ and all $u \in C(\mathbb{R}^2/\Gamma)$, there is a $\delta(\varepsilon, \tilde{\varepsilon}, u) > 0$ such that for all $k \in B_{2\varepsilon}(k_0)$ with $k_0 \in \mathcal{R} \cap B_\varepsilon(k^\pm_{\Gamma^*})^c$ and all $\nu \in \Gamma^*_\delta$, there holds*

$$|\det(S^{-1}(k + k^\pm_\nu + \mathcal{A}(k + k^\pm_\nu, u))) - \det(S^{-1}(k + k^\pm_\nu))| < \tilde{\varepsilon}.$$

(b) For all $\varepsilon > 0$ as in Remark 2.10, all $\tilde{\varepsilon} > 0$ and all $u \in C(\mathbb{R}^2/\Gamma)$, there exists a $\delta(\varepsilon, \tilde{\varepsilon}, u) > 0$ such that for all $k \in B_{4\varepsilon}(0)$ and all $\nu \in \Gamma^*_\delta$, there holds

$$|\det(S^{-1}(k + k^\pm_\nu + \mathcal{A}(k + k^\pm_\nu, u))) - \det(S^{-1}(k + k^\pm_\nu))| < \tilde{\varepsilon}.$$

Proof. For (a), one has

$$\lim_{\|\nu\| \to \infty} |\det(S^{-1}(k + k^\pm_\nu) + \mathcal{A}(k + k^\pm_\nu, u)) - \det(S^{-1}(k + k^\pm_\nu))| = \lim_{\|\nu\| \to \infty} \frac{|\hat{u}_0|}{\|4\pi^2(k + k^\pm_\nu)\|} = 0$$

since $\hat{u}_0$ is constant and $\|k + k^\pm_\nu\| \to \infty$ for $\nu \to \infty$. For (b), the same holds since

$$\lim_{\|\nu\| \to \infty} |\det(S^{-1}(k + k^\pm_\nu) + \mathcal{A}(k + k^\pm_\nu, u)) - \det(S^{-1}(k + k^\pm_\nu))| =$$

$$= \lim_{\|\nu\| \to \infty} \frac{|\hat{u}_0^2|}{16\pi^4 \|(k + k^\pm_\nu)^2 (k + k^\mp_\nu)^2\|} = 0.$$

□

Let $\delta > 0$ be sufficiently small such that the statements from Theorem 2.18 can be applied to describe $F(u) \cap \mathbb{C}^2_\delta$. We denote the locally defined holomorphic functions determined in Theorem 2.18, whose zero sets describe $F(u)$ on $k^\pm_\nu + B_{2\varepsilon}(k_0)$ respectively $k^\pm_\nu + B_{4\varepsilon}(0)$ with ε and k_0 as in Remark 2.10 and $\nu \in \Gamma^*_\delta$, as follows: On $k^\pm_\nu + B_{2\varepsilon}(k_0)$, let

$$f^0_{k_0 + k^\pm_\nu} := \det(S^{-1}(k + k^\pm_\nu)) \quad \text{and} \quad f^u_{k_0 + k^\pm_\nu} := \det(S^{-1}(k + k^\pm_\nu) + \mathcal{A}(k + k^\pm_\nu))$$

with S and $\mathcal{A}$ as in Theorem 2.18(a) and on $B_{4\varepsilon}(k^\pm_\nu)$ let

$$f^0_{k^\pm_\nu} := \det(S^{-1}(k + k^\pm_\nu)) \quad \text{and} \quad f^u_{k^\pm_\nu} := \det(S^{-1}(k + k^\pm_\nu) + \mathcal{A}(k + k^\pm_\nu))$$

with S and $\mathcal{A}$ being the 2×2-matrices constructed in Theorem 2.18(b).

Lemma 2.22. *For all $u \in C(\mathbb{R}^2/\Gamma)$, all multi-indices $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_i \in \mathbb{N}_0$, all $\varepsilon > 0$ as defined in Remark 2.10 and all $\tilde{\varepsilon} > 0$, there exists a $\delta(\varepsilon, \tilde{\varepsilon}, \alpha, u) > 0$ such that for all $k + k^\pm_\nu$ with*

$k \in B_{2\varepsilon}(k_0)$ and $k_0 \in \mathcal{R} \cap B_{4\varepsilon}(\Gamma^*)^c$ and all $\nu \in \Gamma^*_\delta$, there holds

$$|\partial^\alpha (f^u_{k_0+k^\pm_\nu} - f^0_{k_0+k^\pm_\nu})(k + k^\pm_\nu)| < \tilde{\varepsilon}.$$

and for $k \in B_{4\varepsilon}(0)$ and all $\nu \in \Gamma^*_\delta$, one has

$$|\partial^\alpha (f^u_{k^\pm_\nu} - f^0_{k^\pm_\nu})(k + k^\pm_\nu)| < \tilde{\varepsilon}. \tag{2.6}$$

Proof. It follows from the proof of Corollary 2.21 that for $\delta > 0$ sufficiently small, the zero sets of the holomorphic functions $f^0_{k_0+k^\pm_\nu}$ and $f^u_{k_0+k^\pm_\nu}$ respectively $f^0_{k^\pm_\nu}$ and $f^u_{k^\pm_\nu}$ are ε-close to each other on $k^\pm_\nu + B_{2\varepsilon}(k_0)$ respectively $k^\pm_\nu + B_{4\varepsilon}(0)$. Corollary 2.21 yields that $|(f^0_{k_0+k^\pm_\nu} - f^u_{k_0+k^\pm_\nu})(k + k^\pm_\nu)| < \tilde{\varepsilon}$ respectively $|(f^0_{k_0+k^\pm_\nu} - f^u_{k_0+k^\pm_\nu})(k + k^\pm_\nu)| < \tilde{\varepsilon}$ for $k + k^\pm_\nu$ on these neighborhoods. Cauchys integral inequality [Gunning and Rossi, 1965, I.A.2] then implies for local coordinates $(z_1, z_2) \in k^\pm_\nu + B_{2\varepsilon}(k_0)$ centered at k_0 and any multi-index α

$$\left|\partial^\alpha (f^u_{k_0+k^\pm_\nu} - f^0_{k_0+k^\pm_\nu})(z_1, z_2)\right| \le c \frac{\alpha!}{(2\varepsilon)^{|\alpha|}} \max_{(z_1,z_2) \in k^\pm_\nu + B_{2\varepsilon}(k_0)} |f^u_{k_0+k^\pm_\nu} - f^0_{k_0+k^\pm_\nu})(z_1, z_2)| \le \tilde{\varepsilon} \frac{c\alpha!}{(2\varepsilon)^{|\alpha|}}.$$

Since 2ε is fixed for all $\nu \in \Gamma^*$ and $f^0_{k_0+k^\pm_\nu}$ as well as $f^u_{k_0+k^\pm_\nu}$ are holomorphic on $k^\pm_\nu + B_{2\varepsilon}(k_0)$, this yields the assertion away from the double points. Analogously, the claim for $k \in B_{4\varepsilon}(0)$ follows. Note that the convergence of the derivatives is not uniform in k, i.e. for each higher derivative one might need a smaller $\delta(\varepsilon, \alpha)$. □

Finally, we can also deduce the following asymptotics for the eigenfunctions of the Schrödinger operator from the considerations in this section.

Lemma 2.23. *(a) For all $\varepsilon > 0$ as in Remark 2.10 and all $u \in C(\mathbb{R}^2/\Gamma)$, there is a $\delta(\varepsilon, u) > 0$ such that for all $k \in B_{2\varepsilon}(k_0)$ with $k_0 \in \mathcal{R} \cap B_\varepsilon(k^\pm_{\Gamma^*})^c$ and $\nu \in \Gamma^*_\delta$, there holds*

$$\|\psi_{k+k^\pm_\nu}(x, y) - \hat{\psi}(0))\|_{W^{1,2}(\mathbb{R}^2/\Gamma)} \le \varepsilon |\hat{\psi}(0)|.$$

(b) For all $\varepsilon > 0$ as in Remark 2.10 and all $u \in C(\mathbb{R}^2/\Gamma)$, there is a $\delta(\varepsilon, u) > 0$ such that for all $k \in B_{4\varepsilon}(0)$ and all $\nu \in \Gamma^*_\delta$, there holds

$$\|\psi_{k+k^\pm_\nu}(x, y) - \hat{\psi}(0) - \hat{\psi}(\nu) e^{2\pi\iota \langle k + k^\pm_\nu, \nu\rangle} \|_{W^{1,2}(\mathbb{R}^2/\Gamma)} \le \varepsilon \left(|\hat{\psi}(0)| + |\hat{\psi}(\nu)| \right).$$

Proof. By formally considering

$$\triangle_{k+k^\pm_\nu} - u = \triangle_{k+k^\pm_\nu} (\mathbb{1} - \triangle^{-1}_{k+k^\pm_\nu} u), \tag{2.7}$$

one sees that formally $(\mathbb{1} - \triangle^{-1}_{k+k^\pm_\nu} u)$ maps the eigenspace of $\triangle_{k+k^\pm_\nu} - u$ to the eigenspace of $\triangle_{k+k^\pm_\nu}$.

Writing this with respect to the partition $E \oplus E^{\perp}$ as in Definition 2.7 yields

$$(\mathbb{1} - \triangle^{-1}_{k+k_\nu^\pm} u) = \begin{pmatrix} \pi(\mathbb{1} - S(k+k_\nu^\pm)u)_E & -\pi S(k+k_\nu^\pm)u|_{E^\perp} \\ -\pi^\perp R(k+k_\nu^\pm)u|_E & \pi^\perp(\mathbb{1} - R(k+k_\nu^\pm))|_{E^\perp}) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Then there is a $\psi^0 \in E$ such that for $\psi_{k+k_\nu^\pm}$ in the eigenspace of $\triangle_{k+k_\nu^\pm} - u$, one has due to (2.7)

$$(\mathbb{1} - \triangle^{-1} u) \begin{pmatrix} \psi_{k+k_\nu^\pm}|_E \\ \psi_{k+k_\nu^\pm}|_{E^\perp} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \psi_{k+k_\nu^\pm}|_E \\ \psi_{k+k_\nu^\pm}|_{E^\perp} \end{pmatrix} = \begin{pmatrix} \psi^0|_E \\ 0 \end{pmatrix}$$

and hence

$$C\psi_{k+k_\nu^\pm}|_E + D\psi_{k+k_\nu^\pm}|_{E^\perp} = 0 \;\Leftrightarrow\; \psi_{k+k_\nu^\pm}|_{E^\perp} = -D^{-1}C\psi_{k+k_\nu^\pm}|_E.$$

For a given $\psi_{k+k_\nu^\pm} = \hat{\psi}(0) + \psi_{k+k_\nu^\pm}|_{E^\perp}$, it is

$$D^{-1}C : E \to E^\perp, \quad \hat{\psi}(0) \mapsto \psi_{k+k_\nu^\pm}|_{E^\perp}.$$

We want to show that for $\psi_{k+k_\nu^\pm} \in L^2(\mathbb{R}^2/\Gamma)$, it is $D^{-1}C\psi_{k+k_\nu^\pm} \in W^{1,2}(\mathbb{R}^2/\Gamma)$. We have seen in Lemma 2.14 that $Ru|_E : L^2(\mathbb{R}^2/\Gamma) \to W^{1,2}(\mathbb{R}^2/\Gamma)$. Furthermore, Lemma 2.14 yields that for $\delta > 0$ sufficiently small $\| \triangle_{k+k_\nu^\pm} u\|_{W^{1,2}(\mathbb{R}^2/\Gamma)} < 1$. Thus, the Neumann series $\sum_{n=0}^\infty (\triangle_{k+k_\nu^\pm} u)^n$ converges and $(\mathbb{1} - \triangle_{k+k_\nu^\pm} u)^{-1} : W^{1,2}(\mathbb{R}^2/\Gamma) \to W^{1,2}(\mathbb{R}^2/\Gamma)$ exists. This shows that $D^{-1}C : L^2(\mathbb{R}^2/\Gamma) \to W^{1,2}(\mathbb{R}^2/\Gamma)$. This operator obeys

$$\|D^{-1}C\| \le \|D^{-1}\|\|C\|.$$

Proposition 2.17 applies to $R(k+k_\nu^\pm)u$ and hence also to the corresponding restrictions to E and $E^\perp$. Therefore, with $k \in B_{2\varepsilon}(k_0)$ for (a) respectively $k \in B_{4\varepsilon}(0)$ for (b), one gets

$$\lim_{\|\nu\|\to\infty} \|D^{-1}C\| = \lim_{\|\nu\|\to\infty} \left\| \left(\mathbb{1}|_E^\perp - R(k+k_\nu^\pm)u|_{E^\perp}\right)^{-1} \pi^\perp R(k+k_\nu^\pm)u|_E \right\| = 0.$$

Moreover, in the case of (a)

$$\psi_{k+k_\nu^\pm} - \hat{\psi}(0) = \psi_{k+k_\nu^\pm}|_{E^\perp}$$

and in the case of (b)

$$\psi_{k+k_\nu^\pm} - \hat{\psi}(0) - \hat{\psi}(\nu)e^{2\pi\iota\langle k+k_\nu^\pm,\nu\rangle} = \psi_{k+k_\nu^\pm}|_{E^\perp}.$$

Due to $\|\psi_{k+k_\nu^\pm}|_{E^\perp}\| = \|D^{-1}C\psi|_E\| \le \|D^{-1}C\|\|\psi_{k+k_\nu^\pm}|_E\|$, the assertion follows from the above considerations. ☐

2.3. Asymptotic freeness of $F(u)/\Gamma^*$

In the rest of this work, we want to associate a compact curve to the Fermi curve. This is only possible if we consider $F(u)/\Gamma^*$. Therefore, it is necessary to formulate the asymptotics for this case. Further, we also want to determine the number of connected components of the regular part $\mathfrak{R}(F(u)/\Gamma^*)$ of $F(u)/\Gamma^*$ and show that this part is complex one-dimensional. Thereby, we use the following definitions.

Definition 2.24 ([Gunning, 1990, Definition F.])**.** A point in a holomorphic variety V at which V is a complex manifold is called a *regular point* of V. A point that is not regular is called a *singular point* of V. The set of all regular points comprises the *regular part* $\mathfrak{R}(V) \subseteq V$ of V while the set of all singular points comprises the *singular part* $S(V) = V \setminus \mathfrak{R}(V) \subseteq V$.

Definition 2.25 ([Gunning, 1990, Definition G.1])**.** The *dimension of a holomorphic variety* V is the dimension of the complex manifold $\mathfrak{R}(V)$.

The question of the number of connected components cannot be answered for $F(u)$ since already the part where $F(0)$ is a complex one-dimensional manifold consists of infinitely many connected components, see Section 1.4. In Lemma 1.8 we have seen that $F(u)$ is invariant under translations by elements of the dual lattice Γ^*. So it would be convenient to consider $F(u)/\Gamma^*$ instead of $F(u)$ to formulate the asymptotics. We know from Corollary 2.6 and the chosen covering of $F(u)$ in Remark 2.10 that the asymptotic behavior of $F(u)$ bounded away from the excluded domains can be considered in

$$\mathcal{U}^{\pm}_{\varepsilon,\delta} := \left\{ k \in \mathbb{C}^2 \;\middle|\; |k_1 \pm \imath k_2| < \varepsilon,\ \|\operatorname{Im} k\| > \frac{1}{\sqrt{2}\delta} \text{ and } \|k - k^{\pm}_{\nu}\| > 2\varepsilon \ \forall\, \nu \in \Gamma^* \setminus \{0\} \right\}.$$

The condition $|k_1 \pm \imath k_2| < \varepsilon$ in the definition of $\mathcal{U}^{\pm}_{\varepsilon,\delta}$ means that all elements $\mathcal{U}^{\pm}_{\varepsilon,\delta}$ are close to the free Fermi curve, the condition $\|\operatorname{Im}(k)\| > \frac{1}{\sqrt{2}\delta}$ reflects that one considers the asymptotics, i.e. the behavior of $F(u) \cap \mathbb{C}^2_{\delta}$, and the condition $\|k - k^{\pm}_{\nu}\| > 2\varepsilon$ ensures that all elements of $\mathcal{U}^{\pm}_{\varepsilon,\delta}$ are staying away from the double points $k^{\pm}_{\nu}$. With $\mathcal{R}$ as in (1.15), we know from Section 1.4 that for every $k \in \mathcal{R} \setminus \{k^{\pm}_{\nu} \mid \nu \in \Gamma^*\}$, the Γ^*-orbit of k contains exactly one point in $\mathcal{R}(\kappa) \setminus \{k^{\pm}_{\nu} + \kappa \mid \nu \in \Gamma^*\}$ for every $\kappa \in \Gamma^*$ and in particular intersects $\mathcal{R}$ only in k. So for ε as in Remark 2.10, the open sets $\mathcal{U}^{\pm}_{\varepsilon,\delta}$ also intersect any Γ^*-orbit of k at most once. A sketch of this situation, projected to $\mathbb{C}$, can also be seen in Figure 2.1. Hence, we may consider $\mathcal{U}^{\pm}_{\varepsilon,\delta}$ as subsets of $\mathbb{C}^2/\Gamma^*$.
Because $F(u)$ is translation invariant under Γ^*, see Lemma 1.8, we can consider the asymptotics for $F(u)/\Gamma^*$. When we consider the quotient, we often write $k = (k_1, k_2)$ instead of $[k]$ respectively (k, λ) instead of $([k], \lambda)$. We mean by this that we consider a representant of the corresponding equivalence class and if we compare two such elements, we also take the representants corresponding to the same translation $\kappa \in \Gamma^*$. Usually, in the asymptotics, we consider the equivalence class which is contained in an $\mathcal{U}^{\pm}_{\varepsilon,\delta}$ around $\mathcal{R}$.

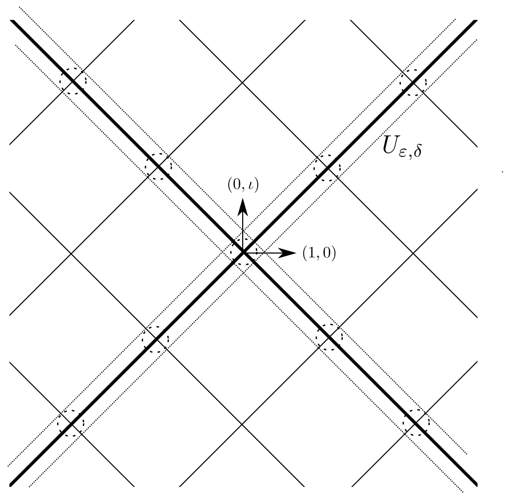


Figure 2.1.: Intersection of $F(0)$ and $\mathcal{U}_{\varepsilon,\delta}$ with respect to the lattice $4\mathbb{Z} \otimes 4\mathbb{Z}$ with the real plane spanned by $(1,0)$ and $(0,\iota)$ and for δ sufficiently small. All double points $k_\nu^\pm$ are excluded from $\mathcal{U}_{\varepsilon,\delta}$ which is indicated by the small circles around them.

A problem that occurs is that even if $F(u)$ is ε-close to $F(0)$ far outside, one does not know what the Fermi curve $F(u)$ looks like in an ε-neighborhood of $k_\nu^\pm$. Interpreting the potential u as a perturbation of the zero potential, we will show in Theorem 2.34 that a double point $k_\nu^\pm \in F(u)$ may decay into two separate branch points on $F(u)$ with respect to the covering $(k_1, k_2) \mapsto k_1$. Furthermore, the two-dimensional eigenspace of the two points constituting the double point may decay into two one-dimensional eigenspaces. Therefore, we define the excluded domains more precisely as before.

Definition 2.26. For given $\varepsilon > 0$ and every $\nu \in \Gamma_\delta^*$ with $\delta(\varepsilon) > 0$ sufficiently small, we call the compact subset

$$\mathfrak{e}_\nu := \left\{ k \in \mathbb{C}^2 \;\middle|\; |k_1 \pm \iota k_2| < \varepsilon,\ \|\operatorname{Im} k\| > \frac{1}{\sqrt{2}\delta} \text{ and } \|k - k_\nu^\pm\| \leq 2\varepsilon \right\}$$

of $\mathbb{C}^2$ an *excluded domain* for the double point $k_\nu^\pm$ of $F(0)/\Gamma^*$. We call the compact subset $\mathfrak{e}_\nu \cap F(u)$ a *handle* of $F(u)$ around the double point $k_\nu^\pm$ if the regular part of this subset is connected.

To describe the parts of $F(u)/\Gamma^*$ at which the eigenspace has more than one dimension, the next definition is necessary.

Definition 2.27. The eigenvalues of the Schrödinger operator at which the corresponding generalized eigenspace has more than one dimension are called *degenerated eigenvalues*.

We want to determine the number of connected components of the regular part of $F(u)/\Gamma^*$ and show with help of this that generically, the eigenspace the Schrödinger operator is one-dimensional on $F(u)$ respectively $F(u)/\Gamma^*$. As already mentioned, the first question can only be answered for the quotient.

Theorem 2.28. *Let $u \in C(\mathbb{R}^2/\Gamma)$.*

(a) $F(u)/\Gamma^*$ is a one-dimensional variety in $\mathbb{C}^2/\Gamma^*$ and for $\varepsilon > 0$ as in Remark 2.10 and $\delta > 0$ sufficiently small, $F(u)/\Gamma^* \cap \mathcal{U}^{\pm}_{\varepsilon,\delta}$ is a one-dimensional connected manifold.

(b) The regular part of the Fermi curve $F(u)/\Gamma^*$ has at most two connected components. Each component contains one of the two sets $F(u)/\Gamma^* \cap \mathcal{U}^{\pm}_{\varepsilon,\delta}$.

(c) The set of singularities and branch points of the covering $\pi_1 : F(u) \to \mathbb{C}$, $(k_1, k_2) \mapsto k_1$ as well as of the covering $\pi_2 : F(u) \to \mathbb{C}$, $(k_1, k_2) \mapsto k_2$ of $F(u)/\Gamma^*$ is discrete.

(d) The set of $k \in F(u)/\Gamma^*$ such that $\lambda = 0$ is a degenerated eigenvalue of $-\triangle_k + u$ contains only discrete points.

Proof. (a) We know already from Lemma 1.18 that $F(0)$ is a one-dimensional subvariety of $\mathbb{C}^2$ and from Corollary 1.15 that $F(u)$ is a holomorphic variety in $\mathbb{C}^2$ which is due to Lemma 1.8 invariant under translations by elements of Γ^*. So $F(u)/\Gamma^*$ is a subvariety of $\mathbb{C}^2/\Gamma^*$. Hence, it is necessary to show that $F(u)/\Gamma^*$ is one-dimensional. Due to Corollary 2.6, for every $\varepsilon > 0$ we can choose $\delta > 0$ sufficiently small such that $F(u)/\Gamma^* \cap \mathbb{C}^2_\delta$ is contained in an ε-tube around $F(0)/\Gamma^*$. Theorem 2.18 (a) yields that for $\delta > 0$ sufficiently small and $\nu \in \Gamma^*_\delta$, the values of $k \in B_{2\varepsilon}(k_0) \cap F(u)/\Gamma^*$ are described by the solution set of

$$f^u_{k_0+k^{\pm}_\nu}(k + k^{\pm}_\nu) = \det(S_0^{-1} + \mathcal{A}_0)(k + k^{\pm}_\nu) = 0.$$

Lemma 2.20 implies that $\partial\mathcal{A}_0/\partial k_1(k + k^{\pm}_\nu)$ vanishes for $\|\nu\| \to \infty$ and direct calculation yields that $\partial S_0^{-1}/\partial k_1(k + k^{\pm}_\nu) = -8\pi^2(k_i + k^{\pm}_{\nu,1})$. Using the chain rule and taking into account that $|k^{\pm}_{\nu,2}| = \frac{1}{2}\|\nu\| \to \infty$ for $\|\nu\| \to \infty$ yields that the partial derivative $\partial f^0_{k_0+k^{\pm}_\nu}/\partial k_2$ does not vanish on any of the open balls $k^{\pm}_\nu + U_{2\varepsilon}(k_0)$ with ε and k_0 defined as in Remark 2.10 and $\delta > 0$ sufficiently small. By the Implicit Function Theorem for several complex variables [de Jong and Pfister, 2012, Theorem 3.3.1], the zero set $F(u)/\Gamma^* \cap (k^{\pm}_\nu + B_{2\varepsilon}(k_0))$ can be represented as a holomorphic function $k_2(k_1)$. Thus, both sets $F(u)/\Gamma^* \cap \mathcal{U}^{\pm}_{\varepsilon,\delta}$ are one-dimensional holomorphic subvarieties of $\mathbb{C}^2_\delta$. This induces that $F(u)/\Gamma^*$ is a one-dimensional holomorphic variety in all of $\mathbb{C}^2/\Gamma^*$ since the alternative would be that there are parts of $F(u)/\Gamma^*$ where it is a two-dimensional submanifold. However, this would imply that there is an open set $U \subset \mathbb{C}^2/\Gamma^*$ on which the zero set of the holomorphic function describing $F(u)/\Gamma^*$ on U is identically zero. Since $F(u)/\Gamma^*$ is a variety, see Corollary 1.15, on all nonempty intersections of open sets V, W of $\mathbb{C}^2/\Gamma^*$ where $V \cap W \cap F(u)/\Gamma^* \neq 0$, the local zero sets describing $F(u)/\Gamma^*$ coincide. Then on all nonempty intersections of open sets with U, the Fermi curve would also be identical to the open set intersected with U. Successively continuing like that would yield that $F(u)/\Gamma^*$ is all of $\mathbb{C}^2/\Gamma^*$ which contradicts the fact that $F(u)/\Gamma^* \cap \mathcal{U}^{\pm}_{\varepsilon,\delta}$ is a one-dimensional holomorphic variety. Determining also the derivative with respect to k_1 on $k^{\pm}_\nu + B_{2\varepsilon}(k_0)$ under the same conditions as above shows that also this derivatives vanishes nowhere on $F(u)/\varGamma^* \cap \mathcal{U}^{\pm}_{\varepsilon,\delta}$. Thus, $F(u)/\varGamma^*$ is a one-dimensional manifold on those sets.

(b) Since $F(u)/\varGamma^*$ is a variety in $\mathbb{C}^2/\varGamma^*$, we can locally consider the Weierstraß coverings

$$\pi_i : F(u) \to \mathbb{C}, \quad (k_1, k_2) \mapsto k_i$$

for $i = 1, 2$. Both of these mappings are, due to the Open Mapping Theorem [Conway, 1978, Theorem 7.5], either open or constant on a connected component of $\mathfrak{R}(F(u))$. If both of these maps are constant on one connected component of $\mathfrak{R}(F(u))$, then this connected component consists only of one point and is a 0-dimensional variety. This contradicts (a), where we have seen that $F(u)$ is a one-dimensional variety. So without loss of generality, let π_1 be open on $\mathfrak{R}(F(u))$. Then the branch points of this covering are discrete on $\mathfrak{R}(F(u))$. Let $(k_{0,1}, k_{0,2}) \in F(u)$ be an arbitrary point of the Fermi curve. We connect $k_{0,1} \in \mathbb{C}$ with an arbitrary point $k'_{0,1} \in \mathbb{C}$ by a continuous path $\gamma(t) : [0, 1] \to \mathbb{C}$ with $\gamma(0) = k_{0,1}$ and $\gamma(1) = k'_{0,1}$. Due to the openness of π_1 and the lifting property of paths, compare [Miranda, 1995, Section III.4], we can lift this path to a continuous path $k(t) = (k_1(t), k_2(k_1(t))) \in F(u)$ such that the lifted path in $F(u)$ is also connected. We have seen in (b) that the singularities of $F(u)$ as well as the branch points of π_1 are discrete, so we can choose γ in such a way that the lifted path passes neither singularities of $F(u)$ nor branch points of π_1. Moreover, the union $\bigcup_{\nu\in\varGamma^*\setminus\varGamma^*_\delta} k^{\pm}_\nu + \varDelta_{\mathbb{C}}$ is compact. So Corollary 2.6 yields that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for $\tilde{k} \in \varDelta_{\mathbb{C}}$ as in Definition 2.1, the value of $\|\operatorname{Im}(\tilde{k} + k^{\pm}_\nu)\|$ with $\nu \in \varGamma^* \setminus \varGamma^*_\delta$ attains it maximum M. Let γ be chosen in such a way that

$$\|\operatorname{Im}(k_1(1))\| > \max\left\{\frac{1}{\sqrt{2}\delta}, M\right\}.$$

Then $\|\operatorname{Im}(k(1))\| > \frac{1}{\sqrt{2}\delta}$, so $k(1) \in (F(u) \cap \mathbb{C}^2_\delta)/\varGamma^*$. Furthermore, it follows from Corollary 2.6 that there exists some $\kappa \in \varGamma^*$ such that $|(k_1(1) + \kappa_1) \pm \iota(k_2(1) + \kappa_2)| < \varepsilon$. Furthermore, for $\varepsilon > 0$ sufficiently small, we can choose γ in such a way that for all lifted values $k(t)$ of $k_1(t)$ contained in γ holds that $\|k - k^{\pm}_\nu\| \geq 2\varepsilon$, because the points $k^{\pm}_\nu$ lie discrete in $\mathbb{C}^2/\varGamma^*$. Then $k(1) \in \kappa + \mathcal{U}^{\pm}_{\varepsilon,\delta}$ for some $\kappa \in \varGamma^*$ and the corresponding equivalence class $[(k(1)] \in F(u)/\varGamma^*$ is either contained in $F(u)/\varGamma^* \cap \mathcal{U}^{+}_{\varepsilon,\delta}$ or in $F(u)/\varGamma^* \cap \mathcal{U}^{-}_{\varepsilon,\delta}$. Both options are possible because Lemma 2.22 implies that none of these two sets is empty. Since $F(u)/\varGamma^*$ is a one-dimensional holomorphic variety, the regular parts of $F(u)/\varGamma^*$ are connected, see [Gunning, 1990, Theorem E.19]. The path γ we constructed is continuous, starts at $(k_1, k_2) \in F(u)$ and connects this point with the part of the Fermi curve which is contained in $\kappa + U^{\pm}_{\varepsilon,\delta}$ for some $\kappa \in \varGamma^*$. Therefore, the number of connected components of the regular part is at most two.

(c) Due to [Gunning, 1990, Theorem E.17], the part where $F(u)/\varGamma^*$ is not regular consists only of discrete points. Let $k = (k_1, k_2) \in F(u) \cap \mathbb{C}^2_\delta$. Due to the asymptotic freeness of $F(u)/\Gamma^*$, we can assume without loss of generality that $|k_1 \pm \imath k_2| < \varepsilon$ on $\Re(F(u)) \cap U^\pm_{\varepsilon,\delta}$. So neither π_1 nor π_2 is a constant map on $\Re(F(u)/\Gamma^*) \cap U^\pm_{\varepsilon,\delta}$. Therefore, by (b) and the Open Mapping Theorem [Conway, 1978, Theorem 7.5], they are open on both of the maximal two connected components of $\Re(F(u)/\Gamma^*)$. Hence, the number of branch points of these coverings is discrete, because $\partial\pi_i/\partial k_i$ with $i = 1, 2$ can only vanish at discrete points on $\Re(F(u)/\Gamma^*)$.

(d) For fixed $u \in C(\mathbb{R}^2/\Gamma)$, we know from Corollary 1.15 that the Bloch variety $B(u)$ is a variety in $\mathbb{C}^3$. So due to Lemma 1.8, $B(u)/\Gamma^*$ is a variety in $\mathbb{C}^3/\Gamma^*$. It follows from Theorem 1.14 that $B(u)/\Gamma^*$ can locally, on a small open set $U \subset \mathbb{C}^2$ which intersects $B(u)$, be described by the zero set of the characteristic polynomial of an $n \times n$-matrix $A(k)$ which depends holomorphically on k, i.e. by the set of points such that $\det(A(k) - \lambda\mathbb{1}) = 0$. Hereby, n is the range of the eigenprojection of the Schrödinger operator on U as defined in the proof of Theorem 1.14. The values k such that λ is degenerated on $B(u)/\Gamma^*$ can be detected by the solutions of the zero set of the discriminant D of the characteristic polynomial which is defined as

$$D(\det(A(k) - \lambda\mathbb{1})) := \prod_{i<j} (\lambda_i(k) - \lambda_j(k))^2. \tag{2.8}$$

The discriminant is a polynomial in the coefficients of the polynomial $\det(A(k) - \lambda\mathbb{1})$ and can be written as the sum of the elementar-symmetric functions over the sheets of the covering over λ in terms of k, compare [Forster, 1981, Section 8.1]. So $D(\det A(k) - \lambda\mathbb{1})$ is holomorphic in k and λ. Thus, the set of (k, λ) such that $D(\det(A(k) - \lambda\mathbb{1})) = 0$ defines a subvariety of $\mathbb{C}^3$. Restricting these holomorphic function on $B(u)/\Gamma^*$ to $\lambda = 0$ yields a subvariety on $F(u)/\Gamma^*$. If we can find an open subset $U \subset F(u)/\Gamma^*$ on one of the maximal two connected components $F(u)/\Gamma^*$ such that $D(\det(A(k) - \lambda\mathbb{1}))|_U \equiv 0$, this would yield that $D(\det(A(k) - \lambda\mathbb{1})) \equiv 0$ on all of the connected component. However, due to (a) and the proof of Theorem 1.14, $F(u)/\Gamma^* \cap \mathcal{U}^\pm_{\varepsilon,\delta}$ can be represented as the zero set of the characteristically polynomial of an 1×1-matrix. So the eigenspace is one-dimensional on $F(u) \cap \mathcal{U}^\pm_{\varepsilon,\delta}$ and thus the discriminant 2.8 cannot vanish identically on each of the maximal two connected components of $F(u)/\Gamma^*$.

□

Corollary 2.29. *Let $u \in C(\mathbb{R}^2/\Gamma)$.*

(a) $B(u)/\Gamma^*$ is a two-dimensional variety in $\mathbb{C}^3/\Gamma^*$.

(b) The regular part of the Bloch variety $B(u)/\Gamma^*$ has at most two connected components.

(c) The set of $([k], \lambda) \in B(u)/\Gamma^*$ such that λ is a degenerated eigenvalues of $- \triangle_k + u$ is a subvariety of codimension 1.

Proof. (a) We know from Corollary 1.15 that $B(u)$ is a variety in $\mathbb{C}^3$ and from Lemma 1.8 that it is invariant under translations by $\kappa \in \Gamma^*$. So $B(u)/\Gamma^*$ is a variety in $\mathbb{C}^3/\Gamma^*$. In the proof of Theorem 1.14 is shown that $B(u)/\Gamma^*$ can locally be represented as the zero set of one holomorphic function. Since considering the zero set of only one function can the reduce the dimension of a variety in $\mathbb{C}^3$ at most by one, compare [Gunning, 1990, Theorem G.5], the dimension of $B(u)/\Gamma^*$ cannot be less than two. $B(u)/\Gamma^*$ can also not be a three-dimensional variety since this would imply that $F(u)/\Gamma^*$ is a two-dimensional variety.

(b) Since $B(u)/\Gamma^*$ is a variety in $\mathbb{C}^3/\Gamma^*$, the map $B(u)/\Gamma^* \to \lambda$ is – due to the Open Mapping Theorem [Conway, 1978, Theorem 7.5] – either constant or open. If this map is constant, then $B(u)/\Gamma^* = F(u)/\Gamma^*$. This contradicts the fact that $F(u)/\Gamma^*$ is a two-dimensional variety as shown in Theorem 2.28(a) and $B(u)/\Gamma^*$ a three-dimensional variety as shown in (a). Thus, every path in $\mathbb{C}$ containing λ can be lifted to a path on $B(u)/\Gamma^*$. So for $\lambda \neq \lambda'$, two Fermi curves $F_\lambda(u)/\Gamma^*$ and $F_{\lambda'}(u)/\Gamma^*$ with $\lambda \neq \lambda'$ are connected. In Theorem 2.28(b) it is shown that $\Re(F(u)/\Gamma^*)$ has at most two connected components. The same also holds for the regular part of the translated Fermi curves $F_\lambda(u)/\Gamma^*$ and thus $\Re(B(u)/\Gamma^*)$ has at most two connected components.

(c) We have seen in Theorem 2.28(d) that the subvariety of the degenerated eigenvalues has codimension one on $F(u)/\Gamma^*$. Since $F(u)/\Gamma^*$ is a subset of $B(u)/\Gamma^*$, this implies that the codimension of the subvariety of degenerated eigenvalues on $B(u)/\Gamma^*$ is bigger or equal to one. However, if it was two, then also the codimension on $F(u)/\Gamma^*$ of this subvariety would be two which contradicts the statement in Theorem 2.28(d).

□

The results of the previous section are now very helpful to gain insight into the fact that for $\delta > 0$ sufficiently small, $F(u) \cap \mathbb{C}^2_\delta$ can locally be represented as a Weierstrass covering of maximal degree two. This means that there is a covering $F(u) \cap \mathbb{C}^2_\delta \to \mathbb{C}$, $(k_1, k_2) \mapsto k_1$ such that locally around $(k_{0,1}, k_{0,2}) \in F(u) \cap \mathbb{C}^2_\delta$, there is a Weierstraß polynomial in k_2 of degree one or two with holomorphic coefficients $a_i(k_1)$ such that the highest coefficient is equal to one and all lower coefficients vanish at $k_{0,1}$ and the zero set of this polynomial coincides locally with $F(u)$, see [de Jong and Pfister, 2012, Weierstraß Preparation Theorem 3.2.4]. It is clear from the above considerations that the degree of the Weierstraß polynomial on $F(u) \cap \mathcal{U}^{\pm}_{\varepsilon,\delta}$ equals one. For $\delta > 0$ sufficiently small, it remains to analyze the Weierstraß polynomial of $F(u)$ inside of the excluded domains $\mathfrak{e}_\nu$ with $\nu \in \Gamma^*_\delta$. We already know from Section 1.4 that the degree of the polynomial representing $F(0)$ at $k^{\pm}_\nu$ with $\nu \in \Gamma^*$ equals two.

Remark 2.30. To ensure that we can find Weierstraß coverings of both parts of $F(u)/\Gamma^*$, the part contained in the excluded domains and the part bounded away from these, we consider coverings over k_1 of $F(u) \cap \mathbb{C}^2_\delta$ which can be realized by choosing neighborhoods $B_\varepsilon(k_{0,1}) := \{k_1 \in \mathbb{C} \mid$

$|k_1 - k_{0,1}| < \varepsilon\}$ with $k_0 = (k_{0,1}, k_{0,2})$ and ε chosen as in Remark 2.10, $\delta > 0$ sufficiently small and $\nu \in \Gamma^*_\delta$. We know already from Theorem 2.28(a) that in this case $F(u) \cap \mathcal{U}^\pm_{\varepsilon,\delta}$ can be represented by a one-sheeted Weierstraß covering over k_1 on $k^\pm_\nu + B_\varepsilon(k_{0,1})$. Because $F(0)/\Gamma^*$ is described by planes in this area, for fixed $k_1 \in k^\pm_{\nu,1} + B_\varepsilon(k_{0,1})$, the minimal distance between $(k_1, k_2) \in F(u)/\Gamma^*$ and $(k_1, \tilde{k}_2) \in F(0)/\Gamma^*$ is given by $|k_2 - \tilde{k}_2| < \varepsilon$. Then $(k_1, k_2) \in k^\pm_\nu + B_\varepsilon(k_{0,1}) \times B_\varepsilon(k_{0,2})$. Let $\nu \in \Gamma^*_\delta$ with $\delta > 0$ small. To describe $F(u)$ also in the neighborhoods of $k^\pm_\nu$ by Weierstraß coverings, note that we have chosen $\varepsilon > 0$ such that 8ε is smaller than half of the distance between the generators of $\Gamma^*_\mathbb{C}$ as in Definition 2.1. Therefore, we can transfer the asymptotics in the excluded domains with 4ε-balls around $k^\pm_\nu$ also on larger excluded domains with radius 8ε. We consider $k^\pm_\nu + k_1$ with $k_1 \in B_{4\varepsilon}(0)$. Since $\varepsilon > 0$ is chosen such that $F(u)$ is contained in an ε-neighborhood of $F(0)$ and the two sheets $\mathcal{R}^\pm$ and $\mathcal{R}^\mp(\nu)$ intersect in $k^\pm_\nu$, this ensures that the Weierstraß covering in the neighborhood of the $k^\pm_\nu$ yields values $k^\pm_\nu + (k_1, k_2) \in F(u)$ such that $(k_1, k_2) \in B_{4\varepsilon}(k^\pm_{\nu,1}) \times B_{4\varepsilon}(k^\pm_{\nu,2})$. By this choice, all open neighborhoods in which we describe $(F(u) \cap \mathbb{C}^2_\delta)/\Gamma^*$ by Weierstraß coverings overlap. We denote the polynomials which we obtain to describe $F(u)$ in local coordinates (z_1, z_2) centered at $k^\pm_\nu$ in analogy to (2.6) by $p^0_{k^\pm_\nu}$ and $p^u_{k^\pm_\nu}$.

In order to tell more about the correspondence of the number of branch points of this covering in open neighborhoods of $\mathbb{C}^2$, the following definitions are necessary:

Definition 2.31. The *discriminant* of

$$p(z_1, z_2) = z_2^2 + a(z_1)z_2 + b(z_1) = 0$$

with is defined as

$$D_p(z_1) := a(z_1)^2 - 4b(z_1).$$

Since for a Weierstraß polynomial p the coefficients a and b are holomorphic in z_1, also D_p is.

Definition 2.32. We call a point $(z_{0,1}, z_{0,2})$ of the zero set defined by

$$p(z_1, z_2) := z_2^2 + a(z_1)z_2 + b(z_1) = 0$$

on an open polydisc $U := U_1 \times U_2$ of $\mathbb{C}^2$ with $a, b \in \mathcal{O}(U_1)$ an *ordinary branch point* of the Weierstraß covering over z_1 if

$$\frac{\partial p}{\partial z_1}(z_{0,1}, z_{0,2}) \neq 0, \quad \text{and} \quad \frac{\partial p}{\partial z_2}(z_{0,1}, z_{0,2}) = 0$$

and an *ordinary double point* of this covering if

$$\frac{\partial p}{\partial z_1}(z_{0,1}, z_{0,2}) = 0, \quad \frac{\partial p}{\partial z_2}(z_{0,1}, z_{0,2}) = 0 \text{ and } D''(z_{0,1}) \neq 0.$$

There is also another way to characterize the branch and double points of a locally two-sheeted Weierstraß covering which uses the discriminant.

Lemma 2.33. *(a) The Weierstraß covering over z_2 defined by $p = 0$ as in Definition 2.32 has an ordinary branch point at $(z_{0,1}, z_{0,2}) \in \{z \in \mathbb{C}^2 \mid p(z) = 0\}$ if and only if*

$$D_p(z_{0,1}) = 0 \text{ and } D'_p(z_{0,1}) \neq 0.$$

(b) The Weierstraß covering over z_1 defined by $p = 0$ as in Definition 2.32 has an ordinary double point at $(z_{0,1}, z_{0,2}) \in \{z \in \mathbb{C}^2 \mid p(z) = 0\}$ if and only if

$$D_p(z_{0,1}) = 0, \quad D'_p(z_{0,1}) = 0 \quad \text{and } D''_p(z_{0,1}) \neq 0.$$

Proof. Let $(z_{0,1}, z_{0,2}) \in \{z \in \mathbb{C}^2 \mid p(z) = 0\}$. First of all, $D_p(z_{0,1}) = 0$ if and only if p has a zero of higher order in z_2 at $(z_{0,1}, z_{0,2})$, i.e. if $\frac{\partial p}{\partial z_2}(z_{0,1}, z_{0,2}) = 0$: $D_p(z_{0,1}) = 0$ is equivalent to $a(z_{0,1})^2 = 4b(z_{0,1})$. Inserting this into p yields that $z_{0,2} = -\frac{a(z_{0,1})}{2}$ is a zero of second order of p. Conversely, p has only one zero of second order at $z_{0,2} = -\frac{a(z_{0,1})}{2}$. Furthermore,

$$\begin{aligned}
\frac{\partial p}{\partial z_2}(z_{0,1}, z_{0,2}) &= 2z_{0,2} + a(z_{0,1}),\\
\frac{\partial p}{\partial z_1}(z_{0,1}, z_{0,2}) &= a'(z_{0,1})z_{0,2} + b'(z_{0,1}),\\
\frac{\partial^2 p}{\partial z_1^2}(z_{0,1}, z_{0,2}) &= a''(z_{0,1})z_{0,2} + b''(z_{0,1}),\\
D'_p(z_{0,1}) &= -2a'(z_{0,1})a(z_{0,1}) + 4b'(z_{0,1}),\\
D''_p(z_{0,1}) &= -2a''(z_{0,1})a(z_{0,1}) - 2(a'(z_{0,1}))^2 + 4b''(z_{0,1}).
\end{aligned}$$

We need to show that $\frac{\partial p}{\partial z_1} \neq 0 \Leftrightarrow D'_p(z_{0,1}) \neq 0$ for $D_p(z_{0,1}) = p(z_{0,1}, z_{0,2}) = \frac{\partial p}{\partial z_2}(z_{0,1}, z_{0,2}) = 0$. Since $\frac{\partial p}{\partial z_2}(z_{0,1}, z_{0,2}) = 0$ one has $z_{0,2} = -\frac{a(z_{0,1})}{2}$. Hence,

$$\begin{aligned}
\frac{\partial p}{\partial z_1}\left(z_{0,1}, -\frac{a(z_{0,1})}{2}\right) = -a'(z_{0,1})\frac{a(z_{0,1})}{2} + b'(z_{0,1}) \neq 0 \\
\Leftrightarrow \quad D'_p(z_{0,1}) = -2a'(z_{0,1})a(z_{0,1}) + 4b'(z_{0,1}) \neq 0.
\end{aligned}$$

□

With this at hand, we can describe $F(u) \cap \mathbb{C}^2_\delta$ for $\delta > 0$ in the neighborhood of the double points more precisely.

Theorem 2.34. *For $\varepsilon > 0$ as in Remark 2.10 and $u \in C(\mathbb{R}^2/\Gamma)$, there exists a $\delta(\varepsilon, u) > 0$ such that for all $k_1 \in B_{4\varepsilon}(0)$ and all $\nu \in \Gamma^*_\delta$ the part of $F(u) \cap \left(\left(k^\pm_{\nu,1} + B_{4\varepsilon(0)}(0)\right) \times \left(k^\pm_{\nu,2} + B_{4\varepsilon}(0)\right)\right)$ is a complex curve consisting of two sheets with respect to the covering $(k_1, k_2) \mapsto k_1$ and for every $\nu \in \Gamma^*_\delta$, each open set $(k^\pm_{\nu,1} + B_{4\varepsilon}(0)) \times (k^\pm_{\nu,2} + B_{4\varepsilon}(0))$ contains either two branch points of this covering or a double point $k^\pm_\nu(u)$.*

Proof. Note that we are considering neighborhoods of $k^\pm_\nu \in F(0)$ with $\nu \in \Gamma^*_\delta$ and $\delta > 0$ sufficiently small. In case that $F(u) \cap \mathfrak{e}_\nu$ decays into two sheets, we abuse of the common notation in the sense that we define the product of the two Weierstraß polynomials – which are both linear in k_2 in this neighborhood – as the Weierstraß polynomial of $F(u)$ in the excluded domain.
To see that the number of sheets of $F(u)$ considered as a covering $k_2(k_1)$ over $k^\pm_{\nu,1} + k_1$ with $k_1 \in B_{\varepsilon,1}(0)$ is two, remember that we have seen in Theorem 2.18 that

$$\det S^{-1}_{\pm\nu}(k + k^\pm_\nu) = 16\pi^4 \left((k + k^\pm_\nu)^2\right) \left((k + k^\pm_\nu + \nu)^2\right) = 0 \tag{2.9}$$

describes the free Fermi curve in a neighborhood of $k^\pm_\nu$ for these k_1 and for every $\nu \in \Gamma^*_\delta$ with $\delta > 0$ sufficiently small. This is equivalent to one of the four factors in this product being equal to zero. At k^+_ν it is

$$\begin{gathered}\left(k_2 + k^\pm_{\nu,2} + \iota(k_1 + k^\pm_{\nu,1})\right) = 0 \text{ and } \left(k_2 + k^\pm_{\nu,2} + \nu_2 - \iota(k_1 + k^\pm_{\nu,1} + \nu_1)\right) = 0,\\ \left(k_2 + k^\pm_{\nu,2} - \iota(k_1 + k^\pm_{\nu,1})\right) \neq 0 \text{ and } \left(k_2 + k^\pm_{\nu,2} + \nu_2 + \iota(k_1 + k^\pm_{\nu,1} + \nu_1)\right) \neq 0\end{gathered}$$

and at k^-_ν this is just the other way around. Hence, the factors unequal to zero in the product (2.9) can be assumed to be approximately constant for $k^\pm_{\nu,1} + k_1$ with $k_1 \in B_{\varepsilon,1}(0)$, and so equation (2.9) yields exactly two solutions of $k_2(k_1)$ for $k_1 \in (k^\pm_{\nu,1} + B_{r_1}(0)) \setminus \{k^\pm_{\nu,1}\}$ which are of multiplicity one and one solution at $k^\pm_{\nu,1}$ with multiplicity two. For $k^\pm_\nu + k$ with $k \in B_\varepsilon(0)$, Lemma 2.19 yields for $\|\nu\| \to \infty$

$$\mathcal{A}(u, k + k^\pm_\nu) = \begin{pmatrix} a(u, k + k^\pm_\nu) & b(u, k + k^\pm_\nu) \\ c(u, k + k^\pm_\nu) & d(u, k + k^\pm_\nu) \end{pmatrix} \to \begin{pmatrix} \hat{u}_0 & 0 \\ 0 & \hat{u}_0 \end{pmatrix},$$

where a, b, c, d are holomorphic in k. So for $\delta > 0$ sufficiently small and $k^\pm_{\nu,1} + k_1$ with $k_1 \in B_{\varepsilon,1}(0)$, f_u can approximately be described by the zero set of

$$\det(S^{-1}_{\pm\nu}(k + k^\pm_\nu) + \mathcal{A}(u, k + k^\pm_\nu)) \approx \det \begin{pmatrix} (k + k^\pm_\nu)^2 + \hat{u}_0 & 0 \\ 0 & (k + k^\pm_\nu + \nu)^2 + \hat{u}_0 \end{pmatrix}.$$

We get from this equation that asymptotically

$$((k + k^\pm_\nu)^2 + \hat{u}_0)((k + k^\pm_\nu + \nu)^2 + \hat{u}_0) = 0.$$

Using again the assumption that on $(k^+_{\nu,1} + B_{4\varepsilon}(0)) \times (k^+_{\nu,2} + B_{4\varepsilon}(0))$ it is $k_2 + k^+_{\nu,2} + \iota(k_1 + k^+_{\nu,1}) \approx c_1 \neq 0$

and $k_2 + k^+_{\nu,2} + \nu_2 - \iota(k_1 + k^+_{\nu,1} + \nu_1) \approx c_2 \neq 0$ as well as $k_2 + k^+_{\nu,2} - \iota(k_1 + k^+_{\nu,1}) \approx 0$ and $k_2 + k^+_{\nu,2} + \nu_2 + \iota(k_1 + k^+_{\nu,1} + \nu_1) \approx 0$ and vice versa for k^-_ν, one gets two solutions $k_2(k_1) + k^\pm_{\nu,2}$ counted with multiplicity for every $k^\pm_{\nu,1} + k_1$ with $k_1 \in B_{4\varepsilon}(0)$. So the covering is locally two sheeted.

As shown for the one-sheeted part of $F(u) \cap \mathcal{U}^\pm_{\varepsilon,\delta}$, these solutions are for all $\nu \in \Gamma^*_\delta$ contained in $k^\pm_{\nu,2} + B_\varepsilon(0)$. They can either coincide a discrete number of times or not at all, so the number of intersection points of these two sheets is discrete or empty in $k^\pm_\nu + (B_{4\varepsilon}(0) \times B_{4\varepsilon}(0))$. Note that here, the open balls are one-dimensional balls around each component of $(k^\pm_{\nu,1}, k^\pm_{\nu,2})$

To see that the number of branch points of the local Weierstraß covering of $F(u)/\Gamma^*$ in the neighborhood of these double points equals two, we first show that $F(0)$ has an ordinary double point at $k^\pm_\nu$ for $\nu \in \Gamma^*$ and consider without loss of generality k^+_ν: choosing local coordinates $(z_{\nu,1}, z_{\nu,2}) = (k^+_{\nu,1} - k_1, k^+_{\nu,2} - k_2)$ on $(k^+_{\nu,1} + B_{4\varepsilon}(0)) \times (k^+_{\nu,2} + B_{4\varepsilon}(0))$ yields that on these open neighborhoods

$$\begin{aligned} p^0_{k^+_\nu}(z_{\nu,1}, z_{\nu,2}) &= ((k_{\nu,1} - z_{\nu,1}) + \iota(k_{\nu,2} - z_{\nu,2}))((k_{\nu,1} - z_{\nu,1}) + \nu_1 - \iota((k_{\nu,2} - z_{\nu,2}) + \nu_2)) \\ &= (k_{\nu,1} - z_{\nu,1})^2 + (k_{\nu,1} - z_{\nu,1})(\nu_1 - \iota\nu_2) + (k_{\nu,2} - z_{\nu,2})(\nu_2 + \iota\nu_1) + (k_{\nu,2} - z_{\nu,2})^2, \end{aligned}$$

and therefore

$$\frac{\partial p^0_{k^+_\nu}}{\partial z_{\nu,1}} = 2(k^\pm_{\nu,1} - z_{\nu,1}) + \nu_1 - \iota\nu_2 \quad \text{and} \quad \frac{\partial p^0_{k^+_\nu}}{\partial z_{\nu,2}} = 2(k^\pm_{\nu,2} - z_{\nu,2}) + \nu_2 + \iota\nu_1.$$

So $(\partial p^0_{k^+_\nu}/\partial z_{\nu,1})(z_{\nu,1}, z_{\nu,2}) = (\partial p^0_{k^\pm_\nu}/\partial z_{\nu,2})(z_{\nu,1}, z_{\nu,2}) = 0$ if and only if $(z_{\nu,1}, z_{\nu,2}) = (0, 0)$. For the second derivative of the discriminant of $p^0_{k\nu^\pm}$ holds $D''_{p^0_{k\nu^\pm}}(0) = 2 \neq 0$. Therefore, Lemma 2.33(b) yields that every $k^\pm_\nu$ is an ordinary double point of $F(0)$. The definition of the open neighborhoods of k_0 and $k^\pm_\nu$ in Remark 2.10 ensures that $k^{\pm,1}_\nu + B_{4\varepsilon}(0)$ contains only points k_1 such that both coverings $(k_1, k_2) \mapsto k_1$ of $F(0)$ and $F(u)$ restricted to $k_1 \in \gamma_\nu := k^\pm_{\nu,1} + \partial B_{4\varepsilon}(0)$ are contained in $\Re(F(u))$ respectively $\Re(F(0))$. Next, we consider the difference of the zero-counting integrals of the discriminants $D_{p^u_{k^\pm_\nu}}(z_{\nu,1})$ and $D_{p^0(k^\pm_\nu)}(z_{\nu,1})$ inside of γ_ν to determine the number of branch points. For brevity, we set $D_{p^u_{k^\pm_\nu}} := D_u$ and $D_{p^0_{k^\pm_\nu}} := D_0$. The discriminants D_u and D_0 are polynomials in the coefficients of the Weierstraß polynomial $p^u_{k_n u^\pm}$ and $p^0_{k^\pm_\nu}$ and D'_u and D'_0 derivatives of those. Therefore, all these four functions are holomorphic in $z_{\nu,1}$. Since $F(0)$ has an ordinary double point at all $k^\pm_\nu$, the value of the zero counting integral is

$$\oint_{\gamma_\nu} \frac{D'_0}{D_0}(z_{\nu,1})\ dz_{\nu,1} = 2.$$

Furthermore,

$$\left| \int_{\gamma_\nu} \frac{D'_u}{D_u}(z_{\nu,1})\, \mathrm{d}z_{\nu,1} - \int_\gamma \frac{D'_0}{D_0}(z_{\nu,1})\, \mathrm{d}z_{\nu,1} \right| \leq \int_{\gamma_\nu} \left| \frac{D'_u D_0 - D'_0 D_u}{D_0 D_u}(z_{\nu,1}) \right| dz_{\nu,1}.$$

It is

$$\begin{aligned}\|D'_u D_0 - D'_0 D_u\|_\infty &\le \|D'_u D_0 - D'_0 D_0\| + \|D'_0 D_0 - D'_0 D_u\|_\infty \\ &= \|D_0\|_\infty \|D'_u - D'_0\|_\infty + \|D'_0\|_\infty \|D_0 - D_u\|_\infty.\end{aligned}$$

Due to the first part of this proof, for all $\varepsilon'' > 0$ there exists a $\delta'' > 0$ such that $\|D_0 - D_u\|_\infty < \varepsilon''$ as well as $\|D'_0 - D'_u\|_\infty < \varepsilon''$ for $(z_{\nu,1}, z_{\nu,2}) \in \gamma_\nu$ with $\nu \in \Gamma^*_{\delta''}$. Moreover, neither D_0 nor D_u is identically zero on γ_ν, both are holomorphic in $z_{\nu,1}$ and γ_ν is compact. So there exists a $c > 0$ such that $\|D_0 D_u|_{\gamma_\nu}\|_\infty \ge c$ and $\|D_u|_{\gamma_\nu}\|_\infty$ as well as $\|D_0|_{\gamma_\nu}\|_\infty$ are bounded. Thus, for $\nu \in \Gamma^*_{\delta''}$,

$$\left\| \int_{\gamma_\nu} \frac{D'_u}{D_u}(z_{\nu,1})\, \mathrm{d}z_{\nu,1} - \int_{\gamma_\nu} \frac{D'_0}{D_0}(z_{\nu,1})\, \mathrm{d}z_{\nu,1} \right\|_\infty \le \int_{\gamma_\nu} \left\| \frac{D'_u D_0 - D'_0 D_u}{D_0 D_u}(z_{\nu,1}) \right\|_\infty \mathrm{d}z_{\nu,1} \le \tilde{c}(c,\gamma,u)\varepsilon''$$

Since the values of each of these zero-counting integrals are integer, for $\delta > 0$ sufficiently small, this integral must equal. Hence, the number of branch points of $F(u)$ and $F(0)$ coincides in the neighborhood of the double point $k^\pm_\nu$ with $\nu \in \Gamma^*_{\delta''}$. Since $k^\pm_\nu$ is the only branch point of $F(0)$ and counted with multiplicity two, $F(u)$ has in each excluded domain around $k^\pm_\nu$ either two branch points of the covering $(k_1, k_2) \mapsto k_1$ or a double point $k^\pm_\nu(u)$. Due to Corollary 2.6 and since the two sheets $\mathcal{R}^\pm$ and $\mathcal{R}^\mp(\nu)$ meet in $k^\pm_\nu$, these two solutions of k_2 are contained in $k^\pm_{\nu,2} + B_{4\varepsilon,2}(0)$. □

Taking all the above together finally leads us to the following Theorem which summarizes all results concerning the asymptotics of $F(u)/\Gamma^*$.

Theorem 2.35 (Trisection of $F(u)/\Gamma^*$)**.** *Let $u \in C(\mathbb{R}^2/\Gamma)$ be fixed.*

(a) For all $\varepsilon > 0$ as in Remark 2.10 and all $u \in C(\mathbb{R}^2/\Gamma)$, there exists a $\delta > 0$ such that the two open sets $F(u)/\Gamma^* \cap \mathcal{U}^\pm_{\varepsilon,\delta}$ are isomorphic to connected one-dimensional complex submanifolds of $\mathbb{C}^2/\Gamma^*$. These submanifolds look like two real planes from which one huge hole and infinitely many small holes are cut out. The eigenspaces of $-\triangle_k + u$ over $F(u)/\Gamma^* \cap \mathcal{U}^\pm_{\varepsilon,\delta}$ are purely one-dimensional.

(b) The relative complement of $(F(u)/\Gamma^* \cap \mathcal{U}^+_{\varepsilon,\delta}) \cup (F(u)/\Gamma^* \cap \mathcal{U}^-_{\varepsilon,\delta})$ decomposes into two different parts:

(i) A compact set contained in

$$\left\{ k \in \mathbb{C}^2/\Gamma^* \;\middle|\; \|\operatorname{Im}(k)\| \le \frac{1}{\sqrt{2}\delta} \right\}.$$

(ii) Infinitely many small excluded domains $\mathfrak{e}_\nu$ indexed by $\Gamma^*_\delta = \{\nu \in \Gamma^* \mid \|k^\pm_\nu\| \ge (\sqrt{2}\delta)^{-1}\}$. The excluded domain $\mathfrak{e}_\nu$ is contained in

$$\{k \in \mathbb{C}^2 \mid \|k - k^+_\nu\| \le 2\varepsilon\} = \{k \in \mathbb{C}^2 \mid \|k - k^-_\nu\| \le 2\varepsilon\}$$

and connects the small disc around k_ν^+ excluded from $\mathcal{F}(u)/\Gamma^* \cap \mathcal{U}^+_{\varepsilon,\delta}$ with the small disc around k_ν^- excluded from $F(u)/\Gamma^* \cap U^-_{\varepsilon,\delta}$. The eigenspaces of $-\triangle_k + u$ over these excluded domains has at most two dimensions.

Proof. We can summarize the foregoing considerations as follows:

(a) That $F(u) \cap \mathcal{U}^\pm_{\varepsilon,\delta}$ is a one-dimensional manifold is the statement of Theorem 2.28(a) and that the eigenspace of the Schrödinger operator over these parts of the Fermi curve is one-dimensional follows from Lemma 2.23(a). The second statement also follows from Theorem 1.14 together with Theorem 2.34 since they imply that $-\triangle_k + u$ can be represented as a 1×1-matrix on $U^\pm_{\varepsilon,\delta}$ and thus has a one-dimensional eigenspace.

(b) (i) The fundamental domain $\varDelta_{\mathbb{C}}$ is compact and $\Gamma^* \setminus \Gamma^*_\delta$ contains for any $\delta > 0$ only a finite set of points. Due to the definition of $\mathbb{C}^2_\delta$ in (2.1), the set $(\mathbb{C}^2 \setminus \mathbb{C}^2_\delta)/\Gamma^*$ is for any $\delta > 0$ the union of finitely many compact sets and thus compact. Since the Fermi curve $F(u)/\Gamma^*$ is closed, $(F(u) \cap (\mathbb{C}^2 \setminus \mathbb{C}^2_\delta))/\Gamma^*$ is compact.

(ii) That also follows from Theorems 1.14 and 2.34 since they imply that $-\triangle_k + u$ can be represented as a 2×2-matrix on each $\mathfrak{e}_\nu$ and thus has at most a two-dimensional eigenspace.

□

3. The eigenfunctions

The meaning of this chapter is two-fold. The Fermi curve $F(u)/\Gamma^*$ is not a Riemann surface, but a singular curve. Hereby, we call one-dimensional singular complex analytic spaces X' in the sense of [de Jong and Pfister, 2012, Definition 6.1.36] singular curves in the sequel. Moreover, we keep in mind that we can understand all one-sheeted coverings of $F(u)/\Gamma^*$ as desingularizations $F(u)/\Gamma^*$. Then the 'most desingularized' one-sheeted covering is the normalization of $F(u)/\Gamma^*$. Because $F(u)/\Gamma^*$ is a one-dimensional variety in $\mathbb{C}^2/\Gamma^*$, the normalization is a Riemann surface and thus all singular points of $F(u)/\Gamma^*$ are resolved on the normalization. Furthermore, the Fermi curve $F(u)/\Gamma^*$ is, as a variety, a topological space. So we can consider the sheaves of functions on small open neighborhoods of every $[k] \in F(u)/\Gamma^*$. In the first section of this chapter, we define the normalization of $F(u)/\Gamma^*$, introduce the sheaf of holomorphic functions on $F(u)/\Gamma^*$ and define regular differentials on $F(u)/\Gamma^*$ which can be understood as the analogon for singular curves to the holomorphic differentials on a Riemann surface. These are defined as in [Serre, 1988, Chapter IV §3] or in [Rosenlicht, 1952].

On singular curves, one cannot define divisors as the 'classical' divisors are defined on Riemann surfaces. Nevertheless, we want to be able to consider an object on $F(u)/\Gamma^*$ which can – on the regular part of $F(u)/\Gamma^*$ – be understood at the pole divisor of the eigenfunction when the latter is normalized in an appropriate way. This is the first meaning of this chapter and will be introduced in the second section in terms of a generalized divisor on $F(u)/\Gamma^*$. Generalized divisors are finitely generated subsheaves of the meromorphic functions. We will consider the subsheaf of the meromorphic functions which is generated by normalized the eigenfunction of $-\triangle + u$ and show that this defines a so-called generalized divisor on $F(u)/\Gamma^*$.

The second meaning of this chapter is to construct a regular operator-valued 1-form on the Fermi curve. The construction of this 1-form is done in two steps: First, we consider the spectral projection of the Schrödinger operator on the Bloch variety and show that this defines a regular operator-valued 2-form on $B(u)/\Gamma^*$. However, it will not be possible to restrict this 2-form to a regular-valued 1-form on the Fermi curve because it is defined with respect to the covering $(k, \lambda) \mapsto k$. So in a second step we will introduce another projection and deduce from the first step that this defines a regular 2-form on $B(u)/\Gamma^*$ with respect to the covering $(k, \lambda) \mapsto (k_1, \lambda)$. Note that for local considerations, it does not make a difference whether we consider $(k, \lambda) \in B(u)$ or $([k], \lambda) \in B(u)/\Gamma^*$ because Γ^* is discrete. We will see that the latter 2-form can be restricted to $F(u)/\Gamma^*$ by setting $\lambda = 0$ and obtain the desired regular operator-valued 1-form on the Fermi curve. This 1-form will be crucial to show in Section 4.2.1 that the eigendivisor of the Schrödinger

operator with a so-called regular finite type potential has to obey a certain symmetry with respect to the holomorphic involution σ introduced in Section 1.3. This symmetry is very important in the whole remainder of this work. We will go into this in more detail in Chapter 4.2 after we have introduced regular finite potentials u.
From now on we assume more regularity on the eigenfunctions, i.e. we define

$$C^\infty_{[k]}(\varDelta) := \{f \in C^\infty(\mathbb{R}^2, \mathbb{C}) \mid \forall \gamma \in \varGamma : f((x,y)+\gamma) = e^{2\pi\iota\langle k,\gamma\rangle} f(x,y)\}$$

and assume that $\psi([k]) \in C^\infty_{[k]}(\varDelta)$ and $\varphi([k]) \in C^\infty_{[-k]}(\varDelta)$. This is no obstruction since we will see in the inverse problem in Chapter 5 that the reconstruction of the eigenfunctions out of some given data yields indeed elements of this space and we can also apply the results of the foregoing sections since $C^\infty(\varDelta) \subset L^2(\varDelta)$. Note that $fg \in L^2(\mathbb{R}^2/\varGamma)$ for $f \in C^\infty_{[-k]}(\varDelta)$ and $g \in C^\infty_{[k]}(\varDelta)$. Hence, $\langle\!\langle \partial_y f, g \rangle\!\rangle = -\langle\!\langle f, \partial_y g\rangle\!\rangle$. Analogously, we assume that the eigenfunctions of $-\triangle_k + u$ are elements of $C^\infty(\mathbb{R}^2/\varGamma, \mathbb{C})$. In the inverse problem in Chapter 5, we will see that even more regularity holds for the eigenfunctions corresponding to regular finite type potentials.

3.1. The Fermi curve as a singular curve

As a first step to define finite type potentials in the following chapter, the normalization $\pi : X^\circ(u) \to F(u)/\varGamma^*$ is necessary, where $X^\circ(u)$ is a Riemann surface and π is a one-sheeted covering over $F(u)/\varGamma^*$. For brevity, we set $X' = X'(u) := F(u)/\varGamma^*$ and $X^\circ := X^\circ(u)$ in all of this chapter and we omit the dependency of u if it is clear from the context which Fermi curve we consider. By abuse of notation, we sometimes denote the elements of X° by k and the elements of X' by k' instead of by $[k]$ and $[k']$, respectively.
Corollary 1.15 together with Theorem 2.28 (a) and (c) yields that $X'(u)$ is a one-dimensional variety in $\mathbb{C}^2/\varGamma^*$ with two open ends and hence at any point $k' \in X'(u)$, a germ of an analytic space as defined in [de Jong and Pfister, 2012, Definition 3.4.2 (a)]. So the existence of the normalization X° follows by [de Jong and Pfister, 2012, Theorem 4.4.8]. The direct image $\pi_* \mathcal{O}_{X^\circ}$ of the sheaf of holomorphic functions $\mathcal{O}_{X^\circ}$ on $X^\circ(u)$ equals $\bar{\mathcal{O}}_{X'}$ which is the sheaf of locally bounded functions on $X'(u)$, compare [de Jong and Pfister, 2012, Theorem 4.4.15 and proof of Theorem 4.4.8]:

$$\bar{\mathcal{O}}_{X'} = \pi_* \mathcal{O}_{X^\circ}, \qquad \bar{\mathcal{O}}_{X',k'} := \bigoplus_{k \in \pi^{-1}[\{k'\}]} \mathcal{O}_{X^\circ,k}.$$

We identify the meromorphic functions $\mathcal{M}_{X'}$ on $X'(u)$ with the meromorphic functions $\mathcal{M}_{X^\circ}$ on $X^\circ(u)$ via $f \mapsto f \circ \pi$ as in [Klein et al., 2016, Section 2.1] and denote them by the same symbol. This induces an isomorphism of sheaves $\mathcal{M}_{X'} \simeq \pi_* \mathcal{M}_{X^\circ}$. It is shown in [Klein et al., 2016, Proposition 2.1] that the set S of non-regular points of $X'(u)$ is given by the set of $k' \in X'(u)$ for which $\bar{\mathcal{O}}_{k'}/\mathcal{O}_{k'} \neq 0$ and that this is a discrete subvariety of $X'(u)$. These non-regular points we call the singularities of $X'(u)$, because they coincide with $X'(u) \setminus \mathfrak{R}(X'(u))$. Since $X'(u)$ is a one-dimensional variety in $\mathbb{C}^2/\Gamma^*$, the normalization space $X^\circ(u)$ is smooth, i.e. has no singularities, compare [de Jong and Pfister, 2012, Corollary 4.4.10]. Moreover, the one-sheeted covering $\pi : X^\circ(u) \setminus \pi^{-1}[S] \to X'(u) \setminus S$ is biholomorphic. Together with some statements shown in the foregoing chapter, we obtain the following Lemma.

Lemma 3.1. *Let $u \in C(\mathbb{R}^2/\Gamma)$ and let $X'(u)$ be the corresponding Fermi curve. Then there exists a covering $\pi : X^\circ(u) \to X'(u)$, where $X^\circ(u)$ is a Riemann surface, the image of π is $X'(u)$ and π is biholomorphic on $X^\circ(u) \setminus S$. The covering $(X^\circ(u), \pi)$ is called the normalization of $X'(u)$.*

(i) For $u = 0$, the singularities are exactly the double points $\{k_\nu^\pm \mid \nu \in \Gamma^*\}$ of $X'(0)$.

(ii) For $\delta > 0$ sufficiently small, the singularities $S \cap \mathbb{C}^2_\delta$ of $X'(u)$ are contained in the excluded domains $\mathfrak{e}_\nu$ around $k_\nu^\pm$ as introduced in Definition 2.26 for all $\nu \in \Gamma^*_\delta$. These singularities are only double points.

(iii) The singularities in $X'(u) \cap (\mathbb{C}^2 \setminus \mathbb{C}^2_\delta)/\Gamma^*$ may be of higher order, but are discrete.

Proof. The existence and smoothness of the normalization for $X'(u)$ was already justified above. Equation (1.15) yields that the double points $k_\nu^\pm$ are the only non-regular points of $X'(0)$. Theorems 2.28(b) and 2.34 are just the statement in (ii) and Theorem 2.35 b(i) together with the fact that the set of singularities is a discrete set in $X'(u)$ yields (iii). □

On singular curves X', we define generalized divisors as in [Klein et al., 2016, Definition 3.1] or [Hartshorne, 1986, §1].

Definition 3.2. [Klein et al., 2016, Definition 3.1] A *generalized divisor* on a singular curve X' is a finitely generated subsheaf $\mathcal{S}$ of the sheaf of meromorphic functions $\mathcal{M}$ on X'. The *support* $\operatorname{supp} \mathcal{S}$ of a generalized divisor $\mathcal{S}$ is the set of all $k' \in X'$ such that $\mathcal{S}_{k'} \neq \mathcal{O}_{k'}$. Two generalized divisors $\mathcal{S}$ and $\mathcal{S}'$ are *linear equivalent* if there exists an $f \in \mathcal{M}$ such that $f \cdot \mathcal{S} = \mathcal{S}'$.

For a generalized divisor $\mathcal{S}$ with finite support on a singular curve X', we define the degree as it is done in [Klein et al., 2016, Definition 3.5]:

Definition 3.3. Let $\mathcal{S}'$ be any generalized divisor with finite support containing $\mathcal{S}$ and $\mathcal{O}_{X'}$. Then

$$\deg(\mathcal{S}) := \dim H^0(X', \mathcal{S}'/\mathcal{O}_{X'}) - \dim H^0(X', \mathcal{S}'/\mathcal{S}).$$

Since the support of $\mathcal{S}$ is discrete, see [Klein et al., 2016, Proposition 3.3], there exists for each $k' \in \operatorname{supp} \mathcal{S}$ a generalized divisor $\mathcal{S}(k')$ such that $\operatorname{supp} \mathcal{S}(k') = \{k'\}$ and $\mathcal{S}(k')_{k'} = \mathcal{S}_{k'}$. We call $\deg_{k'}(\mathcal{S}) := \deg(\mathcal{S}(k'))$ the *degree of $\mathcal{S}$ at k'*. In a sufficiently small open neighborhood of a regular point of X' a generator of maximal pole order respectively minimal zero order alone suffices to generate $\mathcal{S}$, compare [Klein et al., 2016, § 6]. In particular on $\mathfrak{R}(X')$, a generalized divisor $\mathcal{S}$

equals $\mathcal{O}_D$ for some classical divisor D. Furthermore, the regular differential forms on X' are defined as follows.

Definition 3.4 ([Klein et al., 2016, Definition 6.1])**.** Let X' be a complex one-dimensional curve with normalization $\pi : X \to X'$. Then a meromorphic differential form ω on X' is *regular* at $k' \in X'$ if

$$\sum_{k\in\pi^{-1}[\{k'\}]} \mathrm{Res}_k(\pi^*(f \cdot \omega)) = 0 \text{ for all } f \in \mathcal{O}_{X',k'},$$

where the residue Res_k is defined as in [Forster, 1981, §9.9] We say that ω is a *regular* 1*-form* if it is regular at every $k' \in X'$. $\Omega_{X'}$ is the sheaf of regular differential forms on X'.

Every global meromorphic function f which does not vanish identically on a connected component of X' has an inverse meromorphic function. For such an f, the map $g \mapsto g \cdot df$ is an isomorphism from the sheaf of meromorphic functions onto the sheaf of meromorphic 1-forms and thus identifies finitely generated $\mathcal{O}_{X'}$-submodules of the sheaf of meromorphic 1-forms with generalized divisors on X'. The degree of such a submodule is defined as the degree of the corresponding generalized divisor, compare [Klein et al., 2016, Section 6]. The arithmetic genus of X' is defined as $g' := \dim H^1(X', \mathcal{O}_{X'})$, the geometric genus as $g := \dim H^1(X, \mathcal{O}_X)$ and δ as $\sum_{k'\in S} \delta_{k'}$ with $\delta_{k'} := \dim(\bar{\mathcal{O}}_{k'}/\mathcal{O}_{k'})$ as defined in [Klein et al., 2016, Proposition 2.1(a)]. We remark that $\delta_{k'} > 0$ is equivalent to $k' \in S$: if $k' \in S$, then $\bar{\mathcal{O}}_{k'} \neq \mathcal{O}_{k'}$, so $\delta_{k'} > 0$ and conversely, for $\delta_{k'} = 0$ one has $\bar{\mathcal{O}}_{k'} = \mathcal{O}_{k'}$, so $k' \notin S$. Moreover, is shown in [Klein et al., 2016, Lemma 5.1(b)] that the arithmetic and the geometric genus of X' are correlated by $g' = g + \delta$. For the degree of the regular 1-forms it is shown in [Klein et al., 2016, Corollary 6.6] that $\deg(\Omega'_X) = 2g' - 2$.

Definition 3.5 ([Hartshorne, 2013, Section 5, page 109])**.** Let $(X', \mathcal{O}_{X'})$ be a ringed space. A sheaf of modules $\mathcal{S}$ over $\mathcal{O}_{X'}$ is said to be *locally free* if for every point $k' \in X'$, there is an open neighborhood $U \subset X'$ of k' such that the restriction $\mathcal{S}|_U$ is a free sheaf of modules over $\mathcal{O}_{X'}|_U$, i.e. it is isomorphic to the direct sum of a set $\ell(X')$ of finitely many copies of the structure sheaf $\mathcal{O}_{X'}|_U$. If X' is connected and $\ell(X')$ consists of n elements, then n does not depend on the point k' and is called the *rank* of the locally free sheaf.

For hypersurfaces, one can show that the sheaf of regular differential forms is locally free. The two main ingredients for this proof are that a hypersurface is the zero set of only one holomorphic function R and that one can describe the curve locally by Weierstraß coverings. More precisely, the following Lemma – taken from [Schmidt, 2002, Lemma 2.20] – yields that on a hypersurface in $\mathbb{C}^{n+1}$, the regular differential form

$$\omega := \frac{1}{\partial R/\partial z_1}\, dz_2 \wedge \cdots \wedge dz_n$$

locally generates all regular differential forms since all regular forms can be represented as $f \cdot \omega$ with f being holomorphic. Hence, the regular forms are an invertible sheaf, i.e. a sheaf which is – as a module – locally free of rank 1 in the sense of Definition 3.5. This means in particular that every stalk over a point on X of this sheaf is, as a vector space, isomorphic to the stalk of the holomorphic functions at this point. We decided to give the proof of the Lemma because the proof which can be found in [Schmidt, 2002] is rather short.

Lemma 3.6. *[Schmidt, 2002, Lemma 2.20] Let $R(z_1, \dots, z_{n+1})$ be a holomorphic function on some open subset $U \subset \mathbb{C}^{n+1}$ whose partial derivative $\partial R/\partial z_1$ is not identically zero on the connected components of the subvariety of U defined by the equation $R(z_1, \dots, z_{n+1}) = 0$. Therefore, this subvariety can locally be considered as a covering space over $\tilde{z} := (z_{0,2}, \dots, z_{n+1}) \in \mathbb{C}^n$. On every open subset $V \subset \mathbb{C}^n$ where such a local representation of the subvariety in $\mathbb{C}^{n+1}$ exists, we define*

$$O := \{(z_1, \dots, z_{n+1}) \in \mathbb{C} \times V \mid R(z_1, \dots, z_{n+1}) = 0\}.$$

We assume in the sequel that V is always chosen in such a way that it contains at least one connected component of O. The following conditions on a meromorphic function f on this subvariety are equivalent:

(i) $f \in \mathcal{O}_O$.

(ii) Let us consider the subvariety as a covering space over $\tilde{z} \in \mathbb{C}^n$. Then for all $g \in \mathcal{O}_O$, the local sum of $gf/(\partial R/\partial z_1)$ over all sheets of this covering contained in O which contain an arbitrary element of this subvariety is a holomorphic function.

Proof. Since the statement of the Lemma is a local statement, it suffices to show that for all $z_0 := (z_{0,1}, \dots, z_{0,n+1}) = (z_{0,1}, \tilde{z}_0) \in O$, there exists a ball $B_\varepsilon(\tilde{z}_0) \subset V$ such that the lemma holds on the restriction of the Weierstraß covering $O \to V$ to the preimage $\tilde{O}$ of $B_\varepsilon(\tilde{z}_0)$. In particular, one can choose $B_\varepsilon(\tilde{z}_0)$ small enough such that all connected components of the preimage of $\tilde{z}_0$ consists only of one point in $\tilde{O}$. Since the holomorphic functions on the connected components are independent, we can assume that $\tilde{O} \to B_\varepsilon(\tilde{z}_0)$ only consists of the one point z_0 over $\tilde{z}_0$. Let us choose some element z_0 of the subvariety $\tilde{O}$. Due to the Weierstraß Preparation Theorem [de Jong and Pfister, 2012, Theorem 3.2.4], locally, there exists a Weierstraß polynomial

$$Q[z_1](\tilde{z}) = z_1^d + \sum_{i=0}^{d-1} q_i(\tilde{z}) z_1^i$$

with respect to z_1 whose coefficients q_i are holomorphic functions depending on $B_\varepsilon(\tilde{z}_0)$ such that $R = Q \cdot u$, where u is a locally holomorphic unit. Moreover, the degree d of Q equals the number of all sheets of $\tilde{O}$ considered as a covering space over $\tilde{z} \in B_\varepsilon(\tilde{z}_0)$ which contain the element $(z_1, \tilde{z})$ of $\tilde{O}$. In a neighborhood of z_0, condition (ii) is equivalent to an analogous condition, where R is replaced by Q. Let Q be an arbitrary polynomial in z_1 of degree d with leading coefficient 1 whose coefficients are complex numbers and let $p_1, \dots, p_d$ be the zeros of Q. Assume that the values of an arbitrary polynomial g in z_1 with complex-valued coefficients and of degree less than d are given in these zeros, i.e. $(p_i, g(p_i))$ for $i = 1, \dots, d$. Then the Lagrange interpolation [Roman, 2007, page 248] is given by

$$g(z_1) = \sum_{i=1}^{d} g(p_i)\ell_i(z_1), \quad \text{where } \ell_i(z_1) := \prod_{\substack{j=0 \\ j\neq i}}^{d} \frac{z_1 - p_j}{p_i - p_j}.$$

This leads to

$$g(z_1) = \sum_{i=1}^{d} \frac{g(p_i)}{\partial Q(p_i)/\partial z_1} \prod_{j\neq i} (z_1 - p_j)$$

if at least $\deg(g)$ zeros of Q are pairwise different since

$$\frac{\partial Q}{\partial z_1}(z_1) = \sum_{i=1}^{d} \prod_{\substack{j=1 \\ j\neq i}}^{d} (z_1 - p_j), \quad \text{so} \quad \frac{\partial Q}{\partial z_1}(p_i) = \prod_{\substack{j=1 \\ j\neq i}}^{d} (p_i - p_j),$$

In particular, the sum

$$\sum_{i=1}^{d} \frac{g}{\partial Q/\partial z_1}(p_i)$$

is equal to the coefficient of the monomial z_1^{d-1} in $g(z_1)$. We conclude that we can also apply Lagrange interpolation as above if the coefficients of Q are holomorphic functions depending on $\tilde{z}$ such that the discriminant of Q does not vanish identically because in this case all zeros of the polynomial are still pairwise different. Due to the Weierstraß Division Theorem [de Jong and Pfister, 2012, Theorem 3.2.3], each meromorphic function on the subvariety which is defined by the zero set of Q can be written locally as a polynomial with respect to z_1 of degree smaller or equal to $d-1$ whose coefficients are meromorphic functions depending on $\tilde{z}$. Hence, the local sum of this function over all sheets of the subvariety considered as a covering space over $\tilde{z} \in B_\varepsilon(\tilde{z}_0)$ which contain z_0 is locally a holomorphic function if and only if the coefficient of z_1^{d-1} of the polynomial is locally holomorphic. With this at hand, we show that (i) and (ii) are equivalent.

(i)⇒(ii): Let

$$\begin{aligned} f(z_1, \tilde{z}) &= z_1^{d-1} f_1(\tilde{z}) + \cdots + f_d(\tilde{z}), \\ g(z_1, \tilde{z}) &= z_1^{d-1} g_1(\tilde{z}) + \cdots + g_d(\tilde{z}), \\ Q(z_1, \tilde{z}) &= z_1^d + z_1^{d-1} q_1(\tilde{z}) + \cdots + q_d(\tilde{z}) \end{aligned}$$

be holomorphic functions on $\mathbb{C} \times B_\varepsilon(\tilde{z}_0)$. That means all coefficients of each of these functions are holomorphic in $\tilde{z}$ on $B_\varepsilon(\tilde{z}_0)$. Note that $f(z_1,\tilde{z})\cdot g(z_1,\tilde{z})$ has degree $2d-2$. To consider $(f\cdot g)(z_1,\tilde{z})$ on $\tilde{O}$, we have do divide it by Q. Due to the normalization of Q, it is

$$(f\cdot g)(z_1,\tilde{z}) = (s_1(\tilde{z})z_1^{g-2}+\dots+s_{g-2}(\tilde{z})z_1+s_{g-1}(\tilde{z}))Q(z_1,\tilde{z})+r_1(\tilde{z})z_1^{d-1}+\dots+r_d(\tilde{z}),$$

where $s_1(\tilde{z})z_1^{g-2}+\dots+s_{g-1}(\tilde{z})$ and $r_1(\tilde{z})z_1^{d-1}+\dots+r_d(\tilde{z})$ are polynomials in z_1 whose coefficients are linear combinations of the coefficients of $f(z_1,\tilde{z})$, $g(z_1,\tilde{z})$ and $Q(z_1,\tilde{z})$ and thus holomorphic on $B_\varepsilon(\tilde{z}_0)$. Since we consider $f\cdot g$ on $\tilde{O}$, i.e. $Q(z_1,\tilde{z})=0$, one has .

$$(f\cdot g)|_{\tilde{O}}(z_1,\tilde{z}) = z_1^{d-1}r_1(\tilde{z})+\dots+r_d(\tilde{z}).$$

Due to the above considerations,

$$r_1(\tilde{z}) = \sum_{\text{sheets of }\tilde{O}} \frac{g\cdot f}{\partial R/\partial z_1}(\tilde{z})$$

is holomorphic on $B_\varepsilon(\tilde{z}_0)$.

(ii)⇒(i): Let $\sum_{\text{sheets of }\tilde{O}} \frac{g(\tilde{z})f(\tilde{z})}{\partial Q/\partial z_1}$ be holomorphic on $B_\varepsilon(\tilde{z}_0)$ for every $g\in\mathcal{O}_{\tilde{O}}$. Inserting $g=z_1^0$ leads to

$$(g\cdot f)(z_1,\tilde{z}) = f(z_1,\tilde{z}) = z_1^{d-1}f_1(\tilde{z})+\dots+f_d(\tilde{z}).$$

Then $f_1(\tilde{z})=\sum\frac{gf}{\partial Q/\partial z_1}$, and therefore, f_1 is holomorphic on $B_\varepsilon(\tilde{z}_0)$. For $g=z_1^1$ one gets

$$(g\cdot f)(z_1,\tilde{z}) = z_1 f(z_1,\tilde{z}) = z_1^d f_1(\tilde{z})+\dots+f_d(\tilde{z})z_1.$$

Moreover,

$$\begin{aligned}(g\cdot f)(z_1,\tilde{z}) - f_1(\tilde{z})\cdot Q(z_1,\tilde{z}) &= z_1^d f_1(\tilde{z}) + z_1^{d-1}f_2(\tilde{z})+\dots+z_1 f_d(\tilde{z})-\\ &\quad - f_1(\tilde{z})\cdot(z_1^d+z_1^{d-1}q_1(\tilde{z})+\dots+q_d(\tilde{z}))\\ &= z_1^{d-1}(f_2(\tilde{z})-f_1(\tilde{z})q_1(\tilde{z}))+\tilde{z}^{d-2}(f_3(\tilde{z})-f_1(\tilde{z})q_2(\tilde{z}))+\dots\\ &\quad \dots + z_1(f_d(\tilde{z})-f_1(\tilde{z})q_{d-1}(\tilde{z})+f_1(\tilde{z})q_d(\tilde{z}).\end{aligned}$$

Hence, $f_2(\tilde{z})-f_1(\tilde{z})q_1(\tilde{z})=\sum\frac{gf}{\partial Q/\partial z_1}$. Since q_1 and f_1 are holomorphic on $B_\varepsilon(\tilde{z})$, also f_2 is holomorphic on $B_\varepsilon(\tilde{z}_0)$. Repeating this procedure for $g=z_1^\ell$ with $\ell\in\{2,\dots,d-1\}$ finally yields that all coefficients of f are holomorphic on $B_\varepsilon(\tilde{z}_0)$, so f is holomorphic on $\mathbb{C}\times B_\varepsilon(\tilde{z}_0)$.

□

This Lemma shows that on n-dimensional varieties O, the regular n-forms are defined as those meromorphic n-forms whose products with the holomorphic functions in $\mathcal{O}_O$ have no residue in some neighborhood of any point of O. So Definition 3.4 of regular 1-forms on a one-dimensional variety coincides with the conditions given in Lemma 3.6 for $\omega = f\ dz_1$. Let the subvariety O of $U \subset \mathbb{C}^{n+1}$ be a hypersurface, defined as in Lemma 3.6, and let not only $\partial R/\partial z_1 \not\equiv 0$, but also some other derivative $\partial R/\partial z_j \not\equiv 0$ for $j = 2, \dots, n+1$ on a connected components of O. Then one can consider this subvariety locally either as a covering space over $(z_2, \dots, z_{n+1}) \in \mathbb{C}^n$ or as a covering space over $(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_{n+1}) \in \mathbb{C}^n$. Due to $(\partial R/\partial z_1)\mathrm{d}z_1 + \dots + (\partial R/\partial z_{n+1})\mathrm{d}z_{n+1} = 0$, it is

$$\frac{1}{\partial R/\partial z_1}\,\mathrm{d}z_2 \wedge \dots \wedge \mathrm{d}z_{n+1} = \frac{(-1)^{j+1}}{\partial R/\partial z_j}\,\mathrm{d}z_1 \wedge \dots \wedge \mathrm{d}z_{j-1} \wedge \mathrm{d}z_{j+1} \wedge \dots \wedge \mathrm{d}z_{n+1}. \tag{3.1}$$

So Lemma 3.6 yields that an n-form

$$\omega = \frac{f(z_1, \dots, z_{n+1})}{\partial R(z_1, \dots, z_{n+1})/\partial z_1}\mathrm{d}z_2 \wedge \dots \wedge \mathrm{d}z_{n+1} = \\ = (-1)^j \frac{f(z_1, \dots, z_{n+1})}{\partial R(z_1, \dots, z_{n+1})/\partial z_j}\mathrm{d}z_1 \wedge \dots \wedge \mathrm{d}z_{j-1} \wedge \mathrm{d}z_{j+1} \wedge \dots \wedge \mathrm{d}z_{n+1}$$

on a hypersurface is regular in the sense of Definition 3.4 if and only if f satisfies one of the equivalent conditions (i) or (ii) from Lemma 3.6 either for z_1 or the analogous condition to (ii) with z_1 replaced by z_j.

3.2. The eigendivisor

Let $u \in C(\mathbb{R}^2/\Gamma)$ be fixed. We now want to introduce a generalized divisor which corresponds to the pole divisor of a normalized eigenfunction in the smooth case. Remember that $\pi : X^\circ \to X'$ is the covering map defined in Lemma 3.1.

Since we now talk in the language of germs, it will be convenient to indicate the eigenfunctions with an index k', where $k' \in X'$ denotes the point at which the stalk of a germ is considered. So far, we indicated the eigenfunctions $\psi_{k'}$ of the Schrödinger equation (1.6) in the formulation with help of the trivializations as in Definition 1.5 with an index k' in order to distinguish them from the eigenfunctions ψ in the formulation in Definition 1.4 on the fundamental domain Δ. In this section, we only consider the formulation of the fundamental domain, and therefore, we want to point out that the index k' in this section is used to indicate the corresponding germ at $k' \in X'$. We choose the fundamental domain Δ in (1.4) in such a way that it always contains $(x, y) = (0, 0)$. To define a generalized divisor on the subvariety X' which corresponds to the pole divisor of a normalized eigenfunction in the smooth case, we normalize the locally defined, holomorphic eigenfunction ψ on X° as

$$\psi_N(\,\cdot, (x, y)) := \frac{\psi(\cdot, (x, y))}{\psi(\,\cdot, (0, 0))} \tag{3.2}$$

and consider for every $\xi \in L^2(\mathit{\Delta})$

$$\langle\!\langle \xi, \psi_N(\cdot) \rangle\!\rangle = \frac{\langle\!\langle \xi, \psi(\cdot) \rangle\!\rangle}{\psi(\,\cdot, (0,0))}. \tag{3.3}$$

For every $\xi \in L^2(\mathit{\Delta})$, the latter defines a meromorphic function which is globally defined on X°. This induces a germ in $\mathcal{M}$ and motivates the following definition:

Definition 3.7. We define the generalized divisor $\mathcal{S}$ as the subsheaf which is generated by the meromorphic functions in (3.3) with any $\xi \in L^2(\mathit{\Delta})$. The corresponding germs at $k' \in X'$ we denote by $\mathcal{S}_{k'}$.

From [Klein et al., 2016, Section 2.1], we know that an open neighborhood of $k' \in S$ is defined as the disjoint union of the elements $k \in \pi^{-1}[\{k'\}]$. So for $k' \in S$, one has

$$\mathcal{M}_{X',k'} = \bigtimes_{k \in \pi^{-1}[\{k'\}]} \mathcal{M}_{X^\circ(u),k}$$

and for the germ of a holomorphic function $\psi \in \mathcal{O}_{X',k'}$ holds that the values of $\pi^*\psi$ coincide at all $k \in \pi^{-1}[\{k'\}]$. In particular, the covering $\pi : X^\circ \to X'$ is locally biholomorphic on $X^\circ \setminus \pi^{-1}[S]$. So let $k' \in \mathfrak{R}(X')$. The eigenfunction ψ can be chosen holomorphic on any sufficiently small open neighborhood of k' and is, as an eigenfunction of the Schrödinger operator, not identically zero. So we can always find a $\xi \in L^2(\mathit{\Delta})$ such that $\mathcal{S}_{k'}$ is generated by $\left(\frac{\langle\!\langle \xi,\psi \rangle\!\rangle}{\psi(0,0)}\right)_{k'}$ with $\xi \in L^2(\mathit{\Delta})$. Then $\mathcal{S}_{k'} \neq \mathcal{O}_{k'}$ if and only if $\psi(k',(0,0)) = 0$. For $k' \in S$, let $\pi^{-1}[\{k'\}] = \{k_1, \dots, k_n\}$ with $n \in \mathbb{N}$. Moreover, let $O_{k'}$ be a small open neighborhood of k' and let $\pi^{-1}[O_{k'}]$ consist of open sets O_{k_i} with $i = 1, \dots, n$ such that each O_{k_i} is a small open neighborhood of $k_i \in \pi^{-1}[\{k'\}]$. We always choose $O_{k'}$ sufficiently small such that $O_{k_i} \cap O_{k_j} = \emptyset$ for $i \neq j$. Then $O_{k_1} \,\dot\cup\, \dots \,\dot\cup\, O_{k_n}$ is an open neighborhood of $\pi^{-1}[\{k'\}]$. In this case,

$$\psi|_{O_{k'}} : O_{k_1} \,\dot\cup\, \dots \,\dot\cup\, O_{k_n} \to L^2(\mathit{\Delta}).$$

For every $\xi \in L^2(\mathit{\Delta})$, the germ $\left(\frac{\langle\!\langle \xi,\psi \rangle\!\rangle}{\psi(0,0)}\right)_{k'}$ is contained in $\mathcal{M}_{X^\circ(u),k_1} \times \dots \times \mathcal{M}_{X^\circ(u),k_n}$ and $\left.\frac{\langle\!\langle \psi,\xi \rangle\!\rangle}{\psi(0,0)}\right|_{O_{k'}}$ is generated by

$$\left.\frac{\langle\!\langle \xi, \psi(k) \rangle\!\rangle}{\psi(k,(0,0))}\right|_{O_{k_1}}, \dots, \left.\frac{\langle\!\langle \xi, \psi(k) \rangle\!\rangle}{\psi(k,(0,0))}\right|_{O_{k_n}}.$$

Nevertheless, the germ of the generalized divisor $\mathcal{S}$ from Definition 3.7 at k' contains the germ of the holomorphic functions of this point, as it is shown in the next proposition.

Proposition 3.8. *$\mathcal{O}_{k'} \subseteq \mathcal{S}_{k'}$ for all $k' \in X'$.*

Proof. Let $(\xi_n)_{n \in \mathbb{N}}$ be a sequence of smooth functions with $\operatorname{supp}(\xi_n) \subset B_{\frac{1}{n}}(0,0)$ and $\int_{\mathbb{R}^2} \xi_n \, dA = 1$ for all $n \in \mathbb{N}$. For $n > 0$ sufficiently large, this defines a sequence with $\xi_n \in L^2(\mathit{\Delta})$ which converges to the δ-function. Because we have assumed for every $k' \in X'(u)$ that the eigenfunction $\psi(k', \cdot)$ is continuous on $\mathbb{R}^2$, it is

$$\lim_{n\to\infty} |\langle\!\langle \xi_n, \psi(k', \cdot)\rangle\!\rangle - \langle\!\langle \xi_n, \psi(k', (0,0))\rangle\!\rangle| = \lim_{n\to\infty} \left| \langle\!\langle \xi_n, \psi(k', \cdot)\rangle\!\rangle - \psi(k', (0,0)) \int_\Delta \xi_n \, dA \right| = 0.$$

Now let $k' \in X'(u)$ with $\pi^{-1}[\{k'\}] = k_1, \ldots, k_n$ and $O_{k'}$ be a simply connected, small open neighborhood of k' such that $\pi^{-1}[O_{k'}]$ is the disjoint union of small disks $O_{k_1}, \ldots, O_{k_n}$ where O_{k_i} contains k_i for $i = 1, \ldots, n$.
It follows from the Cauchy Integral Formula that the bounded holomorphic functions on $\pi^{-1}[O_{k'}]$ are closed in the Banach space $C(\pi^{-1}[O_{k'}], \mathbb{C})$. Let $\mathcal{O}_{O_{k'}}$ denote the holomorphic functions on $O_{k'}$. The codimension of $\mathcal{O}_{O_{k'}}$ is finite in $\bar{\mathcal{O}}_{O_{k'}}$ because $\mathcal{O}_{k'}$ has finite codimension in $\bar{\mathcal{O}}_{k'}$, compare [Klein et al., 2016, Proposition 2.1(a)]. Therefore, $\mathcal{O}_{O_{k'}}$ is closed in $C(\pi^{-1}[u], \mathbb{C})$. So $\{\pi^* f \mid f \in \mathcal{O}_{O_{k'}}\} \subseteq \mathcal{O}_{k'}$ is a closed set in $C(\pi^{-1}[O_{k'}], \mathbb{C})$. Let $\mathcal{S}_{O_{k'}}$ be the set of all sections of $\mathcal{S}$ on $O_{k'}$. Then $\mathcal{S}_{O_{k'}} \subseteq \mathcal{S}_{k'}$. By definition of $\mathcal{S}$, it is $\psi(0,0)\mathcal{S}_{k'} \subseteq \mathcal{O}_{k'}$. In the neighborhood of every element in $\pi^{-1}[\{k'\}]$, there is a generator of $\psi(0,0)\mathcal{S}_{k'} \not\equiv 0$. So the generators of $\psi(0,0)S_{k'}$ are a finitely generated submodule of the meromorphic function over the ring $\mathcal{O}_{X,k'}$. This corresponds to a divisor D of finite degree on X. Then D is positive since it is generated by holomorphic functions. So the codimension of the above submodule in the stalk of the holomorphic functions equals the degree of all points of D contained in $\pi^{-1}[\{k'\}]$ and thus is finite. Because $\delta_{k'} < \infty$, $\psi(0,0)\mathcal{S}_{k'}$ has finite codimension in $\pi_*\mathcal{O}_D$. So $\psi(0,0)\mathcal{S}_{k'}$ has finite codimension in $\bar{\mathcal{O}}_X$, and therefore also in $\mathcal{O}_{k'}$, compare [Klein et al., 2016, Proposition 2.1(a)]. Then the closedness of $\mathcal{O}_{O_{k'}}$ yields that also $\psi(0,0)\mathcal{S}_{O_{k'}}$ is closed.
For every $n \in \mathbb{N}$, it follows from the definition of $\mathcal{S}$ that $\langle\!\langle \xi_n, \psi\rangle\!\rangle \in \psi(0,0)\mathcal{S}_{O_{k'}}$ and we have seen above that $(\langle\!\langle \xi_n, \psi\rangle\!\rangle)_{n\in\mathbb{N}}$ converges in $\psi(0,0)\mathcal{S}_{O_{k'}}$ to $\psi(0,0)$. Therefore, $\psi(0,0)\mathcal{O}_{O_{k'}} \subseteq \psi(0,0)\mathcal{S}_{O_{k'}}$. Because this holds for arbitrarily small $O_{k'}$ defined as above and every element of the germ $\mathcal{O}_{k'}$ is defined on such a small open neighborhood, this yields that $\psi(0,0)\mathcal{O}_{k'} \subseteq \psi(0,0)\mathcal{S}_{k'}$. Since $\psi(0,0)$ is – as a meromorphic function in k' – invertible, this yields that $\mathcal{O}_{k'} \subseteq \mathcal{S}_{k'}$ for all $k' \in X'$. □

We have already seen that $\mathcal{S}_{k'}$ has only one generator for $k' \in \mathfrak{R}(X')$. One might think that $\mathcal{S}_{k'}$ has infinitely many generators for $k' \in S$ because $\mathcal{S}_{k'}$ contains all functions of the form $\frac{\langle\!\langle \xi, \psi(k')\rangle\!\rangle}{\psi(k',(0,0))}$ with $\xi \in L^2(\Delta)$. However, the next Lemma shows that $\mathcal{S}_{k'}$ is also finitely generated for all $k' \in S$.

Lemma 3.9. *The divisor $\mathcal{S}$ as in Definition 3.7 is a finitely generated submodule of $\mathcal{M}$ over $\mathcal{O}$.*

Proof. It remains to show that $\mathcal{S}_{k'}$ is also finitely generated for $k' \in S$. We normalize the eigenfunction ψ in such a way that $\psi|_{O_{k'}} : O_{k_1} \dot\cup \ldots \dot\cup O_{k_n} \to L^2(\Delta)$ is a holomorphic function which has no zeros in $L^2(\Delta)$. Then for every $\xi \in L^2(\Delta)$, $\langle\!\langle \xi, \psi\rangle\!\rangle_{k'} \in \bar{\mathcal{O}}_{k'}$. Let $\hat{\psi}(0)$ be the zeroth Fourier coefficient of ψ. We know from Lemma 2.23 that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for $k \in \mathfrak{R}(X') \cap \mathbb{C}^2_\delta$, there holds $\|\psi(k,(0,0)) - \hat{\psi}(0)\| < \varepsilon$ with $\hat{\psi}(0) \neq 0$, wherefore $\psi(\cdot,(0,0)) \not\equiv 0$ on both of the maximal two connected components of $\Re(X')$. So for every $\xi \in L^2(\Delta)$, $\frac{\langle\langle \xi,\psi\rangle\rangle}{\psi(0,0)}$ is a submodule of the submodule of the meromorphic functions over $\mathcal{O}$ which is generated by $\frac{1}{\psi_{k'}(0,0)}\bar{\mathcal{O}}_{k'}$. Therefore, also $\mathcal{S}$ is a submodule of the latter submodule. For $k' \in S$, we can decompose $\bar{\mathcal{O}}_{X',k'}$ as

$$\bar{\mathcal{O}}_{X',k'} = \mathcal{O}_{X',k'} \oplus \bar{\mathcal{O}}_{X',k'}/\mathcal{O}_{X',k'}.$$

Accordingly, the number of generators of $\bar{\mathcal{O}}_{X',k'}$ is at most $1+\delta_{k'} < \infty$, compare [Klein et al., 2016, Proposition 2.1(a) and Lemma 5.1(a)]. So $\frac{1}{\psi_{k'}(0,0)}\bar{\mathcal{O}}_{k'}$ is a finitely generated submodule of $\mathcal{O}_{k'}$. From this follows that the sheaf $\mathcal{S}_{k'}$ is also finitely generated: Due to Proposition 3.8, $\mathcal{O}_{k'} \subseteq \mathcal{S}_{k'}$. If there exists a $\xi_1 \in L^2(\Delta)$ such that $\left(\frac{\langle\langle \xi_1,\psi\rangle\rangle}{\psi(0,0)}\right)_{k'} \notin \mathcal{O}_{k'}$, then we take $\left(\frac{\langle\langle \xi_1,\psi\rangle\rangle}{\psi(0,0)}\right)_{k'}$ as an additional generator of $\mathcal{S}_{k'}$. Successively continuing like this, we can find additional generators $\left(\frac{\langle\langle \xi_n,\psi\rangle\rangle}{\psi(0,0)}\right)_{k'}$ of $\mathcal{S}_{k'}$. However, $\mathcal{S}_{k'}$ is a submodule of a module of at most $\delta_{k'}+1$ dimensions, so $\mathcal{S}_{k'}$ cannot contain more that $\delta_{k'}+1$ generators. □

Corollary 3.10. *Let $\mathcal{S}^T$ be the generalized divisor which is generated by the normalized eigenfunction of the transposed Schrödinger operator. This is defined analogously to $\mathcal{S}$ in Definition 3.7. Then $\mathcal{S}^T = \sigma^*\mathcal{S}$. For $u \in C(\mathbb{R}^2/\Gamma,\mathbb{R})$ one has $\tau_1^*\mathcal{S} = \mathcal{S}$.*

Proof. For the transposed Schrödinger operator, we let $\mathcal{S}^T$ be the generalized divisor generated by $\langle\langle \varphi_N(\cdot),\xi\rangle\rangle$ with $\xi \in L^2(\Delta)$ and dual normalized eigenfunction defined as

$$\varphi_N(\,\cdot\,) := \frac{\varphi(\cdot,(x,y))}{\varphi(\,\cdot,(0,0))}.$$

Then this is a direct consequence of Lemma 1.17(a) and (b). □

3.3. The Spectral Projection

We have already seen in Theorem 1.14 that the spectral projection of $-\triangle+u$ gives us insight why it is possible to represent $B(u)/\Gamma^*$ respectively $F(u)/\Gamma^*$ locally as the zero set of the characteristical polynomial of a matrix. Now, we want to take a different viewpoint on the spectral projection, i.e. we want to consider it as a meromorphic function on $B(u)/\Gamma^*$ and deduce from this another projection P_∂ which can be restricted to $F(u)/\Gamma^*$. This second projection will be very helpful in various ways for the following considerations concerning the eigenfunctions. As already mentioned at the beginning of this section, the most important result is that it will help us to correlate the eigendivisor of the Schrödinger operator with the eigenfunction of its transposed.

For the Dirac operator, an analogous construction of this modified projection together with its properties is shown in [Schmidt, 2002, Chapter 2.4]. In this section, we transfer these results to the Schrödinger operator with some modifications. Therefore, the Schrödinger equation (1.1) with eigenfunctions which are quasiperiodic with respect to Γ and periodic with respect to Γ^* as in

Definition 1.1 is considered.
In the neighborhood of a regular point $([k], \lambda) \in B(u)/\varGamma^*$, there exist meromorphic functions $\psi([k], \lambda)$ and $\varphi([k], \lambda)$ from $B(u)/\varGamma^*$ to $C^\infty_{[k]}(\varDelta, \mathbb{C})$ respectively $C^\infty_{[-k]}(\varDelta, \mathbb{C})$ which map $([k], \lambda)$ onto an eigenfunction of the Schrödinger operator $-\triangle + u$ with eigenvalue λ and boundary condition $[k]$ respectively onto an eigenfunction of the transposed Schrödinger operator $(-\triangle + u)^T$ with eigenvalue λ and boundary condition $[-k]$. Due to Corollary 2.29(c), the set of degenerated eigenvalues on $B(u)/\varGamma^*$ has codimension one. So generically, the eigenspaces for a certain eigenvalue are one-dimensional on $B(u)/\varGamma^*$ and the above maps are unique up to multiplication by some invertible function which is meromorphic in $([k], \lambda)$. The images of these mappings, i.e. the eigenfunctions of $-\triangle + u$ and $(-\triangle + u)^T$, will also be denoted by $\psi([k], \lambda)$ and $\varphi([k], \lambda)$, respectively. Sometimes we omit λ for clarity of notation if λ is fixed. If we want to point out the dependency of the image of these functions on $(x, y) \in \varDelta$, then we write $\psi([k], (x, y))$, $\psi([k], \lambda, (x, y))$ or only $\psi(x, y)$ when it is clear at which point $([k], \lambda) \in B(u)/\varGamma^*$ we consider these images. Sometimes we leave all arguments away. Then it is made clear before to which values these functions belong.
With this notation, we will take a look at the spectral projection mentioned above which is defined as a meromorphic function on $B(u)/\varGamma^*$ with values in the finite rank operators on $L^2(\varDelta)$ by

$$P([k], \lambda) : \chi \mapsto \frac{\langle\!\langle \varphi([k], \lambda), \chi \rangle\!\rangle}{\langle\!\langle \varphi([k], \lambda), \psi([k], \lambda) \rangle\!\rangle} \psi([k], \lambda). \tag{3.4}$$

This definition does not depend on the normalization of the functions ψ and φ, i.e. let f and g be non-vanishing, meromorphic functions in $([k], \lambda)$, then

$$\frac{\langle\!\langle g\varphi([k], \lambda), \chi \rangle\!\rangle}{\langle\!\langle g\varphi([k], \lambda), f\psi([k], \lambda) \rangle\!\rangle} f\psi([k], \lambda) = \frac{\langle\!\langle \varphi([k], \lambda), \chi \rangle\!\rangle}{\langle\!\langle \varphi([k], \lambda), \psi([k], \lambda) \rangle\!\rangle} \psi([k], \lambda).$$

Hence, we can assume that locally in $([k], \lambda)$, the functions ψ and φ in $P([k], \lambda)$ are always normalized in such a way that they have neither poles nor zeros and such that the enumerator of $P([k], \lambda)$ has also neither poles nor zeros. The second assumption is justified by the non-degeneracy of the L^2-scalar product. However, the denominator of P can vanish. To see that, consider the holomorphic covering

$$B(u)/\varGamma^* \to \{[k] \in \mathbb{C}^2/\varGamma^* \mid \exists\, \lambda \in \mathbb{C} \text{ such that } ([k], \lambda) \in B(u)/\varGamma^*\}, \quad ([k], \lambda) \mapsto [k]. \tag{3.5}$$

We have seen in Corollary 2.29(c) that the degenerated eigenvalues λ are generically discrete. So this covering is well-defined. The next Lemma shows that the denominator of P can only vanish at the branch points of $B(u)/\varGamma^*$ with respect to this covering, i.e. at the points where λ is a degenerated eigenvalue. Therefore, P is a well defined global meromorphic function on $B(u)/\varGamma^*$.

Lemma 3.11. *The poles of P are contained in the branching divisor of $B(u)/\varGamma^*$ with respect to the covering* (3.5)*. Hereby, the branching divisor contains all branch points of this covering as well as the singularities of $B(u)/\varGamma^*$. In particular, all regular branch points of this covering are poles* *of P.*

Proof. Due to Corollary 2.29(c), the denominator of P does not vanish identically on any of the maximal two connected components of $B(u)/\Gamma^*$. This is since for $([k_0], \lambda_0)$ such that λ_0 is non-degenerate, $P = \check{P}$ and it follows from the proof of Theorem 1.14 that the rank of $\check{P}$ equals the number of sheets meeting in the neighborhood of $([k_0], \lambda_0)$. In this case, the rank is one, and therefore $\check{P}$ has no zero. If λ is not a degenerated eigenvalue, then we can normalize ψ, φ in such a way that $\langle\!\langle \varphi([k], \lambda), \psi([k], \lambda) \rangle\!\rangle \neq 0$. So if the denominator of P vanishes, then only at the degenerate eigenvalues.

To see that P can indeed have a pole which is contained in the branching divisor of $B(u)/\Gamma^*$ with respect to the covering in (3.5), we first remember that the preimage with respect to this covering may contain several points of $B(u)/\Gamma^*$. More precisely, due to the Weierstraß Preparation Theorem [de Jong and Pfister, 2012, Theorem 3.2.4], there are locally finitely many values $\lambda(k)$ over each k. Only at a degenerated eigenvalue λ, the corresponding sheets meet and those are precisely the ramification points of the covering (3.5) and the singularities of $B(u)/\Gamma^*$. Let $([k_0], \lambda) \in B(u)/\Gamma^*$ be contained in the branching divisor. Without loss of generality, we assume that the preimage of $[k_0] \in \mathbb{C}^2/\Gamma^*$ with respect to the covering (3.5) contains only one point $([k_0], \lambda_0)$, i.e. $\lambda_0 = \pi^{-1}[\{[k_0]\}]$ and that two sheets corresponding to the eigenvalues λ_i and λ_j meet in this point. Since these are only local considerations, it follows as in the proof of Lemma 3.6 that this is no obstruction. For $k \to k_0$ and hence $\lambda_i([k])$ and $\lambda_j([k])$ converging to λ_0, two different cases for the eigenfunctions can occur: either the two eigenfunctions $\psi([k], \lambda_i(k))$ and $\psi([k], \lambda_j(k))$ are linearly independent for $k \to k_0$ or they are linearly dependent. If they are linearly dependent, then also $\varphi([k], \lambda_i)$ and $\varphi([k], \lambda_j)$ are linearly dependent for $k \to k_0$ since we can locally represent the Schrödinger operator as a matrix and the transposed Schrödinger operator as the transposed matrix of the first one, see Corollary 1.16. For brevity, we can even assume that the eigenfunctions become equal for $k \to k_0$, which can be reached by normalizing the eigenfunctions. From this we want to deduce that the denominator of P vanishes at $([k_0], \lambda_0)$. Let us consider a small neighborhood $U \subset \mathbb{C}^2/\Gamma^*$ of $[k_0]$ such that no other sheets than the sheets corresponding to λ_i and λ_j are contained in the preimage of U with respect to the covering (3.5) and such that the preimage of $[k_0]$ is also the only intersection point of the two sheets contained in U. Then $\lambda_i([k]) \neq \lambda_j([k])$ for $[k] \in U \setminus \{[k_0]\}$ and with $A([k])$ being the local matrix representation of the Schrödinger operator with fixed potential u introduced in Theorem 1.14, one has for $[k] \in U \setminus \{[k_0]\}$

$$\begin{aligned} &\langle\!\langle (\lambda_i([k]) - \lambda_j([k])\varphi([k], \lambda_i), \psi([k], \lambda_j) \rangle\!\rangle = \\ &\qquad = \langle\!\langle A^T([k])\varphi([k]\lambda_i), \psi([k], \lambda_j) \rangle\!\rangle - \langle\!\langle \varphi([k], \lambda_i), A([k])\psi([k], \lambda_j) \rangle\!\rangle = 0. \end{aligned} \tag{3.6}$$

Ergo, $\langle\!\langle \varphi([k]\lambda_i), \psi([k], \lambda_j) \rangle\!\rangle = 0$ for $k \in U \setminus \{[k_0]\}$. Since the eigenfunctions $\psi([k], \lambda_j)$ and $\varphi([k], \lambda_i)$ are holomorphic in k, so in particular continuous, we can extend $\langle\!\langle \varphi([k], \lambda_i), \psi([k], \lambda_j) \rangle\!\rangle$ by zero to

$[k_0]$. Thus, the denominator of P vanishes at $[k_0]$, and so P has a pole at $([k_0], \lambda_0)$.

□

Note that in the above proof, linear independence of the eigenfunctions at the considered point $([k_0], \lambda([k_0]))$ would yield that this point is a singularity of $B(u)/\Gamma^*$. Because if $B(u)/\Gamma^*$ is smooth at this point, then we can normalize the eigenfunction in such a way that they are holomorphic and then the limit considered in this proof would be continuous which would contradict the assumed linear independence.
The following Lemma is shown in [Schmidt, 2002, Lemma 2.21] for the Dirac operator. In this Lemma is, among others, shown that the operator-valued 1-form $P\,dk_1 \wedge dk_2$ is regular on $B(u)/\Gamma^*$. Later on, we also want to show that another operator-valued 1-form is regular on $B(u)/\Gamma^*$ respectively $F(u)/\Gamma^*$. Therefore, the following definition is necessary.

Definition 3.12. We call an *operator-valued* 1 *form* $P\,dk_1$ from $L^2(\Delta)$ to the finite rank operators on $L^2(\Delta)$ *regular* on a subvariety X if for all $\chi, \xi \in L^2(\Delta)$, there holds that $\langle\langle \xi, P(\chi)\rangle\rangle\, dk_1$ is regular on X. Analogously, *regular* operator valued two-forms $P\,dk_1 \wedge dk_2$ or $P\,dk_1 \wedge d\lambda$ are defined.

Note that evaluating P for different $\chi \neq 0$ does not influence the poles or zeros of this map on $B(u)/\Gamma^*$ since χ is constant in k and λ. For the spectral projection in (3.4), this yields that we want to show that the form

$$\langle\langle \xi, P([k], \lambda)(\chi)\rangle\rangle\, dk_1 \wedge dk_2 = \frac{\langle\langle \varphi([k], \lambda), \chi\rangle\rangle}{\langle\langle \varphi([k], \lambda), \psi([k], \lambda)\rangle\rangle} \langle\langle \xi, \psi([k], \lambda)\rangle\rangle\, dk_1 \wedge dk_2$$

is a regular 2-form on $B(u)/\Gamma^*$ with $\chi, \xi \in L^2(\Delta)$.

Lemma 3.13. *The function P has the following properties:*

(i) The values of P are projections of rank one on $\mathfrak{R}(B(u)/\Gamma^*)$.

(ii) If $([k], \lambda)$ and $([k], \lambda')$ are two different elements of the $B(u)/\Gamma^*$ such that $\lambda \neq \lambda'$, then

$$P([k], \lambda') \circ P([k], \lambda) = 0 = P([k], \lambda) \circ P([k], \lambda'). \tag{3.7}$$

(iii) The local sum of the projection P over all sheets of the Weierstraß covering over $[k]$ which contain an element $([k], \lambda) \in B(u)/\Gamma^*$ is a holomorphic function in a neighborhood of $[k] \in \mathbb{C}^2/\Gamma^*$. The values of this function are projections whose rank is equal to the number of sheets over which that sum is taken. The value of this local sum of P over all sheets containing $([k], \lambda)$ is equal to the generalized spectral projection $\check{P}([k], \lambda)$ as defined in Theorem 1.14 of the Schrödinger operator $-\triangle + u$ with boundary condition $[k]$ which projects onto the generalized eigenspace associated with λ.

(iv) The projection-valued form $P\ dk_1 \wedge dk_2$ is regular.

Proof. (i) Assume that $P([k],\lambda)$ has no pole. Then $P^2([k],\lambda)$ also has no pole and so, by Lemma 3.11, one has on $\mathfrak{R}(B(u)/\Gamma^*)$

$$\begin{aligned}P^2(\chi) &= P\left(\frac{\langle\!\langle\varphi([k],\lambda),\chi\rangle\!\rangle}{\langle\!\langle\varphi([k],\lambda),\psi([k],\lambda)\rangle\!\rangle}\psi([k],\lambda)\right)\\ &= \frac{\langle\!\langle\varphi([k],\lambda),\psi([k],\lambda)\rangle\!\rangle}{\langle\!\langle\varphi([k],\lambda),\psi([k],\lambda)\rangle\!\rangle}\frac{\langle\!\langle\varphi([k],\lambda),\chi\rangle\!\rangle}{\langle\!\langle\varphi([k],\lambda),\psi([k],\lambda)\rangle\!\rangle}\psi([k],\lambda) = P(\chi).\end{aligned}$$

The image of $P([k],\lambda)$ is spanned by $\psi([k],\lambda)$ if λ is a non-degenerated eigenvalue. At the set of points $([k],\lambda)\in B(u)/\Gamma^*$ such that λ is a degenerated eigenvalue, Lemma 3.11 shows that P is not defined. So $P^2 = P$ and P is a projection of rank one on $\mathfrak{R}(B(u)/\Gamma^*)$.

(ii) Let λ_0 be an r-fold eigenvalue at $([k_0],\lambda_0)\in B(u)/\Gamma^*$ and let again $A([k])$ be the matrix-valued, holomorphic function which represents the Schrödinger operator on an open neighborhood U of $k_0\in\mathbb{C}^2$. Due to Corollary 2.29(c), $A([k])$ has generically pairwise different eigenvalues. So we can assume that $A([k])$ is an $r\times r$-matrix with pairwise different eigenvalues $\lambda_1,\dots,\lambda_r$ on $U\setminus\{k_0\}$. Then the eigenfunctions $\psi([k],\lambda_1),\dots,\psi([k],\lambda_r)$ span an r-dimensional vector space for $k\in U\setminus\{k_0\}$ and the eigenfunctions $\varphi([k],\lambda_1),\dots,\varphi([k],\lambda_r)$ of $A^T([k])$ yield the dual basis. Analogous calculations as in (3.6) show that for all $1\le i\ne j\le r$, it is $P([k],\lambda_i)\circ P([k],\lambda_j) = 0 = P([k],\lambda_j)\circ P([k],\lambda_i)$. So (3.7) holds at all $([k],\lambda)$ at which the meromorphic function $P([k],\lambda)$ is defined and thus on all of U. Since we have seen in Corollary 2.29(c) that the set of points $([k],\lambda)\in B(u)/\Gamma^*$ such that λ is non-degenerate is open and dense, we conclude that (3.7) holds on all of $B(u)/\Gamma^*$.

(iii) We use again that $B(u)/\Gamma^*$ is locally an r-sheeted Weierstraß covering over $[k_0]\in\mathbb{C}^2/\Gamma^*$ and define the generalized eigenprojection as in the proof of Theorem 1.14

$$\check{P}([k],\lambda) := \sum_{i=1}^{r} P([k],\lambda_i).$$

We will now use the same notation as in the proof of this theorem. Let λ_0 be a fixed, non-degenerated eigenvalue of $-\triangle+u$ with boundary condition $[k_0]$, i.e. $([k_0],\lambda_0)\in B(u)/\Gamma^*$. Let moreover K be a small open neighborhood of $[k_0]$ and let $B_\varepsilon(\lambda_0)$ be a neighborhood of λ_0 which is so small that no other sheets of $B(u)/\Gamma^*$ are contained in $K\times B_\varepsilon(\lambda_0)$. Then the generalized eigenspace and the 'common' eigenspace coincide, and so there is only one projection in the above sum and $P([k],\lambda) = \check{P}([k],\lambda)$ for all $([k],\lambda)\in K\times B_\varepsilon(\lambda_0)$.

For λ_0 a degenerated eigenvalue of $A([k])$, i.e. at a zero of order r of $\det(A([k])-\lambda\mathbb{1})$, the neighborhoods K and $B_\varepsilon(\lambda_0)$ can be chosen so small that $([k_0],\lambda_0)$ is the only branch point in $K\times B_\varepsilon(\lambda_0)$ and such that only the r sheets colliding at the branch point corresponding to λ_0 are contained inside of $K\times B_\varepsilon(\lambda_0)$. At the branch point λ_0 with respect to the covering (3.5), the image of $\check{P}$ is r-dimensional. Furthermore, Theorem 1.14 yields that P is locally holomorphic in k away from the branch point. It is also shown in the proof of Theorem 1.14 that the dimension of the image of $\check{P}$ stays the same in the neighborhood of the branch point and is spanned by the images of $\sum_{i=1}^{r} P([k],\lambda_i)$. By continuity, $\check{P}([k],\lambda) = \sum_{i=1}^{r} P([k],\lambda_i)$ is holomorphic in K and the rank of $\check{P}$ equals r.

(iv) Let the local holomorphic covering (3.5) of $B(u)/\Gamma^*$ be r-sheeted over $([k_0],\lambda_0) \in B(u)/\Gamma^*$. Then the Weierstraß Preparation Theorem [de Jong and Pfister, 2012, Theorem 3.2.4] yields that we can consider $\det(A([k]) - \lambda\mathbb{1})$ as a polynomial of degree r in λ with coefficients which are holomorphic on an open subset of $[k_0]$, where the leading coefficient is equal to 1 and all other coefficients vanish at $[k_0]$. By the Weierstraß Division Theorem [de Jong and Pfister, 2012, Theorem 3.2.3], every holomorphic function f on $B(u)/\Gamma^*$ can locally be represented as

$$f([k],\lambda) = \sum_{i=0}^{r-1} a_i([k])\lambda^i,$$

where the coefficients $a_i([k])$ are holomorphic functions on an open subset of $[k_0] \in \mathbb{C}^2/\Gamma^*$. With this, we show next that for $\chi \in C^\infty(\varDelta,\mathbb{C})$ and locally around $[k_0] \in \mathbb{C}^2/\Gamma^*$, the sum over the sheets of this covering of $P(\chi)$ times any holomorphic function is a holomorphic function on some open subset of $[k] \in \mathbb{C}^2/\Gamma^*$ with values in the finite rank operators in $C^\infty(\varDelta,\mathbb{C})$. Because of $(-\triangle + u)P_j = P_j$, $P_j^2 = P_j$ and $\sum\limits_{j=1}^{r} P_j = \mathbb{1}$ one has, using the local matrix representation of $-\triangle_k + u$ as in the proof of Theorem 1.14 and leaving the argument $([k],\lambda)$ of the projections as well as the evaluation of P at χ away,

$$\begin{aligned}
\check{P}\cdot f([k],\lambda) &= \sum_{j=1}^{r} P_j f([k],\lambda) = \sum_{j=1}^{r} P_j \sum_{i=0}^{r-1} a_i([k])\lambda^i = \sum_{j=1}^{r}\sum_{i=0}^{r-1} a_i([k])P_j\lambda^i \\
&= \sum_{j=1}^{r}\sum_{i=0}^{r-1} a_i([k])\lambda P_j\lambda^{i-1} == \sum_{j=1}^{r}\sum_{i=0}^{r-1} a_i([k])(-\triangle + u)P_j\lambda^{i-1} = \\
&= \sum_{j=1}^{r}\sum_{i=0}^{r-1} a_i([k])A([k])P_j\lambda^{i-1} = \ldots = \sum_{i=0}^{r-1} a_i([k])A([k])^i \underbrace{\sum_{j=1}^{r} P_j}_{=\mathbb{1}} = \sum_{i=0}^{r-1} A([k])^i a_i([k]).
\end{aligned}$$

Therefore, locally, the whole sum over the sheets is holomorphic and stays holomorphic in $[k]$ when considering $\frac{\langle\!\langle \varphi([k],\lambda),\chi\rangle\!\rangle}{\langle\!\langle \varphi([k],\lambda),\psi([k],\lambda)\rangle\!\rangle}\langle\!\langle \xi,\psi([k],\lambda)\rangle\!\rangle$ for $\xi \in C^\infty(\varDelta,\mathbb{C})$ instead of $P(\chi)$. So by Lemma 3.6, $P\ dk_1 \wedge dk_2$ is regular in sense of Definition 3.12.

□

Until now, we considered the Bloch variety as a covering space over $([(k_1,k_2)],\lambda) \to [(k_1,k_2)]$. We now explain why it follows from Theorem 1.14 that we can also consider the covering $([k_1,k_2],\lambda) \mapsto (k_1,\lambda) \in \mathbb{C}^2$. Since Γ^* is discrete, we can consider $B(u)/\Gamma^*$ locally as $B(u)$, and therefore we can write $([k_1,k_2],\lambda)\mapsto(k_1,\lambda)\in\mathbb{C}^2$. In the proof of this theorem, it is shown that $B(u)/\Gamma^*$ is locally described by the zero set of

$$\{([k],\lambda)\in\mathbb{C}^2/\Gamma^*\times\mathbb{C}\mid\det(A([k])-\lambda\mathbb{1})=0\}.$$

Let us denote this locally defined function as $f([k],\lambda):=\det(A([k])-\lambda\mathbb{1})$. Due to (3.1), $dk_1\wedge dk_2=\frac{\partial f/\partial\lambda}{\partial f/\partial k_2}\,d\lambda\wedge\,dk_1=\frac{\partial k_2}{\partial\lambda}\,d\lambda\wedge\,dk_1$, where the last equality sign holds due to the implicit function Theorem [de Jong and Pfister, 2012, Theorem 3.3.1] and the branch points of the covering $([k],\lambda)\mapsto k_2([k_1],\lambda)$ are exactly those points for which $\partial k_2/\partial k_1=0$ and $\partial k_2/\partial\lambda=0$. We know from Corollary 2.29 and 3.11 that these are discrete. At all other points $([k_0],\lambda_0)\in B(u)/\Gamma^*$, both partial derivatives $\partial f/\partial\lambda$ and $\partial f/\partial k_2$ are not identically zero. So we may consider the subvariety $B(u)/\Gamma^*$ in a small open neighborhood of these point either as a covering space over $[(k_1,k_2)]\in\mathbb{C}^2/\Gamma^*$ or as a covering space over $(k_1,\lambda)\in\mathbb{C}\times\mathbb{C}$. Hence, $([k_1,k_2],\lambda)\mapsto(k_1,\lambda)\in\mathbb{C}^2$ is indeed a covering.

From this we will now start to construct a projection P_∂ which is related to this second covering and has all the properties we wish for to get a regular form on $B(u)/\Gamma^*$ as covering over $([k],\lambda)\mapsto(k_1,\lambda)$ such that we can restrict it to $\lambda=0$ to obtain a regular form on $F(u)/\Gamma^*$.

Lemma 3.14. *Let $\psi([k],\lambda)$ be an eigenfunction of the Schrödinger operator $-\triangle+u$ corresponding to an element $([k],\lambda)\in B(u)/\Gamma^*$ and let $\varphi([k'],\lambda)$ be an eigenfunction of the transposed Schrödinger operator $(-\triangle+u)^T$ corresponding to another element $([k'],\lambda)\in B(u)/\Gamma^*$. Then there holds:*

(i) For fixed $([k],\lambda),([k'],\lambda)\in B(u)/\Gamma^*$, the 1-form

$$\omega:=\big((\partial_y\varphi([k'],\lambda))\psi([k],\lambda)-\varphi([k'],\lambda)(\partial_y\psi_k)\big)\,dx-\\ -\big((\partial_x\varphi([k'],\lambda))\psi([k],\lambda)-\varphi([k'],\lambda)(\partial_x\psi([k],\lambda))\big)\,dy$$

on $\mathbb{R}^2$ is closed, where $\psi([k],\lambda)=\psi([k],\lambda,(x,y))$ and $\varphi([k'],\lambda))=\varphi([k'],\lambda,(x,y))$.

(ii) If $k_1=k_1'\mod\mathbb{Z}$, then for all $p=(x,y)^T\in\mathbb{R}^2$ one has

$$\int_p^{p+\binom{1}{0}}\omega=\frac{\langle\!\langle\partial_y\varphi([k'],\lambda),\psi([k],\lambda)\rangle\!\rangle-\langle\!\langle\varphi([k'],\lambda),\partial_y\psi([k],\lambda)\rangle\!\rangle}{|\check{\gamma}_2|}.\tag{3.8}$$

(iii) If additionally to $k_1=k_1'\mod\mathbb{Z}$ it is $[k]\neq[k']$ for $[k],[k']\in\mathbb{C}^2/\Gamma^*$, then $\int_p^{p+\binom{1}{0}}\omega=0$.

Proof. (i) For all $(x,y)\in\Delta$, it is

$$(-\triangle+u)\psi([k],\lambda)=\lambda\psi([k],\lambda)\quad\text{and}\quad(-\triangle+u)^T\varphi([k'],\lambda)=\lambda\varphi([k'],\lambda).$$

For clarity, we will omit the dependence of ψ on $([k],\lambda)$ as well as the dependence of φ on

$([k'], \lambda)$ in the notation in the rest of this proof. Then ω is closed because

$$\begin{aligned} d\omega =& \left(\partial_y^2\varphi\psi + \partial_y\varphi\partial_y\psi - \partial_y\varphi\partial_y\psi - \varphi\partial_y^2\psi - \varphi\partial_x^2\psi - \partial_x\varphi\partial_x\psi + \partial_x\varphi\partial_x\psi + \partial_x^2\varphi\psi\right)\, dy \wedge dx \\ =& \left(\partial_x^2\varphi + \partial_y^2\varphi\psi - \varphi\partial_x^2\psi + \partial_y^2\psi\right)\, dy \wedge dx = (\triangle\varphi\psi - \varphi \triangle \psi)\, dy \wedge dx \\ =& (u-\lambda)(\varphi\psi - \varphi\psi)\, dy \wedge dx = 0. \end{aligned}$$

(ii) The closedness of ω and applying Stokes' Theorem yields that for $p \neq p' \in \mathbb{R}^2$, there holds

$$0 = \int_p^{p+\binom{1}{0}} \omega + \int_{p+\binom{1}{0}}^{p'+\binom{1}{0}} \omega + \int_{p'+\binom{1}{0}}^{p'} \omega + \int_{p'}^{p} \omega.$$

So by the quasiperiodicity of ψ and φ with respect to Γ together with the preliminary $e^{-2\pi i k_1'} \cdot e^{2\pi i k_1} = 1$,

$$\int_p^{p'} \omega = \int_p^{p'} (\partial_y\varphi\psi - \varphi\partial_y\psi)\, dx = \int_{p+\binom{1}{0}}^{p'+\binom{1}{0}} (\partial_y\varphi\psi - \varphi\partial_y\psi)\, dx = \int_{p+\binom{1}{0}}^{p'+\binom{1}{0}} \omega.$$

Hence,

$$\int_p^{p+\binom{1}{0}} \omega = \int_{p'+\binom{1}{0}}^{p+\binom{1}{0}} \omega + \int_{p'}^{p'+\binom{1}{0}} \omega + \int_p^{p'} \omega = \int_{p'+\binom{1}{0}}^{p+\binom{1}{0}} \omega + \int_{p'}^{p'+\binom{1}{0}} \omega + \int_{p+\binom{1}{0}}^{p'+\binom{1}{0}} \omega = \int_{p'}^{p'+\binom{1}{0}} \omega,$$

so this integral does not depend on the base point p. Let $\gamma : [0,1]$, $t \mapsto t\check{\gamma}$. Then $dy = \check{\gamma}_2 dt$ and so

$$\begin{aligned} \int_p^{p+\binom{1}{0}} (\partial_y\varphi\psi - \varphi\partial_y\psi)\, dx =& \int_0^1 \int_{p+\gamma(t)}^{p+\binom{1}{0}+\gamma(t)} (\partial_y\varphi\psi - \varphi\partial_y\psi)\, dx \wedge dt \\ =& \frac{1}{|\check{\gamma}_2|} \int_{\Delta} (\partial_y\phi\psi - \varphi\partial_y\psi)\, dx \wedge dy = \frac{\langle\!\langle \partial_y\phi, \psi \rangle\!\rangle - \langle\!\langle \varphi, \partial_y\psi \rangle\!\rangle}{|\check{\gamma}_2|}. \end{aligned}$$

(iii) The last statement follows immediately from the independence of the integral from the base point shown in (ii) together with the quasiperiodicity of the eigenfunctions. Due to the first, it is

$$\int_p^{p+\binom{1}{0}} (\partial_y\varphi\psi - \varphi\partial_y\psi)\, dx = \int_{p+\check{\gamma}}^{p+\check{\gamma}+\binom{1}{0}} (\partial_y\varphi\psi - \varphi\partial_y\psi)\, dx \tag{3.9}$$

and due to the second,

$$e^{2\pi\iota\langle k-k', \check{\gamma}\rangle} \int_p^{p+\binom{1}{0}} (\partial_y\varphi\psi - \varphi\partial_y\psi)\, dx = \int_{p+\check{\gamma}}^{p+\binom{1}{0}+\check{\gamma}} (\partial_y\varphi\psi - \varphi\partial_y\psi)\, dx.$$

Both of these equations can only hold simultaneously for $[k] \neq [k']$ if the integrals on both sides of (3.9) vanish.

□

Motivated by the fact that the integral in (3.8) does not depend on the base point p, we define another bilinear form which is "weighted" by the derivatives of the eigenfunctions $\psi([k],\lambda)$ and $\varphi([k'],\lambda)$ in y-direction on $C^\infty_{[-k]}(\Delta,\mathbb{C})\times C^\infty_{[k]}(\Delta,\mathbb{C})$ for two functions f and g the bilinear form

$$\langle\!\langle f,g\rangle\!\rangle_\partial := \langle\!\langle \partial_y f,g\rangle\!\rangle - \langle\!\langle f,\partial_y g\rangle\!\rangle = 2\langle\!\langle \partial y f,g\rangle\!\rangle = -2\langle\!\langle f,\partial_y g\rangle\!\rangle,$$

where $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ is the complex bilinear form introduced in (1.5). This bilinear form will be helpful to obtain a regular 1-form on the Fermi curve. Similarly this is done in [Krichever, 1989, Chapter 3], where the reconstruction of a potential from given data is shown for the two-dimensional Schrödinger operators in terms of θ-functions.

Let P_∂ be the map from $B(u)/\Gamma^*$ into the finite rank operators on $L^2(\Delta)$ which is defined by

$$P_\partial([k],\lambda)(\chi) = \frac{2\langle\!\langle \partial_y\varphi([k],\lambda),\chi\rangle\!\rangle}{\langle\!\langle \varphi([k],\lambda),\psi([k],\lambda)\rangle\!\rangle_\partial}\psi([k],\lambda).$$

Again, it is easy to see that the definition of P_∂ does not depend on the normalization of the ψ and φ on $B(u)/\Gamma^*$ since multiplication of ψ or φ with a non-vanishing function which is meromorphic in $([k],\lambda)$ cancels out in the operator P_∂.

Now, we give reason why P_∂ is meromorphic on $B(u)/\Gamma^*$. P_∂ is invariant under translation of k by $\kappa\in\Gamma^*$ since ψ and φ are invariant under these translations. Note that φ and ψ as well as their derivatives in the direction of y are asymptotically free, compare Lemma 2.23. We will show in the proof of Lemma 4.7 that for every $\varepsilon>0$, the difference between $\langle\!\langle \varphi_N,\psi_N\rangle\!\rangle_\partial$ and $\langle\!\langle e^{-2\pi\iota\langle k,\binom{x}{y}\rangle}, e^{2\pi\iota\langle k,\binom{x}{y}\rangle}\rangle\!\rangle_\partial$ on $\Re(F(u)/\Gamma^*\cap\mathbb{C}^2_\delta)$ becomes smaller than this ε if $\delta>0$ is sufficiently small. Hereby, the index N indicates the eigenfunctions which are normalized in such a way that their values at $(x,y)=(0,0)$ are equal to one. Furthermore, it is also shown in the proof of this theorem that $\langle\!\langle \varphi^0([k]),\psi^0([k])\rangle\!\rangle_\partial\neq 0$ for $\|\operatorname{Im}(k)\|\to\infty$. Since we know from Corollary 2.29(b) that $B(u)/\Gamma^*$ has at most two connected components which each contain one open end of $B(u)/\Gamma^*$, we can deduce from this that $\langle\!\langle \varphi_N,\psi_N\rangle\!\rangle_\partial$ does not vanish identically on any of the maximal two connected components of $B(u)/\Gamma^*$. Because P_∂ is independent of the normalization of ψ and φ, this shows that P_∂ defines a meromorphic function on $B(u)/\Gamma^*$. We now show how the denominators of P_∂ and P are correlated. This is the key ingredient to deduce from the regularity of $P\,dk_1\wedge dk_2$ that also $P_\partial\,dk_1\wedge d\lambda$ is a regular operator-valued 2-form on $B(u)/\Gamma^*$ in the following lemma.

Proposition 3.15. *Let $\psi=\psi([k],\lambda)$ be an eigenfunction of $-\triangle+u$ corresponding to the eigenvalue λ with quasiperiodicity $[k]$ and let $\varphi:=\varphi([k],\lambda)$ be an eigenfunction of $(-\triangle+u)^T$ corresponding to the same eigenvalue with quasiperiodicity $[-k]$. Then*

$$2\pi\iota\langle\!\langle \varphi,\psi\rangle\!\rangle_\partial = \frac{\partial\lambda}{\partial k_2}\langle\!\langle \varphi,\psi\rangle\!\rangle. \tag{3.10}$$

Proof. We consider the Schrödinger operator $S_k := -\triangle_k + u$ as in (1.6) and its transpose $S_k^T := (-\triangle_k + u)^T = -\triangle_{-k} + u$ as in (1.6). Taking formally the derivatives of S_k and S_k^T into the direction of k_2 yields

$$\frac{\partial S_k}{\partial k_2} = -4\pi\iota\partial_y + 8\pi^2 k_2 = 4\pi\iota(-\partial_y - 2\pi\iota k_2),$$
$$\frac{\partial S_k^T}{\partial k_2} = 4\pi\iota\partial_y + 8\pi^2 k_2 = 4\pi\iota(\partial_y - 2\pi\iota k_2).$$

Furthermore, it is shown in Lemma 1.2 that the eigenfunctions $\psi := \psi([k],\lambda)$ of $-\triangle + u = \lambda$ with boundary condition $[k]$ and $\varphi := \varphi([k],\lambda)$ of $(-\triangle + u)^T = \lambda$ with boundary condition $[-k]$ differ from the eigenfunctions $\psi_k := \psi_k(\lambda)$ of S_k and $\varphi_k := \varphi_k(\lambda)$ of S_k^T by multiplication with a phase, i.e. $\psi_k = e^{-2\pi\iota\langle k, \binom{x}{y}\rangle}\psi$ and $\varphi_k = e^{2\pi\iota\langle k, \binom{x}{y}\rangle}\varphi$. Therefore,

$$\partial_y\psi_k = -2\pi\iota k_2 e^{-2\pi\iota\langle k, \binom{x}{y}\rangle}\psi + e^{-2\pi\iota\langle k, \binom{x}{y}\rangle}\partial_y\psi \quad\text{and}\quad \partial_y\varphi_k = 2\pi\iota k_2 e^{2\pi\iota\langle k, \binom{x}{y}\rangle}\varphi + e^{2\pi\iota\langle k, \binom{x}{y}\rangle}\partial_y\varphi.$$

So with $\langle\!\langle \varphi_k, \psi_k\rangle\!\rangle = \langle\!\langle \varphi, \psi\rangle\!\rangle$ it is

$$\begin{aligned}
\frac{1}{2\pi\iota}\frac{\partial\lambda}{\partial k_2}\langle\!\langle \varphi, \psi\rangle\!\rangle &= \frac{1}{2\pi\iota}\frac{\partial\lambda}{\partial k_2}\langle\!\langle \varphi_k, \psi_k\rangle\!\rangle = \frac{1}{4\pi\iota}\left(\left\langle\!\!\left\langle \frac{\partial\lambda}{\partial k_2}\varphi_k, \psi_k\right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \varphi_k, \frac{\partial\lambda}{\partial k_2}\psi_k\right\rangle\!\!\right\rangle\right) \\
&= \frac{1}{4\pi\iota}\left(\left\langle\!\!\left\langle \frac{\partial S_k^T}{\partial k_2}\varphi_k, \psi_k\right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle S_k^T\frac{\partial\varphi_k}{\partial k_2}, \psi_k\right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \varphi_k, \frac{\partial S_k}{\partial k_2}\psi_k\right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \varphi_k, S_k\frac{\partial\psi_k}{\partial k_2}\right\rangle\!\!\right\rangle\right) \\
&= \frac{1}{4\pi\iota}\left(\left\langle\!\!\left\langle \frac{\partial S_k^T}{\partial k_2}\varphi_k, \psi_k\right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \frac{\partial\varphi_k}{\partial k_2}, S_k\psi_k\right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \varphi_k, \frac{\partial S_k}{\partial k_2}\psi_k\right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle S_k^T\varphi_k, \frac{\partial\psi_k}{\partial k_2}\right\rangle\!\!\right\rangle\right) \\
&= \frac{1}{4\pi\iota}\left(\left\langle\!\!\left\langle \frac{\partial S_k^T}{\partial k_2}\varphi_k, \psi_k\right\rangle\!\!\right\rangle + \lambda\left\langle\!\!\left\langle \frac{\partial\varphi_k}{\partial k_2}, \psi_k\right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \varphi_k, \frac{\partial S_k}{\partial k_2}\psi_k\right\rangle\!\!\right\rangle + \lambda\left\langle\!\!\left\langle \varphi_k, \frac{\partial\psi_k}{\partial k_2}\right\rangle\!\!\right\rangle\right) \\
&= \frac{1}{4\pi\iota}\left(\left\langle\!\!\left\langle \frac{\partial S_k^T}{\partial k_2}\varphi_k, \psi_k\right\rangle\!\!\right\rangle + \lambda\left\langle\!\!\left\langle \frac{\partial\varphi_k}{\partial k_2}, \psi_k\right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \varphi_k, \frac{\partial S_k}{\partial k_2}\psi_k\right\rangle\!\!\right\rangle - \lambda\left\langle\!\!\left\langle \frac{\partial\varphi_k}{\partial k_2}, \psi_k\right\rangle\!\!\right\rangle\right) \\
&= \frac{1}{4\pi\iota}\left(\left\langle\!\!\left\langle \frac{\partial S_k^T}{\partial k_2}\varphi_k, \psi_k\right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \varphi_k, \frac{\partial S_k}{\partial k_2}\psi_k\right\rangle\!\!\right\rangle\right) \\
&= \langle\!\langle (\partial_y - 2\pi\iota k_2)\varphi_k, \psi_k\rangle\!\rangle + \langle\!\langle \varphi_k, (-\partial_y - 2\pi\iota k_2)\psi_k\rangle\!\rangle \\
&= \langle\!\langle e^{2\pi\iota\langle k, \binom{x}{y}\rangle}\partial_y\varphi, e^{-2\pi\iota\langle k, \binom{x}{y}\rangle}\psi\rangle\!\rangle - \langle\!\langle e^{2\pi\iota\langle k, \binom{x}{y}\rangle}\varphi, e^{-2\pi\iota\langle k, \binom{x}{y}\rangle}\partial_y\psi\rangle\!\rangle \\
&= \langle\!\langle \varphi, \psi\rangle\!\rangle_\partial.
\end{aligned}$$

□

Even though P_∂ is not a spectral projection anymore, similar properties can be shown. In addition, this new projection can be used to define a regular 1-form $P_\partial\, dk_1$ on the Fermi curve $F(u)/\Gamma^*$, because on the Bloch variety, it is defined with respect of the covering $B(u) \mapsto \mathbb{C}^2$, $(k,\lambda) \mapsto (k_1,\lambda)$. This can be restricted to $\lambda = 0$.

Lemma 3.16. *The function P_∂ has the following properties*

(i) The values of P_∂ are projections of rank 1 on $\mathfrak{R}(B(u)/\Gamma^*)$.

(ii) If $([k],\lambda)$ and $([k'],\lambda)$ are two different elements of $B(u)/\Gamma^*$ such that $k_1 = k_1' \mod \mathbb{Z}$, then

$$P_\partial([k],\lambda) \circ P_\partial([k'],\lambda) = 0 = P_\partial([k'],\lambda) \circ P_\partial([k],\lambda).$$

(iii) Locally, the Bloch variety is a finite-sheeted covering over $(k_1,\lambda) \in \mathbb{C}^2$. The local sum $\check{P}_\partial$ of P_∂ over all sheets which contain an element $([k],\lambda) \in B(u)/\Gamma^*$ is a holomorphic function on some open subset of $(k_1,\lambda) \in \mathbb{C}^2$ with values in the finite rank projections on $C^\infty(\Delta,\mathbb{C})$. Moreover, the rank of any of these projections is equal to the number of sheets over which the sum is taken.

(iv) The 2-form $P_\partial\, dk_1 \wedge d\lambda$ is a regular form on $B(u)/\Gamma^*$ and the 1-form $P_\partial\, dk_1$ is a regular 1-form on $F(u)/\Gamma^*$.

(v) For all elements $([k],\lambda) \in B(u)/\Gamma^*$, there exists a unique projection $\check{P}_\partial([k],\lambda)$ which locally adds all values of $P_\partial([k],\lambda)$ over all sheets of $B(u)/\Gamma^*$ that contain $([k],\lambda)$.
Let $([k],\lambda), ([k'],\lambda) \in B(u)/\Gamma^*$. If $([k],\lambda) \neq ([k'],\lambda)$ and $k_1 = k_1' \mod \mathbb{Z}$, then

$$\check{P}_\partial([k],\lambda) \circ \check{P}_\partial([k'],\lambda) = 0 = \check{P}_\partial([k'],\lambda) \circ \check{P}_\partial([k],\lambda).$$

(vi) If χ is an eigenfunction of $-\triangle + u$ with boundary condition $[k]$ corresponding to $([k],\lambda) \in B(u)/\Gamma^*$, then the range of $\check{P}_\partial$ contains χ.

Proof. (i) This is shown the same way as Lemma 3.13(i).

(ii) This follows from Lemma 3.14 (ii) and (iii) since there, it is shown that for $([k],\lambda), ([k'],\lambda) \in B(u)/\Gamma^*$ with $[k] \neq [k']$ and $k_1 = k_1' \mod \mathbb{Z}$, one has $\langle\!\langle \varphi([k],\lambda), \psi([k'],\lambda) \rangle\!\rangle_\partial = 0$.

(iii) and (iv) We will deduce the regularity of $P_\partial\, dk_1 \wedge d\lambda$ on $B(u)/\Gamma^*$ from the regularity of $P\, dk_1 \wedge dk_2$ shown in Lemma 3.13. So using (3.10)

$$\begin{aligned} P([k],\lambda)\, dk_1 \wedge dk_2 &= \frac{\partial k_2}{\partial \lambda} P([k],\lambda)\, dk_1 \wedge d\lambda = \frac{\partial k_2}{\partial \lambda} \frac{\langle\!\langle \varphi([k],\lambda), \cdot \rangle\!\rangle}{\langle\!\langle \varphi([k],\lambda), \psi([k],\lambda) \rangle\!\rangle} \psi([k],\lambda)\, dk_1 \wedge d\lambda \\ &= \frac{\cancel{\partial k_2}}{\cancel{\partial \lambda}} \frac{\langle\!\langle \varphi, \cdot \rangle\!\rangle}{2\pi\iota \cancel{\left(\frac{\partial \lambda}{\partial k_2}\right)^{-1}} \langle\!\langle \varphi([k],\lambda), \psi([k],\lambda) \rangle\!\rangle_\partial} \psi([k],\lambda)\, dk_1 \wedge d\lambda \\ &= \frac{\langle\!\langle \varphi, \cdot \rangle\!\rangle}{2\pi\iota \langle\!\langle \varphi([k],\lambda), \psi([k],\lambda) \rangle\!\rangle_\partial} \psi([k],\lambda)([k],\lambda)\, dk_1 \wedge d\lambda. \end{aligned} \tag{3.11}$$

To see that
$$P_\partial \, dk_1 \wedge d\lambda = \frac{2\langle\!\langle \partial_y \varphi([k],\lambda), \cdot \rangle\!\rangle}{\langle\!\langle \varphi([k],\lambda), \psi([k],\lambda)\rangle\!\rangle} \psi([k],\lambda) \, dk_1 \wedge d\lambda \tag{3.12}$$
is regular, note that the only difference between the expression on the right hand side of (3.11) and on the right hand side of (3.12) are the enumerators of the fractions. So the next step is to show that deriving φ into the direction of y does not change the pole order of φ on $B(u)/\Gamma^*$. Therefore, remember that the regularity of $P \, dk_1 \wedge dk_2$ does not depend on the normalization of the eigenfunction of $(-\triangle + u)^T$ with boundary condition $[k]$. Let $\varphi([k],(x,y))$ be an eigenfunction of $-\triangle + u$ for fixed λ with the same boundary condition normalized in such a way that it is locally holomorphic and nowhere zero in $([k],\lambda)$, but at an isolated point $([k_0],\lambda) \in B(u)/\Gamma^*$ has a zero at $(x,y) = (0,0)$, i.e. $\psi([k_0],(0,0)) = 0$. We then normalize this eigenfunction as
$$\varphi_N(\cdot,(x,y)) := \frac{\varphi(\cdot,(x,y))}{\varphi(\cdot,(0,0))}.$$
Then the denominator of $\varphi_N([k])$ does not depend on (x,y). Since $\varphi_N([k]) \in C^\infty(\triangle)$, deriving $\varphi_N([k])$ into the direction of y can only cause zeros in the enumerator of $\varphi_N([k],(x,y))$ which can either reduce the pole order of $\varphi_N([k])$ or generate new zeros compared to the zeros $\varphi_N([k])$. These additional zeros could only influence the regularity of the form on the right hand side of (3.12) if $([k],\lambda)$ is a singularity of $B(u)/\Gamma^*$ or a branch point with respect to the covering in (3.5). Due to Lemma 3.6(ii), we can deduce from the regularity of $P \, dk_1 \wedge dk_2$ that for a locally holomorphic function g, also the sum of $g \frac{\langle\!\langle \varphi([k]),\chi\rangle\!\rangle}{\langle\!\langle \varphi([k]),\psi([k])\rangle\!\rangle} \langle\!\langle \psi([k]),\xi\rangle\!\rangle$ over all sheets of the covering (3.5) is holomorphic. Choosing $g \in \mathcal{O}_{F(u)/\Gamma^*}$ such that the zero order of g equals the difference of the zero order of $\partial_y \varphi([k],\lambda)$ minus the zero order of $\varphi([k],\lambda)$ at these points shows that also these additional zeros do not influence the regularity of the form on the right hand side of (3.12). Hence, $P_\partial \, dk_1 \wedge d\lambda$ is regular on $B(u)/\Gamma^*$.
So again due to Lemma 3.6(ii), the values of the local sum of P_∂ over all sheets of $B(u)/\Gamma^*$ which contain $([k],\lambda)$ is holomorphic. The rank of this sum over the projections is equal to the number of sheets of the covering $([k],\lambda) \to [k]$ over which the sum is taken: Formally, every sheet of $B(u)/\Gamma^*$ over $[k]$ has a contribution to the image of the sum over $P_\partial([k],\lambda)$. However, a sheet does not appear in the sum if all of its values are in the kernel of $P_\partial([k],\lambda)$. And we have seen in Lemma 3.14 that this is the case for $[k] \neq [k']$ with $k_1 = k_1' \mod \mathbb{Z}$, i.e. for $([k'],\lambda)$ not on the same sheet as $([k],\lambda)$. Then $\langle\!\langle \varphi([k'],\lambda), \psi([k],\lambda)\rangle\!\rangle = 0$. Due to Theorem 2.28(d), the values of $[k]$ such that $\lambda = 0$ is a degenerated eigenvalue for $([k],0) \in B(u)/\Gamma^*$ are discrete. So we can restrict $P_\partial \, dk_1 \wedge dk_2$ to $F(u)/\Gamma^*$ by setting $\lambda = 0$ and show the regularity of $P_\partial \, dk_1$ as follows: Due to Theorem 2.28(a) and Corollary 2.29, $F(u)/\Gamma^*$ is a one-dimensional variety in $\mathbb{C}^2$ and $B(u)/\Gamma^*$ a two-dimensional variety in $\mathbb{C}^3$, i.e. both are locally defined as the zero set of one holomorphic function. So by Lemma 3.6(i), the regularity

of $P_\partial\, dk_1 \wedge d\lambda$ on $B(u)/\varGamma^*$ yields that $\langle\!\langle \xi, P_\partial([k],\lambda)(\chi)\rangle\!\rangle$ is holomorphic for all $\chi, \xi \in L^2(\varDelta)$. Then also the restriction of this function to $\lambda = 0$ is holomorphic and thus applying again Lemma 3.6(ii) yields that $P_\partial\, dk_1$ is regular on $F(u)/\varGamma^*$.

(v) This property follows immediately from (iii).

(vi) For the proof of the last statement, let χ be an eigenfunction of the Schrödinger operator $-\triangle + u$ corresponding to $([k],\lambda) \in B(u)/\varGamma^*$ and $\varphi := \varphi([k'],\lambda')$ be an eigenfunction of the transposed Schrödinger operator $(-\triangle + u)^T$ corresponding to $([k'],\lambda') \in B(u)/\varGamma^*$ with $k_1 = k_1' \mod \mathbb{Z}$ and $\lambda \neq \lambda'$. Consider

$$\widetilde{\omega} := (\partial_y \phi\chi - \varphi\partial_y\chi)\, dx - (\partial_x\phi\chi - \varphi\partial_x\chi)\, dy$$

on $\mathbb{R}^2$. Similar calculations as the ones in the proof of Lemma 3.14(i) yield that the exterior derivative is given by

$$\mathrm{d}\left[(\partial_y\phi\chi - \varphi\partial_y\chi)\, dx - (\partial_x\phi\psi_k - \chi\partial_x\chi)dy\right] = (\lambda - \lambda')\varphi\chi\, dx \wedge dy.$$

This shows that $\widetilde{\omega}$ is not closed for $\lambda \neq \lambda'$. Applying Stokes Formula leads to

$$(\lambda - \lambda')\langle\!\langle \varphi, \chi\rangle\!\rangle = \int_{\varDelta} d\,\widetilde{\omega} = \int_{\partial\varDelta} \widetilde{\omega} = \int_p^{p+\binom{1}{0}} \widetilde{\omega} + \int_{p+\binom{1}{0}}^{p+\binom{1}{0}+\check{\gamma}} \widetilde{\omega} + \int_{p+\binom{1}{0}+\check{\gamma}}^{p+\check{\gamma}} \widetilde{\omega} + \int_{p+\check{\gamma}}^{p} \widetilde{\omega}.$$

With the quasiperiodicity of φ_k and χ in mind, we get for the first and third term on the right hand side of this sum that

$$\int_{p+\binom{1}{0}+\check{\gamma}}^{p+\check{\gamma}} \widetilde{\omega} = -e^{2\pi\iota\langle k'-k,\check{\gamma}\rangle} \int_p^{p+\binom{1}{0}} \widetilde{\omega}.$$

Taking additionally $k_1 = k_1' \mod \mathbb{Z}$ into account, the second and fourth term of this sum are related by

$$\int_{p+\binom{1}{0}}^{p+\binom{1}{0}+\check{\gamma}} \widetilde{\omega} = -\int_{p+\check{\gamma}}^{p} \widetilde{\omega}.$$

So by Lemma 3.14(ii), it is

$$\begin{aligned}\frac{\left(1 - e^{2\pi\iota\langle k'-k,\check{\gamma}\rangle}\right)}{|\check{\gamma}_2|}\langle\!\langle \varphi, \chi\rangle\!\rangle_\partial &= \left(1 - e^{2\pi\iota\langle k'-k,\check{\gamma}\rangle}\right) \int_p^{p+\binom{1}{0}} \widetilde{\omega} \\ &= \left(1 - e^{2\pi\iota\langle k'-k,\check{\gamma}\rangle}\right) \int_{\varDelta} d\widetilde{\omega} = (\lambda' - \lambda)\langle\!\langle \varphi, \chi\rangle\!\rangle.\end{aligned}$$

This relation is equivalent to

$$\frac{1}{\lambda' - \lambda}\langle\!\langle \varphi, \chi\rangle\!\rangle_\partial = \frac{|\check{\gamma}_2|}{1 - \exp(2\pi\iota\langle k' - k, \check{\gamma}\rangle)}\langle\!\langle \varphi, \chi\rangle\!\rangle.$$

Using the equality $\langle\!\langle\varphi,\psi\rangle\!\rangle_\partial = \frac{1}{2\pi\iota}\frac{\partial\lambda}{\partial k_2}\langle\!\langle\varphi,\psi\rangle\!\rangle$ with $\psi := \psi([k'],\lambda')$ from Proposition 3.15, one sees that this is equivalent to

$$\frac{1}{\lambda'-\lambda}\frac{\langle\!\langle\varphi,\chi\rangle\!\rangle_\partial}{\langle\!\langle\varphi,\psi\rangle\!\rangle_\partial} = \frac{2\pi\iota|\check{\gamma}_2|}{1-\exp(2\pi\iota\langle k'-k,\check{\gamma}\rangle)}\frac{\langle\!\langle\varphi,\chi\rangle\!\rangle}{\frac{\partial\lambda}{\partial k_2}\langle\!\langle\varphi,\psi\rangle\!\rangle}.$$

So finally, we obtain with $d\lambda = \frac{\partial\lambda}{\partial k_2}\,dk_2$ that

$$\frac{1}{\lambda'-\lambda}P_\partial([k'],\lambda')(\chi)\,d\lambda = \frac{1}{1-\exp(2\pi\iota\langle k'-k,\check{\gamma}\rangle)}P([k'],\lambda')(\chi)\,dk_2.$$

The subvariety $\{(k'_1,\lambda')\in\mathbb{C}^2 \mid ([k'],\lambda')\in B(u)/\Gamma^*\}$ can locally be considered as a covering space over $k'_1\in\mathbb{C}$ or over $\lambda'\in\mathbb{C}$. The residue of the right hand side is the value of the local sum of $P(\chi)$ over all sheets which contain (k_1,λ) of the subvariety considered as a covering over $k'_1\in\mathbb{C}$ since it adds up all the values λ over k_1 and the residue of the left hand side is the value of the local sum of $P_\partial(\chi)$ over all sheets which contain (k_1,λ) of the subvariety considered as a covering over $\lambda'\in\mathbb{C}$ since it adds up all the values k_1 over λ. Then the restrictions of the sheets of $B(u)/\Gamma^*$, considered as a covering space over $[k]\in\mathbb{C}^2/\Gamma^*$, to the subvariety $k_1 = \mathit{constant}$ are locally different and also the restriction of the sheets of $B(u)/\Gamma^*$ considered as a covering space over (k_1,λ) with $k_1 = \mathit{constant}$ are locally different. Hence, the residue of the right hand side is equal to $\breve{P}([k],\lambda)(\chi)$ and the residue of the left hand side is equal to $\breve{P}_\partial([k],\lambda)(\chi)$. This proves (vi) because we have seen in Lemma 3.16(vi) that χ is in the range of $\breve{P}([k],\lambda)$ if it is an eigenfunction of $-\triangle+u$ corresponding to $([k],\lambda)$. So it is also in the range of $\breve{P}_\partial([k],\lambda)$.

□

Corollary 3.17. *For every $\chi,\xi\in L^2(\varDelta)$, the form $\frac{\langle\!\langle\varphi,\chi\rangle\!\rangle}{\langle\!\langle\varphi,\psi\rangle\!\rangle_\partial}\langle\!\langle\xi,\psi\rangle\!\rangle\,dk_1$ is regular on $F(u)/\Gamma^*$.*

Proof. This follows exactly by the same means as it is shown in the foregoing proof that $P_\partial\,dk_1$ is a regular operator-valued one-form on $F(u)/\Gamma^*$. □

4. Fermi curves of finite type

As the title of this work suggests, we want to consider so-called 'finite type' potentials $u \in C(\mathbb{R}^2/\Gamma)$. These are potentials such that the normalization $X^\circ(u)$ of $X'(u)$, see Lemma 3.1, can be compactified and such that normalized eigenfunctions of the Schrödinger operator can be lifted to a meromorphic function on this compactified normalization. The second condition we will formulate with help of the so-called middleding which we introduce in Definition 4.4: this is the unique 1-sheeted covering of X' which is the desingularization of X' such that the germs of holomorphic functions on this middleding are defined as the maximal subring of $\mathcal{O}_X$ which acts on the divisor $\mathcal{S}$. It will turn out that a compactifiable middleding leads to the second condition for a potential to be of finite type. We then will show that the lift to the middleding of the regular operator-valued 1-form $P_\partial \, dk_1$, constructed in the foregoing section, is also regular on the latter. After that, we define another class of so-called regular finite type potentials. For these, the middleding and the normalization coincide. Because the normalization of $F(u)/\Gamma^*$ is a Riemann surface, we will be able to define a classical divisor D as in [Forster, 1981, Definition 16.1] on the normalization from the generalized divisor $\mathcal{S}$. For regular finite type potentials, the lift of $P_\partial \, dk_1$ to the normalization is a holomorphic 1-form. In the remainder of this chapter, we then consider only Fermi curves with regular finite type potentials and show several properties of the corresponding divisor D. One central point in this work is that we deduce a connection between the fixed points of the holomorphic involution σ on X, which will turn out to be exactly the two points added to compactify the normalization X°, and the divisor D in terms of a linear equivalence. More precisely, we show that $D + \sigma(D) \simeq K + Q^+ + Q^-$, where K is the canonical divisor on X and $Q^\pm$ are two points which can be added to compactify X° for finite type potentials as we will see hereinafter. That this property holds for the divisor of the normalized eigenfunction is well-known in the common literature, compare for example [Novikov and Veselov, 1984, Veselov, 1984, Novikov and Veselov, 1986], but we could not find a proof of it. We will also show that such a linear equivalence can hold if and only if the involution σ has exactly two fixed points. It is mentioned in [Novikov and Veselov, 1984] that I. R. Shafarevich and V. V. Shokurov pointed out that this relation should hold. For real-valued potentials, we show that the action of τ on D has to leave D invariant. After exploiting the symmetries of the Fermi curve, we also show that the divisor D is non-special and that also all other divisors obtained from normalizing the eigenfunctions to 1 at another point $(x, y) \in \mathbb{R}^2$ then $(0, 0)$ also leads to a non-special divisor.

4.1. Finite type potentials

The normalization $X^\circ(0)$ of $X'(0)$ can be compactified. As before, we consider without loss of generality the representant $\mathcal{R}$ of $X'(0)$ as defined in equation (1.15) with the pairs of distinct points (k_ν^+, k_ν^-) for $\nu \in \Gamma^*$ identified. It has been shown in Theorem 2.34 that $k_\nu^\pm$ is an ordinary double point for all $\nu \in \Gamma^*$ and that these ordinary double points are the only singular points of $F(0)/\Gamma^*$. Then $\Re(\mathcal{R}) = \mathcal{R} \setminus \{k_\nu^\pm \mid \nu \in \Gamma^*\}$ consists of the two connected components $\mathcal{R}^+$ and $\mathcal{R}^-$ as in equation (1.18) with the corresponding double points k_ν^+ respectively k_ν^- taken out. Since $\mathcal{R} \setminus \{k_\nu^\pm \mid \nu \in \Gamma^*\}$ is the product of two linear equations in k_1 and k_2 which are each irreducible, $X^\circ(0)$ has the two connected components

$$X^\circ(0) = (X^+)^\circ \,\dot\cup\, (X^-)^\circ,$$

where

$$(X^+)^\circ := \left\{ \begin{pmatrix} t \\ -\iota t \end{pmatrix} \,\middle|\, t \in \mathbb{C} \right\} \quad \text{and} \quad (X^-)^\circ := \left\{ \begin{pmatrix} s \\ \iota s \end{pmatrix} \,\middle|\, s \in \mathbb{C} \right\}$$

with two different parameters s, t and $(X^+)^\circ \cap (X^-)^\circ = \emptyset$. This is the usual desingularization of double points of singular curves as for example formulated in [de Jong and Pfister, 2012, Example 4.7.7 (2)]. Hence, a small open neighborhood of a double point $k_\nu^\pm$ on $\mathcal{R}$ decays into two disjoint open discs on the normalization $X^\circ(0)$ and each of the two components $(X^\pm)^\circ$ of the normalization is isomorphic to $\mathbb{C}$. Covering each of these components with two open sets

$$U_1^\pm = \mathbb{C} \text{ and } U_2^\pm = (\mathbb{C} \setminus \{0\}) \cup \{Q^\pm\},$$

where

$$Q^+ := \lim_{|t| \to \infty} \begin{pmatrix} t \\ \iota t \end{pmatrix} \quad \text{and} \quad Q^- := \lim_{|s| \to \infty} \begin{pmatrix} s \\ -\iota s \end{pmatrix},$$

yields that the usual one-point compactification [Munkres, 2000, § 29, page 185] of each of these components is given by the local charts $t_1 = t$ on U_1^+, $t_2 = \frac{1}{t}$ on U_2^+ and $s_1 = s$ on U_1^-, $s_2 = \frac{1}{s}$ on U_2^-, where t_2 and s_2 map $U_2^\pm$ homeomorphically to the punctured open disc $\{z \in \mathbb{C} \setminus \{0\} \mid |z| < 1\}$. The respective transition functions $f_{i,j}^\pm : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ with $i, j \in \{1, 2\}$ and $i \neq j$ are defined as

$$f_{1,2}^+ = t_1 \circ t_2^{-1}, \quad f_{2,1}^+ = t_2 \circ t_1^{-1}, \quad f_{1,2}^- = s_1 \circ s_2^{-1}, \quad f_{2,1}^- = s_2 \circ s_1^{-1}.$$

We extend the homeomorphisms t_2 and s_2 to $U_2^\pm \cup \{Q^\pm\} \to \{z \in \mathbb{C} \mid |z| < 1\}$ such that $t_2(Q^+) = 0$ and $s_2(Q^-) = 0$. The new compact Riemann surface $X(0) := X^\circ(0) \cup \{Q^+, Q^-\}$, defined as the union of a complex atlas of $X^\circ(0)$ with the coordinate charts t_2 and s_2 defined on $U_2^\pm \cup \{Q^\pm\}$, is compact. We denote the corresponding compactified connected components of $X(0)$ with $X^+ := (X^+)^\circ \cup \{Q^+\}$ and $X^- := (X^-)^\circ \cup \{Q^-\}$.

In Theorem 2.34, it is shown that for $\delta > 0$ sufficiently small, the only singularities that can occur in $X' \cap (\mathbb{C}^2_\delta/\Gamma^*)$ are double points $k^\pm_\nu(u)$ which are contained in the excluded domains $\mathfrak{e}_\nu$ of $k^\pm_\nu$. It is not possible to compactify the Fermi curve X': Theorem 2.34 implies that the geometric genus of X' will in general be infinite since each handle which contains two branch points contributes by one to the arithmetic genus of X' and there will in general be infinitely many excluded domains containing two branch points accumulating in the two open ends of X'. More precisely, the arithmetic genus of X' is constant under small deformations, compare [Grauert et al., 1994, Theorem III.4.7(c)], and is defined as $g_a = g_g + \delta$, where $\delta := \sum_{q \in S} \delta_q$. So every time a double point is replaced by a handle, the δ-invariant is $\delta - 1$, and so the geometric genus g_g of the former curve gains one in comparison to the latter curve. This motivates the following definition.

Definition 4.1. A potential $u \in C(\mathbb{R}^2/\Gamma)$ is a *compact curve potential* if there exists a $\delta > 0$ such that for all ν in Γ^*_δ every excluded domain $\mathfrak{e}_\nu$ contains a double point $k^\pm_\nu(u)$ of $X'(u)$ and the normalization of the excluded domain $\mathfrak{e}_\nu$ decomposes into two connected components U^+_ν and U^-_ν. With $\pi : X^\circ \to X'$ as defined in Lemma 3.1, each of these components is a disc which is contained in $\pi^{-1}[(\mathbb{C}^2_\delta/\Gamma^*) \setminus \mathcal{U}^\pm_{\varepsilon,\delta}]$ around one of the two points in $\pi^{-1}[\{k^\pm_\nu(u) \mid \nu \in \Gamma^*_\delta\}]$ with $\varepsilon > 0$ as in Remark 2.10 and $U^+_\nu \cap U^-_\nu = \emptyset$.

Lemma 4.2. *For every potential $u \in C(\mathbb{R}^2/\Gamma)$, the normalization $X^\circ(u)$ of the Fermi curve X' can be compactified if and only if u is a compact curve potential as in Definition 4.1. We call the compactified curve $X(u)$.*

Proof. In order to compactify $X^\circ(u)$ by the same means as it is done for $X^\circ(0)$, the two components of $X'(u) \cap \mathbb{C}^2_\delta/\Gamma^*$ which lie in a ε-neighborhood of $X'(0)$ for $\delta > 0$ sufficiently small, compare Corollary 2.6, also need to be separated. This is only possible if all branch points of $X'(u)$ which are contained in $X'(u) \cap (\mathbb{C}^2_\delta/\Gamma^*) \setminus \mathcal{U}^\pm_{\varepsilon,\delta}$, are discrete ordinary double points which is just Definition 4.1 of compact curve potentials. On the other hand, for u not being a compact curve potential, infinitely many branch points do not collide into a double point of X' and the normalization of an open set around these branch points consists only of one connected component. Thus, it is not possible to find a $\delta > 0$ such that $X^\circ(u) \cap \mathbb{C}^2_\delta/\Gamma^*$ is biholomorphic to $X^\circ(0) \cap \mathbb{C}^2_\delta/\Gamma^*$. □

The involutions $\sigma : X' \to X'$ and $\tau_1 : X' \to X'$ as defined in Lemma 1.17 can be lifted to X° and later on analogously also on all other coverings of X'. We will also denote these involutions by σ and τ_1. They can be extended to all of X:

Corollary 4.3. *The involution $\sigma : X' \to X'$ induced by σ in Lemma 1.17(a) can be extended to an involution on X which leaves $Q^\pm$ invariant. The involution $\tau_1 : X^\circ \to X^\circ$ induced by τ_1 in Lemma 1.17(b) can also be extended to X and acts on $Q^\pm$ as $\tau_1(Q^\pm) = Q^\mp$.*

Proof. It is $\sigma(k) = -k$ and $\tau_1(k) = -\bar{k}$. On $X(0)$ that means that if $k \in \mathcal{R}^\pm$, then $-k \in \mathcal{R}^\pm$ and $-\bar{k} \in \mathcal{R}^\mp$. This also holds for $k \in U_\pm$, where $U_\pm \subset \mathfrak{R}(X)$ is a small open neighborhoods of $Q^\pm$, since X is asymptotically free. Since both involutions are continuous in k, we can extend these involutions over $Q^{\pm}$ and the assertion follows. □

It would be convenient if we would find an analogous condition such that the Fermi curve X' can be compactified. Unfortunately this is not possible since infinitely many of the excluded domains 'far outside' might contain handles. So we seek for a unique one-sheeted covering $\pi_M : M^\circ(u) \to X'$ such that $M^\circ(u)$ can be compactified by adding two points Q^+ and Q^- and such that there exists a generalized divisor $\mathcal{S}_M$ on $M^\circ(u)$ with $(\pi_M)_*\mathcal{S}_M = \mathcal{S}$, where $\mathcal{S}$ as in Definition 3.7 and $Q^{\pm} \notin \operatorname{supp} \mathcal{S}_M$. Such a covering corresponding to $(X', \mathcal{S})$ is given by the so-called middelding which is introduced in [Klein et al., 2016, Definition 4.2].

Definition 4.4. For a generalized divisor $\mathcal{S}$ on a singular one-dimensional curve X', let $\pi_M : M^\circ \to X'$ be the unique one-sheeted covering such that

$$(\pi_M)_*\mathcal{O}_{M^\circ} = \{f \in \pi_*\mathcal{O}_{X^\circ} \mid f \cdot g \in \mathcal{S} \text{ for all } g \in \mathcal{S}\},$$

where $\mathcal{O}_{M^\circ}$ is the sheaf of regular functions on M°. We call M° the *middleding* of X'.

It is shown in [Klein et al., 2016, Lemma 4.1] that this covering is unique and that there exists a unique generalized divisor $\mathcal{S}_M$ which obeys $(\pi_M)_*\mathcal{S}_M = \mathcal{S}$. Hereby, $\mathcal{S}_M$ is the $\mathcal{O}_{M^\circ}$-module which is generated by the pullbacks of some choice of local generators of $\mathcal{S}$. Since $X^\circ \to M^\circ \to X'$ are coverings, the asymptotic freeness of X' transfers also to M°. So if M° is compactifiable, then it can also only be compactified by adding two points for $\|\operatorname{Im}(k)\| \to \infty$ as it is done for X°. We will also denote these points by Q^+ and Q^-. With this, we can now give the full definition of finite type potentials.

Definition 4.5. We call a potential $u \in C(\mathbb{R}^2/\Gamma)$ of *finite type* if the corresponding middleding $M^\circ(u)$ can be compactified by adding two smooth points Q^+ and Q^- at infinity. The compactified curve is denoted as M. The corresponding sheaf of holomorphic functions on M is denoted by $\mathcal{O}_M$. We call a finite type potential u *regular* if $M(u)$ equals $X(u)$.

Note that every finite type potential is in particular also a compact curve potential in the sense of Definition 4.1.

Remark 4.6. In general, it is not clear whether the middleding M corresponding to $\mathcal{S}$ equals the middleding corresponding to $\sigma^*\mathcal{S}$. For a regular finite type potential u both middledings equal the normalization of $X'(u)$ since $\sigma : X'(u) \to X'(u)$, so they are also the same.

The definition of a finite type potential is equivalent to the existence of a one-sheeted covering $\tilde{\pi} : \tilde{X} \to X'$ which can be compactified by adding two points Q^+ and Q^- at infinity and on which exists a sheaf $\tilde{\mathcal{S}}$ which is a finitely generated submodule of $\mathcal{M}$ such that $\tilde{\pi}_*\tilde{\mathcal{S}} = \mathcal{S}$: Assuming Definition 4.5 holds, then $\pi_M : M^\circ \to X'$ is this covering. Vice versa, let $\tilde{\pi} : \tilde{X} \to X'$ be such a one-sheeted covering. Since $\tilde{X}$ can be compactified by adding two points Q^+ and Q^-, $\tilde{X} \cap (\mathbb{C}^2_\delta/\Gamma^*)$ is biholomorphic to $X^\circ \cap (\mathbb{C}^2_\delta/\Gamma^*)$ for $\delta > 0$ sufficiently small. Because $\tilde{\pi}_*\tilde{\mathcal{S}} = \mathcal{S}$ and M° is the covering of X' such that $\mathcal{O}_{M^\circ}$ is the maximal subsheaf of $\mathcal{O}_{X^\circ}$ acting on $\mathcal{S}$, M° covers $\tilde{X}$, so also M° can be compactified. Furthermore, $Q^\pm$ are defined in such a way that they are contained in the regular part of the compactified middleding M.

In the rest of this section, we assume that u is a finite type potential in the sense of Definition 4.5 and we often consider the eigenfunctions on small open neighborhoods $U_\pm$ of $Q^\pm$ on X respectively M and especially in the excluded domains contained in $U_\pm$. So we now introduce some notation for these cases. For a finite type potential and $\delta > 0$ sufficiently small, $M^\circ \cap \mathbb{C}^2_\delta/\Gamma^* \simeq X^\circ \cap \mathbb{C}^2_\delta/\Gamma^*$ and all singularities in $X' \cap \mathbb{C}^2_\delta/\Gamma^*$ are double points $k^\pm_\nu(u)$ with $\nu \in \Gamma^*_\delta$ which are desingularized in X° as well as in M°. So we make no difference in the notation for the objects on and subsets of M and X in this case. We denote the two points in $\pi^{-1}[\{k^\pm_\nu(u)\}] \in X^\circ \cap \mathbb{C}^2_\delta/\Gamma^*$ respectively $\pi_M^{-1}[\{k^\pm_\nu(u)\}] \in M^\circ \cap \mathbb{C}^2_\delta/\Gamma^*$ as $k_{\nu,+}$ and $k_{\nu,-}$. Further, we define an open neighborhood U_ν of $k^\pm_\nu(u)$ on X' such that U_ν contains no other double point then $k^\pm_\nu(u)$. The two preimages of U_ν under π respectively π_M we denote as U^+_ν and U^-_ν. These are smooth open neighborhoods of $k_{\nu,+}$ respectively $k_{\nu,-}$ which contain no preimages of other double points. The following Lemma motivates the definition of finite type potentials.

Lemma 4.7. *Let $u \in C(\mathbb{R}^2/\Gamma)$ be a compact curve potential in the sense of Definition 4.1. Then the following three statements are equivalent:*

(i) u is a finite type potential in sense of Definition 4.5.

(ii) There exists a $\delta > 0$ such that the eigenspaces corresponding to the Schrödinger equation (1.6) *are two dimensional at all double points $k^\pm_\nu(u)$ with $\nu \in \Gamma^*_\delta$.*

(iii) The meromorphic function $\langle\langle \xi, \psi_N \rangle\rangle$ on X' as defined in (3.3) *can for every $\xi \in L^2(\Delta)$ be lifted to a meromorphic function in k on X.*

To see the equivalence, the following two helping Lemmata are necessary which we show first:

Lemma 4.8. *The* 1*-forms*

$$\pi_M^* \langle\langle \xi, P_\partial(\chi) \rangle\rangle \, dk_1 \;\; \text{and} \;\; \pi_M^* \left(\frac{\langle\langle \varphi, \chi \rangle\rangle}{\langle\langle \varphi, \psi \rangle\rangle_\partial} \langle\langle \xi, \psi \rangle\rangle \right) dk_1 \tag{4.1}$$

are regular on M° for all $\chi, \xi \in L^2(\Delta)$.

Proof. We only have to show this for $k' \in S$. So let $k' \in S$ with $\pi^{-1}[\{k'\}] = \{k_1, \ldots, k_J\}$. Due to Lemma 3.16, $P_\partial \, dk_1$ is regular on X'. So at $k' \in S$, where J sheets of X' meet, there holds for all $f \in \mathcal{O}_{X'}$ and $\chi, \xi \in L^2(\Delta)$

$$\operatorname{Res} \pi^* \left(f \cdot \langle\langle \xi, P_\partial(\chi) \rangle\rangle\right)_{k'} = \sum_{j=1}^J \operatorname{Res} \left(f \cdot \frac{\langle\langle \varphi, \chi \rangle\rangle_\partial}{\langle\langle \varphi, \psi \rangle\rangle_\partial} \langle\langle \xi, \psi \rangle\rangle \right)_{k_j} = 0,$$

where the index k_j means that we are considering the sheaf generated by the above function at $k_j \in \pi^{-1}(k')$ on X°. Furthermore, since P_∂ is independent of the normalization of the eigenfunctions, we can evaluate this operator-valued 1-form with the germs at k' of the normalized eigenfunction $\psi_{N,k'}$ such that $\langle\!\langle \xi, \psi_N \rangle\!\rangle_{k'} \in \mathcal{S}_{k'}$. By Lemma 3.9, $\mathcal{S}$ is a finitely generated submodule of the meromorphic functions over $\mathcal{O}_{X'}$. For $l = 1, \ldots, L$, we choose $\xi_l \in L^2(\Delta)$ such that the $L \le \delta_{k'} + 1 < \infty$ generators at k' are given by $\langle\!\langle \xi_l, \psi_N \rangle\!\rangle_{k'}$. Let $f \in ((\pi_M)_* \mathcal{O}_{M^\circ})_{k'}$. Definition 4.4 of the middleding M° yields that $f \langle\!\langle \xi, \psi_N \rangle\!\rangle$ is an element of $\mathcal{S}_{k'}$ and thus can be represented as $\sum_{l=1}^{L} (f_l \langle\!\langle \xi_l, \psi_N \rangle\!\rangle)_{k'}$ with $f_l \in \mathcal{O}_{k'}$. So

$$\left(f \frac{\langle\!\langle \varphi, \chi \rangle\!\rangle_\partial}{\langle\!\langle \varphi, \psi_N \rangle\!\rangle_\partial} \langle\!\langle \xi, \psi_N \rangle\!\rangle \right)_{k'} = \left(\sum_{l=1}^{L} \frac{\langle\!\langle \varphi, \chi \rangle\!\rangle_\partial}{\langle\!\langle \varphi, \psi_N \rangle\!\rangle_\partial} f_l \langle\!\langle \xi_l, \psi_N \rangle\!\rangle \right)_{k'} .$$

The regularity of $P_\partial \, dk_1$ on X' yields that

$$\operatorname{Res} \pi^* \left(f_l \frac{\langle\!\langle \varphi, \chi \rangle\!\rangle_\partial}{\langle\!\langle \varphi, \psi_N \rangle\!\rangle_\partial} \langle\!\langle \xi_l, \psi_N \rangle\!\rangle \right)_{k'} = 0$$

for $l = 1, \ldots, L$. Hence, for all $f \in (\pi_M)_* \mathcal{O}_{M^\circ}$ one has

$$\operatorname{Res} \pi^* \left(f \frac{\langle\!\langle \varphi, \chi \rangle\!\rangle_\partial}{\langle\!\langle \varphi, \psi_N \rangle\!\rangle_\partial} \langle\!\langle \xi, \psi_N \rangle\!\rangle \right)_{k'} = 0.$$

Let $\pi'_M : X \to M^\circ$ be the covering such that $\pi_M \circ \pi'_M = \pi$. The existence of this covering is shown in [Klein et al., 2016, Proof of Lemma 4.1]. Recall that $k' \in S$ with $\pi^{-1}[\{k'\}] = \{k_1, \ldots, k_J\} \subset X$. Let further $\pi_M^{-1}[\{k'\}]$ contain a point $\tilde{k} \in M^\circ$, where $(\pi'_M)^{-1}[\{\tilde{k}\}] = \{k_1, \ldots, k_m\}$ with $1 < m \le J$. To see that then also holds

$$\operatorname{Res} \left((\pi'_M)^* (f \langle\!\langle \xi, \pi_M^* P_\partial(\chi) \rangle\!\rangle)_{\tilde{k}} \right) = \sum_{j=1}^{m} \operatorname{Res} \left(f \pi_M^* \langle\!\langle \xi, P_\partial(\chi) \rangle\!\rangle \right)_{k_j} = 0$$

for all $f \in \mathcal{O}_{M^\circ, \tilde{k}}$, we use the fact that $((\pi_M)_* \mathcal{O}_{M^\circ})_{k'}$ contains an element g such that $\pi_M^* g = f$ at $k_1, \ldots, k_m$ but vanishes at all other preimages of k'. So we can use g to separate the points which are separated on M° but identified on X'. Then

$$\sum_{j=1}^{m} \operatorname{Res} \left(f \pi_M^* \langle\!\langle \xi, P_\partial(\chi) \rangle\!\rangle \right)_{k_j} = \sum_{j=1}^{J} \operatorname{Res} \left(g \langle\!\langle \xi, P_\partial(\chi) \rangle\!\rangle \right)_{k_j} = \operatorname{Res} \pi_M^* (g \langle\!\langle \xi, P_\partial(\chi) \rangle\!\rangle)_{k'} = 0.$$

Thus, we have shown that $\pi_M^* P_\partial \, dk_1$ is a regular 1-form on M° in sense of Definition 3.12. The proof for the regularity of $\pi_M^* \left(\frac{\langle\!\langle \varphi, \chi \rangle\!\rangle}{\langle\!\langle \varphi, \psi \rangle\!\rangle_\partial} \langle\!\langle \xi, \varphi \rangle\!\rangle \right) dk_1$ follows analogously. Also this form has no zeros, because $\pi_M^* P_\partial \, dk_1$ contains no zeros and the only difference between the two forms in (4.1) are the enumerators, which is for $\pi_M^* \left(\frac{\langle\!\langle \varphi, \chi \rangle\!\rangle}{\langle\!\langle \varphi, \psi \rangle\!\rangle_\partial} \langle\!\langle \xi, \varphi \rangle\!\rangle \right) dk_1$ given by $\langle\!\langle \varphi, \chi \rangle\!\rangle$ and for $P_\partial \, dk_1$ by $\langle\!\langle \partial_y \varphi, \chi \rangle\!\rangle$. □

In what follows next, we will make a few considerations for germs in small neighborhoods of the preimage of the double points. Therefore, we denote these points as

$$\pi^{-1}[\{k_\nu^\pm(u)\}] = \{k_{\nu,+}, k_{\nu,-}\}$$

and consider the eigenfunctions pulled back to small open neighborhoods of these points for $\nu \in \Gamma_\delta^*$ and $\delta > 0$ sufficiently small. To gain clarity, we make no difference between the germs as well as the pullback of the germs of the eigenfunction at $k_\nu^\pm(u)$ and denote the germs $\psi_{N,k_\nu^\pm(u)}$ and $\pi^*\psi_{N,k_\nu^\pm(u)}$ respectively $\varphi_{N,k_\nu^\pm(u)}$ and $\pi^*\varphi_{N,k_\nu^\pm(u)}$ as $\psi_\pm$ and $\varphi_\pm$. We also omit the dependence on $\nu \in \Gamma_\delta^*$ and point out that these assertions shall always hold for every $\nu \in \Gamma_\delta^*$.

Lemma 4.9. *For $\delta > 0$ sufficiently small, the pullback of the generalized eigenprojection $\pi^* P_\partial$ to X° has a pole at both points $k_{\nu,\pm} = \pi^{-1}(k_{\nu,\pm}(u))$ with $\nu \in \Gamma_\delta^*$ if and only if the values of the eigenfunctions at these two points are linearly dependent. In this case, these poles are each of first order.*

Proof. We assume that $\psi_+(k_\nu^\pm(u))$ and $\psi_-(k_\nu^\pm(u))$ are linearly dependent for all $\nu \in \Gamma_\delta^*$. As in the proof of Lemma 3.11 follows that then also $\varphi_+(k_\nu^\pm(u))$ and $\varphi_-(k_\nu^\pm(u))$ are linearly dependent. So

$$\langle\langle \varphi_+(k_{\nu,+}), \psi_+(k_{\nu,+})\rangle\rangle_\partial = c\langle\langle \varphi_-(k_{\nu,-}), \psi_-(k_{\nu,-})\rangle\rangle_\partial = \tilde{c}\langle\langle \varphi_\pm(k_{\nu,\pm}), \psi_\mp(k_{\nu,\mp})\rangle\rangle_\partial.$$

With the covering $(k_1, k_2) \mapsto k_1$, the two values $k_\pm \in U_\nu^\pm \setminus \{k_{\nu,\pm}\}$ are the preimages of $k, k' \in U_\nu$ with the same k_1 and different values k_2 and k_2'. Let ψ_+ and φ_+ be the eigenfunction and the transposed eigenfunction belonging to $k = (k_1, k_2)$ and ψ_- and φ_- be the eigenfunction and the transposed eigenfunction belonging to $k' = (k_1, k_2')$. Because $\psi_+ \in C^\infty_{[k]}(\Delta)$ and $\varphi_- \in C^\infty_{[-k]}(\Delta)$ for every $k \in X(u)$, Lemma 3.16(ii) yields that

$$0 = P_\partial(k')(\psi_+) = \frac{2\langle\langle \partial_y \varphi_-, \psi_+\rangle\rangle}{\langle\langle \varphi_-, \psi_-\rangle\rangle_\partial}\psi_+ = \frac{2\langle\langle \varphi_-, \psi_+\rangle\rangle_\partial}{\langle\langle \varphi_-, \psi_-\rangle\rangle_\partial}\psi_+.$$

So due to continuity, $\langle\langle \varphi_-, \psi_+\rangle\rangle_\partial = 0$ on all of $U_\nu^\pm$. Evaluating $P_\partial(k)(\psi_-)$ yields that also $\langle\langle \varphi_+, \psi_-\rangle\rangle_\partial = 0$. Since $\pi^* P_\partial$ does not depend on the normalization of the eigenfunctions, we can renormalize $\psi_\pm$ and $\varphi_\pm$ in this operator such that they are holomorphic functions in $k_\pm$ on $U_\nu^\pm \subset X$. Neither deriving $\psi_\pm$ respectively $\varphi_\pm$ into the direction of y nor integrating over Δ influences the holomorphy in k, so $\langle\langle \varphi_\pm, \psi_\mp\rangle\rangle_\partial$ is a holomorphic function in $k_\pm \in U_\nu^\pm$. Thus also for $k_\pm \in U_\nu^\pm \setminus \{k_{\nu,\pm}\}$, there holds

$$\langle\langle \varphi_\pm(k_\pm), \psi_\mp(k_\pm)\rangle\rangle_\partial = \pi^*\langle\langle \varphi_\pm, \psi_\mp\rangle\rangle_\partial(k_\pm) = \langle\langle \varphi_\pm(k), \psi_\mp(k)\rangle\rangle_\partial = 0,$$

where $k = \pi(k_\pm) \in U_\nu \setminus [\{k_\nu^\pm(u)\}]$. Then continuous continuation of $\langle\langle \varphi_\pm(k_\pm), \psi_\mp(k_\pm)\rangle\rangle_\partial$ to $k_\nu^\pm(u)$ yields that for ψ_+ and ψ_- being linearly dependent at $k_\nu^\pm(u)$, the denominator of $\pi^* P_\partial$ at

$\pi^{-1}(k_\nu^\pm(u))$ equals zero. Because the δ-invariant at a double point equals 1, see [Klein et al., 2016, Example 2.5(1)], this pole can only be of first order.
Now assume on the other hand that ψ_+ and ψ_- are linearly independent. As in the proof of Lemma 3.11, then also φ_+ and φ_- are linearly independent at $k_\nu^\pm(u)$, wherefore $\mathcal{O}_{M^\circ,k_{\nu,\pm}} = \mathcal{O}_{X^\circ,k_{\nu,\pm}}$. Since u is a compact curve potential, this yields that $U_\nu^\pm \subset M$ is an open neighborhood of $k_{\nu,\pm} \in \pi_M^{-1}[\{k_\nu^\pm(u)\}]$ which contains no branch point with respect to the Weierstraß covering $(k_1, k_2) \mapsto k_1$ for $\nu \in \Gamma_\delta^*$ and $\delta > 0$ sufficiently small. More precisely, it follows from Theorem 2.34 that the only singularities and branch points on $X' \cap \mathbb{C}_\delta^2/\Gamma^*$ are either two branch points or one double point which are contained in the handles indexed with ν for $\nu \in \Gamma_\delta^*$. Because u is a compact curve potential, all these singularities are double points and hence $U_\nu^\pm \subset X$ contains no branch points for all $\nu \in \Gamma_\delta^*$, so $dk_1 \neq 0$ on these open sets. We then use Lemma 4.8: Since $\pi_M^* P_\partial \, dk_1$ is regular on M°, $\pi_M^* P_\partial$ it is holomorphic on $U_\nu^+ \cup U_\nu^-$. Accordingly, since dk_1 has no zero on $U_\nu^\pm$, there are also no poles of $\pi_M^* P_\partial$ on these open sets. For a finite type potential, there exists a $\delta > 0$ such that $M^\circ \cap \mathbb{C}_\delta^2/\Gamma^*$ is locally biholomorphic to $X^\circ \cap \mathbb{C}_\delta^2/\Gamma^*$. This implies that also $\pi^* P_\partial$ has no pole at $\pi^{-1}[\{k_\nu^\pm(u)\}]$. □

In the next proof, we will often consider objects which are either defined on U_ν^+ or on U_ν^-. For brevity, we write ψ for $\psi_\pm$ as well as φ for $\varphi_\pm$ when no difference between these functions is necessary.

Proof of Lemma 4.7. To see that (i) $\Rightarrow$ (ii), we use again that u is a finite type potential and choose $\nu \in \Gamma_\delta^*$ with $\delta > 0$ sufficiently small. In [Klein et al., 2016, Example 2.5.1], it is shown that the subring $\mathcal{O}_{k_\nu^\pm(u)}$ on $X' \cap (\mathbb{C}_\delta^2/\Gamma^*)$ of $\bar{\mathcal{O}}_{k_\nu^\pm(u)} = \mathcal{O}_{k_{\nu,+}} \times \mathcal{O}_{k_{\nu,-}}$ is given by all elements of the form (f_+, f_-) with $f_+(k_+) = f_-(k_-)$. Then $((\pi_M)_* \mathcal{O}_M)_{k_\nu^\pm(u)} = \bar{\mathcal{O}}_{k_\nu^\pm(u)}$, and therefore contains the functions $(1,0)$ and $(0,1)$ which take different values at the two points $k_{\nu,\pm}$ over $k_\nu^\pm(u)$ on M. Since $k_\nu^\pm(u)$ is an ordinary double point, $\mathcal{S}_{k_\nu^\pm(u)}$ contains two elements ψ_+ and ψ_-. Assume that these are linearly dependent at $k_\nu^\pm(u)$. Then due to the normalization of the eigenfunctions, it is $\psi_+(k_\nu^\pm(u), (0,0)) = \psi_-(k_\nu^\pm(u), (0,0))$, so $\psi_+(k_\nu^\pm, (x,y)) = \psi_-(k_\nu^\pm(u), (x,y))$ for all $(x,y) \in \Delta$ and hence also $\langle\!\langle \xi, \psi_+(k_\nu^\pm(u)) \rangle\!\rangle = \langle\!\langle \xi, \psi_-(k_\nu^\pm(u)), \rangle\!\rangle$ for all $\xi \in L^2(\Delta)$. Then $(1,0)\mathcal{S}$ and $(0,1)\mathcal{S}$ are not contained in $\mathcal{S}$ which contradicts Definition 4.4 of the middleding.
Vice versa, we show that $\neg$(i) implies $\neg$(ii), i.e. we deduce from $M^\circ(u)$ not being compactifiable that there exists a $\delta > 0$ such that the eigenfunctions ψ_+ and ψ_- at $k_\nu^\pm(u)$ are linearly dependent for all $\nu \in \Gamma_\delta^*$. Since $M^\circ(u)$ cannot be compactified, there are infinitely many points in $M^\circ(u) \cap \mathbb{C}_\delta^2/\Gamma^*$ such that at these points, the germs of the holomorphic functions of $M^\circ(u)$ do not equal the germs of holomorphic functions of $X^\circ(u)$ at the corresponding preimages of these points. These points are contained in $\{\pi_M^{-1}[\{k_\nu^\pm(u)\}] \mid \nu \in \Gamma_\delta^*\}$. So linear independence of the eigenfunctions at these points would yield that $((\pi_M)_* \mathcal{O}_M)_{k_\nu^\pm(u)} = \bar{\mathcal{O}}_{k_\nu^\pm(u)}$ for all $\nu \in \Gamma_\delta^*$ which contradicts the assumption that $M^\circ(u)$ cannot be compactified. Consequently, the eigenfunctions are linearly dependent.
Now (i) implies (iii) as follows: Since $M^\circ(u)$ can be compactified, the arithmetic genus $g_{M,a}$ of M is

finite. Ergo, $\deg(\mathcal{S}_M) = \deg(\sigma^*\mathcal{S}_M) < \infty$, compare [Klein et al., 2016, Lemma 5.1(b)]. This implies that also the support of the generalized divisor generated by $\pi_M^*\left(\frac{\langle\langle \xi, \psi(\cdot,\cdot)\rangle\rangle}{\psi(\cdot,(0,0))}\right)$ and the support of the generalized divisor generated by $\pi_M^*\left(\frac{\langle\langle \varphi(\cdot,\cdot),\xi\rangle\rangle}{\varphi(\cdot,(0,0))}\right)$ are finite for each fixed $\xi \in L^2(\Delta)$. Since there exists a covering $\pi'_M : X \to M$ such that $\pi_M \circ \pi'_M|_{X^\circ} = \pi$ and X is smooth, there exists a classical divisor D on X such that $\pi'_M(D)$ equals the generalized divisor generated by $\pi_M^*\left(\frac{\langle\langle \xi, \psi(\cdot,\cdot)\rangle\rangle}{\psi(\cdot,(0,0))}\right)$ and by the definition of $\mathcal{S}$, D corresponds to the pole divisor of $\pi^*\psi_N$. Since $\deg(\mathcal{S}_M) < \infty$, also $\deg(D) < \infty$.

Next we show that ¬(ii) implies ¬(iii). So let $\psi_+(k_{\nu,\pm})$ and $\psi_-(k_{\nu,\pm})$ be linearly dependent for all ν which are contained in a subset $A \subset \Gamma_\delta^*$ which consists of infinitely many points. We have seen in the proof of Lemma 4.9 that then $\langle\langle \varphi(k_\nu^\pm(u)), \psi(k_\nu^\pm(u))\rangle\rangle_\partial$ has a zero of first order for all $\nu \in A$. Thus, also the zeros at the two points $k_{\nu,\pm}$ in the preimage $\pi^{-1}[\{k_\nu^\pm(u)\}]$ are each of first order for every $\nu \in \Gamma_\delta^*$. We now want to deduce from this with Rouchet's Theorem for meromorphic functions [Moskowitz, 2002, Theorem 4.3.1] that $\langle\langle \varphi(\cdot), \psi(\cdot)\rangle\rangle_\partial$ must have a pole of first order in each of the two preimages of the excluded domains $\mathfrak{e}_\nu$ intersected with $X'(u)$ for every $\nu \in A$.

Remember that we can represent $(k_1, k_2) \in X(u)$ as the Weierstraß covering $(k_1, k_2(k_1))$ for $k_1 \in B_{4\varepsilon}(k_{\nu,\pm,1})$ if we choose $\varepsilon > 0$ as in Remark 2.10. The same is done in the proof of Theorem 2.34. For brevity, we define $B_\nu := B_{4\varepsilon}(k_{\nu,\pm,1})$ and $\partial B_\nu := \partial B_{4\varepsilon}(k_{\nu,\pm,1})$. Also remember that the two elements in $\pi^*\psi_{N,k_\nu^\pm(u)}$ respectively $\pi^*\varphi_{N,k_\nu^\pm(u)}$ are denoted by ψ respectively φ. In the sequel of this proof, we will only evaluate them at $(k_1, k_2) \in X(u)$ with $k_1 \in B_\nu$. In the proof of Theorem 2.34 is given reason why for $k_1 \in \partial B_\nu$, the values $(k_1, k_2(k_1)) \in \mathfrak{R}(X'(u))$. One has for every $k \in X(u)$ that

$$\left\| e^{2\pi\iota\langle k, \binom{x}{y}\rangle} \right\| = \left\| e^{-2\pi\iota\langle k, \binom{x}{y}\rangle} \right\| = \mathrm{Vol}(\Delta).$$

Choosing ε as in Remark 2.10 assures by Theorem 2.34 that there is a $\delta(\varepsilon) > 0$ such that for all $\nu \in \Gamma_\delta^*$ and $k_\pm = (k_{\pm,1}, k_{\pm,2}(k_{\pm,1}))$ with $k_{\pm,1} \in \partial B_\nu$, the functions $\langle\langle e^{-2\pi\iota\langle k, \binom{x}{y}\rangle}, e^{2\pi\iota\langle k, \binom{x}{y}\rangle}\rangle\rangle_\partial$ and $\langle\langle \varphi, \psi\rangle\rangle_\partial$ have neither poles nor zeros. Since the only branch points of $X'(u) \cap \mathbb{C}_\delta^2/\Gamma^* \to \mathbb{C}$, $(k_1, k_2) \mapsto k_1$ are contained inside of B_ν, the holomorphic function $\langle\langle \varphi, \psi\rangle\rangle$ is not equal to zero for $(k_1, k_1(k_2)) \in X(u)$ with $k_1 \in \partial B_\nu$, see Lemma 3.11. Let $k_1 \in \partial B_\nu$ and $k = (k_1, k_2(k_1)) \in X(u)$. We want to estimate

$$\left| \langle\langle \varphi, \psi\rangle\rangle_\partial - \left\langle\!\!\left\langle e^{-2\pi\iota\langle k, \binom{x}{y}\rangle}, e^{2\pi\iota\langle k, \binom{x}{y}\rangle} \right\rangle\!\!\right\rangle_\partial \right|$$

and then apply Rouchet's Theorem to this expression to count the poles of $\langle\langle \varphi, \psi\rangle\rangle_\partial$ for $k_1 \in B_\nu$. Note that for $k_1 \in \partial B_\nu$, we have shown in (4.2) that

$$\begin{aligned} \left| \left\langle\!\!\left\langle e^{-2\pi\iota\langle k, \binom{x}{y}\rangle}, e^{2\pi\iota\langle k, \binom{x}{y}\rangle} \right\rangle\!\!\right\rangle_\partial \right| =& \left| \int_\Delta \partial_y \left(e^{-2\pi\iota\langle k, \binom{x}{y}\rangle} \right) e^{2\pi\iota\langle k, \binom{x}{y}\rangle} - e^{-2\pi\iota\langle k, \binom{x}{y}\rangle} \partial_y \left(e^{2\pi\iota\langle k, \binom{x}{y}\rangle} \right) d\mu \right| \\ =& 4\pi |k_2| \,\mathrm{Vol}(\Delta). \end{aligned} \tag{4.2}$$

Remember that $\|\cdot\| = \|\cdot\|_{L^2(\varDelta)}$ and that the eigenfunctions ψ and its duals φ from the fundamental domain-formulation of the Schrödinger equation in Definition 1.4 and the eigenfunction ψ_k and its duals φ_k from the trivialization-formulation of the Schrödinger equation in Definition 1.5 only differ by a phase of length $\left|e^{\pm 2\pi\iota\langle k,\binom{x}{y}\rangle}\right| = 1$. Note moreover that for the normalized eigenfunction $\psi_N(k)$ holds that the zeroth Fourier coefficient $\hat{\psi}(0) = 1$. So by Lemma 2.23(a), it is for $k = (k_1, k_1(k_2))$ with $k_1 \in B_\nu$

$$\left\|\psi - e^{2\pi\iota\langle k,\binom{x}{y}\rangle}\right\| = \|\psi_k - 1\| \quad \text{as well as} \quad \left\|\varphi - e^{-2\pi\iota\langle k,\binom{x}{y}\rangle}\right\| = \|\varphi_k - 1\|.$$

For the derivative with respect to y it is for $k_1 \in \partial B_\nu$ and $k = (k_1, k_2(k_1))$

$$\begin{aligned}\|\partial_y(\psi_k - 1)\| &= \left\|\partial_y\left(e^{-2\pi\iota\langle k,\binom{x}{y}\rangle}\left(\psi - e^{2\pi\iota\langle k,\binom{x}{y}\rangle}\right)\right)\right\| \\ &= 2\pi|k_2|\left\|\psi - e^{2\pi\iota\langle k,\binom{x}{y}\rangle}\right\| + \left\|\partial_y\left(\psi - e^{2\pi\iota\langle k,\binom{x}{y}\rangle}\right)\right\|.\end{aligned}$$

From the asymptotic freeness in Lemma 2.23(a) of the eigenfunctions ψ_k and φ_k in the formulation of Definition 1.5 in $W^{1,2}(\mathbb{R}^2/\varGamma)$ shown in Lemma 2.23(a) follows that on $\mathfrak{R}(X(u))$, there holds

$$\left\|\psi - e^{2\pi\iota\langle k,\binom{x}{y}\rangle}\right\|, \left\|\partial_y\left(\psi - e^{2\pi\iota\langle k,\binom{x}{y}\rangle}\right)\right\| \le \left\|\psi - e^{2\pi\iota\langle k,\binom{x}{y}\rangle}\right\|_{W^{1,2}(\varDelta,\mathbb{C})} < \tilde{\varepsilon}\mathrm{Vol}(\varDelta).$$

All these estimates also transfer to ψ and φ evaluated at the corresponding k on the normalization because $\mathfrak{R}(X'(u))$ is isomorphic to $\pi^{-1}[\mathfrak{R}(X'(u))] \subset X(u)$. For brevity, we set in the next calculation $\psi_0 := e^{2\pi\iota\langle k,\binom{x}{y}\rangle}$ and $\varphi_0 := e^{-2\pi\iota\langle k,\binom{x}{y}\rangle}$ as well as $\psi := \psi(k)$ and $\varphi := \varphi(k)$ with $k = (k_1, k_2(k_1))$ and $k_1 \in \partial B_\nu$. Using the quasiperiodicity of the eigenfunctions and its duals (1.3), we obtain by partial integration, where we take the periodicity of $\psi \cdot \varphi$ on $\varDelta$ into account, that

$$\begin{aligned}|\langle\!\langle \varphi, \psi\rangle\!\rangle_\partial - \langle\!\langle \varphi_0, \psi_0\rangle\!\rangle_\partial| &= |\langle\!\langle \partial_y\varphi, \psi\rangle\!\rangle - \langle\!\langle \varphi, \partial_y\psi\rangle\!\rangle - \langle\!\langle \partial_y\varphi_0, \psi_0\rangle\!\rangle + \langle\!\langle \varphi_0, \partial_y\psi_0\rangle\!\rangle| \\ &= 2|\langle\!\langle \partial_y\varphi, \psi\rangle\!\rangle - \langle\!\langle \partial_y\varphi_0, \psi_0\rangle\!\rangle| \\ &= 2|\langle\!\langle \partial_y(\varphi - \varphi_0), \psi\rangle\!\rangle - \langle\!\langle \partial_y\varphi_0, \psi_0 - \psi\rangle\!\rangle| \\ &= 2|\langle\!\langle \partial_y(\varphi - \varphi_0), \psi - \psi_0\rangle\!\rangle + \langle\!\langle \partial_y(\varphi - \varphi_0), \psi_0\rangle\!\rangle - \langle\!\langle \partial_y\varphi_0, \psi_0 - \psi\rangle\!\rangle| \\ &\le 2\left(|\langle\!\langle \partial_y(\varphi - \varphi_0), \psi - \psi_0\rangle\!\rangle| + |\langle\!\langle (\varphi - \varphi_0), \partial_y\psi_0\rangle\!\rangle| + |\langle\!\langle \partial_y\varphi_0, \psi_0 - \psi\rangle\!\rangle|\right) \\ &\le 2\left(\|\partial_y(\varphi - \varphi_0)\|\|\psi - \psi_0\| + \|\varphi - \varphi_0\|\|\partial_y\psi_0\| + \|\partial_y\varphi_0\|\|\psi - \psi_0\|\right).\end{aligned}$$

Due to the above considerations, it is

$$\|\partial_y(\varphi - \varphi_0)\|\,\|(\psi - \psi_0)\| \le 2\pi|k_{\pm,2}|\tilde{\varepsilon}^2\mathrm{Vol}(\varDelta).$$

Since $\|\partial_y\varphi_0\| = 2\pi|k_2|\mathrm{Vol}(\varDelta)$, one has

$$\|(\varphi - \varphi_0)\|\|\partial_y\psi_0\| < 2\pi|k_{\pm,2}|\tilde{\varepsilon}\mathrm{Vol}(\varDelta)$$

and analogously
$$\|\partial_y \varphi_0\| \|(\psi - \psi_0)\| < 2\pi |k_{\pm,2}| \tilde{\varepsilon} \mathrm{Vol}(\Delta).$$
So altogether, we obtain for $k = (k_1, k_2(k_1))$ with $k_1 \in B_\nu$ that
$$|\langle\!\langle \varphi, \psi \rangle\!\rangle_\partial - \langle\!\langle \varphi_0, \psi_0 \rangle\!\rangle_\partial| \leq 4\pi \mathrm{Vol}(\Delta) \left(|k_{\pm,2}| \tilde{\varepsilon}(\tilde{\varepsilon} + 2)\right) < 4\pi |k_{2,\pm}| \mathrm{Vol}(\Delta)$$
for $\tilde{\varepsilon} > 0$ sufficiently small. By Rouchet's Theorem for meromorphic functions, the differences of the number of poles and the number of zeros of $\langle\!\langle \varphi, \psi \rangle\!\rangle_\partial$ and $\langle\!\langle \varphi_0, \psi_0 \rangle\!\rangle_\partial$ for $k_1 \in B_\nu$ and $k = (k_1, k_2(k_1)) \in X(u)$ coincide. Because $\langle\!\langle \varphi_0, \psi_0 \rangle\!\rangle$ has neither poles nor zeros on B_ν and $\langle\!\langle \varphi, \psi \rangle\!\rangle_\partial$ has one zero on each of the two open subsets on $X(u)$ corresponding to $k_1 \in B_\nu$, the latter has also a pole of first order in each of the sets corresponding to $k_1 \in B_\nu$. Let us denote this pole by $k'_{\nu,\pm}$. Since deriving ψ and φ into the direction of y does not generate new poles in k for $k \in X^\circ(u)$, compare the proof of Lemma 3.16(iii) and (iv), for every $\nu \in A$ one of the germs of $\left(\frac{\langle\!\langle \varphi, \nu \rangle\!\rangle}{\varphi(\cdot,(0,0))}\right)_{k_{\nu,\pm}}$ and $\left(\frac{\langle\!\langle \xi, \psi \rangle\!\rangle}{\psi(\cdot,(0,0))}\right)_{k_{\nu,\pm}}$ has a pole of first order at some $\nu \in L^2(\Delta)$ at $k'_{\nu,\pm}$ with $k'_{\nu,\pm,1} \in B_\nu$.
Now remember that the germs of $\mathcal{S}$ are generated by $\frac{\langle\!\langle \xi, \psi \rangle\!\rangle}{\psi((0,0))}$ and $\frac{\langle\!\langle \varphi, \xi \rangle\!\rangle}{\varphi(0,0)}$, where the zero sets of $\langle\!\langle \cdot, \psi \rangle\!\rangle$ and $\langle\!\langle \varphi, \cdot \rangle\!\rangle$ both have codimension 1 in $L^2(\Delta)$ since $\psi, \varphi \not\equiv 0$. That means that the complement set of their zeros in $L^2(\Delta)$ is an open and dense subset of $L^2(\Delta)$ for each $\nu \in \Gamma^*_\delta$. So if $\psi(k'_{\nu,\pm}, (0,0)) = 0$ respectively $\varphi(k'_{\nu,\pm}, (0,0)) = 0$, the germs of $\mathcal{S}$ have poles on an open and dense subset of $L^2(\Delta)$. As Γ^* is countable, also A is. For each $\nu \in A$, the set of poles of either $\langle\!\langle \xi, \psi_N(k'_{\nu,\pm}) \rangle\!\rangle$ or $\langle\!\langle \varphi_N(k'_{\nu,\pm}), \xi \rangle\!\rangle$ is an open and dense subset of Δ. Then by Baire's Theorem [Reed and Simon, 1980, Theorem III.8], also the intersection of these open and dense subsets over all $\nu \in A$ is open and dense. Hence, there exists an $\xi_0 \in L^2(\Delta)$ such that either $\langle\!\langle \xi_0, \psi_N \rangle\!\rangle$ or $\langle\!\langle \varphi_N, \xi \rangle\!\rangle$ has a pole for all $\nu \in A$. So either $\langle\!\langle \xi_0, \pi^* \psi_N \rangle\!\rangle$ or $\langle\!\langle \pi^* \varphi_N, \xi_0 \rangle\!\rangle$ has infinitely many poles on $U_+ \cup U_-$, where $U_\pm \subset \mathfrak{R}(X)$ are open neighborhoods of $Q^\pm$, respectively. Since $\psi_N = \sigma^* \varphi_N$, the number of poles of these functions coincides and thus there exists a $\xi_0 \in L^2(\Delta)$ such that neither $\langle\!\langle \xi_0, \psi_N \rangle\!\rangle$ nor $\langle\!\langle \varphi_N, \xi_0 \rangle\!\rangle$ can be lifted to a meromorphic function on the compact Riemann surface X. □

For a finite type potential u, the biholomorphy of $X \cap \mathbb{C}^2_\delta / \Gamma^*$ and $M \cap \mathbb{C}^2_\delta / \Gamma^*$ for $\delta > 0$ sufficiently small yields the following Lemma.

Corollary 4.10. *For a finite type potential u, there are small open neighborhoods $U_\pm$ of $Q^\pm$ on M as well as on X such that for all $k \in U_\pm \setminus \{Q^\pm\}$, there holds*
$$\left\| \psi_N(k, (x, y)) - e^{2\pi\iota \langle k, \binom{x}{y} \rangle} \right\|_{W^{1,2}(\mathbb{R}^2/\Gamma)} < \tilde{\varepsilon}.$$

Proof. We show this only for $M(u)$ since the proof for $X(u)$ follows analogously by replacing π_M with π. Recall that for $\psi_N(k)$ holds that the zeroth Fourier coefficient $\hat{\psi}(0) = 1$. Due to Lemma 2.23(a), the above assertion already holds on $\mathcal{U}_{\varepsilon,\delta}$ with $\varepsilon > 0$ as in Remark 2.10, so it only has

to be shown in the excluded domains $\mathfrak{e}_\nu \cap X(u)$ around $k_\nu^\pm$ with $\nu \in \Gamma_\delta^*$ and $\delta(\varepsilon) > 0$ sufficiently small.
Since u is a finite type potential, the equivalence of (i) and (iii) in Lemma 4.7 yields that $\pi^*\psi_N(k)$ is holomorphic for $k = (k_1, k_2(k_1))$ and $k_1 \in B_\nu$. So also $\pi^*\psi_N - e^{2\pi\iota\langle k, \binom{x}{y}\rangle}$ is holomorphic for $k_1 \in B_\nu$ and all $\nu \in \Gamma_\delta^*$. Lemma 2.23 together with Remark 2.10 yields that for all $(k_1, k_2(k_1)) \in X(u) \cap \mathbb{C}_\delta^2$ with $k_1 \notin B_\nu$ and $\nu \in \Gamma_\delta^*$, it is

$$\left\|\psi_N - e^{2\pi\iota\langle k, \binom{x}{y}\rangle}\right\| < \varepsilon \left\|e^{2\pi\iota\langle k, \binom{x}{y}\rangle}\right\| = \varepsilon \int_\Delta \left|e^{2\pi\iota\langle k, \binom{x}{y}\rangle}\right| d\mu = \varepsilon \mathrm{Vol}(\Delta).$$

This also holds for the pullback of these functions to $M \cap \pi^{-1}[\Re(X(u))] \simeq \Re(X(u))$. So let $\delta > 0$ be chosen so small that the above estimate of the eigenfunctions holds for $k_1 \in \partial B_\nu$. This yields that the modulus $\left\|\psi_N - e^{2\pi\iota\langle k, \binom{x}{y}\rangle}\right\|$ cannot archive its maximum in k_1 inside of the open set B_ν but that this maximum must be contained in ∂B_ν. On ∂B_ν, this modulus is smaller then $\varepsilon\mathrm{Vol}(\Delta)$. Since B_ν, defined as in the foregoing proof, is biholomorphic to an open subset of $\mathbb{C}$, the maximum modulus Theorem [Conway, 1978, Theorem 1.1] can be applied. This yields that the above estimate of the eigenfunction holds for all $k_1 \in B_\nu$ and for every $\nu \in \Gamma_\delta^*$. The maps $k \mapsto e^{2\pi\iota\langle k, \binom{x}{y}\rangle}$ and $k \mapsto \psi_N(k)$ are bounded and unequal to zero for $k = (k_1, k_2(k_1))$ with $k_1 \in B_\nu$. By Lemma 2.23(a) also the partial derivatives $\partial_x\psi_N$ and $\partial_y\psi_N$ can be estimated analogously: These are also holomorphic and have no zeros for $k_1 \in \partial B_\nu$, so the above estimate also holds for $\left\|\psi_N - e^{2\pi\iota\langle k, \binom{x}{y}\rangle}\right\|_{W^{1,2}(\mathbb{R}^2/\Gamma)}$ with another ε. □

Of course, the same assertions also hold for the eigenfunction of the transposed Schrödinger operator.

Corollary 4.11. *For a finite type potential u, the support of the generalized divisor $\mathcal{S}_M$ on M which obeys $(\pi_M)_*\mathcal{S}_M = \mathcal{S}$ consists of finitely many points. The marked points $Q^\pm$ on $M(u)$ are not contained in* $\operatorname{supp}\mathcal{S}_M$.

Proof. Since $\mathcal{S}$ is a generalized divisor on X', its support contains only discrete points, see [Klein et al., 2016, Proposition 3.3]. Due to Lemma 4.7(iii), we can choose small open neighborhoods $U_\pm$ of $Q^\pm$ such that $\mathcal{S}_M \cap U_\pm = \emptyset$. Hence, $\operatorname{supp}\mathcal{S}_M$ is contained in a compact subset of M and thus consists of finitely many points. The second assertion holds because the above Lemma yields that there exists open neighborhoods $U_\pm$ of $Q^\pm$ on M on which $|\psi(0,0) - 1| < \varepsilon$, and so by continuity, it is $\psi(Q^\pm, (0,0)) \neq 0$. □

4.2. Properties of the divisor for regular finite type potentials

From now on, we assume that $u \in C(\mathbb{R}^2/\Gamma)$ is a regular finite type potential. That means X equals M and thus the 1-form $\pi^*P_\partial\, dk_1$ is, by Lemma 4.8, a holomorphic 1-form on the normalization X°.

For these potentials, the generalized divisor $\mathcal{S}_M$ can be identified with a classical positive divisor D on X of finite degree as it is done in [Klein et al., 2016, § 6]. The action of the involutions on these divisor is then denoted as $\sigma^*\mathcal{S}_M = \sigma(D)$ and $\tau_1^*\mathcal{S}_M = \tau_1(D)$. Hereby, the involution σ extends to an involution on the divisors on X by $\sigma\Big(\sum_{p\in X} a(p)p\Big) := \sum_{p\in X} a(p)\sigma(p)$ which we also denote as σ and the same holds for τ_1. So the degree of a divisor is conserved under these involutions. We will now summarize some more properties of the eigenfunctions. These will be necessary in Chapter 5 to show that the unique function which we reconstruct out of some given data equals the unique normalized eigenfunction of the Schrödinger operator corresponding to a reconstructed unique potential u.

4.2.1. A linear equivalence

Next, we want to deduce from the holomorphy of $\pi^* P_\partial\, dk_1$ on $X(u)$ which we have seen in Lemma 4.8 that for the pole divisor D of the normalized eigenfunction holds

$$D + \sigma(D) \simeq K + Q^+ + Q^-,$$

where K is the canonical divisor on X and $Q^\pm$ are the two distinguished points added to compactify X° as it is done at the beginning of Section 4. Therefore, it is necessary to show the following proposition:

Proposition 4.12. *Let u be a regular finite type potential. Then $\omega := \frac{1}{\langle\!\langle \varphi_N, \psi_N \rangle\!\rangle_\partial}\, dk_1$ is a holomorphic 1-Form on X°.*

Proof. Lemma 4.8 yields together with the fact that X° is a Riemann surface that

$$\frac{\langle\!\langle \varphi(k), \chi \rangle\!\rangle}{\langle\!\langle \varphi(k), \psi(k) \rangle\!\rangle_\partial} \langle\!\langle \xi, \psi(k) \rangle\!\rangle\, dk_1$$

is a holomorphic 1-form on X° for all $\chi, \xi \in L^2(\varDelta)$. Since this is independent of the normalization of the eigenfunctions, we can assume that for all $k_0 \in X^\circ$ and all k contained in a small open neighborhood $U_{k_0} \subset X^\circ$ of k_0, the germs ψ_{k_0} and φ_{k_0} are elements of $\mathcal{O}_{X^\circ, k_0}$ without zeros on U_{k_0}. Since the L^2-scalar product is non-degenerate, we can always find $\chi, \xi \in L^2(\varDelta)$ such that $\langle\!\langle \varphi_{k_0}, \chi \rangle\!\rangle$ and $\langle\!\langle \xi, \psi_{k_0} \rangle\!\rangle$ are unequal to zero and holomorphic for $k \in U_{k_0}$: Since φ_{k_0} holomorphic on U_{k_0}, also $\varphi_k \cdot \chi$ is and since integration over $\varDelta$ does not influence this holomorphy, also $\langle\!\langle \varphi_{k_0}, \chi \rangle\!\rangle$ is holomorphic and unequal to zero on U_{k_0}. The same holds for $\langle\!\langle \xi, \psi_{k_0} \rangle\!\rangle$. Thus, $\langle\!\langle \varphi_{k_0}, \chi \rangle\!\rangle^{-1}, \langle\!\langle \xi, \psi_{k_0} \rangle\!\rangle^{-1} \in \mathcal{O}_{X^\circ, k_0}$ and due to Lemma 3.6(i), $\frac{1}{\langle\!\langle \varphi_{k_0}, \psi_{k_0} \rangle\!\rangle}\, dk_1$ is holomorphic in a small neighborhood of k_0. Note that on X°, the divisor D coincides with the zeros of $\psi(\cdot, (0,0))$ and the divisor $\sigma(D)$ coincides with the zeros of $\varphi(\cdot, (0,0))$. Since ψ_{k_0} and φ_{k_0} are normalized in such a way that they have no poles on U_{k_0}, also $\psi(0,0)_{k_0}$ and $\varphi(0,0)_{k_0}$ are holomorphic on U_{k_0}. Consequently, Lemma 3.6, the biliniarity of $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\partial$ and the independence of $\psi(0,0)_{k_0}$ and $\varphi(0,0)_{k_0}$ from $(x,y) \in \Delta$ together yield that

$$\psi(0,0)_{k_0}\varphi(0,0)_{k_0}\frac{1}{\langle\!\langle \varphi_{k_0}, \psi_{k_0} \rangle\!\rangle_\partial} = \frac{1}{\frac{1}{\varphi(0,0)_{k_0}\psi(0,0)_{k_0}}(\langle\!\langle \varphi_{k_0}, \partial_y \psi_{k_0} \rangle\!\rangle - \langle\!\langle \partial_y \varphi_{k_0}, \psi_{k_0} \rangle\!\rangle)} = \frac{1}{\langle\!\langle \varphi_{N,k_0}, \psi_{N,k_0} \rangle\!\rangle_\partial}$$

is holomorphic on U_{k_0} for every $k_0 \in X^\circ$. So the assertion follows. □

From this, we can now deduce the well-known linear equivalence for regular finite type potentials. We show it here so explicitly since it is mentioned often in the common literature, see e.g. [Novikov and Veselov, 1984] and [Novikov and Veselov, 1986], but we could not find the explicit proof of this statement. Hereby, K is the canonical divisor of degree $2g-2$, i.e. the divisor of a meromorphic 1-form on X, compare for example [Forster, 1981, § 16.2].

Lemma 4.13. *For a regular finite type potential u, the pole divisor D on the compactified normalization X is positive and of degree g, where g is the genus of X, and fulfills*

$$D + \sigma(D) \simeq K + Q^+ + Q^-. \tag{4.3}$$

Proof. It follows from Theorem 2.28(c) that the covering $k_1 : X^\circ \to \mathbb{C}$ is a multivalued holomorphic map on X whose values over a point in X° only differ by elements of $\mathbb{Z}$. So dk_1 is a meromorphic 1-form on X which is holomorphic and single-valued on X° and vanishes only at the branch points of the above covering. Only at Q^+ and Q^- can be poles of dk_1. We first show that

$$K \simeq (dk_1) = Z - 2Q^+ - 2Q^-, \tag{4.4}$$

where Z is the zero divisor of dk_1 on X. The linear equivalence in (4.4) holds since dk_1 is a meromorphic differential on the compact Riemann surface X. The equality sign in (4.4) follows by determining the pole divisor of dk_1 which can only be contained in Q^+ and Q^-. That dk_1 has poles of second order at both of these points is a consequence of the asymptotic freeness of X: In a neighborhood $U_\pm$ of $Q^\pm$, X can be represented by a local coordinates $z_\pm : U_\pm \to \mathbb{C}$ centered at $Q^\pm$, so the asymptotic freeness yields that the value of k on an open neighborhood $U_\pm$ containing $Q^\pm$ is given by $k_1(z_\pm) = \frac{1}{z_\pm}$ and $k_2(z_\pm) = \frac{\mp\iota}{z_\pm} + \sum_{j=1}^\infty a_j z_\pm^{2j+1}$, where the power series of k_2 only contains uneven powers of $z_\pm$ because $\sigma : X \to X,\ k \mapsto -k$ leaves $Q^\pm$ invariant, so the local coordinates have to be uneven in $z_\pm$. Hence, $dk_1(z_\pm) = -\frac{1}{z_\pm^2}$, and so dk_1 has a pole of second order at both points Q^+ and Q^-. Furthermore, we claim that

$$0 \simeq \left(\frac{1}{\langle\!\langle \varphi_N, \psi_N \rangle\!\rangle_\partial}\right) = -Z + D + \sigma(D) + Q^+ + Q^-, \tag{4.5}$$

where the linear equivalence in (4.5) holds since the divisor of a meromorphic function on a compact Riemann surface is linear equivalent to the zero-divisor. The equality sign we show in three steps: first we consider this meromorphic function only on X°. Then the holomorphy of $\frac{1}{\langle\!\langle \varphi_N, \psi_N \rangle\!\rangle_\partial} dk_1$ on X°, shown in Lemma 4.8, yields immediately that the zeros of $\langle\!\langle \varphi_N, \psi_N \rangle\!\rangle_\partial$ must be contained in Z. To see that even equality holds, note that by the same argumentation how to separate the points on $M(u)$ in the proof of Lemma 4.8, the same properties which hold for P_∂ on $F(u)/\Gamma^*$ shown in Lemma 3.16 also hold for $\pi_M^* P_\partial$. Since u is a regular finite type potential, it is $X(u) = M(u)$. We consider the covering $X(u) \to \mathbb{C}$, $k \mapsto k_1$. Then analogous argumentation as in the proof of Lemma 3.16(iii) yields that if r sheets of this covering meet at a point k of $X(u)$ with $(\pi_M')^{-1}[\{k\}] = k_1, \dots, k_r$, then

$$\sum_{j=1}^{r} \left(\frac{\langle\!\langle \varphi, \chi \rangle\!\rangle}{\langle\!\langle \varphi, \psi \rangle\!\rangle} \right)_{k_j} \tag{4.6}$$

is holomorphic and and of rank r, i.e. unequal to zero since the image contains at least one direction of the eigenspace of $-\triangle + u$. Since $X(u)$ is smooth, this sum can only contain more than one summand if the considered point k is a branch point of the above covering. In this case, we can always find a local coordinate z centered at $k = (k_1, k_2) \in X(u)$ and a local coordinate w centered at k_1 such that $w = z^r$. So we can read this as a special case of Lemma 3.6, and therefore, $P_\partial\, dk_1$ is holomorphic with $P_\partial\, dk_1 = 0$ if and only if the expression in (4.6) equals zero. So $P_\partial\, dk_1$ has no zeros on $X(u)$. So if dk_1 has a zero of order n at some $k \in X(u)$, also $\langle\!\langle \varphi, \psi \rangle\!\rangle_\partial$ has to vanish of order n, and therefore, the zero divisor of $\langle\!\langle \varphi, \psi \rangle\!\rangle_\partial$ equals Z.

Next, we argue why the zeros of this function are just the poles of the normalized eigenfunctions. Lemma 1.17(a) yields that for the normalized eigenfunctions, there holds $\varphi_N = \sigma^* \psi_N$. So $\sigma(D)$ is the divisor of φ_N which is defined in the same manner as D, only that it refers to the eigenfunction of the transposed Schrödinger equation with the same potential. Accordingly, we want to show that

$$(\langle\!\langle \partial_y \varphi_N, \psi_N \rangle\!\rangle_\partial)_{X^\circ} = -Z + D + \sigma(D), \tag{4.7}$$

where ψ_N and φ_N both have no zeros in k since they map k to an eigenfunction of the Schrödinger operator respectively its transpose. We have already seen in the proof of Lemma 4.7 that $\langle\!\langle \varphi_N, \psi_N \rangle\!\rangle_\partial = 2 \langle\!\langle \partial_y \varphi_N, \psi_N \rangle\!\rangle$ and it is shown in the proof of Lemma 3.16(iii) and (iv) that deriving φ_N into the direction of y cannot cause new poles at $k \in X^\circ$. So only new zeros can occur. We have seen above that these are contained in Z, and therefore, also dk_1 equals zero there of the same order. So these new zeros do not influence the poles $\langle\!\langle \varphi_N, \psi_N \rangle\!\rangle_\partial$ on X°. Since

$$\varphi_N(\,\cdot\,, (x,y)) = \frac{\varphi(\,\cdot\,, (x,y))}{\varphi(\,\cdot\,, (0,0))} \quad \text{and} \quad \psi_N(\,\cdot\,, (x,y)) = \frac{\psi(\,\cdot\,, (x,y))}{\psi(\,\cdot\,, (0,0))},$$

the denominator of these functions does not depend on $(x,y) \in \Delta$, so the same argumentation why deriving into the direction of y cannot cause new poles also yields that the poles on X° of the whole integrand $\partial_y \varphi_N \psi_N$ are not influenced by integrating over Δ. Since for each k, there always exist $\chi, \xi \in L^2(\Delta)$ such that $\langle\!\langle \xi, \phi \rangle\!\rangle, \langle\!\langle \varphi, \chi \rangle\!\rangle \not\equiv 0$, the divisor D coincides on X° with the

zero divisor of $\psi(\cdot,(0,0))$ and $\sigma(D)$ coincides on X° with the zero divisor of $\varphi(\cdot,(0,0))$. So the pole divisor of $\langle\!\langle \varphi_N, \psi_N \rangle\!\rangle_\partial$ on X coincides with $D+\sigma(D)$ and equation (4.7) holds.
Due to Corollary 4.10, $\langle\!\langle \varphi_N, \psi_N \rangle\!\rangle_\partial$ has a pole at $Q^\pm$ if and only if $\langle\!\langle \varphi_N^0, \psi_N^0 \rangle\!\rangle_\partial$ has a pole at $\tilde{Q}^\pm$, where $\tilde{Q}^\pm$ are the two points added at infinity to compactify $X(0)^\circ$. In the calculations in (4.2) we have seen that

$$\langle\!\langle \varphi_N^0, \psi_N^0 \rangle\!\rangle_\partial = -4\pi\iota \operatorname{Vol}(\varDelta) k_2 = 4\pi \operatorname{Vol}(\varDelta) k_1 = \frac{\pm 4\pi\iota \operatorname{Vol}(\varDelta)}{z_\pm} = \frac{4\pi \operatorname{Vol}(\varDelta)}{z_\pm},$$

where now $z_\pm$ is a local coordinate on $X(0)$ centered at $\tilde{Q}^\pm$. So $\langle\!\langle \varphi_N^0, \psi_N^0 \rangle\!\rangle_\partial$ has a pole of first order at each $\tilde{Q}^+$ and $\tilde{Q}^-$ and thus also $\langle\!\langle \varphi_N, \psi_N \rangle\!\rangle_\partial$ has simple poles at Q^+ and Q^-. Combining this with (4.7) shows that (4.5) holds and finally combining (4.4) with (4.5) yields

$$0 \simeq K + 2Q^+ + 2Q^- - D - \sigma(D) - Q^+ - Q^- \Leftrightarrow D + \sigma(D) \simeq K + Q^+ + Q^-.$$

The degree is invariant under σ, so $\deg(D) = \deg(\sigma(D))$ and $\deg(K) = 2g-2$, wherefore $\deg(D) = \frac{2g-2+1+1}{2} = g$. □

4.2.2. A connection between the holomorphic involution and the pole divisor

In [Novikov and Veselov, 1984] is remarked without proof that I. R. Shafarevich and V. V. Shokurov pointed out that $D+\sigma(D) \simeq K + Q^+ + Q^-$ can hold if and only if Q^+ and Q^- are the only fixed points of σ. To prove this assertion, we basically use the results reflecting the connection between the Jabobian variety and the Prym variety as defined in A.12 which is shown in [Mumford, 1974]. A more detailed reflection of the Prym variety, leaned on the results in [Mumford, 1974], can be found in Appendix A. Since we are mainly using the ideas shown there and not the whole concept, we will explain in this section how this connection involves here.

A two-sheeted covering

We first define another Riemann surface X_σ such that $\pi_\sigma : X \to X_\sigma$ defines a two-sheeted covering.

Definition 4.14. For $k, \tilde{k} \in X$ let $k \sim_\sigma \tilde{k} :\Leftrightarrow (k = \tilde{k} \vee k = \sigma(\tilde{k}))$ and define $X_\sigma := X/\sim_\sigma$.

In Lemma A.1, it is shown that X_σ is also a compact Riemann surface. Let $\pi_\sigma : X \to X_\sigma$ be the canonical two-sheeted covering map which is holomorphic, compare [Miranda, 1995, Theorem III.3.4]. Due to the construction of X_σ, the fixed points of σ coincide with the ramification points of π_σ.

Definition 4.15. We denote the *set of ramification points* of π_σ on X by r_{π_σ} and define the *ramification divisor* of π_σ on X as $R_{\pi_\sigma} := \sum_{k\in r_{\pi_\sigma}} k$. The *set of branch points* of π_σ on X_σ we define as $b_{\pi_\sigma} := \pi_\sigma[r_{\pi_\sigma}]$.

The map π_σ is locally biholomorphic on $X \setminus r_{\pi_\sigma}$, see [Forster, 1981, Corollary I.2.5]. In general, the ramification divisor is defined as $R_{\pi_\sigma} := \sum_{k\in X}(\mathrm{mult}_k(\pi_\sigma) - 1)\cdot k$, where the multiplicity $\mathrm{mult}_k(\pi_\sigma)$ of π_σ in k denotes the number of sheets which meet in k, compare [Miranda, 1995, Definition II.4.2]. Since $\mathrm{mult}_k(\pi_\sigma) = 1$ for $k \in X \setminus r_{\pi_\sigma}$ and $\mathrm{mult}_k(\pi_\sigma) = 2$ for $k \in r_{\pi_\sigma}$, this coincides with the definition above.

Definition 4.16. We define the *pullback of a point* $k_\sigma \in X_\sigma$ as

$$\pi_\sigma^* k_\sigma := \sum_{k\in\{\pi_\sigma^{-1}[\{k_\sigma\}]\}} \mathrm{mult}_k(\pi_\sigma)k.$$

The *pullback of a divisor* $D := \sum_{k_\sigma\in X_\sigma} a(k_\sigma)k_\sigma$ on X_σ is defined as $\pi_\sigma^* D := \sum_{k_\sigma\in X_\sigma} a(k_\sigma)\pi_\sigma^* k_\sigma$.

Since π_σ is a non-constant holomorphic map between two Riemann surfaces, every meromorphic 1-form on X_σ can be pulled back to a meromorphic 1-form $\omega := \pi_\sigma^*\omega_\sigma$ on X, compare for example [Miranda, 1995, Section IV.2.].

Lemma 4.17. *Let X, X_σ and π_σ be given as above and let ω_σ be a non-constant meromorphic 1-form on X_σ.*

(i) The divisor of $\pi_\sigma^*\omega_\sigma$ on X is given by $(\pi_\sigma^*\omega_\sigma) = \pi_\sigma^*(\omega_\sigma) + R_{\pi_\sigma}$.

(ii) Let g_σ be the genus of X_σ. Then there exists a divisor $\tilde{K}$ on X with $\deg(\tilde{K}) = 2g_\sigma - 2$ such that $(\pi_\sigma^*\omega_\sigma) = \tilde{K} + \sigma(\tilde{K}) + R_{\pi_\sigma}$ where $\pi_\sigma(\tilde{K})$ is the canonical divisor of X_σ.

Proof. (i) Due to [Miranda, 1995, Lemma IV.2.6], there holds for $k \in X$ that

$$\mathrm{ord}_k(\pi_\sigma^*\omega_\sigma) = (1 + \mathrm{ord}_{\pi_\sigma(k)}(\omega_\sigma))\mathrm{mult}_k(\pi_\sigma) - 1$$

with $\mathrm{ord}_k(\pi_\sigma^*\omega_\sigma)$ as defined in [Miranda, 1995, Section IV.1.9]. Inserting this into the definition of $(\pi_\sigma^*\omega_\sigma) = \sum_{k\in X}(\mathrm{ord}_k(\pi_\sigma^*\omega))$ yields the assertion as follows:

$$\begin{aligned}
(\pi_\sigma^*\omega_\sigma) &= \sum_{k\in X}(\mathrm{ord}_k(\pi_\sigma^*\omega))k = \sum_{k\in X}\Big((1+\mathrm{ord}_{\pi_\sigma(\tilde{k})}(\omega_\sigma))\cdot\mathrm{mult}_k(\pi_\sigma) - 1\Big)k \\
&= \sum_{k\in X}\mathrm{ord}_{\pi_\sigma(k)}(\omega_\sigma)\cdot\mathrm{mult}_k(\pi_\sigma)k + \sum_{k\in X}(\mathrm{mult}_k(\pi_\sigma)-1)k \\
&= \sum_{k_\sigma\in X_\sigma}\sum_{k\in\pi_\sigma^{-1}[\{k_\sigma\}]}\mathrm{ord}_{k_\sigma}(\omega_\sigma)\cdot\mathrm{mult}_k(\pi_\sigma)k + R_{\pi_\sigma} \\
&= \sum_{k_\sigma\in X_\sigma}\mathrm{ord}_{k_\sigma}(\omega)\sum_{k\in\pi_\sigma^{-1}[\{k_\sigma\}]}\mathrm{mult}_k(\pi_\sigma)k + R_{\pi_\sigma} = \sum_{k_\sigma\in X_\sigma}\mathrm{ord}_{k_\sigma}(\omega)\pi_\sigma^*k_\sigma + R_{\pi_\sigma} \\
&= \pi_\sigma^*(\omega) + R_{\pi_\sigma}.
\end{aligned}$$

(ii) One has $\deg K_\sigma = \deg(\omega_\sigma) = 2g_\sigma - 2$, where K_σ is the canonical divisor on X_σ. Let $k_\sigma \in X_\sigma$ be a point in the support of (ω_σ) as defined in [Miranda, 1995, Section V.1]. For $k_\sigma \notin b_{\pi_\sigma}$, one has $\pi_\sigma^* k_\sigma = k + \sigma(k)$ with $k \neq \sigma(k) \in X$ and for $k_\sigma \in b_{\pi_\sigma}$, it is $\pi_\sigma^* k_\sigma = 2k$ with $k \in r_{\pi_\sigma}$. For $k_\sigma \notin b_{\pi_\sigma}$, let one of the pulled back points in $\pi_\sigma^* k_\sigma$ be the contribution to $\tilde{K}$ and for $k_\sigma \in b_{\pi_\sigma}$, the pulled back point is counted with multiplicity one in $\tilde{K}$. Then $\pi_\sigma^*(K_\sigma) = \tilde{K} + \sigma(\tilde{K})$ and the claim follows from (i).

□

Now, we are going to construct a symplectic cycle basis of $H_1(X, \mathbb{Z})$ from a symplectic cycle basis of $H_1(X_\sigma, \mathbb{Z})$. The notation and the basic ideas for the construction of the cycles in this section is based on [Adler et al., 2010, Section 5.2.4]. The holomorphic map $\sigma : X \to X$ induces a homomorpism of $H_1(X, \mathbb{Z})$ which we denote as

$$\sigma_\sharp : H_1(X, \mathbb{Z}) \to H_1(X, \mathbb{Z}), \quad \gamma \mapsto \sigma_\sharp \gamma. \tag{4.8}$$

Let g_σ be the genus of X_σ and $A_{\sigma,1}, \ldots, A_{\sigma,g_\sigma}$, $B_{\sigma,1}, \ldots, B_{\sigma,g_\sigma}$ be representatives of a symplectic basis of $H_1(X_\sigma, \mathbb{Z})$, i.e.

$$A_{\sigma,i} \star A_{\sigma,\ell} = B_{\sigma,i} \star B_{\sigma,\ell} = 0 \text{ and } A_{\sigma,i} \star B_{\sigma,\ell} = \delta_{i\ell}, \tag{4.9}$$

where $\star$ is the intersection product between two cycles. From Riemann surface theory, it is known that such a basis exists, compare e.g. [Miranda, 1995, Section VIII.4]. Due to Hurwitz's Formula [Miranda, 1995, Theorem II.4.16], one knows that that $\sharp b_{\pi_\sigma} = 2n$ is even for the two sheeted covering $\pi_\sigma : X \to X_\sigma$ and that the genus g of X is given by $g = 2g_\sigma + n - 1$. Hence, a basis of $H_1(X, \mathbb{Z})$ consists of $4g_\sigma + 2n - 2$ cycles. The aim is to construct a symplectic basis of $H_1(X, \mathbb{Z})$ which we denote as $A_i, \sigma_\sharp A_i, B_i, \sigma_\sharp B_i$ and C_j, D_j which has the following two properties: First of all, the only non-trivial pairwise intersections between elements of the basis of $H_1(X, \mathbb{Z})$ must be given by

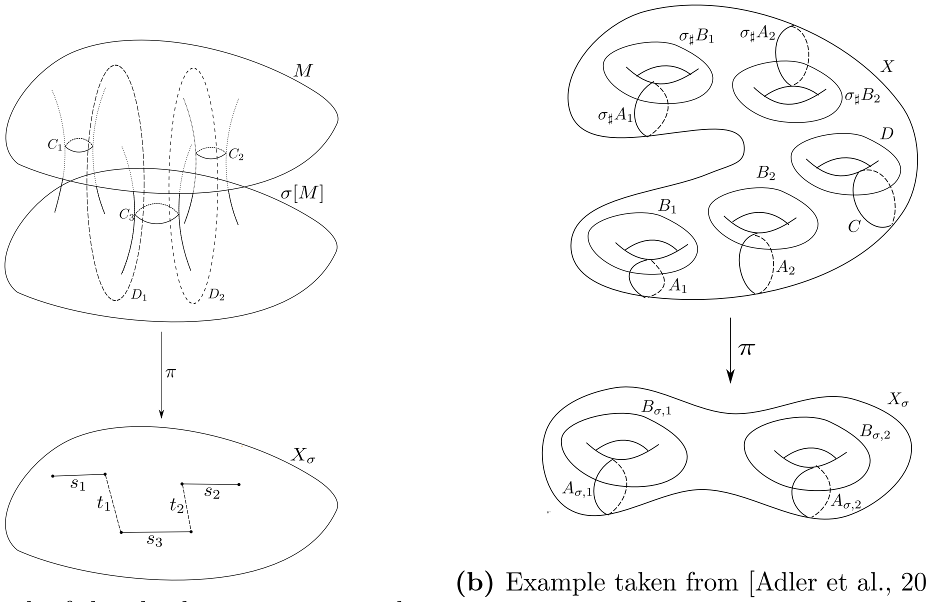
$$A_i \star B_i = \sigma_\sharp A_i \star \sigma_\sharp B_i = C_j \star D_j = 1. \tag{4.10}$$

Secondly, the involution $\sigma_\sharp$ has to map A_i to $\sigma_\sharp A_i$ and vice versa, B_i to $\sigma_\sharp B_i$ and vice versa and has to act on C_j and D_j as $\sigma_\sharp C_j = -C_j$ and $\sigma_\sharp D_j = -D_j$. In the rest of this subsection, let $i, \ell \in \{1, \ldots, g_\sigma\}$ and $j, k \in \{1, \ldots, n-1\}$ as long as not pointed out differently. The difference in the notation of the cycles indicates the origin of these basis elements: the A- and B-cycles on X will be constructed via lifting a certain symplectic cycle basis of $H^1(X_\sigma, \mathbb{Z})$ via π_σ and the C- and D-cycles originate from the branch points of π_σ.

We will start by constructing the C- and D-cycles. A sketch of the idea how to do this is shown for $n = 3$ and $g_\sigma = 0$ in Figure 4.1a . We connect the points in b_{π_σ} pairwise by paths s_j for $j = 1, \ldots, n$. The set of points corresponding to a path $s_j : [0,1] \to X_\sigma$ we denote by $[s_j] := \{s_j(t) \mid t \in [0,1]\}$

and use the same notation for any other path considered as a set of points in X or X_σ. Let $[s_j]^\circ$ be the corresponding set with $t \in (0,1)$. The paths s_j are constructed in such a way that every branch point is connected with exactly one other branch point and such that $s_k \cap s_j = \emptyset$ for $k \neq j$. This is possible because the branch points lie discrete on X_σ: suppose the first two branch points are connected by s_1 such that s_1 contains no other branch point. Then one can find a small open tubular neighborhood $N(s_1)$ of s_1 in X_σ with boundary $\partial N(s_1)$ in X_σ isomorphic to S^1. To see that $X_\sigma \setminus [s_1]$ is path connected, let γ be a path in X_σ which intersects $\partial N(s_1)$ in the two points $p_1, p_2 \in X_\sigma$. Then there is a path $\tilde{\gamma}$ such that $\tilde{\gamma}|_{X_\sigma \setminus N(s_1)} = \gamma|_{X_\sigma \setminus N(s_1)}$ and such that the points p_1 and p_2 are connected via a part of $\partial N(s_1)$. Hence, $X_\sigma \setminus [s_1]$ is path connected. Like that one can gradually choose $s_2, \dots s_n$. To find a path s_j not intersecting $s_1, \dots s_{j-1}$, consider $X_\sigma \setminus ([s_1] \cup \dots \cup [s_{j-1}])$ which is path connected and repeat the above procedure until all branch points are sorted into pairs. The preimage of s_j under π_σ yields two paths in X which both connect the preimage of the connected two branch points. These preimages are ramification points of π_σ and we denote them as b_j^1 and b_j^2. A suitable linear combination of the two paths on X then defines a cycle C_j for $j = 1, \dots, n$. Since π_σ is unbranched on $X \setminus r_{\pi_\sigma}$, i.e. a homeomorphism, and since $\pi_\sigma[r_{\pi_\sigma}] = b_{\pi_\sigma} \subset [s_1] \cup \dots \cup [s_n]$, $\pi_\sigma^{-1}[X_\sigma \setminus ([s_1] \cup \dots \cup [s_n])]$ consists of two disjoint connected manifolds whose boundaries both are equal to $\pi_\sigma^{-1}[s_1] \cup \dots \cup \pi_\sigma^{-1}[s_n]$ and σ interchanges those manifolds. We call them M and $\sigma[M]$. Since the n C-cycles are the boundary of M respectively $\sigma[M]$, they are homologous to another, i.e. $C_n = -\sum_{i=1}^{n-1} C_i$. So this construction yields maximal $n-1$ C-cycles which are not homologous to each other. These n cycles we orientate as the boundary of the Riemann surface M. We will see later on that, due to the intersection numbers, the cycles $C_1, \dots, C_{n-1}$ are not homologous to each other. By construction, each cycle C_j contains the two ramification points b_j^1 and b_j^2 of π_σ and no other ramification points.

The next step is to construct $n-1$ D-cycles such that one has $C_j \star D_k = \delta_{jk}$. We will see that it is possible to connect $\pi_\sigma(b_j^2)$ with $\pi_\sigma(b_{j+1}^1)$ by a path t_j for $j = 1, \dots, n-1$ such that $t_j \cap t_k = \emptyset$ for $j \neq k$. Since $X_\sigma \setminus ([s_1] \cup \dots \cup [s_n])$ is path connected, also $X_\sigma \setminus ([s_1]^\circ \cup [s_2]^\circ \cup [s_3] \cup \dots \cup [s_n])$ is path connected. So one can connect b_1^2 with b_2^1 with a path t_1 in X_σ not intersecting $s_3, \dots, s_n$ and the path $s_1 + t_1 + s_2$ in X_σ contains no loop. As above, one can chose a small open neighborhood $N([s_1] \cup [t_1] \cup [s_2])$ with boundary isomorphic to S^1. Therefore, $X_\sigma \setminus ([s_1] \cup \dots \cup [s_n] \cup [t_1])$ is path connected. Repeating this procedure shows that $X_\sigma \setminus ([s_1] \cup \dots \cup [s_n] \cup [t_1] \cup \dots \cup [t_j])$ remains path connected and that $\sum_{m=1}^{j}(s_m + t_m) + s_{j+1}$ contains no loop for $j = 1, \dots, n-1$. This yields the desired $n-1$ paths t_j in X_σ. Lifting these paths via π_σ yields each $n-1$ paths on M and $n-1$ paths on $\sigma[M]$. The paths on M and $\sigma[M]$ which result from the lift of t_j both start at b_j^2 and end in b_{j+1}^1. Hence, identifying these end points with each other yields a cycle on X which we denote as $\widetilde{D}_j$. We orientate $\widetilde{D}_j$ such that $C_j \star \widetilde{D}_j = 1$ and $C_{j+1} \star \widetilde{D}_j = -1$ for $j \in \{1, \dots, n-1\}$. Due to the construction of $\widetilde{D}_j$, one has $C_i \star \widetilde{D}_j = 0$ for $i \notin \{j, j+1\}$. Defining $D_j := \sum_{i=j}^{n-1} \widetilde{D}_i$



(a) A sketch of the idea how to construct the C-cycles for $n = 3$ and $g_\sigma = 0$.

(b) Example taken from [Adler et al., 2010, Figure 5.1] of how a cycle basis on X and X_σ could look like for $g_\sigma = 2$ and π_σ having two branch points on X.

Figure 4.1.: About constructing the cycle basis of $H_1 X, \mathbb{Z}$ out of $H_1(X_\sigma, \mathbb{Z})$.

yields for $k < j$ that

$$C_j \star D_j = C_j \star \sum_{l=j}^{n-1} \widetilde{D}_l = C_j \star \widetilde{D}_j = 1, \quad C_k \star D_j = C_k \star \sum_{l=j}^{n-1} \widetilde{D}_l = 0,$$
$$C_j \star D_k = C_j \star \sum_{l=k}^{n-1} \widetilde{D}_l = C_j \star (\widetilde{D}_j + \widetilde{D}_{j-1}) = 1 - 1 = 0$$

and hence $n-1$ cycles which obey $C_k \star D_j = \delta_{kj}$. Two cycles cannot be homologous to each other if the intersection number of each one of those cycles with a third cycle is not equal. Consequently, $C_k \star D_j = \delta_{kj}$ implies that the above construction yields $2n-2$ cycles C_j and D_j which are not homologous to each other. To construct the missing $4g_\sigma$ cycles, we choose a symplectic cycle basis $A_{\sigma,i}, B_{\sigma,i}$ of $H_1(X_\sigma, \mathbb{Z})$ such that they intersect none of the paths $s_1, \ldots, s_n$ and $t_1, \ldots, t_{n-1}$. This is possible since all of these paths in X_σ are connected, and therefore can be contracted to a point. On the preimage of $X_\sigma \setminus \{\bigcup_{j=1}^{n-1}([s_j] \cup [t_j]) \cup [s_n]\}$, the map π_σ is a homeomorphism. So each of the cycles in $H_1(X_\sigma, \mathbb{Z})$ is lifted to one cycle in M and one cycle in $\sigma[M]$ via π_σ and those two cycles are interchanged by σ. Thus, lifting the whole basis yields $4g_\sigma$ cycles on X, where we denote the $2g_\sigma$ cycles lifted to M as A_i and B_i and the corresponding cycles lifted to $\sigma[M]$ as $\sigma_\sharp A_i$ and $\sigma_\sharp B_i$. By the universal lifting property of paths, see Lemma A.2, these cycles obey the desired transformation behavior under $\sigma_\sharp$. Since M and $\sigma[M]$ are disjoint, the intersection number of the lifted cycles on X stays the same as the intersection number of the corresponding cycles on X_σ if two cycles are lifted to the same sheet M respectively $\sigma[M]$ or equals zero if they are lifted to different sheets. Furthermore, the construction of these cycles ensures that the lifted A- and B-cycles do not intersect any of the C- and D-cycles on X. Hence, $A_i, \sigma_\sharp A_i, B_i, \sigma_\sharp B_i, C_j$ and D_j are in total $4g_\sigma + 2n - 2$ cycles which obey condition (4.10). So by Hurwitz Formula [Miranda, 1995, Theorem II.4.16], they represent a symplectic basis of $H_1(X, \mathbb{Z})$ and the A- and C-cycles are disjoint. That the C- and D-cycles constructed like this have the desired transformation behavior under $\sigma_\sharp$ is shown in the next lemma.

Lemma 4.18. *For $C_j, D_j \in H_1(X, \mathbb{Z})$ as defined above one has $\sigma_\sharp C_j = -C_j$ and $\sigma_\sharp D_j = -D_j$.*

Proof. Every cycle C_j is the preimage of a path in X_σ and X_σ is invariant under σ. So $\sigma[C_j] = [C_j]$ and the two points b_j^1 and b_j^2 stay fixed. Therefore, $\sigma_\sharp C_j = \pm C_j$. Since σ commutes, the two lifts of the path s_j in X_σ, i.e. b_j^1 and b_j^2 are the only fixed points of σ on C_j, one has $\sigma_\sharp C_j = -C_j$. Since D_j also consists of the two lifts of t_j which are interchanged by σ, it follows by the same means that also $\sigma_\sharp D_j = -D_j$. □

Decomposition of $H_1(X, \mathbb{Z})$

With help of the Abel map Ab one can identify the elements of $H_1(X, \mathbb{Z})$ with a lattice in $\mathbb{C}^g$ such that $\mathrm{Jac}(X) \simeq \mathbb{C}^g/\Lambda$, compare [Miranda, 1995, Section VIII.2]. To do so, let $\omega_1, \ldots, \omega_g \in H^0(X, \Omega)$ be a basis of the $g = 2g_\sigma + n - 1$ holomorphic differential forms on X which are normalized with respect to the A-, $\sigma_\sharp A$- and C-cycles, i.e.

$$\oint_{A_i} \omega_\ell = \delta_{i\ell}, \quad \oint_{\sigma_\sharp A_i} \omega_{g_\sigma+\ell} = \delta_{i\ell}, \quad \oint_{C_j} \omega_{2g_\sigma+k} = \delta_{jk} \tag{4.11}$$

and all other integrals over one of the A- and C-cycles with another element of the basis of $H^0(X, \Omega)$ are equal to zero. Furthermore, note that the construction of the A-cycles yields $\sigma^*\omega_i = \omega_{g_\sigma+i}$ and that Lemma 4.18 implies $\sigma^*\omega_{2g_\sigma+j} = -\omega_{2g_\sigma+j}$. We define

$$\omega_i^\pm := \frac{1}{2}(\omega_i \pm \omega_{g_\sigma+i}) \quad \text{and} \quad \omega_{g_\sigma+j}^- := \omega_{2g_\sigma+j}. \tag{4.12}$$

Direct calculation shows that these differential forms also yield a basis of $H^0(X, \Omega)$, compare the proof of Lemma A.7 and Proposition A.10. For a path γ in X, we define the vectors

$$\Omega_\gamma := \left(\int_\gamma \omega_k \right)_{k=1}^{g}, \; \Omega_\gamma^+ := \left(\int_\gamma \omega_k^+ \right)_{k=1}^{g_\sigma}, \; \Omega_\gamma^- := \left(\int_\gamma \omega_k^- \right)_{k=1}^{g_\sigma+n-1}. \tag{4.13}$$

and the following lattices generated over $\mathbb{Z}$ as

$$\begin{aligned}
\Lambda &:= \langle \Omega_{A_i}, \Omega_{\sigma_\sharp A_i}, \Omega_{C_j}, \Omega_{B_i}, \Omega_{\sigma_\sharp B_i}, \Omega_{D_j} \rangle_{\substack{i=1,\dots,g_\sigma \\ j=1,\dots,n-1}} \\
\Lambda_+ &:= \langle \Omega^+_{A_i+\sigma_\sharp A_i}, \Omega^+_{B_i+\sigma_\sharp B_i} \rangle_{i=1,\dots,g_\sigma} \\
\Lambda_- &:= \langle \Omega^-_{A_i-\sigma_\sharp A_i}, \Omega^-_{B_i-\sigma_\sharp B_i}, \Omega^-_{C_j}, \Omega^-_{D_j} \rangle_{\substack{i=1,\dots,g_\sigma \\ j=1,\dots,n-1}}.
\end{aligned} \tag{4.14}$$

Furthermore, the map

$$\Phi : \mathbb{C}^g \to \mathbb{C}^{g_\sigma} \oplus \mathbb{C}^{g_\sigma+n-1}, \qquad \begin{pmatrix} v_1 \\ \vdots \\ v_g \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{2}(v_1 + v_{g_\sigma+1}) \\ \vdots \\ \frac{1}{2}(v_{g_\sigma} + v_{2g_\sigma}) \end{pmatrix} \oplus \begin{pmatrix} \frac{1}{2}(v_1 - v_{g_\sigma+1}) \\ \vdots \\ \frac{1}{2}(v_{g_\sigma} - v_{2g_\sigma}) \\ v_{2g_\sigma+1} \\ \vdots \\ v_{2g_\sigma+n-1} \end{pmatrix}$$

is obviously linear and bijective. Hence, Φ is a vector space isomorphism.

Lemma 4.19. *For every path γ on X, one has*

$$\Phi(\Omega_\gamma) = \Omega^+_\gamma \oplus \Omega^-_\gamma = \Omega^+_{\frac{1}{2}(\gamma+\sigma_\sharp\gamma)} \oplus \Omega^-_{\frac{1}{2}(\gamma-\sigma_\sharp\gamma)}.$$

Proof. The first equality follows from the definitions of Φ and the differential forms in (4.12):

$$\Phi(\Omega_\gamma) = \Phi \begin{pmatrix} \int_\gamma \omega_1 \\ \vdots \\ \int_\gamma \omega_g \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\int_\gamma \omega_1 + \omega_{g_\sigma+1}) \\ \vdots \\ \frac{1}{2}(\int_\gamma \omega_{g_\sigma} + \omega_{2g_\sigma}) \end{pmatrix} \oplus \begin{pmatrix} \frac{1}{2}(\int_\gamma \omega_1 - \omega_{g_\sigma+1}) \\ \vdots \\ \frac{1}{2}(\int_\gamma \omega_{g_\sigma} - \omega_{2g_\sigma}) \\ \int_\gamma \omega_{2g_\sigma+1} \\ \int_\gamma \omega_{2g_\sigma+n-1} \end{pmatrix} =$$

$$= \begin{pmatrix} \int_\gamma \omega^+_1 \\ \vdots \\ \int_\gamma \omega^+_{g_\sigma} \end{pmatrix} \oplus \begin{pmatrix} \int_\gamma \omega^-_1 \\ \vdots \\ \int_\gamma \omega^-_{g_\sigma} \\ \int_\gamma \omega^-_{g_\sigma+1} \\ \int_\gamma \omega^-_{g_\sigma+n-1} \end{pmatrix}.$$

Since $\omega_k^+ = \sigma^*\omega_k^+$ for $k = 1, \ldots, g_\sigma$ and $\omega_k^- = -\sigma^*\omega_k^-$ for $k = 1, \ldots, g_\sigma + n - 1$ one has

$$\int_\gamma \omega_k^+ = \frac{1}{2}\Big(\int_\gamma \omega_k^+ + \sigma^*\omega_k^+\Big) = \int_{\frac{1}{2}(\gamma+\sigma_\sharp\gamma)} \omega_k^+$$

as well as

$$\int_\gamma \omega_k^- = \frac{1}{2}\Big(\int_\gamma \omega_k^- - \sigma^*\omega_k^-\Big) = \int_{\frac{1}{2}(\gamma-\sigma_\sharp\gamma)} \omega_k^-$$

which implies the second equality. □

Corollary 4.20. *The generators of $\Phi^{-1}(\Lambda_+ \oplus \Lambda_-)$ span a basis of $\mathbb{C}^g$ over $\mathbb{R}$, the generators of Λ_+ span a basis of $\mathbb{C}^{g_\sigma}$ over $\mathbb{R}$ and the generators of Λ_- span a basis of $\mathbb{C}^{g_\sigma+n-1}$ over $\mathbb{R}$.*

Proof. Since $\mathrm{Jac}(X) = \mathbb{C}^g/\Lambda$ is a complex torus, the generators of Λ given in (4.14) are a basis of $\mathbb{C}^g$ over $\mathbb{R}$, compare for example [Lange and Birkenhake, 1992, Section II.2]. Basis transformation yields that $\Omega_{A_i+\sigma_\sharp A_i}$, $\Omega_{A_i-\sigma_\sharp A_i}$,$\Omega_{B_i+\sigma_\sharp B_i}$, $\Omega_{B_i-\sigma_\sharp B_i}$, Ω_{C_j} and Ω_{D_j} are also a basis of $\mathbb{C}^g$ over $\mathbb{R}$. Since Φ is a vector space isomorphism with

$$\begin{aligned} \Phi(\Omega_{A_i+\sigma_\sharp A_i}) &= \Omega^+_{A_i+\sigma_\sharp A_i} \oplus 0, & \Phi(\Omega_{A_i-\sigma_\sharp A_i}) &= 0 \oplus \Omega^-_{A_i-\sigma_\sharp A_i}, \\ \Phi(\Omega_{B_i+\sigma_\sharp B_i}) &= \Omega^+_{B_i+\sigma_\sharp B_i} \oplus 0, & \Phi(\Omega_{B_i-\sigma_\sharp B_i}) &= 0 \oplus \Omega^-_{B_i-\sigma_\sharp B_i}, \\ \Phi(\Omega_{C_j}) &= 0 \oplus \Omega^-_{C_j}, & \Phi(\Omega_{D_j}) &= 0 \oplus \Omega^-_{D_j}, \end{aligned}$$

the generators of $\Phi^{-1}(\Lambda_+ \oplus \Lambda_-)$ yield a basis of $\mathbb{C}^g$ over $\mathbb{R}$. Since Φ is an isomorphism, the generators of Λ_+ are a basis of $\mathbb{C}^{g_\sigma}$ and the generators of Λ_- of $\mathbb{C}^{g_\sigma+n-1}$ over $\mathbb{R}$. □

In the sequel, we will apply Φ and Φ^{-1} to lattices. Note that we abuse the notation to gain clarity in the sense that $\Phi(\Lambda)$ denotes the lattice in $\mathbb{C}^{g_\sigma} \oplus \mathbb{C}^{g_\sigma+n-1}$ spanned by the image of the generators of Λ under Φ and analogously for Φ^{-1} applied to lattices.

Lemma 4.21. *(a) $\Lambda_+ \oplus 0 = \Phi(\Lambda) \cap (\mathbb{C}^{g_\sigma} \oplus 0)$ and $0 \oplus \Lambda_- = \Phi(\Lambda) \cap (0 \oplus \mathbb{C}^{g_\sigma+n-1})$.*

(b) $\Phi(\Lambda)$ decomposes as

$$\Phi(\Lambda) = (\Lambda_+ \oplus \Lambda_-) + M \tag{4.15}$$

with

$$M := \Big\{ \sum_{i=1}^{g_\sigma} \Big(\frac{a_i}{2}\Omega^+_{A_i+\sigma_\sharp A_i} + \frac{b_i}{2}\Omega^+_{B_i+\sigma_\sharp B_i}\Big) \oplus \sum_{i=1}^{g_\sigma} \Big(\frac{a_i}{2}\Omega^-_{A_i-\sigma_\sharp A_i} + \frac{b_i}{2}\Omega^-_{B_i-\sigma_\sharp B_i}\Big) \,\Big|\, a_i, b_i \in \{0,1\} \Big\}.$$

(c) $M \cap (\Lambda_+ \oplus \Lambda_-) = \{0\}$.

Proof. Obviously, $\Lambda_+ \oplus 0$ is contained in $\Phi(\Lambda) \cap (\mathbb{C}^{g_\sigma} \oplus 0)$. To see that $\Phi(\Lambda) \cap (\mathbb{C}^{g_\sigma} \oplus 0)$ is also a subset of $\Lambda_+ \oplus 0$, note that for every $\gamma \in \Lambda$, there exist coefficients $a_i, a_{\sigma,i}, b_i, b_{\sigma,i}, c_j, d_j \in \mathbb{Z}$ such that

$$\gamma = \sum_{i=1}^{g_\sigma} a_i \Omega_{A_i} + a_{\sigma,i}\Omega_{\sigma_\sharp A_i} + b_i \Omega_{B_i} + b_{\sigma,i}\Omega_{\sigma_\sharp B_i} + c_j \Omega_{C_j} + d_j \Omega_{D_j}.$$

The generators of Λ_+ and Λ_- are linearly independent, compare Corollary 4.20. So the second equality in Lemma 4.19 shows that $\Phi(\gamma) \in \mathbb{C}^{g_\sigma} \oplus 0$ can only hold if $c_j = d_j = 0$, $a_i = a_{\sigma_i}$ and $b_i = b_{\sigma_i}$. So for such γ, it is

$$\begin{aligned}\Phi(\gamma) &= 2a_i \Omega^+_{\frac{1}{2}(A_i+\sigma_\sharp A_i)} + 2b_i \Omega^+_{\frac{1}{2}(B_i+\sigma_\sharp B_i)} \oplus 0 \\ &= a_i \Omega^+_{A_i+\sigma_\sharp A_i} + b_i \Omega^+_{B_i+\sigma_\sharp B_i} \oplus 0 \in \Lambda_+ \oplus 0.\end{aligned}$$

The equality $0 \oplus \Lambda_- = \Phi(\Lambda) \cap (0 \oplus \mathbb{C}^{g_\sigma+n-1})$ follows in the same manner. So (a) holds.
To get insight into (b), we will show that for the set of cosets one has

$$\begin{aligned}\Phi(\Lambda)/(\Lambda_+ \oplus \Lambda_-) &= \{\Phi(\lambda) + (\Lambda_+ \oplus \Lambda_-) \mid \lambda \in \Lambda\} \\ &= \{\Phi(\lambda) + (\Lambda_+ \oplus \Lambda_-) \mid \lambda \in M\}.\end{aligned}$$

The lattice Λ is a finitely generated abelian group, so also $\Phi(\Lambda)$, Λ_+ and Λ_- are finitely generated abelian groups and $\Phi(2\Lambda) \subset \Lambda_+ \oplus \Lambda_- \subset \Phi(\Lambda)$, where the second inclusion is obvious and the first inclusion holds since any element $2\Omega_\gamma$ of 2Λ can be decomposed as $2\Omega_\gamma = 2\big(\frac{1}{2}(\Omega_\gamma + \Omega_{\sigma_\sharp\gamma}) + \frac{1}{2}(\Omega_\gamma - \Omega_{\sigma_\sharp\gamma})\big)$. Therefore, $\Phi(\Lambda)/(\Lambda_+ \oplus \Lambda_-) \subset \Phi(\Lambda)/\Phi(2\Lambda)$ and the set of the $(\Lambda : 2\Lambda) = 2^{2g}$ elements contained in $\Phi(\Lambda)/\Phi(2\Lambda)$ is the maximal set of points which are not contained in $\Lambda_+ \oplus \Lambda_-$, but in $\Phi(\Lambda)$. One has $\Phi(\Omega_{C_j}), \Phi(\Omega_{D_j}) \in \Lambda_+ \oplus \Lambda_- \subset \Phi(\Lambda)$. Therefore, all points in M are linear combinations of $\Phi(\Omega_{A_i})$, $\Phi(\Omega_{\sigma_\sharp A_i})$, $\Phi(\Omega_{B_i})$ and $\Phi(\Omega_{\sigma_\sharp B_i})$ with coefficients in $\{0,1\}$. Since $\Omega_{A_i} = \Omega_{A_i+\sigma_\sharp A_i} - \Omega_{\sigma_\sharp A_i}$, it is $[\Phi(\Omega_{A_i})] = [\Phi(\Omega_{\sigma_\sharp A_i})]$ and $[\Phi(\Omega_{B_i})] = [\Phi(\Omega_{\sigma_\sharp B_i})]$ in $\Phi(\Lambda)/(\Lambda_+ \oplus \Lambda_-)$ and thus

$$M \subseteq \Big\{ \sum_{i=1}^{g_\sigma} a_i \Phi(\Omega_{A_i}) + b_i \Phi(\Omega_{B_i}) \,\Big|\, a_i, b_i \in \{0,1\}\Big\}. \tag{4.16}$$

Furthermore, $\Phi(\Omega_{A_i}) = \frac{1}{2}(\Phi(\Omega_{A_i+\sigma_\sharp A_i}) + \Phi(\Omega_{A_i-\sigma_\sharp A_i}))$. Due to Corollary 4.20, these representations of $\Phi(\Omega_{A_i})$ as vectors in $\mathbb{C}^g$ in the basis given by the generators of $\Lambda_+ \oplus \Lambda_-$ is unique, i.e. $\Phi(\Omega_{A_i}) \notin \Lambda_+ \oplus \Lambda_-$ and by the same means $\Phi(\Omega_{\sigma_\sharp A_i})$, $\Phi(\Omega_{B_i})$, $\Phi(\Omega_{\sigma B_i}) \notin \Lambda_+ \oplus \Lambda_-$. The linear independence of the generators of Λ then yields equality in (4.16). Hence, $(\Phi(\Lambda) : \Lambda_+ \oplus \Lambda_-) = 2^{2g_\sigma}$, and so Λ can be seen as finitely many copies of $\Lambda_+ \oplus \Lambda_-$ translated by the points in M. Finally, the linear independence of the generators of Λ_+ and Λ_- and the definition of M imply (c). ☐

Remark 4.22. In [Mumford, 1974], it is shown that $\mathrm{Jac}(X_\sigma) \simeq \mathbb{C}^{g_\sigma}/\Lambda_+$ and that the Prym variety $\mathrm{Prym}(X,\sigma)$ can be identified with $\mathbb{C}^{g_\sigma+n-1}/\Lambda_-$. Furthermore, it is also shown that the direct sum $\mathrm{Jac}(X_\sigma) \oplus \mathrm{Prym}(X,\sigma)$ is only isogenous to $\mathrm{Jac}(X)$, but that the quotient of this direct sum divided by a finite set of points is isomorphic to $\mathrm{Jac}(X)$. The explicit calculations in Lemmata

4.19 and 4.21 are mirroring this connection and the finite set of points which are divided out of the direct sum in [Mumford, 1974, Section 2, Data II] are exactly the points in M. This is illustrated more detailed in Appendix A from on Lemma A.11.

The fixed points of σ and the linear equivalence

Theorem 4.23. *Let X be a Riemann surface of genus g, K be a canonical divisor on X, $\sigma : X \to X$ be a holomorphic involution and $Q^+, Q^- \in X$ be fixed points of σ. Then there exists a divisor D of degree g on X which solves*

$$D + \sigma(D) \simeq K + Q^+ + Q^- \tag{4.17}$$

if and only if σ has exactly the two fixed points Q^+ and Q^-.

Proof. Assume that σ has more fixed points than Q^+ and Q^-, i.e. $n > 1$, and that (4.17) holds. Due to Lemma 4.17, there exists a divisor $\tilde{K}$ of degree $2g_\sigma - 2$ on X such that $K = \tilde{K} + \sigma(\tilde{K}) + R_{\pi_\sigma}$ and hence equation (4.17) yields $D - \tilde{K} + \sigma(D - \tilde{K}) \simeq R_{\pi_\sigma} + Q^+ + Q^-$. We sort the $2n$ ramification points in r_{π_σ} into pairs as it is done in the construction of the C-cycles and denote the two fixed points on C_n as Q^+ and Q^-. Then equation (4.17) reads as $D - \tilde{K} + \sigma(D - \tilde{K}) \simeq \sum_{j=1}^{n-1}(b_j^1 + b_j^2) + 2Q^+ + 2Q^-$. With $\tilde{D} := D - \tilde{K} - \sum_{j=1}^{n-1} b_j^1 - Q^+ - Q^-$ this is equivalent to

$$\tilde{D} + \sigma(\tilde{D}) + \sum_{j=1}^{n-1}(b_j^1 - b_j^2) \simeq 0. \tag{4.18}$$

Furthermore, $\deg(\tilde{D} + \sigma(\tilde{D}) + \sum_{j=1}^{n-1}(b_j^1 - b_j^2)) = 0$ and $\deg(\sum_{j=1}^{n-1}(b_j^1 - b_j^2)) = 0$ since $\sum_{j=1}^{n-1}(b_j^1 - b_j^2)$ contains the same number of points counted with multiplicity 1 as counted with multiplicity -1. Since deg acts linear on divisors and is invariant under σ, this yields $\deg(\tilde{D}) = 0$. So counted without multiplicity, there are as many points with positive sign as with negative sign in $\tilde{D}$, i.e. $\tilde{D} = \sum_{m=1}^{\ell}(p_m^1 - p_m^2)$. Let $\gamma_m : [0,1] \to X$ be a path with $\gamma_m(0) = p_m^1$ and $\gamma_m(1) = p_m^2$. Then $\sigma_\sharp\gamma_m : [0,1] \to X$ is a path with $\sigma_\sharp\gamma_m(0) = \sigma(p_m^1)$ and $\sigma_\sharp\gamma_m(1) = \sigma(p_m^2)$. We define $\gamma_{\tilde{D}} := \sum_{m=1}^{\ell}\gamma_m$ and $\sigma_\sharp\gamma_{\tilde{D}} := \sum_{m=1}^{\ell}\sigma_\sharp\gamma_m$. Analogously, let $\gamma_{R,j}$ be defined as the paths $\gamma_{R,j} : [0,1] \to X$ such that $\gamma_{R,j}(0) = b_j^1$ and $\gamma_{R,j}(1) = b_j^2$. Then, due to the construction of the C-cycles, one has $\gamma_{R,j} - \sigma_\sharp\gamma_{R,j} = C_j$. We define $\gamma_R := \sum_{j=1}^{n-1}\gamma_{R,j}$. Set $\gamma := \gamma_{\tilde{D}} + \sigma_\sharp\gamma_{\tilde{D}} + \gamma_R$ and let $\omega_1, \dots, \omega_g$ be a canonical basis of $H^0(X, \Omega)$ normalized with respect to the A- and C-cycles as in (4.11). Again, we use the identification $\mathrm{Jac}(X) = \mathbb{C}^g/\Lambda$ via the Abel map Ab with the basis of holomorphic 1-forms on X normalized as in (4.11). Due to (4.18), the linear equivalence can also be expressed as

$$\mathrm{Ab}\Big(\tilde{D} + \sigma(\tilde{D}) + \sum_{j=1}^{n-1}(b_j^1 - b_j^2)\Big) = 0 \mod \Lambda.$$

This equation can only hold if $\Omega_\gamma \in \Lambda$. Due to Lemma 4.19, we can split $\Omega_\gamma \in \mathbb{C}^g$ uniquely by considering $\Phi(\Omega_\gamma) = \Omega_\gamma^+ \oplus \Omega_\gamma^-$ and because of the decomposition of Λ in (4.15), $\Omega_\gamma \in \Lambda$ is equivalent to $\Omega_\gamma^+ \oplus \Omega_\gamma^- \in (\Lambda_+ \oplus \Lambda_-) + M$ as defined in Lemma 4.21. So we want to show that $\Omega_\gamma^+ \oplus \Omega_\gamma^-$ is not contained in any of the translated copies of $\Lambda_+ \oplus \Lambda_-$ if $n > 1$. Since it will turn out that it is Ω_γ^- which leads to this assertion, we determine the explicit form of Ω_γ^-. For every $\omega^- \in H^0(X, \Omega)$ such that $\sigma^*\omega^- = -\omega^-$, one has

$$\int_{\gamma_{\tilde{D}}+\sigma_\sharp\gamma_{\tilde{D}}} \omega^- = \int_{\gamma_{\tilde{D}}} \omega^- + \int_{\gamma_{\tilde{D}}} \sigma^*\omega^- = \int_{\gamma_{\tilde{D}}} \omega^- - \int_{\gamma_{\tilde{D}}} \omega^- = 0$$

as well as

$$2\int_{\gamma_{R,i}} \omega^- = \int_{\gamma_{R,i}} \omega^- - \int_{\gamma_{R,i}} \sigma^*\omega^- = \int_{\gamma_{R,i}} \omega^- - \int_{\sigma_\sharp\gamma_{R,i}} \omega^- = \int_{\gamma_{R,i}-\sigma_\sharp\gamma_{R,i}} \omega^- = \oint_{C_i} \omega^-,$$

i.e. $\int_{\gamma_{R,i}} \omega^- = \frac{1}{2}\oint_{C_i} \omega^-$. So

$$\Omega_\gamma^- = \left(\int_{\gamma_R} \omega_i^-\right)_{k=1}^{g_\sigma+n-1} = \frac{1}{2}\left(\sum_{k=1}^{n-1}\oint_{C_k} \omega_k^-\right)_{k=1}^{g_\sigma+n-1} = \frac{1}{2}\sum_{k=1}^{n-1}\left(\oint_{C_k} \omega_k^-\right)_{k=1}^{g_\sigma+n-1} = \frac{1}{2}\sum_{k=1}^{n-1}\Omega_{C_k}^-.$$

Due to the definition of $\Omega_{C_k}^-$ in (4.13), it is $\Omega_\gamma^- = \frac{1}{2}\sum_{k=1}^{n-1}\Omega_{C_k}^-$. If $\Omega_\gamma^+ \oplus \Omega_\gamma^-$ would be contained in one of the translated copies of $\Lambda_+ \oplus \Lambda_-$, then Ω_γ^- would be contained in the second component of the direct sum in one of the translated copies of Λ_- introduced in Lemma 4.21. This is not possible since the generators of $\Lambda_+ \oplus \Lambda_-$ are linearly independent and only integer linear combinations of C-cycles are contained in all translated lattices. Therefore, $\Omega_\gamma \notin \Lambda$ for $n > 1$. If $n \leq 1$, then there are no C-cycles in $H_1(X, \mathbb{Z})$ and equation (4.18) reads as $D + \sigma(D) \simeq 0$. So equation (4.17) can only hold if $n \leq 1$. Since Q^+ and Q^- are fixed points of σ it is $n = 1$.
Let now Q^+ and Q^- be the only fixed points of σ. Then Lemma 4.17 yields that there exists a divisor $\tilde{K}$ on X with $\deg(\tilde{K}) = 2g_\sigma - 2$ such that $K = \tilde{K} + \sigma(\tilde{K}) + Q^+ + Q^-$. Define $D := \tilde{K} + Q^+ + Q^-$. The Hurwitz Formula [Miranda, 1995, Theorem II.4.16] for $n = 1$ yields $\deg(D) = 2g_\sigma = g$ and one has

$$D + \sigma(D) = \tilde{K} + \sigma(\tilde{K}) + 2Q^+ + 2Q^- \simeq K + Q^+ + Q^-.$$

□

4.2.3. A divisor condition for real regular finite type potentials

If the potential u is real-valued, then there is an antiholomorphic involution $\tau_1 : X' \to X'$, $k \mapsto -\bar{k}$, see Lemma 1.17(b). From this one can deduce a 'realness' condition on the divisor.

Lemma 4.24. *Let $u \in C(\mathbb{R}^2/\Gamma, \mathbb{R})$ be a real-valued regular finite type potential, X' the corresponding Fermi curve and D the corresponding divisor of ψ_N on the normalization X. Then $\tau_1(D) = D$.*

Proof. Let $u \in C(\mathbb{R}^2/\Gamma, \mathbb{R})$ be a regular finite type potential and $\psi_N(\cdot)_k$ the eigenfunction of

$-\triangle_k + u$ normalized as $\psi_{N,k}(0,0) = 1$, i.e. we consider the formulation of the Schrödinger equation we called trivialization in Definition 1.5. Then $(-\triangle_k + u)\psi_{N,k} = 0$ yields

$$(\tau_1^* \overline{(-\triangle_k + u)\psi_{N,k}}) = 0 \;\Leftrightarrow\; (-\tau_1^*\overline{\triangle}_k + u)\tau_1^*\bar{\psi}_{N,k}) = 0 \;\Leftrightarrow\; (-\triangle_k + u)\tau_1^*\bar{\psi}_{N,k} = 0.$$

Due to the normalization of the eigenfunctions, it is $\tau_1^*\bar{\psi}_{N,k} = \psi_{N,k}$. So Lemma 1.2 yields

$$\begin{aligned}\tau_1^*\bar{\psi}_N(k) &= \tau_1^*\left(\overline{e^{2\pi\iota\langle k,\binom{x}{y}\rangle}\psi_{N,k}}\right) = \tau_1^*\left(e^{-2\pi\iota\langle\bar{k},\binom{x}{y}\rangle}\bar{\psi}_{N,k}\right)\\ &= e^{2\pi\iota\langle k,\binom{x}{y}\rangle}\tau_1^*\bar{\psi}_{N,k} = e^{2\pi\iota\langle k,\binom{x}{y}\rangle}\psi_{N,k} = \psi_N(k)\end{aligned}$$

and thus $\tau_1(D) = D$. □

4.2.4. Non-speciality of the eigendivisor

Usually, a divisor is called special if $\dim H^1(X, \mathcal{O}_D) > 0$ and non-special otherwise. The Riemann Roch Theorem [Forster, 1981, § 16.10] yields that for a divisor of degree g, it is

$$\dim H^0(X, \mathcal{O}_D) = 1 + \dim H^1(X, \mathcal{O}_D).$$

So the divisors of degree g which are special and belong to a normalized eigenfunction cannot be uniquely assigned to an element of $H^0(X, \mathcal{O}_D)$ since then $\dim H^0(X, \mathcal{O}_D) > 1$. So it will be a crucial assumption in the inverse problem that the considered given divisor is non-special in a certain sense.

Definition 4.25. We call a positive divisor D of degree g on X *non-special* if

$$\dim H^1(X, \mathcal{O}_{D-Q^\pm}) = 0.$$

To show that this modification also holds in our case, the following two Lemmata are necessary. These are direct consequences of Serre Duality [Forster, 1981, §17.9] and the Riemann Roch Theorem [Forster, 1981, § 16.10]. These Lemmata are also necessary to show that our modified version of non-speciality holds for the divisor D. The first Lemma has also been shown for the more general case of singular curves and generalized divisors in [Klein et al., 2016, Lemma 8.1]. However, the proof for the case of classical divisors on a Riemann surface is much shorter, so we formulate it here anyway.

Lemma 4.26. *Let $D' \geq D$ be two divisors on X. Then $H^1(X, \mathcal{O}_D) = 0$ implies $H^1(X, \mathcal{O}_{D'}) = 0$.*

Proof. This follows immediately from Serre Duality [Forster, 1981, §17.9]. Let D, D' be divisors on a Riemann surface X such that $D' \geq D$. Then

$$\underbrace{H^1(X, \mathcal{O}_D)}_{=0} \underset{\Rightarrow}{\simeq} \underbrace{H^0(X, \mathcal{O}_{K-D})}_{=0},$$

where K is the canonical divisor on X. Because of $-D' \leq -D$, there holds $H^0(X, \mathcal{O}_{K-D'}) \subset H^0(X, \mathcal{O}_{K-D})$, so $H^0(X, \mathcal{O}_{K-D'}) = 0$. Using Serre Duality [Forster, 1981, 17.6] again then yields $H^0(X, \mathcal{O}_{K-D'}) \simeq H^1(X, \mathcal{O}_{D'})$. □

Lemma 4.27. *For every positive divisor D of degree g on X, the following equivalences hold:*

$$\dim H^0(X, \mathcal{O}_{D-Q^\pm}) = 0 \;\Leftrightarrow\; \dim H^1(X, \mathcal{O}_D) = 0 \;\textit{ and }\; \operatorname{supp} D \subset X \setminus \{Q^\pm\}.$$

Proof. Let $\dim H^0(X, \mathcal{O}_{D-Q^\pm}) = 0$. If $Q^\pm \in \operatorname{supp} D$ holds, then $D \geq 0$ implies $D - Q^\pm \geq 0$ which contradicts $\dim H^0(X, \mathcal{O}_{D-Q^\pm}) = 0$. So $Q^\pm \notin \operatorname{supp} D$. One has $\deg D = g$, so due to the Riemann Roch Theorem [Forster, 1981, § 16.10], it is $\dim H^1(X, \mathcal{O}_{D-Q^\pm}) = 0$. Then Lemma 4.26 yields that $\dim H^1(X, \mathcal{O}_D) = 0$ since $D \geq D - Q^\pm$.
Conversely, let $\dim H^1(X, \mathcal{O}_D) = 0$ and $\operatorname{supp} D \subset X \setminus \{Q^\pm\}$. Again, the Riemann Roch Theorem [Forster, 1981, § 16.10] for $\deg(D) = g$ and $\dim H^1(X, \mathcal{O}_D) = 0$ yields that $\dim H^0(X, \mathcal{O}_D) = 1$. Since $D \geq 0$, $1 \in H^0(X, \mathcal{O}_D)$, and therefore 1 generates $H^0(X, \mathcal{O}_D)$. Because $\dim H^0(X, \mathcal{O}_{D-Q^\pm}) \leq \dim H^0(X, \mathcal{O}_D)$, it is $1 \in H^0(X, \mathcal{O}_{D-Q^\pm})$ if $\dim H^0(X, \mathcal{O}_{D-Q^\pm}) \neq 0$. Let $1 \in \dim H^0(X, \mathcal{O}_{D-Q^\pm})$. Then $Q^\pm$ must be contained in D since otherwise the corresponding section would not be generated by 1. This contradicts the assumption that $Q^\pm \notin \operatorname{supp} D$, so $H^0(X, \mathcal{O}_{D-Q^\pm}) = 0$. □

We know from Lemma 4.13 that $\deg(D) = g < \infty$. From the finiteness of the degree of the divisor and with help of the asymptotic behavior of ψ_k on X, we show in the next Theorem that the pole divisor of the normalized eigenfunctions is non-special.

Lemma 4.28. *Let the compactified Fermi curve X be a Riemann surface of genus g with two marked points $Q^+ \neq Q^-$ at infinity and let D be the pole divisor of the normalized eigenfunction $\psi_{N,k}$ on X with $\operatorname{supp} D \subset X \setminus \{Q^+, Q^-\}$. Then*

$$\dim H^1(X, \mathcal{O}_D) = 0, \quad \dim H^1(X, \mathcal{O}_{D-Q^+}) = 0 \;\textit{ and }\; \dim H^1(X, \mathcal{O}_{D-Q^-}) = 0.$$

Proof. To show this, we use the Wirtinger operators $\partial = \frac{1}{2}(\partial_x + \iota\partial_y)$ and $\bar{\partial} = \frac{1}{2}(\partial_x - \iota\partial_y)$ again. The divisor D is positive and of degree g, where g is the genus of X and D is defined in such a way that $\psi_N(\,\cdot\,, (x, y)) \in H^0(X, \mathcal{O}_D)$ for all $(x, y) \in \varDelta$. As in the proof of Lemma 4.13, let $U_\pm \subset X$ be small disjoint open neighborhoods of $Q^\pm$, respectively, with local coordinates $z_\pm : U_\pm \to \mathbb{C}$ centered at $Q^\pm$. Then the holomorphic involution σ acts on $U_\pm$ as $\sigma : z_\pm \mapsto -z_\pm$, see Lemma 4.3. Therefore, the images of the local coordinates have to be point-symmetric around $0 \in \mathbb{C}^2$ and together with the asymptotic freeness of X, we can choose $z_\pm$ such that

$$k(z_+) = \left(\frac{1}{z_+}, -\frac{\iota}{z_+} + \sum_{j=0}^{\infty} a_{+,j} z_+^{2j+1}\right) \text{ and } k(z_-) = \left(\frac{1}{z_-}, \frac{\iota}{z_-} + \sum_{j=0}^{\infty} a_{-,j} z_-^{2j+1}\right). \tag{4.19}$$

Since $\psi_N(k,(x,y)) = e^{2\pi\iota\langle k, \binom{x}{y}\rangle}\psi_{N,k}(x,y)$ it is

$$\psi_N(k(z_\pm),(x,y)) = \exp\left(\frac{2\pi\iota x}{z_\pm} \pm \frac{2\pi y}{z_\pm} + 2\pi\iota y \cdot \sum_{j=0}^{\infty} a_{\pm,j} z_\pm^{2j+1}\right)\psi_{N,k(z_\pm)}(x,y).$$

So the Leibniz rule yields for the partial derivatives on $U_\pm$

$$e^{-2\pi\iota\langle k(z_\pm), \binom{x}{y}\rangle}\partial_x\psi_N(k(z_\pm),(x,y)) = \frac{2\pi\iota}{z_\pm}\psi_{N,k(z_\pm)}(x,y) + \partial_x\psi_{N,k(z_\pm)}(x,y)$$

and

$$e^{-2\pi\iota\langle k(z_\pm), \binom{x}{y}\rangle}\partial_y\psi_N(k(z_\pm),(x,y)) = \left(\mp\frac{2\pi}{z_\pm} + 2\pi\iota\sum_{j=0}^{\infty} a_{\pm,z} z_\pm^{2j+1}\right)\psi_{N,k(z_\pm)}(x,y) + \partial_y\psi_{N,k(z_\pm)}(x,y).$$

From this, we obtain for the Wirtinger operators ∂ and $\bar\partial$ applied to $\psi_N(k,\cdot)$ on U_+, that

$$e^{-2\pi\iota\langle k(z_+), \binom{x}{y}\rangle}\partial\psi_N(k(z_+),(x,y)) = \left(-\pi\sum_{j=0}^{\infty} a_{+,j} z_+^{2j+1}\right)\psi_{N,k(z_+)}(x,y) + \partial\psi_{N,k(z_+)}(x,y)$$

and

$$e^{-2\pi\iota\langle k(z_+), \binom{x}{y}\rangle}\bar\partial\psi_N(k(z_+),(x,y)) = \left(\frac{2\pi\iota}{z_+} + \pi\sum_{j=0}^{\infty} a_{+,j} z_+^{2j+1}\right)\psi_{N,k(z_+)}(x,y) + \bar\partial\psi_{N,k(z_+)}(x,y).$$

On U_-, we obtain

$$e^{-2\pi\iota\langle k(z_-), \binom{x}{y}\rangle}\partial\psi_N(k(z_-),(x,y)) = \left(\frac{2\pi\iota}{z_-} - \pi\sum_{j=0}^{\infty} a_{-,j} z_-^{2j+1}\right)\psi_{N,k(z_-)} + \partial\psi_{N,k(z_-)}(x,y)$$

and

$$e^{-2\pi\iota\langle k(z_-), \binom{x}{y}\rangle}\bar\partial\psi_N(k(z_-),(x,y)) = \left(\pi\sum_{j=0}^{\infty} a_{-,j} z_-^{2j+1}\right)\psi_{N,k(z_-)}(x,y) + \bar\partial\psi_{N,k(z_-)}(x,y).$$

Hence, $\bar\partial\psi_N$ has an additional pole of first order at Q^+ and $\partial\psi_N$ has an additional pole of first order at Q^-. Let $\xi \in L^2(\varDelta)$ be such that $\langle\!\langle \xi, \psi_N \rangle\!\rangle \neq 0$. If $\bar\partial\psi_N$ has an additional pole at

Q^+, then also $\langle\!\langle \xi, \bar{\partial}\psi_N \rangle\!\rangle$ has an additional pole at Q^+ in comparison to $\langle\!\langle \xi, \psi_N \rangle\!\rangle$ and the same holds for similar expressions of this form. So $\langle\!\langle \xi, \partial\psi_N \rangle\!\rangle$ with $\xi \in L^2(\Delta)$ such that $\langle\!\langle \xi, \psi_N \rangle\!\rangle \neq 0$ has an additional pole at Q^- in comparison to $\langle\!\langle \xi, \psi_N \rangle\!\rangle$. Analogously as it is shown for $\partial_y \varphi_N$ in the proof of Lemma 3.16, deriving ψ_N in the direction of $\frac{1}{2}(x \pm \iota y)$ does not generate new poles in $k \in X^\circ$. So for $\xi \in L^2(\Delta)$ chosen as above, there holds for the corresponding divisors $(\langle\!\langle \xi, \partial\psi_N \rangle\!\rangle), (\langle\!\langle \xi, \bar{\partial}\psi_N \rangle\!\rangle) \geq -D - Q^- - Q^+$ and $\langle\!\langle \xi, \partial\psi_N \rangle\!\rangle$ as well as $\langle\!\langle \xi, \bar{\partial}\psi_N \rangle\!\rangle$ generate sections in $\mathcal{O}_{D+Q^-+Q^+}$ which are not contained in $\mathcal{O}_D$ since $\operatorname{supp} D \subset X^\circ$. Because all other elements with poles of first order at Q^+ and Q^- can be obtained by a linear combinations of $\langle\!\langle \xi, \partial\psi_N \rangle\!\rangle$ and $\langle\!\langle \xi, \bar{\partial}\psi_N \rangle\!\rangle$, it is

$$\dim H^0(X, \mathcal{O}_{D+Q^-+Q^+}) = \dim H^0(X, \mathcal{O}_D) + 2.$$

The second derivatives of $\partial\bar{\partial}\psi_N$, $\partial^2\psi_N$ and $\bar{\partial}^2\psi_N$ can be determined analogously. Again, using the Leibniz rule yields that $\bar{\partial}^2\psi_N$ has a pole of second order at Q^+, $\partial^2\psi_N$ has a pole of second order at Q^-. For the mixed derivative, we obtain on U_+

$$\begin{aligned}
e^{2\pi\iota\langle k(z_+), \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\rangle} \partial\bar{\partial}\psi_N(k(z_+), (x,y)) = {} & \pi^2 \left(\left(\frac{1}{z_+}\right)^2 + \left(\frac{\iota}{z_+} + \sum_{j=0}^{\infty} a_{+,j} z_+^{2j+1}\right)^2 \right) \psi_{N,k(z_+)}(x,y) + \\
& + \pi\iota \left(\frac{1}{z_+} + \iota \left(\frac{\iota}{z_+} + \sum_{j=0}^{\infty} a_{+,j} z_+^{2j+1} \right) \right) \bar{\partial}\psi_{N,k(z_+)}(x,y) + \\
& + \pi\iota \left(\frac{1}{z_+} - \iota \left(\frac{\iota}{z_+} + \sum_{j=0}^{\infty} a_{+,j} z_+^{2j+1} \right) \right) \partial \\
= {} & \pi^2 \left(2\iota \sum_{j=0}^{\infty} a_{+,j} z_+^{2j} + \left(\sum_{j=0}^{\infty} a_{+,j} z^{2j+1} \right)^2 \right) \psi_{N,k(z_+)}(x,y) + \\
& + \pi\iota \left(\iota \sum_{j=0}^{\infty} a_{+,j} z_+^{2j+1} \right) \bar{\partial}\psi_{N,k(z_+)}(x,y) \\
& + \pi\iota \left(\frac{2}{z_+} - \iota \sum_{j=0}^{\infty} a_{+,j} z_+^{2j+1} \right) \partial\psi_{N,k(z_+)}(x,y) \\
& + \partial\bar{\partial}\psi_{N,k(z_+)}(x,y).
\end{aligned}$$

Analogous calculations on U_- show that $\langle\!\langle \xi, \partial\bar{\partial}\psi_N \rangle\!\rangle \in \mathcal{O}_{D+Q^++Q^-}$ and hence this second derivative does not generate a new section compared to the sections of the first derivatives. So the second derivatives generate sections in $H^0(X, \mathcal{O}_{D+2Q^++2Q^-})$ with $\partial^2\psi_N, \bar{\partial}^2\psi_N \notin H^0(X, \mathcal{O}_{D+Q^++Q^-})$ and all additional poles of second order can be generated by linear combinations of $\langle\!\langle \xi, \partial^2\psi_N \rangle\!\rangle$ and $\langle\!\langle \xi, \bar{\partial}^2\psi_N \rangle\!\rangle$. So

$$\dim H^0(X, \mathcal{O}_{D-2Q^+-2Q^-}) = \dim H^0(X, \mathcal{O}_{D-Q^+-Q^-}) + 2 = \dim H^0(X, \mathcal{O}_D) + 4.$$

Continuing this procedure for $n \in \mathbb{N}$ yields that the n-th derivatives always generate two functions $\langle\!\langle \xi, \partial^n \psi_N \rangle\!\rangle$ and $\langle\!\langle \xi, \bar\partial^n \psi_N \rangle\!\rangle$ which are contained in $H^0(X, \mathcal{O}_{D+nQ^+ + nQ^-})$ but which are not contained in $H^0(X, \mathcal{O}_{D+(n-1)Q^+ + (n-1)Q^-})$ since these derivative always have an extra pole at Q^+ respectively Q^- of one order higher than the derivatives of order $n-1$. With $k + l = n$ and $k, l \in \mathbb{N}$, direct calculations show that for the mixed derivatives, there holds $\bar\partial^k \partial^l \psi_N \in H^0(X, \mathcal{O}_{D+(n-1)Q^+ + (n-1)Q^-})$ since they behave similar as the mixed derivative for $n = 2$. Thus, $\dim H^0(X, \mathcal{O}_{D_{nQ^+ + nQ^-}}) = \dim H^0(X, \mathcal{O}_D) + 2n$. Due to $\deg(D + nQ^+ + nQ^-) = \deg(D) + 2n$, the Riemann Roch Theorem [Forster, 1981, Theorem 16.9] yields for every $n \in \mathbb{N}$

$$\begin{aligned}
\dim H^1(X, \mathcal{O}_{D+nQ^+ + nQ^-}) &= \dim H^0(X, \mathcal{O}_{D+nQ^+ + nQ^-}) - \deg(D + nQ^+ + nQ^-) + g - 1 \\
&= \dim H^0(X, \mathcal{O}_D) + 2n - \deg(D) - 2n + g - 1 \\
&= \dim H^1(X, \mathcal{O}_D).
\end{aligned}$$

Since $g < \infty$, there exists an $n \in \mathbb{N}$ such that $\deg(D + nQ^+ + nQ^-) > 2g - 2$. Then Serre Duality [Forster, 1981, 17.6] implies for this n that $\dim H^1(X, \mathcal{O}_{D+nQ^+ nQ^-}) = 0$, and therefore $\dim H^1(X, \mathcal{O}_D) = 0$.

Finally, by Lemma 4.27 follows that $\dim H^0(X, \mathcal{O}_{D-Q^\pm}) = 0$. Since $\deg(D - Q^\pm) = g - 1$, we obtain by the Riemann Roch Theorem [Forster, 1981, Theorem 16.9] that $\dim H^1(X, \mathcal{O}_{D-Q^\pm}) = 0$. □

4.2.5. Translations of the divisor

The divisor D is defined by the divisor $\mathcal{S}$ which is induced by the subsheaf of the meromorphic functions on X° that is generated by the pullback $\langle\!\langle \xi, \psi_N \rangle\!\rangle$, whereby ψ_N is normalized at $(x, y) = (0, 0)$ as $\psi_N(k, (x, y)) = 1$ for each $k \in X^\circ$. So we consider the germ $\psi_k \in \mathcal{O}_k$ for every $k \in X^\circ$ and define $D \subset X^\circ$ as the pole divisor of $\left(\frac{\langle\!\langle \xi, \psi \rangle\!\rangle}{\psi(0,0)}\right)_k$ with $\langle\!\langle \xi, \psi \rangle\!\rangle_k \neq 0$. Likewise we can define for every $(x, y) \in \varDelta$ the subsheaf $\mathcal{O}_{D(x,y)}$ of the meromorphic functions whose germ is at each $k \in X^\circ$ generated by

$$\left(\frac{\langle\!\langle \xi, \psi \rangle\!\rangle}{\psi(x, y)} \right)_k, \tag{4.20}$$

where $\xi \in L^2(\varDelta)$ is again chosen in such a way that $\langle\!\langle \xi, \psi \rangle\!\rangle \not\equiv 0$. We say that $D(x, y)$ is a translation of the divisor D.

We now want to describe these translations as the tensor product of $\mathcal{O}_D$ with a double-periodic flow $\mathcal{L}_h(x, y)$. To construct this flow, we use a presentation of elements of $H^1(X, \mathcal{O})$ by Mittag Leffler distributions with support in Q^+ and Q^- which is known as the *Krichever construction* [Krichever, 1977]. This is presented here as in [Klein et al., 2016, Chapter 8].

Definition 4.29. Let H be the algebra of germs of functions which are holomorphic in a punctured open neighborhood of $0 \in \mathbb{C}$

$$H := \{(U, h) \mid U \subset \mathbb{C} \text{ open with } 0 \in U \text{ and } h : U \setminus \{0\} \to \mathbb{C} \text{ is holomorphic}/\sim\},$$

where

$$(U, h) \sim (U', h') \Leftrightarrow \exists V \subset U \cap U' \text{ such that } 0 \in V \text{ and } h|_{V\setminus\{0\}} = h'|_{V\setminus\{0\}}.$$

Furthermore, we define the following subset of H:

$$H^-_{\text{finite}} := \{(U, h) \in H \mid h \text{ extends holomorphically to } \mathbb{C}P^1 \setminus \{0\} \text{ with } h(\infty) = 0 \text{ and } h \text{ has a pole at } 0\}.$$

Let X be a compact Riemann surface with two marked points $Q^+ \neq Q^-$ and let z_+ respectively z_- be local coordinates such that $z_+(Q^+) = 0$ and $z_-(Q^-) = 0$. For any $(h_+, h_-) \in H^2$, we choose disjoint open neighborhoods U_+ respectively U_- of Q^+ respectively Q^- such that $z^*_\pm h_\pm$ is defined on $U_\pm \setminus \{Q^\pm\}$. With the open set $X^\circ \subset X$ the set $\mathcal{U} = \{X^\circ, U_+, U_-\}$ is an open covering of X. Since the only non-empty intersections of sets in this covering are $X^\circ \cap U_\pm = U_\pm \setminus \{Q^\pm\}$, $z^*_\pm h_\pm$ are holomorphic on these intersections and thus define an element of the cochain group $C^1(\mathcal{U}, \mathcal{O})$, compare [Forster, 1981, Section 12.1]. Since $X^\circ \cap U_+ \cap U_- = \emptyset$ it is $C^2(\mathcal{U}, \mathcal{O}) = 0$, and so (h_+, h_-) defines a cocycle which induces an element in $H^1(\mathcal{U}, \mathcal{O})$, see [Forster, 1981, Definition 12.2]. Since X° and $U_\pm$ are open and not compact, $\mathcal{U}$ is a Leray cover, compare [Forster, 1981, S.93 and Theorem 26.1], and therefore $H^1(\mathcal{U}, \mathcal{O}) = H^1(X, \mathcal{O})$. Each element $(h_+, h_-) \in (H^-_{\text{finite}})^2$ defines a Mittag-Leffler distribution on X, compare [Forster, 1981, Sections 18.1 to 18.3]. A solution is a meromorphic function f on X such that $f - z^*_\pm h_\pm$ is holomorphic on $U_\pm$ and f is holomorphic on X°, i.e. f and $z^*_\pm h_\pm$ have the same principal parts on $U_\pm$. It is shown in [Klein et al., 2016, Lemma 7.2] that a Mittag-Leffler distribution $(h_+, h_-) \in (H^-_{\text{finite}})^2$ has a solution if and only if for all $\omega \in H^0(X, \Omega)$, there holds

$$\operatorname{Res}_{Q^+} z^*_+ h_+ \omega + \operatorname{Res}_{Q^-} z^*_- h_- \omega = 0.$$

Moreover, it is shown in [Klein et al., 2016, Chapter 7] that $(h_+, h_-) \in (H^-_{\text{finite}})^2$ defines a one-parameter group of cocycles $z^*_\pm \exp(2\pi\iota t h_\pm)$ via the exponential map $\exp : H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}^*) = \operatorname{Pic}(X)$, where $t \in \mathbb{R}$. For given $t \in \mathbb{R}$, these cocycles map the holomorphic functions on X° to holomorphic functions on $U_\pm \setminus \{Q^\pm\}$. We construct a holomorphic line bundle on X whose local sections define a one parameter family $\mathcal{L}_h(t)$ on X with cocycles $z^*_\pm \exp(2\pi\iota t h_\pm)$ out of these cocycles, see [Forster, 1981, Theorem 26.16]. For the trivial line bundle on X° together with this cocycles on $U_\pm \setminus \{Q^\pm\}$, it is shown in [Forster, 1981, Theorem 29.16] that this line bundle has a global meromorphic section and thus $\mathcal{L}_h(t)$ can be identified with a divisor. One-parameter family means that $\mathcal{L}_h(t+t') = \mathcal{L}_h(t) \otimes \mathcal{L}_h(t')$ for $t, t' \in \mathbb{R}$, where $\otimes$ denotes the product in $\mathrm{Pic}(X)$. Since $\mathcal{L}_h(0) = \mathcal{O} = \mathbb{1}_{\mathrm{Pic}(X)}$, $\mathcal{L}_h(t)$ stays in the unit component $\mathrm{Pic}_0(X)$ of $\mathrm{Pic}(X)$ which is the group of isomorphy classes of bundles of degree 0. It is also shown in [Klein et al., 2016, Chapter 7] that every one-parameter group in $\mathrm{Pic}_0(X)$ is obtained this way. Due to [Klein et al., 2016, Lemma 7.3 (ii)], an element $(h_+, h_-) \in (H^-_{\mathrm{finite}})^2$ induces a periodic flow with period $T > 0$, i.e. $\mathcal{L}_h(T) = \mathbb{1}_{\mathrm{Pic}(X)}$, if and only if the Mittag-Leffler distribution can be solved by means of a multivalued function f whose values over a point differ by an element of $T\mathbb{Z}$, i.e. $\int_\gamma df \in T\mathbb{Z}$ for all $\gamma \in H_1(X, \mathbb{Z})$. This method of constructing flows on X is called the *Krichever construction.* Our next aim is to transfer these results to two different Mittag-Leffler distributions with support in Q^+ and Q^- to construct a flow $\mathcal{L}_h(x, y)$ which is double periodic with respect to the two dimensional lattice $\Gamma \subset \mathbb{R}^2$ that corresponds to the periodicity of the Schrödinger operator $-\triangle + u$ with $u \in C(\mathbb{R}^2/\Gamma)$. Let $\hat{\gamma}$ and $\check{\gamma}$ be the generators of Γ. We will deduce two linearly independent elements $\hat{h}$ and $\check{h}$ in $(H^-_{\mathrm{finite}})^2$ from local representations of the Fermi curve as a tuple $(H^-_{\mathrm{finite}})^2 \times (H^-_{\mathrm{finite}})^2$ which represent the Fermi curve on U_+ respectively U_- asymptotically as in (4.19). Since we are only concerned with the Mittag Leffler distributions, i.e. with the pole behavior at $Q^\pm$ of these representations, we consider

$$h_+ := (h^x_+, h^y_+) = \left(\frac{1}{z_+}, -\frac{\iota}{z_+}\right) \text{ and } h_- := (h^x_-, h^y_-) = \left(\frac{1}{z_-}, \frac{\iota}{z_-}\right).$$

With this we define $\hat{h} := (\hat{h}_+, \hat{h}_-)$ and $\check{h} := (\check{h}_+, \check{h}_-)$ with

$$\begin{aligned} \hat{h}_+ &:= \langle h_+, \hat{\gamma}\rangle = \frac{\hat{\gamma}_1 - \iota\hat{\gamma}_2}{z_+}, \quad \hat{h}_- &:= \langle h_-, \hat{\gamma}\rangle = \frac{\hat{\gamma}_1 + \iota\hat{\gamma}_2}{z_-}, \\ \check{h}_+ &:= \langle h_+, \check{\gamma}\rangle = \frac{\check{\gamma}_1 - \iota\check{\gamma}_2}{z_+}, \quad \check{h}_- &:= \langle h_-, \check{\gamma}\rangle = \frac{\check{\gamma}_1 + \iota\check{\gamma}_2}{z_-}. \end{aligned} \tag{4.21}$$

Each of the elements $\hat{h}, \check{h} \in (H^-_{\mathrm{finite}})^2$ induce a line bundle $\mathcal{L}_{\hat{h}}$ and $\mathcal{L}_{\check{h}}$, respectively, such that for the periods $\hat{T} = 1$ and $\check{T} = 1$, there holds $\mathcal{L}_{\hat{h}}(1) = \mathcal{L}_{\check{h}}(1) = \mathbb{1}_{\mathrm{Pic}(X)}$. Tensorating these flows together yields a double-periodic flow which we can express in terms of $(x, y) \in \mathbb{R}^2$ instead of in $(\hat{t}, \check{t}) \in \mathbb{R}^2$: With $(x, y) := \hat{t}\hat{\gamma} + \check{t}\check{\gamma}$ it is $\hat{t} = \langle \hat{\kappa}, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\rangle$ and $\check{t} = \langle \check{\kappa}, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\rangle$, where $\hat{\kappa}$ and $\check{\kappa}$ are the generators of the dual lattice as defined in (1.10). Thus

$$\hat{h}_\pm \hat{t} + \check{h}_\pm \check{t} = \frac{1}{z_\pm}(\hat{\gamma}_1\hat{t} + \check{\gamma}_1\check{t}) \mp \frac{\iota}{z_\pm}(\hat{\gamma}_2\hat{t} + \check{\gamma}_2\check{t}) = \langle h_\pm, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\rangle$$

and we can interpret the two flows generated by $\hat{h}$ and $\check{h}$ also as flows in $(x, y) \in \mathbb{R}^2$ with periods $\hat{\gamma}$ and $\check{\gamma}$. We abbreviate $\mathcal{L}_h(x, y) := \mathcal{L}_{\hat{h}}(x, y) \otimes \mathcal{L}_{\check{h}}(x, y)$ for $(x, y) \in \mathbb{R}^2$. As $\mathcal{L}_h(t)$ defines a one-parameter group on $\mathrm{Pic}^0(X)$ for flows which are periodic in one direction, this defines a two-parameter group on $\mathrm{Pic}^0(X)$ via $\mathcal{L}_h((x, y) + (x' + y')) = \mathcal{L}_h(x, y) \otimes \mathcal{L}_h(x', y')$ for $(x, y), (x', y') \in \mathbb{R}^2$. The corresponding cocycles have the form $z^*_\pm e^{2\pi\iota(xh^x_\pm + yh^y_\pm)}$ and $\mathcal{L}_h(\gamma) = \mathbb{1}_{\mathrm{Pic}(X)}$ for all $\gamma \in \Gamma$.

Transferring [Klein et al., 2016, Lemma 7.3] to double-periodic flows immediately yields the following Lemma:

Lemma 4.30. *The two linearly independent elements $\hat{h}, \check{h} \in (H^-_{\textit{finite}})^2$ which are defined as in equation* (4.21) *induce a double-periodic flow with respect to a non-degenerated lattice $\mathit{\Gamma} \subset \mathbb{R}^2$ with generators $\hat{\gamma}$ and $\check{\gamma}$ if and only if the Mittag-Leffler distribution can be solved by means of two meromorphic functions $\hat{c} := \langle k, \hat{\gamma}\rangle$ and $\check{c} := \langle k, \check{\gamma}\rangle$ whose values over a point differ by an element of $\mathbb{Z}$, i.e. one has*

$$\int_\gamma \mathrm{d}\hat{c} \in \mathbb{Z} \quad \textit{and} \quad \int_\gamma \mathrm{d}\check{c} \in \mathbb{Z} \;\; \textit{for all} \;\; \gamma \in H_1(X, \mathbb{Z}).$$

Hereby, it is $k = \hat{c}\hat{\kappa} + \check{c}\check{\kappa}$, where $\hat{\kappa}$ and $\check{\kappa}$ are generators of $\mathit{\Gamma}^*$ defined in (1.10) *which are dual to the generators $\hat{\gamma}$ and $\check{\gamma}$ of $\mathit{\Gamma}$.*

For fixed $(x, y) \in \mathbb{R}^2$, we consider $\mathcal{L}_h(x, y)$ as a sheaf on X which is at $Q^\pm$ generated by

$$z^*_\pm \exp\left(2\pi\iota \left\langle \begin{pmatrix} h^x_\pm \\ h^y_\pm \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle\right)$$

and on X° by $\mathcal{O}_{X^\circ}$. With that we can show the following proposition about the translated divisors $D(x, y)$.

Proposition 4.31. *Let X be a compact Riemann surface of genus g, $Q^\pm$ two marked points on X and let $D(x, y)$ be the positive divisor as defined in* (4.20)*, where $D = D(0, 0)$. Then*

$$\mathcal{O}_{D(x,y)} \simeq \mathcal{O}_D \otimes \mathcal{L}_h(x, y)$$

with $\mathcal{L}_h(x, y)$ as defined above.

Proof. We claim that the isomorphism is given by

$$H^0(X, \mathcal{O}_{D(x,y)}) \to H^0(X, \mathcal{O}_D \otimes \mathcal{L}_h(x, y)), \quad f \mapsto \frac{\psi(k, (x, y))}{\psi(k, (0, 0))} f \tag{4.22}$$

with inverse map

$$H^0(X, \mathcal{O}_D \otimes \mathcal{L}_h(x, y)) \to H^0(X, \mathcal{O}_{D(x,y)}), \quad f \mapsto \frac{\psi(k, (0, 0))}{\psi(k, (x, y))} f. \tag{4.23}$$

One has to show this for every germ at $k \in X$. Hereby, we consider the two cases $k = Q^\pm$ and $k \neq Q^\pm$. At first, we show that the map (4.22) is a homomorphism between the given spaces. Therefore, note that due to the definition of D and $D(x, y)$ none of these divisors has a contribution at $Q^\pm$. So we have to show that (4.22) maps $\mathcal{O}_{Q^\pm}$ to $(\mathcal{O} \otimes \mathcal{L}_h(x, y))_{Q^\pm}$. Let $f \in H^0(X, \mathcal{O}_{Q^\pm})$. Since the germ $\psi_{Q^\pm} \in \mathcal{O}_{Q^\pm}$ is a not normalized eigenfunction of $- \triangle + u$ and

the eigenfunctions are asymptotically free, compare Lemma 1.19 or Corollary 4.10, $\psi(0,0)_{Q^\pm}$ is the product of the exponential factor $z_\pm^* \exp(2\pi\iota\langle\binom{h_x}{h_y}, \binom{0}{0}\rangle) = 1$ with a non-vanishing element of $\mathcal{O}_{Q^\pm}$ and thus in $\mathcal{O}_{Q^\pm}$. Likewise the germ $\psi(x,y)_{Q^\pm}$ is the product of the exponential factor $z_\pm^* \exp(2\pi\iota\langle\binom{h_x}{h_y}, \binom{x}{y}\rangle)$ with a non-vanishing element of $\mathcal{O}_{Q^\pm}$. So $\psi(x,y)_{Q^\pm} \in (\mathcal{O} \otimes \mathcal{L}_h(x,y))_{Q^\pm}$ and thus also $\left(f \cdot \frac{\psi(x,y)}{\psi(0,0)}\right)_{Q^\pm} \in \mathcal{O}_{Q^\pm} \otimes \mathcal{L}_h(x,y)$.
Let now $k \in X^\circ$ and $\tilde{\xi} \in L^2(\Delta)$ be chosen such that $\langle\!\langle \tilde{\xi}, \psi \rangle\!\rangle \neq 0$. We have already seen before that such an $\tilde{\xi}$ exists because the zero set of $\langle\!\langle \cdot, \psi \rangle\!\rangle$ has codimension one in $L^2(\Delta)$. Then $\mathcal{O}_{D,k}$ is generated by $\left(\frac{\langle\!\langle \tilde{\xi}, \psi \rangle\!\rangle}{\psi(0,0)}\right)_k$ and $\mathcal{O}_{D(x,y),k}$ is generated by $\left(\frac{\langle\!\langle \tilde{\xi}, \psi \rangle\!\rangle}{\psi(x,y)}\right)_k$. We assume that the germ $\psi(0,0)_k$ has a zero of order n and $\psi(x,y)_k$ a zero of order m at k. Then at k it is

$$\frac{\psi(k,(x,y))}{\psi(k,(0,0))} \quad \begin{cases} \text{is holomorphic for } n = m \\ \text{has a pole of order } n - m \text{ for } n > m \\ \text{has a zero or order } m - n \text{ for } n < m. \end{cases}$$

Since $\mathcal{L}_h(x,y)_k = \mathcal{O}_k$, we have to show that the map in (4.22) maps $\mathcal{O}_{D(x,y)}$ to $\mathcal{O}_D$. For every section $f \in H^0(X, \mathcal{O}_{D(x,y)})$, it is $(f) \geq -D(x,y)$. So f_k either has a pole of order $\leq -m$, is holomorphic or has has a zero of arbitrary finite order at k. We denote this order with $p \geq -m$, where $p < 0$ if f_k has a pole at k of order $p \geq -m$, $p = 0$ if f_k is holomorphic and $p > 0$ for f_k having a zero of order p. In any of the three cases one has $p \geq -m$, so

$$\left(f \cdot \frac{\psi(x,y)}{\psi(0,0)}\right)_k = (p + m - n)k$$

and $p + m - n \geq -n$. Therefore, $f_k \cdot \frac{\psi_k(x,y)}{\psi_k(0,0)} \in \mathcal{O}_D$ and the map in (4.22) is indeed a homomorphism from $H^0(X, \mathcal{O}_{D(x,y)})$ to $H^0(X, \mathcal{O}_D \otimes \mathcal{L}_h(x,y))$.
By the same means one sees that the map in (4.23) is a homomorphism from $H^0(X, \mathcal{O}_D \otimes \mathcal{L}_h(x,y))$ to $H^0(X, \mathcal{O}_{D(x,y)})$, where the only differences in the proof are the considerations at $Q^\pm$. Now $\frac{\psi(0,0)}{\psi(x,y)}$ contains the factor $z_\pm^* \exp\left(-2\pi\iota \left\langle\binom{h_x}{h_y}, \binom{x}{y}\right\rangle\right)$, and therefore maps an element of $(\mathcal{O} \otimes \mathcal{L}_h(x,y))_{Q^\pm}$ to $\mathcal{O}_{Q^\pm}$.
Obviously, $\left(\frac{\psi(0,0)}{\psi(x,y)} \frac{\psi(x,y)}{\psi(0,0)}\right)_k = 1$ for all $k \in X$, so these two mappings are inverse to each other. □

The next theorem summarizes the results from this section for the divisor D.

Theorem 4.32. *Let u be a regular finite type potential, X be the compactified normalization of $X'(u)$ with two marked points $Q^\pm$, $D \subset X^\circ$ the divisor as in Definition 3.7 and $D(x,y) \subset X^\circ$ the translated divisor as defined in* (4.20)*, $\sigma : X \to X$ the holomorphic involution from Lemma 1.17(a) which acts on X as $\sigma(k) = -k$ and $\mathcal{L}_h(x,y)$ the sheaf as defined above. Then there holds:*

(i) $D(x,y)$ is positive and of degree g.

(ii) $D(x,y)+\sigma(D(x,y)) \simeq K+Q^++Q^-$, *where K is the canonical divisor on X.*

(iii) $H^1(X,\mathcal{O}_{D(x,y)-Q^\pm})=0$.

(iv) $\mathcal{O}_{D(x,y)} \simeq \mathcal{O}_D \otimes \mathcal{L}_h(x,y)$.

If u is real-valued, there holds additionally $\tau_1(D(x,y))=D(x,y)$.

Proof. By definition, $D(x,y)$ is positive. Since we can translate $D(x,y)$ into the origin by considering the coordinates $(x-x_0,y-y_0)$ instead of (x,y), there also holds $D(x,y)+\sigma(D(x,y)) \simeq K+Q^++Q^-$ as it is shown for D in Lemma 4.13 and $D(x,y)$ is is of degree g which yields (i) and (ii). The same argument also yields (iii) and the realness-condition. Statement (iv) is just the statement of Proposition 4.31. □

4.3. Two meromorphic differentials of second kind

We will now show that for a regular finite type potential u, there exist two Abelian differentials of second kind on the compactified normalization $X(u)$ which have poles of second order at Q^+ and Q^- and are holomorphic on $X^\circ(u)$. Therefore, we first remember the 'coordinate change' which we have already used in the Kritchever construction in the foregoing section, compare Lemma 4.30. We want to use coordinates which depend on the lattice $\varGamma$. On $X^\circ(u)$, we define the maps $c_\gamma : k \mapsto \langle k,\gamma\rangle$ with $\gamma \in \varGamma$. These are multivalued holomorphic functions from $X^\circ(u)$ to $\mathbb{C}$, i.e. $c_\gamma(k+\varGamma^*)=c_\gamma(k)+\mathbb{Z}$ for $k \in F(u)$. We define

$$\hat{c}: X^\circ(u) \to \mathbb{C}, \quad k \mapsto \langle k,\hat{\gamma}\rangle \;\text{ and }\; \check{c}: X^\circ(u) \to \mathbb{C}, \quad k \mapsto \langle k,\check{\gamma}\rangle,$$

where $\hat{\gamma}$ and $\check{\gamma}$ are the generators of $\varGamma$. Then for every $\gamma \in \varGamma$, c_γ can be generated by a linear combination of these two functions. On $X^\circ(0)$, the components of k at Q^+ are related by $k_1+\iota k_2=0$ and at Q^- by $k_1-\iota k_2=0$. Let $U_\pm$ be a neighborhood of $Q^\pm$ which consists only of smooth points of $X(u)$. Then the asymptotic freeness yields that $k_1 \approx \mp\iota k_2$ for $k=(k_1,k_2) \in U_\pm$. Hence, each of the two components of k has a pole of first order at Q^+ and a pole of first order at Q^-. Choosing local coordinates z_+ resp. z_- in a neighborhood $U_\pm$ of $Q^\pm$ such that $z_\pm(Q^\pm)=0$, we can find local representations $h^1_\pm$ in k_1-direction and $h^2_\pm$ in k_2-direction of $X(u)$ on $U_\pm \setminus \{Q^\pm\}$, similar as it is done in the proof of Lemma 4.28, such that

$$z_+^* h_+^1(z_+) = \frac{1}{z_+}, \qquad z_+^* h_+^2(z_+) = -\frac{\iota}{z_+} + \sum_{i=0}^{\infty} c_i z_+^{2i+1},$$
$$z_-^* h_-^1(z_-) = \frac{1}{z_-}, \qquad z_-^* h_-^2(z_-) = \frac{\iota}{z_-} + \sum_{i=1}^{\infty} d_i z_-^{2i+1}.$$

With this notation, one has on U_+ respectively U_- that

$$z_+^* h_+ = \begin{pmatrix} z_+^* h_+^1 \\ z_+^* h_+^2 \end{pmatrix} \quad \text{respectively} \quad z_-^* h_- = \begin{pmatrix} z_-^* h_-^1 \\ z_-^* h_-^2 \end{pmatrix},$$

so

$$\hat{c}|_{U_\pm} = \langle z_\pm^* h_\pm, \hat{\gamma} \rangle \quad \text{and} \quad \check{c}|_{U_+} = \langle z_\pm^* h_\pm, \check{\gamma} \rangle.$$

Since k_1 and k_2 are locally holomorphic on $X^\circ(u)$ also $\hat{c}$ and $\check{c}$ are locally holomorphic on $X^\circ(u)$. The principal parts of $\hat{c}$ respectively $\check{c}$ on U_+ can be represented as $\frac{\hat{\gamma}_1 - \iota\hat{\gamma}_2}{z_+}$ respectively $\frac{\check{\gamma}_1 - \iota\check{\gamma}_2}{z_+}$ and on U_- as $\frac{\hat{\gamma}_1 + \iota\hat{\gamma}_2}{z_-}$ respectively $\frac{\check{\gamma}_1 + \iota\check{\gamma}_2}{z_-}$. We deduce from this that $d\hat{c}$ and $d\check{c}$ are two abelian differentials of second kind which are holomorphic on $X^\circ(u)$ with poles of second order at $Q^\pm$, more precisely on $U_\pm$

$$\begin{aligned} d\hat{c}|_{U_+} &= \left(\frac{-\hat{\gamma}_1 + \iota\hat{\gamma}_2}{z_+^2} + \sum_{i=1}^\infty \tilde{c}_i z_+^i \right) \mathrm{d}z_+ \quad \text{as well as} \quad d\check{c}|_{U_+} = \left(\frac{-\check{\gamma}_1 + \iota\check{\gamma}_2}{z_+^2} + \sum_{i=1}^\infty c_i' z_+^i \right) \mathrm{d}z_+, \\ d\hat{c}|_{U_-} &= \left(\frac{-\hat{\gamma}_1 - \iota\hat{\gamma}_2}{z_-^2} + \sum_{i=1}^\infty \tilde{d}_i z_-^i \right) \mathrm{d}z_- \quad \text{as well as} \quad d\check{c}|_{U_-} = \left(\frac{-\check{\gamma}_1 - \iota\check{\gamma}_2}{z_-^2} + \sum_{i=1}^\infty \tilde{d}_i' z_-^i \right) \mathrm{d}z_-. \end{aligned} \tag{4.24}$$

Furthermore, the multivaluedness of $\hat{c}$ and $\check{c}$ on $X(u)$ yields that they generate differential forms over $\mathbb{Z}$ such that

$$\int_\gamma d\hat{c}, \int_\gamma d\check{c} \in \mathbb{Z} \quad \text{for all closed curves } \gamma \in H_1(X, \mathbb{Z}),$$

i.e. for all $\gamma \in \Gamma$ one has that $\mathrm{d}\langle k, \gamma \rangle$ is an integer linear combination of $d\hat{c}$ and $d\check{c}$, so $\int_\gamma \mathrm{d}\langle k, \gamma \rangle \in \mathbb{Z}$ for all $\gamma \in \Gamma$. Furthermore, the definition of these 1-forms yields immediately that the involution $\sigma : X(u) \to X(u)$, $k \mapsto -k$ acts as $\sigma^* \, d\hat{c} = -\, d\hat{c}$ and $\sigma^* \, d\check{c} = -\, d\check{c}$. Moreover, for $u \in C(\mathbb{R}^2/\Gamma, \mathbb{R})$, the involutions $\tau_1 : X(u) \to X(u)$, $k \mapsto -\bar{k}$ respectively $\tau_2 : X(u) \to X(u)$, $k \mapsto \bar{k}$ act as $\tau_1^* \overline{d\hat{c}} = -\, d\hat{c}$ and $\tau_1^* \overline{d\check{c}} = -\, d\check{c}$ respectively $\tau_2^* \overline{d\hat{c}} = \, d\hat{c}$ and $\tau_2^* \overline{d\check{c}} = \, d\check{c}$.

Part II.

The inverse problem for finite type potentials

5. Reconstruction of the eigenfunctions and the potential

In the direct problem, we have so far deduced some properties that hold for the Fermi curve $X'(u)$ and the pole divisor D of the pullback of normalized eigenfunction of $-\triangle + u$ on $X'(u)$ to the normalization $X^\circ(u)$ for a regular finite type potential u. Conversely, we now assume that some so-called spectral data is given – which reflect the properties shown in the direct problem – to reconstruct a unique potential $u \in C(\mathbb{C}^2/\Gamma, \mathbb{C})$ and a unique normalized function $\psi \in C^\infty_{[k]}(\Delta, \mathbb{C})$ for every $k \in X^\circ(u)$ which lies in the kernel of the Schrödinger operator with potential u. Therefore, we introduce first the necessary spectral data. Let X be a compact Riemann surface of genus $g < \infty$ with the following properties:

(F1) On X, there are two marked points Q^+ and Q^-.

(F2) (i) There exist two multivalued functions $\hat{c}$ and $\check{c}$ on X which are holomorphic on $X^\circ := X \setminus \{Q^+, Q^-\}$. The differentials $\mathrm{d}\hat{c}$ and $\mathrm{d}\check{c}$ are meromorphic differentials of second kind with double poles at $Q^\pm$, linear independent principal parts over $\mathbb{R}$ and with vanishing residues. Additionally, there are disjoint, small open neighborhoods $U_\pm$ of $Q^\pm$ such that in local coordinates on these neighborhoods centered at $Q^\pm$, it is on $U_\pm$

$$d\hat{c}|_{U_\pm} = \frac{1}{z_\pm^2}, \quad d\check{c}|_{U_+} = \frac{b}{z_+^2} + \sum_{j=1}^\infty \check{a}_{+,j} z_+^{2j} \text{ and } \quad d\check{c}|_{U_-} = \frac{\bar{b}}{z_-^2} + \sum_{j=1}^\infty \check{a}_{-,j} z_\pm^{2j}.$$

(ii) $\int_\gamma \mathrm{d}\hat{c} \in \mathbb{Z}$ and $\int_\gamma \mathrm{d}\check{c} \in \mathbb{Z}$ for all closed curves $\gamma \in H_1(X, \mathbb{Z})$.

(F3) On X, there exists a holomorphic involution σ with exactly two fixed points $Q^\pm$ and $\sigma^*\hat{c} = -\hat{c}$ as well as $\sigma^*\check{c} = -\check{c}$.

We have shown in the direct problem that $\mathfrak{R}(X)$ consist of at most two connected components and that every connected component contains one of the marked points Q^+ and Q^-. This property is implicitly contained the above conditions. Because if there would be a connected component on which the differentials $d\hat{c}$ and $d\check{c}$ have no poles, then the anti-derivatives $(\hat{c}, \check{c})$ would be holomorphic function on this connected component. X is assumed to be compact, this would yield that $(\hat{c}, \check{c})$ are constant and thus the differentials $d\hat{c}$ and $d\check{c}$ would be identically zero. However, this contradicts condition (F2)(ii). Therefore, every connected component of $\mathfrak{R}(X)$ must contain either Q^+ or Q^-. So the number of connected components of $\mathfrak{R}(X)$ is at most two.

Next, we want to define a double periodic flow $\mathcal{L}_h(x,y)$ on X for $(x,y) \in \mathbb{R}$ from the given data $(X, Q^+, Q^-, d\hat{c}, d\check{c}, \sigma)$, similar as it was done in Section 4.2.5.
Therefore, note first that for $b \in \mathbb{C} \setminus \{0\}$ as in condition (F2)(i), we can set $b = \check{\gamma}_1 - \iota\check{\gamma}_2$ with $\check{\gamma}_2 \neq 0$ because otherwise the principal parts of $d\hat{c}$ and $d\check{c}$ would be linearly dependent over $\mathbb{R}$. Moreover, condition (F3) yields that there are only even powers in the representation of $d\hat{c}$ and $d\check{c}$ in (F2)(i). In case that one searches for a real-valued potential $u \in C(\mathbb{C}^2/\Gamma, \mathbb{R})$, additionally a reality condition for u is necessary. This can be expressed as follows:

(R1) On X, there is an antiholomorphic involution τ_1 with $\tau_1(Q^\pm) = Q^\mp$ and $\tau_1^*\hat{c} = -\bar{\hat{c}}$ as well as $\tau_1^*\check{c} = -\bar{\check{c}}$.

We now show that the periodicity conditions on $d\hat{c}$ and $d\check{c}$ in (F2)(ii) induce a real 2-dimensional lattice Γ with linear independent generators $\hat{\gamma} = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $\check{\gamma} = \left(\begin{smallmatrix}\check{\gamma}_1\\\check{\gamma}_2\end{smallmatrix}\right)$ and we introduce the real 2-dimensional subgroup of the Picard group corresponding to the translations of $(x,y) \in \mathbb{R}^2$ by $\mathbb{R}^2/\Gamma$. As in Section 4.2.5, we define an open covering of X by $X^\circ := X \setminus \{Q^+, Q^-\}$ and the two small open neighborhoods $U_\pm$ of $Q^\pm$. The only non-empty intersections of each two sets in this covering are $U_\pm^* := U_\pm \setminus \{Q^+, Q^-\}$. We define a cocycle on X via its transition functions on $U_\pm^*$. These are for any $(\hat{t}, \check{t}) \in \mathbb{R}^2$ given by $e^{2\pi\iota(\hat{t}\hat{c}+\check{t}\check{c})}$, compare [Forster, 1981, Theorem 29.16]. We set $(x,y) := \hat{t}\hat{\gamma} + \check{t}\check{\gamma}$ and $k := \hat{\kappa}\hat{c} + \check{\kappa}\check{c}$, whereby $\hat{\gamma} = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $\check{\gamma} = \left(\begin{smallmatrix}\check{\gamma}_1\\\check{\gamma}_2\end{smallmatrix}\right)$. Hereby, $\check{\gamma}_1$ and $\check{\gamma}_2$ are defined in condition (F2)(ii). Due to $\check{\gamma}_2 \neq 0$, these two vectors are obviously linearly independent. Let Γ be the lattice generated by $\hat{\gamma}$ and $\check{\gamma}$. Then the corresponding dual lattice Γ^* as in (1.10) is generated by $\hat{\kappa} = \left(\begin{smallmatrix}1\\-\frac{\check{\gamma}_1}{\check{\gamma}_2}\end{smallmatrix}\right)$ and $\check{\kappa} = \left(\begin{smallmatrix}0\\\frac{1}{\check{\gamma}_2}\end{smallmatrix}\right)$. With this we obtain from the definition of k that $k_1 = \hat{\kappa}_1\hat{c} + \check{\kappa}_1\check{c} = \hat{c}$ and $k_2 = \hat{\kappa}_2\hat{c} + \check{\kappa}_2\check{c} = -\frac{\check{\gamma}_1}{\check{\gamma}_2}\hat{c} + \frac{1}{\check{\gamma}_2}\check{c}$. So in the local coordinates from condition (F2)(i), it is

$$dk_1|_{U_\pm} = \frac{1}{z_\pm^2} \tag{5.1}$$

and

$$dk_2|_{U_\pm} = -\frac{\check{\gamma}_1}{\check{\gamma}_2}\frac{1}{z_\pm^2} + \frac{1}{\check{\gamma}_2}\left(\frac{\check{\gamma}_1 \mp \iota\check{\gamma}_2}{z_\pm^2} + \sum_{j=1}^{\infty} \check{a}_{\pm,j} z_\mp^{2i}\right) = \mp\frac{\iota}{z_\pm^2} + \sum_{j=1}^{\infty} \frac{\check{a}_{\pm,j}}{\check{\gamma}_2} z_\pm^{2i} \tag{5.2}$$

which shows that on the neighborhoods $U_\pm$, dk_1 and dk_2 are correlated as $dk_1|_{U_\pm} \approx \mp\iota dk_2|_{U_\pm}$. Moreover, direct calculation yields that

$$e^{2\pi\iota(\hat{t}\hat{c}+\check{t}\check{c})} = e^{2\pi\iota\langle k, \left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\rangle}. \tag{5.3}$$

For all $(x,y) \in \mathbb{R}^2$, we denote the locally free sheaf of rank 1 on X which is defined by this cocycles as $\mathcal{L}_h(x,y)$. If $(x,y) \in \Gamma$, i.e. $(\hat{t}, \check{t}) \in \mathbb{Z}^2$, then this transition function extends to a global non-vanishing holomorphic function on X°. Due to condition (F1)(ii) and Lemma 4.30, the corresponding sheaf is equal to $\mathcal{O}_X$ in this case.
So far we have only given properties on the curve X. To make sure that there exists a unique

potential u and a unique normalized eigenfunction ψ_N, more information is necessary. Therefore, we consider in addition a divisor D on X with the following properties:

(D1) D is a positive divisor.

(D2) $D + \sigma(D) \simeq K + Q^+ + Q^-$, where K is the canonical divisor on X.

(D3) For all $(x, y) \in \mathbb{R}^2$ it is $\dim H^1(X, \mathcal{O}_{D-Q^\pm} \otimes \mathcal{L}_h(x, y)) = 0$.

Note that due to the Riemann Roch Theorem [Forster, 1981, § 16.10] and Lemma 4.27, the condition $\dim H^1(X, \mathcal{O}_{D-Q^\pm}) = 0$ immediately yields that $Q^\pm \notin \operatorname{supp} D$. Furthermore, condition (D2) yields that deg $D = g$ and due to Theorem 4.23, the genus of X must be even. The reality condition for the divisor D reads as

(R2) $\tau_1(D) = D$.

Definition 5.1. Let $(X, Q^+, Q^-, d\hat{c}, d\check{c}, \sigma, D)$ obey conditions (F1) to (F3) as well as (D1) to (D3). We denote this set as *spectral data* if we seek for a complex potential u. If we seek for a real-valued potential u, this spectral data is given by the set $(X, Q^+, Q^-, \hat{c}, \check{c}, \sigma, \tau_1, D)$ and this data additionally has to obey $(R1)$ and $(R2)$.

Due to the next lemma, the demand that D is a positive divisor is actually obsolete: the choice of the representant D in $H^1(X, \mathcal{O}_{D-Q^\pm})$ depends only on the isomorphy class of D and the lemma shows that if there exists a divisor which obeys (D3), then there always exists a unique divisor D with the properties in (D1) as well.

Lemma 5.2. *Every divisor D of degree g satisfying*

$$\dim H^1(X, \mathcal{O}_{D-Q^+}) = 0 \quad \textit{and} \quad \dim H^1(X, \mathcal{O}_{D-Q^-}) = 0$$

is linearly equivalent to a unique positive divisor $\widetilde{D}$ with $\dim H^1(X, \mathcal{O}_{\widetilde{D}}) = 0$ and $\operatorname{supp} \widetilde{D} \subset X^\circ$.

Proof. For a divisor D with $\dim H^1(X, \mathcal{O}_{D-Q^\pm}) = 0$, there holds due to Lemma 4.26 that $\dim H^1(X, \mathcal{O}_D) = 0$. Since $\deg D = g$, the Riemann Roch Theorem [Forster, 1981, § 16.10] yields that $\dim H^0(X, \mathcal{O}_D) = 1$. Therefore, there exists a function $f \in \mathcal{O}_D$, unique up to multiplication by $c \in \mathbb{C} \setminus \{0\}$ such that $(f) + D \geq 0$ and such that $(f) \simeq D - \widetilde{D}$. So $\widetilde{D} \simeq (f) + D \geq 0$ is unique and because of $\deg(f) = 0$, it is $\deg \widetilde{D} = g$. Then Lemma 4.27 yields that $\operatorname{supp} D \in X^\circ$ and the assertion follows. □

The theory presented now can most likely also be formulated for the more general case of a compact complex curve X' together with a generalized divisor $\mathcal{S}$ to reconstruct a finite type potential u instead of a regular finite type potential. Most of the corresponding proofs necessary for the more general construction in a slightly different setting can be found in [Klein et al., 2016, Chapter 8].

However, we have shown the condition $D + \sigma(D) \simeq K + Q^+ + Q^-$ only under the assumption that u is a regular finite type potential and not for the generalized divisors $\mathcal{S}_M$ on the middleding of the Fermi curve corresponding to a finite type potential. So we restrict ourselves to reconstruction of regular finite type potentials and refer the reader to [Klein et al., 2016, Chapter 8] to get insight into the first part of the reconstruction of a Schrödinger operator with probably magnetic field and without this linear equivalence for the divisors. Our next aim is to show the following theorem in several steps.

Theorem 5.3. *If for a compact Riemann surface X conditions (F1) to (F3) hold and additionally, there exists a divisor D such that conditions (D1) to (D3) hold, then there exists a unique, real-analytic regular finite type potential $u : \mathbb{R}^2/\Gamma \to \mathbb{C}$ such that the compactified normalization $X(u)$ of the corresponding Fermi curve $X'(u)$ equals X and such that the pole divisor of the pullback of the corresponding unique normalized eigenfunction ψ_N of $-\triangle + u$ as in* (1.1) *to X equals D. The reconstructed eigenfunction also obeys the quasiperiodicity condition in* (1.3) *with respect to Γ. If additionally properties (R1) and (R2) hold, then the corresponding potential u is real-valued.*

To do so, we show that the given spectral data defines a unique Baker-Akhiezer function [Akhiezer, 1961] which we define next. Hereby, the double-periodic flow $\mathcal{L}_h(x, y)$ will yield the quasiperiodicity of the reconstructed Baker-Akhiezer function. Hereinafter, we show that the Baker-Akhiezer function is just the seeked normalized eigenfunction corresponding to a real-analytic potential $u : \mathbb{R}^2/\Gamma \to \mathbb{C}$. We also deduce this potential u from the given spectral data. Baker-Akhiezer functions combine the two equivalent concepts divisors and cocycles to describe line bundles on Riemann surfaces. They describe sections of families of line bundles, see [Dubrovin et al., 1990, Chapter 2, §2]. Baker-Akhiezer functions for general complex curves have been constructed in [Klein et al., 2016, Chapter 8]. Let now $(X, Q^+, Q^-, d\hat{c}, d\check{c}, \sigma, D)$ be given spectral data as in Definition 5.1. For non-special divisors in the sense of Lemma 4.28, i.e. with $\dim H^1(X, \mathcal{O}_{D-Q^\pm}) = 0$, one can show that the value of the Baker-Akhiezer function at Q^+ determines the eigenfunction uniquely. To see this, the next lemma is necessary.

Lemma 5.4. *Let X be a compact Riemann surface of genus g, $Q^\pm$ two marked points on X and D a divisor on X of degree g such that $\operatorname{supp} D \subset X^\circ$. Then the linear map*

$$H^0(X, \mathcal{O}_D) \to \mathbb{C}, \qquad f \mapsto f(Q^\pm) \tag{5.4}$$

is an isomorphism if and only if

$$H^1(X, \mathcal{O}_{D-Q^\pm}) = 0.$$

Proof. Let the mapping in (5.4) be an isomorphism. Then its kernel has to be zero. The kernel equals $H^0(X, \mathcal{O}_{D-Q^\pm})$ since $(f) \geq -D + Q^\pm$ yields $f(Q^\pm) = 0$. Because $\deg(D - Q^\pm) = g - 1$, the Riemann Roch Theorem [Forster, 1981, § 16.10] implies that $\dim H^0(X, \mathcal{O}_{D-Q^\pm}) =$

$\dim H^1(X, \mathcal{O}_{D-Q^\pm})$. Hence, $H^0(X, \mathcal{O}_{D-Q^\pm}) = 0$ holds if and only if $H^1(X, \mathcal{O}_{D-Q^\pm}) = 0$. So $H^1(X, \mathcal{O}_{D-Q^\pm}) = 0$.

Conversely, let $H^1(X, \mathcal{O}_{D-Q^\pm}) = 0$. Then also $H^0(X, \mathcal{O}_{D-Q^\pm}) = 0$, and so the map in (5.4) is injective. Because of $\dim H^1(X, \mathcal{O}_{D-Q^\pm}) = 0$, Lemma 4.26 yields that $\dim H^1(X, \mathcal{O}_D) = 0$ and so, by the Riemann-Roch Theorem [Forster, 1981, § 16.10] , $\dim H^0(X, \mathcal{O}_D) = 1$. Therefore, the map in (5.4) is an isomorphism of one-dimensional vector spaces. □

This lemma justifies the modified non-speciality from Definition 4.25. Motivated by this, we define the set of parameters T which belong to special divisors in the above sense as $T := T^+ \cup T^-$, where

$$T^\pm := \{(x, y) \in \mathbb{C}^2 \mid \dim H^1(X, \mathcal{O}_{D-Q^\pm} \otimes \mathcal{L}_h(x, y)) \neq 0\}.$$

Hereby, we extend the sheaf $\mathcal{L}_h(x, y)$ defined above from $(x, y) \in \mathbb{R}^2$ to $(x, y) \in \mathbb{C}^2$. We can apply [Klein et al., 2016, Theorem 8.6] with $n = 2$ to see that T^+ and T^- are subvarieties of $\mathbb{C}^2$ in the sense of Definition 1.11. Hence, also T is a subvariety of $\mathbb{C}^2$, and so the complement of T is open and dense in $\mathbb{C}^2$. The set T describes the set of parameters for which the Baker-Akhiezer function with pole divisor D is not uniquely defined since in this case $\dim H^0(X, \mathcal{O}_D \otimes \mathcal{L}_h(x, y)) > 1$, see Lemma 5.4. As already mentioned in Section 4.2.5, we consider $\mathcal{L}_h(x, y)$ as a sheaf on X for fixed $(x, y) \in \mathbb{C}^2$. The same cocycles with variable $(x, y) \in \mathbb{C}^2$ induce a sheaf $\mathcal{L}_h$ on $X \times \mathbb{C}^2$ and we consider $\mathcal{O}_D \otimes \mathcal{L}_h$ as a sheaf on $X \times \mathbb{C}^2$. From [Klein et al., 2016, Lemma 8.5] it follows that one can consider $\mathcal{O}_D \otimes \mathcal{L}_h$ as a deformation of $\mathcal{O}_D$ on X for every $(x, y) \in \mathbb{C}^2$. So one can use the theory of deformations of sheaves to control the dependence of the cohomology groups $H^q(X, \mathcal{O}_D \otimes \mathcal{L}_h(x, y))$ for $(x, y) \in \mathbb{C}^2 \setminus T$. The dimension of these spaces does not change for $(x, y) \notin T$, see [Grauert et al., 1994, Theorem III.4.7]. We will see in Corollary 5.10, that the two sets T^+ and T^- as defined above coincide if the divisor D obeys (D2).

Definition 5.5. For $c \in \mathbb{C} \setminus \{0\}$, a *Baker-Akhiezer function* is a function

$$\psi : X^\circ \times (\mathbb{C}^2 \setminus T) \to \mathbb{C},$$

with the following properties:

(i) For $(x, y) \in \mathbb{C}^2 \setminus T$, the map $k \mapsto \psi(k, (x, y))$ is a holomorphic section of $\mathcal{O}_D$ on X°.

(ii) For $(x, y) \in \mathbb{C}^2 \setminus T$, the map

$$U_+^* \to \mathbb{C}, \quad k \mapsto \psi(k, (x, y)) \cdot \exp\left(-2\pi\iota \left\langle k|_{U_+}, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right\rangle\right)$$

extends to a holomorphic function on U_+ with value $c : \mathbb{C}^2/\Gamma \to \mathbb{C}$ at Q^+, where c is holomorphic and not identically zero and the map

$$U_-^* \to \mathbb{C}, \quad k \mapsto \psi(k, (x, y)) \cdot \exp\left(-2\pi\iota \left\langle k|_{U_-}, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right\rangle\right)$$

extends to a holomorphic function on U_-.

Hereby, ψ is a holomorphic section in $\mathcal{O}_D \otimes \mathcal{L}_h$, so especially holomorphic in $(x, y) \in \mathbb{C}^2 \setminus T$, and therefore real-analytic when restricted to $(x, y) \in \mathbb{R}^2 \setminus (T|_{\mathbb{R}^2})$. Also c is real analytic in this case. In the above definition, one could also normalize the map $k \mapsto \psi(k, (x, y))$ at Q^- instead of at Q^+. The standard normalization for Baker-Akhiezer functions on complex curves with two marked points is to determine the values of the Baker-Akhiezer function at both of these points, whereby the degree of the given divisor is $g + 1$, compare [Klein et al., 2016, Chapter 8]. Then the set T comprises the points $(x, y) \in \mathbb{C}^2$ such that $\dim H^1(X, \mathcal{O}_{D-Q^+-Q^-}) \neq 0$. In this work, due to the degree g of the divisor and the periodic flow in two directions, it suffices to normalize the Baker-Akhiezer function at one of these two points to obtain the desired isomorphism in Lemma 5.4 which is the only difference in the proof of the existence of a unique eigenfunction. We will see in the sequel that in our case, condition (D2) enforces that the value of the Baker-Akhiezer function at Q^- is determined by the value at Q^+. Theorem [Klein et al., 2016, Theorem 8.8] transfers to our situation, where $n = 1$ is the number of marked points on X and $L = 2$ the number of linear independent directions of the considered linear flow $\mathcal{L}_h$.

Theorem 5.6. *Let $(X, Q^\pm, d\hat{c}, d\check{c}, \sigma, D)$ be given spectral data. Then for every holomorphic $c : \mathbb{C}^2 \to \mathbb{C}$ which is not identically zero, there exists one and only one Baker-Akhiezer function ψ such that* $\exp\left(2\pi\iota \left\langle k, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right\rangle\right) \psi(Q^+, (x, y)) = c(x, y)$ *and* $\psi(x, y) \in H^0(X^\circ \times (\mathbb{C}^2 \setminus T), \mathcal{O}_D \otimes \mathcal{L}_h(x, y))$, *where k is defined by the anti-derivatives of the Mittag Leffler distributions in* (5.1) *and* (5.2).

Besides a different definition of T and a different isomorphism indicated by Lemma 5.4, our situation concerning the above Theorem is identical to the situation in [Klein et al., 2016] and the proof of [Klein et al., 2016, Theorem 8.8] to obtain the existence of a unique Baker-Akhiezer function corresponding to given spectral data applies here. So we only show where the proof in our case differs from the proof in [Klein et al., 2016, Theorem 8.8]. The rest transfers identically to our situation and can be found in [Klein et al., 2016].

Proof of Theorem 5.6. An element of $H^0(X, \mathcal{L}_h(x, y))$ is given by a triple of holomorphic functions (ψ_0, ψ_+, ψ_-) with $\psi_0, \psi_\pm \in H^0(U_\pm, \mathcal{O}_{U_\pm})$ and $\psi_0 \in H^0(X^\circ, \mathcal{O}_{X^\circ})$ such that on $U^*_\pm$

$$\psi_\pm = \psi_0 \cdot z^*_\pm \exp\left(-2\pi\iota \left\langle \begin{pmatrix} h_{x,\pm} \\ h_{y,\pm} \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle\right). \tag{5.5}$$

Consequently, for $(x, y) \in \mathbb{C}^2 \setminus T$, an element ψ of $H^0(X, \mathcal{O}_D \otimes \mathcal{L}_h(x, y))$ is given by functions $\psi_0 \in H^0(X^\circ, \mathcal{O}_D)$ and $\psi_\pm \in H^0(U_\pm, \mathcal{O}_{U_\pm})$ such that on $U_\pm \setminus \{Q^\pm\}$ again (5.5) holds.
By Lemma 5.4, the map

$$H^0(X, \mathcal{O}_D \otimes \mathcal{L}_h(x, y)) \to \mathbb{C}, \qquad \psi \mapsto \psi(Q^+)$$

is an isomorphism for $(x, y) \in \mathbb{C}^2 \setminus T$. Therefore, there exists a unique element $\psi \in H^0(X, \mathcal{O}_D \otimes \mathcal{L}_h(x, y))$ which is mapped to $\psi(Q^+)$ by the above isomorphism. Then $\psi := \psi_0$ is the unique function with the properties of the Baker-Akhiezer function ψ given in Definition 5.5. The holomorphy of ψ in $(x, y) \in \mathbb{C}^2 \setminus T$ now follows analogously as shown in the proof of [Klein et al., 2016, Theorem 8.8]. □

Condition (ii) in Definition 5.5 yields that the unique Baker-Akhiezer function obeys the quasiperiodicity condition (1.3) if we chose the normalization $c(x, y)$ in the right way:

Proposition 5.7. *Let $\psi \in \mathcal{O}_D \otimes \mathcal{L}_h$ be a Baker-Akhiezer function which is normalized with a holomorphic function $c : \mathbb{C}^2/\Gamma \to \mathbb{C}$. Then for all $\gamma \in \Gamma$ and $(x, y) \in \mathbb{C}^2 \setminus T$, there holds*

$$\psi(k, (x, y) + \gamma) = e^{2\pi\iota\langle k,\gamma\rangle}\psi(k, (x, y)).$$

and there exists a holomorphic function $\tilde{\psi}$ which is periodic with respect to Γ such that

$$\psi(k, (x, y)) = e^{2\pi\iota\langle k,\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\rangle}\tilde{\psi}(x, y).$$

Proof. It follows from [Klein et al., 2016, Lemma 7.3(ii)] that $\mathcal{L}_h(\gamma)$ induces the trivial flow for all $\gamma \in \Gamma$. Then the definition of the Baker-Akhiezer function yields that $\psi((x, y)+\gamma) = g(k, \gamma)\psi(x, y)$, where $g : X(u) \times \Gamma \to \mathbb{C}$ can be interpreted as the change of the normalization of the unique function $\psi(x, y)$. This is since $g(k, \gamma)\psi(x, y)$ obeys the same conditions as ψ in Definition 5.5 and due to Theorem 5.6, ψ is unique. Because of (ii) in Definition 5.5, this normalization is given by $\exp(2\pi\iota \langle k, \gamma\rangle)$ and thus $\psi(k, \cdot)$ is quasiperiodic with respect to Γ. Defining $\tilde{\psi}$ at $k \in X^\circ$ as $\tilde{\psi}(k, (x, y)) := \exp(-2\pi\iota \langle k, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\rangle)\, \psi$ for all $(x, y) \in \mathbb{C}^2 \setminus T$ yields then for all $\gamma \in \Gamma$ that

$$\begin{aligned}\tilde{\psi}(k, (x, y) + \gamma) = \exp\left(-2\pi\iota \left\langle k, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) + \gamma\right\rangle\right) \psi(k, (x, y) + \gamma) = \\ = \exp\left(-2\pi\iota \left\langle k, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right\rangle\right) \psi(k, (x, y)) = \tilde{\psi}(k, (x, y)).\end{aligned}$$

□

Since we are later on only interested in reconstruction of Baker-Akhiezer functions which obey the quasiperiodicity condition we have just shown for Γ-periodic normalization, we assume from now on that $c(x, y)$ is a holomorphic function which is periodic on $\mathbb{C}^2$ with respect to Γ.

Proposition 5.8. *Let X be a Riemann surface with two marked points $Q^\pm$ and $D, \tilde{D}$ be positive divisors of degree g with support in X° which obey $D + \tilde{D} \simeq K + Q^+ + Q^-$. Then*

$$\dim H^0(X, \mathcal{O}_{D-Q^\pm}) = 0 \;\Leftrightarrow\; \dim H^0(X, \mathcal{O}_{\tilde{D}-Q^\pm}) = 0.$$

Proof. It is $\deg D = g$, so the Riemann Roch Theorem [Forster, 1981, § 16.10] and Serre Duality [Forster, 1981, §17.9] yield

$$\begin{aligned} &\dim H^0(X, \mathcal{O}_{D-Q^\pm}) = 0 \;\Leftrightarrow\; \dim H^1(X, \mathcal{O}_{D-Q^\pm}) = 0 \;\Leftrightarrow\; \dim H^1(X, \mathcal{O}_{K-D+Q^\pm}) = 0 \\ \Leftrightarrow\; &\dim H^1(X, \mathcal{O}_{K-D+Q^\pm+Q^\mp-Q^\mp}) = 0 \;\Leftrightarrow\; \dim H^1(X, \mathcal{O}_{\tilde{D}-Q^\mp}) = 0 \\ \Leftrightarrow\; &\dim H^0(X, \mathcal{O}_{\tilde{D}-Q^\mp}) = 0. \end{aligned}$$

□

To see how condition (D2) involves in the reconstruction of the Schrödinger operator, we introduce a Schrödinger operator with magnetic field

$$H := -4\partial\bar{\partial} + A^z(x,y)\partial + A^{\bar{z}}(x,y)\bar{\partial} + u(x,y), \tag{5.6}$$

where ∂ and $\bar{\partial}$ are again the Wirtinger operators. We want to deduce an analytic potential $u : \mathbb{C}^2 \setminus T \to \mathbb{C}$ which is double-periodic with respect to the lattice Γ as well as a vector valued magnetic field $(A^z, A^{\bar{z}})$ with $A^z, A^{\bar{z}} : \mathbb{C}^2 \setminus T \to \mathbb{C}^2$ such that ψ lies in the kernel of H. It will turn out that the magnetic field is zero for divisors which obey the linear equivalence in (D2) and constant c. Moreover, we show that u is periodic with respect to Γ if the chosen normalization at Q^+ is periodic with respect to Γ.

A first step is to see the influence of the normalization of ψ at Q^+ on H and why we could also have chosen a normalization at Q^- in Definition 5.5. So let $H\psi = 0$, where ψ is normalized at Q^+ as $\psi(Q^+, (x,y)) = c_+(x,y)$. Direct calculations yield for an analytic function $f : \mathbb{C}^2 \to \mathbb{C}$ that

$$\begin{aligned} \left(-4\partial\bar{\partial} + \tilde{A}^z\partial + \tilde{A}^{\bar{z}}\bar{\partial} + \tilde{u}\right)(f\psi) = -4\left(f\partial\bar{\partial}\psi - \partial\bar{\partial}f\psi - \bar{\partial}f\partial\psi - \partial f\bar{\partial}\psi\right) + \\ + \tilde{A}^z\left(\partial f\psi + f\partial\psi\right) + \tilde{A}^{\bar{z}}\left(\bar{\partial}f\psi + f\bar{\partial}\psi\right) + \tilde{u}f\psi. \end{aligned}$$

Inserting

$$\tilde{A}^z := A^z + 4\bar{\partial}\ln(f),\; \tilde{A}^{\bar{z}} := A^{\bar{z}} + 4\partial\ln(f),\; \tilde{u} := u - A^z\partial\ln(f) - A^{\bar{z}}\bar{\partial}\ln(f) - 8\frac{\partial f\bar{\partial}f}{f^2} + 4\frac{\partial\bar{\partial}f}{f}$$

into this equation yields

$$
\begin{aligned}
\left(-4\partial\bar{\partial}+\tilde{A}^{z}\partial+\tilde{A}^{\bar{z}}\bar{\partial}+\tilde{u}\right)(f\psi) = & -4\left(f\partial\bar{\partial}\psi-\cancel{\partial\bar{\partial}f\psi}-\bar{\partial}f\partial\psi-\partial f\bar{\partial}\psi\right)+\\
& +\left(A^{z}+4\bar{\partial}\ln(f)\right)(\partial f\psi+f\partial\psi)+\\
& +\left(A^{\bar{z}}+4\partial\ln(f)\right)\left(\bar{\partial}f\psi+f\bar{\partial}\psi\right)+\\
& +\left(u-A^{z}\partial\ln(f)-A^{\bar{z}}\bar{\partial}\ln(f)-8\frac{\partial f\bar{\partial}f}{f^{2}}+4\cancel{\frac{\partial\bar{\partial}f}{f}}\right)f\psi\\
& +\left(A^{z}\partial f+A^{\bar{z}}\bar{\partial}f+4(\ln\partial f\bar{\partial}f+\ln\bar{\partial}f\partial f)\right)\psi+\\
& +4\left(\bar{\partial}\ln(f)f\partial\psi+\partial\ln(f)f\bar{\partial}\psi\right)-\\
& -\left(A^{z}\partial\ln(f)+A^{\bar{z}}\bar{\partial}\ln(f)+8\frac{\partial f\bar{\partial}f}{f^{2}}\right)f\psi
\end{aligned}
$$

and since $\partial\ln(f)=\frac{\partial f}{f}$ as well as $\bar{\partial}\ln(f)=\frac{\bar{\partial}f}{f}$, this is equivalent to

$$
\left(-4\partial\bar{\partial}+\tilde{A}^{z}\partial+\tilde{A}^{\bar{z}}\bar{\partial}+\tilde{u}\right)(f\psi)=f\left(-4\partial\bar{\partial}+A^{z}\partial+A^{\bar{z}}\bar{\partial}+u\right)\psi. \tag{5.7}
$$

So however one normalizes ψ at Q^{+}, there is always an element $f\psi$ normalized differently at Q^{+} which lies in the kernel of an operator $\tilde{H}$ with possibly different potential and different magnetic field and $\tilde{H}$ is obtained by gauging with f as in equation (5.7). In particular, one can change the normalization at Q^{+} by considering $f=\frac{\tilde{c}_{+}(x,y)}{c_{+}(x,y)}$, where $f\psi(Q^{+},(x,y))=\tilde{c}_{+}(x,y)$ is the normalization $f\psi$ at Q^{+}. Likewise one can also normalize $f\psi$ at Q^{-} as $f\psi(Q^{-},(x,y))=\tilde{c}_{-}(x,y)$ and then $f=\frac{\tilde{c}_{-}(x,y)}{\psi(Q^{-},(x,y))}$. In the proof of the next Theorem it will become clear how one can determine the value $\psi(Q^{-},(x,y))$ out of $\psi(Q^{+},(x,y))$ if the corresponding divisor fulfills $D+\sigma(D)\simeq K+Q^{+}+Q^{-}$. It will moreover turn out that one has to choose the normalization as periodic with respect to $\mathit{\Gamma}$ to obtain an operator which is periodic with respect to $\mathit{\Gamma}$. This will be made more precise in the proof of part (a) of the following Theorem.

Theorem 5.9. *Let $(X,Q^{+},Q^{-},d\hat{c},d\check{c},\sigma,D)$ be given spectral data as in Definition 5.1 obeying all conditions but (D2) and (D3). Let $z_{\pm}$ be local coordinates centered at $Q^{\pm}$ and let the corresponding Mittag-Leffler distribution induced by $d\hat{c}$ and $d\check{c}$ be given by a local parametrization of X in a neighborhood of $Q^{\pm}$, that is*

$$
k|_{U_{+}^{*}}=\left(\frac{1}{z_{+}},-\frac{\iota}{z_{+}}+\sum_{j=0}^{\infty}a_{+,j}z_{+}^{2j+1}\right)\quad \textit{and}\quad k|_{U_{-}^{*}}=\left(\frac{1}{z_{-}},\frac{\iota}{z_{-}}+\sum_{j=0}^{\infty}a_{-,j}z_{-}^{2j+1}\right).
$$

For every $(x,y)\in\mathbb{C}^{2}\setminus T$, let $\psi_{N}(k,(x,y))$ be the unique Baker-Akhiezer function associated to this data as in Theorem 5.6 with holomorphic $c:\mathbb{C}^{2}\to\mathbb{C}$ and $c\not\equiv 0$. Then there exists an analytic function $u:\mathbb{C}^{2}\setminus T\to\mathbb{C}$ and a magnetic field $(A^{z},A^{\bar{z}}):\mathbb{C}^{2}\setminus T\to\mathbb{C}^{2}$ such that ψ_{N} lies in the kernel of (5.6) *and D is the pole divisor of ψ_{N}. Furthermore, there holds:*

(a) $(A^z, A^{\bar{z}}) = (0,0)$ *if and only if (D2) holds and c is constant.*

(b) If additionally to (D2) also (R1) and (R2) hold, then the potential u is real-valued.

Proof. With $\psi_0 = \psi$ and $\psi_\pm$ defined as in the proof of Theorem 5.6, one has

$$\psi_0|_{U_\pm^*} = \exp\left(2\pi\iota \left\langle k|_{U_\pm^*}, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right\rangle\right) \cdot \psi_\pm.$$

Thereby, the representation of k on $U_\pm$ is a consequence of the principal parts of $\hat{c}$ and $\check{c}$ given by the pole behavior of $d\hat{c}$ and $d\check{c}$, compare (5.1) and (5.2). Due to (F3), only the uneven powers of $z_\pm$ occur in the power set in a neighborhood of $Q^\pm$. To determine $\partial\psi$, $\bar{\partial}\psi$ and $\partial\bar{\partial}\psi$ at Q^+ and Q^-, note that with the triple $(\psi, \psi_+, \psi_-) \in H^0(\mathcal{O}_D \otimes \mathcal{L}_h)$, it is on $U_\pm^*$

$$\begin{aligned}
\exp\left(-2\pi\iota \left\langle k, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right\rangle\right) \partial\psi &= \pi\iota(k_1 + \iota k_2)\psi_\pm + \partial\psi_\pm, \\
\exp\left(-2\pi\iota \left\langle k, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right\rangle\right) \bar{\partial}\psi &= \pi\iota(k_1 - \iota k_2)\psi_\pm + \bar{\partial}\psi_\pm, \\
\exp\left(-2\pi\iota \left\langle k, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right\rangle\right) \partial\bar{\partial}\psi &= -\pi^2(k_1^2 + k_2^2)\psi_\pm + \pi\iota(k_1 + \iota k_2)\bar{\partial}\psi_\pm \\
&\quad + \pi\iota(k_1 - \iota k_2)\partial\psi_\pm + \partial\bar{\partial}\psi_\pm.
\end{aligned}$$

In the following calculations, we ignore the fact that $\frac{1}{z_\pm}$ does not exist at $Q^\pm$. This abuse of notation is justified by the fact that we are only interested in the poles at $Q^\pm$ and the constants in front of them when evaluating the above derivatives at $Q^\pm$ to find out the right values for the magnetic field $(A^z, A^{\bar{z}})$ and the potential u of the corresponding operator H such that u is holomorphic on $\mathbb{C}^2 \setminus T$ and periodic with respect to Γ. This is done in such a way that the terms with poles which occur in $\partial\bar{\partial}\psi_\pm$ are canceled out by the right choice of A^z and $A^{\bar{z}}$. We know that $\psi_\pm(x,y)$ is holomorphic on $U_\pm$ for every $(x,y) \in \mathbb{C}^2 \setminus T$, i.e. we can write these functions as power series

$$\psi_\pm(x,y) = \sum_{j=0}^{\infty} \psi_{\pm,j}(x,y) z^j$$

on the corresponding neighborhoods, where $\psi_{\pm,j} \in C^\infty(\mathbb{C}^2/\Gamma, \mathbb{C})$ due to Proposition 5.7. In the following calculations, we omit the dependence of ψ on (x,y) and the dependence of $k|_{U_\pm^*}$ on z_+ and z_-,respectively. Inserting $k|_{U_+^*}$ into the above derivatives and evaluating the obtained expressions at Q^+, i.e. setting $z_+ = 0$, yields formally

$$\begin{aligned}
&\exp\left(-2\pi\iota\left\langle k, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right\rangle\right)\partial\psi = -\pi\left(\sum_{j=0}^{\infty} a_{+,j} z_+^{2j+1}\right)\psi_+ + \partial\psi_+ \\
\Rightarrow\ &\exp\left(-2\pi\iota\left\langle k, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right\rangle\right)\partial\psi(Q^+) = \partial\psi_{+,0}, \\
&\exp\left(-2\pi\iota\left\langle k, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right\rangle\right)\bar\partial\psi = \pi\iota\left(\frac{2}{z_+} + \iota\sum_{j=0}^{\infty} a_{+,j} z_+^{2j+1}\right)\psi_+ + \bar\partial\psi_+ \\
\Rightarrow\ &\exp\left(-2\pi\iota\left\langle k, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right\rangle\right)\bar\partial\psi(Q^+) = 2\pi\iota\left(\frac{\psi_{+,0}}{z_+} + \psi_{+,1}\right) + \bar\partial\psi_{+,0}, \\
&\exp\left(-2\pi\iota\left\langle k, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right\rangle\right)\partial\bar\partial\psi = -\pi^2\underbrace{\left(\frac{1}{z_+^2} + \left(\frac{\iota}{z_+} + \sum_{j=0}^{\infty} a_{+,j} z_+^{2j+1}\right)^2\right)}_{=2\iota\sum_{j=0}^{\infty} a_{+,j} z_+^{2j} + \left(\sum_{j=0}^{\infty} a_{+,j} z_+^{2j+1}\right)^2}\psi_+ \\
&\qquad - \pi\left(\sum_{j=0}^{\infty} a_{+,j} z_+^{2j+1}\right)\bar\partial\psi_+ \\
&\qquad + \pi\iota\left(\frac{2}{z_+} - \iota\sum_{j=0}^{\infty} a_{+,j} z_+^{2j+1}\right)\partial\psi_+ + \bar\partial\partial\psi_+ \\
\Rightarrow\ &\exp\left(-2\pi\iota\left\langle k, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right\rangle\right)\partial\bar\partial\psi(Q^+) = -2\pi^2\iota a_{+,0}\psi_{+,0} + 2\pi\iota\left(\frac{\partial\psi_{+,0}}{z_+} + \partial\psi_{+,1}\right) + \bar\partial\partial\psi_{+,0}.
\end{aligned}$$

Analogously one obtains at Q^-

$$\begin{aligned}
&\exp\left(-2\pi\iota\left\langle k, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right\rangle\right)\partial\psi(Q^-) = 2\pi\iota\left(\frac{\psi_{-,0}}{z_-} + \psi_{-,1}\right) + \partial\psi_{-,0} \\
&\exp\left(-2\pi\iota\left\langle k, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right\rangle\right)\bar\partial\psi(Q^-) = \bar\partial\psi_{-,0},
\end{aligned}$$

$$\exp\left(-2\pi\iota\left\langle k, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right\rangle\right)\partial\bar\partial\psi(Q^-) = 2\pi^2\iota a_{-,0}\psi_{-,0} + 2\pi\iota\left(\frac{\bar\partial\psi_{-,0}}{z_-} + \bar\partial\psi_{-,1}\right) + \bar\partial\partial\psi_{-,0}.$$

Let now

$$A^z(x,y) := 4\bar\partial\ln(\psi_{-,0})(x,y) \quad\text{and}\quad A^{\bar z} := 4\partial\ln(\psi_{+,0})(x,y). \tag{5.8}$$

Then

$$\begin{aligned}
&\exp\left(-2\pi\iota\left\langle k, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right\rangle\right)\left(4\partial\bar{\partial} - A^z\partial - A^{\bar{z}}\bar{\partial}\right)\psi(Q^+) = \\
&\qquad = -8\pi^2\iota a_{+,0}\psi_{+,0} + 8\pi\iota\left(\frac{\partial\psi_{+,0}}{z_+} + \partial\psi_{+,1}\right) + 4\bar{\partial}\partial\psi_{+,0} - \\
&\qquad\quad - 4\bar{\partial}\ln(\psi_{-,0})\partial\psi_{+,0} - 8\pi\iota\partial\ln(\psi_{+,0})\left(\frac{\psi_{+,0}}{z_+} + \psi_{+,1}\right) + 4\bar{\partial}\psi_{+,0} \\
&\qquad = -8\pi^2\iota a_{+,0}\psi_{+,0} + 8\pi\iota\partial\psi_{+,1} + 4\bar{\partial}\partial\psi_{+,0} - \\
&\qquad\quad - 4\bar{\partial}\ln(\psi_{-,0})\partial\psi_{+,0} - 8\pi\iota\partial\ln(\psi_{+,0})\psi_{+,1} + 4\bar{\partial}\psi_{+,0}
\end{aligned}$$

and

$$\begin{aligned}
&\exp\left(-2\pi\iota\left\langle k, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right\rangle\right)\left(4\partial\bar{\partial} - A^z\partial - A^{\bar{z}}\bar{\partial}\right)\psi(Q^-) = \\
&\qquad = 8\pi^2\iota a_{-,0}\psi_{-,0} + 8\pi\iota\left(\frac{\bar{\partial}\psi_{-,0}}{z_-} + \bar{\partial}\psi_{-,1}\right) + 4\bar{\partial}\partial\psi_{-,0} - \\
&\qquad\quad - 8\pi\iota\bar{\partial}\ln(\psi_{-,0})\left(\frac{\psi_{-,0}}{z_-} + \psi_{-,1}\right) + 4\partial\psi_{-,0} - 4\bar{\partial}\ln(\psi_{+,0})\bar{\partial}\psi_{-,0} \\
&\qquad = 8\pi^2\iota a_{-,0}\psi_{-,0} + 8\pi\iota\bar{\partial}\psi_{-,1} + 4\partial\bar{\partial}\psi_{-,0} - \\
&\qquad\quad - 8\pi\iota\bar{\partial}\ln(\psi_{-,0})\psi_{-,1} + 4\partial\psi_{-,0} - 4\bar{\partial}\ln(\psi_{+,0})\bar{\partial}\psi_{-,0}.
\end{aligned}$$

We define

$$u(x,y)\psi := \left(4\partial\bar{\partial} - A^z(x,y)\partial - A^{\bar{z}}(x,y)\bar{\partial}\right)\psi. \tag{5.9}$$

Then by definition of $u(x,y)$, $A^z(x,y)$ and $A^{\bar{z}}(x,y)$, ψ is in the kernel of H. Moreover, due to the above calculations, $\exp\left(2\pi\iota\left\langle k, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right\rangle\right)u(x,y)\psi$ is holomorphic on $U_\pm$ for $(x,y) \in \mathbb{C}^2 \setminus T$: we have given reason above that the dimensions of the cohomology groups are constant under deformations of $\mathcal{O}_D$ by tensorating it with $\mathcal{L}_h(x,y)$ for $(x,y) \in \mathbb{C}^2 \setminus T$. Therefore, $\dim H^0(X, \mathcal{O}_{D-Q^+} \otimes \mathcal{L}_h(x,y)) = 0$ and $\dim H^0(X, \mathcal{O}_{D-Q^-} \otimes \mathcal{L}_h(x,y)) = 0$ for all $(x,y) \in \mathbb{C}^2 \setminus T$. Accordingly, Lemma 4.27 yields that $\psi_{0,\pm} \not\equiv 0$. Since the right hand side of (5.9) multiplied with $\exp(-2\pi\iota\langle k, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\rangle)$ is holomorphic on $U_\pm$, the same holds for the left hand side of (5.9). So $\exp(-2\pi\iota\langle k, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\rangle)u(x,y)|_{U_\pm}$ has no pole. Furthermore, $u\psi(x,y) \in H^0(X^\circ, \mathcal{O}_D \otimes \mathcal{L}_h(x,y))$ for $(x,y) \in \mathbb{C}^2 \setminus T$ since deriving ψ into the direction of $(x \pm \iota y)$ does not change the pole order of ψ in k which can be seen by the same means as it is shown for the normalized eigenfunction in the proof of Lemma 3.16. Hence, $u\psi$ is a Baker-Akhiezer function with a different normalization then ψ at Q^+ which is uniquely determined by Theorem 5.6. Furthermore, Proposition 5.7 yields that $\psi_{\pm,0}$ are periodic with respect to Γ on $\mathbb{C}^2 \setminus T$ for periodic c. Thus, also A^z and $A^{\bar{z}}$ as in (5.8) are periodic with respect to Γ, wherefore finally also u is.

(a) We know from the direct problem, more precisely from Lemma 4.13, that for the divisor

D corresponding to the Schrödinger operator without magnetic field, the linear equivalence $D + \sigma(D) \simeq K + Q^+ + Q^-$ holds.

Let now D be a positive divisor which obeys $D + \sigma(D) \simeq K + Q^+ + Q^-$ and let ψ be the Baker-Akhiezer function, unique up to normalization at Q^+ and solving (5.6) with pole divisor D. Since $D + \sigma(D) \simeq K + Q^+ + Q^-$ holds, there exists a 1-form with divisor $D + \sigma(D) - Q^+ - Q^-$ which is holomorphic on X° and has $2g$ zeroes and poles of first order at Q^+ and Q^-. Hence, the Residue Theorem yields that $\mathrm{Res}_{Q^+}(\omega) + \mathrm{Res}_{Q^-}(\omega) = 0$. Condition (F2)(i), Property (ii) of the Baker-Akhiezer function and Proposition 5.7 together yield, as in the proof of Lemma 4.13, that such a 1-form is given by

$$\omega = \frac{\psi(x,y)\sigma^*\psi(x,y)}{\langle\!\langle \sigma^*\psi, \psi \rangle\!\rangle_\partial} d\hat{c}$$

with $\psi(x,y) \neq 0$. Since $\sigma(k) = -k$ due to condition (F3), the exponential factors in ψ and $\sigma^*\psi$ have the same absolute value but opposite signs. So taking into account that $\sigma(Q^\pm) = Q^\pm$, one has with $\sigma^*\psi(Q^\pm) = \psi(Q^\pm)$

$$\sigma^*\psi(Q^\pm)(x,y)\psi(Q^\pm)(x,y) = \sigma^*\psi_{\pm,0}\psi_{\pm,0} = \psi^2_{\pm,0}$$

and hence

$$0 = \mathrm{Res}_{Q^+}(\omega) + \mathrm{Res}_{Q^-}(\omega) = c_+\psi^2_{+,0} + c_-\psi^2_{-,0},$$

where $c_\pm \in \mathbb{C}$ are constants depending on the evaluated integral over the fundamental domain $\langle\!\langle \sigma^*\psi, \psi \rangle\!\rangle_\delta$, and therefore are independent from $(x,y) \in \mathbb{C}^2$. So if we normalize ψ at Q^+ as $\psi(Q^+,(x,y)) = 1$ for all $(x,y) \in \mathbb{C}^2$, then also

$$\psi_{-,0} = \pm\sqrt{\frac{c_+}{c_-}}\psi_{+,0}$$

is constant. Here, we can choose the sign of $\psi_{-,0}$ arbitrary since $\psi_{-,0}$ is constant in (x,y), so the choice of the sign of $\psi_{-,0}$ at one fixed $(x,y) \in \mathbb{C}^2 \setminus T$ fixes this choice for all $(x,y) \in \mathbb{C}^2 \setminus T$. Hence, if ψ is normalized as constant at Q^+, it is also constant at Q^- and vice versa. Since $\psi_{\pm,0}$ is a constant functions on the compact surface $\mathbb{C}^2/\Gamma$, it is $\partial \ln(\varphi_{\pm,0}) = \bar{\partial} \ln(\varphi_{\pm,0}) = 0$. So the corresponding operator H has zero magnetic field $(A^z, A^{\bar{z}})(x,y) = (0,0)$, and therefore corresponds to the Schrödinger operator (1.1).

Vice versa, let $A(x,y) = 0$ for all $(x,y) \in \mathbb{C}^2 \setminus T$. This implies also that $\psi_{+,0}$ and $\psi_{-,0}$ are constant. More precisely, we can interpret $\mathbb{C}^2/\Gamma$ via $\mathbb{C}^2 \simeq \mathbb{C}$ as the one-dimensional torus, i.e a compact Riemann surface and

$$\bar{\partial} \ln \psi_{-,0} = \frac{\bar{\partial}\psi_{-,0}}{\psi_{-,0}} = 0 \;\Leftrightarrow\; \bar{\partial}\psi_{-,0} = 0.$$

So $\psi_{-,0}$ is a double-periodic, holomorphic function on a compact Riemann surface and thus constant. We have seen above that $\psi_{-,0}$ is constant if and only if $\psi_{+,0}$ is, so also $\varphi_{+,0}$ is constant.

(b) To see that $\tau_1(D) = D$ implies that $u(x,y) \in \mathbb{R}$ for all $(x,y) \in \mathbb{C}^2 \setminus T$, note that $\tau_1^* \bar{\psi} = \frac{\psi_{+,0}}{\psi_{-,0}} \psi$ since $\tau_1(Q^+) = Q^-$, so $\tau_1^* \bar{\psi}$ is the eigenfunction such that $\bar{\psi}_{-,0} = 1$ and ψ the eigenfunction normalized as $\psi_{+,0} = 1$. Due to the above considerations on gauging the eigenfunctions and the operator, these eigenfunctions are both unique and differ only by the constant $\frac{\bar{\psi}_{-,0}}{\psi_{+,0}}$. Therefore, $\tau_1^* \bar{H} \tau_1^* \bar{\psi} = \frac{\bar{\psi}_{-,0}}{\psi_{+,0}} H\psi$, i.e. $\ker \tau_1^* \bar{H} = \ker H$. If ψ is known, also u is known since then $\triangle\psi = u\psi$. Since ψ is not identically to zero on U_+ if normalized as $\psi(Q^+, (x,y)) = 1$, $\psi \not\equiv 0$, so

$$\tau_1^* \bar{u} = \frac{\triangle \tau_1^* \bar{\psi}}{\tau_1^* \bar{\psi}} = \frac{\triangle\psi}{\psi} = u.$$

and thus $u = \bar{u}$.

□

Note that the potential u is only real-valued due to both symmetries on the divisor $D + \sigma(D) \simeq K + Q^+ + Q^-$ and $\tau_1(Q^+) = Q^-$. Assuming (D2) does not hold, then – as we have seen in the discussion about gauging ψ to obtain an eigenfunction with another normalization – τ_1^* applied to $\bar{\psi}$ normalized at Q^+ only equals $\frac{\tilde{\psi}_{0,-}}{\psi_{0,+}} \tilde{\psi}$ with $\tilde{\psi}$ a Baker-Akhiezer function normalized at Q^-. The above proof shows that for divisors D obeying the linear equivalence in (D2) one has that $\frac{\tilde{\psi}_{0,-}}{\psi_{0,+}}$ is constant.
From now on, we choose the normalization $c \equiv 1$ in Theorem 5.6. In that case, the above proof yields that the two sets T^+ and T^- coincide.

Corollary 5.10. *Let $(X, Q^+, Q^-, d\hat{c}, d\check{c}, \sigma, D)$ be spectral data as in Definition 5.1 and $c \equiv 1$ in Definition 5.5. Then*

$$H^0(X, \mathcal{O}_{D-Q^+}) = 0 \;\Leftrightarrow\; H^0(X, \mathcal{O}_{D-Q^-}) = 0.$$

Proof. Note that $H^0(X, \mathcal{O}_{D-Q^\pm}) \subset H^0(X, \mathcal{O}_D)$. $H^0(X, \mathcal{O}_{D-Q^+}) = 0$ and $\dim H^0(X, \mathcal{O}_D) = 1$ can hold if and only if all elements in $H^0(X, \mathcal{O}_D)$ are unequal to zero at Q^+. In the foregoing proof, it is shown that this implies that this element is also unequal to zero at Q^-. So the assertion follows. □

In Part I, we considered a normalization of the eigenfunction (3.2) to obtain its uniqueness which on the first view might differ from the normalization $c \equiv 1$. However, it follows that the Baker-Akhiezer function tinkered above with normalization $c \equiv 1$ is just the same as a Baker-Akhiezer function not normalized at Q^+ but as $\psi(k, (0,0)) = 1$.

Corollary 5.11. *Let $(X, d\hat{c}, d\check{c}, Q^+, Q^-, \sigma, D)$ be given spectral data. Then the Baker-Akhiezer function (ψ_0, ψ_+, ψ_-) is normalized with $c \equiv 1$ if and only if $\psi(k,(0,0)) = 1$.*

Proof. That $c \equiv 1$ for $\psi(k,(0,0)) = 1$ is obvious. Conversely, let $c \equiv 1$. Due to Lemma 4.27, $\dim H^0(X, \mathcal{O}_D) = 1$. So the normalization of the Baker-Akhiezer function at Q^+ as $\psi_{0,+} = 1$ yields that the Baker-Akhiezer function is identically to 1 in k for $(x,y) = (0,0)$. Therefore, also $\frac{\psi_{-,0}}{\psi_{+,0}} = 1$ since this expression does not depend on $(x,y) \in \mathbb{C}^2 \setminus T$ and the assertion follows. □

Taking all the above together, Theorem 5.3 follows because the Baker-Akhiezer function is unique.

Proof of Theorem 5.3. We have seen in Theorems 5.6 and 5.9 that normalizing the Baker-Akhiezer function at Q^+ as equal to one yields for given spectral data $(X, Q^+, Q^-, d\hat{c}, d\check{c}, \sigma, D)$ and for every $k \in X^\circ$ a unique real-analytic function $\psi(k) : \mathbb{R}^2 \setminus T \to \mathbb{C}$. Since we assumed (D3) holds, it is $T = \emptyset$. In Proposition 5.7, it is shown that $\psi(k)$ obeys (1.3). From Theorem 5.9 follows the existence of a unique real-analytic potential $u : \mathbb{R}^2/\varGamma \to \mathbb{C}$ such that $(-\triangle + u)\psi = 0$. Also from Theorem 5.9 we obtain that if conditions (R1) and (R2) hold, then u is real-valued.
It follows from Theorem 2.28(b) and Section 4 that $X(u)$ obeys (F1) and Lemma 1.17 yields that (F3) holds for $\hat{c} = \langle k, \hat{\gamma} \rangle$ and $\check{c} = \langle k, \check{\gamma} \rangle$. Furthermore, in Section 4.3 is shown that the Mittag Leffler distributions we used to generate the double-periodic flow correspond to the lattice $\varGamma$ with respect to which u is periodic and which obey (F2). Moreover, the differentials in condition (F2), which are uniquely defined by their periods and their pole behavior at $Q^\pm$, enable us to reconstruct a map $(\hat{c}, \check{c}) : X \to \mathbb{C}^2$ whose image in $\mathbb{C}^2$ equals the Fermi curve $X'(u)$ and X is the normalization of $X'(u)$. Points which are separated on X are then again identified under the map $(\hat{c}, \check{c})$. The structure sheaf $\mathcal{O}_X$ might contain more functions then the holomorphic function on the Fermi curve. The latter are precisely the holomorphic functions which are generated by $\hat{c}$ and $\check{c}$.
The conditions on the divisor which make the Baker-Akhiezer function unique are also all fulfilled by the pole divisor of the normalized eigenfunction of $-\triangle + u$ pulled back to the normalization if u is a regular finite type potential, compare Theorem 4.32. That (R1) and (R2) hold for u real-valued we have seen in Lemmata 1.17(b) and 4.24. Moreover, the Baker-Akhiezer function corresponding to some given spectral data is unique and also the normalization of the unique eigenfunction ψ_N in the direct problem as $\psi_N(k,(0,0)) = 1$ coincides with the chosen normalization of the Baker-Akhiezer function. So due to Corollary 5.11, Theorem 5.3 follows. □

6. The isospectral set for regular finite type potentials

Formally, the isospectral set in our setting is the set of regular finite type potentials u which yield the same Fermi curve. We know from the results in Chapter 5 that on a fixed Fermi curve, these potentials can be identified with the pole divisors of the corresponding eigenfunction. So we consider here the isospectral set as the set of divisors D which lead to the same Fermi curve. Our intuition tells us that for fixed genus g, small deformations of the divisor D from given spectral data $(X, Q^+, Q^-, \hat{c}, \check{c}, \sigma, D)$ as in Definition 5.1 will also yield spectral data which is a solution of the inverse problem of the Schrödinger operator with a different potential $\tilde{u}$. So the aim of the inverse problem would be to 'move' the divisor D – interpreted as an element in a translation of the Jacobian variety – to a different divisor $\tilde{D}$. The questions that arise are how to parameterize such translations and in which cases all translated divisors are non-special. In this work, we will only give partly answers to these questions in the sense that we show that the Prym variety parametrizes the isospectral set and that the latter is an open set in the Prym variety. Hereby, the focus is on real-valued potential.

6.1. Complex potential

Lemma 6.1. *The set of divisors of degree 0 which obey $D' = -\sigma(D')$ acts transitively on the set of positive divisors D obeying $D + \sigma(D) \simeq K + Q^+ + Q^-$.*

Proof. Let D and D' both be divisors obeying (4.17). Since σ acts linearly on divisors, taking the difference of these two equations yields that $D - D' + \sigma(D - D') \simeq 0$, and therefore $\sigma(D - D') = -(D - D')$. □

Note that the Prym variety $\mathrm{Prym}(X, \sigma)$ consists of the equivalence classes of the images of this group of divisors under the Abel map. We are mainly interested in non-special divisors D of degree g obeying (4.17). So let D be such a divisor and let D' be another non-special divisor of degree g such that $\sigma(D - D') = -(D - D')$, i.e. $D - D'$ is the preimage of an element in the Prym variety under the Abel map. Then these two relations define D' uniquely as a positive non-special divisor of degree g. Therefore, the isospectral set for complex-valued potentials has only one connected component which is parametrized by the Prym variety. Since the Prym variety is isomorphic to a complex torus $\mathbb{C}^{g/2}/\Lambda_-$, compare Definition A.12, one immediately obtains the following corollary.

Corollary 6.2. *The isospectral set for a Fermi curve X is contained in a complex torus of dimension $g/2$.*

Hereby, we use that $\mathrm{Prym}(X,\sigma)$ can be embedded into $\mathrm{Jac}(X)$. Actually one can embed $2^{g/2}$ differently translated Prym varieties int $\mathbb{C}^g$, compare Lemma A.13. However, these are all translations via elements in Λ with $\mathrm{Jac}(X) \simeq \mathbb{C}^g/\Lambda$. So there is only one Prym variety embedded into $\mathrm{Jac}(X)$.

Lemma 6.3. *Let X be a compact Riemann surface endowed with a holomorphic involution σ and let $Q^\pm \in X$ be exactly the fixed points of σ. Then the set of all positive divisors D of degree g on X which obey $D + \sigma(D) \simeq K + Q^+ + Q^-$ and $\dim H^1(X, \mathcal{O}_{D-Q^\pm} \otimes \mathcal{L}_h(x,y)) = 0$ for all $(x,y) \in \mathbb{R}^2$, where $\mathcal{L}_h(x,y)$ is defined in under* (5.3)*, is open in the set of all positive divisors obeying $D + \sigma(D) \simeq K + Q^+ + Q^-$.*

Proof. Let D be a divisor as given in the theorem and let P be the set of divisors $\tilde{D}$ of degree 0 such that $\tilde{D} + \sigma(\tilde{D}) \simeq 0$. The proof of the above theorem is done in two steps. For the first step, let

$$M_1 := \left\{\tilde{D} \in P \mid \dim H^1(X, \mathcal{O}_{D+\tilde{D}-Q^\pm}) = 0\right\}.$$

Due to [Farkas and Kra, 2012, Proposition III.6.5], the image of the set of divisors such that $\dim H^1(X, \mathcal{O}_{D-Q^\pm}) \neq 0$ under the Abel map is a subvariety in the Jacobian variety of X and its complement is open and dense in $\mathrm{Jac}(X)$. So also the restriction of this open set to the Prym variety is open. Since the Abel map is continuous and maps P into $\mathrm{Prym}(X,\sigma)$, this yields that M_1 is an open set in P. Proposition 4.31 implies that for each $(x,y) \in \mathbb{R}^2$, it is $\dim H^1(X, \mathcal{O}_D \otimes \mathcal{L}_h(x,y)) = \dim H^1(X, \mathcal{O}_D)$. So for every fixed $(x,y) \in \mathbb{R}^2$, also $\{\tilde{D} \in M_1 \mid \dim H^1(X, \mathcal{O}_{D+\tilde{D}-Q^\pm} \otimes \mathcal{L}_h(x,y)) = 0\}$ is an open set in P. In the second step we deduce from the first step that also

$$M_2 := \left\{\tilde{D} \in P \mid \dim H^1(X, \mathcal{O}_{D+\tilde{D}-Q^\pm} \otimes \mathcal{L}_h(x,y)) = 0 \text{ for all } (x,y) \in \mathbb{R}^2\right\}$$

is open. To do so, we define the map

$$F : P \times \mathbb{R}^2/\Gamma \to P, \quad D \times (x,y) \mapsto D(x,y),$$

where $D(x,y)$ is the divisor D such that $\mathcal{O}_{D(x,y)}$ is isomorphic to $\mathcal{O}_D \otimes \mathcal{L}_h(x,y)$. Let $D \in M_2$. We show that also a small open neighborhood of D is contained in M_2. In [Klein et al., 2016, Lemma 8.5], it is shown that the complexification of $\mathcal{L}_h(x,y)$ defines a sheaf on $X \times \mathbb{C}^2$ which is flat with respect to the map $X \times \mathbb{C}^2 \to \mathbb{C}^2$. Therefore, F is continuous and we have seen in the first step that M_1 is open. So for every $(x,y) \in \mathbb{R}^2/\Gamma$, there exists an open neighborhood $U \in \mathbb{R}^2/\Gamma$ of (x,y) and an open neighborhood $V \subset P$ of D such that $F[V \times U] \subset M_1$. Because $\mathbb{R}^2/\Gamma$ is compact, we find finitely many $(x,y) \in \mathbb{R}^2/\Gamma$ such that the union of the corresponding open neighborhoods U yields a finite covering $\mathcal{U}$ of $\mathbb{R}^2/\Gamma$. For each of these finitely many U, there exists an open neighborhood $V \subset P$ of D such that $F[V \times U] \subset M_1$. Intersecting all these finitely many open sets V with each other yields an open set $\tilde{V}$ which contains D. So M_2 is open. □

6.2. Real potentials

The aim of this chapter is to define the real parts of the Prym variety of a Riemann surface X with two marked points $Q^{\pm}$, holomorphic involution σ and antiholomorphic involution τ such that $\tau \circ \sigma = \sigma \circ \tau$, $\sigma(Q^{\pm}) = Q^{\pm}$ and $\tau(Q^{\pm}) = Q^{\mp}$. The isospectral set of a regular finite type potential u is parametrized by the real Prym variety. We also want to describe the connected components of this real Prym variety and analyze whether there are connected components which obtain only non-singular divisors in the sense of Definition 4.25. Due to Lemma 4.27, these are just the positive divisors of degree g with support away from Q^+ and Q^-.
The whole section and Appendix B are elaborations of parts of [Natanzon, 2004, Chapter 1 and 2]. We decided not just to cite the results given there, but to work them out completely because it turned out that in [Natanzon, 2004], many proofs either contain mistakes or are not fully worked out so that we are not able to understand the argumentations given there, even though the final results are – up to small modifications – usually correct.
In order to define the real Prym variety, we consider real curves. These are pairs of a compact Riemann surfaces X and an antiholomorphic involution τ on X. We will see that the fixed point set of τ on X consists of simple closed contours which are called ovals in the sequel. To introduce the real Prym variety on a real curve, a certain 'toolbox', taken from [Natanzon, 2004, Chapters 1 and 2], is necessary. This can be found in Appendix B. This toolbox contains a short introduction to so-called real Arf functions and spinor bundles on real curves. The Arf functions used here are functions from $H_1(X, \mathbb{Z}_2)$ to $\mathbb{Z}_2$ which assign a sign to the ovals of a real curve (X, τ). Real spinor bundles are bundles which square is the cotangent bundle and real spinors are sections of this bundle. They are used here to show the existence of holomorphic differential forms with certain properties on the ovals of a real curve. In Appendix B, a 1-to-1 correspondence between these two objects is established with help of the real Fuchsian groups. These are Fuchsian groups with an additional antiholomorphic structure. This connection can be used to show the existence of spinors which induce a certain orientation on the ovals. A square of such a real spinor then is a holomorphic real differential on the real curve (X, τ), i.e. a 1-form with a certain behavior under the pullback of τ to the space of 1-forms on X.
The first task of this chapter is to take a a closer look at real curves (X, τ) and how they are being constructed, whereby we introduce the topological type of a real curve min order to classify the different kinds of real curves which can occur. After that, we take a closer look at the real differential forms. The existence and non-existence theorems shown there will afterwards be very useful to describe the real part of the Prym variety. We define the realness-condition on these 1-forms with different sign than it can be found in [Natanzon, 2004]. This is done because the way we define it is later on also used in [Natanzon, 2004] to consider the real parts of the Jacobian variety. Moreover, the two last proofs in the section about real M-curves – which are compact Riemann surfaces of genus g with the maximal number of $g+1$ ovals – are not at all understandable

for us in [Natanzon, 2004] and large parts of the necessary argumentation is missing in our eyes. A request to the author about some detail there remained unanswered. So we try to fill in these holes here. After that, we define the real part of the Jacobian variety. Also here, some mistakes are made in the corresponding section in [Natanzon, 2004]. For example the action of the involution on a so-called real basis of $H_1(X, \mathbb{Z})$ is wrong. Also large parts to show the precise form of the real Jacobian variety are missing and what can be found in [Natanzon, 2004] is more the sketch of the proof structure. So we try to complete the missing steps here, where we take the changes due to the different behavior of the real basis of $H_1(X, \mathbb{Z})$ into account. Finally, we take these results to present the real part of the Prym variety. Also in this part, we made many changes compared to the results which can be found in [Natanzon, 2004]. We give proofs of statements which are made but not shown in [Natanzon, 2004], as for example on the existence of a certain two-sheeted covering which is necessary to construct real curves (X, τ) which are equipped with a holomorphic involution. We define clearly how a symplectic basis of a real curve with holomorphic involution should look like and show that such a basis always exists. Finally, in [Natanzon, 2004] a definition for positive respectively negative definiteness of certain types of meromorphic differentials on the ovals of a real curve with involution is made which in our eyes does not make sense. Therefore, we adjust the definition of this definiteness and argue why we think that our definition of this definiteness comprises what is actually meant in [Natanzon, 2004]. We will see that with this new definition, the remaining results from [Natanzon, 2004] on the real Prym variety carry over. Finally, we will be able to show some assertions on the structure of the Prym variety of a real curve with holomorphic involution as well as on the possibility of the existence of connected components in this Prym variety which contain only non-special divisors.

6.2.1. The topological type of real curves

At first, we take a closer look at the so-called real curves, i.e. compact Riemann surfaces X with an antiholomorphic involution τ. These are defined as follows:

Definition 6.4. A *real curve* is a pair $X = (X, \tau)$, where X is a compact Riemann surface of genus g and $\tau : X \to X$ is an antiholomorphic involution. The set of fixed points $X^\tau \subset X$ of this involution is the *set of real points* or the *real part* of the curve X.

We identify two real curves $X_1 = (X_1, \tau_1)$ and $X_2 = (X_2, \tau_2)$ if there is a biholomorphic map $\psi : X_1 \to X_2$ such that $\psi \circ \tau_1 = \tau_2 \circ \psi$. We call curves such that this holds biholomorphically equivalent. We will not distinguish between curves which are biholomorphically equivalent.

Definition 6.5. Two real curves (X_1, τ_1) and (X_2, τ_2) are said to be *topologically equivalent* if there is a homeomorphism $\phi : X_1 \to X_2$ such that $\tau_2 \circ \phi = \phi \circ \tau_1$.

To characterize these curves, the following datum which can be found in [Natanzon, 2004, Section 2.1] is necessary:

Definition 6.6. A real curve X is said to be *separating* if $X \setminus X^\tau$ is not connected. Otherwise X is said to be *non-separating.* We define

$$\varepsilon = \varepsilon(X) = \begin{cases} 0 & \text{if the curve } X \text{ is non-seperating,} \\ 1 & \text{if the curve } X \text{ is seperating.} \end{cases}$$

The *topological type* of X is given by the triple (g, k, ε), where g is the genus of X and $k = k(X)$ is the number of connected components of X^τ.

It will turn out that the data (g, k, ε) is sufficient to describe a real curve up to topological equivalence. Closed paths on X without self-intersections we call simple closed contours in the sequel and smooth simple closed contours denote simple closed contours which are isomorphic to S^1.

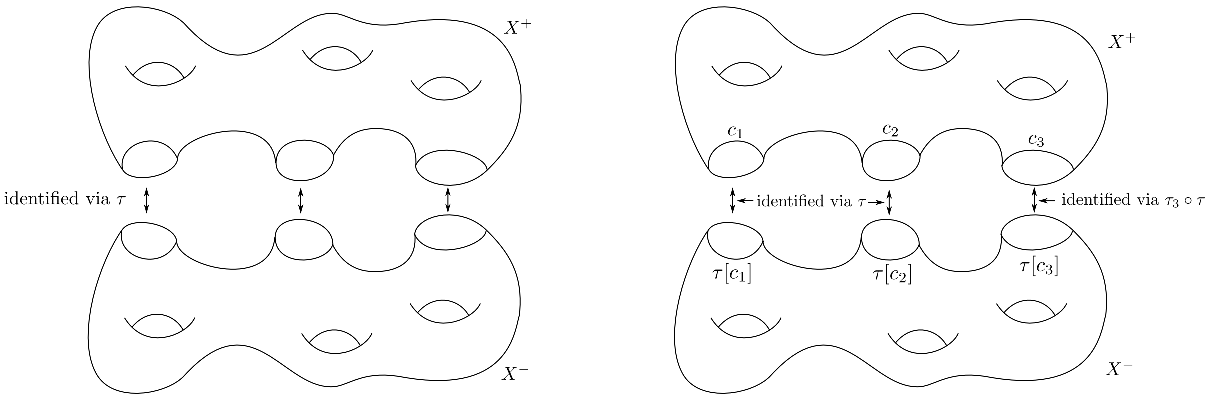
The Uniformization Theorem for simply connected Riemann surfaces, compare e.g. [Forster, 1981, Theorem 27.9], yields that every compact simply connected Riemann surface X with genus $g \geq 2$ is biholomorphically equivalent to a surface of the form $\mathcal{H}/\Lambda$, where $\mathcal{H} := \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$ is the upper half plane and Λ is a discrete group that acts without fixed points on $\mathcal{H}$. The standard metric of constant curvature on $\mathcal{H}$, compare e.g. [Dieck, 2000, Section 1.6], induces a metric of constant curvature on $X = \mathcal{H}/\Lambda$ by the quotient metric. All real curves can be constructed as in the two following examples of real curves. Hereby, we explain more detailed than it can be found in [Natanzon, 2004, Example 2.1.1] how these curves are constructed and cite [Natanzon, 2004, Example 2.1.2]. However, both examples are essential for the further understanding. To keep this work self-contained, we represent both.

Example 6.7. [Natanzon, 2004, Example 2.1.1] Let X^+ be a compact Riemann surface with boundary of genus g with k connected boundary components which are each a one-dimensional compact manifold, and therefore are isomorphic to S^1. We call them *boundary cycles.* Let X^+ be equipped with the structure of a Riemann surface given by the atlas of holomorphic charts

$$\{(U_i, z_i)\}, \; X^+ = \bigcup U_i, \; z_i : U_i \to \mathcal{H},$$

where $\bigcup U_i$ is an open covering of X^+. This atlas is chosen in such a way that the boundary components of X^+ are mapped to $\mathbb{R}$. Moreover, let X^- be the Riemann surface with boundary which is obtained by considering the local charts $\bar{z}_i : U_i \to \iota\mathcal{H}$. Then there exists a unique antiholomorphic map $\tau : X^+ \to X^-$ such that $\tau^* \bar{z}_i = z_i$. Moreover, $X^+ \dot\cup X^-$ is a topological space. From this we want to define a real curve $X_{\tilde{g},k}$ as the surface $X := X^+ \dot\cup X^- / \sim$, where $\sim$ means that we identify the boundary components of X^+ and X^- via τ as indicated in Figure 6.1a. To do so, we use the quotient topology induced by the map $X^+ \dot\cup X^- \to X^+ \dot\cup X^- / \sim$. Let $U_i \subset X^+$ be open. Then $\tau[U_i \cap \partial X^+] = \tau[U_i] \cap \partial X^-$, where $\tau[U_i] \subset X^-$ is also open. The open sets on X are given by $(U_i \dot\cup \tau[U_i]) / \sim \to \mathbb{C}$, where we use the local charts $z_i : U_i \to \mathcal{H}$ and

$\bar{z}_i : \tau[U_i] \to \iota\mathcal{H}$. This map is a homeomorphism and yields an atlas on X such that X is a compact Riemann surface without boundary. The identified boundaries $\partial X^{\pm}$ are fixed points of τ on X. This set of fixed points we denote by X^τ. Due to the selections of the charts at the beginning of this example, it is $z_i(p) \in \mathbb{R}$ for $p \in X^\tau$, i.e. $z_i(p) = \bar{z}_i(p)$. Because τ is an antiholomorphic involution on X, the constructed Riemann surface (X, τ) is a real curve of type $(2g + k - 1, k, 1)$. The complex structure of the Riemann surface of X induces a unique metric of constant sectional curvature on X. The lift of τ to the universal covering $\mathcal{H}$ of X is an antiholomorphic Möbius transformation. Because the metric on $\mathcal{H}$ is invariant under this metric, also τ is an isometry with respect to the metric on X.



(a) Tinkering a real curve of topological type $(8, 3, 1)$

(b) Tinkering a real curve of topological type $(8, 3, 0)$, i.e. and $m = 2$

Figure 6.1.: Examples of the constructions of real curves for $g = k = 3$ and (a) $\varepsilon = 1$, (b) $\varepsilon = 0$.

Example 6.8. [Natanzon, 2004, Example 2.1.2] Repeating the construction of Example 6.7, we take the Riemann surfaces with boundary X^+ and X^- and the antiholomorphic map $\tau : X^+ \to X^-$. The boundary ∂X^+ consists of simple closed contours $c_1, \ldots, c_k$. Let $0 \leq m < k$. For $i \leq m$, we identify the contours c_i and $\tau[c_i]$ by means of the map τ as in Example 6.7. For $i > m$, let us consider fixed-point-free isometries $\tau_i : c_i \to c_i$ such that $\tau_i^2 = 1$. In this case, we identify the simple closed contours c_i and $\tau[c_i]$ by means of the map $\tau \circ \tau_i$ as indicated in Figure 6.1b. For these simple closed contours, local coordinates can be found similarly as in Example 6.7 by considering $\tau_i^* \bar{z}$ instead of $\bar{z}$ as a local coordinate on X^-. This yields simple closed contours on the tinkered curve which are invariant under τ but no ovals. We again obtain a real curve $X = (X, \tau)$ of genus $2\tilde{g} + k - 1$, but in this case $X^\tau = \bigcup_{i=1}^m c_i$. So X is a curve of topological type $(2g + k - 1, m, 0)$.

We show next that any real curve is topologically equivalent to one of the curves in Examples 6.7 and 6.8. Therefore, the following Lemma [Natanzon, 2004, Lemma 2.1.1] is necessary. The proof in [Natanzon, 2004] is complete and we only give it here to for self-containedness because we will frequently use its result in the sequel.

Lemma 6.9 ([Natanzon, 2004, Lemma 2.1.1]). *The set X^τ of real points of a real curve $X = (X, \tau)$ decomposes into pairwise disjoint smooth simple closed contours which are all geodesics on X.*

Proof. As already mentioned, the complex structure of the surface X induces a metric of constant curvature and τ is an isometry with respect to this metric. If $p \in X^\tau$, then the involution $d\tau_p : T_pX \to T_pX$ of the tangent plane T_pX is the reflection with respect to a direction $v \in T_pX$. We denote by $\ell \subset X$ the geodesic [Jost, 2006, Definition 1.4.2] that passes through p in the direction of the line v. All its points are fixed under τ and in a small open neighborhood of p, there are no other fixed points of τ. Thus, by [Jost, 2006, Corollary 1.4.2], each of the points $p \in X^\tau$ belongs to exactly one maximal geodesic $\ell \subset X^\tau$, i.e. a geodesic with maximal domain, without self-intersections. Since X is compact, it follows that each of these geodesics is a smooth simple closed contour. □

Definition 6.10. The pairwise disjoint smooth simple closed contours in X^τ are called *ovals*.

The proof of the next Theorem [Natanzon, 2004, Theorem 2.1.1] is given here completely since in [Natanzon, 2004], the existence of the desired homeomorphism as well as the topological equivalence is not shown explicitly.

Theorem 6.11 ([Natanzon, 2004, Theorem 2.1.1.]). *Let (X, τ) be a real curve of type $(g, k, 1)$. Then $1 \leq k \leq g+1$, $k \equiv g+1 \mod 2$ and (X, τ) is topologically equivalent to the curve $(X_{\tilde{g},k}, \tau_{\tilde{g},k})$ of Example 6.7, where $\tilde{g} = \frac{1}{2}(g + 1 - k)$.*

Proof. By Lemma 6.9, the set $X \setminus X^\tau$ decomposes into two surfaces X^+ and X^- of genus $\tilde{g}$ with k boundary cycles such that $\tau[X^+] = X^-$. Hence, $g = 2\tilde{g}+k-1$, and therefore $k \leq g+1$ with equality if and only if $\tilde{g} = 0$ and $k = g + 1$. Let us consider a homeomorphism $\phi_+ : (X^+ \cup X^\tau) \to \tilde{X}^+$ with $\phi_+|_{X^\tau} : X^\tau \to \partial\tilde{X}^+$, where $\tilde{X}^+$ is defined as in Example 6.7. This homeomorphism exists because the genus and the number of connected components of the boundary of X^+ and $\tilde{X}^+$ are the same. We set

$$\phi : X \to X_{\tilde{g},k}, \quad p \mapsto \begin{cases} \phi_+(p) & \text{for } p \in X^+ \cup X^\tau, \\ \tau_{\tilde{g},k} \circ \phi_+(p) & \text{for } p \in X^-. \end{cases}$$

Then one has for $p \in X^+$ that $\tau(p) \in X^-$, so

$$(\phi \circ \tau)(p) = (\tau_{\tilde{g},k} \circ \phi_+ \circ \tau^2)(p) = (\tau_{\tilde{g},k} \circ \phi_+)(p) = (\tau_{\tilde{g},k} \circ \phi)(p)$$

and for $p \in X^-$, it is $\tau(p) \in X^+$, and therefore

$$(\phi \circ \tau)(p) = (\phi_+ \circ \tau)(p) = (\tau_{\tilde{g},k}^2 \circ \phi_+ \circ \tau)(p) = (\tau_{\tilde{g},k} \circ \phi)(p).$$

The points $p \in X^\tau$ are mapped to $\partial\tilde{X}^+$, which yields the ovals of $X_{\tilde{g},k}$. So ϕ realizes the desired topological equivalence between X and $X_{\tilde{g},k}$. □

Now, we want to give the same characterization by its topological type for non-separating curves. Until the end of this characterization, let Y be a compact Riemann surface of genus g with an even number of n boundary cycles and let $\tau : Y \to Y$ be an antiholomorphic involution without fixed points such that we can sort the n boundary cycles into pairs which are interchanged by τ. For a real curve of type $(g, k, 0)$, we can interpret Y as $X \setminus X^\tau$. In [Natanzon, 2004], it is neither assumed that Y has to be compact nor that the number of boundary components has to be even and that these boundary components have to obey the above symmetry with respect to τ. However, we think that these assumptions are necessary to show the next Lemma. Both of them are also no obstructions in the sequel since we will apply the next Lemma only to curves with these properties. To formulate the next Lemma, we additionally the following definition is necessary.

Definition 6.12. (a) A simple closed contour $\gamma \subset Y$ is called *invariant* if $\tau[\gamma] = \gamma$.

(b) A system $A = (\gamma_1, \ldots, \gamma_m)$ of pairwise disjoint, invariant simple closed contours is said to be *complete* if the set $Y \setminus A$ is disconnected.

If A is complete, then $Y \setminus A$ consists of two surfaces Y^+ and Y^- which are both of genus $\frac{1}{2}(g-m+1)$ with each $m + \frac{n}{2}$ boundary cycles and $\tau[Y^+] = Y^-$. The proof of the following Lemma extends the statements compared to what can be found in [Natanzon, 2004] in such a way that we show besides the existence of a complete system of simple closed contours on Y that these are smooth and contained in the interior of Y. Moreover, we also extend the arguments for the proof of the existence in comparison to [Natanzon, 2004].

Lemma 6.13 ([Natanzon, 2004, Lemma 2.1.2])**.** *There is a complete system formed by $g + 1$ invariant smooth simple closed contours which are contained in Y°.*

Proof. We show this assertion in three steps. At first we show that there always exists at least one invariant contour $\gamma \subset Y$ such that $\gamma \cap \partial Y = \emptyset$. We then show that for $g > 0$, there is an invariant contour $\tilde{\gamma} \subset Y$ such that $Y \setminus \tilde{\gamma}$ is connected. These two facts can then be used to prove the assertion in the last step.
To see that there always exists at least one invariant contour γ, let d be the standard metric of constant curvature on Y. Since Y is a compact Riemann surface with boundary, the continuous function $f : Y \to \mathbb{R},\ p \mapsto d(p, \tau(p))$ attains its minimum on Y at p_0. Then there exists a path ℓ of shortest length which connects p_0 and $\tau(p_0)$. because τ has no fixed points on Y, there holds $\tau[\ell] \neq \ell$. Assume that $\ell \cap \tau[\ell] \neq \emptyset$. Then there exist at least two points p_1 and $\tau(p_1)$ which are contained in ℓ as well as in $\tau_\sharp \ell$ since τ has no fixed points on Y. This yields that the geodesics ℓ and $\tau_\sharp \ell$ connect p_1 and $\tau(p_1)$ with $f(p_1) < f(p_0)$ which contradicts the assumption that p_0 is the minimum of f. Therefore, the geodesics ℓ and $\tau[\ell]$ do not intersect each other. By definition, $\gamma := \ell + \tau_\sharp \ell$ is a closed contour which is invariant under τ. Next, we want to gain insight why γ is smooth. Let $v_\ell(p)$ be the direction of the geodesic at $p \in \ell$ which is contained in $T_p X$ and

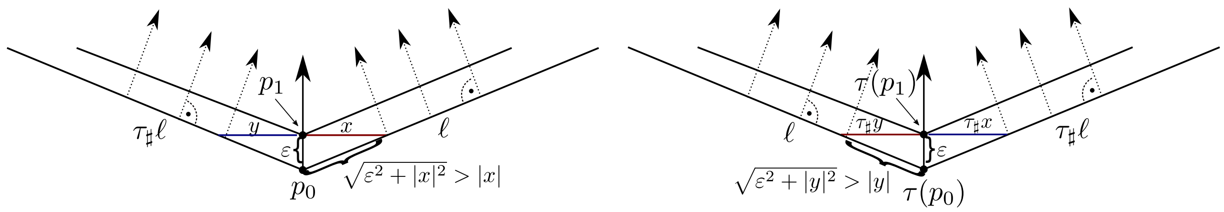


Figure 6.2.: Sketch why the geodesics ℓ and $\tau_\sharp \ell$ cannot meet at p_0 with an angle smaller than π, where the geodesics are linearized for clarity of the sketch.

let $v_{\tau_\sharp \ell}(p)$ be the direction of the geodesic at $\tau(p) \in \tau_\sharp \ell$ contained in $T_{\tau(p)}X$. Then the angle α between $v_\ell(p_0)$ and $v_{\tau_\sharp \ell}(p_0)$ is the same as the angle between $v_\ell(\tau(p_0))$ and $v_{\tau_\sharp \ell}(\tau(p_0))$ since τ is a conformal map. Moreover, $\alpha \in [0, \pi]$. Assume that $\alpha < \pi$ as indicated in Figure 6.2. Then there exists a vector field contained in TX which is attached orthogonally to ℓ and $\tau_\sharp \ell$. On the vector v of this vector field attached at p_0, one can find a point p_1 on v with $d(p_0, p_1) < \varepsilon$ for $\varepsilon > 0$ sufficiently small. The point $\tau(p_1)$ then lies on the corresponding vector $\tilde{v}$ attached at $\tau(p_0)$ and obeys $d(\tau(p_0), \tau(p_1)) = d(p_0, p_1)$ since τ is an isometry. Then there is a small path x, orthogonally attached to v, which connects p_1 and ℓ and a small path y which connects p_1 with a point on $\tau_\sharp \ell$. Likewise $\tau_\sharp y$ connects $\tau(p_1)$ with ℓ and $\tau_\sharp x$ connects $\tau(p_1)$ with $\tau_\sharp \ell$. By the triangle inequality, the length of x is shorter than the length of the segment of ℓ which lies between p_0 and the intersection point of p_0 and x and also the length of y is smaller then the segment of $\tau_\sharp \ell$ which lies between the $\tau(p_0)$ and the intersection point of y with $\tau_\sharp \ell$. So the length of ℓ connecting p_0 and $\tau(p_0)$ is larger than the distance between p_1 and $\tau(p_1)$ which are connected via x, the part of ℓ which lies between the intersection points of x with ℓ and y with ℓ and then via y. Thus, $\alpha < \pi$ contradicts the assumption that p_0 minimizes f, and so $\alpha = \pi$ which implies that $\gamma = \ell \cup \sigma_\sharp \ell$ is a smooth simple closed contour.

If $\partial Y = \emptyset$ this shows the assertion of the first step. So assume that ∂Y contains at least two boundary cycles. We claim that this simple closed contour does not intersect the boundary components of Y. So assume that $p_0, \tau(p_0) \in \partial Y$ such that f attains its minimum at p_0. By the construction of Y, the points p_0 and $\tau(p_0)$ must lay on different boundary cycles which are interchanged by τ. Let again ℓ be the unique geodesic connecting p_0 and $\tau(p_0)$. As before, the angle α between $v_\ell(p_0)$ and $v_{\tau_\sharp \ell}(p_0)$ must lie between 0 and π, whereby $\alpha \in (0, \pi)$ is not possible by the same argumentation as above. Assume that $\alpha = \pi$. Since the boundary cycles are simple closed contours and geodesics, this would imply that ℓ is contained in the boundary contour which contains p_0. This contradicts the fact that p_0 and $\tau(p_0)$ lie on different boundary cycles. So $\pi = 0$ and we can find a point p_1 contained in ℓ with $d(p_0, p_1) < \varepsilon$ such that $\tau(p_1) \in \tau_\sharp \ell$. Let $\tilde{\ell}$ be the geodesic connecting p_1 and $\tau(p_1)$. Then $\tilde{\ell} \subset \ell$ and the length of $\tilde{\ell}$ is 2ε shorter than the length of ℓ. This contradicts again the assumption that p_0 minimizes f, and so $\gamma \cap \partial Y = \emptyset$.

To show the second step, let $\gamma \subset Y$ be the contour constructed in step one. For $Y \setminus \gamma$ connected

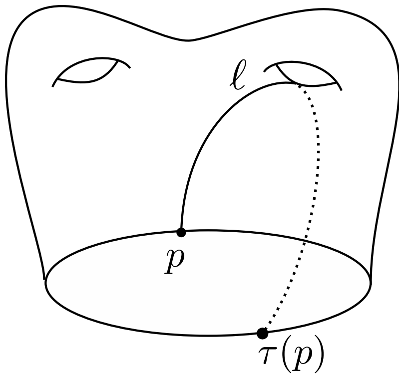


Figure 6.3.: Origin of ℓ depicted as in [Natanzon, 2004, Figure 2.1.1].

there is nothing to show. So let $Y \setminus \gamma$ be disconnected. Then $Y \setminus \gamma = Y^+ \cup Y^-$, where Y^+ and Y^- are surfaces of positive genus and $\tau[Y^+] = Y^-$. By construction, γ is invariant under τ but not an oval of τ. So for $p \in \gamma$, one has $p \neq \tau(p) \in \gamma$. Let us join points p and $\tau(p)$ by a curve $\ell \subset Y^+$ without self-intersections and such that $Y^+ \setminus \ell$ is connected. This is depicted – as in [Natanzon, 2004, Figure 2.1.1] – in Figure 6.3. Then $\tilde{\gamma} = \ell + \tau[\ell]$ is an invariant contour and $Y \setminus \tilde{\gamma}$ is connected. Furthermore, we can choose this curve in such a way that it intersects the boundary cycle orthogonally. So $\tilde{\gamma}$ is a smooth simple closed curve. To see that there always exists a complete system A consisting of $g+1$ invariant smooth simple closed contours, let $\tilde{\gamma}_1$ be the smooth simple closed contour constructed in the second step. The surface $Y \setminus \tilde{\gamma}_1$ is of genus $g-1$. If $g-1 > 0$, one can again apply the assertion from the second step and continue like this successively until the genus of $Y \setminus \tilde{\gamma}_1, \ldots, \tilde{\gamma}_k$ equals zero. Then one applies the first step which yields the assertion. □

The proof of the following Theorem is not given. The only difference to the proof of Theorem 6.11 is that, in addition to X^τ, a complete system A of smooth simple closed contours of $X \setminus X^\tau$ is considered such that $X \setminus (X^\tau \cup A)$ decomposes into two surfaces X^+ and X^-. Also the homeomorphism between an arbitrary real curve of type $(g, k, 0)$ and the corresponding model curve in Example 6.8 is defined analogously as in the proof of Theorem 6.11. In [Natanzon, 2004, proof of Theorem 2.1.2], it is not shown that this yields the desired topological equivalence, but the calculations are the same as the calculations used to show the topological equivalence in the proof of Theorem 6.11 for a real curve of type (g, k, ε) and the corresponding model curve from Example 6.7.

Theorem 6.14 ([Natanzon, 2004, Theorem 2.1.2]). *Let (X, τ) be a real curve of topological type $(g, m, 0)$. Then for any $m < k \leq g+1$ with $k \equiv g+1 \mod 2$, the curve (X, τ) is topologically equivalent to the curve $(X^m_{\tilde{g},k}, \tau^m_{\tilde{g},k})$ in Example 6.8, where $\tilde{g} = \frac{1}{2}(g+1-k)$.*

The results of Examples 6.7 and 6.8 as well as of Theorems 6.11 and 6.14 are summarized in the following corollary.

Corollary 6.15 ([Natanzon, 2004, Corollary 2.1.1])**.** *Real algebraic curves are topologically equivalent if and only if they have the same topological type. A set* (g, k, ε) *is a topological type of a real algebraic curve if and only if either* $\varepsilon = 1$*,* $1 \leq k \leq g+1$ *and* $k \equiv g+1 \mod 2$ *or* $\varepsilon = 0$ *and* $0 \leq k \leq g$*.*

6.2.2. Holomorphic differentials on real curves

From now on, we assume that the ovals of a real curve (X, τ) of type (g, k, ε) are endowed with an orientation. For $\varepsilon = 1$, this orientation is induced by an orientation of one of the connected components of the set $X \setminus X^\tau$ and for $\varepsilon = 0$, it is the orientation induced by $X \setminus X^\tau$. This we call the *original orientation of an oval.*

Definition 6.16. A local chart $z : U \to \mathbb{C}$ in a neighborhood of a real point $p_0 \in X^\tau$ is a *real local chart* if $\tau[U] = U$ and $z(\tau(p)) = \overline{z(p)}$. A real chart *agrees with the orientation* of the set X^τ if z sends an oriented segment $\ell = U \cap X^\tau$ into the segment $z[\ell] \subset \mathbb{R}$ oriented by increasing order of the real numbers.

In deviation from [Natanzon, 2004], let $\tau^* : \Omega(X) \to \Omega(X)$ be the pullback of τ to the space of 1-forms on X. The map τ^* in [Natanzon, 2004] is defined slightly different, but we believe that it shall also just denote the pullback of τ there since it is applied like that in [Natanzon, 2004]. With this pullback, we are able to define a real differential, whereby we made the opposite sign choice in the action of τ^* on ω as in [Natanzon, 2004]. The choice in [Natanzon, 2004] seems to be more natural on the first sight. However, our choice turns out to be more convenient in the sequel of this chapter and is also the one used in [Natanzon, 2004]. It is the involution corresponding to $\tau : X \to X$, $p \mapsto -\bar{p}$ as it will be used hereinafter when we consider real curves with an additional involution σ in Section 6.2.5.

Definition 6.17. (a) A differential ω is said to be a *real differential* if $\tau^*\omega = -\bar{\omega}$. In a real chart, this yields for $\omega = f(z)dz$ that $f(\bar{z}) = -\overline{f(z)}$. In particular, $f(z(p)) \in \iota\mathbb{R}$ for $p \in X^\tau$.

(b) The sign of the value $-\iota f(z(p)) \in \mathbb{R}$ with $p \in X^\tau$ is the same for all real charts that agree with the orientation of the set X^τ. It is called the *sign of the differential* ω *at a point* $p \in X^\tau$.

(c) A real differential ω is *positive* (*non-negative, non-positive, negative*, respectively) *on an oval* $c \subset X^\tau$ if $-\iota\omega$ is positive (non-negative, non-positive, negative, respectively) at any point of c.

In this definition, we have changed the condition for realness of a holomorphic differential in such a way that $\tau^*\omega = -\bar{\omega}$. In the next proof we will frequently use that a spinor can induce an orientation on the ovals of a real curve (X, τ), compare Definition B.38. We will also make use of the statements and the notation used in Theorems B.42 and B.43.

Lemma 6.18 ([Natanzon, 2004, Lemma 2.6.1]). *Let η be a real spinor on the curve (X, τ) as in Definition B.37. Then $\omega = -\iota\eta^2$ is a real differential that is non-negative on the oval $c \subset X^\tau$ if the orientation generated by η coincides original one and non-positive on the oval c if the orientation generated by η is opposite to the original one.*

Proof. If the spinor η is described by a function $f \circ z$ in a local real chart $z : U \to \mathbb{C}$ which agrees with the orientation of an oval $c \subset X^\tau$. Let z be centered at $p \in c$. Then there is an open neighborhood of $p \in c$ on which $\omega = -\iota(f^2 \circ z)dz$. If additionally the orientation of the oval c is generated by η, then it follows from Lemma B.40 that $f \circ z \circ \tau = \overline{f \circ z}$ and f^2 is non-negative on c, so also $\iota(-\iota f^2)$ is non-negative. A change of the orientation of the oval changes the sign of f^2. □

Theorem 6.19 ([Natanzon, 2004, Theorem 2.6.1]). *Let (X, τ) be a real curve of type (g, k, ε) with ovals $c_1, \dots, c_k$, where $k = k_+ + k_- + k_0$. For $\varepsilon = 0$, let additionally $k_0 < g$ and for $\varepsilon = 1$, let additionally $k_+ \cdot k_- \neq 0$. Then there is a real differential on (X, τ) that is non-negative on c_i for $i \leq k_+$, non-positive on c_i for $k_+ < i \leq k_+ + k_-$ and has zeros on c_i for $i > k_+ + k_-$ such that the sum of the orders of the zeros restricted to each one of these ovals divided by two is odd.*

In [Natanzon, 2004], no obstruction for the sum of the orders of zeros is shown. However, this is necessary hereinafter. So we added it in the next theorem.

Proof. We use Theorems B.42 and B.43 to deduce that there exists a real spinor η which has zeros on $c_{k_+ + k_- + 1}, \dots, c_k$ and generates on any other oval c_i an orientation that coincides with that of X^τ for $i \leq k_+$ and is opposite to the orientation of X^τ for $k_+ < i \leq k_+ + k_-$. To ensure that the ovals $c_{k_+ + k_- + i}$ with $i = 1, \dots, k_0$ have a zero such that the sum of the orders of the zeros on this oval divided by two is odd, the corresponding values α_i in these Theorems must equal 1. Then the sum of the orders of the zeros of η on one such oval is odd. Since $\omega = -\iota\eta^2$, the assertion follows. Let us consider $\varepsilon = 0$. We can always obtain $\sum_{i=1}^{k} \alpha_i = g + 1 \mod 2$ since $k_0 < g$. So either k_+ or k_- is unequal to zero and the ovals c_i with $1 \leq i \leq k_+ + k_-$ might also contain elements with $\alpha_i = 1$. Choosing $k_+ \leq m \leq k_+ + k_0$ in Theorem B.42 yields the desired spinor η.
For $\varepsilon = 1$, the assumptions in Theorem B.43 also hold since $k_+ \cdot k_- \neq 0$ implies that there always exist two ovals of opposite orientation, without loss of generality c_1 and c_k, on which we can set $\alpha_1 = \alpha_k = 0$. As for $\varepsilon = 0$, the other conditions on $\alpha_1, \dots, \alpha_k$ can always be fulfilled since the maximal number of ovals of X is $g + 1$ and $k_+, k_- \geq 1$, $k_0 < g$. Choosing $k_+ \leq m \leq k_+ + k_0$ in Theorem B.43 yields the desired spinor η. Applying Lemma 6.18, the differential $\omega = -\iota\eta^2$ has the desired properties. □

6.2.3. Real M-curves

Now we take a closer look at real curves which have the maximal possible number of ovals. Obviously, such a curve is of separating type, and so $\varepsilon = 1$.

Definition 6.20. A *real M-curve* is a real curve of type $(g, g+1, 1)$.

On such curves, the real differentials have some additional properties which we will study now in more detail.

Lemma 6.21 ([Natanzon, 2004, Lemma 2.6.1])**.** *Let $c_1, \dots, c_{g+1}$ be ovals of an M-curve of genus g and let $1 \le \alpha \le n < \beta \le g+1$. Then there is a real differential ω that is positive on c_α, non-negative on $c_1, \dots, c_n$, negative on c_β and non-positive on $c_{n+1}, \dots, c_{g+1}$.*

Proof. By Theorem B.43, there is a real spinor η that generates the orientation of X^+ on $c_1, \dots, c_n$, generates the opposite orientation of X^+ on $c_{n+1}, \dots, c_{g+1}$ and has zeros on the ovals c_i with $i \neq \alpha, \beta$. It is shown in [Atiyah, 1971, Lemma 3.2] that the total number of zeros of a spinor is $g-1$. Hence, η has no zeros on c_α and c_β. Therefore, by Lemma 6.18, the real differential $\omega = -\iota\eta^2$ satisfies all hypotheses of the lemma. □

This immediately yields the following assertion.

Lemma 6.22 ([Natanzon, 2004, Lemma 2.6.2])**.** *Let $c_1, \dots, c_{g+1}$ be the ovals of an M-curve (X, τ) of genus g and let $1 \le n < g+1$. Then there is a real differential ω that is positive on $c_1, \dots, c_n$ and negative on $c_{n+1}, \dots, c_{g+1}$.*

Proof. Let $1 \le n < g+1$. Then by Lemma 6.21, there exists for each (i,j) with $i \in \{1, \dots, n\}$ and $j \in \{n+1, \dots, g+1\}$ a real differentials $\omega_{i,j}$ on X which is non-negative on $c_1, \dots, c_n$, non-positive on $c_{n+1}, \dots, c_{g+1}$, positive on c_i and negative on c_j. Adding up these differentials yields the assertion. □

The next Lemma is concerned with hyperelliptic M-curves. These are curves which are defined by

$$X^\circ := \{(x,y) \in \mathbb{C}^2 \mid y = \sqrt{h(x)}\},$$

where $h(x) = \prod_{i=1}^{2g+2}(x - \alpha_i)$ and $\alpha_1 < \cdots < \alpha_{2g+2}$ are real numbers. We compactify X° in such a way that there are two smooth points over ∞ and the covering $X \to \mathbb{C}P^1, (x,y) \mapsto x$ is unbranched over ∞. The compactification of X° we denote as X. This Lemma will be helpful to show that any real differential is positive on an oval of an M-curve and negative on another one. For this Lemma, we give a very precise proof in comparison to [Natanzon, 2004]. The reason for this is two-fold: First of all, the set of ovals given in [Natanzon, 2004] with the choice $\tau^*\omega = \bar{\omega}$ is just complementary to the set of ovals of a hyperelliptic curve we consider. It is claimed implicitly in [Natanzon, 2004] that these are given by the points $(x,y) \in X$ corresponding to

$x \in (-\infty, a_1) \cup (a_2, a_3) \cup \cdots \cup (a_{2g+2}, \infty)$. However, the latter would imply that there are ovals which are no simple closed curves. So the main modification compared to [Natanzon, 2004] is that, as explained above, we have changed the action of τ^* on real differentials ω. With this choice, the proof works out. Hereby, we find another reason why to change the definition of realness for a holomorphic differential. The second reason why we give this proof so precise compared to [Natanzon, 2004] is that we do not understand the argumentation given there easily without this additional information.

Lemma 6.23 ([Natanzon, 2004, Lemma 2.6.3])**.** *Let $\alpha_1 < \cdots < \alpha_{2g+2}$ be real numbers, let $h(x) = \prod_{i=1}^{2g+2}(x - \alpha_i)$, let X be the hyperelliptic curve corresponding to $y^2 = h(x)$ and let $\tau : X \to X$ be the antiholomorphic involution generated by the correspondence $(x, y) \mapsto (\bar{x}, -\bar{y})$. Then (X, τ) is a real M-curve of genus g each of whose real differentials is positive on one of the ovals and negative on another one.*

Proof. The set

$$X^\circ := \{(x, y) \in \mathbb{C}^2 \mid y = \sqrt{h(x)}\}$$

describes a real hyperelliptic curve of genus g, compare [Farkas and Kra, 2012, Section III.7.4]. Let X be the compactification of X° such that there are two smooth points over ∞ and the covering $X \to \mathbb{C}P^1, (x, y) \mapsto x$ is unbranched over ∞. This is defined as a ramified double cover of $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$, where the branch points of $X \to \mathbb{C}P^1$ are at the zeros $\alpha_1, \ldots, \alpha_{2g+2}$ of h. There exists a holomorphic involution $X \to X,\ (x, y) \mapsto (x, -y)$ on X. Since the coefficients of h are real, $(x, y) \in X$ implies $(\bar{x}, \bar{y}) \in X$, so $\tau : X \to X,\ (x, y) \mapsto (\bar{x}, -\bar{y})$ is an antiholomorphic involution on X. The fixed points of the antiholomorphic involution τ are called the ovals of such a curve. These are precisely the points $(x, y) \in X$ such that $y^2 = h(x)$ has a solution in $\iota\mathbb{R}$, i.e. $h(x) \in \mathbb{R}$ and $h(x) \leq 0$ for $x \in \mathbb{R}$. Since $h(x)$ is a polynomial of even degree in x, one has for $x \in \mathbb{R}$ that $\lim_{x \to \pm\infty} h(x) = \infty$, so $h(x) \leq 0$ for $x \in [\alpha_{2i-1}, \alpha_{2i}]$ with $i = 1, \ldots, g+1$. So X has $g+1$ ovals c_i of τ corresponding to the segments $[\alpha_{2i-1}, \alpha_{2i}]$ for $i = 1, \ldots, g+1$. Let $X^\tau = \bigcup_{i=1}^{g+1} c_i$. Since the two sheets of X are only connected via these ovals, $X \setminus X^\tau$ consists of two oriented Riemann surfaces X^+ and X^- with boundary components $c_1, \ldots, c_{g+1}$ which are interchanged by τ. By [Farkas and Kra, 2012, Corollary III.7.5.1], any holomorphic differential is of the form $f(x)\frac{dx}{y}$, where $f(x)$ is a polynomial of degree at most $g-1$, so the differential

$$\omega_f := f(x)\frac{dx}{y},$$

where f is a polynomial with real coefficients and of degree at most $g-1$ is also regular on (X, τ). For $y \in \iota\mathbb{R}$ and $x \in \mathbb{R}$, τ acts as $\tau(x, y) = (\bar{x}, -\bar{y}) = (x, y)$. Taking also the realness of the

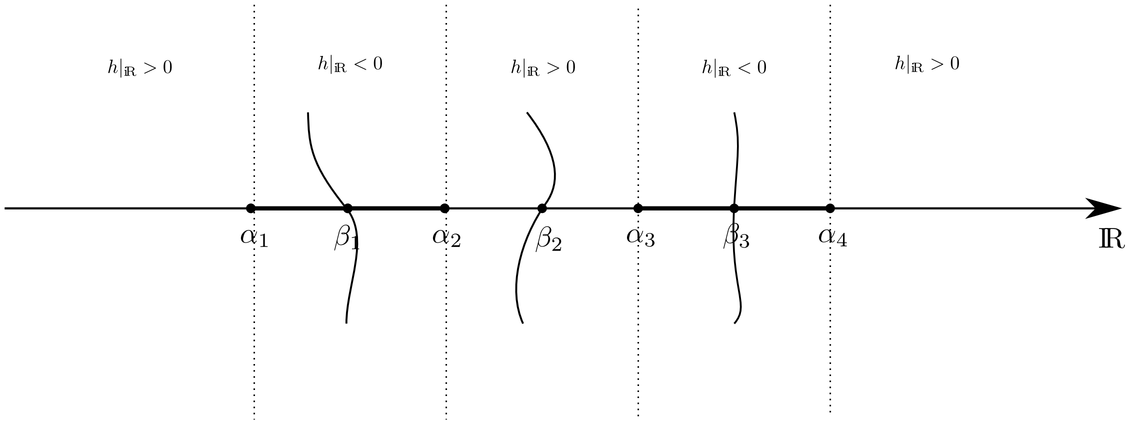


Figure 6.4.: For $g = 2$: Sketch of the four zeros α_i of h and the three zeros β_i of h' as well as the areas in which $\mathrm{Re}(h) > 0$ and in which $\mathrm{Re}(h) < 0$ and the curves of $M = \{(x, h(x)) \in \mathbb{C}^2\}$ starting at β_i on which $h(x) \in \mathbb{R}$.

coefficients of f into account we see that

$$\overline{\left(\frac{f(x)}{y}\right)} = \frac{\overline{f(x)}}{\overline{y}} = \frac{f(x)}{-y} = -\frac{f(x)}{y},$$

and so we obtain

$$\tau^*\omega_f = \tau^*\left(\frac{f(x)}{y}\,dx\right) = \frac{f(\tau(x))}{\tau(y)}\,d\tau(x) = -\frac{\overline{f(x)}}{y}dx = -\bar{\omega}_f.$$

Thus, ω_f is a real differential. We will deduce the sign of $-\iota\omega_f$ on the ovals by considering $\omega := \frac{dx}{y}$ and then multiplying it with the corresponding f. To determine the sign of $-\iota\omega$ on the ovals, we have to consider the sign of $\iota y = \iota\sqrt{h(x)}$ on these ovals. Therefore, we consider another compact variety defined by

$$M^\circ := \{(x, h(x)) \in \mathbb{C}^2\}$$

and compactify the curve M° in such a way that we add one smooth point over $x = \infty$. The derivative h' is a polynomial of degree $2g+1$ with $2g+1$ zeros $\beta_i \in \mathbb{R}$ such that for $i = 1, \ldots, g+1$

$$\alpha_{2i-1} < \beta_{2i-1} < \alpha_{2i}$$

with $h(\beta_i) < 0$ for i odd and $h(\beta_i) > 0$ for i even. Since α_i are zeros of first order of h it is $h'(\alpha_i) \neq 0$. So $\alpha_i \neq \beta_j$ for $i = 1, \ldots, 2g+2$ and $j = 1, \ldots, 2g+1$. The points β_i are the branch points of the covering $(x, h(x)) \mapsto x$. At these points, one has a local coordinate z_i centered at β_i such that one can describe M on a small open neighborhood U_i of β_i by $y = z_i^2$. So on U_i one has $z_i^* h(x) \in \mathbb{R}$ if and only if the image of z_i is contained in $\mathbb{R}$ or $\iota\mathbb{R}$. For brevity, we write $z_i \in \mathbb{R}$

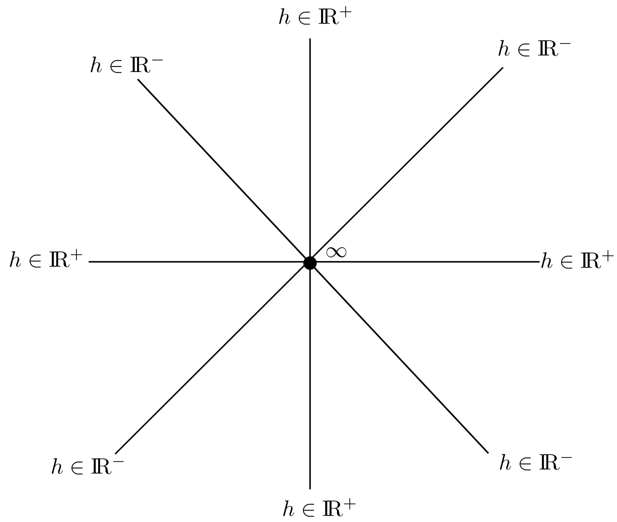


Figure 6.5.: For $g = 2$: The $4g = 8$ curves meeting at ∞ on which $h \in \mathbb{R}$.

respectively $z_i \in \iota\mathbb{R}$. Accordingly, we know that at each β_i, there start four one-dimensional curves on which $h \in \mathbb{R}$. The two curves corresponding to $z_i \in \mathbb{R}$ are contained in the real axis and one of the other curves corresponding to $z_i \in \iota\mathbb{R}$ is contained in the upper half plane $\mathcal{H}$ and the other one in the lower half plane $\{z \in \mathbb{C} \mid \operatorname{Im}(z) < 0\}$. This is depicted in Figure 6.4. In total, this yields $4g$ curves on which $h \in \mathbb{R}$, whereby two of these are contained in the real line and at each β_i with $i \in \{1, \ldots, 2g-1\}$, there start two curves which are not contained in the real line. The set

$$R := \{(x, h(x)) \mid h(x) \in \mathbb{R} \cup \{\infty\}\}$$

is a real, one-dimensional subvariety of M. Furthermore, compactifying M at ∞ by the usual one-point compactification as in [Munkres, 2000, § 29, page 185], there is a local coordinate z centered at ∞ such that $z(\infty) = 0$ and such that on a small open neighborhood U of ∞, one can describe M by $y = z^{2g}$. Hence, there are $4g$ one-dimensional curves starting at ∞ on which $h(x)$ is real: $2g$ different choices for $z \in \mathbb{R}$ and $2g$ different choices for $z \in \iota\mathbb{R}$, so the sign of h varies on two neighbored curves, see Figure 6.5 for $g = 2$. Since $h \in \mathbb{R}^+$ for real $x < \alpha_1$ and real $x > \alpha_{2g+2}$, two of these curves originating at ∞ correspond to $x \in \mathbb{R}$. The set

$$R := \{(x, h(x)) \in M \mid h(x) \in \mathbb{R} \cup \{\infty\}\} \subset M$$

is a real, one-dimensional compact subvariety of M. To see that the $4g$ curves, out of which each two that are not contained in the real line are starting at each β_i, are connected to the $4g$ curves starting at ∞, first note that the only points in which the real subvariety R has a branch point are the points at $x = \beta_i$, so the curves starting at β_i which are not contained in $\mathbb{R}$ can intersect $\mathbb{R}$ only at one of the points β_i with $i \in \{1, \ldots, 2n-1\}$. We assume that a curve which starts at β_i

has a second intersection point β_j with $i \neq j$ on $\mathbb{R}$. Without loss of generality, let this be the curve contained in $\mathcal{H}$. Then $i - j = 0 \mod 2$ since the sign of h on the curve starting at β_i is opposite to the sign of h on the curve starting at $\beta_{i\pm1}$. So we can assume without loss of generality that the second intersection point is in β_{i+2}. Then the curves contained in $\mathcal{H}$ starting at β_{i+1} cannot cross the curve connecting β_i and β_{i+2} since $h(x) \in \mathbb{R}$ and $h(x) \neq 0$ on both curves, whereby $h(x) > 0$ and $h(x) < 0$ cannot hold simultaneously. Hence, the curve starting at β_{i+1} has to remain in the part of $\mathcal{H}$ enclosed by $\mathbb{R}$ and the curve connecting β_i and β_{i+2}. Then the curve starting at β_{i+1} must have an open end. This contradicts the definition of R as a subvariety of M: since M is compact, R is as well. The same holds for the curves starting at β_i contained in the lower half plane. So the assignment of the $4g$ curves starting at infinity to the $4g$ curves out of which $4g-2$ are starting at β_i is unique. This yields in total g simple closed contours intersecting at infinity, where one corresponds to the real line and the others are each crossing the real line in exactly one of the β_i. The sign of $\operatorname{Im}(\sqrt{h})$ changes when crossing a curve contained in R with $h \in \mathbb{R}_+$ because then $\sqrt{h} \in \mathbb{R}^+$, so the imaginary part of $\sqrt{h}$ has a zero. These are all the simple closed curves which intersect the real line at β_{2i} with $i \in \{1, \ldots, g-1\}$. When crossing a simple closed curve containing β_{2i-1} for $i \in \{1, \ldots, g\}$, the real part of $\sqrt{h}$ changes its sign because $h \in \mathbb{R}^-$ on this curve. Since β_{2i} is contained in the complement of the ovals of τ on the real line, the imaginary part of y has opposite signs on two neighboring segments corresponding to ovals of τ. Because on these curves, y is purely imaginary, the sign of ιy is also opposite on these neighboring ovals and hence also $-\iota\omega$ has opposite signs on each two neighboring ovals.

Let us now consider an arbitrary holomorphic 1-form on X given by $\omega_f = \frac{f(x)}{y}dx$. Then $\deg(f) \leq g-1$ assures that there have to be at least two ovals on which $f \neq 0$.

We assume without loss of generality that $f > 0$ and $-\iota\omega > 0$ on the oval c_1 corresponding to $[\alpha_1, \alpha_2]$. If $f > 0$ on the neighboring oval c_2 corresponding to $[\alpha_3, \alpha_4]$, then $-\iota\omega_f > 0$ on c_1 and $-\iota\omega_f < 0$ on c_2.

If f has a zero on the oval corresponding to $[\alpha_3, \alpha_4]$, then the sign of $-\iota\omega$ on $[\alpha_5, \alpha_6]$ is equal to the sign of $-\iota\omega$ on $[\alpha_1, \alpha_2]$, but the sign of f changed and accordingly $-\iota\omega_f$ has opposite signs on c_1 and c_3. Successively repeating this procedure while assuming that every of the $i-2$ ovals in between c_1 and c_i contains at least one zero (since an even number of zeros has the same effect as no zero), one sees that the number of times that $-\iota\omega$ changed sign from c_1 to c_i is i and the maximal number of zeros of f is $i-1$. Hence, the sign of $-\iota\omega_f$ on c_1 and c_i is opposite to each other and the assumption follows. □

In the version of the proof of the following Theorem in [Natanzon, 2004], it is claimed but not shown that two sets which are defined there are open respectively closed. Because we think that this is one of the crucial steps in the following proof, we worked this part out.

Theorem 6.24 ([Natanzon, 2004, Theorem 2.6.2])**.** *For any real differential ω on an M-curve, there is an oval on which this differential is positive and an oval on which it is negative.*

Proof. Let $\widetilde{M}$ be the set of all M-curves of genus g with an ordered set of ovals $c_1, \dots, c_{g+1}$. We define two bundles over $\widetilde{M}$: a fiber bundle $f : F \to \widetilde{M}$, where the fiber over each point of $\widetilde{M}$ is just the corresponding M-curve, i.e. $f^{-1}(X, \tau) = (X, \tau)$, and a vector bundle $e : E \to \widetilde{M}$ which fibers $\tilde{e}^{-1}(X, \tau)$ consists of all real differentials on (X, τ). We take a basis of $e^{-1}(X, \tau)$ which is formed by differentials $\omega_i = \omega_i(X, \tau)$ such that $\oint_{c_i} \omega_j = \delta_{ij}$ for $i, j \leq g$. Then there exists a unique flat connection on e such that the sections ω_i are parallel for $i = 1, \dots, g$. Let us introduce the following classification of real differentials:

- A real differential is called a *differential of type A* if each of the ovals contains at least one points at which the differential is non-positive. The set of M-curves that admit a differential of type A is called M^A.
- A real differential is called a *differential of type B* if each of the ovals contains at least one point at which the differential is negative. The set of M-curves that admit a differential of type B is called M^B.

We first show that M^A is a closed set in $\widetilde{M}$. Therefore, we choose a sequence $(X_n, \tau_n)_{n \in \mathbb{N}}$ of curves in M^A which converges in $\widetilde{M}$. We denote the limit of this sequence by (X, τ) and the corresponding ovals by c_i with $i = 1, \dots, g + 1$. This yields a sequence $\omega_n := \sum_{i=1}^{g} \alpha_{i,n} \omega_{i,n}$ of real differentials of type A. To see that this sequence also converges to a differential of type A, we normalize the coefficients $\alpha_{i,n} \in \mathbb{C}$ such that $\|(\alpha_{1,n}, \dots, a_{g,n})\| = 1$. Then the sequence $(\alpha_{1,n}, \dots, \alpha_{g,n})_{n \in \mathbb{N}}$ is bounded and thus – by the Bolzano Weierstraß Theorem – contains a convergent subsequence $(\omega_m)_{m \in \mathbb{N}}$. This subsequence converges to $\omega \neq 0$. Next, we consider the sequence of real curves (X_m, τ_m) corresponding to ω_m. To see that each oval $c_i \subset X$ contains a point p_i such that ω is non-positive at p_i, note that the set of ovals is ordered on each X_m with $m \in \mathbb{N}$. This defines also a convergent sequence of ovals $(c_{i,m})_{m \in \mathbb{N}}$ with $i = 1, \dots g + 1$ in the fibers of f. Since $(X_m, \tau_m) \in M^A$ for all $n \in \mathbb{N}$, there exists a point $p_{i,m}$ on each oval $c_{i,m}$ such that the differential ω_m is non-positive at the points. On every curve X_m, seen as a fiber of f at $(X_m, \tau_m) \in \widetilde{M}$, the set of ovals $\{c_{1,m}, \dots, c_{g+1,m}\}$ is a compact subset. So the set of ovals

$$\bigcup_{m \in \mathbb{N}} (c_{1,m} \cup \dots \cup c_{g+1,m}) \cup c_1 \cup \dots \cup c_{g+1}$$

is also compact in F. Therefore, the sequence $(p_{1,m}, \dots, p_{g+1,m})_{m \in \mathbb{N}}$ contains a convergent subsequence which we endow with the same index. The limit of this sequence is contained in the ovals of (X, τ). Since f is a proper map, this also defines a convergent subsequences of points in the ovals of $(X_m, \tau_m) \in \widetilde{M}$. So ω is non-positive at the points $p_i \in c_i$ for $i = 1, \dots g + 1$ and thus (X, τ) is a real curve of type A.

Similarly, we show that M^B is open: Let $(X, \tau) \subset M^B$ be a fixed M-curve in $\widetilde{M}$. We have to show that any curve which is arbitrarily close to (X, τ) in $\widetilde{M}$ is also an element of M^B. Let $(\widetilde{X}, \tilde{\tau})$ be

such an M-curve in $\widetilde{M}$. We denote the set of ordered ovals of $(\tilde{X}, \tilde{\tau})$ as $\{\tilde{c}_1, \ldots, \tilde{c}_{g+1}\}$. These are again just the fixed points of $\tilde{\tau}$ in the fiber $(\tilde{X}, \tilde{\tau})$ over $\tilde{M}$. Since $(X, \tau) \in M^B$, there exists a the real holomorphic differential ω on X which is positive at $p_i \in c_i$ for $i = 1, \ldots, g+1$. We can write $\omega = \sum_{i=1}^{g} \alpha_i \omega_i$, where ω_i are the elements of the basis of holomorphic differential forms normalized with respect to $c_1, \ldots, c_g$. Next, we show that there exists a real holomorphic differential $\tilde{\omega}$ and a set of points $\{\tilde{p}_i, \ldots, \tilde{p}_{g+1}\}$ with $\tilde{p}_i \in \tilde{c}_i$ on $\tilde{X}$ such that $\tilde{\omega}$ is positive at these points. Consider $\tilde{\omega} = \sum_{i=1}^{g} \alpha_i \omega_i$, where the coefficients α_i are the same as the coefficients of ω. This defines a real holomorphic differential $\tilde{\omega}$ on $(\tilde{X}, \tilde{\tau})$. Then $\tilde{\omega}$ defines a smooth 1-form ω_F on F. Moreover, τ induces an antiholomorphic involution τ_F on F. Then the set of all fixed points of τ_F such that ω_F is positive is an open set in the fixed point set of τ_F in F. Therefore, for all real curves $(\tilde{X}, \tilde{\tau})$ in a small open neighborhood of (X, τ), the restriction of ω_F to $\tilde{X}$ is a differential form of type B and thus the set M^B is open.
Obviously, $M^B \subset M^A$. If $M^A \subset M^B$, then M^A is an open and closed set in $\widetilde{M}$. Since it is shown in [Natanzon, 2004, Theorem 2.2.1] that $\widetilde{M}$ is connected, this yields that M^A contains either all M-curves of fixed genus g or none. Using the example of the hyperelliptic curve from Lemma 6.23 yields that $\widetilde{M} \setminus M^A \neq \emptyset$ and thus $M^A = \emptyset$ and the assertion follows. So we show that $M^A \subset M^B$. Let $(X, \tau) \subset M^A$ and let ω be a differential of type A on (X, τ). Because we can decompose $X \setminus X^\tau$ into two parts X^+ and X^- such that the boundary of these parts is given by $c_1, \ldots, c_{g+1}$ and this boundary is homologous to zero, it is

$$\sum_{i=1}^{g+1} \oint_{c_i} \omega = 0.$$

It follows that the differential is negative at least at one point of X^τ. Let c be an oval containing such a point. By Lemma 6.22, there is a real differential γ that is positive on c and negative on the other ovals. Then for sufficiently small ε, the differential $\omega + \varepsilon\gamma$ is negative on at least one point of c and since γ is negative on the remaining ovals and ω non-positive on these, $\omega + \varepsilon\gamma$ is negative on these ovals for arbitrary $\varepsilon > 0$. So $\omega + \varepsilon\gamma$ is a differential of type B for sufficiently small $\varepsilon > 0$. Thus, $M^A = M^B$ is an open and closed set in $\widetilde{M}$. □

In the classification of the connected components of the real Prym variety, the following theorem is necessary. Though this is shown in [Natanzon, 2004], we again do not understand the proof given there due to its shortness and moreover found several small mistakes, e.g. in the formula for the genus of that curve or several sign switches in the definition of some involution necessary in that proof. Therefore, we try to give the full picture here.

Theorem 6.25 ([Natanzon, 2004, Theorem 2.6.3])**.** *Let $1 \leq k \leq g+1$, $k \equiv g+1 \mod 2$ and $k > \ell \geq k - \frac{k}{2}$ for k even and $k > \ell \geq k - \frac{k-1}{2}$ for k odd. Then there exists a real curve of type $(g, k, 1)$ with ovals $c_1, \ldots, c_k$ such that any real differential which is non-negative on $c_1, \ldots, c_\ell$ must* *be positive on one of the ovals $c_1, \dots, c_k$ and negative on another.*

Proof. For brevity, we set $p_\alpha(x) := \prod_{i=1}^n (x-\alpha_i)$ and $p_\beta(x) := \prod_{i=1}^m (x-\beta_i)$. Let us consider the Riemann surface X° which is the normalization of the zero set $\{(x,y) \in \mathbb{C}^2 \mid f(x,y) = 0\}$, where

$$f(x,y) := y^4 - 2y^2(p_\alpha(x) - p_\beta(x)) + (p_\alpha(x) + p_\beta(x))^2 = 0,$$

with $\alpha_1 < \dots < \alpha_n \leq \beta_1 < \dots < \beta_m \in \mathbb{R}$, $n > 0$ and $n, m \equiv 0 \mod 2$. Solving the above equation for y^2 yields

$$\begin{aligned} y^2(x) &= p_\alpha(x) - p_\beta(x) \pm \sqrt{(p_\alpha(x) - p_\beta(x))^2 - (p_\beta(x) + p_\alpha(x))^2} \\ &= p_\alpha(x) - p_\beta(x) \pm 2\iota\sqrt{p_\alpha(x)p_\beta(x)} = \left(\sqrt{p_\alpha(x)} \pm \iota\sqrt{p_\beta(x)}\right)^2. \end{aligned}$$

So the surface X° is obtained by considering the normalization of the set

$$\left\{(x,y) \in \mathbb{C}^2 \mid y = \pm\sqrt{p_\alpha(x)} \pm \iota\sqrt{p_\beta(x)} \text{ or } y = \pm\sqrt{p_\alpha(x)} \mp \iota\sqrt{p_\beta(x)}\right\}. \tag{6.1}$$

This surface defines a four-sheeted covering of $X \to \mathbb{C}$, $(x,y) \mapsto x$. At $x = \infty$, the terms of x of highest power in $f(x,y)$ approximate y, i.e.

$$y \approx \pm x^{n/2} \pm \iota x^{m/2} \text{ respectively } y \approx \mp x^{n/2} \pm \iota x^{m/2},$$

where $m, n \geq 2$ are even. So in a small open neighborhood U of ∞, $z : U \to \mathbb{C}$ with $z = \frac{1}{x}$ is a local coordinate of X centered at $x = \infty$. Then X° can be compactified such that the compactification X has four different smooth points over $x = \infty$ and the covering $X \mapsto \mathbb{C}P^1$, $(x,y) \mapsto x$ is unbranched at these points.

We want to determine the branch points of this covering together with their order, compare Definition 2.32. Because this definition is given for a polynomial f of second degree in y, we here also have to assure that the second derivative of f into the direction of y is unequal to zero. So it is necessary to take the following partial derivatives of f into account:

$$\begin{aligned} \frac{\partial f}{\partial y}(x,y) &= 4y^3 - 4y(p_\alpha(x) - p_\beta(x)), \\ \frac{\partial f}{\partial x}(x,y) &= 2y^2(p'_\alpha(x) - p'_\beta(x)) + 2(p_\alpha(x) + p_\beta(x))(p'_\alpha(x) + p'_\beta(x)), \\ \frac{\partial^2 f}{\partial y^2}(x,y) &= 12y^2 - 4(p_\alpha(x) - p_\beta(x)). \end{aligned}$$

For $\alpha_n \neq \beta_1$, $i = 1, \dots, n$ and $j = 1, \dots, m$ it is $p_\alpha(\beta_j), p_\beta(\alpha_i), p'_\alpha(\beta_j), p'_\alpha(\alpha_i), p'_\beta(\alpha_i), p'_\beta(\beta_j) \neq 0$, whereby the formulas for the derivatives hold since p_α, p_β are polynomials which have only zeros of first order, so the interval between each two neighbored zeros of p_α respectively p_β contains a

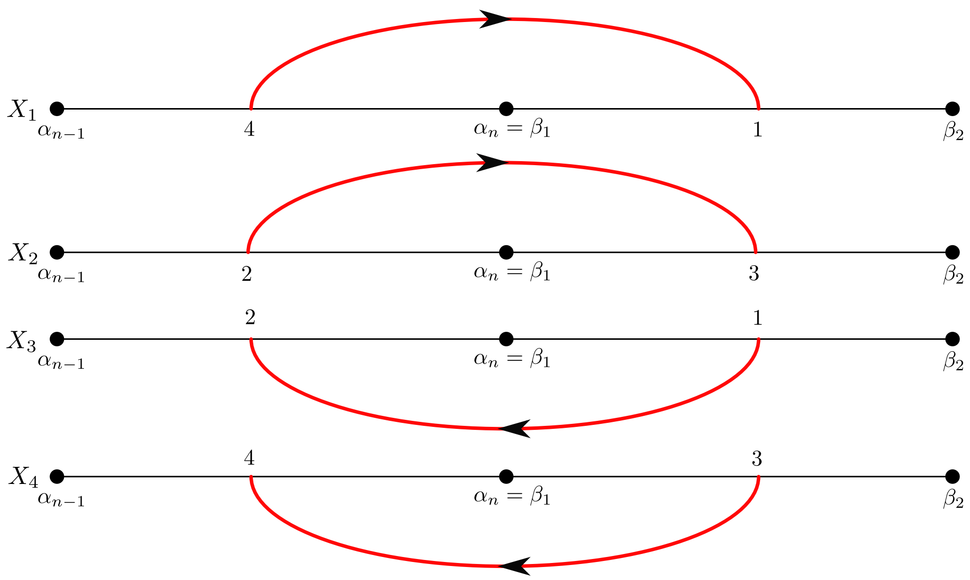


Figure 6.6.: Depicting the cuts between α_{n-1}, $\alpha_n = \beta_1$ and β_2 on all four sheets $X_1, \ldots, X_4$ to determine the winding number of $(\alpha_n, 0)$ as 2.

zero of the derivative and there are no other zeros. Therefore, one has

$$\frac{\partial f}{\partial y}\left(\alpha_i, \pm\iota\sqrt{p_\beta(\alpha_i)}\right) = 4\left(\sqrt{p_\beta(\alpha_i))}\right)^3 - 4\sqrt{p_\beta(\alpha_i)}p_\beta(\alpha_i) = 0,$$
$$\frac{\partial f}{\partial y}\left(\beta_j, \pm\sqrt{p_\alpha(\beta_j)}\right) = 4\left(\sqrt{p_\alpha(\beta_j)}\right)^3 - 4\sqrt{p_\alpha(\beta_j)}p_\alpha(\beta_j) = 0,$$
$$\frac{\partial f}{\partial x}\left(\alpha_i, \pm\iota\sqrt{p_\beta(\alpha_i)}\right) = \begin{cases} 4p_\beta(\alpha_i)p_\beta'(\alpha_i) \neq 0, \\ 4p_\beta(\alpha_i)p_\alpha'(\alpha_i) \neq 0, \end{cases}$$
$$\frac{\partial f}{\partial x}\left(\beta_j, \pm\sqrt{p_\alpha(\beta_j)}\right) = \begin{cases} 4p_\alpha(\beta_j)p_\alpha'(\beta_j) \neq 0, \\ 4p_\alpha(\beta_j)p_\beta'(\beta_j) \neq 0, \end{cases}$$
$$\frac{\partial^2 f}{\partial y^2}\left(\alpha_i, \pm\iota\sqrt{p_\beta(\alpha_i)}\right) = -8p_\beta(\alpha_i) \neq 0,$$
$$\frac{\partial^2 f}{\partial y^2}\left(\beta_j, \pm\sqrt{p_\alpha(\beta_j)}\right) = 8p_\alpha(\beta_j) \neq 0,$$

whereby the degree of $\partial f/\partial y$ shows that $\partial f/\partial y$ has no other zeros then the one above.
Let $\alpha_n \neq \beta_1$. Over an interval of the form $x \in [\alpha_{2i-1}, \alpha_{2i}]$ respectively $[\beta_{2i-1}, \beta_{2i}]$, each two sheets of the four sheets meet. Accordingly, the covering $X \to \mathbb{C}P^1$, $(x, y) \mapsto x$ has two branch points of order one at each $x = \alpha_i$ for $i = 1, \ldots, n$ and each $x = \beta_j$ for $j = 1, \ldots m$. So the total branching order is $b = 2n + 2m$.

Let now $\alpha_n = \beta_1$. Due to $y(\alpha_n) = p_\alpha(\alpha_n) = p_\beta(\beta_1) = 0$, one has

$$\frac{\partial f}{\partial x}(\alpha_n, 0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(\alpha_n, 0) = 0$$

and the point on X corresponding to $x = \alpha_n$ is a singularity of X in which all four sheets meet. By determining the winding number around $\alpha_n = \beta_1$, we show next that the corresponding point on the normalization is a ramification point of order one. For this, take a look at Figure 6.6: To determine the winding number, we cut X at $x \in [\alpha_{n-1}, \alpha_n]$ and $x \in [\beta_1, \beta_2]$. We then encircle $\alpha_n = \beta_1$ once. Let us denote the four sheets of X with all branching lines cut out by $X_1, \ldots, X_4$, whereby

$$\begin{aligned}
X_1 &:= \left\{(x, y) \in \mathbb{C}^2 \mid y = \sqrt{p_\alpha(x)} + \iota\sqrt{p_\beta(x)}\right\}, \\
X_2 &:= \left\{(x, y) \in \mathbb{C}^2 \mid y = -\sqrt{p_\alpha(x)} - \iota\sqrt{p_\beta(x)}\right\}, \\
X_3 &:= \left\{(x, y) \in \mathbb{C}^2 \mid y = \sqrt{p_\alpha(x)} - \iota\sqrt{p_\beta(x)}\right\}, \\
X_4 &:= \left\{(x, y) \in \mathbb{C}^2 \mid y = -\sqrt{p_\alpha(x)} + \iota\sqrt{p_\beta(x)}\right\}.
\end{aligned}$$

The cut in $[\alpha_{n-1}, \alpha_n]$ connects X_1 with X_4 and X_2 with X_3 while the cut in $[\beta_1, \beta_2]$ connects X_1 with X_3 and X_2 with X_4. We consider a simple closed curve as depicted in Figure 6.6 and count how often it has to wind around the singularity until it closes. Without loss of generality, we assume that this circle starts on X_1 at the point indicated by 4 and ends on X_1 at the point indicated by 1, continues on X_3 at the point indicated by 1 to the point indicated by 2, then continues on X_2 at the point indicated by 2 and ending at the point indicated by 3, further continues on X_4, starting at the point indicated by 3 and ending at the point indicated by 4 back to the starting point indicated by 4 on X_1. So the winding number around $(\alpha_n, 0) = (\beta_1, 0)$ equals two. Accordingly, the branching order at this point is one. So the total branching order of the covering $X \to \mathbb{C}P^1$, $(x, y) \mapsto X$ is $b = 2(n-1) + 2m$. Due to the Riemann-Hurwitz Formula [Forster, 1981, Theorem 17.4], it is

$$2g - 2 = b - 8 = \begin{cases} 2n + 2m - 8 & \text{for } \alpha_n \neq \beta_1, \\ 2n + 2m - 10 & \text{for } \alpha_n = \beta_1, \end{cases}$$

and so the genus of X is given by

$$g = \begin{cases} n + m - 3 & \text{for } \alpha_n < \beta_1, \\ n + m - 4 & \text{for } \alpha_n = \beta_1, \end{cases}$$

where $g > 0$ since $n, m \geq 2$. The coefficients of p_α and p_β are real and $(x, y) \mapsto (x, -y)$ is an

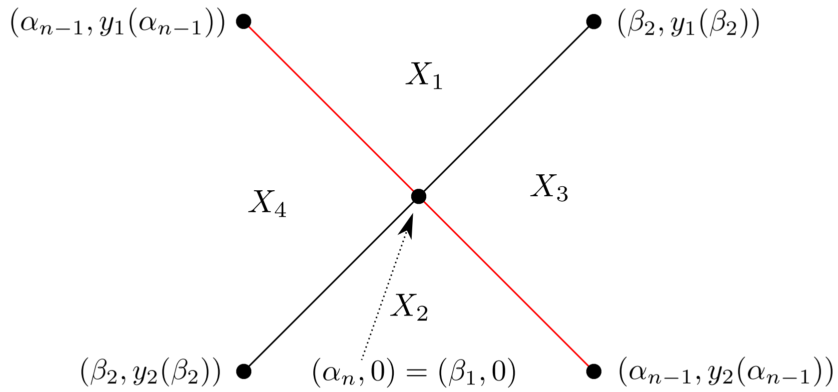


Figure 6.7.: Schematic sketch in a local coordinate in $\mathbb{C}$ of the four cuts meeting in $(\alpha_n, 0) = (\beta_1, 0)$. The cut between $(\alpha_{n-1}, y_1(\alpha_{n-1}))$ and $(\alpha_{n-1}, y_2(\alpha_{n-1}))$ (red line) yields the oval over $[\alpha_{n-1}, \alpha_n]$.

involution of X. Thus, also the antiholomorphic map

$$\tau : X \to X, \quad (x, y) \mapsto (\bar{x}, -\bar{y}),$$

defines an involution on X. The connected components of the fixed point set of τ in X are exactly the ovals for which $x \in \mathbb{R}$ and $y \in \iota\mathbb{R}$. This is the case for $p_\alpha \leq 0$ and $p_\beta \geq 0$, i.e. the ovals are corresponding to the intervals $[\alpha_{2i-1}, \alpha_{2i}]$ for $i = 1, \dots, n/2$. If $\alpha_1 \neq \beta_n$, then there are n ovals on X. For $\alpha_1 = \beta_n$, each two contours on X corresponding to $[\alpha_{2i-1}, \alpha_{2i}]$ for $i = 1, \dots, \frac{n}{2} - 1$ yield $n - 2$ ovals. However, it is necessary to take a closer look on the form of X over the interval $[\alpha_{n-1}, \alpha_n]$: At α_{n-1}, only the sign of p_α changes while at α_n, p_α and p_β are equal to zero, so all four sheets meet in this point. Following the gluing structure as indicated in Figure 6.6, one sees that this yields precisely one oval on X over $[\alpha_{n-1}, \alpha_n]$, compare Figure 6.7. So the total number of ovals in this case equals $n - 1$. Furthermore, τ interchanges the sheets given by X_1 and X_4 as well as the sheets given by X_2 and X_3 since $\pm\sqrt{p_\alpha(\alpha_i)} + \iota\sqrt{p_\beta(\alpha_i)} = \iota\sqrt{p_\beta(\alpha_i)}$ and $\pm\sqrt{p_\alpha(\alpha_i)} - \iota\sqrt{p_\beta(\alpha_i)} = -\iota\sqrt{p_\beta(\alpha_i)}$.
We denote the set of ovals of X by X^τ. Then $X \setminus X^\tau = (X_1 \cup X_3)^\circ \cup (X_2 \cup X_4)^\circ$ with $(X_1 \cup X_3)^\circ \cap (X_2 \cup X_4)^\circ = \emptyset$. So (X, τ) is a real curve of separating type and the topological type of (X, τ) equals $(g, k, 1)$, where

$$k = \begin{cases} n & \text{for } \alpha_n < \beta_1, \\ n - 1 & \text{for } \alpha_n = \beta_1. \end{cases}$$

On X, we define the correspondence

$$\tau_\beta : X \to X, \quad \left(x, \pm\sqrt{p_\alpha(x)} \pm \iota\sqrt{p_\beta(x)}\right) \mapsto \left(x, \pm\sqrt{p_\alpha(x)} \mp \iota\sqrt{p_\beta(x)}\right)$$

which maps X_1 to X_3 and X_2 to X_4. By the definition of X in (6.1), we see that this map defines an involution which commutes with τ. The involution τ_β pairwise transposes the ovals for $\alpha_n < \beta_1$ and preserves exactly the oval corresponding to $[\alpha_{n-1}, \alpha_n]$ for $\alpha_n = \beta_1$ and transposes the other ones. Let us number the ovals $c_1, \dots, c_k$ such that $\tau_\beta[c_i] = c_{k+1-i}$ for $i = 1, \dots, \frac{k}{2}$ in the case $\alpha_n \neq \beta_1$ and for $i = 1, \dots, \frac{k-1}{2}$ in the case $\alpha_n = \beta_1$, whereby then $\tau_\beta[c_k] = c_k$. We assume that there is a real differential ω that is non-negative on the ovals $c_1, \dots, c_\ell$ with $\ell \geq \frac{n}{2}$ and is positive or non-negative on the other ovals. Then there is no oval on which the differential $\omega + \tau_\beta^*\omega$ is negative. The involution τ induces an antiholomorphic involution $\tilde{\tau} : \tilde{X} \to \tilde{X}$ on the surface $\tilde{X} = X/\sim_{\tau_\beta}$, where $p, q \in X$ obey $p \sim_{\tau\beta} q$ if and only if $p = q$ or $p = \tau_\beta(q)$. Then $(\tilde{X}, \tilde{\tau})$ is an M-curve of genus $(n/2) - 1$. Since $\omega + \tau_\beta^*\omega$ is invariant under τ_β, the differential $\omega + \tau_\beta^*\omega$ induces a real differential on the curve $(\tilde{X}, \tilde{\tau})$ that is negative on no oval. This contradicts Theorem 6.24 which says that any holomorphic real differential is always negative on one oval and positive on another one. Thus, we have shown that there is no such differential ω on X. □

Remark 6.26. On X as in the foregoing theorem, one can also define another correspondence

$$\tau_\alpha : X \to X, \quad \left(x, \pm\sqrt{p_\alpha(x)} \pm \iota\sqrt{p_\beta(x)}\right) \mapsto \left(x, \mp\sqrt{p_\alpha(x)} \pm \iota\sqrt{p_\beta(x)}\right)$$

which commutes with τ as well as τ_β. The involution τ_α preserves each of the ovals. This is also indicated in the proof given in [Natanzon, 2004]. Because we cannot see where this is necessary for the proof, we remark it here to describe the full symmetry of X.

6.2.4. The Jacobian variety of a real curve

Let X be a compact Riemann surface of genus g and

$$\{A_i, B_i \,|\, i = 1, \dots, g\} \in H_1(X, \mathbb{Z})$$

a symplectic homology basis as in (4.9), i.e for the intersection numbers of the elements of this basis holds

$$A_i \star A_j = B_i \star B_j = 0 \quad \text{and} \quad A_i \star B_j = \delta_{ij}.$$

Let $\omega_1, \dots, \omega_g$ be a basis of the space of holomorphic differentials on X which is in this section normalized as

$$\oint_{a_k} \omega_j = 2\pi\iota\delta_{kj}. \tag{6.2}$$

In this case, the matrix $B = (B_{kj})_{k,j=1}^g$, given by $B_{kj} = \oint_{b_k} \omega_j$, is symmetric and has negative-definite real part $\operatorname{Re}(B) = (\operatorname{Re} B_{kj})_{j,k=1}^g$, compare [Farkas and Kra, 2012, Proposition III.2.8].

Therefore, one can define a θ-function $\theta : \mathbb{C}^g \to \mathbb{C}$ by

$$\theta(z) = \theta(z \mid B) = \sum_{N \in \mathbb{Z}^g} \exp\left\{\frac{1}{2}\langle BN, N\rangle + \langle N, z\rangle\right\},$$

compare [Farkas and Kra, 2012, Section VI.1.1 with $\tau = B$]. As in Section 4.2.2, let Λ be the group generated over $\mathbb{Z}$ by the vectors

$$\Omega_{A_k} := 2\pi\iota(\delta_{k1}, \dots, \delta_{kg})^T \quad \text{and} \quad \Omega_{B_k} := (B_{k1}, \dots, B_{kg})^T \quad \text{for } k = 1, \dots, g$$

such that $\mathrm{Jac}(X) = \mathbb{C}^g/\Lambda$. Let $\Phi : \mathbb{C}^g \to \mathrm{Jac}(X)$ be the natural projection. In the rest of this section, we use some notation which is for example introduced in [Farkas and Kra, 2012, Sections III.11.8 and III.11.9]. Let S_k be the set of all positive divisors of degree k. Since X is compact, S_k is a complex compact manifold, see [Farkas and Kra, 2012, Section III.11.9]. We further define

$$S_k^r := \{D \in S_k \mid \dim H^0(X, \mathcal{O}_D) \geq r + 1\}.$$

Due to the Riemann-Roch Theorem, $\dim H^0(X, \mathcal{O}_D) - \dim H^1(X, \mathcal{O}_D) = \deg D - g + 1$. Accordingly,

$$\begin{aligned} S_g &= S_g^0, & S_{g-1}^0 &= \{D \in S_{g-1} \mid \dim H^1(X, \mathcal{O}_D) \geq 1\}, \\ S_g^1 &= \{D \in S_g \mid \dim H^1(X, \mathcal{O}_D) \geq 1\}, & S_{g-1}^1 &= \{D \in S_{g-1} \mid \dim H^1(X, \mathcal{O}_D) \geq 2\}, \\ S_g \setminus S_g^1 &= \{D \in S_g \mid \dim H^1(X, \mathcal{O}_D) = 0\}, & S_{g-1}^0 \setminus S_{g-1}^1 &= \{D \in S_{g-1} \mid \dim H^1(X, \mathcal{O}_D) = 1\}. \end{aligned}$$

As already done in Section 4.2.2, we can use the Abel map to map $S_k \to \mathrm{Jac}(X)$. Since this map is not independent of the choice of a basepoint $q \in X$ unless $k = 0$, compare [Miranda, 1995, p. 250], and we will hereinafter use that we can choose $q \in X^\tau$, we introduce this map a bit more detailed than before: For arbitrary $p, q \in X$, let γ_{qp} denote a fixed path starting at q and ending at p. We then fix a path which connects each point of a divisor $D = \sum_i p_i$ with q and denote it by $\gamma_{qD} := \sum_i \gamma_{qp_i}$. Let us choose a point q on X as base point of the map $\widetilde{\mathrm{Ab}}_q : S_k \to \mathbb{C}^g$ which maps a divisor $D = \sum_{i=1}^k p_i \in S_k$ to

$$\widetilde{\mathrm{Ab}}_q(D) := \left(\int_{\gamma_{qD}} \omega_1, \dots, \int_{\gamma_{qD}} \omega_g\right) = \sum_{i=1}^k \left(\int_{\gamma_{qp_i}} \omega_1, \dots, \int_{\gamma_{qp}} \omega_g\right).$$

This map is not unique on $\mathbb{C}^g$ since it depends on the chosen paths γ_{qp_i}. However, if two different paths γ_{qp_i} and $\tilde{\gamma}_{qp_i}$ both start at q and end at p_i, then $\gamma_{qp_i} - \tilde{\gamma}_{qp_i} \in H_1(X, \mathbb{Z})$, where the chosen basis of $H_1(X, \mathbb{Z})$ corresponds to the basis elements of Λ in $\mathbb{C}^g$. Let again $\Phi : \mathbb{C}^g \to \mathrm{Jac}(X) = \mathbb{C}^g/\Lambda$ be the natural projection. With this, the Abel map $\mathrm{Ab}_q : S_k \to \mathrm{Jac}(X)$ is uniquely defined as

$$\mathrm{Ab}_q(D) := \Phi\left(\int_{\gamma_{qD}} \omega_1, \dots, \int_{\gamma_{qD}} \omega_g\right) = \Phi\left(\sum_{i=1}^k \left(\int_{\gamma_{qp_i}} \omega_1, \dots, \int_{\gamma_{qp_i}} \omega_g\right)\right) \in \mathrm{Jac}(X).$$

By the Jacobi Inversion Theorem, $\mathrm{Ab}_q(S_g) = \mathrm{Jac}(X)$ and the Abel map Ab_q is invertible at a generic point $D \in S_g \setminus S_g^1$, see for example [Farkas and Kra, 2012, Proposition III.11.11 (a) and Proposition III.11.12].
We further define $W_n := \mathrm{Ab}_q[S_n] \subset \mathrm{Jac}(X)$. Then $W_n \subset W_{n+1}$ for all $n \in \mathbb{N}_0$ since $\mathrm{Ab}_q(D) = \mathrm{Ab}_q(D+q)$ for every divisor $D \in W_n$. Furthermore, we define W_n^r as the set of points in $\mathrm{Jac}(X)$ which are images of divisors $D \in S_n$ with $\dim H^0(X, \mathcal{O}_D) \geq r+1$ and set $W_n := W_n^0$. Then

$$\begin{aligned} W_g^1 &= \{\mathrm{Ab}_q(D) \mid D \in S_g \text{ and } \dim H^1(X,\mathcal{O}_D) \geq 1\}, \\ W_g \setminus W_g^1 &= \{\mathrm{Ab}_q(D) \mid D \in S_g \text{ and } \dim H^1(X,\mathcal{O}_D) = 0\}, \\ W_{g-1} &= \{\mathrm{Ab}_q(D) \mid D \in S_{g-1} \text{ and } \dim H^1(X,\mathcal{O}_D) \geq 1\}, \\ W_{g-1}^1 &= \{\mathrm{Ab}_q(D) \mid D \in S_{g-1} \text{ and } \dim H^1(X,\mathcal{O}_D) \geq 2\}, \\ W_{g-1} \setminus W_g^1 &= \{\mathrm{Ab}_q(D) \mid D \in S_g \text{ and } \dim H^1(X,\mathcal{O}_D) = 1\}. \end{aligned}$$

For $n \leq g$, W_n is an irreducible subvariety of $W_g = \mathrm{Jac}(X)$ of dimension n, see [Farkas and Kra, 2012, III.11.13]. Furthermore, it follows for example from [Farkas and Kra, 2012, Propotion III.6.5] that $W_g \setminus W_g^1$ is open and dense in W_g and for $n \leq g$, the set of singularities of W_n is a subvariety of W_n which equals W_n^1, compare [Farkas and Kra, 2012, Proposition III.11.11(c)]. Hence, $W_n \setminus W_n^1$ is also for $n < g$ open and dense in W_n. The projection $\Phi(K_q)$ to $\mathrm{Jac}(X)$ of the vector $(K_q^1, \ldots, K_q^g) \in \mathbb{C}^g$ with components

$$K_q^j = \frac{2\pi\iota + B_{jj}}{2} - \frac{1}{2\pi\iota} \sum_{\ell \neq j} \oint_{a_\ell} \left(\omega_\ell(p) \int_q^p \omega_j \right)$$

is called the vector of Riemann constants, see [Farkas and Kra, 2012, Theorem VI.2.4]. It is shown in [Farkas and Kra, 2012, Theorem VI.3.1] that the set $(\theta) = \mathrm{Ab}_q[S_{g-1}] + K_q \subset \mathrm{Jac}(X)$ coincides with the image of the set of zeros of the θ-function in $\mathrm{Jac}(X)$ and is called the θ-divisor.

Definition 6.27. A subset $\Sigma \subset \mathrm{Jac}(X)$ is said to be *singular* if $\Sigma \cap W_{g-1} \neq \emptyset$. Otherwise, Σ is called non-singular.

Recall that due to $\mathrm{Ab}_q(q) = 0$, every divisor D such that $q \in \operatorname{supp} D$ implies that $\mathrm{Ab}_q(D) \subset W_{g-1}$. So $W_{g-1} \subset W_g$. In this case, we write $D = \tilde{D} + q \in S_{g-1} + q$ with $\tilde{D} \in S_{g-1}$. Therefore, a subset $\Sigma \subset \mathrm{Jac}(X)$ is also-called singular if $\Sigma \cap \mathrm{Ab}_q[S_{g-1} + q] \neq \emptyset$. That means that $\Sigma + K_q$ contains a zero of the θ-function if and only if Σ is singular. Because the elements of W_{g-1} are exactly the image of the special divisors in S_g under Ab_q, see [Farkas and Kra, 2012, Proposition III.11.11(c)], a subset of $\mathrm{Jac}(X)$ is called singular if it contains elements $x \in \mathrm{Jac}(X)$ which is the image of a divisor D such that $\dim H^1(X, \mathcal{O}_D) \geq 1$. After these preliminaries, we start now to introduce the objects necessary to describe the real part of the Jacobian variety.
Let (X, τ) be a real curve such that $X^\tau \neq \emptyset$ and choose the base point of the Abel map as $q \in X^\tau$.

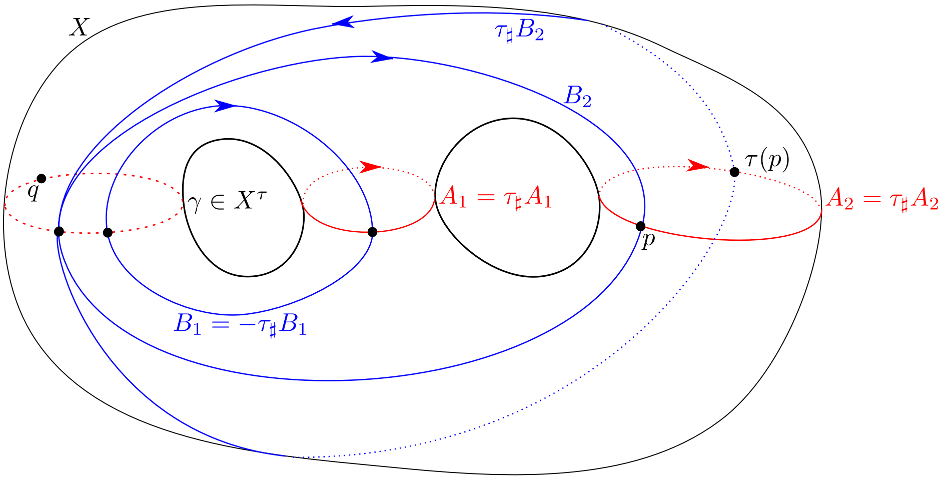


Figure 6.8.: Sketch of the construction of a real homology basis for a curve X of type $(2, 1, 0)$, whereby the A-cycles are depicted red and the B-cycles blue.

We define a symplectic basis $\{A_i, B_i \mid i = 1, \dots, g\}$ of $H_1(X, \mathbb{Z})$ that agrees with τ.

Definition 6.28. For curves of type $(g, k, 0)$, a basis $\{A_i, B_i \mid i = 1, \dots, g\}$ of $H_1(X, \mathbb{Z})$ is called *real homology basis* if its elements have the following properties:

(a) $\tau[A_i] = A_i$ for $i = 1, \dots, g$, $\tau[B_i] = -B_i$ for $i = 1, \dots, k-1$ and $\tau[B_i] = -B_i + A_i$ for $i = k, \dots, g$.

(b) The oval containing the point q is homologous to $\sum_{i=1}^{g} A_i$.

For curves of type $(g, k, 1)$, a basis $\{A_i, B_i \mid i = 1, \dots, g\}$ of $H_1(X, \mathbb{Z})$ is called *real homology basis* if its elements have the following properties:

(a) $\tau[A_i] = A_i$, $\tau[B_i] = -B_i$ for $i = 1, \dots, k-1$, $\tau[A_i] = A_{i+m}$ and $\tau[B_i] = -B_{i+m}$ for $i = k, \dots, k+m-1$, $\tau[A_i] = A_{i-m}$ and $\tau[B_i] = -B_{i-m}$ for $i = k+m, \dots, g$, where $m := \frac{1}{2}(g+1-k)$.

(b) The oval containing the point q is homologous to $\sum_{i=1}^{k-1} A_i$.

We show next that such a real homology basis exists for every possible topological type (g, k, ε) of a real curve (X, τ). Hereby, we extended the proof given in [Natanzon, 2004] by also showing this assertion for $\varepsilon = 1$.

Lemma 6.29 ([Natanzon, 2004, Lemma 2.8.1])**.** *For every real curve of admissible type* (g, k, ε)*, a real homology basis exists.*

Proof. Let (X, τ) be a real curve of type $(g, k, 0)$. Then by Lemma 6.13, there is a set of pairwise disjoint simple closed contours $c_0, c_1, \dots, c_g$ such that $\tau[c_i] = c_i$, $X^\tau = \sum_{i=0}^{k-1} c_i$ and $X \setminus (\sum_{i=0}^{g} c_i)$ decomposes into two disjoint sets X^+ and X^-. Let us number the contours such that $q \in c_0$

and define $A_i := [c_i] \in H_1(X, \mathbb{Z})$ for $i = 1, \ldots, g$. So let $\gamma_i \subset X^+$ join $q \in c_0$ and a point $p_i \in c_i$. We set $B_i := [\gamma_i - \tau_\sharp \gamma_i] \in H_1(X, \mathbb{Z})$ for $i = 1, \ldots, k-1$ and for $i = k, \ldots, g$, we set $B_i := [\gamma_i + r_i - \tau_\sharp \gamma_i] \in H_1(X, \mathbb{Z})$, where $r_i \subset c_i$ joins p_i and $\tau(p_i)$. Similar as in the proof of Lemma B.16, this yields a basis of $H_1(X, \mathbb{Z})$. By definition, these cycles obey condition (a) for $\varepsilon = 0$. Since the boundary of X^+ is homologous to zero, the oval c_0 containing q is homologous to $\sum_{i=1}^{g} c_i$, so the same holds for the corresponding equivalence classes A_i.
The case $(g, k, 1)$ can be treated similarly. Let again $q \in c_0$. Then for $i = 1, \ldots, k-1$, the cycles $A_i \in H_1(X, \mathbb{Z})$ are the equivalence classes of ovals $c_1, \ldots, c_{k-1}$ of τ and the cycles B_i are constructed in exactly the same manner as for $\varepsilon = 0$. Furthermore, $X \setminus X^\tau$ decomposes into two disjoint compact Riemann surfaces X^+ and X^- with boundary $\sum_{j=0}^{k-1} c_j$, whereby each is of genus $m = (g+1-k)/2$ and $\tau[X^+] = X^-$, compare Example 6.7. Each of these two surfaces has a symplectic cycle basis given by $\{A_i^+, B_i^+, C_j^+ \mid i = 1, \ldots, m \text{ and } j = 1, \ldots, k-1\}$ for X^+ and $\{A_i^-, B_i^-, C_j^- \mid i = 1, \ldots, m \text{ and } j = 1, \ldots, k-1\}$ for X^-, where the C-cycles correspond to the boundary components of X^+ respectively X^-. We enumerate the A- and B-cycles of this basis in such a way that $\tau(A_i^+) = A_i^-$ and $\tau(B_i^+) = B_i^-$. The A- and B-cycles of these bases can be interpreted as elements of a real symplectic basis of $H_1(X, \mathbb{Z})$: We set $A_{k-1+i} := A_i^+$, $B_{k-1+i} := B_i^+$, $A_{k-1+m+i} := A_i^-$ and $B_{k-1+m+i} := B_i^-$ for $i = 1, \ldots, m$. These elements obey the relations for a symplectic basis: obviously, it is $A_i \star B_j = \delta_{i,j}$ and all other intersections are zero. Moreover, this basis obeys by definition property (a) in Definition 6.28 and the boundary of X^+ equals $\sum_{i=0}^{k-1} c_i$. So c_0 is homologous to $\sum_{i=1}^{k-1} c_i$. Thus, the same holds for the corresponding equivalence classes in $H_1(X, \mathbb{Z})$. □

From now on, we will make no notation difference anymore whether we consider the equivalence classes of paths in $H_1(X, \mathbb{Z})$ or the paths as representants of an equivalence class and write A_i, B_i respectively C_j for both. Furthermore, we assume in the rest of this section that all considered homology bases are real. To express how complex conjugation acts on Ω_{A_j} and Ω_{B_j} for $\varepsilon = 1$, it is convenient to define the $g \times g$-matrix

$$M := \begin{pmatrix} \mathbb{1}_{(k-1)\times(k-1)} & 0 & 0 \\ 0 & 0 & \mathbb{1}_{m\times m} \\ 0 & \mathbb{1}_{m\times m} & 0 \end{pmatrix}, \tag{6.3}$$

where $m = \frac{1}{2}(g+1-k)$. We will see on the action of τ^* on the basis of the holomorphic differential forms normalized as in (6.2) that this is necessary. This is not mentioned in [Natanzon, 2004]. Therefore, we modified and extended the version of the following lemma in comparison to [Natanzon, 2004].

Lemma 6.30 (Extension of [Natanzon, 2004, Lemma 2.8.2]). *Let (X, τ) be a real curve of type (g, k, ε), $\{A_i, B_i \mid i = 1, \ldots, g\}$ a real basis of $H_1(X, \mathbb{Z})$ and let $\omega_1, \ldots, \omega_g$ be the basis of* *holomorphic differential forms on (X,τ) which are normalized with respect to $A_1,\dots,A_g$ as in (6.2).*

(a) For $\varepsilon=0$, there holds $\tau^*\bar\omega_j=-\omega_j$ as well as $\overline{\Omega}_{A_j}=-\Omega_{A_j}$ for $j=1,\dots,g$ and

$$\overline{\Omega}_{B_j}=\begin{cases}\Omega_{B_j} & \text{for } j=1,\dots,k-1,\\ \Omega_{B_j}-\Omega_{A_j} & \text{for } j=k,\dots,g.\end{cases}$$

(b) For $\varepsilon=1$, one has with $m=\frac{1}{2}(g+1-k)$ that

$$\tau^*\bar\omega_j=\begin{cases}-\omega_j & \text{for } j=1,\dots,k-1,\\ -\omega_{j+m} & \text{for } j=k,\dots,k+m-1,\\ -\omega_{j-m} & \text{for } j=k+m,\dots,g,\end{cases}$$

$$\overline{\Omega}_{A_j}=-\Omega_{A_j}=\begin{cases}-M\cdot\Omega_{A_j} & \text{for } j=1,\dots,k-1,\\ -M\cdot\Omega_{A_{j+m}} & \text{for } j=k,\dots,k+m-1,\\ -M\cdot\Omega_{A_{j-m}} & \text{for } j=k+m,\dots,g\end{cases}$$

and

$$\overline{\Omega}_{B_j}=\begin{cases}M\cdot\Omega_{B_j} & \text{for } j=1,\dots,k-1,\\ M\cdot\Omega_{B_{j+m}} & \text{for } j=k,\dots,k+m-1,\\ M\cdot\Omega_{B_{j-m}} & \text{for } j=k+m,\dots,g.\end{cases}$$

Proof. We first show the transformation behavior of the 1-forms which follows immediately from the fact that the chosen basis of $H_1(X,\mathbb{Z})$ is real: Since τ and $\bar\omega_i$ are both antiholomorphic, $\tau^*\bar\omega_i$ is a holomorphic 1- form and can be written as a linear combination of the chosen basis of holomorphic 1-forms as

$$\tau^*\bar\omega_j=\sum_{l=1}^{g}c_l\omega_l,$$

where $c_l\in\mathbb{C}$. For $\varepsilon=0$ with $i,j=1,\dots,g$ and for $\varepsilon=1$ with $j=1,\dots,k-1$ and $i=1,\dots,g$, it is

$$\oint_{A_j}\tau^*\bar\omega_i=\oint_{\tau_\sharp A_j}\bar\omega_i=\overline{\oint_{A_j}\omega_i}=-2\pi\iota\delta_{ij}=-\oint_{A_j}\omega_i,$$

so $c_l=0$ for $l\neq j$ and $c_j=-1$. For $\varepsilon=1$, $j=k,\dots,g+m-1$ and $i=1,\dots,g$, one has

$$\oint_{A_j}\tau^*\bar\omega_i=\oint_{\tau_\sharp A_j}\bar\omega_i=\overline{\oint_{A_{j+m}}\omega_i}=-2\pi\iota\delta_{(j+m),i}=-\oint_{A_{j+m}}\omega_i,$$

so $c_l=0$ for $l\neq j+m$ and $c_{j+m}=-1$. Analogously one obtains for $\varepsilon=1$ and $j=k+m,\dots,g$

that $c_l = 0$ for $l \neq j-m$ and $c_{j-m} = -1$. So the asserted action of τ^* on $\bar{\omega}_i$ follows. Obviously, because $\Omega_{A_j} = 2\pi\iota e^{(j)}$, where $e^{(j)}$ is the j-th unit vector $(0,\dots,0,1,0,\dots,0)$ in $\mathbb{C}^g$, it is $\overline{\Omega}_{A_j} = -\Omega_{A_j}$ for $\varepsilon = 0$. Likewise we obtain for $\varepsilon = 1$ and $j = 1,\dots,k-1$, that

$$\overline{\Omega}_{A_j} = \left(\oint_{A_j} \bar{\omega}_i\right)_{i=1}^g = \left(\oint_{\tau_\sharp A_j} \tau^*\bar{\omega}_i\right)_{i=1}^g = \left(\oint_{A_j} \tau^*\bar{\omega}_i\right)_{i=1}^g = \begin{pmatrix} -\oint_{A_j} \omega_i|_{i=1}^{k-1} \\ -\oint_{A_j} \omega_i|_{i=k+m}^{g} \\ -\oint_{A_j} \omega_i|_{i=k}^{k+m-1} \end{pmatrix} = -M \cdot \Omega_{A_j},$$

where the first equality holds since applying $\tau_\sharp$ to the path of integration and τ^* on the differential form simultaneously leaves the value of the integral invariant. For $j = k,\dots,k+m-1$, it is

$$\overline{\Omega}_{A_j} = \left(\oint_{A_j} \tau^*\bar{\omega}_i\right)_{i=1}^g = \begin{pmatrix} -\oint_{A_{j+m}} \omega_i|_{i=1}^{k-1} \\ -\oint_{A_{j+m}} \omega_i|_{i=k+m}^{g} \\ -\oint_{A_{j+m}} \omega_i|_{i=k}^{k+m-1} \end{pmatrix} = -M \cdot \Omega_{A_{j+m}}.$$

Analogously $\overline{\Omega}_{A_j} = -M \cdot \Omega_{A_{j-m}}$ for $j = k+m,\dots,g$. Furthermore,

$$\overline{\Omega}_{B_j} = \left(\overline{\oint_{B_j} \omega_i}\right)_{i=1}^g = \left(\oint_{B_j} \bar{\omega}_i\right)_{i=1}^g = \left(\oint_{\tau_\sharp B_j} \tau^*\bar{\omega}_i\right)_{i=1}^g .$$

So the transformation behavior of a real symplectic cycle basis of $H_1(X,\mathbb{Z})$ yields for $\varepsilon = 0$ and $j = 1,\dots,k-1$

$$\overline{\Omega}_{B_j} = \left(\oint_{-B_j} -\omega_i\right)_{i=1}^g = \left(\oint_{B_j} \omega_i\right)_{i=1}^g = \Omega_{B_j}$$

and for $j = k,\dots,g$

$$\overline{\Omega}_{B_j} = \left(\oint_{-B_j+A_j} -\omega_i\right)_{i=1}^g = \left(\oint_{B_j} \omega_i - \oint_{A_j} \omega_i\right)_{i=1}^g = \Omega_{B_j} - \Omega_{A_j}.$$

For $\varepsilon = 1$, the transformation behavior of a real symplectic cycle basis of $H_1(X,\mathbb{Z})$ gives for $j = 1,\dots,k-1$ that

$$\overline{\Omega}_{B_j} = \begin{pmatrix} \oint_{-B_j} -\omega_i\,|_{i=1}^{k-1} \\ \oint_{-B_j} -\omega_{i+m}\,|_{i=k}^{k+m-1} \\ \oint_{-B_j} -\omega_{i-m}\,|_{i=k+m}^{g} \end{pmatrix} = \begin{pmatrix} \oint_{B_j} \omega_i\,|_{i=1}^{k-1} \\ \oint_{B_j} \omega_i\,|_{i=k+m}^{g} \\ \oint_{B_j} \omega_i\,|_{i=k}^{k+m-1} \end{pmatrix} = M \cdot \Omega_{B_j}.$$

Because of $\tau(B_j) = B_{j-m}$, it is for $j = k,\dots,k+m-1$

$$\overline{\Omega}_{B_j} = \begin{pmatrix} \oint_{-B_{j+m}} -\omega_i\,|_{i=1}^{k-1} \\ \oint_{-B_{j+m}} -\omega_{i+m}\,|_{i=k}^{k+m-1} \\ \oint_{-B_{j+m}} -\omega_{i-m}\,|_{i=k+m}^{g} \end{pmatrix} = \begin{pmatrix} \oint_{B_{j+m}} \omega_i\,|_{i=1}^{k-1} \\ \oint_{B_{j+m}} \omega_i\,|_{i=k+m}^{g} \\ \oint_{B_{j+m}} \omega_i\,|_{i=k}^{k+m-1} \end{pmatrix} = M \cdot \Omega_{B_{j+m}}$$

and $\overline{\Omega}_{B_j} = M \cdot \Omega_{B_{j-m}}$ for $j = k+m,\dots,g$. □

We denote the natural involution $S_g \to S_g$ implied by τ also by τ. This is defined at the beginning of Section 4.2. For both, $\varepsilon = 0$ and $\varepsilon = 1$, we consider an involution $\tau_{\mathbb{R}} : \mathbb{C}^g \to \mathbb{C}^g$ which is defined by its action on the basis $\{\Omega_{A_i}, \Omega_{B_i} \mid i = 1, \dots, g\}$ of the space $\mathbb{R}^{2g} \simeq \mathbb{C}^g$. For $\varepsilon = 0$, we define this involution via the $\mathbb{R}$-linear map

$$\Omega_{A_j} \mapsto -\overline{\Omega}_{A_j} = \Omega_{A_j} \text{ for } j = 1, \dots, g, \quad \Omega_{B_j} \mapsto -\overline{\Omega}_{B_j} = \begin{cases} -\Omega_{B_j} & \text{for } j = 1, \dots, k-1, \\ -\Omega_{B_j} + \Omega_{A_j} & \text{for } j = k, \dots, g \end{cases} \tag{6.4}$$

and for $\varepsilon = 1$ by the $\mathbb{R}$-linear map

$$\begin{aligned} \Omega_{A_j} &\mapsto -M \cdot \overline{\Omega}_{A_j} = \begin{cases} \Omega_{A_j} & \text{for } j = 1, \dots, k-1, \\ \Omega_{A_{j+m}} & \text{for } j = k, \dots, k+m-1, \\ \Omega_{A_{j-m}} & \text{for } j = k+m, \dots, g, \end{cases} \\ \Omega_{B_j} &\mapsto -M \cdot \overline{\Omega}_{B_j} = \begin{cases} -\Omega_{B_j} & \text{for } j = 1, \dots, k-1, \\ -\Omega_{B_{j+m}} & \text{for } j = k, \dots, k+m-1, \\ -\Omega_{B_{j-m}} & \text{for } j = k+m, \dots, g, \end{cases} \end{aligned} \tag{6.5}$$

where the last equality as well as the property that $\tau_{\mathbb{R}} : \mathbb{C}^g \to \mathbb{C}^g$ is an antiholomorphic involution follows from $M^2 = \mathbb{1}$.

Proposition 6.31. *Let (X, τ) be a real curve of genus g and $D \in S_g$. Then there exists an unique antiholomorphic involution $\tau_{\mathbb{R}} : \mathrm{Jac}(X) \to \mathrm{Jac}(X)$ such that $\mathrm{Ab}_q(\tau(D)) = \tau_{\mathbb{R}}(\mathrm{Ab}_q(D))$ for $D \in S_g$. Let $x \in \mathrm{Jac}(X)$. Then for $\varepsilon = 0$, this map is given by $\tau_{\mathbb{R}}(x) = -\bar{x}$ and for $\varepsilon = 1$ by $\tau_{\mathbb{R}}(x) = -M \cdot \bar{x}$.*

Proof. It is

$$\widetilde{\mathrm{Ab}}_q(\tau(D)) = \sum_{p \in D} \left(\int_{\tau_\sharp \gamma_{qp}} \omega_i \right)_{i=1}^g = \sum_{p \in D} \left(\int_{\gamma_{qp}} \tau^* \omega_i \right)_{i=1}^g .$$

Hence, Lemma 6.30(a) implies for $\varepsilon = 0$ that

$$\widetilde{\mathrm{Ab}}_q(\tau(D)) = \sum_{p \in D} \left(-\overline{\int_{\gamma_{qp}} \omega_i} \right)_{i=1}^g = -\overline{\widetilde{\mathrm{Ab}}_q(D)}.$$

For $\varepsilon = 1$, Lemma 6.30(b) yields

$$\widetilde{\mathrm{Ab}}_q(\tau(D)) = \sum_{p \in D} \begin{pmatrix} -\overline{\int_{\gamma_{qp}} \omega_i} \mid_{i=1}^{k-1} \\ -\overline{\int_{\gamma_{qp}} \omega_{i+m}} \mid_{i=k}^{k+m-1} \\ -\overline{\int_{\gamma_{qp}} \omega_{i-m}} \mid_{i=k+m}^{g} \end{pmatrix} = -M \cdot \overline{\widetilde{\mathrm{Ab}}_q(D)}.$$

So the involution $\tau_{\mathbb{R}}$ on $\mathbb{C}^g$ is defined as claimed, but it still depends on the chosen path γ_{qp}. In

(6.4) and (6.5), we have seen that $\tau_{\mathbb{R}}[\Lambda] \subset \Lambda$, so $\tau_{\mathbb{R}} : \mathbb{C}^g \to \mathbb{C}^g$ induces an involution on $\mathrm{Jac}(X)$ which we also denote by $\tau_{\mathbb{R}} : \mathrm{Jac}(X) \to \mathrm{Jac}(X)$. □

Definition 6.32. The *real part* or *real points* $\mathrm{Jac}_{\mathbb{R}}(X)$ *of the Jacobian variety* $\mathrm{Jac}(X)$ of the curve (X, τ) is defined as the set of fixed points of the involution $\tau_{\mathbb{R}}$ on $\mathrm{Jac}(X)$.

Theorem 6.33 ([Natanzon, 2004, Theorem 2.8.1])**.** *The real part* $\mathrm{Jac}_{\mathbb{R}}(X)$ *of the Jacobian variety of a real curve* (X, τ) *of type* (g, k, ε) *with* $k > 0$ *decomposes into* 2^{k-1} *real tori of the form* $\Phi(T_{\mathbb{R}} + \delta)$*, where*

$$\delta = \frac{1}{2} \sum_{j=1}^{k-1} \delta_j \Omega_{B_j}, \quad \delta_j \in \{0, 1\}, \tag{6.6}$$

$T_{\mathbb{R}} = \iota\mathbb{R}^g$ *for* $\varepsilon = 0$ *and for* $\varepsilon = 1$

$$T_{\mathbb{R}} = \{(x_1, \ldots, x_g) \in \mathbb{C}^g \mid x_j \in \iota\mathbb{R} \text{ for } 1 \le j \le k-1,\ \bar{x}_j = -x_{j+m} \text{ for } k \le j \le k+m-1\}.$$

Such a torus is non-singular in sense of Definition 6.27 if and only if $\varepsilon = 1$*,* $k = g + 1$ *and* $\delta_1 = \cdots = \delta_g = 1$*.*

In the proof of this theorem given in [Natanzon, 2004], most steps shown here are completely missing. Neither the exact form of the real tori $T_{\mathbb{R}}$ is shown there nor an argumentation why there is only one possibility for a non-singular torus is given in a way that we understand it. So we extended that proof in hope that these steps are now more comprehensible.

Proof. The fixed points of $\tau_{\mathbb{R}}$ on $\mathrm{Jac}(X)$ are a submanifold of real dimension g, compare [Schneps, 2003, Lemma 3.5]. Let $\varepsilon = 0$. Then the fixed points of $\tau_{\mathbb{R}}$ are spanned by the basis elements Ω_{A_j} since $\tau_{\mathbb{R}}$ leaves $\iota\mathbb{R}^g$ invariant. Due to $\tau(q) = q$ and Lemma 6.30(a), it is

$$\left(\textstyle\int_{\gamma_{qp}} \omega_i + \int_{\tau_\sharp \gamma_{qp}} \omega_i,\right)_{i=1}^{g} = \left(\textstyle\int_{\gamma_{qp}} (\omega_i + \tau^* \omega_i)\right)_{i=1}^{g} = \left(\textstyle\int_{\gamma_{qp}} (\omega_i - \overline{\omega}_i)\right)_{i=1}^{g} \in T_{\mathbb{R}}.$$

Recall that the ovals of (X, τ) are given by $\{c_1, \ldots, c_k\}$ and that these are representants of the A-cycles $A_1, \ldots, A_k$. One sees in the construction of the B-cycles of a real homology basis as in the proof of Lemma 6.29 that there is a $p' \in c_j$ such that $B_j = \gamma_{qp'} - \tau_\sharp \gamma_{qp'}$ for $j = 1, \ldots, k-1$. Since

$$\left(\int_{\gamma_{qp'}} \omega_i\right)_{i=1}^{g} = \mathrm{Re}\left(\int_{\gamma_{qp'}} \omega_i\right)_{i=1}^{g} + \iota\,\mathrm{Im}\left(\int_{\gamma_{qp'}} \omega_i\right)_{i=1}^{g}$$

and

$$\begin{aligned}\mathrm{Re}\left(\textstyle\int_{\gamma_{qp'}} \omega_i\right)_{i=1}^{g} &= \frac{1}{2}\left(\textstyle\int_{\gamma_{qp'}} \omega_i + \overline{\textstyle\int_{\gamma_{qp'}} \omega_i}\right)_{i=1}^{g} = \frac{1}{2}\left(\textstyle\int_{\gamma_{qp'}} \omega_i + \int_{\tau_\sharp \gamma_{qp'}} \tau^* \bar{\omega}_i\right)_{i=1}^{g} \\ &= \frac{1}{2}\left(\textstyle\int_{\gamma_{qp'}} \omega_i - \int_{\tau_\sharp \gamma_{qp'}} \omega_i\right)_{i=1}^{g} = \frac{1}{2}\left(\textstyle\int_{\gamma_{qp'} - \tau_\sharp \gamma(qp')} \omega_i\right)_{i=1}^{g} = \frac{1}{2}\Omega_{B_j},\end{aligned}$$

it is $\left(\int_{\gamma_{qp'}}\omega_i\right)_{i=1}^g = \frac{1}{2}\Omega_{B_j} + T_{\mathbb{R}}$. To see that the same holds for arbitrary $p \in c_j$ with $j = 0, \ldots, k-1$, let $p, p' \in c_j$ and $\gamma_{pp'}$ be one of the two paths on c_j connecting p and p'. Then $\mathrm{Re}(\int_{\gamma_{p'p}}\omega_i)_{i=1}^g = 0$ because $\tau = \mathbb{1}$ on c_j and thus Lemma 6.30(a) yields that $\omega_i|_{c_j} = -\bar{\omega}_i|_{c_j}$ for $i = 1, \ldots, g$. Again, due to Lemma 6.30 (a), one has $\left(\int_{\gamma_{pq}}\omega_i\right)_{i=1}^g \in \frac{1}{2}\Omega_{B_j} + T_{\mathbb{R}}$ for arbitrary $p \in c_j$ with $j = 1, \ldots, k-1$ and $\left(\int_{\gamma_{pq}}\omega_i\right)_{i=1}^g \in T_{\mathbb{R}}$ for $j = 0$.

The determination of the fixed point set of $\tau_{\mathbb{R}}$ in the case $\varepsilon = 1$ follows also directly from the definition of $\tau_{\mathbb{R}}$: since $\tau_{\mathbb{R}}$ leaves Ω_{A_i} invariant for $i = 1, \ldots, k-1$ and maps the j-th coordinate to the negative and complex conjugate of the $(j+k)$-th coordinate for $j = k, \ldots, k+m-1$, $T_{\mathbb{R}}$ is given as above. For $\varepsilon = 1$, Lemma 6.30(b) gives for a fixed path γ_{qp} that

$$\left(\int_{\gamma_{qp}}\omega_i + \int_{\tau_\sharp\gamma_{qp}}\omega_i,\right)_{i=1}^g = \left(\int_{\gamma_{qp}}\omega_i + \tau^*\omega_i\right)_{i=1}^g =$$

$$= \begin{pmatrix} \int_{\gamma_{qp}}\omega_i - \overline{\omega}_i \,|_{i=1}^{k-1} \\ \int_{\gamma_{qp}}\omega_i - \overline{\omega}_{i+m} \,|_{i=k}^{k+m-1} \\ \int_{\gamma_{qp}}\omega_i - \overline{\omega}_{i-m} \,|_{i=k+m}^{g} \end{pmatrix} = \begin{pmatrix} \int_{\gamma_{qp}}\omega_i - \overline{\omega}_i \,|_{i=1}^{k-1} \\ \int_{\gamma_{qp}}\omega_i - \overline{\omega}_{i+m} \,|_{i=k}^{k+m-1} \\ -\overline{\int_{\gamma_{qp}}\omega_i - \overline{\omega}_{i+m}} \,|_{i=k}^{k+m-1} \end{pmatrix} \in T_{\mathbb{R}}.$$

Due to the action of τ on the A-cycles shown in Lemma 6.30(b), it is $\omega_i|_{A_j} = -\bar{\omega}_i|_{A_j}$ for $j = 1, \ldots, k-1$, $\omega_i|_{A_j} = -\bar{\omega}_{i+m}|_{A_j}$ for $i = k, \ldots, k+m-1$ and $\omega_i|_{A_j} = -\bar{\omega}_{i-m}|_{A_j}$ for $i = k+m, \ldots, g$. Hence, for $p, p' \in a_j$ with $j = 0, \ldots, k-1$, $(\int_{\gamma_{p'p}}\omega_i)_{i=1}^g = 0 + T_{\mathbb{R}}$. So also in this case $\left(\int_{\gamma_{qp}}\omega_i\right)_{i=1}^g$ does not depend on the chosen point $p \in c_j$ and for arbitrary $p \in c_j$, it is

$$\left(\int_{\gamma_{qp}}\omega_i\right)_{i=1}^g = \frac{1}{2}\left(\int_{\gamma_{qp}}\omega_i + \int_{\tau_\sharp\gamma_{qp}}\tau^*\omega_i\right)_{i=1}^g = \frac{1}{2}\begin{pmatrix} \left(\int_{\gamma_{qp}}\omega_i - \int_{\tau_\sharp\gamma_{qp}}\bar{\omega}_i\right) \Big|_{i=1}^{k-1} \\ \left(\int_{\gamma_{qp}}\omega_i - \int_{\tau_\sharp\gamma_{qp}}\bar{\omega}_{i+m}\right) \Big|_{i=k}^{k+m-1} \\ \left(\int_{\gamma_{qp}}\omega_i - \int_{\tau_\sharp\gamma_{qp}}\bar{\omega}_{i-m}\right) \Big|_{i=k+m}^{g} \end{pmatrix} =$$

$$= \frac{1}{2}\begin{pmatrix} \left(\int_{\gamma_{qp}}\omega_i + \int_{-\tau_\sharp\gamma_{qp}}\omega_i + \int_{\tau_\sharp\gamma_{qp}}(\omega_i - \bar{\omega}_i)\right) \Big|_{i=1}^{k-1} \\ \left(\int_{\gamma_{qp}}\omega_i + \int_{-\tau_\sharp\gamma_{qp}}\omega_i + \int_{\tau_\sharp\gamma_{qp}}(\omega_i - \bar{\omega}_{i+m})\right) \Big|_{i=k}^{k+m-1} \\ \left(\int_{\gamma_{qp}}\omega_i + \int_{-\tau_\sharp\gamma_{qp}}\omega_i + \int_{\tau_\sharp\gamma_{qp}}(\omega_i - \bar{\omega}_{i-m})\right) \Big|_{i=k+m}^{g} \end{pmatrix} =$$

$$= \frac{1}{2}\Omega_{B_j} + \frac{1}{2}\begin{pmatrix} \int_{\tau_\sharp\gamma_{qp}}(\omega_i - \bar{\omega}_i) \,|_{i=1}^{k-1} \\ \int_{\tau_\sharp\gamma_{qp}}(\omega_i - \bar{\omega}_{i+m}) \,|_{i=k}^{k+m-1} \\ -\overline{\int_{\tau_\sharp\gamma_{qp}}(\omega_i - \bar{\omega}_{i+m})} \,|_{i=k}^{k+m-1} \end{pmatrix},$$

so again $\left(\int_{\gamma_{qp}}\omega_i\right)_{i=1}^g = \frac{1}{2}\Omega_{B_j} + T_{\mathbb{R}}$. We define for $\delta_j \in \{0,1\}$ and $j = 1, \ldots, k-1$

$$R_\delta := \{D \in S_g \mid \tau D = D \text{ and } \deg(D \cap c_j) = \delta_j \mod 2 \text{ for } j = 1, \ldots, k-1\}.$$

We show next that $x \in \Phi(T_{\mathbb{R}} + \delta)$ if and only if $x = \mathrm{Ab}_q(D)$ with $D \in R_\delta$. Let $x = \mathrm{Ab}_q(D)$ with $D \in R_\delta$. Every divisor $D \in R_\delta$ can contain two different kinds of points: either points contained in the ovals c_i for $i = 0, \dots, k-1$ or points on $X \setminus X^\tau$. The equality $\tau(D) = D$ implies that $p \in D \cap (X \setminus X^\tau)$ if and only if $\tau(p) \in D \cap (X \setminus X^\tau)$. Accordingly, every divisor D can be decomposed as

$$D = \widetilde{D} + \tau(\widetilde{D}) + D'$$

where $D' = D \cap \sum_{i=0}^{k-1} c_i$ and $\mathrm{supp}(\widetilde{D})$, $\mathrm{supp}(\tau(\widetilde{D})) \in (X \setminus X^\tau)$. It follows from the above calculations that $x \in \Phi(T_{\mathbb{R}} + \delta)$ since for $p \in D \cap c_0$, it is $\left(\int_{\gamma_{qp}} \omega_i\right)_{i=1}^g \in T_{\mathbb{R}}$, and so

$$x = \underbrace{\sum_{p \in \tilde{D}} \left(\int_{\gamma_{qp}} \omega_i + \int_{\tau_\sharp \gamma_{qp}} \omega_i \right)_{i=1}^g}_{\in T_{\mathbb{R}}} + \underbrace{\sum_{p \in D'} \left(\int_q^p \omega_i \right)_{i=1}^g}_{\in \delta + T_{\mathbb{R}}} \mod \Lambda \in \Phi(T_{\mathbb{R}} + \delta),$$

which shows that $\mathrm{Ab}_q[R_\delta] \subset \Phi(T_{\mathbb{R}} + \delta)$.

To see that the other inclusion also holds, we show that $\mathrm{Ab}_q|_{R_\delta} : R_\delta \to \Phi(T_{\mathbb{R}} + \delta)$ is surjective. This is done in two steps. Remember that the compactness of X yields that S_g is also a compact, complex manifold, compare [Farkas and Kra, 2012, Section III.11.9]. Every closed subset of a compact set is compact and Ab_q is continuous, so the image of any closed subset of S_g is also closed in $\mathrm{Jac}(X)$. Moreover, the compactness of S_g yields that the restriction of Ab_q to S_g is a closed map. So to see that the image of R_δ under Ab_q is closed in $\Phi(T_{\mathbb{R}} + \delta)$, it is necessary to show that R_δ is a closed subset of S_g: For this purpose, let $(D_n)_{n \in \mathbb{N}}$ be a sequence such that $D_n \in R_\delta$ for every $n \in \mathbb{N}$ which converges in S_g. Since there are only finitely many ovals and the support of D_n is also finite for every $n \in \mathbb{N}$, there are only finitely many possibilities for the distribution of the divisor points on $D_n \cap c_i$ such that δ_i does not change. So we can find a convergent subsequence $(D_m)_{m \in \mathbb{N}}$ of $(D_n)_{n \in \mathbb{N}}$ such that the number of divisor points on all ovals is constant. So δ_i remains constant on all D_m with $m \in \mathbb{N}$. Furthermore, the limit of this subsequence restricted to c_i, i.e. the limit of $(D_m \cap c_i)_{m \in \mathbb{N}}$, is contained in c_i because the ovals are compact. Accordingly, δ_i also remains constant in the limit of this sequence. Since τ is continuous on X, and so also continuous on S_g, compare [Farkas and Kra, 2012, Proposition III.11.9], $\tau(D) = \tau(\lim_{m \to \infty} D_m) = \lim_{m \to \infty} \tau(D_m) = \lim D_m = D$. So R_δ is compact, and therefore closed in S_g.

Secondly, we show that $\mathrm{Ab}_q[R_\delta \cap (S_g \setminus S_g^1)]$ is dense in $\Phi(T_{\mathbb{R}} + \delta) \cap (W_g \setminus W_g^1)$. Note that $W_g \setminus W_g^1$ is open and dense in $W_g = \mathrm{Jac}(X)$. We use this to show that also $\widetilde{\mathrm{Jac}_{\mathbb{R}}}(X) := \mathrm{Jac}_{\mathbb{R}}(X) \cap (W_g \setminus W_g^1)$ is non-empty and dense in $\mathrm{Jac}_{\mathbb{R}}(X)$. Moreover, we exploit that W_g^1 equals the zeros of the θ-divisor on $\mathrm{Jac}(X)$. Due to [Schneps, 2003, Lemma 3.5], $\mathrm{Jac}_{\mathbb{R}}(X)$ is a non-empty, connected real submanifold of $\mathrm{Jac}(X)$. Let us assume that $\theta(x + K_q)|_{\mathrm{Jac}_{\mathbb{R}}(X)} \equiv 0$. By [Farkas and Kra, 2012, Theorem VI.3.1], the values x are then contained in W_{g-1}, and therefore describe the special divisors. Since $\theta : \mathrm{Jac}(X) \to \mathbb{C}$ is holomorphic, see [Farkas and Kra, 2012, VI.2.3], it can be represented as a Taylor series, where all coefficients vanish identically in every open neighborhood in $\mathrm{Jac}_{\mathbb{R}}(X)$ around any point $p \in \mathrm{Jac}_{\mathbb{R}}(X)$. Taking open neighborhoods of $p \in \mathrm{Jac}_{\mathbb{R}}(X)$ in $\mathrm{Jac}(X)$ yields that also all Taylor coefficients of θ vanish on this open neighborhood of p. Hence, also the restriction of θ to this open neighborhood in $\mathrm{Jac}(X)$ is identically zero. Then, by the identical principle for holomorphic functions in several variables, $\theta \equiv 0$ on $\mathrm{Jac}(X)$. However, $\theta \not\equiv 0$ on $\mathrm{Jac}(X)$ and thus θ does also not vanish on any connected component of $\mathrm{Jac}_{\mathbb{R}}(X)$. So the set $\widetilde{\mathrm{Jac}_{\mathbb{R}}}(X)$ is open and dense in $\mathrm{Jac}_{\mathbb{R}}(X)$ and there exist non-special points in $\Phi(T_{\mathbb{R}} + \delta)$ for every admissible δ. Furthermore, $\mathrm{Ab}_q|_{S_g \setminus S_g^1} : S_g \setminus S_g^1 \to W_g \setminus W_g^1$ is an isomorphism, and therefore injective, see [Farkas and Kra, 2012, Proposition III.11.11(c)]. This yields that also $\mathrm{Ab}_q|_{R_\delta \cap (S_g \setminus S_g^1)}$ is injective. Let $x \in \widetilde{\Phi}(T_{\mathbb{R}} + \delta) := \Phi(T_{\mathbb{R}} + \delta) \cap (W_g \setminus W_g^1)$. Then there exists exactly one element $D \in S_g \setminus S_g^1$ such that $\mathrm{Ab}_q(D) = x$. Furthermore, the injectivity of $\mathrm{Ab}_q|_{R_\delta \cap (S_g \setminus S_g^1)}$ implies $\tau(D) = D$ for $x \in \tilde{\Phi}(T_{\mathbb{R}} + \delta)$ and $\mathrm{Ab}_q(D) = x$. Due to the definition of δ in (6.6), one has

$$\tau_{\mathbb{R}}(\delta) = \tau_{\mathbb{R}}\left(\frac{1}{2}\sum_{j=1}^{k-1} \Omega_{B_j}\delta_j\right) = \frac{1}{2}\sum_{j=1}^{k-1} \tau_{\mathbb{R}}(\Omega_{B_j})\delta_j = -\frac{1}{2}\sum_{j=1}^{k-1} \Omega_{B_j}\delta_j = -\delta = \delta \mod \Lambda.$$

Altogether we see that $D \in R_\delta \cap (S_g \setminus S_g^1)$. Thus, $\tilde{\Phi}(T_{\mathbb{R}} + \delta) \subset \mathrm{Ab}_q[R_\delta \cap (S_g \setminus S_g^1)]$ and $\Phi(T_{\mathbb{R}} + \delta)$ equals the closure $\overline{\mathrm{Ab}_q[R_\delta \cap (S_g \setminus S_g^1)]}$ of $\mathrm{Ab}_q[R_\delta \cap (S_g \setminus S_g^1)]$, where

$$\overline{\mathrm{Ab}_q[R_\delta \cap (S_g \setminus S_g^1)]} = \mathrm{Ab}_q[\overline{R_\delta \cap (S_g \setminus S_g^1)}] = \mathrm{Ab}_q[R_\delta].$$

Next, we show that only for $k = g + 1$, the torus $T_{\mathbb{R}} + \delta$ with $\delta = (1, \dots, 1)$ is non-singular. To do so, we first prove that $R_\delta \cap S_{g-1} = \emptyset$ if and only if $\mathrm{Ab}_q[R_\delta] \cap \mathrm{Ab}_q[S_{g-1}] = \emptyset$.

The first step of this is to show that $\mathrm{Ab}_q[R_\delta] \cap W_{g-1} = \emptyset$ if and only if $R_\delta \cap (S_{g-1} + q) = \emptyset$: Note that $\mathrm{Ab}_q(q) = 0$ implies $\mathrm{Ab}_q[R_\delta \cap (S_{g-1} + q)] \subset \mathrm{Ab}_q[R_\delta] \cap W_{g-1}$. In particular, $\mathrm{Ab}_q[R_\delta] \cap W_{g-1} = \emptyset$ implies $R_\delta \cap (S_{g-1} + q) = \emptyset$.

To see that the other implication also holds, we show the surjectivity of the map $\mathrm{Ab}_q|_{R_\delta \cap (S_{g-1}+q)} : R_\delta \cap (S_{g-1} + q) \to \mathrm{Ab}_q[R_\delta] \cap W_{g-1}$. This implies that $\mathrm{Ab}_q[R_\delta \cap (S_{g-1} + q)] \supset \mathrm{Ab}_q[R_\delta] \cap W_{g-1}$, wherefore emptiness of $R_\delta \cap (S_{g-1} + q)$ implies that also $\mathrm{Ab}_q[R_\delta] \cap W_{g-1} = \emptyset$. As in the proof of the surjectivity of $\mathrm{Ab}_q|_{R_\delta} : R_\delta \to \Phi(T_{\mathbb{R}} + \delta)$ above, one sees that $R_\delta \cap (S_{g-1} + q)$ is a closed subset of the compact set R_δ and hence also compact. So the image of $R_\delta \cap (S_{g-1} + q)$ under the continuous map Ab_q is compact, and therefore closed in $\mathrm{Ab}_q[R_\delta] \cap W_{g-1}$. We use that the set of singularities of W_{g-1} is – as a subvariety of $\mathrm{Jac}(X)$ – given by W_{g-1}^1 and $\mathrm{Ab}_q|_{S_{g-1} \setminus S_{g-1}^1}$ is an isomorphism to $W_{g-1} \setminus W_{g-1}^1$, compare [Farkas and Kra, 2012, Proposition III.11.11(c)]. Hence, the set of points $x \in W_{g-1}$ with preimage $D + q \in S_{g-1} + q$ under Ab_q such that $\dim H^1(X, \mathcal{O}_D) = 1$ is open and dense in W_{g-1} and again, the injectivity of $\mathrm{Ab}_q|_{S_{g-1} \setminus S_{g-1}^1}$ yields that also $\mathrm{Ab}_q|_{R_\delta \cap (S_{g-1} \setminus S_{g-1}^1)}$ is injective. Now, let $x = \tau(x) \in \mathrm{Ab}_q[R_\delta \cap (W_{g-1} \setminus W_{g-1}^1)]$. Due to the isomorphy of $S_{g-1} \setminus S_{g-1}^1$

and $W_{g-1} \setminus W^1_{g-1}$ via Ab_q, there exists a divisor $D = \mathrm{Ab}_q^{-1}(x) \in (S_{g-1} \setminus S^1_{g-1}) + q$. Since $\mathrm{Ab}_q(D) = x = \tau_{\mathbb{R}}(x) = \mathrm{Ab}_q(\tau(D))$, the injectivity of Ab_q on $S_{g-1} \setminus S^1_{g-1}$ yields $\tau(D) = D$, and so $\tau(S_{g-1} + q) \subset S_{g-1} + q$. Furthermore, $D \in R_\delta \cap (S_{g-1} + q)$ implies that $\tau(D) \in R_\delta \cap (S_{g-1} + q)$. Due to the closeness of $\mathrm{Ab}_q[R_\delta \cap (S_{g-1} + q)]$ in $\mathrm{Ab}_q[R_\delta] \cap W_{g-1}$ and since $\mathrm{Ab}_q[R_\delta] \cap (W_{g-1} \setminus W^1_{g-1})$ is dense in W_{g-1}, it follows as in the step before that $\mathrm{Ab}_q[R_\delta] \cap W_{g-1} \subset \mathrm{Ab}_q[R_\delta \cap (S_{g-1} + q)]$. Thus, $\mathrm{Ab}_q|_{R_\delta \cap (S_{g-1}+q)}$ is surjective to $\mathrm{Ab}_q[R_\delta] \cap W_{g-1}$. Altogether we see that $R_q(S_{g-1} + q) = \emptyset$ implies that $\mathrm{Ab}_q[R_\delta] \cap W_{g-1} = \emptyset$.
Finally, we show that $R_\delta \cap (S_{g-1} + q) = \emptyset$ if and only if

$$\sum_{i=1}^{k-1} \delta_i > g - 1,$$

i.e. if and only if $k = g + 1$ and $\delta_1 = \cdots = \delta_g = 1$. That $R_\delta \cap S_{g-1} = \emptyset$ for $k = g + 1$ and $\delta_1 = \cdots = \delta_g = 1$ is obvious because in this case all divisor points are contained in $c_1, \ldots, c_g$. This means that $q \notin D$, and so $D \notin S_{g-1} + q$. On the other hand, let $R_\delta \cap (S_{g-1} + q) = \emptyset$. Then no divisor in R_δ contains q. To avoid the existence of such a divisor in R_δ via linear equivalence, for all divisors in R_δ must hold $\operatorname{supp} D \in \sum_{j=i}^{k-1} c_i$. This is only ensured for $k = g + 1$ and $\delta_1 = \cdots = \delta_g = 1$. □

6.2.5. Prym varieties of real curves

From now on, we consider real curves which are additionally equipped with a holomorphic involution. The next two definitions can be found at the beginning of [Natanzon, 2004, Section 9.1], whereas the two next Lemmata and the corollary afterwards are implicitly claimed to be valid, but not shown.

Definition 6.34. A *real curve with involution* (X, τ_1, σ) is a compact Riemann surface X of genus $2g$ with an antiholomorphic involution τ_1 and a holomorphic involution σ such that $\tau_1 \circ \sigma = \sigma \circ \tau_1$ and such that σ has exactly two fixed points Q^+ and Q^- with $\tau_1(Q^+) = Q^-$. We set $\tau_2 := \tau_1 \circ \sigma$.

Let X be a real curve with involution (X, τ_1, σ) of genus $2g$ as in Definition 6.34. We assume that among the ovals of the involution τ_i with $i \in \{1, 2\}$, there are r_i ovals that are invariant with respect to σ and $2t_i$ ovals that are pairwise transposed by σ. These are the only cases that can occur since otherwise (X, τ) is not invariant under σ. Let X_σ be the quotient surface as in Definition 4.14 and $\pi_\sigma : X \to X_\sigma$ be the corresponding two-sheeted covering. Then τ_1 induces an involution on X_σ as follows: The involutions τ_1 and σ commute. Let $p, q \in X$ with $p \sim_\sigma q$, i.e. either $p = q$ or $p = \sigma(q)$. Then $\tau_1(p)$ is either $\tau_1(q)$ or $\tau_1(\sigma(q)) = \sigma(\tau_1(q))$. Hence, also $\tau(p) \sim_\sigma \tau(q)$, wherefore τ induces an involution τ_σ on X_σ.

Lemma 6.35. *For a real curve with involution (X, τ_1, σ) of genus $2g$, the curve (X_σ, τ_σ) is a real curve of type (g, k, ε), where $k = t_1 + r_1 + t_2 + r_2$. The preimage of the set $X_\sigma^{\tau_\sigma}$ under the* *two-sheeted covering $\pi_\sigma : X \to X_\sigma$ coincides with $X^{\tau_1} \cup X^{\tau_2}$. This preimage decomposes X into two parts if and only if $\varepsilon = 1$.*

Proof. The surface X_σ is a Riemann surface of genus g since σ has exactly two fixed points on X, compare Proposition A.1 and Lemma 4.13. By definition, X_σ is invariant under σ and for $i \in \{1, 2\}$, each pair of the $2t_i$ ovals on X – which are interchanged by τ_i – are mapped to one oval of τ_σ on X_σ and the r_i ovals of τ_i are each mapped to one oval of τ_σ on X_σ. So X_σ has $t_1 + t_2 + r_1 + r_2$ ovals in total and (X_σ, τ_σ) is a real curve of topological type (g, k, ε).
It remains to show that $\varepsilon = 1$ if and only if $X^{\tau_1} \cup X^{\tau_2}$ decomposes X into two parts. Let $\varepsilon = 1$. Then $X_\sigma \setminus X_\sigma^{\tau_\sigma}$ consists of two disjoint sets which are interchanged by τ_σ. Each of these components has a preimage in X. Assume that the intersection of these preimages is not empty, i.e. that $X \setminus (X^{\tau_1} \cup X^{\tau_2})$ does not compose into two parts. Then there exists an element in $X \setminus (X^{\tau_1} \cup X^{\tau_2})$ which is contained in the intersection of the preimage of these two sets. The covering $X \to X_\sigma$ maps each point in X to exactly one point in X_σ. Accordingly, this element is mapped to one element on X_σ which has to be in both connected disjoint components of $X_\sigma \setminus X_\sigma^{\tau_\sigma}$. This contradicts $\varepsilon = 1$.
Conversely, let $X \setminus (X^{\tau_1} \cup X^{\tau_2})$ consist of two open connected components X^+ and X^- with $Q^+ \in X^+$ and $Q^- \in X^-$. Then $\sigma : X^\pm \to X^\pm$ does not interchange those components, and so π_σ maps them to disjoints sets X_σ^+ and X_σ^-, where $X^{\tau_1} \cup X^{\tau_2}$ is mapped to $X_\sigma^{\tau_\sigma}$ under this map. So $\varepsilon = 1$. □

Definition 6.36. The set $(g, \varepsilon, t_1, r_1, t_2, r_2)$ is the *topological type of the real curve with involution* (X, τ_1, σ).

As in Examples 6.7 and 6.8, one can construct models of real curves with involution (X, τ_1, σ) of a certain topological type and show that all real curves with involution of this type are topological equivalent to that model if the genus of X is bigger or equal to four. So our next aim is to consider the construction given in [Natanzon, 2004, Example 2.9.1]. Therefore, a certain two-sheeted covering $\tilde{\pi} : X \to \tilde{X}$ is necessary, where $\tilde{X}$ and X are Riemann surfaces and the genus of $\tilde{X}$ is bigger or equal to two. Recall that every Riemann surface of genus $g \geq 2$ has the upper half space $\mathcal{H}$ as universal covering with a group of deck transformations Λ such that $\tilde{X} = \mathcal{H}/\Lambda$. Furthermore, there is an isomorphism between Λ and $\pi_1(X, q)$, compare [Hatcher, 2002, Proposition 1.40]. For a more precise explanation, see Appendix B.

Lemma 6.37. *Let $\tilde{X}$ be a compact Riemann surface of genus g with boundary which consists of the simple closed contours $c_1, \ldots, c_J, d_1, \ldots, d_L$. Then there exists a two-sheeted covering $\pi : X \to \tilde{X}$ such that the genus of X is $2g$ and the preimages of each of the boundary components $c_1, \ldots, c_J$ consist of each two simple closed contours on which the covering is one-sheeted and the preimages of each of the $d_1, \ldots, d_L$ consist of each one simple closed contour on which the covering is two-sheeted if and only if $L = 0 \mod 2$.*

Proof. Let
$$\{a_i, b_i, c_j, d_\ell \mid i = 1, \dots, g,\ j = 1, \dots, J,\ \ell = 1, \dots, L\}$$
be a set of generators of the fundamental group $G := \pi_1(\tilde{X}, \tilde{q})$ with $\tilde{q} \in d_1$. This basis is chosen such that the representatives a_i, b_i correspond to the cycles which originate from the genus g of $\tilde{X}$ and c_j, d_ℓ correspond to the boundary cycles of $\tilde{X}$. We first show that

$$\{a_i, d_1 a_i^{-1} d_1^{-1}, b_i, d_1 b_i^{-1} d_1^{-1}, c_j, d_1 c_j d_1^{-1}, d_m d_n \mid i = 1, \dots, g,\ j = 1, \dots, J,\ m, n = 1, \dots, L\}$$

generates a normal subgroup H of G with $(G : H) = 2$. Then this subgroup generates a covering $\tilde{\pi} : X \to \tilde{X}$, see [Hatcher, 2002, Theorem 1.38]. Hereby, let $\tilde{\pi} : X \to \tilde{X}$ be a map taking the basepoint $q \in X$ of $H_1(X, q)$ to the basepoint $\tilde{q} \in \tilde{X}$ of $H_1(\tilde{X}, \tilde{q})$. Then $\tilde{\pi}$ induces a homomorphism $\pi_* : \pi_1(X, q) \to \pi_1(\tilde{X}, \tilde{q})$, defined by composing loops $f : I \to X$ based at q with $\tilde{\pi}$, that is $\tilde{\pi}_*[f] = [\tilde{\pi} \circ f]$. A more detailed description on this homomorphism can be found in [Hatcher, 2002, Chapter 1 – Induced Homomorphisms, page 34]. Then $\tilde{\pi}_* \pi_1(X, q) = H$ with $q \in \pi^{-1}[\{\tilde{q}\}]$, compare [Hatcher, 2002, Theorem 1.38]. We can deduce two things if we have shown that the index of H in G equals two: First of all, it implies that the covering $\tilde{\pi}$ generated by H is two-sheeted, compare [Hatcher, 2002, Proposition 1.32]. Secondly, it implies that H is normal. To see the last assertion, let G be a group and H be a subgroup of G with index 2 and let g be any element of G. If $g \in H$, then $gH = H = Hg$. If $g \notin H$, then the two left cosets are given by H and gH and the two right cosets are given by H and Hg. Since $(G : H) = 2$ and $g \notin H$, i.e. $gH \neq H$ as well as $Hg \neq H$, one has $gH = Hg$. Thus, the left and right cosets of H coincide, so H is normal. Therefore, the associated covering $\tilde{\pi} : X \to \tilde{X}$ is a normal covering, i.e. for each $\tilde{p} \in \tilde{X}$ and each pair of lifts $p, p' \in X$ of $\tilde{p}$, there is a deck transformation in Λ carrying p to p'. Then the group of deck transformations of X on $\mathcal{H}$ is isomorphic to G/H, compare [Hatcher, 2002, Proposition 1.39]. In other words, normality implies that the choice of this subgroup is independent from the choice of q since usually, as q varies over the fiber $\tilde{\pi}^{-1}[\{\tilde{q}\}]$, the set of subgroups $\tilde{\pi}_* \pi_1(X, q) \subset \pi_1(\tilde{X}, \tilde{q})$ is exactly one conjugacy class of H, see [Lee, 2010, Theorem 11.19], and the only subgroup conjugate to a normal subgroup H is H itself.

G fulfills equation (B.4), i.e. $\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} \prod_{j=1}^{J} c_j \prod_{\ell=1}^{L} d_\ell = 1$. Because H is a subgroup of G, it has to be compatible with this relation. Moreover, remember that $H \neq G$. Assume that $L = 1 \mod 2$. This yields that for $m \in \{1, \dots, l\}$, it is

$$d_m = \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} \prod_{j=1}^{J} c_j \prod_{\ell=1}^{L} d_\ell \cdot d_m \in H$$

since the product of all d-factors can be decomposed into products of the form $d_i d_j \in H$ with $i, j \in \{1, \dots, L\}$. In that case, H is a subgroup of G containing all generators of G, and therefore equal to G. This contradicts the assumption that $H \neq G$. Conversely, $l = 0 \mod 2$ implies that equation (B.4) applied to H contains an uneven number of factors of d_m and thus d_m is not contained in H for $m = 1, \dots, L$. So H is a subgroup of G which does not equal G if and only if $\ell = 0 \mod 2$.

This can now be used to show that $(G : H) = 2$ since $d_1 \in G \setminus H$. We claim that $H \cup d_1 H = G$. To see this, we first show that $H \cup d_1 H$ is a group. Therefore, we only have to show that $H \cup d_1 H$ is closed. Note that all generators of $d_1^{-1} H d_1$ are contained in H: since $d_1 a_i d_1^{-1} = (d_1 a_i^{-1} d_1^{-1})^{-1} \in H$ and $d_1^2, d_1^{-2} \in H$, also $d_1^{-1} a_i d_1 \in H$ and $d_1^{-1} d_1 a_i^{-1} d_1^{-1} d_1 = a_i^{-1} \in H$ for $i \in \{1, \dots, g\}$. Analogously $d_1^{-1} b_i d_1, d_1^{-1} d_1 b_i d_1^{-1} d_1 \in H$ for $i \in \{1, \dots, g\}$ and $d_1^{-1} c_j d_1, d_1^{-1} d_1 c_j d_1^{-1} d_1 \in H$ for $j \in \{1, \dots, J\}$. Finally, $d_1^{-1} d_m d_n d_1 = (d_m d_1)^{-1} d_m^2 (d_n d_1) \in H$ for $m, n \in \{1, \dots, L\}$, and so $d_1^{-1} H d_1 \subset H$. Repeating these observations the other way around yields that also all generators of H can be written as elements of $d_1^{-1} H d_1$, wherefore $d_1^{-1} H d_1 = H$. We use this equality to show the closedness of $H \cup d_1 H$: It is $h d_1 h' = d_1 d_1^{-1} h d_1 h \in d_1 H$ and $d_1 h d_1 h' = d_1 h d_1^{-1} d_1^2 h'$. Accordingly, for all $h, h' \in H$, one has

$$hh' \in H \subset H \cup d_1 H, \qquad d_1 h d_1 h' \in H \subset H \cup d_1 H,$$
$$h d_1 h' \in d_1 H \subset H \cup d_1 H, \qquad d_1 h' h \in d_1 H \subset H \cup d_1 H,$$

whereby $H \cup d_1 H$ is a group. This group contains all generators of G because $d_1 \in H \cup d_1 H$, and so also $d_m^{-1} = d_1 (d_m d_1)^{-1} \in d_1 H$ for $m = 2, \dots l$. So $H \cup d_1 H$ is a subgroup of G which contains all generators of G, i.e. $H \cup d_1 H = G$. Since $H \cap d_1 H = \emptyset$, this yields $(G : H) = 2$. The preimage of the basepoint $\tilde{q} \in d_1$ of $\pi_1(\tilde{X}, \tilde{q})$ contains two elements q_1, q_2 which are connected by d_1 which is not contained in H, and therefore does not correspond to a loop on X. Furthermore, H contains the elements a_i and $d_1 a_i d_1^{-1}$. These correspond to the two elements in $\pi_1(X)$ which cover $a_i \in \pi_1(\tilde{X})$. The same holds for the preimages of b_i and c_j under $\tilde{\pi}$. The elements d_ℓ for $\ell = 1, \dots l$ are not contained in H, so each preimage of these under $\tilde{\pi}$ yields a path which is not closed in X. However, d_ℓ^2 is contained in H, so the covering over d_ℓ is two-sheeted. □

Corollary 6.38. *Let $\tilde{X}$ be a compact Riemann surface of genus g with boundary consisting of the simple closed contours $c_1, \dots, c_J, d_1, \dots, d_M$ and a marked point $\tilde{Q}$ which is not contained in the boundary. Then there exists a two-sheeted covering $\tilde{\pi} : X \to \tilde{X}$ with exactly one ramification point in X such that the genus of X is $2g$ and the preimages of each of the $c_1, \dots, c_J$ consist of each two simple closed contours on which the covering is one-sheeted and the preimages of each of the $d_1, \dots, d_M$ consist of each one simple closed contour on which the covering is two-sheeted if and only if $M = 1 \mod 2$.*

Proof. Let D be a small closed disc containing $\tilde{Q}$ which does not intersect $\partial \tilde{X}$ and let $z : D \to \mathbb{C}$ be a local coordinate centered at $\tilde{Q}^+$. Then the surface $\tilde{X} \setminus D^\circ$ has $J + M + 1$ boundary components, where the new boundary component is just ∂D. This means that we are exactly in the situation of Lemma 6.37 if we set $L = M + 1$ in the notation of the Lemma. So there is a two sheeted covering $X \to \tilde{X}$ such that the preimage of ∂D on X consists of one simple closed contour. We glue the disc D into the preimage of the boundary component ∂D and use $z^2 : D \to \mathbb{C}$ as local coordinate on this disc. This runs twice through the boundary of ∂D, and therefore also is a chart describing the preimage of ∂D – remember that we have seen in the proof of the foregoing Lemma that the boundary contours with one preimage in X are covered twice by this preimage. Moreover, the choice of the local chart yields that $\tilde{Q}$ is a simple branch point on $\tilde{X}$ of the covering $X \to \tilde{X}$, and therefore the preimage of $\tilde{Q}$ with respect to this covering is a simple ramification point on X. □

Example 6.39 ([Natanzon, 2004, Example 2.9.1]). Let $(\tilde{X}, \tilde{\tau})$ be a real curve of type $(g, k, 1)$ and let $k = t_1 + r_1 + t_2 + r_2$, where $r_1 + r_2 = 1 \mod 2$. Let us consider a connected component $\tilde{X}^+$ of the set $\tilde{X} \setminus \tilde{X}^{\tilde{\tau}}$. This is a compact Riemann surface with boundary $\partial\tilde{X}^+ = \cup_{i=1}^{k} c_i$. Due to Corollary 6.38, there exists a two-sheeted covering $\pi_+ : X^+ \to \tilde{X}^+$ with a unique ramification point $Q^+ \in X^+$ which is two-sheeted on the $r_1 + r_2$ boundary contours $c_1, \dots, c_{r_1+r_2} \in \partial X^+$ and one-sheeted on the other boundary contours $c_{r_1+r_2+1}, \dots, c_{\tilde{k}}$, where $\tilde{k} = r_1 + r_2 + 2t_1 + 2t_2$. Using the construction of Example 6.7, we can tinker a real curve $(\hat{X}, \hat{\tau})$ such that $\hat{X}^{\hat{\tau}} = \sum_{i=1}^{\tilde{k}} c_i$ decomposes $\hat{X}$ into $\hat{X}^+$ and $\hat{X}^- = \hat{\tau}[X^+]$. The covering π_+ induces a two-sheeted covering $\hat{\pi} : \hat{X} \to \tilde{X}$ such that $\hat{\pi} \circ \tau = \tilde{\tau} \circ \pi_+$. Let $\sigma : \hat{X} \to \hat{X}$ be the involution which is defined by the transposition of the two sheets of this covering. This involution commutes with $\hat{\tau}$ and has exactly the two fixed points Q^+ and $Q^- = \hat{\tau}(Q^+)$. As it is done in Example 6.7, we cut the surface $\hat{X}$ along the contours $c_{r_1+1}, \dots, c_{r_1+r_2}$ and $c_{r_1+r_2+2t_1+1}, \dots, c_{\tilde{k}}$ and paste these boundary contours back together in accordance with the map $\sigma \circ \hat{\tau}$. On the surface X obtained like this, the involution $\hat{\tau}$ induces an involution $\tau_1 : X \to X$ that commutes with σ. We set $\tau_2 := \sigma \circ \tau_1$. It follows from the construction of (X, τ_1, σ) that this is a real curve with involution of type $(g, 1, t_1, r_1, t_2, r_2)$.

Example 6.40 ([Natanzon, 2004, Example 2.9.2]). Let $(\tilde{X}, \tilde{\tau})$ be a real curve of type $(g, k, 0)$ and let $k = t_1 + r_1 + t_2 + r_2$, where $r_1 + r_2 = 1 \mod 2$. Using Lemma 6.13, we construct a set of pairwise disjoint simple closed contours $c_1, \dots, c_{g+1}$ such that $\tilde{\tau}[c_i] = c_i$ and $\tilde{X}^{\tilde{\tau}} = \sum_{i=1}^{k} c_i$. Let us consider a connected component $\tilde{X}^+$ of the set $\tilde{X} \setminus \bigcup_{i=1}^{g+1} c_i$ and a two-sheeted covering $\pi_+ : X^+ \to \tilde{X}^+$ with a single ramification point $Q^+ \in X^+$ that is two-sheeted on the contours $c_1, \dots, c_{r_1+r_2}$ and one-sheeted on the other contours $c_{r_1+r_2+1}, \dots, c_{r_1+r_2+2t_1+2t_2}$ which again exists due to Corollary 6.38. Using the construction of Example 6.8, we form a real curve $(\hat{X}, \hat{\tau})$ such that $\hat{X} \setminus \sum_{i=1}^{g+1} c_i$ decomposes $\hat{X}$ into $\hat{X}^+$ and $\hat{X}^- = \hat{\tau}[\hat{X}^+]$ with $\hat{X}^{\hat{\tau}} = \sum_{i=1}^{\hat{k}} c_i$, where $\hat{k} = r_1 + r_2 + 2t_1 + 2t_2$. Repeating the cuts and pastings together described in Examples 6.8 and 6.39, we obtain a real curve with involution (X, τ_1, σ) of type $(g, 0, t_1, r_1, t_2, r_2)$.

The topological equivalence of real curves with involutions follows from the topological equivalence of the real curves X_σ shown in Theorems 6.11 and 6.14. The proof of the following lemma is missing in [Natanzon, 2004].

Lemma 6.41 ([Natanzon, 2004, Lemmata 2.9.1 and 2.9.2])**.** *The construction of Example 6.39 produces all real curves with involution of type* $(g, 1, t_1, r_1, t_2, r_2)$*. The construction of Example 6.40 produces all real curves with involution of type* $(g, 0, t_1, r_1, t_2, r_2)$*.*

Proof. Lemma 6.35 yields that for every real curve with involution, the quotient (X_σ, τ_σ) is a real curve of topological type (g, k, ε), where $k = t_1 + r_1 + t_2 + r_2$ and we know from Theorems 6.11 and 6.14 that the topological type of X_σ determines this curve uniquely up to topological equivalence. So let X_σ be a real curve of type $(g, k, 1)$. Then X_σ^+, whereby $X_\sigma \setminus X_\sigma^{\tau_\sigma} = X_\sigma^+ \cup X_\sigma^-$, is a Riemann surface with boundary which obeys the preliminaries of Corollary 6.38. We use the notation of this Corollary. Hereby, all pairs $(J, M) \in \mathbb{N}_0 \times \mathbb{N}$ with M odd are admissible. Then $J = t_1 + t_2$ and $M = r_1 + r_2$. So we can apply the construction of a real curve with involution from one of the two foregoing examples and choose whether one of the boundary components is covered by one or by two cycles. Repeating the construction as in Example 6.39 first yields a curve $\hat{X}$ which has $L + 2M$ ovals of τ_1. Then there are finitely many combinations such that we can – by the same construction as in Example 6.39 – cut an arbitrary number of these ovals open again and identify the generated boundary contours via $\tau_2 = \sigma \circ \tau_1$. So we can achieve all admissible combinations of topological types. Lifting the homomorphism between the curve (X_σ, τ_σ) of a certain topological type and the model curve in Example 6.7 to a homomorphism between X and the model space by appropriate combination with σ and the respective covering maps from Corollary 6.38 yields that also all real curves with involution of one admissible topological type are topologically equivalent. The proof for $\varepsilon = 0$ is analogous. □

For the rest of this section, let (X, τ_1, σ) be a real curve with involution of type $(g, \varepsilon, t_1, r_1, t_2, r_2)$. A symplectic basis $\{A_i, \sigma_\sharp A_i, B_i, \sigma_\sharp B_i \mid i = 1, \ldots, g\}$ of $H_1(X, \mathbb{Z})$ is said to be symmetric if $\sigma_\sharp$ maps A_i to $\sigma_\sharp A_i$ and B_i to $\sigma_\sharp B_i$ for $i = 1, \ldots, g$. Let $k = t_1 + r_1 + t_2 + r_2$ and $m = \frac{1}{2}(g + 1 - k)$.

Definition 6.42. A symplectic, symmetric basis of $H_1(X, \mathbb{Z})$ of a real curve with involution (X, τ_1, σ) of type $(g, \varepsilon, t_1, r_1, t_2, r_2)$ is a *symmetric real basis* if its generators $\{A_i, \sigma_\sharp A_i, B_i, \sigma_\sharp B_i \mid i = 1, \ldots, g\}$ obey for $\varepsilon = 0$

$$\begin{aligned}
&\tau_{1\sharp} A_i = A_i, && \tau_{1\sharp}\sigma_\sharp A_i = \sigma_\sharp A_i && \text{for } i = 1, \ldots, g,\\
&\tau_{1\sharp} B_i = -B_i, && \tau_{1\sharp}\sigma_\sharp B_i = -\sigma_\sharp B_i && \text{for } i = 1, \ldots, k-1,\\
&\tau_{1\sharp} B_i = -B_i + A_i, && \tau_{1\sharp}\sigma_\sharp B_i = -\sigma_\sharp B_i + \sigma_\sharp A_i && \text{for } i = k, \ldots, g
\end{aligned}$$

and for $\varepsilon = 1$

$$\begin{aligned}
&\tau_{1\sharp} A_i = A_i, && \tau_{1\sharp}\sigma_\sharp A_i = \sigma_\sharp A_i && \text{for } i = 1, \ldots, k-1,\\
&\tau_{1\sharp} A_i = A_{i+m}, && \tau_{1\sharp}\sigma_\sharp A_i = \sigma_\sharp A_{i+m} && \text{for } i = k, \ldots, k+m-1,\\
&\tau_{1\sharp} A_i = A_{i-m}, && \tau_{1\sharp}\sigma_\sharp A_i = \sigma_\sharp A_{i-m} && \text{for } i = k+m, \ldots, g,\\
&\tau_{1\sharp} B_i = -B_i, && \tau_{1\sharp}\sigma_\sharp B_i = -\sigma_\sharp B_i && \text{for } i = 1, \ldots, k-1,\\
&\tau_{1\sharp} B_i = -B_{i+m}, && \tau_{1\sharp}\sigma_\sharp B_i = -\sigma_\sharp B_{i+m} && \text{for } i = k, \ldots, k+m-1,\\
&\tau_{1\sharp} B_i = -B_{i-m}, && \tau_{1\sharp}\sigma_\sharp B_i = -\sigma_\sharp B_{i-m} && \text{for } i = k+m, \ldots, g.
\end{aligned}$$

Lemma 6.43. *Let (X, τ_1, σ) be a real curve with holomorphic involution. Then a real basis of $H_1(X, \mathbb{Z})$ exists.*

Proof. We have seen in Proposition A.1 and Lemma 4.13 that X_σ is a Riemann surface of genus g and that $\pi_\sigma : X \to X_\sigma$ is a two-sheeted covering with the only ramification points being Q^+ and Q^-. Furthermore, we have also seen in Section 4.2.2 that a symplectic, symmetric basis of $H_1(X,\mathbb{Z})$ can be obtained by pulling back a symplectic basis of $H_1(X_\sigma,\mathbb{Z})$ via π. Then a basis of $H_1(X,\mathbb{Z})$ consists of each two preimages of the chosen basis of $H_1(X_\sigma,\mathbb{Z})$ since Q^+ and Q^- are the only ramification points of π. Because (X_σ,τ_σ) is a real curve of genus g, all the $2g$ elements in $H_1(X_\sigma,\mathbb{Z})$ obey the relations in Definition 6.28. Hence, denoting and enumerating the pulled back cycles in $H_1(X,\mathbb{Z})$ in the obvious way, the elements of $H_1(X,\mathbb{Z})$ obey the relations given in Definition 6.42. □

The map $\sigma : X \to X$ induces an involution $\sigma : S_g \to S_g$ as defined at the beginning of Section 4.2. Here, we consider the Abel map A_{Q^+} which transfers this involution to an involution σ on $\mathrm{Jac}(X)$. The subset

$$\mathrm{Prym}(X,\sigma) = \{x \in \mathrm{Jac}(X) \mid \sigma(x) = -x\}$$

as in Definition A.12 is called the Prym variety of the surface X with holomorphic involution σ. Remember that $\mathrm{Prym}(X,\sigma) \simeq \mathbb{C}^g/\Lambda_-$, compare Definition A.12 where Λ_- is a g-dimensional lattice. We now remind again how the lattice Λ^- is obtained: It is shown in Proposition A.9, that $H_1(X,\mathbb{Z})_-$ – the part of the first homology group which is antisymmetric with respect to σ – is generated by $A_i^- := A_i - \sigma_\sharp A_i$ and $B_i^- := B_i - \sigma_\sharp B_i$ for $i = 1,\dots,g$. The corresponding basis of antisymmetric holomorphic differential forms is for $i = 1,\dots,g$, as in (4.12), given by $\omega_i^- := \frac{1}{2}(\omega_i - \sigma^*\omega_i)$, whereby $\sigma^*\omega_i = \omega_{g+i}$. So the lattice Λ_- is, as in (4.14), generated by $\{\Omega^-_{A_j}, \Omega^-_{B_j} \mid j = 1,\dots,g\}$, where

$$\Omega^-_{A_j} = \left(\oint_{A_j - \sigma_\sharp A_j} \omega_i^-\right)_{i=1}^g \quad \text{and} \quad \Omega^-_{B_j} = \left(\oint_{B_j - \sigma_\sharp B_j} \omega_i^-\right)_{i=1}^g .$$

The basis of the $2g$ holomorphic differential forms on X shall now be normalized as

$$\oint_{A_j} \omega_i = 2\pi\iota\delta_{ij}, \quad \oint_{A_j} \omega_{i+g} = 0, \quad \oint_{\sigma_\sharp A_j} \omega_i = 0, \quad \oint_{\sigma_\sharp A_j} \omega_{i+g} = 2\pi\iota\delta_{i+g,j} \tag{6.7}$$

for $i,j = 1,\dots,g$.

Lemma 6.44. *Let (X,τ_1,σ) be a real curve with involution of type $(g,\varepsilon,t_1,r_1,t_2,r_2)$, let the set $\{A_i,B_i,\sigma_\sharp A_i,\sigma_\sharp B_i \mid i = 1,\dots,g\}$ be a symmetric, real basis of $H_1(X,\mathbb{Z})$ and let $\omega_1,\dots,\omega_{2g}$ be the basis of holomorphic differential forms on (X,τ) which are normalized with respect to $A_1,\dots,A_g$, $\sigma_\sharp A_1,\dots,\sigma_\sharp A_g$ as in* (6.7).

(a) For $\varepsilon = 0$, $\tau_1^*\overline{\omega_j^-} = -\omega_j^-$ as well as $\overline{\Omega^-_{A_j}} = -\Omega_{A_j}$ for $j = 1,\dots,g$ and

$$\overline{\Omega^-_{B_j}} = \begin{cases} \Omega^-_{B_j} & \text{for } j = 1,\dots,k-1, \\ \Omega^-_{B_j} - \Omega^-_{A_j} & \text{for } j = k,\dots,g. \end{cases}$$

(b) For $\varepsilon = 1$*,*

$$\tau_1^* \overline{\omega_j^-} = \begin{cases} -\omega_j^- & \text{for } j = 1, \ldots, k-1, \\ -\omega_{j+m}^- & \text{for } j = k, \ldots, k+m-1, , \\ -\omega_{j-m}^- & \text{for } j = k+m, \ldots, g, \end{cases}$$

$$\overline{\Omega_{A_j}^-} = -\Omega_{A_j}^- = \begin{cases} -M \cdot \Omega_{A_j}^- & \text{for } j = 1, \ldots, k-1, \\ -M \cdot \Omega_{A_{j+m}}^- & \text{for } j = k, \ldots, k+m-1, \\ -M \cdot \Omega_{A_{j-m}}^- & \text{for } j = k+m, \ldots, g \end{cases}$$

and

$$\overline{\Omega_{B_j}^-} = \begin{cases} M \cdot \Omega_{B_j}^- & \text{for } j = 1, \ldots, k-1, \\ M \cdot \Omega_{B_{j+m}}^- & \text{for } j = k, \ldots, k+m-1, \\ M \cdot \Omega_{B_{j-m}}^- & \text{for } j = k+m, \ldots, g, \end{cases}$$

where M *is the matrix defined in equation* (6.3)*.*

Proof. Due to $\oint_{A_i} \omega_{j+g} = 0$ and $\oint_{A_j} \omega_i = 2\pi\iota\delta_{i,j}$, it is for $i, j = 1, \ldots, g$

$$\begin{aligned} \Omega_{A_j}^- &= \left(\oint_{A_j - \sigma_\sharp A_j} \omega_i^- \right)_{i=1}^g = \frac{1}{2} \left(\oint_{A_j - \sigma_\sharp A_j} (\omega_i - \sigma^* \omega_i) \right)_{i=1}^g \\ &= \frac{1}{2} \left(\oint_{A_j} \omega_i - \oint_{A_j} \sigma^* \omega_i - \oint_{\sigma_\sharp A_j} \omega_i + \oint_{\sigma_\sharp A_j} \sigma^* \omega_i \right)_{i=1}^g = \left(\oint_{A_j} \omega_i \right)_{i=1}^g . \end{aligned}$$

Repeating this calculation for $\Omega_{B_j}^-$ yields

$$\Omega_{B_j}^- = \left(\oint_{B_j - \sigma_\sharp B_j} \omega_i^- \right)_{i=1}^g = \left(\oint_{B_j} (\omega_i - \sigma^* \omega_i) \right)_{i=1}^g .$$

Using this together with the transformation behavior of ω_i^- under σ^* and repeating the proof of Lemma 6.30 gives the assertions. □

We define an involution $\tau_{1,\mathbb{R}} : \mathbb{C}^g \to \mathbb{C}^g$ via its action on the basis $\{\Omega_{A_i}^-, \Omega_{B_i}^- \mid i = 1, \ldots, g\}$ of the space $\mathbb{R}^{2g} = \mathbb{C}^g$. As before, we define this involution for $\varepsilon = 0$ as the $\mathbb{R}$-linear map

$$\begin{aligned} \Omega_{A_j}^- &\mapsto -\overline{\Omega_{A_j}^-} = \Omega_{A_j}^- && \text{for } j = 1, \ldots, g, \\ \Omega_{B_j}^- &\mapsto -\overline{\Omega_{B_j}^-} = \begin{cases} -\Omega_{B_j}^- & \text{for } j = 1, \ldots, k-1, \\ -\Omega_{B_j}^- + \Omega_{A_j}^- & \text{for } j = k, \ldots, g. \end{cases} \end{aligned} \tag{6.8}$$

and for $\varepsilon = 1$ with M as in (6.3) by the $\mathbb{R}$-linear map

$$\begin{aligned}
\Omega^-_{A_j} &\mapsto -M \cdot \overline{\Omega^-_{A_j}} = \begin{cases} \Omega^-_{A_j} & \text{for } j = 1, \dots, k-1, \\ \Omega^-_{A_{j+m}} & \text{for } j = k, \dots, k+m-1, \\ \Omega^-_{A_{j-m}} & \text{for } j = k+m, \dots, g, \end{cases} \\
\Omega^-_{B_j} &\mapsto -M \cdot \overline{\Omega^-_{B_j}} = \begin{cases} -\Omega^-_{B_j} & \text{for } j = 1, \dots, k-1, \\ -\Omega^-_{B_{j+m}} & \text{for } j = k, \dots, k+m-1, \\ -\Omega^-_{B_{j-m}} & \text{for } j = k+m, \dots, g, \end{cases}
\end{aligned} \tag{6.9}$$

Definition 6.45. Let (X, τ_1, σ) be a real curve with involution. The intersection of the Prym variety $\mathrm{Prym}(X, \sigma) \subset \mathrm{Jac}(X)$ with $\mathrm{Jac}_{\mathbb{R}}(X)$ is called the *real part of the Prym variety* $\mathrm{Prym}_{\mathbb{R}}(X, \sigma)$ of (X, τ_1, σ). The connected components of this part are called *real tori of the Prym variety* of (X, τ_1, σ).

The real tori of $\mathrm{Prym}_{\mathbb{R}}(X, \sigma)$ equals the set of fixed points of the involution $\tau_{1,\mathbb{R}}|_{\mathrm{Prym}(X,\sigma)} : \mathrm{Prym}(X, \sigma) \to \mathrm{Prym}(X, \sigma)$.

Theorem 6.46 ([Natanzon, 2004, Theorem 2.9.1])**.** *The real part of the Prym variety of a real curve with involution (X, τ_1, σ) of type $(g, \varepsilon, t_1, r_1, t_2, r_2)$ decomposes into 2^{k-1} real tori of dimension g, where $k = t_1 + r_1 + t_2 + r_2 > 0$.*

Proof. Let $\{A_i, B_i, \sigma_\sharp A_i, \sigma_\sharp B_i \mid i = 1, \dots, g\}$ be a symmetric real basis of $H_1(X, \mathbb{Z})$ as in Definition 6.42 such that the projections of these cycles to X_σ yield a real basis of $H_1(X_\sigma, \mathbb{Z})$ in sense of Definition 6.28. We denote the set of the basis of $H_1(X_\sigma, \mathbb{Z})$, mapped to $\mathbb{C}^g$ analogously as it is done for Λ in Section 4.2.2, as Λ_σ. This lattice over $\mathbb{Z}$ is generated by $\Omega_{A_{\sigma,i}}, \Omega_{B_{\sigma,i}}$ with $i = 1, \dots, g$ and it is the lattice of the Jacobian variety $\mathrm{Jac}(X_\sigma) \simeq \mathbb{C}^g / \Lambda_\sigma$. As we have seen in Lemma 6.35, the real curve (X_σ, τ_σ) is of type (g, k, ε). As above, let $\{\Omega^-_{A_i}, \Omega^-_{B_i} \mid i = 1, \dots, g\}$ be the generators of the lattice Λ_- of the Prym variety of a real curve with involution (X, τ_1, σ) that corresponds to the given basis of $H_1(X, \mathbb{Z})$. In these bases, the involution $\tau_{1,\mathbb{R}}|_{\mathrm{Prym}} : \mathbb{C}^g \to \mathbb{C}^g$ acts on the generators of Λ_- as in (6.8) for $\varepsilon = 0$ and as in (6.9) for $\varepsilon = 1$. Likewise the action of $\tau_{\sigma,\mathbb{R}} : \mathbb{C}^g \to \mathbb{C}^g$ on the generators of Λ_σ is given in (6.4) for $\varepsilon = 0$ and in (6.5) for $\varepsilon = 1$. So the maps $\tau_{1,\mathbb{R}}$ and $\tau_{\sigma,\mathbb{R}}$ are both $\mathbb{R}$-linear mappings on $\mathbb{C}^g$, where $\tau_{1,\mathbb{R}}$ acts on the given basis $\{\Omega^-_{A_i}, \Omega^-_{B_i} \mid i = 1, \dots, g\}$ of $\mathbb{C}^g$ in the same way as $\tau_{\sigma,\mathbb{R}}$ acts on the basis given by $\{\Omega_{A_{\sigma,i}}, \Omega_{B_{\sigma,i}} \mid i = 1, \dots, g\}$. We use $\mathbb{C}^g \simeq \mathbb{R}^{2g}$. Then a real-linear vector space isomorphism which maps Λ_σ to Λ^- is given by

$$\mathrm{Jac}(X_\sigma) \to \mathrm{Prym}(X, \sigma), \quad \Omega_{A_{\sigma,i}} \mapsto \Omega^-_{A_i} \ \text{ and } \ \Omega_{B_{\sigma,i}} \mapsto \Omega^-_{B_i}.$$

Because the $\mathbb{R}$-linear map $\tau_{1,\mathbb{R}} : \mathbb{C}^g \to \mathbb{C}^g$ acts on the elements of Λ_- and $\tau_{\sigma,\mathbb{R}} : \mathbb{C}^g \to \mathbb{C}^g$ acts on the elements of Λ_σ, the above isomorphism yields a bijection from the fixed point set of $\tau_{\sigma,\mathbb{R}}$ on

$\mathrm{Jac}(X_\sigma)$ to the fixed point set of $\tau_{1,\mathbb{R}}$ on $\mathrm{Prym}(X_\sigma)$. So the fixed point sets of the involutions $\tau_{1,\mathbb{R}} : \mathrm{Prym}(X,\sigma) \to \mathrm{Prym}(X,\sigma)$ and $\tau_{\sigma,\mathbb{R}} : \mathrm{Jac}(X_\sigma) \to \mathrm{Jac}(X_\sigma)$ have equally many connected components. □

Let (X, τ_1, σ) be a real curve with involution of type $(g, \varepsilon, t_1, r_1, t_2, r_2)$. The ovals $a_1^j, \dots, a_{2t_j+r_j}^j$ of the involution τ_j for $j = 1, 2$ shall be enumerated in such a way that $\sigma_\sharp a_i^j = a_{t_j+i}^j$ for $i \le t_j$. We define

$$\mathcal{E} := \{D \subset S_g \mid \tau_1(D) = D \text{ and either } D + \sigma(D) \text{ is the divisor of zeros of a meromorphic differential } \omega_D \text{ on } X \text{ that is holomorphic away from } Q^+ \text{ and } Q^- \text{ and has poles of order 1 at these points or } D + \sigma(D) - Q^+ - Q^- \text{ is the zero divisor of a holomorphic differential } \omega_D \text{ on } X\}.$$

In [Natanzon, 2004], the definition of the above set $\mathcal{E}$ is a bit different, but we think we interpreted the definition given here in the way it is meant in [Natanzon, 2004]. We want to define positive respectively negative definiteness of a differential form on an oval of (X, τ_1, σ). This is also done in [Natanzon, 2004] and then used to analyze whether the connected component of the real Prym variety are singular or not. We do not understand the definition of definiteness of a 1-form on the ovals of a real curve which is given in [Natanzon, 2004]. We modified it in a way that fits to the setup such that the parts hereinafter, for which this definition is necessary, can be shown with our definition. To motivate our definition, note that if $D + \sigma(D) - Q^+ - Q^-$ is the divisor of a holomorphic differential ω_D on X, then ω_D has zeros of odd order at Q^+ and Q^- since then $Q^+, Q^- \in \operatorname{supp} D$ and thus also in $\operatorname{supp}(\sigma(D))$. Accordingly, $D + \sigma(D)$ contains an even number of Q^+ and of Q^-. The following two Lemmata show that the definition of positive respectively negative definiteness we will give hereinafter is feasible.

Lemma 6.47. *Let (X, τ_1, σ) be a real curve with involution of type $(g, \varepsilon, t_1, r_1, t_2, r_2)$, let $Q^+ \neq Q^- \in X$ be the only fixed points of σ and $\tau(Q^+) = Q^-$ and let $\pi_\sigma : X \to X_\sigma$ be the two-sheeted covering, where (X_σ, τ_σ) is a real curve of type (g, k, ε) with $k = t_1 + r_1 + t_2 + r_2$. Let the ovals of X_σ be denoted by c_i with $i \in \{1, \dots, k\}$ such that the following holds:*

- *For $j \in \{1, 2\}$ and $i = 1, \dots, t_j$, the preimage of c_i contains two ovals a_i^j and $a_{t_1+i}^j$ of τ_j,*
- *For $j \in \{1, 2\}$ and $i = t_1 + 1, \dots, t_1 + r_1$ respectively $i = t_1 + r_1 + t_2 + 1, \dots, k$, the preimage of c_i contains one oval a_i^j of τ_j.*

Furthermore, let ω_σ be a differential on X_σ which is non-negative on an oval c_i with $i \in \{t_1 + r_1 + 1, \dots, k\}$ and non-negative on an oval c_i with $i \in \{t_1 + 1, \dots, t_1 + r_1\}$. Then for the zero divisor of the pulled back differential $\omega := \pi^*\omega_\sigma$, which is due to Proposition A.4 of the form $D + \sigma(D)$, holds:

(a) *ω is non-negative on the ovals a_i^2 and $a_{i+t_2}^2$ for $i \in \{1, \dots t_2\}$ and non-negative on the oval a_i^2 for $i \in \{2t_2+1, \dots, 2t_2+r_2\}$ as an oval of the real curve (X, τ_2, σ).*

(b) *(i) Let $p \in D \cap a_i^1$ for $i \in \{2t_1+1, \dots, 2t_1+r_1\}$ and let γ_p be the open arc on a_i^1 between p and $\sigma(p)$ which starts in counter-clockwise direction at p such that $a_i^1 \setminus \{p, \sigma(p)\} = \gamma_p + \sigma_\sharp \gamma_p$. Then*

$$\chi_i(D) := \begin{cases} (-1)^{\deg(D \cap \gamma_p)} = -(-1)^{\deg(D \cap \sigma_\sharp \gamma_p)} & \text{if } \omega \text{ is positive in a small neighborhood of } p \text{ restricted to } \gamma_p, \\ -(-1)^{\deg(D \cap \gamma_p)} = (-1)^{\deg(D \cap \sigma_\sharp \gamma_p)} & \text{if } \omega \text{ is negative in a small neighborhood of } p \text{ restricted to } \gamma_p \end{cases}$$

is independent of the choice of $p \in D \cap a_i^1$.

a) If $D \cap a_i^1 \neq \emptyset$ for an $i \in \{2t_1+1, \dots, 2t_1+r_1\}$, then $D \cap a_i^1$ can always be transformed in such a way that the corresponding transformed divisor $\tilde{D}$ obeys $\tilde{D} = \sigma(\tilde{D})$ and $\chi_i(D) = \chi_i(\tilde{D})$.

Proof. (a) We have to show that ω can only have zeros of even order on a_i^2 with $i = 1, \dots, 2t_2+r_2$. Note that $\tau_2(D) = \sigma(\tau_1(D)) = \sigma(D)$. So for $p \in D \cap a_i^2$, there holds

$$\sigma(p) = \tau_2(p) = p$$

and since the divisor of zeros of the pulled back differential ω is of the form $D + \sigma(D)$, p is a zero of even order of ω. Since we consider ω as a real differential of the curve (X, τ_2, σ), the values of ω on a_i^2 are purely imaginary and the assertion follows.

In the proofs of the next two assertions, we consider the r_1 ovals of τ_1 which are invariant under σ. So let $i \in \{2t_1+1, \dots, 2t_1+r_1\}$. We assume without loss of generality that the multiplicity of all points in $(D + \sigma(D)) \cap a_i^1$ equals one. This situation can always be obtained by a small deformation of a given divisor D.

(b) A 1-form ω_σ is globally defined on X_σ, and therefore has to have an even number of sign changes on any simple closed contour. So the number of zeros of ω_σ on an oval on X_σ is always even. Moreover, we know from Corollary 6.38 that $\pi_\sigma : X \to X_\sigma$ restricted to a_i^1 is two-sheeted. Thus, every point $p_\sigma \in c_i$ has two preimages $p, \sigma(p) \in a_i^1$. Together this yields that the number of zeros of ω on a_i^1 is divisible by four.
Let γ_p be the open arc on a_i^1 as defined in the lemma. Then $\gamma_p \cup \sigma_\sharp \gamma_p = a_i^1 \setminus \{p, \sigma(p)\}$ as in Figure 6.9a. To see that

$$\deg((D + \sigma(D)) \cap \gamma) = \deg((D + \sigma(D)) \cap \sigma_\sharp \gamma) = 1 \mod 2,$$

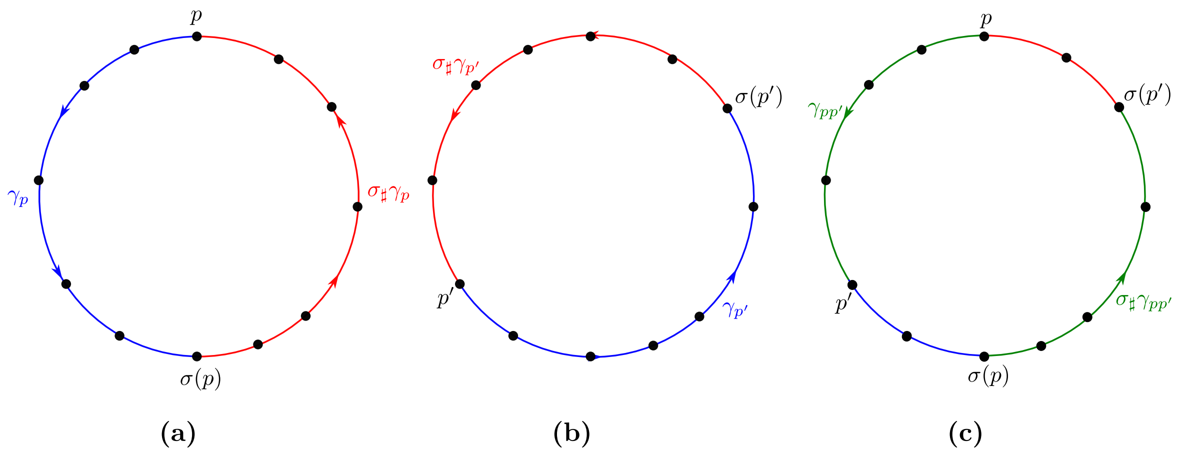


Figure 6.9.: Depicting the different curves on an oval a_i^1 for $i = 1, \dots, r_1$.

let $n \in \mathbb{N}$ such that $4n = \deg((D+\sigma(D))\cap a_i^1)$. Then $\deg((D+\sigma(D))\cap a_i^1 \setminus \{p, \sigma(p)\}) = 4n-2$. Since $\deg((D+\sigma(D))\cap\gamma) = \deg((D+\sigma(D))\cap\sigma_\sharp\gamma)$, the number of points of $D+\sigma(D)$ on each one of the arcs γ_p and $\sigma_\sharp\gamma_p$ equals $2n-1$. Therefore, either

$$\deg(D\cap\gamma) = \deg(\sigma(D)\cap\sigma_\sharp\gamma) = 0 \mod 2 \quad \text{and} \quad \deg(D\cap\sigma_\sharp\gamma) = \deg(\sigma(D)\cap\gamma) = 1 \mod 2$$

or conversely,

$$\deg(D\cap\gamma) = \deg(\sigma(D)\cap\sigma_\sharp\gamma) = 1 \mod 2 \quad \text{and} \quad \deg(D\cap\sigma_\sharp\gamma) = \deg(\sigma(D)\cap\gamma) = 0 \mod 2.$$

The divisor points lie discrete and we have assumed that the multiplicity of all points in $(D+\sigma(D))\cap a_i^1$ is one. So there is a small open neighborhood U_p of p such that ω is either positive or negative on $U_p\cap\gamma_p$ and has opposite sign on $U_p\cap\sigma_\sharp\gamma_p$. We define

$$\chi_{i,p}(D) := \begin{cases} (-1)^{\deg(D\cap\gamma_p)} = -(-1)^{\deg(D\cap\sigma_\sharp\gamma_p)} & \text{if } \omega \text{ is positive in } U_p\cap\gamma_p, \\ -(-1)^{\deg(D\cap\gamma_p)} = (-1)^{\deg(D\cap\sigma_\sharp\gamma_p)} & \text{if } \omega \text{ is negative in } U_p\cap\gamma_p. \end{cases}$$

To show that this sign is independent from the chosen reference point p, let $p' \neq p \in D\cap a_i^1$. Then p' and $\sigma(p')$ also decompose a_i^1 into two open arcs $\gamma_{p'}$ and $\sigma_\sharp\gamma_{p'}$, where $\gamma_{p'}$ starts again in counter-clockwise direction from p', compare Figure 6.9b. Without loss of generality we assume that $p'\in\gamma_p$. We denote the open arc between p and p' on a_i^1 as $\gamma_{pp'}$. Then $\sigma_\sharp\gamma_{pp'}$ is the open arc between $\sigma(p)$ and $\sigma(p')$ and there is also a small open neighborhood $U_{p'}$ of p' such that that ω does not change its sign on $\gamma_{p'}\cap U_{p'}$ and $\sigma_\sharp\gamma_{p'}\cap U_{p'}$. To see that $\chi_{i,p}(D) = \chi_{i,p'}(D)$, it remains to determine $\deg(D\cap\gamma_p')$. It is $\deg(D\cap\sigma_\sharp\gamma_{pp'}) = \deg(\sigma(D)\cap\gamma_{pp'})$ as depicted in

Figure 6.9c. Furthermore, p' is in $D \cap \gamma_p$, but $p \in D \cap \sigma_\sharp \gamma_{p'}$. So

$$\begin{aligned}\deg(D \cap \gamma_{p'}) &= \deg(D \cap \gamma_p) - \deg(D \cap \gamma_{pp'}) + \deg(D \cap \sigma_\sharp \gamma_{pp'}) + 1 \\ &= \deg(D \cap \gamma_p) - \deg(D \cap \gamma_{pp'}) + \deg(\sigma(D) \cap \gamma_{pp'}) + 1.\end{aligned}$$

Therefore,

$$\chi_{i,p'}(D) = -\chi_{i,p}(D) \frac{(-1)^{(\deg(D+\sigma(D)) \cap \gamma_{pp'})+1}(-1)^{\deg(\sigma(D) \cap \gamma_{pp'})}}{(-1)^{\deg(D \cap \gamma_{pp'})}} = \chi_{i,p}(D)$$

and $\chi_{i,p}(D)$ does not depend on $p \in D \cap a_i^1$. So we can set $\chi_i(D) := \chi_{i,p}(D)$.

(c) To show the last assertion, let $D \in \mathcal{E}$ and ω be a differential corresponding to D with $D \cap a_i \neq \emptyset$ as defined in $\mathcal{E}$. We first transform D such that

- the number of divisor points on a_i^1 remains constant,
- the transformed divisor stays in $\mathcal{E}$,
- the multiplicity of all deformed divisor points stays one.

Let us denote a divisor transformed like this by $\tilde{D}$ and the corresponding differential by $\tilde{\omega}$. We will see that $\chi_i(\tilde{D}) = \chi_i(D)$. Let $\tilde{D}$ be a transformed divisor such that only the points inside of γ_p and $\sigma_\sharp \gamma_p$ are transformed. In this case, it follows from the definition of $\chi_i(D)$ that $\chi_i(D) = \chi_i(\tilde{D})$. So it can only happen that $\chi_i(\tilde{D}) \neq \chi_i(D)$ if a point $q \in a_i^1 \cap D$ passes p respectively $\sigma(p)$ under the transformation, whereby p is the point which is used to determine $\chi_i(D)$. In this case, $\sigma(q)$ passes $\sigma(p)$ respectively p. If q passes p, the sign of $\tilde{\omega}$ is opposite to the sign of ω on $\gamma_p \cap U_p$, but also

$$\deg(\tilde{D} \cap \gamma_p) = \deg(D \cap \gamma_p) + 1 \mod 2 \quad \text{as well as} \quad \deg(\tilde{D} \cap \sigma_\sharp \gamma_p) = \deg(D \cap \sigma_\sharp \gamma_p) + 1 \mod 2 \tag{6.10}$$

So $\chi_i(\tilde{D}) = \chi_i(D)$. If q passes $\sigma(p)$, $\sigma(q)$ passes p, and so the sign of $\tilde{\omega}$ on $\gamma_p \cap U_p$ is opposite to the sign of ω and (6.10) also holds. Accordingly, $\chi_i(\tilde{D}) = \chi_i(D)$. Thus, $\chi_i(D)$ is invariant under all considered transformations of D.

We have seen in the proof of (b) that $\deg((D + \sigma(D)) \cap a_i^1)$ is divisible by four. So by the above transformations one can always sort the points in $(D + \sigma(D)) \cap a_i^1$ into pairs $(q', \sigma(q))$ with $q \neq q'$ such that $\chi_i(D)$ remains constant. Now moving these points together yields a divisor $\tilde{D}$ such that $\sigma(\tilde{D}) = \tilde{D}$ on a_i^1. That means the divisor D can be transformed into a divisor $\tilde{D}$ which originates from the square of a spinor and which is non-positive, respectively non-negative on a_i^1.

□

Lemma 6.47 justifies the following definition.

Definition 6.48. Let (X, τ_1, σ) be a real curve with involution of type $(g, \varepsilon, t_1, r_1, t_2, r_2)$. A differential ω_D on X with $D \in \mathcal{E}$ is *positive definite on an oval* a_i^1 *with* $i = 2t_1 + 1, \ldots, 2t_1 + r_1$ if one of the following conditions holds:

(i) ω_D is non-negative on a_i^1.

(ii) The total number of points in $D \cap a_i^1$ divided by two is even and $\chi_i(D) = 1$.

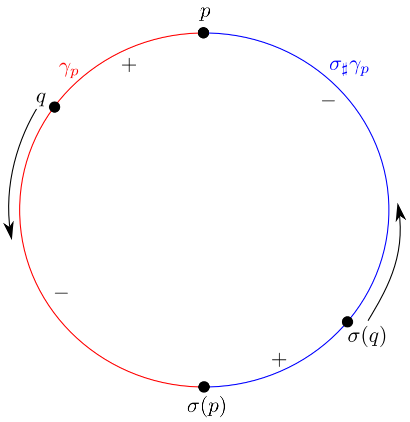
(iii) The total number of points in $D \cap a_i^1$ divided by two is odd and $\chi_i(D) = -1$.

Otherwise, we say that ω_D is *negative definite on* a_i^1. The differential ω_D is *positive* respectively *negative definite on an oval* a_i^2 *with* $i = 1, \ldots, 2t_2 + r_2$ if it is non-negative respectively non-positive on this oval as a real differential of the curve (X, τ_2, σ).

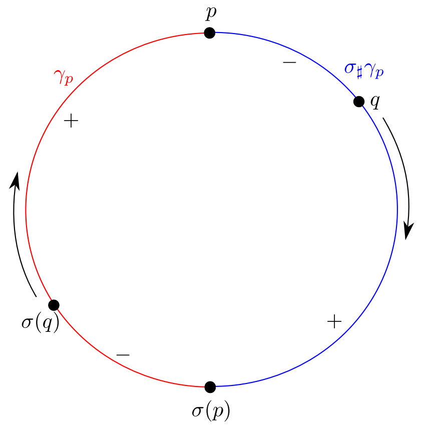
Statement (a) in comparison to (b) and (c) of Lemma 6.47 shows that the conditions for definiteness on the ovals of τ_2 is easier than the condition on the ovals a_i^1 of τ_1 with $i = 2t + 1 + 1, \ldots 2t_1 + r_1$. This is because for D holds $\tau_2(D) = \sigma(D)$. Since we do not have this additional structure on the ovals of τ_1, we can only define the complicated version of definiteness as above and for the ovals a_i^1 with $i = 1, \ldots, 2t_1$ we cannot at all define definiteness because any real holomorphic differential on X_σ lifted to X can also have zeros of first order on these ovals.
Since every real spinor η obeys that $(\eta^2) = 2(\eta) \in \mathcal{E}$, the question arises how $\chi_i(D)$ yields the orientation which a spinor η induces on an oval a_i^1. We show in the following Lemma why we think that our modified version of the definiteness makes sense.

Lemma 6.49. *Let (X, τ_1, σ) be a real curve of type $(g, \varepsilon, t_1, r_1, t_2, r_2)$ and $D \in \mathcal{E}$. Then for $i = 2t_1 + 1, \ldots, 2t_1 + r_1$, the divisor $D \cap a_i^1$ can be transformed into the divisor of a non-negative square of a spinor on a_i^1 if and only if any ω_D associated with $D \in \mathcal{E}$ is positive definite on a_i^1.*

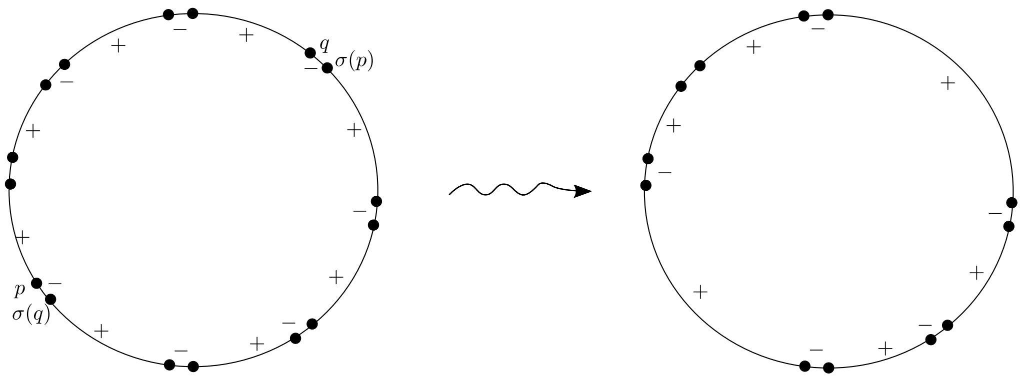
Proof. Let D_m be the zero divisor of a differential $\omega_m = \pi^* \omega_{\sigma,m}$ such that the number of points in $(D + \sigma(D)) \cap a_i^1$ equals $4m$ with $m \geq 1$. As in the foregoing proof of Lemma 6.47, we assume without loss of generality that the multiplicity of all points in $D_m + \sigma(D_m)$ equals one. We have also seen in this proof that one always can deform D_m into a divisor $\tilde{D}_m$ with $\chi_i(D_m) = \chi_i(\tilde{D}_m)$ and all points on $(\tilde{D}_m + \sigma(\tilde{D}_m)) \cap a_i^1$ are sorted into pairs of pairs $((p, \sigma(q)), (q, \sigma(p)))$ with $p \neq q$. Again, let U_p be a small open neighborhood around p such that the sign of ω_m is constant on $U_p \cap \gamma_p$ as well as on $U_p \cap \sigma_\sharp \gamma_p$. For $m = 1$, one has that $D_1 = p + q$. Then either $q \in \gamma_p$ or $q \in \sigma_\sharp \gamma_p$. Moving q into the direction of p and $\sigma(q)$ into the direction of $\sigma(p)$ is not possible since the transformed divisor $\tilde{D}_1$ can only be obtained as the square of a real spinor. So to transform $D_1 = p + q$ in such a way that the transformed divisor $\tilde{D}_1$ is the divisor of a differential $\tilde{\omega}_1$ which is obtained by squaring a spinor, one has to move q towards $\sigma(p)$ and $\sigma(q)$ to p. This square has the divisor $2\tilde{D}_1$. Then $\tilde{D}_1$ can only be contained in $\mathcal{E}$ if $\tilde{D}_1 = \sigma(\tilde{D}_1)$. For $q \in \gamma_p$, the sign of ω_1 on $U_p \cap \gamma_p$ is opposite to $\chi_i(D_1)$. The corresponding situation is depicted in Figure 6.10a on the next page. Moving p and $\sigma(q)$ as well as q and $\sigma(p)$ together yields a divisor $\tilde{D}_1$ of the form $\tilde{D}_1 = \sigma(\tilde{D}_1)$ and the sign of ω_1 on $\gamma_p \cap U_p$ equals the sign of the corresponding square of a spinor $\tilde{\omega}_1$ with divisor



(a) Depicting the case $m = 1$ for $q \in \gamma_p$ with $\omega > 0$ on $\gamma_p \cap U_p$. Then $\chi_i(D_1) = -1$.



(b) Depicting the case $m = 1$ for $q \in \sigma_\sharp\gamma_p$ with $\omega > 0$ on $\gamma_p \cap U_p$. Then $\chi_i(D_1) = 1$.



(c) Depicting that the sign of the square of the spinor by which D_m is induced equals the sign of the spinor by which D_{m-1} is induced. The $\pm$-signs indicate the sign of ω_m respectively ω_{m-1} on the oval.

Figure 6.10.: Sketch corresponding to Lemma 6.49: In (a) and (b), we sketch the two possible situations to deform a divisor $D_1 + \sigma(D_1)$ on an oval a_i^1 with $D_1 \cap a_i^1 = p + q$ into a divisor which is the square of a spinor. The arrows denote the direction in which q and $\sigma(q)$ are transformed and the sign of the differential form ω_1 with divisor $D_1 + \sigma(D_1)$ on the different parts of a_i^1 is indicated by the $\pm$-signs.
In (c) is sketched why the sign of ω_{D_k} on a small open neighborhood $U_{\tilde{p}} \cap a_i^1$ of some point $\tilde{p} \in D \cap a_i^1$ with $\tilde{p} \neq p, q, \sigma(p)$ equals the sign of $\omega_{D_{k-1}}$ on this neighborhood.

$2\tilde{D}_1$. For $q \in \sigma_\sharp\gamma_p$ as depicted in Figure 6.10b, $\chi_i(D)$ equals the sign of ω_1 on $U_p \cap \gamma_p$. However, in case $\sigma(q) \in \gamma_p$, the sign of ω_1 on $\gamma_p \cap U_{\sigma(p)}$ equals the sign of the corresponding square of a spinor $\tilde{\omega}_1$. So in both cases, $\tilde{\omega}_1$ is the square of a spinor which is non-positive on a_i^1 if $\chi_i(D_1) = 1$ and which is non-negative on a_i^1 if $\chi_i(D_1) = -1$.
For $m > 1$, we assume that the points of the divisors D_m are already sorted into pairs of pairs of the form $((p, \sigma(q)), (q, \sigma(p)))$ with $p \neq q, \sigma(q)$ and denote the divisor D_m with one such pair of

pairs taken away as D_{m-1}. The sign of the differential $\omega_{D_{m-1}}$ switches four times less on a_i^1 than the sign of ω_{D_m} on a_i^1. Accordingly, the sign of the square of the spinor by which D_m is induced equals the sign of the spinor by which D_{m-1} is induced, compare Figure 6.10c. So if we choose a pair $(\tilde{p}, \sigma(\tilde{p}))$ with $\tilde{p}, \sigma(\tilde{p}) \neq q, p$, then the sign of ω_{D_m} equals the sign of $\omega_{D_{m-1}}$ on $\gamma_{\tilde{p}} \cap U_{\tilde{p}}$ with $U_{\tilde{p}}$ a sufficiently small neighborhood of $\tilde{p}$. Together with $\deg(D_{m-1} \cap \gamma_{\tilde{p}}) = \deg(D_m \cap \gamma_{\tilde{p}}) - 1$, this yields

$$\chi_i(D_m) = -\chi_i(D_{m-1}).$$

Successively leaving in total $m-1$ pairs of pairs of points in D_m away yields a divisor D_1 such that

$$\chi_i(D_m) = (-1)^{m-1}\chi_i(D_1)$$

Hence, for m even, $\chi_i(D_m) = -\chi_i(D_1)$ and for m odd, $\chi_i(D_m) = \chi_i(D_1)$, so the assertion follows. $\square$

Let us decompose the set $\{a^1_{2t_1+1}, \dots, a^1_{2t_1+r_1}, a^2_1, \dots, a^2_{2t_2+r_2}\}$ into subsets A_+ and A_- such that

$$A_+ \cup A_- = \{a^1_{2t_1+1}, \dots, a^1_{2t_1+r_1}, a^2_1, \dots, a^2_{2t_2+r_2}\} \quad \text{and} \quad A_+ \cap A_- = \emptyset.$$

We define

$$\delta := (\delta_1, \dots, \delta_{t_1}) \in (\mathbb{Z}_2)^{t_1}$$

and denote by $\mathcal{E}(\delta, A_+, A_-)$ the subset of $\mathcal{E}$ consisting of the divisors $D \in \mathcal{E}$ such that ω_D or $-\omega_D$ is positive definite on all ovals in A_+, negative definite on all ovals in A_- and such that for $i = 1, \dots, t_1$, there holds

$$\frac{\deg(D \cap a_i^1)}{2} \mod 2 = \delta_i.$$

Corollary 6.38 yields that $r_1 + r_2 = 1 \mod 2$, so $A_+ = A_- = \emptyset$ is not possible. Here, we extended the proof given in [Natanzon, 2004] – which comprises a bit more than half a page – a lot.

Lemma 6.50 ([Natanzon, 2004, Lemma 2.9.3])**.** *Each of the sets $\mathcal{E}(\delta, A_+, A_-)$ is non-empty if not simultaneously $A_+ = \emptyset$ and $A_- = \emptyset$.*

Proof. Let (X, τ_1, σ) be a real curve with involution of type $(g, \varepsilon, t_1, r_1, t_2, r_2)$ with $r_1 + r_2 = 1 \mod 2$. To see these assertions, we have to show the existence of holomorphic differential forms which have the properties encoded by $\mathcal{E}(\delta, A_+, A_-)$ on X. This will be done by showing the existence of certain holomorphic 1-forms ω_σ on (X_σ, τ_σ) which we can pull back to X such that $\omega := \sigma^*\omega_\sigma$ has the desired properties. To do so, we first show the existence of certain holomorphic 1-forms for any real curve $(\tilde{X}, \tilde{\tau})$ and then apply these results to (X_σ, τ_σ).
So let $(\tilde{X}, \tilde{\tau})$ be a real curve of type (g, k, ε) with ovals $c_1, \dots, c_k$ and let $k := k_+ + k_- + k_0$ with $k_0 < g$ and $k_+ \cdot k_- \neq 0$. We show that for any pair of points $\tilde{Q}^+ \neq \tilde{Q}^- \in \tilde{X}$ such that $\tilde{Q}^- = \tilde{\tau}(\tilde{Q}^+)$, there is a real differential $\tilde{\omega}$ with the following properties:

(i) $\tilde{\omega}$ is either a holomorphic differential on X or a meromorphic differential on X which is holomorphic on $X \setminus \{\tilde{Q}^+, \tilde{Q}^-\}$ and has at most poles of degree one at $\tilde{Q}^+$ and $\tilde{Q}^-$.

(ii) $\tilde{\omega}$ is non-negative on c_i for $i \leq k_+$, non-positive on c_i for $k_+ < i \leq k_+ + k_-$ and has zeros on c_i for $i > k_+ + k_-$ such that the order of these zeros divided by two is odd.

For this purpose, we will now construct another real curve out of $\tilde{X}$ with $k+1$ ovals. This is done by cutting out small open discs around $\tilde{Q}^+$ and $\tilde{Q}^-$ which yields a surface with two boundary components. These boundary components we glue back together such that the 'glueing contour' yields an additional oval of $\tilde{\tau}$ on the new real curve. To this new real curve we can apply the results from Section 6.2.2. We then degenerate the new oval to obtain the above assertions.

For the glueing procedure, it is necessary to define the small discs and a local chart to identify the boundaries of these discs with each other. So let z be a local coordinate centered at $\tilde{Q}^+$. Then $\tilde{\tau}^*\bar{z}$ is a local coordinate centered at $\tilde{Q}^-$. We define disjoint open disks $D_t(\tilde{Q}^\pm) \subset \tilde{X}$ as the preimage of $\{z \in \mathbb{C} \mid |z| < t\}$ under z respectively $\tilde{\tau}^*\bar{z}$, where $t \in (0, \varepsilon + \tilde{\varepsilon})$ and $0 < \tilde{\varepsilon} < \varepsilon$. Since $\tilde{\tau}$ is an isometry on $\tilde{X}$, these discs obey $\tilde{\tau}[D_t(\tilde{Q}^\pm)] = D_t(\tilde{Q}^\mp)$. In order to identify the boundaries $\partial D_t(\tilde{Q}^+)$ and $\partial D_t(\tilde{Q}^-)$ of the surface $\tilde{X} \setminus (D_t(\tilde{Q}^+) \cup D_t(\tilde{Q}^-))$ for $t \in (0, \varepsilon)$ by means of the involution $\tilde{\tau}$ such that the surface $\tilde{X}_t$ obtained like this is a Riemann surface, it is necessary to find local charts on the boundary components $\partial D_t(\tilde{Q}^\pm)$ such that we can glue $\partial D_t(\tilde{Q}^+)$ and $\partial D_t(\tilde{Q}^-)$ together. The local coordinate z from above defines a local chart on $D_{t+\tilde{\varepsilon}}(\tilde{Q}^+) \setminus D_t(\tilde{Q}^+)$. Then $\frac{t^2}{\tilde{\tau}^*\bar{z}}$ defines a local chart for $D_{t+\tilde{\varepsilon}}(\tilde{Q}^-) \setminus D_t(\tilde{Q}^-)$. The local chart z maps $\partial D_t(\tilde{Q}^+)$ isomorphically to $\{z \in \mathbb{C} \mid |z| = t\}$ and the image of $D_{t+\tilde{\varepsilon}}(\tilde{Q}^+) \setminus D_t(\tilde{Q}^+)$ under z is contained in $\{z \in \mathbb{C} \mid |z| > t\}$. Likewise the local chart $\frac{t^2}{\tilde{\tau}^*\bar{z}}$ maps $\partial D_t(\tilde{Q}^-)$ also isomorphically to $\{z \in \mathbb{C} \mid |z| = t\}$, but the image of $D_{t+\tilde{\varepsilon}}(\tilde{Q}^-) \setminus D_t(\tilde{Q}^+)$ is contained in $\{z \in \mathbb{C} \setminus \{0\} \mid |z| < t\}$. By means of these local charts, we can identify the boundaries $\partial D_t(\tilde{Q}^+)$ and $\partial D_t(\tilde{Q}^-)$ with each other, whereby we identify the images of $p \in \partial D_t(\tilde{Q}^+)$ and $\tilde{\tau}(p) \in \partial D_t(\tilde{Q}^-)$ with each other. For each $t \in (0, \varepsilon)$, we denote the compact Riemann surface constructed like this by $\tilde{X}_t$ and the involution indicated by $\tilde{\tau}$ on $\tilde{X}_t$ by $\tilde{\tau}_t$. This involution acts on the local charts as $z \mapsto \frac{t^2}{\bar{z}}$ and the 'glueing contour' is constructed in such a way that it is an oval $c_{t,g+1}$ of $\tilde{\tau}_t$ on $\tilde{X}_t$. So for $t \in (0, \varepsilon)$, we have tinkered a family of real curves $(\tilde{X}_t, \tilde{\tau}_t)$ with an additional oval $c_{t,g+1}$ and ovals $c_{t,i} = c_i$ for $i = 1, \ldots, g$. Each of these curves is a real curve of type $(g+1, k+1, \varepsilon)$.

Let $t \in (0, 1)$. Without loss of generality, we assume that $k_+ \neq 0$ on $(\tilde{X}_t, \tilde{\tau}_t)$ and set $k_{t,-} := k_- + 1$. With $k_{t,0} = k_0$, $k_{t,+} = k_+$ and $k_t = k_{t,+} + k_{t,-} + k_{t,0} = k + 1$, it is $k_{t,+} \cdot k_{t,-} \neq 0$ and $k_{t,0} < g + 1$. By Theorem 6.19, there exists a real holomorphic differential $\tilde{\omega}_t$ on $\tilde{X}_t$ which is non-positive on $c_{k+1,t}$ and with the desired properties on the ovals $c_{t,1}, \ldots, c_{t,k}$ as described in (ii). In the proof of Theorem 6.19 it becomes clear that these 1-forms are squares of real spinors in the sense of Definition B.37.

Next we consider the limit $(\tilde{X}_0, \tilde{\tau}_0)$ of $(\tilde{X}_t, \tilde{\tau}_t)$ for $t \to 0$. On $(\tilde{X}_0, \tilde{\tau}_0)$, the oval c_{k+1} is degenerated to an ordinary double point $\tilde{Q}$. By the construction of $(\tilde{X}_0, \tilde{\tau}_0)$, the normalization of this curve equals $(\tilde{X}, \tilde{\tau})$ which is a real curve of type (g, k, ε) and the preimages of $\tilde{Q}$ on $\tilde{X}$ are just $\tilde{Q}^+$ and $\tilde{Q}^-$. Let us now consider the behavior of $\tilde{\omega}_t$ in this limit.

For $t \in (0, 1)$, there exists a real differential $\tilde{\omega}_t$ on each curve $(\tilde{X}_t, \tilde{\tau}_t)$ which is holomorphic on $\tilde{X}_t$ with the behavior prescribed by (ii) on the ovals. We normalize these differentials such that $\|\tilde{\omega}_t\|_{L^2(X_t)} = 1$. This does not influence the properties in (ii). We want to show that these normalized $\tilde{\omega}_t$ converge to a real differential $\tilde{\omega}_0$ on $(\tilde{X}_0, \tilde{\tau}_0)$. Since $\tilde{\omega}_t$ with $t \in (0, \varepsilon)$ defines a uniformly bounded family of holomorphic functions, Montel's Theorem [Krantz, 2012, Theorem 8.4.3] assures that this sequence has a convergent subsequence. The limit $\tilde{\omega}_0$ of this subsequence is a real differential form on $(\tilde{X}_0, \tilde{\tau}_0)$. We claim that $\tilde{\omega}_0$ is a regular differential form on X_0. This means that we have to show that the residue of $\tilde{\omega}_0$ at the double point $\tilde{Q}$ multiplied with an arbitrary regular function on $\tilde{X}_0$ equals zero. As already mentioned, $\pi : \tilde{X} \to \tilde{X}_0$ is the normalization of $\tilde{X}_0$ with $\pi^{-1}(\tilde{Q}) = \{\tilde{Q}^+, \tilde{Q}^-\}$. The regular functions at the ordinary double point $\tilde{Q}$ are precisely the holomorphic functions of the normalization of this double point which take the same values at $\tilde{Q}^+$ and $\tilde{Q}^-$, compare [Klein et al., 2016, Example 2.5.1]. Let f be a regular function on $\tilde{X}_0$. Then it is $\pi^* f(Q^+) = \pi^* f(Q^-)$. We have to show that

$$\mathrm{Res}_{\tilde{Q}} \,(f \lim_{t\to 0} \tilde{\omega}_t) = 0. \tag{6.11}$$

The function $f|_{\tilde{X}_t}$ is not a holomorphic function on $\tilde{X}_t$, but $f_t := f|_{\tilde{X}_t \setminus c_{k+1,t}}$ is holomorphic. Let $\gamma^\pm$ be the boundary components of a small tubular neighborhood of $c_{k+1,t}$ on X_t. Representing f_t by its Taylor series in an open neighborhood of $\tilde{Q}$ yields that for $p \in \gamma^\pm$, $f_t(p)$ converges due to Riemann's Theorem of Removable Singularities to the holomorphic function f on X_0 and $\lim_{t\to 0} \oint_{\gamma^\pm} f_t \tilde{\omega}_t = f(\tilde{Q}) \oint_{\gamma^\pm} \tilde{\omega}_t$. Inserting this into (6.11) yields

$$\mathrm{Res}_{\tilde{Q}} \,(f \lim_{t\to 0} \tilde{\omega}_t) = f(\tilde{Q}) \mathrm{Res}_{\tilde{Q}} \lim_{t\to 0} \tilde{\omega}_t.$$

By the Residue Theorem, it is $\oint_{\gamma_+} \tilde{\omega}_t + \oint_{\gamma_-} \tilde{\omega}_t = 0$. Then $\mathrm{Res}_{\tilde{Q}}(f \tilde{\omega}_t) = 0$ for all $t \in (0, \varepsilon)$. By continuity, also $\mathrm{Res}_{\tilde{Q}} \lim_{t\to 0}(f \tilde{\omega}_t) = 0$. Accordingly, $\tilde{\omega}_0$ is a regular differential form on $\tilde{X}_0$. Because $\tilde{Q}$ is an ordinary double point of $\tilde{X}_0$, the regular 1-forms at this point are, in a local coordinate z centered at $\tilde{Q}$, generated by $\frac{1}{z}$ and z. So the differential $\tilde{\omega}_0$ is either holomorphic at $\tilde{Q}$ or has a pole of first order at this double point. Since the deformation of X_t for $t \to 0$ is continuous, also the differentials on all deformed curves can only change continuously. We know already that $\tilde{\omega}_\varepsilon$ is the square of a real spinor on $\tilde{X}_\varepsilon$ for $t \in (0, \varepsilon)$. The set of real spinors on $\tilde{X}_t$ only contains elements whose square is an element of the cotangent bundle, so it is discrete on $\tilde{X}_t$. Therefore, the number of zeros of $\tilde{\omega}_t$ on the ovals does not change. So if property (ii) holds for $\tilde{\omega}_\varepsilon$, then it also holds for $\tilde{\omega}_t$ with $t \in (0, \varepsilon)$. Altogether, this yields that the differential $\tilde{\omega}_0 := \lim_{t\to 0} \omega_t$ obeys property (ii) and has at most a pole of first order at $\tilde{Q}$.

Summarizing, we have now seen that the normalization of $(\tilde{X}_0, \tilde{\tau}_0)$ equals the curve $(\tilde{X}, \tilde{\tau})$, that

the preimage of $\tilde{Q}$ on this normalization are exactly the points $\tilde{Q}^+$ and $\tilde{Q}^-$ and that the pullback of $\tilde{\omega}_0$ to $\tilde{X}$ is a real differential $\tilde{\omega}$ with the desired properties (i) and (ii).
We now consider a real curve with holomorphic involution (X, τ_1, σ) of type $(g, \varepsilon, t_1, r_1, t_2, r_2)$ and apply the foregoing observations on the existence of a real differential on $(\tilde{X}, \tilde{\tau})$ to (X_σ, τ_σ). For $i = 1, \dots, t_1$, let the ovals c_i be the images of the ovals a_i^1 and $\sigma_\sharp a_i^1$ under $\pi_\sigma : X \to X_\sigma$. The remaining ovals of (X, τ_1, σ) are either contained in A_+ or in A_-. We denote the set of the images of the ovals contained in A_+ under π_σ by A_+^σ and the set of the images of the ovals contained in A_- by A_-^σ. Applying the above result to the real curve (X_σ, τ_σ), we find a differential ω_σ on X_σ that is either meromorphic on X_σ and holomorphic away from $Q_\sigma^+ := \pi_\sigma(Q^+)$ and $Q_\sigma^- := \pi_\sigma(Q^-)$ with at most simple poles at these points or which is holomorphic on all of X_σ. Furthermore, by appropriate enumeration of the ovals of X_σ, ω_σ is non-negative on the ovals in A_+^σ and non-positive on the ovals in A_-^σ and for $i \in \{1, \dots, t_1\}$, the images of the ovals a_i^1 and $\sigma_\sharp a_i^1$ with $\delta_i = 1$ are ovals $c_i \subset X_\sigma$ such that $k_+ + k_- < i < k_+ + k_- + k_0$ and the ovals which are images of a_i^1 and $\sigma_\sharp a_i^1$ with $\delta_i = 0$ are somehow contained in the set of ovals c_i with $i \leq k_+ + k_-$.
Using the two-sheeted covering π_σ, we can pull back ω_σ to a 1-form $\omega = \pi_\sigma^* \omega_\sigma$ on X as in Proposition A.4. The pullback $\omega = \pi_\sigma^* \omega_\sigma$ is a differential which is – due to Lemmata 6.47 and 6.49 – positive definite on the ovals in A_+ and negative definite on the ovals in A_-. Moreover, ω_σ is the square of a real spinor, compare the proof of Theorem 6.19. Therefore, all zeros of ω_σ on $X_\sigma \setminus \{Q_\sigma^+, Q_\sigma^-\}$ are of even order. Accordingly, the divisor of zeros of $\omega = \pi_\sigma^* \omega_\sigma$ intersected with $a_i^1 \cup \sigma_\sharp a_i^1$ for $i \leq t_1$ has positive degree which is divisible by four and is symmetric with respect to σ. So the number of zeros on a_i^1 respectively $\sigma_\sharp a_i^1$ is even for $i \in \{1, \dots, t_1\}$. This number divided by two is odd if the index i of the image c_i of these ovals on X_σ obeys $k_+ + k_- < i < k_+ + k_- + k_0$.
Let z be a local coordinate centered at Q_σ^+ respectively Q_σ^-. Then ω_σ reads in these coordinates as $f(z)dz$, see the proof of Proposition A.1. Since $Q_\sigma^\pm$ are branch points of the covering π_σ, we can choose these local coordinates in such a way that $\omega = \pi_\sigma^* \omega_\sigma$ can be represented locally around $Q^\pm$ as $f(z^2)dz^2 = 2f(z^2)zdz$. We know from the above considerations that ω either is holomorphic or has a pole of first order at $Q_\sigma^\pm$, i.e. $f(z)$ either is holomorphic or has a pole of first order at $z = 0$. If $f(z)$ is holomorphic with $f(0) \neq 0$, then also $f(z^2)$ is holomorphic and has no zero at $z = 0$. However, in this case $f(z^2)z$ has a zero of first order at $z = 0$. If $f(0) = 0$ and this zero is of order n, then $f(z^2)$ has a zero of order $2n$ at $z = 0$, and so $f(z^2)z$ has a zero of order $2n + 1$ at $z = 0$. Analogously if $f(z)$ has a pole of first order at $z = 0$, then $f(z^2)$ has a pole of second order at $z = 0$ and thus $f(z^2)z$ has a pole of first order at $z = 0$. So $\omega = \pi_\sigma^* \omega_\sigma$ is either a meromorphic differential that is holomorphic away from Q^+ and Q^- and has simple poles at these points or is holomorphic on X with zeros of odd order at Q^+ and Q^-.
Next, we show that one can apply Theorem 6.19 to obtain a differential for every choice of $\mathcal{E}(\delta, A_+, A_-) \neq \mathcal{E}(\delta, \emptyset, \emptyset)$. Therefore, let $(X_\sigma, \tau_\sigma) \to (\tilde{X}_0, \tilde{\tau}_0)$ be the normalization of the real curve with one double point constructed at the beginning of this proof and let $(\tilde{X}_t, \tilde{\tau}_t)$ be the corresponding curves with one extra oval. We want to show that certain real spinors as in Definition

B.37(b) with prescribed zero order respectively orientation on the ovals $c_{t,i}$ exists on $\tilde{X}_t$ for $t \in (0, \varepsilon)$ to obtain by Theorem 6.19 that there exists a real differential $\tilde{\omega}_t$ with certain properties on the ovals of $\tilde{X}_t$. From this, we can deduce the existence of a differential $\tilde{\omega}_0$ on $\tilde{X}_0$ with the same properties on the ovals $c_1, \dots c_k$ as $\tilde{\omega}_t$ has on $c_{1,t}, \dots, c_{k,t}$. This is possible because only the oval $c_{k+1,t}$ is deformed in the limit $t \to 0$. Then also the pullback ω_σ of this differential to (X_σ, τ_σ) under the normalization map has the same properties on the ovals $c_1, \dots, c_k$. We then consider the divisor of ω_σ on X_σ and show that for any admissible configuration of $\mathcal{E}(\delta, A_+, A_-)$, there exists an ω_σ constructed like above, such that the corresponding divisor is an element of $\mathcal{E}(\delta, A_+, A_-)$. We define $\tilde{A}_+^\sigma := A_+^\sigma$ and $\tilde{A}_-^\sigma := A_-^\sigma \cup \{c_{k+1}\}$. Furthermore, we use the notation for α_i and m as well as the results from Theorems B.42 and B.43. These assert the following for fixed $t \in (0, \varepsilon)$:

- Let the real curve $(\tilde{X}_t, \tilde{\tau}_t)$ be of type $(g+1, k+1, 0)$ with oriented ovals $c_{1,t}, \dots, c_{k+1,t}$ and let $0 \leq m \leq k+1$, $\alpha_1, \dots, \alpha_{k+1} \in \mathbb{Z}_2$ and $\sum_{i=1}^{k+1} \alpha_i \equiv g+1 \mod 2$. Then it is shown in Theorem B.42 that there exists a real spinor η on $(\tilde{X}_t, \tilde{\tau}_t)$ such that the orientation of the oval $c_{i,t}$ generated by η coincides with the original orientation of $\tilde{X}_t \setminus \tilde{X}_t^{\tilde{\tau}_t}$ if and only if $i \leq m$ and such that the number of zeros of the spinor η modulo 2 on the oval $c_{i,t}$ is equal to α_i.
- Let the real curve $(\tilde{X}_t, \tilde{\tau}_t)$ be of type $(g+1, k+1, 1)$, let its ovals $c_{1,t}, \dots, c_{k+1,t}$ be oriented as parts of the boundary of a connected component $\tilde{X}_\sigma^+$ of the set $\tilde{X}_\sigma \setminus \tilde{X}_\sigma^{\tau_\sigma}$ and let the set $\{\alpha_1, \dots, \alpha_{k+1}\}$ with $\alpha_i \in \mathbb{Z}_2$, $\alpha_1 = \alpha_{k+1} = 0$ contain evenly many zeros. Then it is shown in Theorem B.43 that for every m with $1 \leq m < k+1$ and $\sum_{i=1}^{m} \alpha_i \equiv m+1 \mod 2$, there exists a real spinor η on $(\tilde{X}_t, \tilde{\tau}_t)$ such that the orientation generated on the oval $c_{i,t}$ by η coincides with the orientation of $c_{i,t}$ induced by the orientation of $\tilde{X}_t^+$ if and only if $i \leq m$ and the number of zeros of η modulo 2 on $c_{i,t}$ is equal to α_i.

Note that on the ovals in $\tilde{X}_t$ corresponding to the ovals in X_σ which are images of a_i^1 and $\sigma_\sharp a_i^1$ with $i \leq t_1$ under π_σ, the values α_j in Theorems B.42 and B.43 equals δ_i for $i = 1, \dots t_1$ and for the remaining ovals in A_+^σ and A_-^σ, the value α_j is not determined by this classification. Hence, we can choose them arbitrarily in $\mathbb{Z}_2$.
Let us first consider $\varepsilon = 0$: If $\sum_{i=1}^{t_1} \delta_i = g+1 \mod 2$, we set $\alpha_i = 0$ for $i = t_1+1, \dots, k+1$. If $\sum_{i=1}^{t_1} \delta_i = g \mod 2$, we set $\alpha_{k+1} = 1$ and $\alpha_i = 0$ for $i = 1, \dots, k$. In both cases, this yields $\sum_{i=1}^{k+1} \alpha_i = g+1 \mod 2$. We also have $k_0 \leq t_1 < g+1$ since $r_1 + r_2 = 1 \mod 2$. Hence, Theorem B.42 can be applied and yields a spinor η on the curve $\tilde{X}_t$ such that $\tilde{\omega}_t = \eta^2$. Degenerating the oval c_{k+1} on $\tilde{X}_t$ and considering the differential on the normalization X_σ of $\tilde{X}_0$ yields a differential ω_σ on X_σ such that $\omega = \pi_\sigma^* \omega_\sigma$ has the desired properties. Thus, for $\varepsilon = 0$, any admissible configuration of $\mathcal{E}(\delta, A_+, A_-)$ can be achieved.
For $\varepsilon = 1$, there has to hold

$$\sum_{i=1}^{m} \alpha_i = m+1 \mod 2, \quad \sum_{i=1}^{k+1} \alpha_i = k+1 \mod 2 \text{ and } \alpha_1 = \alpha_{k+1} = 0.$$

We will now consider all admissible cases for the choices of (δ, A_+, A_-). Hereby, it is sufficient to consider only $A_- = \emptyset$ since the considerations for $A_+ = \emptyset$ follow analogously by considering $-\omega$ instead of ω. For brevity, we write $\sum_{\tilde{A}_j^\sigma} \alpha_i$ if we take the sum over the elements α_i corresponding to the ovals in $\tilde{A}_j^\alpha$ for $j = 1, 2$.

- Let $\delta = \emptyset$ and $A_- = \emptyset$. Then the number of ovals in $\tilde{A}_+^\sigma$ equals $k \geq 1$ since we assumed that $X^\tau \neq \emptyset$, and therefore also $X_\sigma^{\tau_\sigma} \neq \emptyset$, compare Lemma 6.35. Hence, $k_0 = 0 < g + 1$. The number of ovals of $\tilde{X}_t$ is $k + 1$. Setting $m = k$ in Theorem 6.19 yields that for k_+ and k_- in Theorem 6.19 holds $k_+ \cdot k_- \neq 0$. We further set $\alpha_1 = 0$ and $\alpha_{k+1} = 0$ and all other values $\alpha_i = 1$. This yields a combination of $\alpha_1, \ldots, \alpha_k$ such that the total number of zeros of them is even and such that $\sum_{i=1}^m \alpha_i = \sum_{i=1}^{k+1} \alpha_i = k + 1 \mod 2 = m + 1 \mod 2$. So applying Theorem B.43 yields a real spinor η on $\tilde{X}_t$ such that, by Theorem 6.19, η^2 is a holomorphic differential on $\tilde{X}_t$. Due to the explanations above, the corresponding differential ω_σ on X_σ can then be pulled back to X such that the divisor of $\omega = \pi_\sigma^* \omega_\sigma$ is an element of $\mathcal{E}(\emptyset, A_+, \emptyset)$.
- Let $\delta = \emptyset$ and $A_+, A_- \neq \emptyset$. Again, $k_0 = 0 < g + 1$. Let us enumerate the ovals of $\tilde{X}_\sigma$ in such a way that $c_1, \ldots, c_m \in A_+^\sigma$ and $c_{m+1}, \ldots, c_k \in A_-^\sigma$ with $1 < m < k$. Since there are no obstructions on the number of zeros on these ovals, we can set set $\alpha_1 = \alpha_{k+1} = 0$. Then $\sum_{i=1}^m \alpha_i = m + 1 \mod 2$ can always be obtained as follows: For $m = 1$, this equality holds due to $\alpha_1 = 0$ and for $m \geq 2$, we set $\alpha_i = 1$ for $1 < i \leq m$. To see that also $\sum_{i=1}^{k+1} \alpha_i = k + 1 \mod 2$ can be obtained, we have to consider two different cases: For $m + 1 = k + 1 \mod 2$, we set $\alpha_i = 0$ for $m < i \leq k$. For $m = k \mod 2$, we use the fact that $A_- \neq \emptyset$. Then also $A_-^\sigma \neq \emptyset$, and so we can set $\alpha_{m+1} = 1$ and $\alpha_i = 0$ for $m + 1 < i \leq k$. Then there is at least one negatively orientated oval with assigned α-value 0 and the preliminaries of Theorem B.43 are fulfilled As in the first case, this yields a differential ω on X whose divisor in an element of $\mathcal{E}(\emptyset, A_+, A_-)$.

For the other cases, i.e. $\delta \neq \emptyset$, note that we seek a spinor which also induces an orientation on the ovals a_i^1 for $i = 1, \ldots, 2t_1$ corresponding to $c_1, \ldots, c_{t_1}$. This orientation is not specified by the given δ on these ovals, so we can choose it. That means we can determine freely whether an oval belongs to the part of the oval with index smaller, equal or larger than m, where m is defined in Theorem B.42.

- Let $\delta \neq \emptyset$ and $A_- = \emptyset$. We set $\tilde{A}_-^\sigma = \{c_{k+1}\}$. Then $k_0 \leq t_1 < k + 1$ since $k_+ \cdot k_- \neq 0$. Because $\tilde{A}_+^\sigma = A_+^\sigma$ contains at least one element and the total number of ovals of $\tilde{X}_\sigma$ is $k + 1 < g + 2$, it is $k_0 < g + 1$. Let the elements in $\tilde{A}_+^\sigma$ be denoted by $c_1, \ldots, c_{k-t_1}$. For $i \leq t_1$, let the ovals which are images of a_i^1 and $\sigma_\sharp a_i^1$ be denoted by $c_{k-t_1+1}, \ldots, c_k$. We set $\alpha_1 = 0$ and m equals the number of ovals contained in $\tilde{A}_+^\sigma$. Again, there has to hold

$$\sum_{i=1}^{k+1} \alpha_i = k + 1 \mod 2 \quad \text{and} \quad \sum_{i=1}^{m} \alpha_i = m + 1 \mod 2$$

and for at least one oval c_i with $m < i \leq k+1$, there has to hold $\alpha_i = 0$. As before, we set $\alpha_2 = \cdots = \alpha_{k-1} = 1$. Then the equality on the left hand side holds. To see that the equality on the right hand side can also always be achieved, note that one of the two cases

$$\sum_{i=1}^{t_1} \delta_i = \sum_{i=k-t_1+1}^{k} \alpha_i = \begin{cases} t_1 \mod 2, \\ t_1 + 1 \mod 2 \end{cases}$$

has to hold, whereby the second case implies that there exists at least one $i \leq t_1$ such that $\delta_i = 0$. In both cases, the spinor we construct shall induce negative orientation on c_i with $i = k - t_1 + 1, \ldots, k+1$. In the first case, we set $\alpha_{k+1} = 0$, so

$$\sum_{i=1}^{k+1} \alpha_i = \sum_{i=1}^{t_1} \alpha_i + \sum_{i=t_1+1}^{k+1} \alpha_i = t_1 + m + 1 = k + 1 \mod 2.$$

If $\sum_{i=1}^{t_1} \delta_i = t_1 + 1 \mod 2$, we know that there exists at least one δ_i which equals zero. So we can set $\alpha_{k+1} = 1$ which yields

$$\sum_{i=1}^{k+1} \alpha_i = \sum_{i=1}^{k-t_1} \alpha_i + \sum_{i=k-t_1+1}^{k} \alpha_i + 1 = m + 1 + t_1 + 1 + 1 = k + 1 \mod 2.$$

As before, this yields a differential ω on X whose divisor is an element of $\mathcal{E}(\delta, A_+, \emptyset)$.

- Let now $\delta \neq \emptyset$ as well as $A_+, A_- \neq \emptyset$. We enumerate the ovals nearly as in the previous step, where the ovals in $\tilde{A}_-^\sigma$ are indexed by $i = \sharp(\tilde{A}_+^\sigma), \ldots, \sharp(\tilde{A}_+^\sigma + \tilde{A}_-^\sigma)$ and the ovals on $\tilde{X}_\sigma$ corresponding to a_i^1 and $\sigma_\sharp a_i^1$ on X with $i \leq t_1$ by $\sharp(\tilde{A}_+^\sigma + \tilde{A}_-^\sigma) + 1, \ldots, k$. Hereby, $\sharp$ shall denote the number of elements in the respective sets. Furthermore, we orientate all ovals corresponding to $i < t_1$ negatively, set $\alpha_1 = \alpha_{k+1} = 0$, $m = \sharp A_+^\sigma$ and $\alpha_2, \ldots, \alpha_{\sharp A_+^\sigma} = 1$. Then $\sum_{\tilde{A}_+^\sigma} \alpha_i = m + 1 \mod 2$ and one of the following two cases has to hold:

$$\sum_{\tilde{A}_+^\sigma} \alpha_i = \begin{cases} k + 1 \mod 2, \\ k \mod 2. \end{cases}$$

  If $\sum_{\tilde{A}_+^\sigma} \alpha_i = k + 1 \mod 2$, we set $\alpha_{\sharp A_+^\sigma + 1}, \ldots, \alpha_k = 0$. If $\sum_{\tilde{A}_+^\sigma} \alpha_i = k \mod 2$, we set $\alpha_{\sharp(\tilde{A}_+^\sigma)+1} = 1$ and the remaining $\alpha_i = 0$. The latter is possible since A_-^σ contains at least one element. In both cases, it is $\sum_{i=1}^{k+1} \alpha_i = k + 1 \mod 2$. As before, this yields a differential ω on X whose divisor is an element of $\mathcal{E}(\delta, A_+, A_-)$.

So for all choices of (δ, A_+, A_-) with not both $A_+ = A_- = \emptyset$, it is $\mathcal{E}(\delta, A_+, A_-) \neq \emptyset$. □

Theorem 6.51 ([Natanzon, 2004, Theorem 2.9.2]). *Let (X, τ_1, σ) be a real curve with involution of type $(g, \varepsilon, t_1, r_1, t_2, r_2)$, where $k = t_1 + r_1 + t_2 + r_2 > 0$. Then the following assertions hold:*

(a) For $\varepsilon = 0$, all real tori of the Prym variety are singular.

(b) For $\varepsilon = 1$, there is at most one non-singular real torus of the Prym variety.

(c) For $\varepsilon = 1$ and $k = g + 1$, a non-singular real torus of the Prym variety always exists.

(d) For $\varepsilon = 1$ and $t_1 + r_1 < \frac{k}{2}$ for k even and $t_1 + r_1 < \frac{k-1}{2}$ for k odd, there are curves (X, τ_1, σ) of type $(g, \varepsilon, t_1, r_1, t_2, r_2)$ such that there is a non-singular torus among the real tori of their Prym varieties.

Proof. By definition, it is $\mathcal{E}(\delta, A_+, A_-) \cap \mathcal{E}(\delta', A'_+, A'_-) \neq \emptyset$ if and only if $\delta' = \delta$, $A'_+ = A_-$ and $A'_- = A_+$ and in this case, these sets coincide. Because $\delta \in (\mathbb{Z}_2)^{t_1}$ and all ovals in $A_+ \cup A_-$ can be in either one of these sets, the number of disjoint sets of the form $\mathcal{E}(\delta, A_+, A_-)$ is equal to 2^{k-1}. On the other hand, the real part of the Prym variety of a real curve with involution (X, τ_1, σ) coincides with $A_{Q^+}(\mathcal{E}(\delta, A_+, A_-)) - K_{Q^+}$. By Theorem 6.46, $\mathrm{Prym}_{\mathbb{R}}(X, \sigma)$ consists of 2^{k-1} connected components. By Lemma 6.50, none of the sets $\Omega(\delta, A_+, A_-)$ is empty, so each real torus of the Prym variety is of the form $A_{Q^+}(\mathcal{E}(\delta, A_+, A_-)) - K_{Q^+}$. The torus is singular if and only if there is a $D \in \mathcal{E}(\delta, A_+, A_-)$ such that $D + \sigma(D)$ is the divisor of zeros of a holomorphic differential on X, i.e. if Q^+ and Q^- are contained in D since then $A_{Q^+}(D) \in W_g^1$, see Definition 6.35 and the discussion after that.

(a) Let $\varepsilon = 0$ and let $T = A_{Q^+}(\mathcal{E}(\delta, A_+, A_-)) - K_{Q^+}$ be an arbitrary real torus of the Prym variety. Let again $A_\pm^\sigma$ be the image of the set $A_\pm$ under π_σ on the real curve (X_σ, τ_σ). By Theorem 6.19, there exists a holomorphic real differential ω_σ on (X_σ, τ_σ) that is non-negative on A_+^σ, non-positive on A_-^σ and may have zeros of second order on the other ovals. Lemmata 6.47 and 6.49 show that its preimage $\omega = \pi^* \omega_\sigma$ on X is a holomorphic differential that is positive definite on the ovals of A_+ and negative definite on the ovals of A_-. The divisor of the zeros of this differential intersected with $a_i^1 \cup \sigma_\sharp a_i^1$ for $i < t_1$ has positive degree which is divisible by four and is symmetric with respect to σ. Hence, there is a differential $D \in \mathcal{E}(\delta, A_+, A_-)$ such that $\omega_D = \omega$ and T is a singular torus.

(b) Let $\varepsilon = 1$ and let $T = A_{Q^+}(\mathcal{E}(\delta, A_+, A_-)) - K_{Q^+}$ be a torus that differs from $A_{Q^+}(\mathcal{E}(\delta, A_+, \emptyset)) - K_{Q^+}$, where $\delta = (1, \ldots, 1)$. Repeating the arguments used for $\varepsilon = 0$ yields that T is a singular torus.

(c) Let $\varepsilon = 1$ and $k = g+1$. We prove that for $\delta = (1, \ldots, 1)$, the real torus $A_{Q^+}(\mathcal{E}(\delta, A_+, \emptyset)) - K_{Q^+}$ is non-singular. Indeed, otherwise there must be a real holomorphic differential ω on X that is positive definite on all ovals of the involutions τ_1 and τ_2 and such that $\sigma^* \omega = \omega$. This differential induces a holomorphic real differential ω_σ on the M-curve (X_σ, τ_σ) that is positive on all ovals on which it has no zeros. However, by Theorem 6.24, there are no such differentials.

(d) Let $\varepsilon = 1$, $t_1 + r_1 < \frac{k}{2}$ for k even respectively $t_1 + r_1 < \frac{k-1}{2}$ for k odd and let T be a real torus of the form $A_{Q^+}(\mathcal{E}(\delta, A_+, \emptyset)) - K_{Q^+}$, where $\delta = (1, \ldots, 1)$. If T is singular, then we find – repeat the reasoning used in the case of $k = g + 1$ – a holomorphic differential ω_σ on the real curve (X_σ, τ_σ) that is positive on the $t_2 + r_2 \geq \frac{k}{2}$ images of the ovals in A_+ for k even respectively $t_2 + r_2 \geq \frac{k-1}{2}$ images for k odd and either has zeros or is positive on the other ovals of the curve. Lemma 6.41 together with example 6.39 shows that we can take (X_σ, τ_σ) to be any real curve to obtain a real curve with involution from it. In particular, also the curve constructed in Theorem 6.25 on which there are no such differentials. So there are curves (X, τ_1, σ) such that there is a non-singular torus among the real tori of their Prym varieties.

□

To give a more complete picture of the topological type of the normalizations $F(u)/\Gamma^*$, we now consider the topological type of Fermi curves with constant potential. The considerations we made so far are only valid for $g \geq 4$. The three types of real curves which remain to consider are real curves of genus 0, where the normalization has one or two connected components – this corresponds to the normalization of Fermi curves with zero or constant potential – and real curves with involution of genus 2, such that X_σ is a hypersurface. We here only consider the first part. So let $X(0)$ be the compactified normalization of $F(0)/\Gamma^*$ and $X(u_0)$ be the compactified normalization of $F(4\pi^2 u_0)/\Gamma^*$, see Section 1.4. Hereby, the shape of the curve $X(u_0)$ depends on the absolute value of $u_0 \neq 0$. In particular, the constant potential leads to a shape of $X(u_0)$ which differs from the shape of $X(0)$. The latter curve has a double point at $k = (0, 0)$, whereas $X(u_0)$ has a handle in the neighborhood of $k = (0, 0)$ which leads to a simple closed curve in the shape of a circle around $(0, 0)$ intersected with $\mathbb{R}^2$ respectively $\iota\mathbb{R}^2$. The radius of this circle depends on $|u_0|$ and which precise position in $\mathbb{C}^2$ it has depends on the choice of u_0. This is illuminated in the following lemma.

Lemma 6.52. *For $u_0 \in \mathbb{R}$, the compactified normalization $X(u_0)$ of the Fermi curve $X'(u_0)$ is a real curve with holomorphic involution in sense of Definition 6.34.*

(a) For $u_0 = 0$, the topological type is $(0, 1, 0, 0, 0, 0)$ and for the preimages $k_{\nu,\pm} \in X(0)$ of the double points $k_\nu^\pm \in X'(0)$, there holds $\tau_1(k_{\nu,\pm}) = k_{\nu,\mp}$ for $\nu \in \Gamma^*$ that.

Let now $u_0 \neq 0$, $\nu \in \Gamma^* \setminus \{0\}$ and $k_{\nu,\pm} \in X(u_0)$ be the two preimages of $k_\nu^\pm(u_0) \in X'(u_0)$ as defined in (1.20).

(b) For $u_0 > 0$, the topological type of $X(u_0)$ is given by $(0, 1, 0, 1, 0, 0)$ and $\tau_1(k_{\nu,\pm}(u_0)) = k_{\nu,\mp}(u_0)$. Moreover, there exists no $\nu \neq \tilde{\nu} \in \Gamma \setminus \{0\}$ such that $k_{\nu,\pm}(u_0) = k_{\tilde{\nu},\pm}(u_0)$ or $k_{\nu,\pm}(u_0) = k_{\tilde{\nu},\mp}(u_0)$.

(c) For $-\frac{\min_{\nu\in\Gamma^*\setminus\{0\}}\|\nu\|^2}{4} < u_0 < 0$*, the topological type of* $X(u_0)$ *is given by* $(0,1,0,0,1,0)$ *and* $\tau_1(k_{\nu,\pm}(u_0)) = k_{\nu,\mp}(u_0)$*. Moreover, there exists no* $\nu \neq \tilde\nu \in \Gamma^*$ *such that* $k_{\nu,\pm}(u_0) = k_{\tilde\nu,\pm}(u_0)$ *or* $k_{\nu,\pm}(u_0) = k_{\tilde\nu,\mp}(u_0)$*.*

(c) For $u_0 < -\frac{\min_{\nu\in\Gamma^*\setminus\{0\}}\|\nu\|^2}{4}$*, the topological type of* $X(u_0)$ *is given by* $(0,1,0,0,1,0)$*. For* $\nu \in \Gamma^* \setminus \{0\}$ *with* $\|\nu\|^2 > -\frac{u_0}{4}$*, it is* $\tau_1(k_{\nu,\pm}(u_0)) = k_{\nu,\pm}(u_0)$ *and* $\tau_2(k_{\nu,\pm}(u_0)) = k_{\nu,\pm}(u_0)$ *for* $\|\nu\|^2 < -\frac{u_0}{4}$*. Furthermore, the latter set of double points lays on the oval of* τ_2 *which is described by* $\{(k_1,k_2)\in\mathbb{R}^2 \mid k_1^2+k_2^2 = -u_0\}$*. For* $\nu \neq \tilde\nu$ *with* $\|\nu\|^2 \leq -\frac{u_0}{4}$ *and*

$$u_0 = -\frac{\|\nu\|^2\|\tilde\nu\|^2}{2\|\nu\|^2|\tilde\nu|^2 - 2\langle\nu,\tilde\nu\rangle}\left(\|\nu\|^2 - 2\langle\nu,\tilde\nu\rangle + \|\tilde\nu\|^2\right),$$

it is $k_\nu^\pm(u_0) = k_{\tilde\nu}^\pm(u_0) = k_{\nu-\tilde\nu}^\pm(u_0)$*.*

Proof. For $u_0 = 0$, we know from Lemma 1.18 that $X(0)$ is given by two complex planes which can be described by the set of all $(k_1,k_2)\in\mathbb{C}^2$ such that either $k_1 + \iota k_2 = 0$ or $k_1 - \iota k_2 = 0$. The involutions τ_1 and τ_2 interchange these sheets, so there are no ovals on $X(0)$ and the topological type of $(X(0),\tau_1,\sigma)$ in sense of Definition 6.36 is given by $(0,1,0,0,0,0)$ Furthermore, with $k_\nu^\pm$ as in (1.16), it is

$$\tau_1(k_\nu^\pm) = -\frac{1}{2}(\pm\nu_1 - \iota\nu_2, \iota\nu_1 \pm \nu_2) = \frac{1}{2}(\mp\nu_1 + \iota\nu_2, -\iota\nu_1 \mp \nu_2) = k_\nu^\mp.$$

Next, we consider the topological type for the different cases in (b) to (d). We know from Lemma 1.20 that the Fermi curve for constant potential $u_0 \in \mathbb{R}$ is given by the set

$$\{(k_1,k_2)\in\mathbb{C}^2 \mid k_1^2+k_2^2 = -u_0\}.$$

Hence, for $u_0 > 0$, the involution $\tau_2 : k \mapsto \bar k$ has no fixed points and there is one oval of τ_1 given by

$$c := \{(k_1,k_2)\in\iota\mathbb{R}^2 \mid k_1^2+k_2^2 = -u_0\}. \tag{6.12}$$

This oval is left invariant by $\sigma : k \mapsto -k$ and $X(u_0)\setminus c$ decays into two disjoint connected components. So the topological type of $(X(u_0),\tau_1,\sigma)$ in sense of Definition 6.36 is given by $(0,1,0,1,0,0)$ for $u_0 > 0$.

Analogously for $u_0 < 0$, there are no fixed points of $\tau_1 : k \mapsto -\bar k$ on $X(u_0)$ and there is one σ-invariant oval of τ_2 given by the set

$$c := \{(k_1,k_2)\in\mathbb{R}^2 \mid k_1^2+k_2^2 = -u_0\}. \tag{6.13}$$

Accordingly, for $u_0 < 0$, the topological type of $(X(u_0),\tau_1,\sigma)$ in sense of Definition 6.36 is given by $(0,1,0,0,0,1)$.

Next, we show the assertion on the preimages $k_{\nu,\pm}(u_0) \in X(u_0)$ of the double points $k_\nu^\pm(u_0) \in X'(u_0)$ determined in (1.20). The behavior of those depends on whether the square root $\xi(\nu, u_0) := \sqrt{1 + \frac{4u_0}{\|\nu\|^2}}$ is contained in $\mathbb{R}$ or $\iota\mathbb{R}$. We first consider the setting of (b) and (c), i.e. $u_0 > 0$ respectively $u_0 \in \left(-\min_{\nu\in\Gamma^*\setminus\{0\}} \frac{\|\nu\|^2}{4}, 0\right)$. In both cases, it is $\xi(\nu, u_0) \in \iota\mathbb{R}$. Thus,

$$\tau_1(k_\nu^\pm(u_0)) = -\frac{1}{2}(\pm\nu_1 - \iota\xi(\nu, u_0)\nu_2, \iota\xi(\nu, u_0)\nu_1 \pm \nu_2) = k_\nu^\mp(u_0). \tag{6.14}$$

So the double points are interchanged by τ_1. It is impossible that $k_{\nu,\pm}$ with $\nu \in \Gamma^* \setminus \{0\}$ is contained in the oval of τ_1 in (6.12) since

$$\|k_\nu^\pm(u_0)\| = \frac{1}{2}\sqrt{\|\nu\|^2 + \|\nu\|^2\left(1 + 4\frac{u_0}{\|\nu\|^2}\right)} = \frac{1}{2}\sqrt{2\|\nu\|^2 + 4u_0} > \sqrt{|u_0|}.$$

Assume that $k_{\nu,\pm}(u_0) = k_{\tilde{\nu}(u_0),\pm}$ for some $\nu, \tilde{\nu} \in \Gamma^* \setminus \{0\}$ with $\nu \neq \tilde{\nu}$. Then the real parts of the corresponding double points $k_\nu^\pm(u_0)$ would have to be equal, i.e. $(\nu_1, \pm\nu_2) = (\tilde{\nu}_1, \pm\tilde{\nu}_2)$. This can hold if and only if $\nu = \tilde{\nu}$. So in both cases (b) and (c), it is not possible that the preimage of two different double points $k_{\nu,\pm}(u_0)$ and $k_{\tilde{\nu},\pm}(u_0)$ are identical for $\nu, \tilde{\nu} \in \Gamma^* \setminus \{0\}$ with $\nu \neq \tilde{\nu}$. Analogously one shows that there also exist no $\nu \neq \tilde{\nu} \in \Gamma^* \setminus \{0\}$ such that $k_{\nu,\pm}(u_0) = k_{\tilde{\nu},\mp}(u_0)$.
Let now $u_0 < -\min_{\nu\in\Gamma^*\setminus\{0\}} \frac{\|\nu\|^2}{4}$. As in (6.14), one sees that for $\|\nu\|^2 \geq -\frac{u_0}{4}$, the involution τ_1 interchanges the points $k_{\nu,-}(u_0)$ and $k_{\nu,+}(u_0)$. Let $\nu \in \Gamma^*$ with $\|\nu\|^2 < -\frac{u_0}{4}$. Then $\xi(\nu, u_0) \in \iota\mathbb{R}$ and $k_{\nu,\pm}(u_0) \in \mathbb{R}^2$ are fixed points of τ_2.
Let us now take a closer look at the set $\{k_\nu^\pm(u_0) \mid \|\nu\|^2 \leq -\frac{u_0}{4}\}$. This set comprises all preimages of double points of $X'(u_0)$ which are contained in the oval of τ_2 in (6.13). For these, it can happen that two double points $k_\nu^\pm(u_0)$ and $k_{\tilde{\nu}}^\pm(u_0)$ with $\nu \neq \tilde{\nu} \in \Lambda^*$ are equal. Obviously, this can only occur if ν and $\tilde{\nu}$ are linearly independent. Let $\nu \in \Lambda^*$ with $\|\nu\|^2 \geq -\frac{u_0}{4}$ be given. To determine $\tilde{\nu}$ such that $k_{\tilde{\nu}}^\pm(u_0) = k_\nu^\pm(u_0)$, the following three equations with $k = (k_1, k_2)$ have to hold simultaneously:

$$k^2 = -u_0, \quad (k + \nu)^2 = -u_0, \quad (k + \tilde{\nu})^2 = -u_0.$$

Subtracting the first equation from the second as well as from the third yields that these three equations can hold simultaneously if and only if

$$2\langle k, \nu\rangle + \|\nu\|^2 = 0 \text{ and } 2\langle k, \tilde{\nu}\rangle + \|\tilde{\nu}\|^2 = 0. \tag{6.15}$$

Due to the linear independence of ν and $\tilde{\nu}$, there exists a dual basis $\gamma, \tilde{\gamma} \in \Lambda$ corresponding to ν and $\tilde{\nu}$ such that

$$\langle \nu, \gamma\rangle = 1, \quad \langle \nu, \tilde{\gamma}\rangle = 0, \quad \langle \tilde{\nu}, \tilde{\gamma}\rangle = 1, \quad \langle \tilde{\nu}, \gamma\rangle = 0.$$

Then every $k \in \mathbb{C}^2$ which is of the form $k = -\frac{\|\nu\|^2}{2}\gamma - \frac{\|\tilde{\nu}\|^2}{2}\tilde{\gamma}$ solves the two equations in (6.15) simultaneously. Inserting this into $k^2 = -u_0$ yields

$$k^2 = \frac{\|\nu\|^4}{4}\|\gamma\|^2 + \frac{\|\nu\|^2\|\tilde{\nu}\|^2}{2}\langle\gamma, \tilde{\gamma}\rangle + \frac{\|\tilde{\nu}\|^4}{4}\|\tilde{\gamma}\|^2 = -u_0. \tag{6.16}$$

To represent k in terms of the dual lattice, we have to determine $\|\gamma\|^2$, $\|\tilde{\gamma}\|^2$ and $\langle\gamma, \tilde{\gamma}\rangle$ in dependency of ν and $\tilde{\nu}$. Since $\gamma, \tilde{\gamma}$ are a dual basis of $\nu, \tilde{\nu}$, we have

$$\begin{aligned}
\|\nu\|^2\|\gamma\|^2 &= \langle\gamma, \gamma\rangle\langle\nu, \nu\rangle = \langle\langle\gamma, \gamma\rangle\nu, \nu\rangle = \langle\langle\gamma, \nu\rangle\nu, \gamma\rangle = 1,\\
\langle\nu, \tilde{\nu}\rangle\langle\gamma, \tilde{\gamma}\rangle &= \langle\langle\nu, \tilde{\nu}\rangle\gamma, \tilde{\gamma}\rangle = \langle\langle\nu, \gamma\rangle\tilde{\nu}, \tilde{\gamma}\rangle = 1,\\
\|\nu\|^2\langle\gamma, \tilde{\gamma}\rangle &= \langle\nu, \nu\rangle\langle\gamma, \tilde{\gamma}\rangle = \langle\langle\nu, \nu\rangle\gamma, \tilde{\gamma}\rangle = \langle\langle\nu, \gamma\rangle\nu, \tilde{\gamma}\rangle = 0
\end{aligned}$$

and analogously $\|\tilde{\nu}\|^2\|\tilde{\gamma}\|^2 = 1$ as well as $\|\tilde{\nu}\|^2\langle\gamma, \tilde{\gamma}\rangle, \|\gamma\|^2\langle\nu, \tilde{\nu}\rangle, \|\tilde{\gamma}\|^2\langle\nu, \tilde{\nu}\rangle = 0$. Therefore,

$$\frac{1}{\|\nu\|^2\|\tilde{\nu}\|^2 - 2\langle\nu, \tilde{\nu}\rangle}\begin{pmatrix} \|\tilde{\nu}\|^2 & -\langle\tilde{\nu}, \nu\rangle \\ -\langle\nu, \tilde{\nu}\rangle & \|\nu\|^2 \end{pmatrix} = \begin{pmatrix} \|\nu\|^2 & \langle\nu, \tilde{\nu}\rangle \\ \langle\tilde{\nu}, \nu\rangle & \|\tilde{\nu}\|^2 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} \|\gamma\|^2 & \langle\gamma, \tilde{\gamma}\rangle \\ \langle\tilde{\gamma}, \gamma\rangle & \|\tilde{\gamma}\|^2 \end{pmatrix}.$$

Comparing the coefficients of the matrices on the left hand side and on the right hand side of this equation yields that

$$\|\gamma\|^2 = \frac{2\|\tilde{\nu}\|^2}{\|\nu\|^2\|\tilde{\nu}\|^2 - 2\langle\nu, \tilde{\nu}\rangle}, \quad \|\tilde{\gamma}\|^2 = \frac{2\|\nu\|^2}{\|\nu\|^2\|\tilde{\nu}\|^2 - 2\langle\nu, \tilde{\nu}\rangle}, \quad \langle\tilde{\gamma}, \gamma\rangle = \frac{-2\langle\nu, \tilde{\nu}\rangle}{\|\nu\|^2\|\tilde{\nu}\|^2 - 2\langle\nu, \tilde{\nu}\rangle}.$$

By inserting this into equation (6.16), we finally obtain that for

$$u_0 = -\frac{1}{\|\nu\|^2|\tilde{\nu}|^2 - 2\langle\nu, \tilde{\nu}\rangle}\left(\frac{\|\nu\|^4}{2}|\tilde{\nu}|^2 - \|\nu\|^2|\tilde{\nu}|^2\langle\nu, \tilde{\nu}\rangle + \frac{|\tilde{\nu}|^4}{2}\|\nu\|^2\right) = -\frac{1}{2}\frac{\|\nu\|^2|\tilde{\nu}|^2|\nu - \tilde{\nu}|^2}{\|\nu\|^2|\tilde{\nu}|^2 - 2\langle\nu, \tilde{\nu}\rangle},$$

it is $k^{\pm}_{\nu}(u_0) = k^{\pm}_{\tilde{\nu}}(u_0) = k^{\pm}_{\nu-\tilde{\nu}}(u_0)$. □

Moreover, the following Lemma holds by the same arguments as in Lemmata 6.1 and 6.3.

Lemma 6.53.

(a) The set of divisors D of degree 0 such that $D + \sigma(D) \simeq K + Q^+ + Q^-$ and $\tau_1(D) = (D)$ acts transitively on the set of positive divisors D obeying $D + \sigma(D) \simeq K + Q^+ + Q^-$, where K is the canonical divisor on X.

(b) Let (X, τ_1, σ) be a real curve with involution and let D be a positive divisor of degree g on X which obeys additionally to the conditions in (a) that $\dim H^1(X, \mathcal{O}_{D-Q^{\pm}} \otimes \mathcal{L}_h(x, y)) = 0$ for all $(x, y) \in \mathbb{R}^2$, where $\mathcal{L}_h(x, y)$ is defined below (5.3). The set of all such D is open in the set of all positive divisors obeying $D + \sigma(D) \simeq K + Q^+ + Q^-$ and $\tau_1(D) = D$.

Outlook

Even though this thesis is answered many questions concerning the direct and the inverse problem of finite type Fermi curves for the double-periodic Schrödinger equation, we see that the problems are of such rich variety that there are several possibilities to explore further from this point.
One way to go would be to try to transfer the results for finite type potentials to the more general case of arbitrary potentials. Concerning this direction, we mention here only that one big step towards this generalization would be to show that the finite type potentials are dense in the set of all potentials.
Another direction to proceed is to answer the questions for finite type potentials which are either yet not answered in this thesis or raised by the results in this work. Below, we mention the points which we consider of most interest for finite type potentials to continue the research.
First of all, it would be very interesting to figure out whether it is possible to extend the discussion presented in this work for regular finite type potentials to the broader class of all finite type potentials. For that, one would have to show an analogon to the linear equivalence $D + \sigma(D) \simeq K + Q^+ + Q^-$ from Lemma 4.13 for the generalized divisor $\mathcal{S}$ as in Definition 3.7 respectively a generalized divisor on some unique one-sheeted covering of the Fermi curve similar to the middleding in 4.4. One possibility would be to analyze whether or in which cases σ acts on the middleding. If that is not the case, we suspect that it could be a good idea to consider another unique one-sheeted covering $\tilde{X}$, distinct from the middleding, to define an analogon $\tilde{\mathcal{S}}$ to the generalized divisor $\mathcal{S}_M$ from Definition 4.4. A good candidate for this might be the unique one-sheeted covering so that the holomorphic functions of this new covering act on both, $\mathcal{S}$ and $\sigma^*\mathcal{S}$. Instead of the canonical divisor one should consider the sheaf of regular 1-forms $\Omega(\tilde{X})$ on this covering, compare Definition 3.4. Then the linear equivalence itself would be expressed by an isomorphism of vector spaces between the sheaf $\Omega_{\tilde{X}} + Q^+ + Q^-$ and the sheaf which is generated by all multiples of $\tilde{\mathcal{S}}$ and $\sigma^*\tilde{\mathcal{S}}$. However, if the analogon to the linear equivalence can be shown, then the results on the construction of the Baker Akhiezer function should also hold for the Baker Akhiezer function on the corresponding singular curve $\tilde{X}$ as defined in [Klein et al., 2016, Chapter 8].
To transfer also the statement that the number of fixed points of σ on $\tilde{X}$ is two if and only if the above sheaves are isomorphic, one would also have to take the fundamental group of the singular curve $\tilde{X}$ into account. We suspect that this could become complicated.
In Section 3.3, concerning the construction of a regular operator 1-form on the Fermi curve out of the spectral projection, we see at least one more possibility for an interesting question to be

answered. Before we formulate this question, we want to explain the analogous setup for integrable systems with one periodic and one trivial flow as described in [Klein et al., 2016, Case 2 on page 32]. To do so, let $k \in \mathbb{C}$, $A(k)$ be a holomorphic $n \times n$-matrix and let X be the curve which is parametrized by all values $(k, \lambda) \in \mathbb{C}^2$ so that there exists a ψ which obeys $(A(k) - \lambda)\psi = 0$. For this situation, a general procedure is known to obtain a regular operator-valued 1-form with help of the spectral projection as it was done in this work in Section 3.3, i.e. by taking the eigenfunction and the dual eigenfunction into account, compare [Schmidt, 1996, Chapter 3]. Out of this 1-form, one usually obtains a linear equivalence similar to the one shown in Lemma 4.13. More precisely, one can show a relation between the divisor of the eigenfunction and of the dual eigenfunction of the considered differential operator such that the sum of these is linear equivalent to the canonical divisor on X plus maybe some points at infinity. Here, the number of marked points which have to be added in this linear equivalence depends on the considered differential equation.
However, besides the Schrödinger equation considered in this thesis, there are many other infinite dimensional integrable systems with two periodic flows, compare [Klein et al., 2016, Case 3 on page 32]. In these cases, a similar situation occurs in which one takes the kernel of a holomorphic $n \times n$-matrix $A(k_1, k_2)$ with $k_1, k_2 \in \mathbb{C}$ into account. So the question one can ask for these situations are the following: First of all, is there also a standard procedure, similar to the case of one periodic and one constant flow, to obtain a regular operator-valued 1-form if X is the curve which is defined by all values $(k_1, k_2) \in \mathbb{C}^2$ such that there exists a ψ with $A(k_1, k_2)\psi = 0$? And secondly, does such an operator-valued 1-form always lead to a relation between the divisor D of the normalized eigenfunction and its transposed similarly to (4.3)? We suspect that both questions can be answered positively. More precisely, our expectation is that it should be possible to generalize the method presented in Section 3.3 to construct this 1-form for the Schrödinger operator. The standard procedure should comprise the following steps: First, one considers the Bloch variety i.e. another curve Y which is parametrized by all (k_1, k_2, λ) such that there exists a ψ that obeys $(A(k_1, k_2) - \lambda \mathbb{1})\psi = 0$ with $\lambda \in \mathbb{C}$. On this Bloch variety, one can define the spectral projection P as it is done for the Schrödinger operator in (3.4) and show that $P\, dk_1 \wedge dk_2$ is an operator-valued regular 2-form on Y. From this, it should be possible to define a modified projection-valued 2-form $\tilde{P}\, dk_1 \wedge d\lambda$ and show its regularity in the same as we have defined $P_\partial\, dk_1 \wedge d\lambda$, i.e. by exploiting a connection between the enumerators of $\tilde{P}$ and P, whereby $\frac{\partial \lambda}{\partial k_2}$ is taken into account as it was done in Proposition 3.15. With this, the regularity of the corresponding operator-valued 2-form $\tilde{P} dk_1 \wedge d\lambda$ on Y should follow from the regularity of the operator-valued 2-form $P dk_1 \wedge dk_2$. We suspect that this can be shown in a similar way as for P_∂ in Lemma 3.16. Then restricting $\tilde{P}\, dk_1 \wedge d\lambda$ to X should yield the desired regular 1-form.
Other interesting questions in the case of regular finite type potentials which still remain open concern the non-speciality of the divisors which are contained in the isospectral set. In the case of complex-valued potentials, it would be interesting to find out whether there is a divisor D such that all translations by divisors $\tilde{D}$ of degree 0 with $\tilde{D} = -\sigma(\tilde{D})$ are non-special. Similar assertions

have been shown in the unpublished paper [Schmidt, 2002, Section 2.5.1] for the Dirac operator. However, the Fermi curve of the Dirac operator obeys additional symmetries which do not hold for the Fermi curve corresponding to the Schrödinger operator. Therefore, it is not possible to transfer these results to the case considered in this work. Still, it would be interesting to figure out statements on the codimension of the subvariety of special divisors in the Prym variety – considered as a subset of the Picard group – such that the space of global sections is at least two dimensional. Like this, it might be possible to show the existence of a divisor such that the divisor itself and all its translations by $\mathcal{L}_h(x, y)$, as defined in Section 4.2.5, are non-special. This question is closely related to the inquiry whether the set of non-special divisors is dense in the set of all divisors parametrizing the isospectral set of a given potential.
In the real case, it is shown in Theorem 6.51 that there exists at most one connected component of the real Prym variety of a given real curve with involution (X, τ, σ) which contains no special divisors if $g \geq 4$, where g is the genus of X. Here, it would be interesting to consider several questions: First of all, is it possible to give a more detailed description for which types of potentials u there always exists a connected component that does not contain special divisors? And if this could be answered positively: Which of the several connected components is that? In case this connected component could be determined, how to see whether all divisors D which correspond to a Schrödinger operator with regular finite type potential are contained in this connected component? One approach which might at least lead to the beginning of an answer to these questions is to consider small deformations of the Fermi curves with constant or zero potential and to determine for which deformations the corresponding quotient X_σ of the normalization X of the deformed curve is an M-curve. In that case, we know at least for sure that exactly one non-singular connected component of the real Prym variety exists.
Also in this context, one could try to answer the questions we answered in this work for the real isospectral sets of Fermi curves of finite type whose compactified normalization has genus larger than 2 for the case where the normalization has genus 2. In this case, the quotient surface X_σ has genus 1 and thus is a hyperelliptic curve. The connection between the real spinors and the liftings of real Fuchsian groups employed in this thesis to explore the existence or non-existence of certain holomorphic 1-forms on a real curve X of genus $g > 2$ do not apply anymore, because in this case the universal covering of X_σ is not given by $\mathcal{H}$ anymore. However, we think that the necessary statements can be shown directly like the ones for hyperelliptic curves in Section 6.2.
Another question that arises is how to transfer the results we have made for a real curve as a Riemann surface to the more general case of complex curves such that one can embed the whole setup for finite type potentials as considerations on the modified middleding we introduced above.
Finally, we ask ourselves one more question concerning the moduli problem: Let $F(u)/\Gamma^*$ be a given Fermi curve and let X be the corresponding curve to deform, i.e. on X exists a non-special divisor so that also all of its translations by $\mathcal{L}_h(x, y)$ are non-special for $(x, y) \in \mathbb{R}^2$ as defined in Section 4.2.5. Does such a divisor always exist on the fibers of the deformed spaces, at least in a

small open neighborhood of X? We believe that one has to understand the deformations of the Jacobian variety to get insight into this.

Summarizing, we can say that still many interesting questions concerning the direct and the inverse problem of the two-dimensional Schrödinger operator of finite type remain open, whereby for some we cannot predict whether there is an answer. We hope that this thesis provides the foundations to continue research on this topic and that the results presented here can be helpful to answer at least some of the above questions.

Acknowledgments

First and most importantly, I thank Prof. Dr. Martin U. Schmidt for his extensive and intense mentoring and support. He opened up the path for me that allowed me to delve deeply into such an extraordinary and interesting field of mathematics. I would also like to thank Prof. Dr. Horst Knörrer, who has kindly agreed to serve as a referee for my thesis. I am also grateful to Prof. Dr. Claus Hertling, who gave me a deeper insight into the theory of singularities on complex curves.

I thank the members of my working group, most of all Tobias Simon for a great co-operation and for always having an open ear to discuss many of my questions on this topic and with whom I could share many ideas. I also thank Dr. Sebastian Klein, without whom I would not have started to write my thesis in this topic and who often gave me inspiring input.

Moreover, I want to thank our secretary Sabine Braak and other staff members of University Mannheim. Their help was not only supportive when I faced bureaucratic or technical difficulties, I especially appreciate their friendliness to share time for short coffee breaks together, which was always refreshing and relaxing. I would like to mention a few persons who specially made my daily commute easier by sharing their parking cards whenever it was necessary. Thanks to Gabi, Karin, Jürgen and Wolfgang.

A huge thanks to my parents Barbara and Bernd. I consider this achievement as a great gift from my parents who are always there for me. They have supported me at every stage of my life from behind and provided me with opportunities to achieve good education. Their presence and caring makes it much easier for me to reach my goals. I appreciate and thank also my lovely brothers, Peter and Christian, and Peters wife Abby, for their support in this course of time. The discussions with Peter on various topic about working life at universities often influenced me positively in completing this work.

And of course, I would like to convey my heartfelt thanks to all of my friends and colleagues from other working groups. I am grateful for the support they gave me and that they believed in me during this important chapter of my life. Although I could not mention everyone with names, a special thanks for a few special ones for being with me during last six years. Thank you! Antony, Juliane, Leif, Marion, Mike, Olga, Peter, Robin and Vlodymir. You guys were there for me personally and emotionally at various stages during the time of writing this thesis.

A. The Jacobian and the Prym Variety

To understand what the Prym variety is, certain basics about Abelian Varieties and Complex Tori are helpful. These can be found in [Lange and Birkenhake, 1992, Chapters 1,4,10 and 11]. However, the concrete results and definitions that are used in this work are mainly oriented on [Mumford, 1974]. We will now summarize some properties of X and X_σ, where X_σ is defined at the beginning of Section 4.2.2.

Proposition A.1. *Let X be a compact Riemann surface and $\sigma : X \to X$ a holomorphic involution with $\sigma \neq \mathbb{1}$. Let further X_σ be the space defined by the quotient $X_\sigma := X/\sim_\sigma$, where $p \sim_\sigma q$ if $p = q$ or $p = \sigma(q)$ for $p, q \in X$. Then X_σ is also a compact Riemann surface and the natural map $\pi_\sigma : X \to X_\sigma$ is a two sheeted covering.*

Proof. Because $\sigma \neq \mathbb{1}$, there exists at least one $p \in X$ such that $p \neq \sigma(p)$. Then the holomorphy of σ yields that the map $p \mapsto p - \sigma(p)$ has discrete roots on X, and therefore the set of fixed points is also discrete. Equipping X_σ with the quotient topology which is induced by the definition of X_σ yields that X_σ is a topological space. Obviously, as X is compact, the quotient space X_σ is also compact. The quotient topology defines an atlas on X_σ, where all points that are no fixed points of σ have two charts which are induced by the open sets on X. Since the involution σ is holomorphic, these charts are compatible with each other. Accordingly, it suffices to define charts around the images of the fixed points of σ on X_σ under the natural map $\pi_\sigma : X \to X_\sigma$ to show that X_σ is a Riemann surface. To do so, let $p \in X$ be a fixed point of σ, U a small open neighborhood of p and let $z : U \to \mathbb{C}$ be a local chart centered at p, i.e. $z(p) = 0$. Since $p = \sigma(p)$, it is also $\sigma^* z(p) = 0$. Let $\tilde{z} := \frac{z - \sigma^* z}{2}$. Then $\tilde{z} : U \to \mathbb{C}$ is a local coordinate centered at p which obeys $\sigma^* \tilde{z} = -\tilde{z}$ and $\tilde{z}^2$ is invariant under σ. Moreover, only $(\pm\tilde{z})^2 = \tilde{z}^2$. Therefore, only elements which are in the same equivalence class of $\sim_\sigma$ are mapped to the same element in $\mathbb{C}$ by $\tilde{z}^2$. This map is an injective map from a small open neighborhood of $p_\sigma := \pi_\sigma(p)$ to $\mathbb{C}$. It is also an open map because all small discs around $0 \in \mathbb{C}$ contain both, $\tilde{z}$ and $-\tilde{z}$. So $z_\sigma := (\tilde{z})^2$ is a bijective map from a small open neighborhood of $p_\sigma = \pi_\sigma(p)$ to a small open ball $B_r(0) \subset \mathbb{C}$. This defines a chart around the image of a fixed point of σ. Therefore, X_σ is a Riemann surface. By the definition of X_σ, the map π_σ is a two-sheeted covering which is unbranched on $X \setminus \{p \in X \mid \sigma(p) = p\}$ and branched over the fixed points of σ. □

Whenever we speak of $\pi_\sigma : X \to X_\sigma$, this denotes the compact Riemann surfaces X and X_σ and the covering from the previous proposition. We have shown in Section 4.2.2 how to construct a basis of $H_1(X, \mathbb{Z})$ from a cycle basis of $H_1(X_\sigma, \mathbb{Z})$ and the branch points of σ. Thereby, is is used that the intersection number of the lifted cycles on X can only become smaller compared to the intersection number of the non-lifted cycles on X_σ. The proof of this standard assertion can for general lifting of paths be found in [Miranda, 1995, Section III.4]. It is repeated in the next Lemma for the situation described in Section 4.2.2. We use the notation of the subsection 'A two-sheeted covering'.

Lemma A.2. *The absolute value of the intersection number of two paths $\gamma_{\sigma,1}, \gamma_{\sigma,2}$ in $X_\sigma \setminus ([s_1] \cup \cdots \cup [s_{n-1}] \cup [t_1] \cup \cdots \cup [t_{n-1}])$ cannot raise due to lifting these via π^* to X, where s_i and t_i are the paths connecting the two branch points in X_σ as defined in the construction of a basis of $H_1(X, \mathbb{Z})$ from a basis of $H_1(X_\sigma, \mathbb{Z})$, compare Section 4.2.2.*

Proof. By assumption, none of the lifts of the curves $\gamma_{\sigma,1}$ and $\gamma_{\sigma,2}$ contains a ramification point of X. The covering π is unbranched on the two sheets U and $\sigma[U]$ of $X \setminus \pi^{-1}[[s_1] \cup \cdots \cup [s_{n-1}] \cup [t_1] \cup \cdots \cup [t_{n-1}]]$ and the restriction to each of these sheets is locally biholomorphic to $X_\sigma \setminus ([s_1] \cup \cdots \cup [s_{n-1}] \cup [t_1] \cup \cdots \cup [t_{n-1}])$. Hence, over every path in $X_\sigma \setminus ([s_1] \cup \cdots \cup [s_{n-1}] \cup [t_1] \cup \cdots \cup [t_{n-1}])$, there are two paths in X: one on U and one on $\sigma[U]$. The intersection number of two paths γ_1 and γ_2 lifted to different sheets equals zero, since neither γ_1 nor γ_2 contains a ramification point of X, and so they cannot change from one sheet to another.
Let now γ_1 and γ_2 be two paths which originate from lifting $\gamma_{\sigma,1}$ and $\gamma_{\sigma,2}$ to the same sheet U respectively $\sigma[U]$. Without loss of generality, we consider the lifts to U. Then the intersection number of γ_1 and γ_2 equals the intersection number of $\gamma_{\sigma,1}$ and γ_{σ_2}. This can be seen by considering the lift

$$\begin{array}{ccc} & & X \\ & \overset{\gamma_1,\gamma_2}{\nearrow} & \Big\downarrow \pi \\ [0,1] & \xrightarrow{\gamma_{\sigma,1},\gamma_{\sigma,2}} & X_\sigma \end{array}$$

where $t \in [0,1]$ and $\pi \circ \gamma_i = \gamma_{\sigma,i}$ for $i \in \{1,2\}$. We define the set of intersection points as $I := \{t \in [0,1] \mid \gamma_{\sigma,1}(t) = \gamma_{\sigma,2}(t)\}$. On the one hand, there holds $\pi \circ \gamma_1(t) = \pi \circ \gamma_2(t)$ for all $t \in I$ and because π is locally biholomorphic on U and γ_1, γ_2 both lie in U, one has $\gamma_1(t) = \gamma_2(t)$. On the other hand, let $t \in [0,1]$ such that $\gamma_1(t) = \gamma_2(t)$, but $\gamma_{\sigma,1}(t) \neq \gamma_{\sigma,2}(t)$. Then there exists a $p_\sigma \in X_\sigma$ such that $\pi(\gamma_i(t)) = p_\sigma \in \operatorname{supp}(\gamma_{\sigma,i})$ for $i = 1, 2$ because $\gamma_{\sigma,i} = \pi \circ \gamma_i$. In other words, $p_\sigma \in \operatorname{supp}(\gamma_{\sigma,1} \cap \gamma_{\sigma,2})$ which contradicts the assumption that $\gamma_1 \cap \gamma_2 = \emptyset$. So the absolute value of the intersection number of two cycles cannot raise if the cycles are lifted to the same sheet and do not contain a branch point. □

Therefore, the lifted A- and B-cycles do not intersect any of the C- and D-cycles on X in the cycle basis constructed in Section 4.2.2.
Furthermore, we will consider meromorphic functions f as well as meromorphic 1-forms ω on X which are the pullback of meromorphic functions f_σ respectively 1-forms ω_σ on X_σ by π. We

remember that $\mathcal{M}(X)$ is the set of the global meromorphic functions on X and $\mathcal{M}(X_\sigma)$ is the set of global meromorphic functions on X_σ.

Definition A.3. The set of *symmetric global meromorphic functions* on X is defined as $\mathcal{M}_+(X) := \{f \in \mathcal{M}(X) \mid f = \sigma^* f\}$.

Note that $\mathcal{M}(X_\sigma) \simeq \mathcal{M}_+(X)$, compare [Forster, 1981, Section 8.1]. Furthermore, all meromorphic 1-forms on X which are invariant under σ are the pullback of a meromorphic 1-form on X_σ.

Proposition A.4. *A meromorphic 1-form ω on X obeys $\omega = \sigma^*\omega$ if and only if there exists a meromorphic 1-form ω_σ on X_σ such that $\omega = \pi^*\omega_\sigma$. This assignment is unique.*

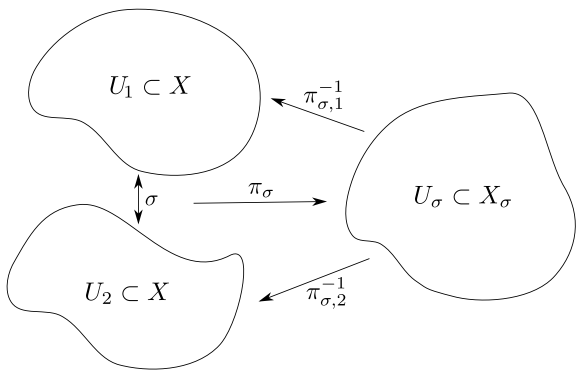


Figure A.1.: The locally biholomorphic mappings π_i for $i = 1, 2$ from the proof of proposition A.4 with $U_i = \sigma[U_j]$ and $\sigma^*\omega|_{U_i} = \omega|_{U_j}$ for $i \neq j \in \{1, 2\}$.

Proof. Let ω_σ be a meromorphic 1-form on X_σ. Then $\omega := \pi^*\omega_\sigma$ is uniquely defined. The definition of X_σ yields

$$\sigma^*\omega = \sigma^*\pi^*\omega_\sigma = (\pi \circ \sigma)^*\omega_\sigma = \pi^*\omega_\sigma = \omega.$$

Conversely, let ω be a meromorphic 1-form on X which is invariant under σ. We show that there exists a meromorphic 1-form ω_σ on X_σ such that $\omega = \pi^*\omega_\sigma$. Therefore, let $\mathcal{U}$ be an open covering of $X_\sigma \setminus b_{\pi_\sigma}$, where b_{π_σ} is the set of branch points of the covering π_σ as in Definition 4.15. We define ω_σ by defining $\omega_\sigma|_U$ for all $U \in \mathcal{U}$ and then showing that these local definitions coincide on all non-empty intersections of open sets contained in $\mathcal{U}$. This yields that ω_σ is a global 1-form on $X_\sigma \setminus b_\pi$. So let $U \subset X_\sigma \setminus b_\pi$. Since the branch points are discrete, one can choose U small enough such that $\pi^{-1}[U]$ consists of two disjoint sets in X, compare Figure A.1. Hence, for all $U \in \mathcal{U}$, there are two disjoint embeddings

$$\pi_{\sigma,1}^{-1} : U \to X \text{ and } \pi_{\sigma,2}^{-1} : U \to X$$

which invert π_σ locally. The set $\pi_\sigma^{-1}[U]$ consists of two disjoint open sets $U_1 := \pi_{\sigma,1}^{-1}[U]$ and $U_2 := \pi_{\sigma,2}^{-1}[U]$ such that $\sigma[U_i] = U_j$ for $i \neq j \in \{1, 2\}$. That means for all points $p \in U \subset X_\sigma \setminus b_{\pi_\sigma}$, one has $\pi_{\sigma,1}^{-1}(p) = \sigma(\pi_{\sigma,2}^{-1}(p))$. These maps can be chosen such that $\sigma \circ \pi_{\sigma,i}^{-1}|_U = \pi_{\sigma,j}^{-1}|_{\sigma[U]}$ for $i \neq j \in \{1,2\}$. With this choice, one has as above

$$(\pi_{\sigma,1}^{-1})^*\omega|_{U_1} = (\sigma \circ \pi_{\sigma,2}^{-1})^*\omega|_{U_1} = (\pi_{\sigma,2}^{-1})^*\sigma^*\omega|_{U_2} = (\pi_{\sigma,2}^{-1})^*\omega|_{U_2}.$$

So on each $U \in \mathcal{U}$, there exists a unique form $\omega_\sigma|_U$. Furthermore, there holds for $i \in \{1,2\}$

$$\omega|_{U_i} = (\pi_{\sigma,i}^{-1} \circ \pi_\sigma)^*\omega|_{U_i} = \pi_\sigma^*(\pi_{\sigma,i}^{-1})^*\omega|_{U_i} = (\pi_\sigma^*\omega_\sigma|_U)_{U_i}.$$

Therefore, $\omega|_{\pi_\sigma^{-1}[U]} = \pi_\sigma^*\omega_\sigma|_U$. From the construction of ω_σ, it is clear that for $W \subset U$, one has $\omega_\sigma|_W = (\omega_\sigma|_U)|_W$. Accordingly, for all $U, V \in \mathcal{U}$, it is $(\omega_\sigma|_U)_{U\cap V} = \omega_\sigma|_{U\cap V} = (\omega_\sigma|_V)|_{U\cap V}$ and hence $\omega_\sigma|_U$ is the restriction of a global meromorphic 1-form on $X_\sigma \setminus b_{\pi_\sigma}$. Since the meromorphic 1-form ω_σ is known on $X_\sigma \setminus b_{\pi_\sigma}$, it can be continued uniquely to all of X_σ, and so the claim follows. □

We now want to define mappings between the Jacobi varieties of X and X_σ which are defined as

$$\mathrm{Jac}(X) := \frac{H^0(X,\varOmega)^*}{H_1(X,\mathbb{Z})} \quad \text{and} \quad \mathrm{Jac}(X_\sigma) := \frac{H^0(X_\sigma,\varOmega)^*}{H_1(X_\sigma,\mathbb{Z})},$$

where $H^0(X,\varOmega)^* := \mathcal{L}(H^0(X,\varOmega),\mathbb{C})$ and $H^0(X_\sigma,\varOmega)^* := \mathcal{L}(H^0(X_\sigma,\varOmega),\mathbb{C})$. The elements of $H_1(X,\mathbb{Z})$ can be interpreted as elements of $H^0(X,\varOmega)^*$ with help of the injective mapping

$$H_1(X,\mathbb{Z}) \to H^0(X,\varOmega)^*, \quad \gamma \mapsto \left(\omega \mapsto \int_\gamma \omega\right) := \int_\gamma.$$

The interpretation of elements of $H_1(X_\sigma,\mathbb{Z})$ as elements of $H^0(X_\sigma,\varOmega)^*$ is defined analogously. Let $\gamma_{p_0,p}$ be an arbitrary path starting at p_0 and ending at p.

Theorem A.5 (Abels Theorem, [Forster, 1981, Satz II.21.7])**.** *Let X be a compact Riemann surface of finite genus $g < \infty$ and let $D \in \mathrm{Div}_0(X)$ a divisor of degree 0 on X. For a fixed base point $p_0 \in X$, we consider the map*

$$\mathrm{Div}_0(X) \to \mathrm{Jac}(X), \quad \sum_{p\in D} n_p p \mapsto \sum_{p\in D} \int_{\gamma_{p_0,p}} \mod H_1(X,\mathbb{Z}).$$

This mapping induces an isomorphism on the classes of divisors modulo principal divisors and the Jacobian variety $\mathrm{Jac}(X)$ *by*

$$\mathrm{Pic}^0(X) \xrightarrow{\sim} \mathrm{Jac}(X), \quad \left[\sum_{p\in D} n_p p\right] \mapsto \sum_{p\in D} \int_{\gamma_{p_0,p}} \mod H_1(X,\mathbb{Z}).$$

With this isomorphism, a Riemann surface X can be embedded into $\mathrm{Jac}(X)$ respectively $\mathrm{Pic}^0(X)$

by

$$\mathrm{Ab} : X \longrightarrow \mathrm{Jac}(X),\ p \mapsto \int_{p_0}^{p} \mod H_1(X, \mathbb{Z})$$

respectively

$$\mathrm{Ab} : X \to \mathrm{Pic}^0(X),\ p \mapsto [p - p_0].$$

For $\pi_\sigma : X \to X_\sigma$, we define the respective maps for X with base point $p_0 \in X$ and for X_σ with base point $p_{\sigma,0} := \pi_\sigma(p_0) \in X_\sigma$.
With help of the holomorphic involution σ, one can split $\mathrm{Jac}(X)$ into two parts: a symmetric part $\mathrm{Jac}(X)_+$ and an antisymmetric part we call the Prym variety $\mathrm{Prym}(X, \sigma)$. We want to give insight why the symmetric part is isomorphic to $\mathrm{Jac}(X_\sigma)$ and why the direct sum of $\mathrm{Jac}(X)_+$ and $\mathrm{Prym}(X, \sigma)$ is not isomorphic to $\mathrm{Jac}(X)$. Hereby, we orientate our presentation on [Adler et al., 2010, Section 2.5.4].

Definition A.6. An element $\omega_+ \in H^0(X, \Omega)$ is called *symmetric* if $\omega_+ = \sigma^*\omega_+$. The space of all symmetric elements of $H^0(X, \Omega)$ is denoted as $H^0(X, \Omega)_+$. An element $\omega_- \in H^0(X, \Omega)$ is called *antisymmetric* if $\omega_- = -\sigma^*\omega_-$. The space of all antisymmetric elements of $H^0(X, \Omega)$ is denoted as $H^0(X, \Omega)_-$. We define the projections

$$P_\pm : H^0(X, \Omega) \to H^0(X, \Omega)_\pm, \quad \omega \mapsto \frac{\omega \pm \sigma^*\omega}{2}$$

to the symmetric respectively antisymmetric part of $H^0(X, \Omega)$. Furthermore, we define

$$H^0(X, \Omega)^*_\pm := \{\ell \in H^0(X, \Omega)^* \mid \ell(\omega_\mp) = 0 \text{ for all } \omega_\mp \in H^0(X, \Omega)_\mp\}$$

and

$$P_\pm : H^0(X, \Omega) \to H^0(X, \Omega)^*_\pm, \quad \omega \mapsto \ker(\ell \circ (\omega \mp \sigma^*\omega)).$$

Classically, the *even divisors* are defined as all divisors D on X such that $D - \sigma(D) \simeq 0$ and the *odd divisors* as D on X such that $D + \sigma(D) \simeq 0$, compare [Adler et al., 2010, Section 5.2.4]. Let now Ab be the abelian mapping $\mathrm{Ab} : \mathrm{Pic} \to \mathrm{Jac}(X_\sigma) \times \mathbb{Z}$ as defined in [Miranda, 1995, Section XI.4, The Jacobian]. With help of this map one can express the above relations on the Jacobian variety as $\mathrm{Ab}(D - \sigma(D)) = (0, 2\deg(D)) \in \mathrm{Jac}(X_\sigma) \times \mathbb{Z}$ for the even divisors respectively $\mathrm{Ab}(D + \sigma(D)) = (0, 2\deg(D)) \in \mathrm{Jac}(X_\sigma) \times \mathbb{Z}$ for the odd divisors.

The subsets of the even respectively uneven divisors are subgroups of $\mathrm{Div}(X)$. For $H^0(X, \Omega)^*$, there is the following decomposition.

Lemma A.7.

$$H^0(X, \Omega)^* \simeq H^0(X, \Omega)^*_+ \oplus H^0(X, \Omega)^*_-.$$

Proof. Every $\omega \in H^0(X, \Omega)$ can be written as $\omega = P_+(\omega) + P_-(\omega) = \omega_+ + \omega_-$. This decomposition is direct since

$$\mathrm{im}(P_+) \cap \mathrm{im}(P_-) = \{\omega \in H^0(X,\Omega) \mid (\omega = \sigma^*\omega) \wedge (\omega = -\sigma^*\omega)\} = \{0\} \in H^0(X,\Omega).$$

Moreover, $\mathrm{im}(P_\pm) = \ker(P_\mp)$ because $P_\pm(\omega_\mp) = 0$ and for $\omega \in \ker(P_\pm)$, one has

$$\omega \pm \sigma^*\omega = 0 \Leftrightarrow \omega = \mp\sigma^*\omega \Leftrightarrow \omega \in H^0(X,\Omega)^*_\mp,$$

so $H^0(X,\Omega) = H^0(X,\Omega)_+ \oplus H^0(X,\Omega)_-$. This direct sum carries over to the dual space

$$H^0(X,\Omega)^* = (H^0(X,\Omega)_+ \oplus H^0(X,\Omega)_-)^* = H^0(X,\Omega)^*_+ \oplus H^0(X,\Omega)^*_-$$

since $H^0(X,\Omega)_+ \oplus H^0(X,\Omega)_-$ is a direct sum of vector spaces and every linear mapping $\ell_\pm \in H^0(X,\Omega)^*_\pm$ is extendable to a linear mapping $\bar{\ell}_\pm \in H^0(X,\Omega)^*$ by setting $\bar{\ell}_\pm[H^0(X,\Omega)_\mp] = 0$. Then

$$\bar{\ell}_\pm \in H^0(X,\Omega)^\circ_\mp := \{\ell \in H^0(X,\Omega)^* \mid \ell[H^0(X,\Omega)_\mp] = 0\},$$

where $H^0(X,\Omega)^\circ_\mp$ is the annihilator of $H^0(X,\Omega)_\mp$. The maps

$$H^0(X,\Omega)^*_\pm \to H^0(X,\Omega)^\circ_\mp,\ \ell_\pm \mapsto \bar{\ell}_\pm$$

are isomorphisms which inverse maps are the restrictions of $\bar{\ell}_\pm$ onto $H^0(X,\Omega)_\pm$. Therefore, $H^0(X,\Omega)^\circ_\pm \simeq H^0(X,\Omega)^*_\mp$. Every linear function in $H^0(X,\Omega)^*$ can be written as a sum of a linear function annihilating $H^0(X,\Omega)_-$ and a linear function annihilating $H^0(X,\Omega)_+$. More precisely, $(H^0(X,\Omega)_+ \oplus H^0(X,\Omega)_-)^* = H^0(X,\Omega)^\circ_- \oplus H^0(X,\Omega)^\circ_+$. The sum on the right hand side is direct since $H^0(X,\Omega)^\circ_+ \cap H^0(X,\Omega)^\circ_- = \{0\}$. In addition $P_+ + P_- = \mathbb{1}$. So

$$\ell = \ell \circ (P_+ + P_-) = (\ell \circ P_+) + (\ell \circ P_-) \in H^0(X,\Omega)^\circ_- \oplus H^0(X,\Omega)^\circ_+.$$

Consequently, $H^0(X,\Omega)^* \subseteq H^0(X,\Omega)^*_+ \oplus H^0(X,\Omega)^*_-$. As also $H^0(X,\Omega)^*_+ \oplus H^0(X,\Omega)^*_- \subseteq H^0(X,\Omega)^*$, the assertion follows. ☐

To get a decomposition of $H_1(X,\mathbb{Z})$ similar to the decomposition of $H^0(X,\Omega)^*$ into symmetric and antisymmetric part, we define

Definition A.8. The *symmetric* and *antisymmetric cycles* of $H_1(X,\mathbb{Z})$ are defined as

$$H_1(X,\mathbb{Z})_\pm := \{\gamma \in H_1(X,\mathbb{Z}) \mid \sigma_\sharp\gamma = \pm\gamma\}.$$

Proposition A.9. *Let g_σ be the genus of X_σ and $2n$ be the number of branch points of π_σ on X_σ. Then the cycles $A_i^+ := A_i + \sigma_\sharp A_i$ and $B_i^+ := B_i + \sigma_\sharp B_i$ with $i = 1, \ldots g_\sigma$ define a symplectic basis of $H_1(X,\mathbb{Z})_+$ and* $\dim H_1(X,\mathbb{Z})_+ = 2g_\sigma$. *The cycles $A_i^- := A_i - \sigma_\sharp A_i$, $B_i^- := B_i - \sigma_\sharp B_i$, $C_j^- := C_j$*

and $D_j^- := D_j$ *with* $i = 1, \dots g_\sigma$ *and* $j = 1, \dots, n-1$ *define a symplectic basis of* $H_1(X, \mathbb{Z})_-$ *and* $\dim H_1(X, \mathbb{Z})_- = 2(g_\sigma + n - 1)$*. Furthermore,* $H_1(X, \mathbb{Z})_\pm \subseteq H_1(X, \mathbb{Z}) \cap H^0(X, \Omega)^*_\pm$*.*

Proof. The intersection numbers of the two cycle bases can be calculated immediately with the results from the construction of the cycle basis of $H_1(X, \mathbb{Z})$ in Section 4.2.2. By definition, the A^+- und B^+-cycles are contained in $H_1(X, \mathbb{Z})_+$ and for $\omega_- \in H^0(X, \mathbb{Z})_-$, it is

$$\int_{A_i^+} \omega_- = \int_{A_i} \omega_- + \int_{\sigma_\sharp A_i} \omega_- = \int_{A_i} \omega_- + \int_{A_i} \sigma^* \omega_- = \int_{A_i} \omega_- - \omega_- = 0$$

and analogously $\int_{B_i^+} \omega_- = 0$. So $H_1(X, \mathbb{Z})_+ \subseteq H_1(X, \mathbb{Z}) \cap H^0(X, \Omega)^*_+$. Let $\gamma \in H_1(X, \mathbb{Z})_+$, i.e. $\sigma_\sharp \gamma = \gamma$. Since $\gamma \in H_1(X, \mathbb{Z})$, it can be represented as

$$\gamma = \sum_{i=1}^{g_\sigma} (a_i A_i + b_i \sigma_\sharp A_i + c_i B_i + d_i \sigma_\sharp B_i) + \sum_{j=1}^{n-1} (e_j C_j + f_j D_j)$$

with $a_i, b_i, c_i, d_i, e_j, f_j \in \mathbb{Z}$. Due to Lemma 4.18, one has

$$\begin{aligned} \sigma_\sharp \gamma &= \sum_{i=1}^{g_\sigma} (a_i \sigma_\sharp A_i + b_i A_i + c_i \sigma_\sharp B_i + d_i B_i) + \sum_{j=1}^{n-1} (e_j \sigma_\sharp C_j + f_j \sigma_\sharp D_j) \\ &= \sum_{i=1}^{g_\sigma} (a_i \sigma_\sharp A_i + b_i A_i + c_i \sigma_\sharp B_i + d_i B_i) - \sum_{j=1}^{n-1} (e_j C_j + f_j D_j) \end{aligned}$$

and since $A_i, \sigma_\sharp A_i, B_i, \sigma_\sharp B_i, C_j$ and D_j form a symplectic cycle basis of $H_1(X, \mathbb{Z})$, it is $a_i = b_i$, $c_i = d_i$ and $e_j = f_j = 0$. Therefore, a basis of $H_1(X, \mathbb{Z})_+$ is given by $\left\{A_i^+, B_i^+ \mid i = 1, \dots, g_\sigma\right\}$. We have already seen in Lemma 4.18 that for $j = 1, \dots, n-1$, it is $C_j, D_j \in H_1(X, \mathbb{Z})_-$. As above, also $A_i - \sigma_\sharp A_i$ and $B_i - \sigma_\sharp B_i$ are elements of $H_1(X, \mathbb{Z})_- \subset H^0(X, \Omega)^*_-$. The rest of the proof for the antisymmetric parts is shown analogously. □

Likewise for the symmetric and antisymmetric 1-forms the following proposition holds.

Proposition A.10. *A basis of* $H^0(X, \Omega)_+$ *is given by* $\omega_1^+, \dots, \omega_{g_\sigma}^+$ *and a basis of* $H^0(X, \Omega)_-$ *is given by* $\omega_1^-, \dots, \omega_{g_\sigma+n-1}^-$ *which are defined in* (4.12)*.*

Proof. One has

$$\dim(H^0(X, \Omega)_+) = \dim(H^0(X_\sigma, \Omega)) =: g_\sigma$$

Using the Riemann-Hurwitz Theorem [Forster, 1981, Theorem 17.14], $\dim(H^0(X_\sigma, \Omega)) = g_\sigma$ and $\dim(H^0(X, \Omega)) = g = 2g_\sigma + n - 1$ yields

$$\dim(H^0(X, \Omega)_-) = \dim(H^0(X, \Omega)) - \dim(H^0(X, \Omega)_+) = g - g_\sigma = g_\sigma + n - 1.$$

For $i = 1, \dots, g_\sigma$, it is

$$\sigma^* \omega_i^+ = \sigma^*(\omega_i + \omega_{g+i}) = \sigma^*\omega_i + \sigma^*\omega_{g+i}) = \omega_{g+i} + \omega_i = \omega_i^+.$$

Hence, $\omega_i^+ \in H^0(X, \Omega)_+$ and analogously $\sigma^*\omega_i^- = -\omega_i^-$. The normalization of the differential forms yields that also the remaining $n-1$ 1-forms $\omega_{g_\sigma+j}$ with $j = 1, \dots, n-1$ are elements of $H^0(X, \Omega)_-$:

$$1 = \int_{C_j} \omega_{g_\sigma+j}^- = -\int_{\sigma_\sharp C_j} \omega_{g_\sigma+j}^- = \int_{C_j} -\sigma^*\omega_{g_\sigma+j}^-.$$

The proof that every element $\omega_\pm \in H^0(X, \Omega)_\pm$ can be generated by the claimed bases follows by similar calculations as done in the corresponding part of the proof of proposition A.9. ☐

The next Lemma shows that $\mathrm{Jac}(X_\sigma)$ can be embedded into $\mathrm{Jac}(X)$.

Lemma A.11.

$$\mathrm{Jac}(X_\sigma) \simeq \frac{H^0(X, \Omega)_+^*}{H_1(X, \mathbb{Z})_+} = \left(\frac{H^0(X, \Omega)^*}{H_1(X, \mathbb{Z})} \right)_+ =: \mathrm{Jac}(X)_+.$$

Proof. We know from Proposition A.4 that $H^0(X_\sigma, \Omega) \simeq H^0(X, \Omega)_+$ and since $H^0(X, \Omega)$ and $H^0(X_\sigma, \Omega)$ are finite dimensional, it is

$$H^0(X_\sigma, \Omega)^* \simeq H^0(X_\sigma, \Omega) \simeq H^0(X, \Omega)_+ \simeq H^0(X, \Omega)_+^*.$$

Furthermore, it follows from the considerations in the construction of the cycle basis in Section 4.2.2 that $H_1(X, \mathbb{Z})_+ \simeq H_1(X_\sigma, \mathbb{Z})$. We combine these two results to prove that there is an embedding $\mathrm{Jac}(X_\sigma) \hookrightarrow \mathrm{Jac}(X)$ by showing that they lead to

$$\mathrm{Jac}(X_\sigma) = \frac{H^0(X_\sigma, \Omega)^*}{H_1(X_\sigma, \mathbb{Z})} \simeq \frac{H^0(X, \Omega)_+^*}{H_1(X, \mathbb{Z})_+}.$$

Now, it is clear that $\frac{H^0(X)_+^*}{H_1(X,\mathbb{Z})_+} \subset \left(\frac{H^0(X)^*}{H_1(X,\mathbb{Z})}\right)_+$. Conversely, for $\left[\int_\gamma\right] \in \left(\frac{H^0(X)^*}{H_1(X,\mathbb{Z})}\right)_+$, one has $[\gamma] = [\sigma_\sharp \gamma]$. Accordingly, it is with some linear combination $\delta \in H_1(X, \mathbb{Z})$

$$2\left[\int_\gamma\right] = \int_{\gamma+\delta+\sigma_\sharp(\gamma+\delta)} = \int_{\gamma+\sigma_\sharp\gamma} + \int_{\delta+\sigma_\sharp\delta},$$

where $\delta + \sigma_\sharp\delta \in H_1(X, \mathbb{Z})_+$. So $\left(\frac{H^0(X)^*}{H_1(X,\mathbb{Z})}\right)_+ \subset \frac{H^0(X)_+^*}{H_1(X,\mathbb{Z})_+}$. Taking these two inclusions together gives $\mathrm{Jac}(X_\sigma)_+ = \frac{H^0(X)_+^*}{H_1(X,\mathbb{Z})_+}$. ☐

Moreover, we can define the Prym variety with the above notation.

Definition A.12. The *Prym variety* of X with respect to the holomorphic involution σ is defined as

$$\mathrm{Prym}(X, \sigma) := \frac{H^0(X, \Omega)_-^*}{H_1(X, \mathbb{Z})_-} \subset \mathrm{Jac}(X).$$

It is shown in [Adler et al., 2010, Section 5.2.4] that $\mathrm{Prym}(X,\sigma)$ is a polarized Abelian variety which inherits a polarization of type $(1,\dots,1,2\dots,2)$ with $n-1$ times 1 and g_σ times 2 from $\mathrm{Jac}(X)$. So especially in the case of exactly two branch points, it follows from the considerations in Section 4.2.2 that $\mathrm{Prym}(X,\sigma) \simeq \mathrm{Jac}(X_\sigma)$ as $\mathbb{R}$-linear vector spaces. Seen as divisors, we know from $\mathcal{M}(X)_+ \simeq \mathcal{M}(X_\sigma)$ that every divisor D such that $D - \sigma(D) \simeq 0$ is the pullback of a divisor D_σ on $\mathrm{Jac}(X_\sigma)$ under π_σ. Furthermore, for a divisor D such that $D + \sigma(D) \simeq 0$, one has

$$\mathrm{Ab}(D) = \left(\int_D \omega_i\right)_{i=1}^g = \left(\int_D \frac{\omega_i + \sigma^*\omega_i}{2} + \frac{\omega_i - \sigma^*\omega_i}{2}\right)_{i=1}^g$$

The first summand vanishes since

$$\int_D \frac{\omega_i + \sigma^*\omega_i}{2} = -\int_{\sigma(D)} \frac{\omega_i + \sigma^*\omega_i}{2} = -\int_D \frac{\omega_i + \sigma^*\omega_i}{2}.$$

So $\mathrm{Ab}(D) \subset \mathrm{Prym}(X,\sigma)$. Conversely, let $P_+(D) = 0$. Then the definition of P_+ yields that $D + \sigma(D) \simeq 0$.
Now one sees why

$$\mathrm{Jac}(X) \not\simeq \mathrm{Jac}(X_\sigma) \oplus \mathrm{Prym}(X,\sigma)$$

by interpreting all elements of $H_1(X,\mathbb{Z})$ as a lattice in $H^0(X,\Omega)^*$ as it is done in (4.14). The elements of $H_1(X,\mathbb{Z})_+ \oplus H_1(X,\mathbb{Z})_-$ cannot generate all elements of $H_1(X,\mathbb{Z})$, compare figure A.2. In fact,

$$H_1(X,\mathbb{Z}) \supsetneq H_1(X,\mathbb{Z})_+ \oplus H_1(X,\mathbb{Z})_-. \tag{A.1}$$

The inclusion is obvious. To see the inequality, take for example an element A_i of the canonical basis of $H_1(X,\mathbb{Z})$. Then there is no linear combination of the cycles generating $H_1(X,\mathbb{Z})_+$ and $H_1(X,\mathbb{Z})_-$ which is equal to A_i since all linear combinations of these cycles either contain even multiples of A_i respectively $\sigma_\sharp A_i$ or combinations of A_i and $\sigma_\sharp A_i$. Therefore, $\mathrm{Jac}(X_\sigma) \oplus \mathrm{Prym}(X,\sigma) \not\simeq \mathrm{Jac}(X)$. This problem can be removed by dividing out additional points on the

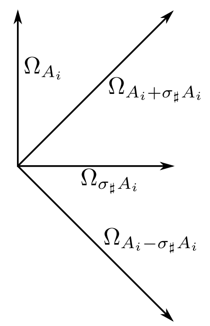


Figure A.2.: Sketch of one complex dimension in $\mathbb{C}^g$ spanned by Ω_{A_i} and $\Omega_{\sigma_\sharp A_i}$ as defined in (4.14), This sketch shows which elements of $H_1(X,\mathbb{Z})$ are missing in $H_1(X,\mathbb{Z})_+ \oplus H_1(X,\mathbb{Z})_-$, where A_i and $\sigma_\sharp A_i$.

right hand side of (A.1) to add additional points to the the lattice generated by the direct sum, such that this quotient is just fine enough to get an isomorphism between the finer lattice obtained like this and $H_1(X,\mathbb{Z})$. Which points are appropriate to divide out is proven in [Mumford, 1974, Theorem 2.1]. In [Mumford, 1974, Section I.2], the linear injection $\psi : H_1(X_\sigma,\mathbb{Z}) \to H_1(X,\mathbb{Z})_-$ is defined as

$$\psi\left(\frac{A_{\sigma,i}}{2}\right) = \frac{A_i - \sigma_\sharp A_i}{2}, \qquad \psi\left(\frac{B_{\sigma,i}}{2}\right) = \frac{B_i - \sigma_\sharp B_i}{2},$$

where $\{A_{\sigma,i}, B_{\sigma,i} \mid i = 1,\dots,g_\sigma\}$ is a symplectic basis of $H_1(X_\sigma,\mathbb{Z})$. To gain an isomorphism relating the two spaces $H_1(X,\mathbb{Z})$ and $H_1(X_\sigma,\mathbb{Z}) \oplus H_1(X,\mathbb{Z})_-$, we need to take into account that $H_1(X,\mathbb{Z})_+ \simeq H_1(X_\sigma,\mathbb{Z})$ and that one has to divide the cycles A_i, $\sigma_\sharp A_i$, B_i and $\sigma_\sharp B_i$ for $i \in \{1,\dots,g_\sigma\}$ from the direct sum. Those can be split up into a cycle belonging to $H_1(X_\sigma,\mathbb{Z})$ and a cycle belonging to $H_1(X,\mathbb{Z})_-$:

$$\gamma = \underbrace{\frac{\gamma + \sigma_\sharp\gamma}{2}}_{\in H_1(X,\mathbb{Z})_+} + \underbrace{\frac{\gamma - \sigma_\sharp\gamma}{2}}_{\in H_1(X,\mathbb{Z})_-} \qquad \text{for all } \gamma \in H_1(X,\mathbb{Z}).$$

Thus, $\gamma \in H_1(X,\mathbb{Z})$ can always be represented as $(\gamma_\sigma, \psi(\gamma_\sigma))$ with $\gamma_\sigma \in H_1(X_\sigma,\mathbb{Z})$ as it is done in [Mumford, 1974]. The element corresponding to $\frac{\gamma+\sigma_\sharp\gamma}{2}$ in $H_1(X_\sigma,\mathbb{Z})$ then equals $\pi_{\sigma,\sharp}\left(\frac{\gamma+\sigma_\sharp\gamma}{2}\right) = \frac{\gamma}{2}$. Let

$$\mathrm{Jac}_2(X_\sigma) := \{p \in \mathrm{Jac}(X_\sigma) \mid 2p = 0\},$$

i.e. the set which contains all elements in $(\Lambda_{X_\sigma}/2, \Lambda_-/2)$ which are not contained in $(\Lambda_{X_\sigma}, \Lambda_-)$. With this notation and $\alpha \in \mathrm{Jac}_2(X_\sigma)$, it is

$$H_1(X,\mathbb{Z}) = \frac{(H_1(X_\sigma,\mathbb{Z}) \oplus H_1(X,\mathbb{Z})_-)}{(\alpha, \psi(\alpha))}.$$

In [Mumford, 1974, Theorem 2.1] this is proven as

Lemma A.13 (Part of Corollary 1 in Mumford [Mumford, 1974].)**.** *There is a surjective mapping*

$$\mathrm{Jac}(X_\sigma) \times \mathrm{Prym}(X,\sigma) \to \mathrm{Jac}(X)$$

with kernel $\{(\alpha, \psi(\alpha)) \mid \alpha \in \mathrm{Jac}_2(X_\sigma)\}$.

B. Toolbox for real holomorphic 1-forms

This appendix contains the main tools which are applied in Section 6.2.2 to show the existence of holomorphic differentials which either have a certain orientation or a prescribed number of zeros on the ovals of a real curve (X, τ). Large parts of the corresponding parts in [Natanzon, 2004] were more or less correct despite they are not always worked out entirely. So we decided to attach a worked-out version in this appendix to give a more complete picture of the used method to describe the real Prym variety. We will give a review of some results from [Natanzon, 2004] concerning a 1-1 connection between so-called real Arf-functions and spinors, i.e. sections of a rank 1 spinor bundle, on a real curve (X, τ). Spinors are section of bundles whose square is the cotangent bundle and Arf functions are used to indicate an orientation on the ovals of the real curve (X, τ). What exactly we understand under these objects and what the meaning of the respective additional structures due to the realness-condition is we will explain hereinafter in Sections B.3 and B.4. In this work, we will only show the mentioned 1-1-connection to the extend that we think is necessary to understand intuitively why it exists. The bridge between these two objects is build by so-called real Fuchsian groups. So we will introduce the latter first.

B.1. Real Fuchsian groups

Fuchsian models are a well known concept to represent Riemann surfaces. A good introduction is for example given in [Imayoshi and Taniguchi, 2012, Chapter 2.4]. However, for self-containedness, we will now give a brief introduction of Fuchsian groups and Fuchsian models of Riemann surfaces of genus $g \geq 2$ before we define what a real Fuchsian group is.

For $g \geq 2$, the universal covering space [Imayoshi and Taniguchi, 2012, Section 2.2.1] is the upper half plane $\mathcal{H}$. For a given universal covering map $\pi : \mathcal{H} \to X$ the universal covering group Λ on $\mathcal{H}$ is isomorphic to the fundamental group $\pi_1(X, p_0)$, compare [Munkres, 2000, § 52, page 331], which is defined as the set of simple closed curves on X starting and ending at a given fixed point p_0 up to homotopy. The universal covering $\mathcal{H}$ is constructed from a given Riemann surface by considering the homotopy classes of all paths on X also starting at p_0. We do not make any difference whether we consider an element as an equivalence class or as a representant of this equivalent class. The fundamental group $\pi_1(X, p_0)$ acts on $\mathcal{H}$ as follows: Let $\gamma \in X$ be a path starting at p_0 and ending at p and $\alpha \in \pi_1(X, p_0)$. Then α acts on γ as $\gamma \mapsto \alpha + \gamma$ and for two elements $\alpha, \beta \in \pi_1(X, p_0)$, one has $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$, and therefore $\pi_1(X, p_0)$ defines a group action on $\mathcal{H}$. Obviously, this group action is discrete and free. To get insight into how τ

acts on $\pi_1(X, p_0)$ respectively $\mathcal{H}$, let δ be a path starting at p_0 and ending at $\tau(p_0)$. Then

$$\tau(\alpha) = \delta + \tau \circ \alpha \text{ and } \tau(\gamma) = \delta + \tau \circ \gamma \tag{B.1}$$

for $\alpha \in \pi_1(X, p_0)$ and $\gamma \in \mathcal{H}$. We extend $\pi_1(X, p_0)$ as follows:

$$\tilde{\pi}_1(X, p_0) := \{a \in \mathrm{Aut}(\mathcal{H}) \mid \pi \circ a = \pi \text{ or } \pi \circ a = \tau \circ \pi\}.$$

Because the two maps τ and $\mathbb{1}$ generate a group, also $\tilde{\pi}_1(X, p_0)$ generates a group which contains $\pi_1(X, p_0)$ as a subgroup and the cardinality of $\tilde{\pi}_1(X, p_0)$ is just twice the cardinality of $\pi_1(X, p_0)$.

Proposition B.1. *Let $a \in \pi_1(X, p_0)$ and $\delta \in \mathcal{H}$ an element which connects p_0 with $\tau(p_0)$. Then there is an element $\alpha \in \tilde{\pi}_1(X, p_0) \setminus \pi_1(X, p_0)$ such that*

$$\alpha \circ a \circ \alpha^{-1} = \delta + \tau(a) + \delta^{-1}.$$

Proof. To see this, we write α, a and α^{-1} as diffeomorphisms on $\mathcal{H}$, i.e. as diffeomorphisms on homotopy classes of paths on X. Let γ be an arbitrary element of $\mathcal{H}$, i.e. a finite and connected path starting at p_0. As already mentioned, any $a \in \pi_1(X, p_0)$ acts on $\mathcal{H}$ as $\gamma \mapsto a + \gamma$. Due to equation (B.1), α acts as $\gamma \mapsto \delta + \tau(\gamma)$, where $\delta \in \mathcal{H}$ connects p_0 with $\tau(p_0)$. To determine α^{-1}, we consider the map

$$\alpha^2 : \gamma \mapsto \delta + \tau(\delta + \tau(\gamma)) = \delta + \tau(\delta) + \gamma = \beta + \gamma,$$

where β is a closed path in X starting and ending at p_0, and therefore a representant of an element of $\pi_1(X, p_0)$ and α^2 is invertible with inverse map $\beta^{-1} : \gamma \mapsto -\beta + \gamma$. Hence,

$$\alpha^{-1} = \alpha \circ \alpha^{-2} : \gamma \mapsto \delta + \tau(-\beta + \gamma).$$

This yields together with $\tau(\delta) - \beta = -\delta$ that

$$\alpha \circ a \circ \alpha^{-1} : \gamma \mapsto \delta + \tau(a + \delta + \tau(-\beta + \gamma)) = (\delta + \tau(a) - \delta) + \gamma.$$

So τ acts on elements $\gamma \in \mathcal{H}$ as conjugation with the element $\alpha \in \tilde{\pi}_1(X, p_0) \setminus \pi_1(X, p_0)$. □

Furthermore, we can define a map $\pi_1(X, p_0) \to H_1(X, \mathbb{Z})$ since $H_1(X, \mathbb{Z})$ is just the abelianization of $\pi_1(X, p_0)$, compare [Hatcher, 2002, Theorem 2A.1]. Also because $H_1(X, \mathbb{Z})$ is the abelianization of $\pi_1(X, p_0)$, it is $\alpha \circ a \circ \alpha^{-1} = \tau(a)$ in $H_1(X, \mathbb{Z})$. By taking the quotient $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, we can define a projection which is necessary for the sequel as

$$\mathrm{Pr} : \Lambda \to \pi_1(X, \mathbb{Z}) \to H_1(X, \mathbb{Z}) \to H_1(X, \mathbb{Z}_2). \tag{B.2}$$

In case that X carries a metric, the isometries of X always constitute a group of transformations. For the upper half space, this group is defined as follows.

Theorem B.2 ([Jost, 2013, Theorem 2.3.4]). *We define* $\mathrm{PSL}(2,\mathbb{R})$ *by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{R}) := \mathrm{SL}(2,\mathbb{R})/\{\pm \mathbb{1}\},$$

where

$$\mathrm{SL}(2,\mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbb{R} \quad \text{and } ad - bc = 1 \right\}.$$

Then $\mathrm{PSL}(2,\mathbb{R})$ *is a transformation group of* $\mathcal{H}$. *The operation is transitive, i.e. for any* $z_1, z_2 \in \mathcal{H}$, *there is a* $g \in \mathrm{PSL}(2,\mathbb{R})$ *with* $g(z_1) = z_2$, *and effective, i.e. if* $g(z) = z$ *for all* $z \in \mathcal{H}$, *then* $g = e$.

Note that for G acting properly discontinuous, the orbit $\{g(p) \mid g \in G\}$ of every $p \in X$ is discrete, compare [Jost, 2013, Definition 2.3.7 and Lemma 2.4.1]. In the sequel, we will mainly consider properly discontinuous subgroups Λ of $\mathrm{PSL}(2,\mathbb{R})$, where Λ acts on $\mathcal{H}$ as a group of isometries. It is basic knowledge of Möbius transformations that $\mathrm{SL}(2,\mathbb{R})$ decomposes into three different types of elements, compare [Jost, 2013, Lemma 2.4.2]: Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{R})$ and define $\mathrm{tr}(\gamma) := a + d$. Then the following classification is for example illuminated nicely after [Jost, 2013, Lemma 2.4.2]:

γ **is elliptic:** There is one fixed point of γ in $\mathcal{H}$ and $\mathrm{tr}(\gamma) < 2$.

γ **is parabolic:** There is one fixed point of γ on the extended real line $\partial\mathcal{H} := \mathbb{R} \cup \{\infty\}$ and $\mathrm{tr}(\gamma) = 2$.

γ **is hyperbolic:** There are two fixed points of γ on $\partial\mathcal{H}$ and $\mathrm{tr}(\gamma) > 2$. Furthermore, every hyperbolic element is $\mathrm{Aut}(\mathcal{H})$-conjugate to an element of the form $\tilde{\gamma} = \left(\begin{smallmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{\lambda}^{-1} \end{smallmatrix}\right)$ with $\lambda > 1$ and $\tilde{\gamma}$ leaves the geodesic connecting these two points, namely the imaginary axis, invariant. On that geodesic, $\tilde{\gamma}$ operates as a translation, i.e. shifts points along it by the distance $\log(\lambda^2)$.

In the sequel, we will mainly consider hyperbolic elements of $\mathrm{PSL}(2,\mathbb{R})$. The reason for that is formulated in the next Lemma.

Lemma B.3 ([Jost, 2013, Lemma 2.4.4]). *Let* $\mathcal{H}/\Lambda$ *be a compact Riemann surface for a subgroup* Λ *of* $\mathrm{PSL}(2,\mathbb{R})$. *Then all elements of* Λ *are hyperbolic.*

So let the Λ be the universal covering transformation group of $\mathcal{H}$. Then Λ is a discrete subgroup of the automorphisms $\mathrm{Aut}(\mathcal{H})$ of $\mathcal{H}$ and called a *Fuchsian model* of X, compare [Imayoshi and Taniguchi, 2012, Theorem 2.15]. Hereby, $\mathrm{Aut}(\mathcal{H})$ denotes the holomorphic automorphisms of $\mathcal{H}$

which are called *real Möbius transformations*. The *hyperbolic* or *real* elements of $\gamma \in \mathrm{Aut}(\mathcal{H})$ are of the form

$$\gamma(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$, see [Imayoshi and Taniguchi, 2012, Lemma 2.8(iv)], i.e. $\gamma \in \mathrm{PSL}(2, \mathbb{R})$. The two fixed points $\alpha, \beta \in \mathbb{R} \cup \{\infty\}$ are determined by $C(z) = z$ which yields

$$\alpha, \beta = \frac{(a-d) \pm \sqrt{(a-d)^2 + 4bc}}{2c}.$$

Then γ can equivalently be represented as

$$\gamma(z) = \frac{(\lambda\alpha - \beta)z + (1-\lambda)\alpha\beta}{(\lambda - 1)z + (\alpha - \lambda\beta)}, \tag{B.3}$$

whereas $0 < \lambda = \frac{1}{(\alpha-\beta)^2} \neq 1$ since

$$1 = \det \begin{pmatrix} \lambda\alpha - \beta & (1-\lambda)\alpha\beta \\ \lambda - 1 & \alpha - \lambda\beta \end{pmatrix} = (\lambda\alpha - \beta)(\alpha - \lambda\beta) + (1-\lambda)^2\alpha\beta = \lambda(\alpha - \beta)^2.$$

It is

$$\begin{aligned} \gamma'(z) &= \frac{(\lambda\alpha - \beta)(\lambda z - z + \alpha - \lambda\beta) - (\lambda - 1)(\lambda\alpha z - \beta z + \alpha\beta - \lambda\alpha\beta)}{((\lambda-1)z + (\alpha - \lambda\beta))^2} \\ &= \frac{\lambda(\alpha-\beta)^2}{((\lambda-1)z + (\alpha - \lambda\beta))^2} = \frac{1}{((\lambda-1)z + (\alpha - \lambda\beta))^2}, \end{aligned}$$

and therefore $|\gamma'(\alpha)| = \frac{1}{\lambda}$ and $|\gamma'(\beta)| = \lambda$. So if we consider $\lambda > 1$, then the repelling fixed point of γ is given by β and the attracting fixed point by α, compare [Morosawa, 2000, § 2.6, Definition page 209]. Since γ is a Möbius transformation with two fixed points, it is known that γ is $\mathrm{Aut}(\mathcal{H})$-conjugate to a Möbius transformations $\gamma_0(z) = \lambda z$ for some $\lambda \in \mathbb{R} \cup \{\infty\}$, i.e. it acts as a pure rotation-dilation, which corresponds to the matrix $\left(\begin{smallmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{\lambda}^{-1} \end{smallmatrix}\right)$. This can be seen by considering another Möbius transformation $\phi \in \mathrm{PSL}(2, \mathbb{R})$ which maps α to 0 and β to ∞ such that $\phi \circ \gamma \circ \phi^{-1}$ has the two fixed points 0 and ∞ and can be represented as $z \mapsto \lambda z$. Let $\ell(\gamma) \subset \mathcal{H}$ be the unique geodesic in the metric of $\mathcal{H}$ which joins α and β and is oriented from β to α, compare [Jost, 2013, Lemma 2.4.4]. The automorphism γ preserves the line $\ell(\gamma)$ while shifting it in the direction of the orientation. We explain the geometric correspondence between X and its group of universal covering transformations Λ as it is done in [Imayoshi and Taniguchi, 2012, Section 2.4.2]. Hereby, we use a fundamental domain F for Λ. An open set F of $\mathcal{H}$ is a *fundamental domain* of Λ if it satisfies the following three properties:

(i) For every $C \in \Lambda$ with $C \neq \mathbb{1}$, it is $C(F) \cap F = \emptyset$.

(ii) $\mathcal{H} = \bigcup_{C \in \Lambda} C(\overline{F})$ where $\overline{F}$ is the closure of F.

(iii) The boundary of F in $\mathcal{H}$ has measure zero with respect to the two-dimensional Lebesgue measure.

Then $X = \mathcal{H}/\Lambda$ can be considered as $\overline{F}$, where the points of ∂F are identified under the covering group Λ, compare [Imayoshi and Taniguchi, 2012, Section 2.4.2]. For a Fuchsian model of X, this fundamental domain is constructed by cutting X along suitable smooth paths on X to get a simply connected domain X_0. Then F is the inverse image $\pi^{-1}[X_0]$ under the covering map $\pi : \mathcal{H} \to X$. Since we only consider Riemann surfaces of genus g with n boundary components which are all isomorphic to S^1, every fundamental domain is a combination of one of the following two examples in (a) and (b). This can be seen in example (c).

Example B.4 (Fundamental Group and Fuchsian Group of a Riemann surface).

(a) ([Imayoshi and Taniguchi, 2012, Example 5 of Section 2.4.2]) Let X be a compact Riemann surface of genus g and let $\{A_i, B_i \mid i = 1, \ldots, g\}$ be a canonical system of generators of the fundamental group $\pi_1(X, p_0)$ with base point $p_0 \in X$. Then all A_i and B_i are smooth simple closed curves starting and ending at p_0. Cutting X along all these contours yields a simply connected domain X_0 which preimage $F = \pi^{-1}[X_0]$ is again a fundamental domain, compare Figure B.1a for $g = 2$. The lattice Λ is generated by $\widetilde{A}_1, \widetilde{B}_1, \ldots, \widetilde{A}_g, \widetilde{B}_g$. The generators of $\pi_1(X, p_0)$ are called canonical if they satisfy the fundamental relation

$$\prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1} = \mathbb{1},$$

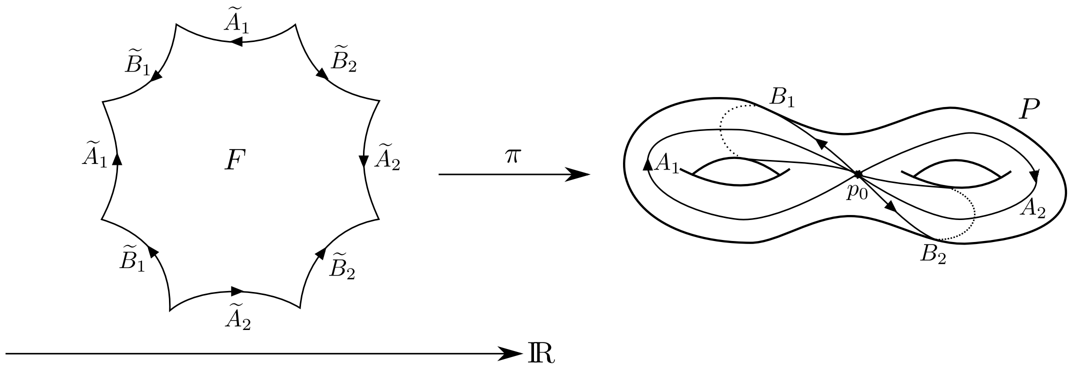
so the generators of Λ satisfy the analogon

$$\prod_{i=1}^{g} \widetilde{A}_i \widetilde{B}_i \widetilde{A}_i^{-1} \widetilde{B}_i^{-1} = \mathbb{1}.$$

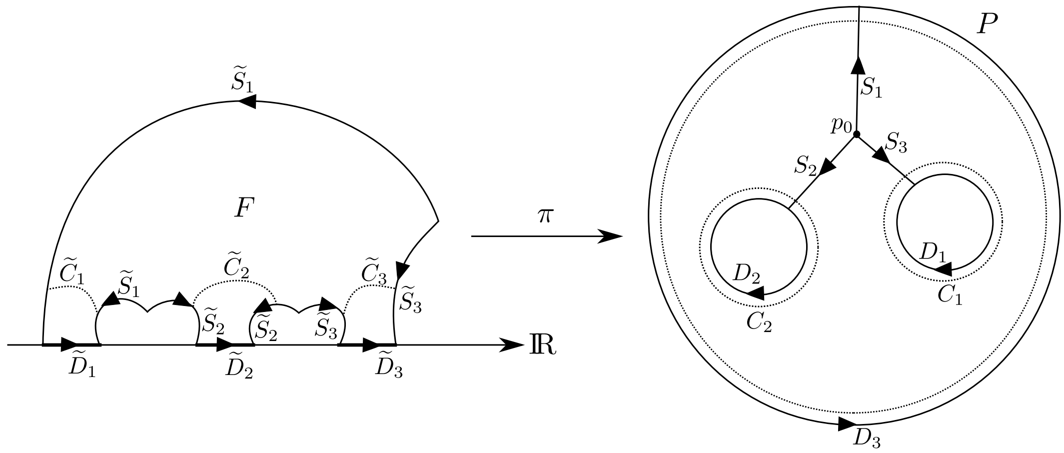
(b) ([Imayoshi and Taniguchi, 2012, Example 6 of Section 2.4.2]) Let X be a domain in $\mathbb{C}$ bounded by n boundary cycles $C_1, \ldots, C_n$ of X. Then one takes a point $p_0 \in X$ and cuts X along n smooth curves $D_1, \ldots, D_n$. This yields a simply connected domain X_0 which preimage $F = \pi^{-1}[X_0]$ is again a fundamental domain. Compare Figure B.1b for $n = 3$. In this case, Λ is generated by $C_1, \ldots, C_{n-1}$ and the fundamental relation in this case is given by

$$\prod_{j=1}^{n-1} C_j = 1 \quad \text{respectively} \quad \prod_{j=1}^{n-1} \widetilde{C}_j = 1.$$

(c) Combining examples (a) and (b) yields a Riemann surface of genus g with n boundary cycles. Thus, the fundamental group $\pi_1(X, p_0)$ is generated by $A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_{n-1}$, where



(a) Fundamental domain F of Example B.4(a) with $g = 2$ as in [Imayoshi and Taniguchi, 2012, Fig. 2.2].



(b) Fundamental domain F of Example B.4 (b) with $n = 3$. as in [Imayoshi and Taniguchi, 2012, Fig. 2.3].

Figure B.1.: Sketch how to construct the fundamental domain F of a Fuchsian model.

the canonical generators satisfy the fundamental relation

$$\prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1} \prod_{j=1}^{n-1} C_j \quad \text{respectively} \quad \prod_{i=1}^{g} \widetilde{A}_i \widetilde{B}_i \widetilde{A}_i^{-1} \widetilde{B}_i^{-1} \prod_{j=1}^{n-1} \widetilde{C}_j. \tag{B.4}$$

Next, we want to define a Fuchsian groups on real curve. More precisely, we want to introduce some extra structure which reflects the realness condition on the universal covering. Therefore, we need some definitions and notation from [Natanzon, 2004, § 1-§ 5]. We will see that real curves of genus $g \geq 2$ can be uniformized by discrete groups of isometries of the metric $\frac{|dz|}{\operatorname{Im} z}$ of the upper half space $\mathcal{H}$, see [Imayoshi and Taniguchi, 2012, Remark in Section 3.1.2, page 53].

Definition B.5. The group $\widetilde{\operatorname{Aut}}(\mathcal{H})$ of isometries on $\mathcal{H}$ consists of the holomorphic automorphisms that form the group $\operatorname{Aut}(\mathcal{H})$ and of the antiholomorphic automorphisms.

We only consider discrete subgroups $\widetilde{\Lambda} \subset \widetilde{\operatorname{Aut}}(\mathcal{H})$ with certain properties which are given in the next definition.

Definition B.6 ([Natanzon, 2004, Beginning of Section 2.2.1])**.** A *real Fuchsian group* is a discrete subgroup $\widetilde{\Lambda} \subset \widetilde{\mathrm{Aut}}(\mathcal{H})$ such that

- $\Lambda := \widetilde{\Lambda} \cap \mathrm{Aut}(\mathcal{H})$ is a Fuchsian group that consists of hyperbolic automorphisms,
- $\Lambda \neq \widetilde{\Lambda}$ and $X = \mathcal{H}/\Lambda$ is a compact Riemann surface.

A real Fuchsian group $\tilde{\Lambda}$ must have twice the cardinality of Λ. Otherwise, $\tilde{\Lambda}$ would not be a group. Let $\gamma_1, \gamma_2 \in \widetilde{\Lambda} \setminus \Lambda$. Then $\gamma_2 \circ \gamma_1^{-1} \in \Lambda$ acts trivially on $X = \mathcal{H}/\Lambda$ and $\gamma_2 = \gamma_2 \circ \gamma_1^{-1} \circ \gamma_1$. So all elements of $\widetilde{\Lambda} \setminus \Lambda$ induce the same map on X. Since $\gamma_1^2 \in \Lambda$, this map is an involution on X. We denote it by $\tau : X \mapsto X$. By definition, τ is antiholomorphic. Thus, a real Fuchsian group $\tilde{\Lambda}$ generates a real curve (X, τ).

Lemma B.7 ([Natanzon, 2004, Lemma 2.2.1])**.** *Every real curve is generated by a real Fuchsian group.*

Proof. Let $\Lambda \subset \mathrm{Aut}(\mathcal{H})$ be a Fuchsian group uniformizing the Riemann surface X and let $\Phi : \mathcal{H} \to X = \mathcal{H}/\Lambda$ be the natural projection defined by $p \mapsto [p]$. Since $\mathcal{H}$ is simply connected, the paths corresponding to $\tau \in \widetilde{\mathrm{Aut}}(X) \setminus \mathrm{Aut}(X)$ can be lifted to $\mathcal{H}$. This is is an element $\alpha \in \widetilde{\mathrm{Aut}}(\mathcal{H}) \setminus \mathrm{Aut}(\mathcal{H})$ such that $\Phi \circ \alpha = \tau \circ \Phi$, compare [Imayoshi and Taniguchi, 2012, Lemma 2.3]. Then the group $\widetilde{\Lambda}$ which is generated by Λ and τ corresponds to (X, τ) and thus is a real Fuchsian group. □

B.2. Arf functions on real curves

Let X be a compact Riemann surface with boundary of genus $g \geq 2$ with k boundary components.

Definition B.8. A basis $\mathcal{B} = \{A_i, B_i, C_j \mid i = 1, \dots, g, j = 1, \dots, k\}$ of the group $H_1(X, \mathbb{Z}_2)$ is said to be *standard* if the generators C_j correspond to the boundary cycles of the surface X and if the only non-trivial intersections of elements of $\mathcal{B}$ are given by $A_i \star B_j = 1$ for $i \neq j = 1, \dots, g$, where

$$\star : H_1(X, \mathbb{Z}_2) \times H_1(X, \mathbb{Z}_2) \to H_0(X, \mathbb{Z}_2) = \mathbb{Z}_2$$

is the homology intersection number, compare [Bredon, 2010, Section VI.11].

Definition B.9. (a) An *Arf function* on X is a function $\omega : H_1(X, \mathbb{Z}_2) \to \mathbb{Z}_2$ such that for all $A, B \in H_1(X, \mathbb{Z}_2)$, there holds

$$\omega(A + B) = \omega(A) + \omega(B) + A \star B. \tag{B.5}$$

(b) An Arf function ω is *even* if there is a standard basis $\mathcal{B}$ such that for all $A_i, B_i \in \mathcal{B}$, there holds

$$\sum_{i=1}^{g} \omega(A_i)\omega(B_i) \equiv 0 \mod 2. \tag{B.6}$$

For an even Arf function we set $\delta = \delta(X, \omega) = 0$. Otherwise we set $\delta = \delta(X, \omega) = 1$ and say that ω is *odd*.

(c) We denote by $k_\alpha = k_\alpha(X, \omega)$ with $\alpha \in \mathbb{Z}_2$ the cardinality of the set of elements C_j of a standard basis $\mathcal{B}$ such that $\omega(C_j) = \alpha$.

(d) The quadruplet (g, δ, k_0, k_1) is called the *topological type* of the Arf function ω.

(e) Two Arf functions ω_1 and ω_2 on X are *topologically equivalent* if there is a homeomorphism $\psi : X \to X$ that induces an automorphism $\tilde{\psi} : H_1(X, \mathbb{Z}_2) \to H_1(X, \mathbb{Z}_2)$ satisfying the relation $\omega_1 = \omega_2 \circ \tilde{\psi}$.

That two Arf functions are topological equivalent if and only if they have the same topological type is shown in [Natanzon, 2004, Theorem 1.8.1].

Proposition B.10. *Let $\omega : H_1(X, \mathbb{Z}_2) \to \mathbb{Z}_2$ be an Arf function. Then for all $\gamma \in H_1(X, \mathbb{Z}_2)$, there holds*

$$\omega(0) = \omega(2\gamma) = 0 \mod 2 \quad \textit{and} \quad \omega(-\gamma) = \omega(\gamma) \mod 2.$$

Proof. Assume that $\omega(0) = 1$. Then (B.5) yields that for all $\gamma \in H_1(X, \mathbb{Z}_2)$, it is

$$\omega(\gamma) = \omega(\gamma + 0) = \omega(\gamma) + \omega(0) + \gamma \star 0 = \omega(\gamma) + 1.$$

So $\omega(0) = 0$ in $\mathbb{Z}_2$. Likewise,

$$\omega(2\gamma) = \omega(\gamma) + \omega(\gamma) = 0 \mod 2.$$

Due to $\omega(0) = 0 \mod 2$, it is

$$0 = \omega(\gamma - \gamma) = \omega(\gamma) + \omega(-\gamma) + \gamma \star (-\gamma) \;\Leftrightarrow\; \omega(-\gamma) = -\omega(\gamma)$$

and since $-1 = 1 \mod 2$, the second assertion follows. □

Lemma B.12 gives insight into when a quadruplet (g, δ, k_0, k_1) is the topological type of an Arf-function. In the proof of this Lemma, certain transformations on the basis elements of a standard basis $\mathcal{B} = \{A_i, B_i, C_j \mid i = 1, \dots, g, j = 1, \dots k\}$ of $H_1(X, \mathbb{Z}_2)$ are used such that the transformed cycles again yield a standard basis of $H_1(X, \mathbb{Z}_2)$ and such that δ corresponding to $\mathcal{B}$ is invariant under most of these transformations. In [Natanzon, 2004] it is neither shown that the image of $\mathcal{B}$ under these transformations yields again a standard basis nor the invariance of δ under some of these transformations. Furthermore, the transformation in [Natanzon, 2004, Lemma 1.8.1(3)] has wrong indices. So we catch up on this proof in the next Lemma.

Lemma B.11. *Let $\mathcal{B} := \{A_i, B_i, C_j \mid i = 1, \ldots, g,\ j = 1, \ldots, k\}$ be a standard basis of $H_1(X, \mathbb{Z}_2)$ as in Definition B.8. For fixed $i \neq l \in \{1, \ldots, g\}$ and $j, m \in \{1, \ldots, k\}$ we consider the transformations*

(i) $\tilde{A}_i = A_i + B_i$,

(ii) $\tilde{A}_i = B_i$ *and* $\tilde{B}_i = A_i$,

(iii) $\tilde{A}_i = A_i + A_l$, $\tilde{B}_l = B_i + B_l$,

(iv) $\tilde{A}_i = A_i + C_j$,

(v) $\tilde{C}_j = C_m$ *and* $\tilde{C}_m = C_j$,

where the other elements of the basis remain unchanged. Then all transformations yield again a standard system of generators of $H_1(X, \mathbb{Z}_2)$ and transformations (i)-(iii) as well as (v) preserve δ in (B.6).

Proof. Let $\tilde{\mathcal{B}} := \{\tilde{A}_i, \tilde{B}_i, \tilde{C}_j \mid i = 1, \ldots, g, j = 1, \ldots, k\}$ be the image of $\mathcal{B}$ under the particular transformation in (i) to (v).

(i) For the transformed elements of $\tilde{\mathcal{B}}$, there holds $\tilde{A}_i \star \tilde{B}_i = (A_i + B_i) \star B_i = A_i \star B_i$ since $A_i, B_i \in \mathcal{B}$. All other intersections between elements of $\tilde{\mathcal{B}}$ remain the same as the for the non-transformed elements in $\mathcal{B}$, so $\tilde{\mathcal{B}}$ is again a standard system of generators of $H_1(X, \mathbb{Z}_2)$. To see that δ does not change under this transformation, note that for $i \in \{1, \ldots, g\}$, it is

$$\sum_{l=1}^{g} \omega(\tilde{A}_l)\omega(\tilde{B}_l) = \sum_{l=1}^{g} \omega(A_l)\omega(B_l) + \omega(A_i + B_i)\omega(B_i) - \omega(A_i)\omega(B_i), \tag{B.7}$$

where due to (B.5)

$$\omega(A_i+A_j)\omega(B_i) = \omega(A_i)\omega(B_i)+\omega(B_i)\omega(B_i)+(A_i\star B_i)\omega(B_i) = \omega(A_i)\omega(B_i)+\omega^2(B_i)+\omega(B_i)$$

and $\omega(B_i)^2 + \omega(B_i) = 0 \mod 2$. Inserting this into (B.7) yields

$$\sum_{l=1}^{g} \omega(\tilde{A}_l)\omega(\tilde{B}_l) = \sum_{l=1}^{g} \omega(A_l)\omega(B_l) + \omega(A_i)\omega(B_i) - \omega(A_i)\omega(B_i) = \sum_{l=1}^{g} \omega(A_l)\omega(B_l) \mod 2.$$

(ii) Since $A_i \star B_i = B_i \star A_i \mod 2$ for $i \in \{1, \ldots, g\}$, $\tilde{\mathcal{B}}$ is again a standard system of generators of $H_1(X, \mathbb{Z}_2)$ and equation (B.6) holds.

(iii) For $i, l \in \{1, \ldots, g\}$, it is

$$\begin{aligned}
\tilde{A}_i \star \tilde{B}_i &= (A_i + A_l) \star B_i = A_i \star B_i + A_l \star B_i = A_i \star B_i, \\
\tilde{A}_l \star \tilde{B}_l &= A_l \star (B_i + B_l) = A_l \star B_l, \\
\tilde{A}_i \star \tilde{B}_l &= (A_i + A_l) \star (B_i + B_l) = A_i \star B_i + A_l \star B_i + A_i \star B_l + A_l \star B_l = 0 \mod 2.
\end{aligned}$$

For $n \in \{1, \ldots, g\}$ with $n \neq i, l$, the intersection product of $\tilde{A}_n$ and $\tilde{B}_n$ with $\tilde{A}_i$ respectively $\tilde{B}_l$ is equal to zero. So $\tilde{\mathcal{B}}$ is a standard system of generators of $H_1(X, \mathbb{Z}_2)$. To see that δ is preserved under this deformation, note that for $i, l \in \{1, \ldots, g\}$, one has

$$\sum_{n=1}^{g} \omega(\tilde{A}_n)\omega(\tilde{B}_n) = \sum_{n=1}^{g} \omega(A_n)\omega(B_n) + \omega(\tilde{A}_i)\omega(\tilde{B}_i) + \omega(\tilde{A}_l)\omega(\tilde{B}_l) - \omega(A_i)\omega(B_i) - \omega(A_l)\omega(B_l),$$

where

$$\begin{aligned}
\omega(\tilde{A}_i)\omega(\tilde{B}_i) + \omega(\tilde{A}_l)\omega(\tilde{B}_l) &= \omega(A_i + A_l)\omega(B_i) + \omega(A_l)\omega(B_i + B_l) \\
&= \omega(A_i)\omega(B_i) + \underbrace{\omega(A_l)\omega(B_i) + \omega(A_l)\omega(B_i)}_{=0 \mod 2} + \omega(A_l)\omega(B_l) \\
&= \omega(A_i)\omega(B_i) + \omega(A_l)\omega(B_l) \mod 2.
\end{aligned}$$

(iv) That $\tilde{\mathcal{B}}$ is a standard system of generators of $H_1(X, \mathbb{Z}_2)$ is obvious since $C_j \star B_i = 0$ for $j \in \{1, \ldots, k\}$ and $i \in \{1, \ldots, g\}$.

(v) Again, $\tilde{\mathcal{B}}$ is a standard system of generators of $H_1(X, \mathbb{Z}_2)$. This holds since the transformation only interchanges C-cycles. Moreover, δ is independent of the transformed C_j and C_m with $j, m \in \{1, \ldots, k\}$, so the transformation preserves δ.

□

Now, we gathered the tools which are necessary deduce the conditions that have to hold such that (g, δ, k_0, k_1) is the topological type of an Arf function. Since the proof given in [Natanzon, 2004] is not very precise, we will formulate it here as well:

Lemma B.12. *[Natanzon, 2004, Lemma 1.8.1] A set (g, δ, k_1, k_0) is the topological type of an Arf function if and only if*

(i) $k_1 = 0 \mod 2$, *(ii)* $\delta = 0$ *for* $k_1 > 0$.

If these conditions are satisfied, then there exists a standard basis $\mathcal{B} = \{A_i, B_i, C_j \mid i = 1, \ldots, g,\ j = 1, \ldots, k\}$ of $H_1(X, \mathbb{Z}_2)$ with $k_0 + k_1 = k$ such that $\omega(A_i) = \omega(B_i) = 0$ for $i > 1$, $\omega(A_1) = \omega(B_1) = \delta$, $\omega(C_j) = 0$ for $j = 1, \ldots, k_0$ and $\omega(C_j) = 1$ for $j = k_0 + 1, \ldots, k$.

Proof. Due to Proposition B.10, it is $\omega(0) = 0$. Moreover, the $\sum_{i=1}^{k} C_i$ is the sum over the boundary components of X, i.e. homologous to zero, and so

$$\sum_{i=1}^{k} \omega(C_i) = \omega\left(\sum_{i=1}^{k} C_i\right) = \omega(0) = 0.$$

This implies $k_1 = 0 \mod 2$. Let $i \in \{1, \dots, g\}$. To obtain a basis such that $\omega(A_i) = \omega(B_i) = 0$ and $\omega(A_1) = \omega(B_1) = \delta$, one can use transformations (i)-(iii) from Lemma B.11: If $\omega(A_i) = \omega(B_i) = 0$ for all $i \in \{1, \dots, g\}$, the assertion holds. If $\omega(A_i) = 1$ and $\omega(B_i) = 0$, one can use transformation (i) to obtain a new basis $\mathcal{B}$ such that $\omega(\tilde{A}_i) = 0$ and $\omega(\tilde{B}_i) = 0$. If conversely $\omega(B_i) = 1$ and $\omega(A_i) = 0$, then we use transformation (ii) to get into the former situation. So let us assume there exists at least one $i \in \{1, \dots, g\}$ such that $\omega(A_i) = \omega(B_i) = 1$. We are done if $i = 1$. We impose that the basis elements are enumerated in such a way that $\omega(A_1) = \omega(B_1) = 0$ and assume that $\omega(A_i) = \omega(B_i) = 1$ for $i \neq 1$. If there are evenly many i such that the above relation holds for the corresponding basis elements, then $\delta = 0$. For oddly many i, it is $\delta = 1$. So we have to distinguish between an even and an odd number of i such that the above assumption holds for the corresponding basis elements. Let $\omega(A_i) = \omega(B_i) = \omega(A_j) = \omega(B_j) = 1$ for $i \neq j \in \{2, \dots, g\}$. Applying transformation (iii) from Lemma B.11 yields a new standard basis $\mathcal{B}$ of $H_1(X, \mathbb{Z})$ such that $\omega(\tilde{A}_i) = \omega(\tilde{B}_j) = 0$ and $\omega(\tilde{B}_i) = \omega(\tilde{A}_j) = 1$. As above, we can apply transformation (i) and (ii) to this new standard basis $\mathcal{B}$ which yields again another basis of $H_1(X, \mathbb{Z}_2)$ which we also denote by $\mathcal{B}$ and which obeys $\omega(\tilde{A}_i) = \omega(\tilde{B}_j) = \omega(\tilde{B}_i) = \omega(\tilde{A}_j) = 0$. If the total number of i such that $\omega(A_i) = \omega(B_i) = 1$ holds is odd, we set all values of ω for A_i and B_i with $i > 1$ to zero with this procedure.

To see that there has to hold $\delta = 0$ for $k_1 > 0$, assume that $\delta = 1$ and that there exists at least one basis element C_j such that $\omega(C_j) = 1$. Without loss of generality, we can assume that we have already transformed the given basis $\mathcal{B}$ in such a way that $\omega(A_1) = \omega(B_1) = 1$ and $\omega(A_i) = \omega(B_i) = 0$ for $i \in \{2, \dots, g\}$. We apply transformation (iv) to A_1 and B_1 and obtain $\tilde{A}_1 = A_1 + C_j$ and $\tilde{B}_1 = B_1 + C_j$. Using again (B.5) yields

$$\begin{aligned}
1 &= \omega(A_1)\omega(B_1) = \omega(A_1 + C_j - C_j)\omega(B_1 + C_j - C_j) \\
&= (\omega(A_1 + C_j) + \omega(C_j))(\omega(B_1 + C_j) + \omega(C_j)) \\
&= \omega(A_1 + C_j)\omega(B_1 + C_j) + \omega(A_1 + C_j) + \omega(B_1 + C_j) \\
&= \omega(A_1 + C_j)\omega(B_1 + C_j) + \underbrace{\omega(A_1) + \omega(C_j) + \omega(B_1) + \omega(C_j)}_{=0 \mod 2} \\
&= (\omega(A_1) + \omega(C_j))(\omega(B_1) + \omega(C_j)) \\
&= \omega(A_1)\omega(B_1) + \omega(C_j)\omega(B_1) + \omega(A_1)\omega(C_j) + \omega^2(C_j) = 1 + 1 + 1 + 1 = 0 \mod 2.
\end{aligned}$$

Therefore, the type $(g, 1, k_0, k_1)$ with $k_1 > 0$ is not realizable. Conversely, for $k_1 = 0$ or $\delta = 0$, all transformations in Lemma B.11 preserve δ and their application leads to a standard basis of the given type. Therefore, under the constraints (i) and (ii), one can define an Arf function on the constructed cycle basis which can be extended to all of $H_1(X, \mathbb{Z}_2)$ by postulating that (B.5) holds. $\square$

Next, we define an Arf function on a real curve (X, τ) and analyze some of its properties. In what follows, a simple contour and the homology class of this contour in $H_1(X, \mathbb{Z}_2)$ are denoted by the same symbol. The involution $\tau_\sharp : H_1(X, \mathbb{Z}_2) \to H_1(X, \mathbb{Z}_2)$ is induced by the involution $\tau : X \to X$. This is defined analogous to $\sigma_\sharp$ on $H_1(X, \mathbb{Z})$ in (4.8). If we consider the image of a representant γ of an element of $H_1(X, \mathbb{Z})$ under τ as a set of points on X, then we write $\tau[\gamma]$. The next definitions are taken from [Natanzon, 2004, Section 2.3.1]

Definition B.13. A *real Arf function*, i.e. an Arf function on a real curve (X, τ), is an Arf function $\omega : H_1(X, \mathbb{Z}_2) \times H_1(X, \mathbb{Z}_2) \to \mathbb{Z}_2$ such that $\tau^*\omega = \omega$.

Definition B.14. An Arf function ω on (X, τ) is said to be *singular* if there is a simple closed contour c such that $\tau[c] = c$ with $c \cap X^\tau = \emptyset$ and $\omega(c) = 0$.

For a real curve (X, τ) of type $(g, 1, k)$, one can see in the constructions of these curves in Example 6.7 that there are no invariant simple closed contours which are no ovals on X. So all Arf functions are non-singular on such a real curve. The remaining part of this section can also be found completely in [Natanzon, 2004]. For self-containedness of this work, we present these results here anyways.

Lemma B.15 ([Natanzon, 2004, Lemma 2.3.2])**.** *If $X^\tau \neq \emptyset$, then any real Arf function on (X, τ) is non-singular.*

Proof. Let $c \subset X$ be a simple closed contour such that $\tau[c] = c$ and $c \cap X^\tau = \emptyset$, where X is a real curve of type $(g, 0, k)$. Let $c' \subset X^\tau$ be an oval of the real curve (X, τ). By Theorem 6.14, there exists a set of pairwise disjoint simple closed contours $c_1, \ldots, c_r \in X \setminus (c \cup c')$ such that $\tau[c_i] = c_i$ and the set $X \setminus (c \cup c' \cup \bigcup_{i=1}^r c_i)$ decomposes into the surfaces X^+ and X^- with $\tau[X^+] = X^-$. Let us join the τ-invariant simple closed contour c and the oval c' of τ by a curve $\gamma \subset X^+$ without self intersections.
Let moreover d be a simple closed contour of the form $d = \gamma + r - \tau_\sharp\gamma$, where $r \subset c$. Analogous calculations as in (B.8) yield $\tau_\sharp d = -d + c$. By construction of d, it is $c \star d = 1 \mod 2$. Together with the realness of ω this yields $\omega(d) = \omega(d + c) = \omega(d) + \omega(c) + 1$ and hence $\omega(c) = 1 \mod 2$. $\square$

Lemma B.16 ([Natanzon, 2004, Lemma 2.3.3])**.** *A singular real Arf function vanishes on all τ-invariant simple closed contours.*

Proof. Let ω be a singular Arf function on a real curve (X, τ) of type $(g, 0, \varepsilon)$. Then $X^\tau = \emptyset$ by Lemma B.15. Suppose that there is a τ-invariant simple closed contour c_0 on X such that

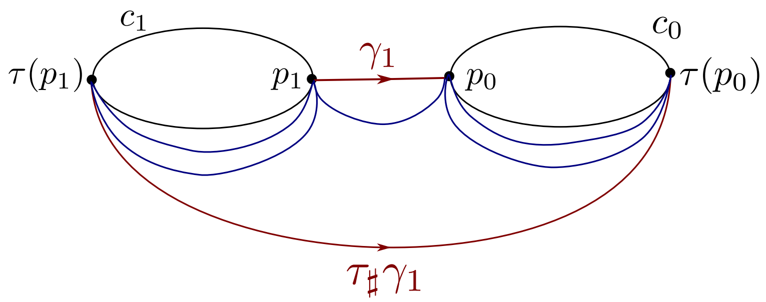


Figure B.2.: Sketch how to get insight that $\{c_i, d_i \mid i = 1, \dots, g+1\}$ defines a basis for $H_1(X, \mathbb{Z}_2)$.

$\omega(c_0) = 1$. By Lemma 6.13 there exists a complete system of τ-invariant simple closed contours $c_0, \dots, c_g$ such that $X \setminus (c_0 \cup \dots \cup c_g)$ decomposes into two disjoint curves X^+ and X^- with $\tau[X^+] = X^-$. Let $\gamma_i \subset X^+$ be the path which starts at some point $p_i \in c_i$ and ends at some point $p_0 \in c_0$ for $i = 1, \dots, g$. Again, we denote the path in c_i starting at p_i and ending at $\tau(p_i)$ as r_i and the path in c_0 starting at p_0 and ending at $\tau(p_0)$ by r_0. We then define the simple closed contours

$$d_i := \gamma_i + r_0 - \tau_\sharp\gamma_i - r_i.$$

Let $\mathcal{D} \subset X^+$ be a disk. As in the proof of Lemma 6.50, identifying the boundary cycles of the surface $X \setminus (\mathcal{D} \cup \tau[\mathcal{D}])$ via the involution τ yields another real curve $(\tilde{X}, \tilde{\tau})$ with exactly one oval $c_{g+1} = \partial\mathcal{D}$. We denote the path joining the simple closed contours c_{g+1} and c_0 by γ_{g+1}. This path starts at $p_{g+1} \in c_{g+1}$ and ends at $p_0 \in c_0$. The path in c_0 starting at p_0 and ending at $\tau(p_0)$ we denote by r_0. Then

$$d_{g+1} := \gamma_{g+1} + r_0 - \tau_\sharp\gamma_{g+1}$$

is a simple closed contour. One has $c_i \star c_j = 0$ and $d_i \star d_j = 0$ for $i, j = 1, \dots g+1$. Next, we show that $c_i \star d_j = \delta_{ij}$ for $i, j = 1, \dots, g+1$. As indicated in Figure B.2, one can add 0-homologous simple closed contours starting at p_i, passing through $\tau(p_i)$ before returning to p_i to d_i for $i = 1, \dots, g+1$ and also 0-homologous simple closed contours starting at p_0 and passing $\tau(p_0)$ before returning to p_0. This procedure leads to closed curves b_i for $i = 1, \dots g+1$ such that by the construction of b_j, one has $c_i \star b_j = \delta_{ij}$. Because the only difference between b_j and d_j are zero-homologous cycles, also $c_i \star d_j = \delta_{ij}$. So the simple closed contours $\{c_i, d_i \mid i = 1, \dots, g+1\}$ define a basis of $H_1(\tilde{X}, \mathbb{Z}_2)$. We define an Arf-function $\tilde{\omega}$ on $H_1(\tilde{X}, \mathbb{Z}_2)$ by defining the images of this basis as

$$\tilde{\omega}(c_i) = \omega(c_i) \text{ for } i = 1, \dots, g \text{ and } \omega(c_{g+1}) = 0,$$
$$\tilde{\omega}(d_i) = \omega(d_i) \text{ for } i = 1, \dots, g \text{ and } \omega(d_{g+1}) = 0$$

and by setting $\tilde{\omega}(a+b) = \tilde{\omega}(a) + \tilde{\omega}(b) + a \star b$ for $a, b \in H_1(\tilde{X}, \mathbb{Z}_2)$. Then

$$\tilde{\omega}(c_{g+1}) = \sum_{i=1}^{g} \tilde{\omega}(c_i) = \sum_{i=1}^{g} \omega(c_i) = \omega(c_{g+1}) = 1.$$

Moreover, it is

$$\tau_\sharp d_{g+1} = \tau_\sharp \gamma_{g+1} + \tau_\sharp r_0 - \gamma_{g+1} = -\gamma_{g+1} - r_0 - \tau_\sharp \gamma_{g+1} + r_0 + \tau_\sharp r_0 = -d_{g+1} + c_0, \tag{B.8}$$

and therefore

$$\tilde{\omega}(\tau_\sharp d_{g+1}) = \omega(-d_{g+1} + c_0) = \omega(d_{g+1}) + \omega(c_0) + d_{g+1} \star c_0 = \omega(d_{g+1}) + 1 + 1 = \omega(d_{g+1}).$$

Together with $c_i = \tau[c_i]$ for $i = 1, \ldots, g+1$ and $\tilde{\omega}(d_i) = \omega(d_i) = \omega(\tau_\sharp d_i) = \tilde{\omega}(\tau_\sharp d_i)$ for $i = 1, \ldots g$, the above calculation shows that $\tilde{\omega}$ is real. So by Lemma B.15, $\tilde{\omega}$ equals 1 on all τ-invariant simple closed contours on $\tilde{X} \setminus \{c_{g+1}\}$. These are $c_1, \ldots, c_g$. Since $\tilde{\omega}(c_i) = \omega(c_i)$ for $i = 1, \ldots, g$, this yields that ω is non-singular. □

Lemmata B.15 and B.16 immediately imply the following theorem:

Theorem B.17 ([Natanzon, 2004, Theorem 2.3.1])**.** *A singular Arf function on a real curve (X, τ) of type (g, k, ε) exists if and only if $k = \varepsilon = 0$. This singular Arf function is even.*

Proof. Lemma B.15 yields the condition $k = \varepsilon = 0$ for a singular Arf-function. To see that the other direction also holds, suppose that $k = \varepsilon = 0$. Let us consider the standard basis $\{c_i, d_i \mid i = 1, \ldots g\}$ of $H_1(X, \mathbb{Z}_2)$ as constructed in the proof of Lemma B.16. The elements of this basis obey $\tau_\sharp c_i = c_i$ and $\tau_\sharp d_i = -d_i + c_i + c_0 = -d_i + c_i - \sum_{i=1}^{g} c_i$. For $i = 1, \ldots, g$, we set $\omega(c_i) = 0$ and assign arbitrary values in $\mathbb{Z}_2$ to $\omega(d_i)$. As before, ω extends to an Arf-function on $H_1(X, \mathbb{Z}_2)$ by imposing equation (B.5). By analogous calculations as in (B.8), it is $\tau_\sharp d_i = -d_i + c_i + c_0$. Together with the fact that $\sum_{j=0}^{g} c_j$ is homologous to zero, this yields

$$\begin{aligned}\omega(\tau_\sharp d_i) &= \omega(-d_i + c_i + c_0) = \omega(d_i) + \omega(c_i + c_0) + \underbrace{(-d_i \star c_i)}_{=1} + \underbrace{(-d_i \star c_0)}_{=1} \\ &= \omega(d_i) + \underbrace{\omega(c_i)}_{=0} + \underbrace{\omega\left(\sum_{j=1}^{g} c_j\right)}_{=0} = \omega(d_i).\end{aligned}$$

Accordingly, ω is a singular real Arf-function. Due to $\omega(c_i) = 0$ for $i = 1, \ldots, g$, it is $\sum_{i=1}^{g} \omega(c_i)\omega(d_i) = 0$, so ω is even. □

Similar as for Arf functions without reality condition one can also define the topological type of a non-singular real Arf function on (X, τ). In this case, one has to distinguish between the

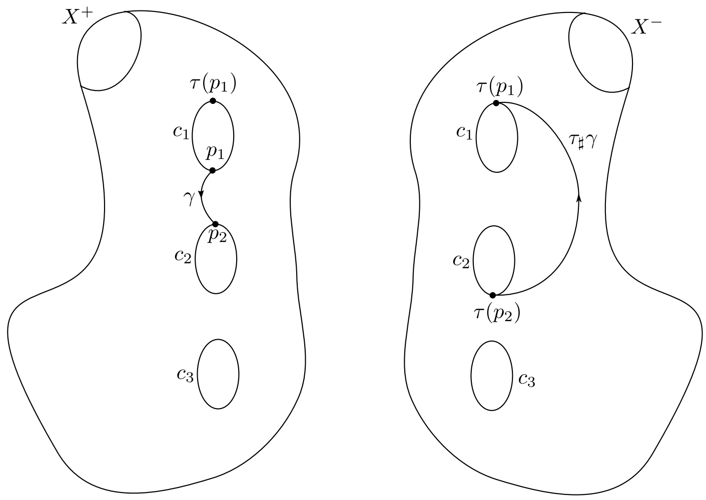


Figure B.3.: Figure taken from [Natanzon, 2004, Figure 2.3.1] depicting the construction of the contour d.

classification for curves of type (g, k, ε) with $\varepsilon = 0$ and with $\varepsilon = 1$, which results from the different conditions on the existence of Arf functions on a real curve given in Lemma B.12 and the role which k_0 and k_1 play now in terms of the ovals and the τ-invariant simple closed contours. To do so, we need the following Lemma.

Lemma B.18 ([Natanzon, 2004, Lemma 2.3.1]). *Let (X, τ) be a real curve, ω an arbitrary Arf function on (X, τ) and let $c_1, c_2 \subset X$ be simple closed contours such that $\tau[c_i] = c_i$ and $c_i \cap X^\tau = \emptyset$ for $i = 1, 2$ as well as $c_1 \cap c_2 = \emptyset$. Then $\omega(c_1) = \omega(c_2)$.*

Proof. By the construction of (X, τ) for $\varepsilon = 1$, it is clear that there are no simple closed contours c with $\tau[c] = c$ and $c \cap X^\tau = \emptyset$. So let $\varepsilon = 0$. By Theorem 6.14, there is a set of pairwise disjoint simple closed contours $c_3, \ldots, c_r$ belonging to $X \setminus (c_1 \cup c_2)$ with $\tau[c_i] = c_i$ for $i = 3, \ldots, r$ such that the set $X \setminus \sum_{i=1}^r c_i$ decomposes into two surfaces X^+ and X^- with $\tau[X^+] = X^-$. Let us join the contours c_1 and c_2 by a path $\gamma \in X^+$ without self-intersections which starts at $p_1 \in c_1$ and ends at $p_2 \in c_2$, compare Figure B.3. Furthermore, let r_1 be a path from p_1 to $\tau(p_1)$ which is contained in c_1 and r_2 be a path from p_2 to $\tau(p_2)$ which is contained in c_2. We define d as the simple closed contour

$$d := \gamma + r_2 - \tau_\sharp\gamma - r_1.$$

Without loss of generality we assume that $c_1 = -r_1 - \tau_\sharp r_1$ and $c_2 = r_2 + \tau_\sharp r_2$ since c_1 and c_2 are oriented into the same direction as boundary contours of X^+. Analogous calculations to (B.8) yield $\tau_\sharp d = -d + c_1 + c_2$. Due to Proposition B.10, it is

$$\begin{aligned}\omega(\tau_\sharp d) &= \omega(-d + c_1 + c_2) = \omega(-d) + \omega(c_1 + c_2) + \underbrace{(-d) \star (c_1 + c_2)}_{=0} \\ &= \omega(d) + \omega(c_1) + \omega(c_2) + \underbrace{c_1 \star c_2}_{=0 \mod 2} \mod 2\end{aligned}$$

and hence

$$\omega(\tau_\sharp d) = \omega(d) \quad \Leftrightarrow \quad \omega(c_1) = \omega(c_2).$$

□

Theorem B.19 ([Natanzon, 2004, Theorem 2.3.2]). *A set (g, δ, k_0, k_1) is the topological type of a non-singular Arf function on (X, τ) if and only if $k = k_0 + k_1 \leq g$ and $k_0 = g + 1 \mod 2$.*

Proof. One has $k \leq g$ since $\varepsilon = 1$, compare Theorem 6.14. Basically, we prove this assertion by exploiting that there exists a set of $g+1$ closed contours $c_1, \dots, c_{g+1}$ such that $X \setminus \{c_1, \dots, c_{g+1}\}$ decomposes into two spheres X^+ and X^- with each $g+1$ boundary cycles. Then $\omega|_{X^+}$ is considered. It is shown in Lemma B.12 that ω takes the value 1 on evenly many boundary cycles. The number on τ-invariant closed contours which are not ovals of τ is $g + 1 - k$. Since we assumed that ω is non-singular, the value of ω on all these τ-invariant contours equals one. So $k_1 + (g + 1 - k) = 0 \mod 2$, and therefore $k_0 = k - k_1 = g + 1 \mod 2$. Let now (g, δ, k_0, k_1) be an arbitrary set such that $k_0 + k_1 = k \leq g$ and such that $k_0 = g + 1 \mod 2$. The rest of the proof follows with help of simple closed contours d_i which are constructed by connecting c_i with c_{g+1} as it is for done in Lemma B.18. One has $\tau_\sharp d_i = -d_i + c_{g+1}$ for $i = 1, \dots, k$ and $\tau_\sharp d_i = -d_i + c_i + c_{g+1}$ for $i = k+1, \dots, g+1$. Next, we set $\omega(c_i) = 0$ for an arbitrary choice of k_0 many contours out of $\{c_1, \dots, c_k\}$ and $\omega(c_i) = 1$ for the remaining contours in $\{c_1, \dots, c_g\}$. Since $k_0 = g + 1 \mod 2$, it is $g - k_0 = 1 \mod 2$. So there exists at least one $c_r \in \{c_1, \dots, c_g\}$ such that $\omega(c_r) = 1$. For $i \neq r$, we assign arbitrary values of ω to d_i and define

$$\omega(d_r) := \delta - \sum_{\substack{i=1 \\ i \neq r}}^{g} \omega(c_i)\omega(d_i).$$

Then $\sum_{i=1}^{g} \omega(c_i)\omega(d_i) = \delta$. Again, by imposing (B.5), ω can be extended to the entire space $H_1(X, \mathbb{Z}_2)$. This yields the assertion. □

For $\varepsilon = 1$, one gets another classification. In this case, $X \setminus X^\tau = X^+ \cup X^-$. Connecting two ovals $c_i, c_j \in X^\tau$ by a path $\gamma_{ij} \subset X^+$ yields a simple closed contour $d_{ij} = \gamma_{ij} - \tau_\sharp \gamma_{ij}$. We call c_i and c_j *ω-similar* on (X, τ) if $\omega(d_{ij}) = 0$. In [Natanzon, 2004, Theorem 2.3.3], it is shown that this defines an equivalence relation which splits the ovals into at most two equivalence classes. With this definition, the topological type of a non-special real Arf function on a separating real curve is defined as follows, compare [Natanzon, 2004, Section 2.3.3].

Definition B.20. (a) Let $c \in X^\tau$ and let B_c be the set of ovals c_i which are ω-similar to c and let $\alpha \in \{0, 1\}$. We denote the number of ovals in B_c with $\omega(c_i) = \alpha$ by k_α^0. and the number of ovals $c_i \in X^\tau \setminus B_c$ such that $\omega(c_i) = \alpha$ holds by k_α^1.

(b) The *topological type* of a real Arf function ω on a real curve (X, τ) of type (g, k, ε) is the set $(g, \tilde{\delta}, k_0^0, k_1^0, k_0^1, k_1^1)$, where $\tilde{\delta} := \delta(X^+, \omega|_{X^+})$ with $X \setminus X^\tau = X^+ \cup X^-$.

Theorem B.21. *[Natanzon, 2004, Theorem 2.3.4] A set* $(g, \delta, k_0^0, k_1^0, k_0^1, k_1^1)$ *is the topological type of an Arf function on the real curve* (X, τ) *of type* $(g, k, 1)$ *if and only if* $(\tilde{g}, \tilde{\delta}, k_0^0 + k_0^1, k_1^0 + k_1^1)$ *is the topological type of an Arf function on a surface of genus* $g_+ = \frac{1}{2}(g - k + 1)$ *with* k *boundary cycles. Furthermore,* $\delta = k_1^0 \mod 2$.

Proof. Obviously, if $(g, \delta, k_0^0, k_1^0, k_0^1, k_1^1)$ is the topological type of an Arf function on a real curve (X, τ) of type $(g, k, 1)$, then the set $(g_+, \delta, k_0^0 + k_1^1, k_1^0 + k_1^1)$ is the topological type of an Arf function $\omega|_{X^+} : H_1(X^+, \mathbb{Z}_2) \to \mathbb{Z}_2$, where $X^+ \cup X^- = X \setminus X^\tau$ and g_+ is the genus of X^+, compare Example 6.7.
Conversely, suppose (X, τ) is a real curve of type $(g, k, 1)$. Then $X \setminus X^\tau = X^+ \cup X^-$. Let $\omega_+ : H_1(X^+, \mathbb{Z}_2) \to \mathbb{Z}_2$ be an Arf function on X^+ of type $(g_+, \delta, k_0^0 + k_0^1, k_1^0 + k_1^1)$. We now show how to extend ω_+ to $H_1(X, \mathbb{Z}_2)$. Therefore, $\{A_i, B_i, C_j \mid i = 1, \ldots, \tilde{g}, j = 1, \ldots, k\}$ be a standard basis of $H_1(X^+, \mathbb{Z}_2)$, where the oval c_j is a representant of the equivalence class C_j. We arbitrarily sort the ovals c_i into groups G_0^0, G_0^1, G_1^0 and G_1^1, where G_α^γ contains k_α^γ contours in an arbitrary way for $\gamma, \alpha \in \{0, 1\}$. For $i = 1, \ldots, g-1$, we connect the ovals c_i and c_k by a line segment $\gamma_i \subset X^+$ and set $d_i = \gamma_i - \tau_\sharp \gamma_i$. We impose that $\omega(c_i) = \alpha$ if $c_i \in G_\alpha^0 \cup G_\alpha^1$ and $\omega(d_i) = 0$ if c_i and c_k belong to the same subset $G_0^\gamma \cup G_1^\gamma$ and otherwise $\omega(d_i) = 1$. Let $\omega : H_1(X, \mathbb{Z}) \to \mathbb{Z}_2$ be defined by the values of the basis $\{A_i, B_i, \tau_\sharp A_i, \tau_\sharp B_i \mid i = 1, \ldots g_+\}$ of $H_1(X, \mathbb{Z})$ as $\omega(\tau_\sharp A_i) = \omega(A_i) := \omega_+(A_i)$ and $\omega(\tau_\sharp B_i) = \omega(B_i) := \omega_+(B_i)$. We then again impose that (B.5) holds for ω. Again, this extends ω to $H_1(X^+, \mathbb{Z}_2)$. Since the classification of the c_k into the sets G_α^γ was arbitrary,t his construction yields all non-singular real Arf functions on (X, τ). The Arf function is even if $k_1 = 0$ and for $k_1 > 0$, δ coincides with the number of elements in $G_1^0 \mod 2$ since $k_1^0 + k_1^1$ is even. □

B.3. Real Arf functions and liftings of real Fuchsian groups

Let $J : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{PSL}(2, \mathbb{R}) = \mathrm{Aut}(\mathcal{H})$ be the natural projection. Moreover, let $\Lambda \subset \mathrm{Aut}(\mathcal{H})$ be a Fuchsian group that consists of hyperbolic automorphisms. As in [Natanzon, 2004, Section 2.4.1], we introduce the lifting of Λ.

Definition B.22. A subgroup $\Lambda^* \subset \mathrm{SL}(2, \mathbb{R})$ is called a *lifting* of Λ if $J(\Lambda^*) = \Lambda$ and $J|_{\Lambda^*} : \Lambda^* \to \Lambda$ is an isomorphism.

By [Natanzon, 2004, § 7], there corresponds a unique Arf function

$$\omega_{\Lambda^*} : H_1(\mathcal{H}/\Lambda, \mathbb{Z}_2) \to \mathbb{Z}_2$$

to the lifting Λ^* which can be defined as follows: Let $a' \in \Lambda$ and let $a \in H_1(\mathcal{H}/\Lambda, \mathbb{Z}_2)$ be the image of a' under the projection in (B.2). Note that $J^{-1}(a')$ contains two elements which differ by their orientation and that only one of these elements is contained in $\varLambda^*$. Therefore, let

$$A := J^{-1}(a') \cap \varLambda^*.$$

With this choice, we can associate exactly one matrix $A \in \mathrm{SL}(2,\mathbb{R})$ with each $a' \in \varLambda$. Let further $\mathrm{Tr}(A)$ be the trace of the matrix $A \in \mathrm{SL}(2,\mathbb{R})$. We set

$$\omega_{\varLambda^*}(a) = \begin{cases} 0 & \text{for } \mathrm{Tr}(A) < 0, \\ 1 & \text{for } \mathrm{Tr}(A) > 0. \end{cases} \tag{B.9}$$

By [Natanzon, 2004, Theorem 7.2], the correspondence $\varLambda^* \mapsto \omega_{\varLambda^*}$ between the liftings of the group $\varLambda$ and the Arf functions on $X = \mathcal{H}/\varLambda$ defined as in (B.9) is 1-to-1. To transfer this to analogous results for real curves (X, τ), we consider the group

$$\mathrm{SL}_{\pm}(2,\mathbb{R}) = \{A \in \mathrm{GL}(2,\mathbb{R}) \mid \det A = \pm 1\}.$$

The projection J extents to a homomorphism $J : \mathrm{SL}_{\pm}(2,\mathbb{R}) \to \widetilde{\mathrm{Aut}}(\mathcal{H})$ by setting

$$J(A) = \frac{a\bar{z} + b}{c\bar{z} + d} \quad \text{for} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \det A = -1.$$

Let $\widetilde{\varLambda}$ be a real Fuchsian group.

Definition B.23. (a) A subgroup $\widetilde{\varLambda}^* \subset \mathrm{SL}_{\pm}(2,\mathbb{R})$ is called a *lifting* of $\widetilde{\varLambda}$ if $J(\widetilde{\varLambda}^*) = \widetilde{\varLambda}$ and $J|_{\widetilde{\varLambda}^*} : \widetilde{\varLambda}^* \to \widetilde{\varLambda}$ is an isomorphism.

(b) Two liftings $\widetilde{\varLambda}^*_+$ and $\widetilde{\varLambda}^*_-$ of a real Fuchsian group $\widetilde{\varLambda}$ are said to be *similar* if $\widetilde{\varLambda}^*_- \setminus \varLambda^* = -\widetilde{\varLambda}^*_+ \setminus \varLambda^*$.

Obviously, a lifting $\widetilde{\varLambda}^*$ of the group $\widetilde{\varLambda}$ induces a lifting $\varLambda^* = \widetilde{\varLambda}^* \cap \mathrm{SL}(2,\mathbb{R})$ of the group $\varLambda = \widetilde{\varLambda} \cap \mathrm{Aut}(\mathcal{H})$ and hence an Arf function $\omega_{\widetilde{\varLambda}^*} = \omega_{\varLambda^*} : H_1(\mathcal{H}/\varLambda, \mathbb{Z}_2) \to \mathbb{Z}_2$.

Lemma B.24 ([Natanzon, 2004, Lemma 2.4.1.])**.** *The Arf function $\omega_{\widetilde{\varLambda}^*}$ is a non-singular Arf function on the real curve (X, τ) induced by $\widetilde{\varLambda}$.*

Proof. We first show that the Arf function $\omega_{\widetilde{\varLambda}^*}$ is real. Let $\alpha \in \widetilde{\varLambda}^* \setminus \varLambda^*$ arbitrary, $a' \in \varLambda$ and $a = \mathrm{Pr}(a')$ with Pr as defined in (B.2). Due to Proposition B.1, $\tau_\sharp a \in H_1(X, \mathbb{Z}_2)$ corresponds to $\alpha^{-1} \circ (J^{-1}(a') \cap \varLambda^*) \circ \alpha$. Because the trace is invariant under conjugation, this yields

$$\mathrm{Tr}(\alpha^{-1} \circ (J^{-1}(a') \cap \varLambda^*) \circ \alpha) = \mathrm{Tr}(J^{-1}(a') \cap \varLambda^*) \;\Rightarrow\; \omega_{\widetilde{\varLambda}^*}(\tau_\sharp a) = \omega_{\widetilde{\varLambda}^*}(a).$$

Next we prove that $\omega_{\widetilde{\varLambda}^*}$ is non-singular. Therefore, let $c \subset X \setminus X^\tau$ be a simple closed contour such that $\tau[c] = c$ and let $C \in \varLambda$ be its image under the natural isomorphism $\pi_1(X, p) \to \varLambda$, where

$X = \mathcal{H}/\Lambda$. For a given real curve (X,τ) with covering $\mathcal{H}$ and Fuchsian group Λ, τ can be lifted to an antiholomorphic self-mapping $\tau : \mathcal{H} \to \mathcal{H}$, see Proposition B.1. This lifting is not necessarily an involution anymore, but it satisfies $\tau^2 \in \Lambda$. In particular, the map $\tau : \mathcal{H} \to \mathcal{H}$ is a glide reflection of $\mathcal{H}$ onto itself, i.e. a hyperbolic Möbius transformation $\sqrt{C}$ followed by a reflection at the geodesic corresponding to the two fixed points of this Möbius transformation $C = \sqrt{C}^2$, see [Seppälä, 2001, Section 1]. This composition is an orientation-reserving isometry of $\mathcal{H}$. So let $\bar{C}$ be the reflection at the geodesic corresponding to C, i.e. connecting the two fixed points $\alpha, \beta \in \mathbb{R}$ of C, $\sqrt{C}$ be a hyperbolic automorphism such that $(\sqrt{C})^2 = C$ and $\tilde{C} := \bar{C}\sqrt{C}$. Then the corresponding lift of $\tilde{C}$ is given by

$$\tilde{C}^* = J^{-1}(\tilde{C}) \cap \widetilde{\Lambda}^* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and

$$J^{-1}(C) \cap \Lambda^* = (\tilde{C}^*)^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 .$$

Since $\sqrt{C} \in \mathrm{PSL}(2,\mathbb{R})$, it is $ad - bc = 1$ and $a + d > 2$, where the latter because $\sqrt{C}$ is a holomorphic hyperbolic automorphism, compare [Jost, 2013, Lemma 2.4.2]. Then $\omega(c) = 1$ because

$$\mathrm{Tr}(J^{-1}(C) \cap \Lambda^*) = a^2 + d^2 + 2bc = (a+d)^2 - 2 > 0. \qquad \square$$

Obviously, there are always exactly two elements in each similarity class of a lift of a real Fuchsian group Λ^*.

Lemma B.25 ([Natanzon, 2004, Lemma 2.4.2])**.** *Let ω be a non-singular Arf function on (X,τ) generated by $\widetilde{\Lambda}$. Then there are exactly two liftings $\widetilde{\Lambda}^*$ of the group $\widetilde{\Lambda}$ for which $\omega_{\widetilde{\Lambda}^*} = \omega$ and these liftings are similar.*

Proof. By [Natanzon, 2004, § 7], there exists a unique lifting $\Lambda^* \subset \mathrm{SL}(2,\mathbb{R})$ of the group $\Lambda = \widetilde{\Lambda} \cap \mathrm{Aut}(\mathcal{H})$ with $\omega_{\Lambda^*} = \omega$. Because ω_{Λ^*} is a real Arf function, any lifting $\widetilde{\Lambda}^*$ of the group $\widetilde{\Lambda}$ with $\omega_{\widetilde{\Lambda}^*} = \omega$ is generated by Λ^* and a matrix α such that $J(\alpha) \in \widetilde{\Lambda} \setminus \Lambda$. If $J(\alpha)(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$, then $\alpha = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The invariance of the trace under conjugation yields $\mathrm{Tr}(\alpha A \alpha^{-1}) = \mathrm{Tr}(A)$ for $A \in \Lambda^*$ and hence $\alpha \Lambda^* \alpha^{-1} = \Lambda^*$. Thus, the group $\widetilde{\Lambda}^*$ generated by Λ^* and α is a lifting of the group $\widetilde{\Lambda}$. $\square$

Lemmata B.24 and B.25 imply the following assertion.

Theorem B.26 ([Natanzon, 2004, Theorem 4.1.])**.** *The correspondence $\widetilde{\Lambda}^* \mapsto \omega_{\widetilde{\Lambda}^*}$ between similarity classes of liftings of a real Fuchsian group $\widetilde{\Lambda}$ and non-singular Arf functions on a real curve (X,τ) generated by $\widetilde{\Lambda}$ is 1-to-1.*

Remark B.27. The isomorphism $\pi_1(\mathcal{H}/\Lambda, p) \to \Lambda$ sends each free homotopy class of a contour $c \in X = \mathcal{H}/\Lambda$ to a conjugacy class $\Lambda_c \subset \Lambda$ which does not depend on the choice of p. Thus, to each geodesic simple closed contour $c \in X$ there corresponds a set $\Lambda_c \subset \Lambda$ with $\Phi(\ell(C)) = c$ if $C \in \Lambda_c$ and $\Phi : \mathcal{H} \to X$ is the natural projection.

Let now $\widetilde{\Lambda}$ be a real Fuchsian group and c an oval of a curve (X, τ) which is generated by $\widetilde{\Lambda}$. We consider $C \in \Lambda_c$. Replacing the group $\widetilde{\Lambda}$ by a conjugate group, we may assume that $\ell(C) = I = \{z \in \mathcal{H} \mid \mathrm{Re}(z) = 0\}$. Then $\widetilde{\Lambda}$ contains the involution $\beta(z) = -\bar{z}$. A lifting $\widetilde{\Lambda} \to \widetilde{\Lambda}^*$ maps β into a matrix of the form $\alpha \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, where $\alpha = \pm 1$. If $\alpha = 1$, then we endow the half-line I with the orientation in which $\mathrm{Im}(z)$ increases and for $\alpha = -1$ with the opposite orientation. The projection $\Phi : \mathcal{H} \to X$ transfers the orientation to the contour $c = \Phi(I)$. The latter's orientation is completely determined by the lifting $\widetilde{\Lambda}^*$ of $\widetilde{\Lambda}$.

Definition B.28. The orientation on c which is induced by the lifting $\widetilde{\Lambda}^*$ is called the *orientation generated on the oval by the lifting* $\widetilde{\Lambda}^*$.

Theorem B.29 ([Natanzon, 2004, Theorem 2.4.2])**.** *Let $\widetilde{\Lambda}^*$ be a lifting of a real Fuchsian group $\widetilde{\Lambda}$ and let (X, τ) be a real curve of type $(g, k, 0)$ generated by $\widetilde{\Lambda}$. Let $(c_1, \ldots, c_g)$ be a set of pairwise disjoint simple closed contours such that $X^\tau = \bigcup_{i=1}^k c_i$ and $\tau[c_i] = c_i$ for $i = 1, \ldots, g$. Then there is an invariant simple closed contour c_{g+1} which is disjoint from the above contours so that $X \setminus \bigcup_{i=1}^g \cup c = X^+ \cup X^-$, where $X^\pm$ are spheres with $g+1$ boundary cycles. Hereby, c can be chosen in such a way that the orientation of $c_1, \ldots, c_g$ generated by $\tilde{\Lambda}^*$ coincides with their orientation as parts of the boundary of one of the surfaces X^+ and X^-.*

Proof. Without loss of generality, we show that there exists a curve c_{g+1} such that the orientation of $c_1, \ldots, c_g$ generated by $\tilde{\Lambda}^*$ coincides with the orientation as parts of the boundary of X^+. By Lemma 6.13, there is a set of pairwise disjoint invariant contours $c_1, \ldots, c_{g+1}$ belonging to X such that $X^\tau = \bigcup_{i=1}^k c_i$ and the set $X \setminus \left(\bigcup_{i=1}^{g+1} c_i\right)$ decomposes into two spheres X^+ and X^- with each $g+1$ boundary cycles. Let us endow the contours $c_1, \ldots, c_g$ with the orientation generated by the lifting $\widetilde{\Lambda}^*$ in the sense of Definition B.28. Their images on the surface $\tilde{X} := X \setminus \bigcup_{i=1}^g c_i$ are represented by pairs of simple closed contours c_i' and c_i'' of opposite orientation, where c_i' and c_j' belong to the same connected components of the surface $\tilde{X} \setminus c$, compare Figure B.4. We then modify the simple closed contour c symmetrically as shown in Figure B.4. More precisely, if the orientation of one of the ovals c_i' as a connected component of the boundary of $\tilde{X}^+$ does not coincide with the orientation generated by Λ^*, we pass from c to a symmetric simple closed contour $\tilde{c}$ that separates the simple closed contours of different orientation, i.e. it surrounds c_i' and c_i'' in such a way that $X \setminus (\bigcup_{i=1}^g c_i \cup \tilde{c})$ also decomposes into two parts X^+ and X^-, where c_i'' is a connected component of the boundary of the new curve X^+ and c_i' is a connected component of the boundary of the new X^-. Successively repeating this modification yields after at most $g - 1$ steps the desired orientation on the ovals as boundaries of X^+ and X^-. □

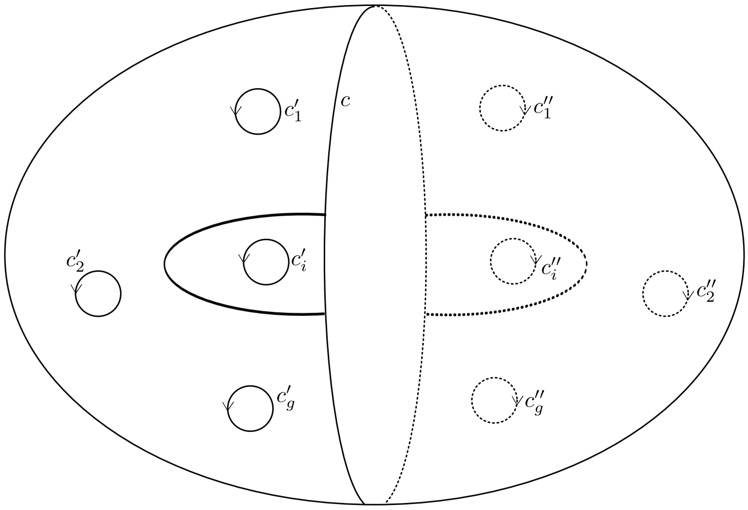


Figure B.4.: Depicting the symmetric modification of c from the proof of Theorem B.29, taken from [Natanzon, 2004, Figure 2.4.2] .

Lemma B.30 ([Natanzon, 2004, Lemma 2.4.3])**.** *Let $\widetilde{\Lambda}^*$ be a lifting of a real Fuchsian group $\widetilde{\Lambda}$ and let (X,τ) be the real curve corresponding to $\widetilde{\Lambda}$. Let further c_1 and c_2 be ovals of the involution τ endowed with the orientation generated by $\widetilde{\Lambda}^*$ as in Definition B.28 and let $a \subset X$ be an oriented simple closed contour which intersects c_1 and c_2 in such a way that $\tau_\sharp a = -a$. Then a has the same intersection numbers with c_1 and c_2 if and only if $\omega_{\widetilde{\Lambda}^*}(a) = 1$.*

Proof. Replacing the group $\widetilde{\Lambda}$ by a conjugate group, we may assume that $A \subset \Lambda_a$ where $A(z) = \lambda z$ and $\lambda > 1$. Because $c_i \cap a \neq \emptyset$, $c_1 \cap c_2 = \emptyset$, we can further assume without loss of generality that the attracting fixed point of the hyperbolic automorphism corresponding to c_i is given by $\alpha_i \in \mathbb{R}$ and the repelling fixed point by $-\alpha_i$. Since a has the same intersection numbers with c_1 and c_2, we can further assume that $0 < \alpha_2 < \alpha_1$. This situation is depicted in Figure B.5. Also without loss of generality, we can assume that $\sqrt{\lambda} = \frac{\alpha_1}{\alpha_2} > 1$. Then the corresponding element in $\mathrm{PSL}(2,\mathbb{R})$ is given by

$$A(z) = \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{\lambda}^{-1} \end{pmatrix} = \begin{pmatrix} \frac{\alpha_1}{\alpha_2} & 0 \\ 0 & \frac{\alpha_2}{\alpha_1} \end{pmatrix}$$

and $\tau_\sharp a = -a$ holds because a corresponds to the map $z \mapsto \lambda z$ and hence $\tau_\sharp a$ to $z \mapsto \lambda(-\bar{z}) = -\lambda\bar{z}$. In this case, we have $\Lambda_{c_i} \supset C_i$, where Λ_{c_i} is the element of Λ induced by $c_i \in P^\tau$ under the isomorphism $H_1(X,\mathbb{Z}) \to \Lambda$ as in Remark B.27. By equation (B.3), it is

$$C_i(z) = \frac{\alpha_i(\lambda_i+1)z + \alpha_i^2(\lambda_i - 1)}{(\lambda_i - 1)z + \alpha_i(\lambda_i+1)}, \quad \lambda_i > 1.$$

Since $\ell(C_i)$ is a half circle in $\mathcal{H}$ with radius α_i, the automorphism of $\mathcal{H}$ which mirrors points of $\mathcal{H}$

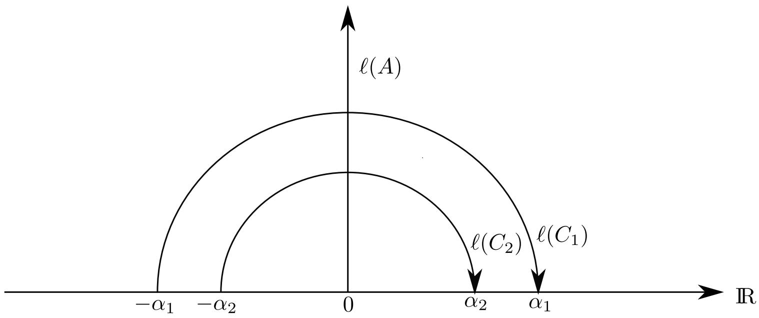


Figure B.5.: Depicting the images of the geodesics in $\mathcal{H}$ corresponding to A, C_1 and C_2 for $c_1 \star a = c_2 \star a$. Taken from [Natanzon, 2004, Figure 2.4.1].

at $\ell(C_i)$ with respect to the hyperbolic metric is described by

$$\bar{C}_i(z) = \frac{\alpha_i^2}{|z|^2} z = \frac{\alpha_i^2}{\bar{z}}.$$

and

$$A(z) = \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{\lambda}^{-1} \end{pmatrix} = \begin{pmatrix} \frac{\alpha_1}{\alpha_2} & 0 \\ 0 & \frac{\alpha_2}{\alpha_1} \end{pmatrix} = \begin{pmatrix} 0 & -\alpha_1 \\ \alpha_1^{-1} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -\alpha_2 \\ \alpha_2^{-1} & 0 \end{pmatrix} = \overline{C}_1 \circ \overline{C}_2.$$

We set $A^* = J^{-1}(A) \cap \widetilde{\Lambda}^*$ and $C_i^* = J^{-1}(C_i) \cap \widetilde{\Lambda}^*$. By Definition B.28, we obtain with the orientation induced by $\widetilde{\Lambda}^*$ that

$$\overline{C}_i^* = -\begin{pmatrix} 0 & \alpha_i \\ \alpha_i^{-1} & 0. \end{pmatrix}$$

Hence, $A^* = \overline{C}_1^* \overline{C}_2^*$. Then $\mathrm{Tr}(A^*) = \frac{\alpha_1}{\alpha_2} + \frac{\alpha_2}{\alpha_1} > 0$, and so $\omega_{\widetilde{\Lambda}^*}(a) = 1$.
Conversely, let $\omega_{\widetilde{\Lambda}^*}(a) = 1$. By the definition of $\omega_{\widetilde{\Lambda}^*}$ in (B.9), this implies $\mathrm{Tr}(A^*) = \frac{\alpha_1}{\alpha_2} + \frac{\alpha_2}{\alpha_1} > 0$. So the attracting fixed points α_1 and α_2 must have the same sign and thus also the repelling fixed points $-\alpha_1$ and $-\alpha_2$ have the same sign. Therefore, $c_1 \star a = c_2 \star a$, compare Figure B.5. □

B.4. Rank one spinors on real curves

Let $e : E \to X$ be a locally trivial line bundle over a Riemann surface $X = \mathcal{H}/\Lambda$ with $\Lambda \in \mathrm{PSL}(2, \mathbb{R})$ a Fuchsian group. This bundle can be pulled back to a bundle $\tilde{e} : \widetilde{E} \to \mathcal{H}$ which is called the induced bundle, see [Steenrod, 1951, § 10]. The latter admits a trivialization, i.e. there is a biholomorphic map $\tilde{\Phi} : \widetilde{E} \to \mathcal{H} \times \mathbb{C}$ taking $\tilde{e}$ to the natural projection $\tilde{\lambda} : (\mathcal{H} \times \mathbb{C}) \to \mathcal{H}$. So e is isomorphic to a bundle that can be obtained by a factorization of the trivial bundle on $\mathcal{H} \times \mathbb{C}$

modulo an action of the group Λ. Then $\gamma \in \Lambda \subset \mathrm{PSL}(2, \mathbb{R})$ acts on $\mathcal{H} \times \mathbb{C}$ according to the rule

$$\tilde{\gamma}(z, x) := \left(\frac{az + b}{cz + d}, f(\gamma, z) \cdot x \right),$$

where $f : \Lambda \times \mathcal{H} \to \mathbb{C} \setminus \{0\}$ is the transition function and $\gamma(z) = \frac{az+b}{cz+d}$. If E is the cotangent bundle, one can choose the projection $\tilde{\lambda}$ such that

$$f(\gamma, z) = \left(\frac{d\gamma}{dz} \right)^{-1} = (cz + d)^2,$$

where the last equality follows from $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$, see [Jost, 2006, Definition 1.5.9 and the abstract after that]. To see that this bundle induces a bundle on $X = \mathcal{H}/\Lambda$, we have to consider how the group Λ acts on the fibers of e. Therefore, we use the group representation of Λ as hyperbolic Möbius transformations. So let $\gamma_1, \gamma_2 \in \Lambda$ be represented as

$$\gamma_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \quad \text{and} \quad \gamma_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2}.$$

Then $(\gamma_1 \circ \gamma_2)(z) = \frac{\tilde{a}z+\tilde{b}}{\tilde{c}z+\tilde{c}}$ with

$$\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}.$$

Therefore,

$$\tilde{\gamma_1}\tilde{\gamma_2}(x, z) = \left(\frac{\tilde{a}z + \tilde{b}}{\tilde{c}z + \tilde{d}}, (\tilde{c}z + \tilde{d})^2 \right).$$

Conversely, it is

$$\begin{aligned} \tilde{\gamma_1}(\tilde{\gamma_2}(z, x)) &= \tilde{\gamma_1}(\gamma_2(z), (c_2 z + d_2)^2 x) \\ &= \left(\frac{\tilde{a}z + \tilde{b}}{\tilde{c}z + \tilde{d}}, \left(c_1 \left(\frac{a_2 z + b_2}{c_2 z + d_2} \right) + d_1 \right)^2 (c_2 z + d_2)^2 x \right), \end{aligned}$$

where

$$\begin{aligned} &\left(c_1 \left(\frac{a_2 z + b_2}{c_2 z + d_2} \right) + d_1 \right)^2 (c_2 z + d_2)^2 = c_1^2 (a_2 z + b_2)^2 + 2c_1 (a_2 z + b_2)(c_2 z + d_2) d_1 + d_1^2 (c_2 z + d_2)^2 \\ &= c_1^2((a_2 z)^2 + 2a_2 b_2 z + b_2^2) + 2c_1 d_1 (a_2 c_2 z^2 + a_2 d_2 z + b_2 c_2 z + b_2 d_2) + d_1^2((c_2 z)^2 + 2c_2 d_2 z + d_2^2) \\ &= (c_1^2 a_2^2 + 2c_1 a_2 d_1 c_2 + d_1^2 c_2^2) z^2 + 2(c_1^2 a_2 b_2 + c_1 a_2 d_1 d_2 + d_1 c_2 c_1 b_2 + d_1^2 c_2 d_2) z + c_1^2 b_2^2 + 2c_1 b_2 d_1 d_2 + d_1^2 d_2^2 \\ &= ((c_1 a_2 + d_1 c_2) z + (c_1 b_2 + d_1 d_2)^2 \\ &= (\tilde{c}z + \tilde{d})^2. \end{aligned}$$

That means that the representation of Λ acts on E, and therefore induces a bundle on $\mathcal{H}/\Lambda$, compare [Gunning, 1967, §9].

Definition B.31. A line bundle $E \to X$ is called a *spinor bundle* if its tensor square $E \otimes E \to X$ is isomorphic to the cotangent bundle.

If the transition functions f and g correspond to two mappings $E \to X$, then the transition functions corresponding to the tensor product $E \otimes E \to X$ is $f \cdot g$. Thus, the transition function of a spinor bundle has the form $f(\gamma, z) = \alpha(a, b, c, d)(cz + d)$ with $\alpha(a, b, c, d) \in \{-1, 1\}$. Then $f(\gamma, z) \cdot f(\gamma, z) = (cz + d)^2$ which is just the transition function of the cotangent bundle introduced above. We associate the matrix

$$J_f^*(\gamma) = \alpha(a, b, c, d) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$$

to a map $\gamma(z) = \frac{az+b}{cz+d} \in \Lambda$. The next lemma is concerned with this map. We just cite it without repeating its proof since the corresponding calculations are the same as the ones above which show that e induces a bundle on $\mathcal{H}/\Lambda$.

Lemma B.32 ([Natanzon, 2004, Lemma 1.10.1])**.** *The map $J_f^* : \Lambda \to \mathrm{SL}(2, \mathbb{R})$ is well defined and it is a monomorphism.*

Due to this Lemma, $J_f^*(\Lambda)$ is a lifting of Λ. The next theorem [Natanzon, 2004, Theorem 1.10.1] then establishes the $1-1$-correspondence between liftings Λ^* of a Fuchsian group Λ and the spinor bundles on $X = \mathcal{H}/\Lambda$:

Theorem B.33 ([Natanzon, 2004, Theorem 1.10.1])**.** *The map $f \mapsto J_f^*$ establishes a $1-1$-correspondence between spinor bundles and liftings Λ^* of Λ.*

Proof. To show this, it suffices to associate to each lifting $J^* : \Lambda \to \mathrm{SL}(2, \mathbb{R})$ a unique spinor bundle with transition function f so that $J_f^* = J^*$. Since line bundles are uniquely determined by an open covering and the corresponding transition functions, see [Jost, 2013, Definition 5.6.2], every spinor bundle can be reconstructed from its transition functions f. For $J^*(\gamma) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, let the seeked transition function be given by $f(\gamma, z) := cz + d$. □

Due to Theorem B.26, there is a unique Arf function $\omega_{\Lambda^*} : H_1(\mathcal{H}/\Lambda, \mathbb{Z}_2) \to \mathbb{Z}_2$ corresponding to Λ^*. Combining this with the 1-to-1 connection between real spinors and Λ^* shown in Theorem B.33, we see that also the correspondence $e \to \omega_{\Lambda^*}$ between spinor bundles and Arf functions on $X = \mathcal{H}/\Lambda$ is 1-to-1, where $e : E \to X$ is a spinor bundle. The next step is to transfer this to real curves (X, τ). This leads to a modified version of this correspondence which takes the additional structure given by the realness of X into account.

Definition B.34 ([Natanzon, 2004, Section 2.5.2])**.** (a) A *real spinor bundle*, i.e. a spinor bundle on a real curve (X, τ), is a pair (e, β), where $e : E \to X$ is a spinor bundle and $\beta : E \to E$ is an antilinear involution such that $e \circ \beta = \tau \circ e$.

(b) Two spinor bundles (e_1, β_1) and (e_2, β_2) on real curves (X_1, τ_1) and (X_2, τ_2), respectively, are isomorphic if there are biholomorphic maps $\phi_E : E_1 \to E_2$ and $\phi_X : X_1 \to X_2$ such that

$$e_2 \circ \phi_E = \phi_X \circ e_1, \quad \beta_2 \circ \phi_E = \phi_E \circ \beta_1, \quad \tau_2 \circ \phi_X = \phi_X \circ \tau_1.$$

We do not distinguish between isomorphic bundles. With any lifting $\widetilde{\Lambda}^*$ of a real Fuchsian group $\widetilde{\Lambda}$, we associate a spinor bundle $e_{\widetilde{\Lambda}^*}$ on the real curve (X, τ) corresponding to $\widetilde{\Lambda}$. By definition, the bundle $e_{\widetilde{\Lambda}^*}$ is of the form $(e_{\widetilde{\Lambda}^*}, \beta_{\widetilde{\Lambda}^*})$, where $\beta_{\widetilde{\Lambda}^*} : (\mathcal{H} \times \mathbb{C})/\Lambda^* \to (\mathcal{H} \times \mathbb{C})/\Lambda^*$ is generated by the map

$$(z, x) \mapsto \left(\frac{a\bar{z} + b}{c\bar{z} + d}, (c\bar{z} + d)\bar{x}\right), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \widetilde{\Lambda}^* \setminus \tilde{\Lambda}^*. \tag{B.10}$$

As shown above for the line bundle on X without holomorphic involution, this also defines an action on $\mathcal{H} \times \mathbb{C}$.

Lemma B.35 ([Natanzon, 2004, Lemma 2.5.1])**.** *The correspondence $\widetilde{\Lambda}^* \mapsto \tilde{e}_{\Lambda^*}$ between similarity classes of liftings $\widetilde{\Lambda}^*$ of a real Fuchsian group $\widetilde{\Lambda}$ and real spinor bundles on (X, τ) corresponding to $\widetilde{\Lambda}$ is* 1*-to-*1*.*

Proof. Let (e, β) be an arbitrary spinor bundle on (X, τ). By Theorem B.33, there is a unique lifting Λ^* of the group $\Lambda = \widetilde{\Lambda} \cap \mathrm{Aut}(\mathcal{H})$ such that

$$e : (\mathcal{H} \times \mathbb{C})/\Lambda^* \to \mathcal{H}/\Lambda.$$

Remember that $g \geq 2$, and therefore only the hyperbolic Möbius transformations are deck transformations of $\mathcal{H}$. Thus, we can replace the group $\widetilde{\Lambda}$ by a conjugate group and assume that $\widetilde{\Lambda}$ contains a map of the form

$$z \mapsto -\mu\bar{z},$$

where $\mu \geq 1$. Let μ_* be the minimal value of all these possible μ's. We set $\nu = \sqrt{\mu_*}$. Then the group Λ^* and the matrices $\pm \begin{pmatrix} -\nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}$ generate some liftings $\widetilde{\Lambda}^*_+$ and $\widetilde{\Lambda}^*_-$ of the group $\widetilde{\Lambda}$ in sense of Definition B.23. These are the only liftings of $\widetilde{\Lambda}$ that contain Λ^*. Moreover, $e_{\widetilde{\Lambda}^*_\pm} = e$ and an isomorphism between $e_{\widetilde{\Lambda}^*_+}$ and $e_{\widetilde{\Lambda}^*_-}$ is generated by the involution $(z, x) \mapsto (z, -x)$. □

Together with Lemma B.7 and Theorem B.26, this finally yields the desired connection between real spinors and non-singular Arf functions.

Theorem B.36 ([Natanzon, 2004, Theorem 2.5.1]). *The correspondence* $e \mapsto \omega_e$ *between spinor bundles and non-singular Arf functions on a real curve* (X, τ) *is* 1*-to-*1*.*

Let (e, β) be a spinor bundle on a real curve (X, τ). Applying Lemmata B.7 and B.35, we construct an isomorphism

$$(e, \beta) \to (e_{\widetilde{\Lambda}^*}, \beta_{\widetilde{\Lambda}^*}),$$

where $\widetilde{\Lambda}^*$ is a lifting of a real Fuchsian group $\widetilde{\Lambda}$ and (X, τ) is the real curve generated by $\widetilde{\Lambda}$. Let us endow the ovals and the invariant simple closed contours of (X, τ) which are disjoint from the ovals with the orientation induced by $\widetilde{\Lambda}^*$ as in Definition B.28. Thus, a spinor bundle (e, β) on a real curve (X, τ) generates an orientation on the ovals and on the invariant simple closed contours of (X, τ) which are disjoint from the ovals. This orientation is defined up to its simultaneous reversal on all ovals and invariant simple closed contours.

Definition B.37. (a) A holomorphic section $\eta : X \to E$ of a spinor bundle $e : E \to X$ is called a *spinor*.

(b) A section η of a spinor bundle (e, β) on a real curve (X, τ) is called a *real spinor* if $\beta \circ \eta = \eta \circ \tau$.

Let $\{\widetilde{\Lambda}_1^*, \widetilde{\Lambda}_2^*\}$ be the similarity classes that correspond to the bundle (e, β) by Lemma B.35. Then the spinor η can be regarded as a section of the spinor bundle induced by $\Lambda^* = \widetilde{\Lambda}_+^* \cap \widetilde{\Lambda}_-^*$, i.e. the intersection of the two liftings of $\tilde{\Lambda}$ with $\mathrm{Aut}(\mathcal{H}/\Lambda) \cap \widetilde{\Lambda}_+^* = \mathrm{Aut}(\mathcal{H}/\Lambda) \cap \widetilde{\Lambda}_-^*$. Moreover, η is invariant with respect to one of the involutions $\beta_{\widetilde{\Lambda}_\pm^*}$ and anti-invariant with respect to the other one. Without loss of generality, let $\beta_{\widetilde{\Lambda}_1^*} \circ \eta = \eta \circ \tau$.

Definition B.38. The orientation generated by the lifting $\widetilde{\Lambda}_+^*$ on the ovals and invariant simple closed contours of (X, τ) as in Definition B.28 is called the orientation generated by the spinor η.

Remember that Definition 6.16 of a real chart implies $\overline{z(p)} = z(\tau(p)) = z(p)$ for $p \in U \cap X^\tau$ and hence $z(U \cap X^\tau) \subset \mathbb{R}$.

Definition B.39. The local chart z on an open neighborhood of $p_0 \in c$ with $c \in X^\tau$ agrees with the spinor η if the spinor generates an orientation of the oval c which contains p_0 that passes under the action of z into the orientation of increasing real values on $\mathbb{R} \subset \mathbb{C}$.

A local chart on a Riemann surface defines a local trivialization of the cotangent bundle, and therefore a local trivialization of the spinor bundle. Thus, in the local chart z, a complex-valued transition function $f \circ z$ corresponds to the spinor.

Lemma B.40 ([Natanzon, 2004, Lemma 2.5.2]). *Let* (e, β) *be a spinor bundle on a real curve* (X, τ) *and let* η *be a real spinor of this bundle. Then the spinor* η *is described by a function* $f \circ z$ *such that* $f \circ z \circ \tau = \overline{f \circ z}$ *in any real chart* $z : U \to \mathbb{C}$ *that agrees with the spinor* η*.*

Proof. We set $\iota_e : (z, x) \mapsto (\iota z, x)$. By Lemma B.7, we may assume that (X, τ) corresponds to $\widetilde{\Lambda}$ and by Lemma B.35, we may assume that $e : (\mathcal{H} \times \mathbb{C})/\Lambda^* \to (X, \tau)$. As before, we can replace the group $\widetilde{\Lambda}$ by a conjugate group and assume that $\widetilde{\Lambda}$ contains a map of the form

$$\begin{pmatrix} -\nu & 0 \\ 0 & \nu^{-1} \end{pmatrix} \in \widetilde{\Lambda}^* \setminus \Lambda^* \quad \text{with} \quad \nu \geq 1,$$

where this matrix is either element of the similarity class Λ^*_+ or Λ^*_-. Furthermore, we may assume that $e \circ \iota_e : (-\iota\mathcal{H} \times 0) \to X$ generates a real chart z that agrees with η in sense of Definition B.39. In this chart, η can be represented in the form $(z, f(z))$ for every $z \in \mathbb{R}$. Thus, $\eta \circ \tau$ can be represented as $(z \circ \tau, f \circ z \circ \tau)$ and since $z \circ \tau = \bar{z}$, $\beta \circ \eta$ can be represented as $(z \circ \tau, \overline{f \circ z}) = \beta_{\widetilde{\Lambda}}(z, f \circ z)$. So due to equation (B.10), it is $f \circ \bar{z} : z \mapsto \frac{az+b}{cz+d} = \overline{\frac{a\bar{z}+b}{c\bar{z}+d}}$, where the last equality holds due to $a, b, c, d \in \mathbb{R}$. Therefore, the relation $\beta \circ \eta = \eta \circ \tau$ reads as

$$(z \circ \tau, \overline{f \circ z}) = \beta_{\widetilde{\Lambda}^*}(z, f \circ z) = (z \circ \tau, f \circ z \circ \tau),$$

and hence $\overline{f \circ z} = f \circ z \circ \tau$. A passage to any other real chart that agrees with η preserves this relation. □

Theorem B.41 ([Natanzon, 2004, Theorem 2.5.2])**.** *Let (e, β) be a spinor bundle on a real curve (X, τ), let η be a real spinor of this bundle and let c be an oval of the curve (X, τ). Then the number of zeros of η on c equals $1 - \omega_e(c) \mod 2$, where ω_e is the unique Arf function corresponding to the spinor bundle e.*

Proof. As in the proof of the Lemma before, we may assume by Lemmata B.7 and B.35 that (X, τ) corresponds to $\widetilde{\Lambda}$,

$$e : (\mathcal{H} \times \mathbb{C})/\Lambda^* \to X \quad \text{and} \quad \begin{pmatrix} -\nu^{-1} & 0 \\ 0 & \nu \end{pmatrix} \in \widetilde{\Lambda}^* \setminus \Lambda^* \quad \text{with} \quad \nu \geq 1.$$

Let $I := \{z \in \mathcal{H} \mid \operatorname{Re}(z) = 0\}$. Since c is an oval, there exists at least one $\gamma \in \Lambda$ such that $\gamma[I] \subset I$. So we may assume further that

$$c = I/\{\gamma \mid \gamma \in \Lambda \text{ and } \gamma[I] \subset I\}.$$

In the local chart z generated by the projection $e : (\mathcal{H} \times 0) \to X$, the spinor η is represented in the form $(z, f \circ z)$, where $z \in \mathcal{H}$ and f is a holomorphic function. Due to Lemma B.32, it is

$\gamma(x,z) = \left(\frac{\alpha z+\beta}{\gamma z+\delta}, (\gamma z+\delta)x\right)$, where $z \in \mathcal{H}$. So for any element $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Lambda^*$, there holds

$$f\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) = f(z)(\gamma z+\delta). \tag{B.11}$$

Since the oval c induces a hyperbolic element of $\mathrm{PSL}(2,\mathbb{R})$, c corresponds to the matrix

$$C = \alpha(c)\begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{\lambda}^{-1} \end{pmatrix} \in \Lambda^*$$

with $\sqrt{\lambda} > 1$ and the sign $\alpha(c)$ is determined by the value of the corresponding value of the unique Arf function $\omega_{\Lambda^*}(c)$. The latter is given by Λ^*. Then ω_{Λ^*} determines the sign of the trace of C as in (B.9). So

$$\alpha(c) = \begin{cases} 1 & \text{for } \omega(c) = 1, \\ -1 & \text{for } \omega(c) = 0. \end{cases}$$

Since $C \in \Lambda^*$ and f obeys equation (B.11), this implies that $f(\lambda z) = \alpha(a)f(z)\sqrt{\lambda}^{-1}$ with $\sqrt{\lambda} \geq 0$. Moreover, the natural projection $\mathcal{H} \to \mathcal{H}/\Lambda$ establishes a 1-to-1 correspondence between the interval $(v, \lambda v] \in I$ and the contour c. Hence, the number of zeros of the spinor η on c is equal to the numbers of zeros of the function $f(z)$ on the interval $(v, \lambda v] \in I$. Conversely, the map $e : (\mathcal{H} \times 0) \to X$ generates a real chart in a neighborhood of each point of the oval c and hence, by Lemma B.40, $f(z)$ is real and continuous on $(v, \lambda v] \in I$. For $\alpha(c) = 1$, the signs of $f(v)$ and $f(\lambda v)$ are the same and these signs differ for $\alpha(c) = -1$. So by the Intermediate Value Theorem, the number of zeros of f in $(v, \lambda v] \in I$ is even for $\alpha(c) = 1$ and odd for $\alpha(c) = -1$. □

Theorem B.42 ([Natanzon, 2004, Theorem 2.5.3])**.** *Let $c_1, \ldots, c_k$ be oriented ovals of a real curve (X, τ) of type $(g, k, 0)$. Let $0 \leq m \leq k$, $\alpha_1, \ldots, \alpha_k \in \mathbb{Z}_2$ and let $\sum_{i=1}^k \alpha_i \equiv g + 1 \mod 2$. Then there is a real spinor η on (X, τ) such that*

(a) the orientation of the oval c_i generated by η coincides with the original orientation if and only if $i \leq m$,

(b) the number of zeros of the spinor η modulo 2 on the oval c_i is equal to α_i.

Proof. By Theorem 6.14, there is a set $\{c_1, \ldots, c_{g+1}\}$ of pairwise disjoint and τ-invariant simple closed contours which decomposes X into spheres X^+ and X^- with $g+1$ boundary cycles. The orientation of X^+ generates an orientation on $\partial X^+ = \{c_1, \ldots, c_{g+1}\}$ which can be different from the original orientation given on $c_1, \ldots, c_k$, compare Lemma B.30. Without loss of generality, we may assume that the orientation of c_1 oriented as a part of ∂X^+ coincides with the orientation induces by $\widetilde{\Lambda}$ corresponding to (X, τ).

We define a real Arf-function on (X, τ) to show that there always exists an odd real Arf-function on (X, τ). The existence of this odd Arf-function then implies that there is a real spinor bundle on (X, τ) which has a non-trivial holomorphic section. This section can be used to construct a non-trivial real holomorphic section, see [Atiyah, 1971, Proposition 3.2]. We then show that this real section has the properties claimed in the theorem.
To define an Arf-function $\omega : H_1(X, \mathbb{Z}_2) \to \mathbb{Z}_2$, we define this function on the elements of a basis of $H_1(X, \mathbb{Z}_2)$. One part of this basis consists of the contours $c_1, \dots, c_g$. It is shown in Lemma B.16 that one obtains the other basis elements of $H_1(X, \mathbb{Z}_2)$ by joining the contours c_i and c_{g+1} by a path $\gamma_i \in X^+$ to define

$$d_i := \gamma_i + r_{g+1} - \tau_\sharp \gamma_i \text{ for } 1 \leq i \leq k$$

and

$$d_i := \gamma_i + r_{g+1} - \tau_\sharp \gamma_i - r_i \text{ for } k < i \leq g.$$

So the set $\{c_i, d_i \mid i = 1, \dots g\}$ yields a basis of the vector space $H_1(X, \mathbb{Z}_2)$. We define a unique Arf-function ω on $H_1(X, \mathbb{Z}_2)$ through the images of these basis elements under ω and by assuming that equation (B.5) holds. We define these values as

$$\begin{aligned}
\omega(c_i) &= 1 - \alpha_i && \text{for } i \leq k, \\
\omega(c_i) &= 1 && \text{for } k < i \leq g, \\
\omega(d_i) &= 0 && \text{for } 1 \leq i \leq m, \\
\omega(d_i) &= 1 && \text{for } m < i \leq k, \\
\omega(d_i) &= 0 && \text{for } k < i \leq g.
\end{aligned}$$

Due to $\sum_{i=1}^{k} \alpha_i \equiv g + 1 \mod 2$, it is

$$\omega(c_{g+1}) = \omega\left(\sum_{i=1}^{g} c_i\right) = \sum_{i=1}^{g} \omega(c_i) = \sum_{i=1}^{k}(1 - \alpha_i) + \sum_{i=k+1}^{g} 1 = g - (g+1) = 1 \mod 2.$$

As in the proof of Lemma B.16, one has $\tau_\sharp d_i = -d_i + c_{g+1}$ for $1 \leq i \leq k$ and $\tau_\sharp d_i = -d_i + c_{g+1} + c_i$ for $k < i \leq g$. So in the former case, it is

$$\omega(\tau_\sharp d_i) = \omega(-d_i + c_{g+1}) = \omega(d_i) + \omega(c_{g+1}) + (-d_i, c_{g+1}) = \omega(d_i) + 1 + 1 = \omega(d_i) \mod 2$$

and in the latter case one also obtains

$$\omega(\tau_\sharp d_i) = \omega(-d_i + c_i + c_{g+1}) = \omega(d_i) + \omega(c_i + c_{g+1}) + \underbrace{(-d_i, c_i + c_{g+1})}_{=0 \mod 2} = \omega(d_i) + \omega(c_i) + \omega(c_{g+1})$$
$$= \omega(d_i) + 1 + 1 = \omega(d_i) \mod 2.$$

Therefore, the Arf-function ω is real and of type (g, δ, k_0, k_1) in the sense of Definition B.9(d) with $k_0 = \sum_{i=1}^{k} \alpha_i = g + 1 \mod 2$ and $k = k_0 + k_1 \leq g$. Hence, ω exists due to Theorem B.17.
By Lemma B.16, ω is non-singular since it does not vanish on all ovals of X. By Lemma B.7, (X, τ) is the real curve corresponding to the real Fuchsian group $\widetilde{\Lambda}$. Then the 1-to-1 correspondence between spinor bundles on real curves and real Arf-functions in Theorem B.36 yields also a spinor bundle $(e_{\widetilde{\Lambda}^*}, \beta_{\widetilde{\Lambda}^*})$ which corresponds uniquely to $\omega_{\widetilde{\Lambda}^*}$.
Along with ω, we consider another real Arf function ω' such that $\omega'(c_i) = \omega(c_i)$ and $\omega'(d_i) = 1 - \omega(d_i)$. Another real spinor bundle (e', β') corresponds uniquely to ω'. Moreover, it is $\sum_{i=1}^{k} \alpha_i \equiv g+1 \mod 2$ and $\omega(d_i) = 1$ holds if and only if $\omega'(d_i) = 0$. Inserting this into the definition of the evenness of an Arf-function B.9 (b) yields

$$\delta(\omega) + \delta(\omega') = \sum_{i=1}^{g} \omega(c_i)\omega(d_i) + \sum_{i=1}^{g} \omega(c_i)(1 - \omega(d_i)) = \sum_{i=1}^{g} \omega(c_i)$$
$$= \sum_{i=1}^{k} (1 - \alpha_i) + \sum_{i=k+1}^{g} 1 = g - g + 1 = 1 \mod 2.$$

Hence, either $\delta(\omega) = 1$ or $\delta(\omega') = 1$. Without loss of generality, let $\delta(\omega) = 1$. By [Atiyah, 1971, Section 5.2], this implies that the bundle e has a non-trivial holomorphic section ω. So one of the sections $\eta = \omega + \beta\omega$ and $\tilde{\eta} = \iota(\omega - \beta\omega)$ is a non-zero real section of the bundle (e, β). Now $d_i - d_{i+1}$ is a simple closed curve which connects c_i and c_{i+1} for $i = 1, \ldots, k-1$. The involution τ acts on these simple closed curves as

$$\tau_\sharp(d_i - d_{i+1}) = \tau_\sharp d_i - \tau_\sharp d_{i+1} = -d_i + c_{g+1} + d_{i+1} - c_{g+1} = -(d_i - d_{i+1})$$

and hence obey the assumptions on the simple closed curve connecting two ovals in the preliminaries of Lemma B.30. Due to $d_i \star d_{i+1} = 0$ for $i = 1, \ldots, g-1$, one has $\omega(d_i - d_{i+1}) = \omega(d_i) + \omega(d_{i+1})$ and hence $\omega(d_i - d_{i+1}) = 0$ for $1 \leq i < m$ and for $m < i < g$, whereas $\omega(d_m - d_{m+1}) = 1$. Together with Lemma B.30 this shows that the section obeys property (a) since $\omega(d_i - d_{i+1}) = 0$ implies that the orientation of the ovals c_i and c_{i+1} induced by η are equal and $\omega(d_i - d_{i+1}) = 1$ implies that these orientations are opposite to each other. Accordingly, the ovals c_i have the same orientation as c_1 for $1 \leq i \leq m$ and the opposite orientation of c_1 for $m < i \leq k$. Property (b) of η follows from Theorem B.41 which says that the number of zeros of η on an oval c_i modulo 2 equals $1 - \omega(c_i) = 1 - 1 + \alpha_i = \alpha_i$. □

Theorem B.43 ([Natanzon, 2004, Theorem 2.5.4]). *Let (X, τ) be a real curve of type $(g, k, 1)$. Let its ovals $c_1, \ldots, c_k$ be oriented as parts of the boundary of a connected component X^+ of the set $X \setminus X^\tau$. Consider a set $\{\alpha_1, \ldots, \alpha_k\} \in \mathbb{Z}_2^k$ that has evenly many zeros and for which $\alpha_1 = \alpha_k = 0$. Let $1 \leq m < k$ and let $\sum_{i=1}^m \alpha_i \equiv m + 1 \mod 2$. Then there is a real spinor η on (X, τ) such that*

(a) the orientation generated on the oval c_i by η coincides with the orientation induced on c_i by the orientation as a boundary part of X^+ if and only if $i \leq m$,

(b) the number of zeros of η modulo 2 on c_i is equal to α_i.

Proof. The proof of this theorem equals in wide parts the proof of Theorem B.42, i.e. we are seeking for a non-singular Arf function on (X, τ) such that there is a unique real spinor due to the 1-to-1 correspondence shown in Theorem B.36. The only essential difference is the choice of the values of the Arf-function since the conditions for the existence of an Arf-function of a certain type for $\varepsilon = 1$ differ from the conditions for $\varepsilon = 0$. Due to Theorem B.21, a real Arf function on a real curve (X, τ) with $\varepsilon = 1$ only exists if $k_\alpha^0 + k_\alpha^1 = 0 \mod 2$ for $\alpha \in \{0, 1\}$, where we use the notation from this theorem. To ensure that this holds for arbitrary choices of $k_1^0 + k_1^1 > 0$, there has to hold that at least one α_i for $i \leq m$ and one α_i for $i > m$ equals zero. Therefore, we choose $\alpha_1 = \alpha_k = 0$. Since $\varepsilon = 1$, the ovals $c_1, \ldots, c_k$ decompose X into two Riemann surfaces X^+ and X^- of genus $g_+ = \frac{1}{2}(g - k + 1)$ with boundary cycles $c_1, \ldots, c_k$. We now define an Arf-function ω_+ on $H_1(X^+, \mathbb{Z})$ by setting $\omega_+(c_i) = 1 - \alpha_i$. Since $\alpha_1, \ldots, \alpha_k$ contains evenly many zeros, i.e. $\{c_i \mid \tilde{\omega}(c_i) = 1\}$ consists of $k_1 \geq 2$ elements and $\{c_i \mid \omega_+(c_i) = 0\}$ consists of k_0 elements with $k_0 + k_1 = 1$ and $k_1 = 0 \mod 2$, Lemma B.12 implies the existence of such an Arf function as well as the existence of a standard basis $\{a_i, b_i, c_j \mid i = 1, \ldots, g_+, j = 1, \ldots, k-1\}$ of $H_1(X^+, \mathbb{Z}_2)$ such that $\omega(a_i) = \omega(b_i) = 0$ and $\omega(c_i) = \omega_+(c_i)$. To extend this to a basis of $H_1(X, \mathbb{Z}_2)$, join the ovals c_i and c_k by a path $\gamma_i \subset X^+$ starting at c_i and ending at c_k and set $d_i := \gamma_i - \tau_\sharp \gamma_i$ for $i = 1, \ldots, k-1$. Then $H_1(X, \mathbb{Z}_2)$ consists of cycles $a_i, b_i, \tau(a_i), \tau(b_i)$ with $i = 1, \ldots, g_+$ and c_j, d_j with $j = 1, \ldots k-1$. As in the proof of Theorem B.21 one can extend ω_+ to an Arf function ω on $H_1(X, \mathbb{Z}_2)$ by setting $\omega(\tau_\sharp a_i) = \omega(a_i) = 0$, $\omega(\tau_\sharp b_i) = \omega(b_i) = 0$, $\omega(c_j) = 1 - \alpha_i$ and $\omega(d_j) = 1$ if and only if $i \leq m$ and assuming that equation (B.5) holds. Due to $\sum_{i=1}^k c_i = 0$, there has to hold $\sum_{i=1}^k \omega(c_i) = 0$. This is ensured by the assumption that the number of zeros in $(\alpha_1, \ldots, \alpha_k)$ is even because

$$\sum_{i=1}^k \omega(c_i) = \sum_{i=1}^k (1 - \alpha_i) = k - (k - 2j) = 2j = 0 \mod 2.$$

Then $\omega(\tau_\sharp c_i) = \omega(c_i)$ and $\omega(\tau_\sharp d_i) = \omega(d_i)$ for $i = 1, \ldots, k$, so ω is real, i.e. $\omega(\tau w) = \omega(w)$ for $w \in H_1(X^+, \mathbb{Z}_2)$. Furthermore, ω is odd since

$$\delta(\omega) = \sum_{i=1}^{\tilde{g}} (\omega(a_i)\omega(b_i) + \omega(\tau_\sharp a_i)\omega(b_i) + \omega(a_i)\omega(\tau_\sharp b_i) + \omega(\tau_\sharp a_i)\omega(\tau_\sharp b_i)) + \sum_{i=1}^{g} \omega(c_i)\omega(d_i) = \sum_{i=1}^{m} \omega_{c_i} = 1.$$

The rest of the proof coincides with the corresponding part of the proof of Theorem B.42. □

C. The moduli space

The question how the module space of a given Fermi curve looks like is answered in a more general setting in the so far unpublished paper [Carberry and Schmidt, 2017]. So we cannot just cite it here. To give a rather full picture of the inverse problem of the two-dimensional Schrödinger operator, we attach the necessary theory here in accordance with one of the authors. We add the corresponding citations with respect to the current preprint version of [Carberry and Schmidt, 2017].

The aim of this appendix is to describe the moduli space of deformations for Fermi curves with arithmetic genus $g_a < \infty$ that obey the conditions (F1) to (F3) which are formulated at the beginning of Chapter 5. That means the space of data $(X, Q^+, Q^-, d\hat{c}, d\check{c}, \sigma)$, where X is a one-dimensional compact complex analytic space of arithmetic genus $g_a < \infty$ with two smooth marked points Q^+ and Q^-, $d\hat{c}$ as well as $d\check{c}$ are meromorphic differentials on X which are holomorphic on $X \setminus \{Q^+, Q^-\}$ with poles of second order at Q^+ and Q^- and with prescribed periods and σ is a holomorphic involution on X such that Q^+ and Q^- are the only fixed points of this involution. To consider also deformations of Fermi curves X corresponding to real-valued potentials, X shall furthermore be endowed with an antiholomorphic involution $\tau_2 = \tau_1 \circ \tau_2$ with $\tau_2(Q^{\pm}) = Q^{\mp}$. We have decided to consider τ_2 in this chapter instead of τ_1 because otherwise, we would have to be more careful with some sign in front of the considered Weierstraß polynomials under the action of τ_1 hereinafter. These remain unchanged under τ_2. It is explained hereinafter why it is better to consider $\tau_2 = \sigma \circ \tau_1 = \tau_1 \circ \sigma : X \to X$, $k \mapsto \bar{k}$ instead of τ_1 in this appendix. In [Carberry and Schmidt, 2017], it is shown that the space of universal local deformations is parametrized by a finite-dimensional manifold. Universal means that up to isomorphisms, every deformation can be described by it and local means in the neighborhood of the given data.

In Chapter 5, we reconstructed a unique potential and the corresponding eigenfunctions for spectral data obeying conditions (F1) to (F3) on a given Riemann surface which could then be considered as the normalization of the Fermi curve. Thereby, the additional assumption is necessary that on X, there exists a divisor D which obeys conditions (D1) to (D3) from Chapter 5. Here, deformations which conserve the conditions (F1) to (F3) and the arithmetic genus g_a are considered. On these deformations, one can always find a divisor D which obeys (D1) and (D2). However, it is unclear whether there always exists a divisor obeying all three conditions on the deformed curves. But the divisors obeying conditions (D1) to (D3) form an open set of the divisors which are mapped by the Abel mapping to the Prym variety, see Lemma 6.3 for complex-valued potentials and Lemma 6.53 for real-valued potentials. In the sequel, we assume that this condition is also open on the

space of deformations. Due to this assumption, there always exists a divisor D obeying (D1) to (D3) on small deformations of a given Fermi curve.
The key to the theory presented in [Carberry and Schmidt, 2017] is to consider first a larger class of deformations, so-called $\hat{c}$-deformations, where only the periods of one of the two differentials is conserved. The space of these $\hat{c}$-deformations is comparatively easy to construct and we will see that it is just $\mathbb{C}^r$ with some $r > 0$. The deformations which conserve two differentials are a subfamily of the former ones. So it is possible to determine the tangent space of these deformations inside the tangent space of the $\hat{c}$-deformations. Integrating the directions of the tangent space of this subfamily then yields the desired space of deformations.
We will not repeat the proofs of the results shown in [Carberry and Schmidt, 2017]. It will for sure be published in near future. So we only explain roughly how these deformations take place and add how we can embed the Fermi curve into the setting used in [Carberry and Schmidt, 2017]. Moreover, we make a short remark on the isomorphism classes of infinitesimal deformations which conserve not only both differentials, but also the lattice Γ.

C.1. The curve to be deformed

In this appendix, elements in the following spaces are frequently considered.

Definition C.1 ([de Jong and Pfister, 2012, Introduction of Chapter 1 and Definition 7.3.6])**.** The ring of convergent series in $\hat{c}$ and $\check{c}$ over $\mathbb{C}$ is denoted by $\mathbb{C}\{\hat{c}, \check{c}\}$, $\mathbb{C}\{\hat{c}\}[\check{c}]$ denotes the ring over $\mathbb{C}$ which elements are polynomials in $\check{c}$ and convergent series in $\hat{c}$. For a germ $\mathcal{B}_0$ of a complex space $\mathcal{B}$ at $0 \in \mathcal{B}$, we denote by $(\mathbb{C}\{\hat{c}, \check{c}\} \hat{\otimes} \mathcal{O}_{\mathcal{B}})_{(0,0)}$ the stalk of the holomorphic functions on $\mathbb{C}^2 \times \mathcal{B}$ at $(\hat{c}, \check{c}, b) = (0, 0, 0)$.

Another central idea presented in [Carberry and Schmidt, 2017] is that the curve to be deformed obeys the following property.

Definition C.2 ([Carberry and Schmidt, 2017, Definition 2.6])**.** An open complex curve X° with two regular coordinate functions $\hat{c}, \check{c} : X^\circ \to \mathbb{C}$ is called *locally planar* if for each $p \in X^\circ$, the germs $((\hat{c} - \hat{c}(p))_p, (\check{c} - \check{c}(p))_p)$ map the space germ X_p of X at p biregularly onto the zero set of some $f_p \in \mathbb{C}\{\hat{c}, \check{c}\}$.

The Fermi curve $X'(u) = F(u)/\Gamma^*$ is a one-dimensional variety in $\mathbb{C}^2/\Gamma^*$, see Corollary 1.15 and Theorem 2.28(a). Hence, it is a locally planar curve. As already discussed in Section 4.2, the arithmetic genus of $X'(u)$ is generically infinite. In this case, $X'(u)$ cannot be compactified. For a finite type potential u, it might be considered as a good idea to take the middleding $M(u)$ as in Section 4.2 as the curve to be deformed. However, there are two reasons which speak against this: first of all, it might happen that $M(u)$ is not locally planar. Secondly, the deformation theory presented in [Carberry and Schmidt, 2017] is based on local deformations around the singularities of a curve. And the more desingularized the considered curve to deform is, the smaller is the moduli space obtained by these deformations. Hence, the moduli space of $M(u)$ might not describe the moduli space of $X'(u)$ completely.

One appropriate choice which we consider in this work is the following: Let u be a finite type potential and $\delta > 0$ be sufficiently small such that the only singularities of $X'(u) \cap \mathbb{C}^2_\delta/\Gamma^*$ are double points, compare Theorem 2.34. Remember that for a finite type potential, $X^\circ(u)$ denotes the normalization of $X'(u)$ with normalization map $\pi : X^\circ(u) \to X'(u)$. We define $X \cap (\mathbb{C}^2 \setminus \mathbb{C}^2_\delta)/\Gamma^* := X'(u) \cap (\mathbb{C}^2 \setminus \mathbb{C}^2_\delta)/\Gamma^*$ and $X \cap \mathbb{C}^2_\delta := X^\circ(u) \cap \mathbb{C}^2_\delta/\Gamma^*$. In other words, we glue the normalization $X^\circ(u)$ to the open ends of the Fermi curve $X'(u)$. This is possible since for $\delta > 0$ sufficiently small, all singularities of $X'(u)$ are double points. Out of these, at most finitely many are contained in $X'(u) \cap \mathbb{C}^2_\delta/\Gamma^*$. Note that for two representants $k, k' \in [k]$, there always holds $\operatorname{Im}(k) = \operatorname{Im}(k')$, because Γ^* is a real two-dimensional lattice. By varying δ a bit if necessary, we can achieve that no double points are contained in $X'(u) \cap \{[k] \in \mathbb{C}^2/\Gamma^* \mid \|\operatorname{Im}([k])\| = \delta^{-1}\}$ since they are contained in the excluded domains around the double points $k^\pm_\nu$ of the free Fermi curve which are discrete and equidistant, compare Section 1.4 and Theorem 2.34. Therefore, an appropriate choice of δ yields that the Fermi curve and its normalization can be considered as the same on a small open tube around $X'(u) \cap \{[k] \in \mathbb{C}^2/\Gamma^* \mid \|\operatorname{Im}([k])\| = \delta^{-1}\}$. Hence, we can glue these two curves together along this boundary. The curve we obtain like this we call X° in the sequel and its compactification we denote as X. As in Chapter 4, $X = X^\circ \cup \{Q^+, Q^-\}$. We also denote X as a Fermi curve in the rest of this chapter.

Furthermore, at any preimage $k_{\nu,\pm}$ of a double point $k^\pm_\nu(u) \in X'(u)$ under π with $\nu \in \Gamma^*_\delta$ and $\delta > 0$ sufficiently small, there exists a biregular map $(\hat{c}, \check{c})$ from the normalization to $\mathbb{C}^2$. This is because in a small open neighborhood $U_{\nu,\pm}$ of a preimage $k_{\nu,\pm}$ of $k^\pm_\nu(u)$, there are maps such that one of the derivatives into the direction of $\hat{c}$ or $\check{c}$ of the germ describing the normalization is unequal to zero. We then can apply the Inverse Function Theorem to obtain that the normalization of U_ν can be described by the Weierstraß covering corresponding to $(\hat{c}, \check{c}) \mapsto \hat{c}$. So also the normalization is locally planar in the preimage of a double point. Since the only preimages of singularities contained in the part of $X^\circ(u) \cap \mathbb{C}^2_\delta$ are the preimages of double points, the corresponding part of the curve X° is locally planar. So the curve X tinkered above is compact and locally planar. Another advantage of this choice is that by shrinking $\delta > 0$, one can add more double points of $X'(u)$ to the considered X, i.e. increase the number of singularities in X. Like this, one can increase the size of the moduli space, whereas the possible deformations of the compact part of $X'(u)$ which also contains other singularities then double points remains unchanged in the modified X. The constructed curve X together with the involutions σ and τ_2 always obeys the following definition.

Definition C.3 ([Carberry and Schmidt, 2017, Definition 2.9])**.** Let X be a compact one-dimensional complex space with smooth points Q^+ and Q^-, σ a holomorphic involution on X and let $d\hat{c}$ and $d\check{c}$ be meromorphic differentials on X. Then $(X, Q^+, Q^-, d\hat{c}, d\check{c}, \sigma)$ is called *locally planar Fermi curve data* if the following conditions hold:

(A) The meromorphic differentials $d\hat{c}$ and $d\check{c}$ are holomorphic on $X^\circ = X \setminus \{Q^+, Q^-\}$ with poles of second order at Q^+ and Q^- with no residues.

(B) X° is locally planar with respect to the local antiderivatives $\hat{c}, \check{c} \in \mathcal{O}_{X^\circ,p}$ of $d\hat{c}$ and $d\check{c}$.

(C) The integrals of $d\hat{c}$ and $d\check{c}$ along closed path of X° take values in $\mathbb{Z}$.

(D) The holomorphic involution σ has exactly the two fixed points $\sigma(Q^\pm) = Q^\pm$ and transforms $d\hat{c}$ and $d\check{c}$ as $\sigma^* d\hat{c} = -d\hat{c}$ and $\sigma^* d\check{c} = -d\check{c}$.

For real-valued potential $u : \mathbb{R}^2/\Gamma \to \mathbb{R}$, there holds additionally

(E) On X, there exists an antiholomorphic involution τ_2 with $\tau_2(Q^\pm) = Q^\mp$ which transforms $d\hat{c}$ and $d\check{c}$ as $\tau_2^* d\hat{c} = \overline{d\hat{c}}$ and $\tau_2^* d\check{c} = \overline{d\check{c}}$.

From Lemma 4.2 and Sections 4.3 and 1.3 we know that for a finite type potential u, the local planar curve X as constructed above together with the two smooth points $Q^\pm$, the meromorphic differentials $d\hat{c}$ and $d\check{c}$ as defined in (4.24) and the involution σ from Lemma 1.17(a) with fixed points $Q^\pm$, see Corollary 4.3, yield locally planar Fermi curve data.

C.2. Definitions of the deformations

Before we start to explain the construction of the deformation space, we give an overview of the necessary definitions of the considered deformations used hereinafter. These are oriented on [Greuel et al., 2007, Chapter II], whereas the definitions in [Greuel et al., 2007] are formulated for deformations of space germs. In the upcoming definitions, we will assume that certain maps are flat. Let Y be a fiber bundle over a base space $\mathcal{B}$. Then flatness used here is a concept from algebraic geometry which ensures that the fibers of the map $Y \twoheadrightarrow \mathcal{B}$ depend in a regular way on the points in $\mathcal{B}$. Therefore, these fibers can be considered as deformations of a special fiber which is the curve X in our case, compare [Grauert et al., 1994, Chapter II, §2]. We will point out more advantages of the flatness-property when they show up hereinafter.

Definition C.4. (a) A *deformation* of a compact complex analytic space X is a pair of complex analytic spaces, the *total space* Y and the *base space* $\mathcal{B}$, together with a marked point $b_0 \in \mathcal{B}$ and a flat and proper map $Y \twoheadrightarrow \mathcal{B}$ such that the preimage of the point b_0 in Y is isomorphic to X. The fibers over $b \in \mathcal{B}$ in Y are denoted by $X(b)$, where $X = X(b_0)$ is called the *special fiber*.

(b) A *morphism* from the deformation $X \hookrightarrow Y \twoheadrightarrow \mathcal{B} \ni b_0$ to $X \hookrightarrow Z \twoheadrightarrow \mathcal{T} \ni t_0$ is a commutative diagram of holomorphic maps

$$\begin{array}{ccccc} X & \hookrightarrow & Y & \twoheadrightarrow & \mathcal{B} \ni b_0 \\ \| & & \downarrow \psi & & \downarrow \varphi \quad \downarrow \varphi \\ X & \hookrightarrow & Z & \twoheadrightarrow & \mathcal{T} \ni t_0 \end{array}$$

such that the surjective horizontal maps are flat.

(c) Two deformations are *isomorphic* if there exist two morphisms which are inverse to each other.

(d) For any deformation $X \hookrightarrow Y \twoheadrightarrow \mathcal{B} \ni b_0$ and any open neighborhood $O \subset \mathcal{B}$ of b_0, let $U \subset Y \twoheadrightarrow \mathcal{B}$ be the preimage of O. The resulting deformation $X \hookrightarrow U \twoheadrightarrow O \ni b_0$ is called *restriction* of $X \hookrightarrow Y \twoheadrightarrow \mathcal{B} \ni b_0$ to O.

(e) A *local deformation* $X \hookrightarrow Y \twoheadrightarrow \mathcal{B}_0$ is a deformation for which the base space is the germ of a complex space $\mathcal{B}$ at a point $b_0 \in \mathcal{B}$.

(f) A deformation $X \hookrightarrow Z \twoheadrightarrow \mathcal{T} \ni t_0$ is called *complete* if for any deformation $X \hookrightarrow Y \twoheadrightarrow \mathcal{B} \ni b_0$, there exists a germ of a map φ which maps b_0 to t_0 such that the pullback of the flat map $Z \twoheadrightarrow \mathcal{T}$ under this germ is isomorphic to $X \hookrightarrow Y \twoheadrightarrow \mathcal{B} \ni b_0$.

(g) A deformation $X \hookrightarrow Z \twoheadrightarrow \mathcal{T}$ is called *universal* if it is complete and if for any deformation $X \hookrightarrow Y \twoheadrightarrow \mathcal{B}$, there exists a unique germ of a map $\varphi : \mathcal{B} \to \mathcal{T}$ such that the pullback of $Z \twoheadrightarrow \mathcal{T}$ with respect to this germ is isomorphic to $X \hookrightarrow Y \twoheadrightarrow \mathcal{B}$.

All these definitions can be found in [Carberry and Schmidt, 2017, Section 3.1]. Two deformations defined on subsets of $\mathcal{B}$ which both contain 0 define the same local deformation at 0 if and only if their restrictions to some open neighborhood of 0 are isomorphic. If we deform a space germ, we only consider local deformations. These are, as in [Carberry and Schmidt, 2017], denoted by *deformations of the space germ*. The fact that the map $Y \twoheadrightarrow \mathcal{B}$ in Definition C.4(a) is flat ensures that the arithmetic genus of all fibers $X(b) \subset Y$ over $b \in \mathcal{B}$ is constant, compare [Grauert et al., 1994, Chapter III, Theorem 4.7 (b)]. Moreover, all deformed curves are also compact because we assumed that the covering map $Y \twoheadrightarrow \mathcal{B}$ is proper. Then the preimage of the points in $\mathcal{B}$, i.e. the fibers in Y, are compact.

By pulling a deformation back with $\varphi : \mathcal{B} \to \widetilde{\mathcal{B}}$, we mean that the fibers in the preimage of the map $\widetilde{Y} \to \{\tilde{b} \in \widetilde{\mathcal{B}} \mid \exists\, b \in \mathcal{B} : \varphi(b) = \tilde{b}\}$ are attached to the corresponding point on $b \in \mathcal{B}$. By saying that the pullback of a deformation is isomorphic to another one, we mean that the fiber attached to b by the pullback has to be isomorphic to the corresponding fiber in the preimage of $Y \twoheadrightarrow \mathcal{B}$ at b.

Furthermore, the deformation $X \hookrightarrow Y \twoheadrightarrow \mathcal{B} \ni b_0$ and its restriction to neighborhoods U of b_0 are not distinguished. In particular, the base space $\mathcal{B}$ turn out to be a complex analytic space germ in $\mathbb{C}^r$ at b_0. Without loss of generality, b_0 is always considered to be $0 \in \mathbb{C}^r$ and will often be neglected. To define the deformations of data $(X, d\hat{c}, d\check{c})$ such that only the periods of $d\hat{c}$ are preserved, the following concept of deformations of space germs of complex analytic spaces relative to the trivial deformation $\mathbb{C}_0 \hookrightarrow \mathbb{C}_0 \twoheadrightarrow \{0\}$ is necessary. Moreover, the space germ which is defined by the zero set of a germ $f \in \mathbb{C}\{\hat{c}, \check{c}\}$ respectively $F \in \mathbb{C}\{\hat{c}, \check{c}\} \hat{\otimes} \mathcal{O}_{\mathcal{B}}$ is denoted by $V(f)$ respectively $V(F)$.

Definition C.5. (a) For $f \in \mathbb{C}\{\hat{c}, \check{c}\}$, a *$\hat{c}$-deformation* of $V(f)$ is a space germ $\mathcal{B}$ at $0 \in \mathcal{B}$ and $F \in \mathbb{C}\{\hat{c}, \check{c}\} \hat{\otimes} \mathcal{O}_{\mathcal{B}}$ such that $V(F)$ is at $b = 0$ equal to $V(f)$. The deformation is endowed with the following morphism of deformations of space germs:

$$\begin{array}{ccccc} V(f) & \hookrightarrow & V(F) & \twoheadrightarrow & \mathcal{B} \\ \downarrow \hat{c} & & \downarrow (\hat{c}, 0) & & \downarrow \\ \mathbb{C}_0 & \hookrightarrow & \mathbb{C}_0 & \twoheadrightarrow & \{0\} \end{array}$$

(b) A *$\hat{c}$-morphism* from the $\hat{c}$-deformation $V(f) \hookrightarrow V(F) \twoheadrightarrow \mathcal{B}$ with $f \in \mathbb{C}\{\hat{c}, \check{c}\}$ and $F \in \mathbb{C}\{\hat{c}, \check{c}\} \hat{\otimes} \mathcal{O}_{\mathcal{B}}$ to the $\hat{c}$-deformation $V(g) \hookrightarrow V(G) \twoheadrightarrow \mathcal{T}$ with $g \in \mathbb{C}\{\hat{c}, \check{c}\}$ and $G \in \mathbb{C}\{\hat{c}, \check{c}\} \hat{\otimes} \mathcal{O}_{\mathcal{T}}$ is a morphism of deformations

$$\begin{array}{ccccc} V(f) & \hookrightarrow & V(F) & \twoheadrightarrow & \mathcal{B} \\ \downarrow & & \downarrow \psi & & \downarrow \varphi \\ V(g) & \hookrightarrow & V(G) & \twoheadrightarrow & \mathcal{T} \end{array}$$

which composition with the morphims of $V(g) \hookrightarrow V(G) \twoheadrightarrow \mathcal{T}$ to $\mathbb{C}_0 \hookrightarrow \mathbb{C}_0 \twoheadrightarrow \{0\}$ is equal to the corresponding morphism of $V(f) \hookrightarrow V(F) \twoheadrightarrow \mathcal{B}$.

Due to [Greuel et al., 2007, Corollary II.1.6], all deformations of $V(f)$ with $f \in \mathbb{C}\{\hat{c}, \check{c}\}$ are of the form $V(f) \hookrightarrow V(F) \twoheadrightarrow \mathcal{B}$ with $F \in \mathbb{C}\{\hat{c}, \check{c}\} \hat{\otimes} \mathcal{O}_{\mathcal{B}}$. Furthermore, the map $V(F) \to V(G)$ is of the form $(\hat{c}, \check{c}, b) \mapsto (\hat{c}, u(\hat{c}, \check{c}, b), \varphi(b))$ with $u \in \mathbb{C}\{\hat{c}, \check{c}\} \hat{\otimes} \mathcal{O}_{\mathcal{B}}$ and there exists a unit $H \in \mathbb{C}\{\hat{c}, \check{c}\} \hat{\otimes} \mathcal{O}_{\mathcal{B}}$ with

$$H(\hat{c}, \check{c}, b) G(\hat{c}, u(\hat{c}, \check{c}, b), \varphi(b)) = F(\hat{c}, \check{c}, b)$$

Since we can consider Hf instead of f, we may assume that $F(\hat{c}, \check{c}, 0) = f(\hat{c}, \check{c})$ such that $V(f) \hookrightarrow V(F)$ is an embedding. Evaluating u and H at $s = 0$ yields the map $V(f) \to V(g)$ such that $(\hat{c}, \check{c}) \to (\hat{c}, u(\hat{c}, \check{c}, 0))$ with $u(\hat{c}, \check{c}, 0), H(\hat{c}, \check{c}, 0) \in \mathbb{C}\{\hat{c}, \check{c}\}$. For $V(f) = V(g)$, we may assume that $f = g$ with $u(\hat{c}, \check{c}, 0) = \check{c}$. Since we assume $F(\hat{c}, \check{c}, 0) = f(\hat{c}, \check{c})$ as well as $G(\hat{c}, \check{c}, 0) = g(\hat{c}, \check{c})$, this yields $H(\hat{c}, \check{c}, 0) = 1$. Furthermore, the construction of a universal deformation of data $(X, Q^+, Q^-, d\hat{c}, d\check{c}, \sigma)$ will be based on so-called infinitesimal deformations of the Fermi curve, i.e. deformations up to first order.

Definition C.6 ([Carberry and Schmidt, 2017, Definition 3.7])**.** An *infinitesimal deformation* is a deformation whose base space has the holomorphic functions $\Pi = \mathbb{C}\{\varepsilon\}/\langle\varepsilon^2\rangle \simeq \mathbb{C}[\varepsilon]/\langle\varepsilon^2\rangle$.

C.3. Deformations of curves with one differential

Let X_p denote the space germ of X at $p \in X$ and $(d\hat{c})_p$ the germ of the 1-form $d\hat{c}$ at p. Analogously, $\mathbb{C}_0$ denotes the space germ of $\mathbb{C}$ at $0 \in \mathbb{C}$. The next definition contains weaker conditions which also hold for Fermi curve data $(X, Q^+, Q^-, d\hat{c}, d\check{c}, \sigma)$ as in Definition C.3.

Definition C.7 ([Carberry and Schmidt, 2017, Definition 3.1])**.** Let X be a compact one-dimensional complex curve with smooth marked points Q^+ and Q^- and let $d\hat{c}$ be a meromorphic 1-form on X and $\sigma : X \to X$ a holomorphic involution. $(X, Q^+, Q^-, d\hat{c}, \sigma)$ is called *locally planar with prescribed poles* if the following conditions hold:

(A1) For both, Q^+ and Q^-, there exists $z_\pm \in \mathcal{O}_{X,Q^\pm}$ which vanishes at $Q^\pm$ and maps $X_{Q^\pm}$ biregularly onto $\mathbb{C}_0$ such that $(d\hat{c})_{Q^\pm} = d(z_\pm^{-1})$.

(B1) For each $q \in X^\circ$, there exists $(\hat{c}_q, \check{c}_q) \in \mathcal{O}_{X,q} \times \mathcal{O}_{X,q}$ which vanishes at q and maps X_q biregularly onto the zero set of some $f_q \in \mathbb{C}\{\hat{c}, \check{c}\}$. Further, it is required that $(d\hat{c})_q = d(\hat{c}_q)$

(D1) On X, there exists a holomorphic involution σ with $\sigma(Q^\pm) = Q^\pm$ and no other fixed points which acts as

$$\hat{c}_{\sigma(q)} = -\sigma^* \hat{c}_q, \quad \check{c}_{\sigma(q)} = -\sigma^* \check{c}_q, \quad z_\pm = -\sigma^* z_\pm.$$

If in addition the following condition holds, $(X, Q^+, Q^-, d\hat{c}, \sigma)$ is said to be real:

(E1) On X, there exists an antiholomorphic involution τ_2 with $\tau_2(Q^+) = Q^-$ which acts as

$$\hat{c}_{\tau_2(q)} = \tau_2^* \bar{\hat{c}}_q, \quad \check{c}_{\tau_2(q)} = \tau_2^* \bar{\check{c}}_q, \quad z_\pm = \tau_2^* \bar{z}_\mp, \quad f_{\tau_2(q)}(\hat{c}, \check{c}) = \bar{f}_q(\bar{\hat{c}}, \bar{\check{c}}). \tag{C.1}$$

Condition (A1) guarantees that $Q^\pm$ are smooth points of X and that X has at most two connected components. The germ $z_\pm$ of the local coordinate in (A1) of X at $Q^\pm$ is determined by the 1-form $d\hat{c}$ and the condition that the germ of $d\hat{c}$ at $Q^\pm$ is equal to $d(z_\pm^{-1})$ at $Q^\pm$. Due to (B1), one has $d(\hat{c}_q - \hat{c}_{q'}) = d\hat{c} - d\hat{c} = 0$ for $q, q' \in X$. Therefore, $\hat{c}_q$ has an analytic continuation along all paths in X° such that $\hat{c}_{q'} = \hat{c}_q - \hat{c}_q(q')$ holds in $\mathcal{O}_{X,q'}$ along the path. The germ of this difference is the germ of a constant function. So $\hat{c}_q$ is holomorphic at all q' nearby q and the case that all integrals over closed paths of $d\hat{c}$ take values in $\mathbb{Z}$ – as formulated in (C) – is included in this condition. (D1) and (E1) ensure that the differentials $d\hat{c}$ and $d\check{c}$ have the behavior as described in (D) respectively (E). It will turn out in Lemma C.11 that the $\hat{c}$-deformations are completely determined by the local deformations in small open neighborhoods of the singularities of X and of the zeros of $d\hat{c}$. Because X is compact, these are finitely many points $q_1, \ldots, q_L$. For brevity, the index q_l is for each $l = 1, \ldots, L$ denoted by l and the space germs in a neighborhood of q_l on X by $X_l := X_{q_l}$. In particular, $\hat{c}_l = \hat{c}_{q_l}$, $\check{c}_l = \check{c}_{q_l}$ and $f_l = f_{q_l}$. Since all Fermi curves X are subvarieties in $\mathbb{C}^2/\Gamma^*$, the Weierstraß Preparation Theorem [de Jong and Pfister, 2012, Theorem 3.2.4] yields that one can describe them locally as Weierstraß coverings over $\hat{c} \in \mathbb{C}$. More precisely, for every $f_l \in \mathbb{C}\{\hat{c}, \check{c}\}$, it follows from the Weierstraß Preparation Theorem [de Jong and Pfister, 2012, Theorem 3.2.4] that for all $p \in X$, there exists a unique Weierstraß polynomial $f_l \in \mathbb{C}\{\hat{c}\}[\check{c}]$ of degree d_l in $\check{c}$ for $l = 1, \dots, L$ with highest coefficient equal to 1 and all lower coefficients vanishing at $\hat{c} = 0$. In [Carberry and Schmidt, 2017], an open ball $\mathcal{B}$ around $0 \in \mathbb{C}^k$ and $\delta_1 > 0$ is chosen such that the polynomials F_l are holomorphic on $(\hat{c}, \tilde{\check{c}}, b) \in B_{\delta_1}(0) \times \mathbb{C} \times \mathcal{B}$. For $l = 1, \dots, L$ and sufficiently small δ_1, there exist disjoint open neighborhoods $O_l := \{(\hat{c}, \tilde{\check{c}}) \in B_{\delta_1}(0) \times \mathbb{C} \mid f_l(\hat{c}, \tilde{\check{c}}) = 0\}$ of q_l in X such that the maps in (B1) extend to biregular maps

$$O_l \to U_l, \quad (\hat{c}, \tilde{\check{c}}) \mapsto \hat{c}. \tag{C.2}$$

Hence, $O_l \to B_{\delta_1}(0)$ is a Weierstraß covering with a fixed number of d_l sheets and a single branch point at $(0, 0)$. The germs $f_l \in \mathbb{C}\{\hat{c}, \check{c}\}$ are replaced by the unique Weierstraß polynomial $f_l \in \mathbb{C}\{\hat{c}\}[\check{c}]$ of degree d_l, compare [de Jong and Pfister, 2012, Theorem 2.3.4]. The involution $\sigma : (\hat{c}, \check{c}) \mapsto (-\hat{c}, -\check{c})$ induces an involution σ^* on $\mathbb{C}\{\hat{c}, \check{c}\}$. One can decompose $f \in \mathbb{C}\{\hat{c}, \check{c}\}$ uniquely into $f = f^+ + f^-$ with $f^\pm = \frac{1}{2}(f \pm \sigma^* f)$. Therefore, $\mathbb{C}\{\hat{c}, \check{c}\} = \mathbb{C}^+\{\hat{c}, \check{c}\} \oplus \mathbb{C}^-\{\hat{c}, \check{c}\}$. Analogously, also $\mathbb{C}\{\hat{c}\}[\check{c}] = \mathbb{C}^+[\hat{c}]\{\check{c}\} \oplus \mathbb{C}^-[\hat{c}]\{\check{c}\}$. So it is necessary to take the behavior of the function f_l under σ into account. Moreover, the function germ locally describing X at $\sigma(q_l)$ is denoted by $f_{\sigma l}$.

Lemma C.8 ([Carberry and Schmidt, 2017, Beginning of Section 6])**.** *Let $f_l \in \mathbb{C}\{\hat{c}\}[\check{c}]$ be the unique Weierstraß polynomial of degree d_l whose zero set locally describes X in a neighborhood of $q_l \in X$. If none of the points $q_1, \dots, q_L$ is a fixed point of σ, then the $q_1, \dots, q_L$ can be sorted into pairs such that $\sigma(q_l) = q_{\sigma l}$ with $l \neq \sigma l \in 1, \dots, L$ and*

$$\sigma^* f_l = (-1)^{d_l} f_{\sigma l}. \tag{C.3}$$

with $d_l = d_{\sigma l}$.

Next, the conditions on $\hat{c}$-deformations as in Definition C.5(a) which correspond to local planar Fermi curve data $(X, Q^+, Q^-, d\hat{c}, \sigma)$ obeying conditions (A1), (B1) and (D1) are defined. Therefore,

$$X \hookrightarrow Y \overset{b}{\twoheadrightarrow} \mathcal{B}$$

is a deformation together with a meromorphic 1-form $d\hat{c}_Y$ on Y. Hereby, b denoted the map which maps a fiber $X(b)$ to the corresponding element $b \in \mathcal{B}$.

Definition C.9 ([Carberry and Schmidt, 2017, Definition 3.4]). A deformation

$$(X, d\hat{c}) \hookrightarrow (Y, d\hat{c}_Y) \twoheadrightarrow \mathcal{B} \tag{C.4}$$

of locally planar $(X, Q^+, Q^-, d\hat{c}, \sigma)$ with prescribed poles is given by a deformation $X \hookrightarrow Y \overset{b}{\twoheadrightarrow} \mathcal{B}$ of the complex space X together with a meromorphic 1-form $d\hat{c}_Y$ on Y such that the following conditions hold:

(A1') For Q^+ and Q^-, the germ $z_\pm$ of Definition C.7(A1) extends to $z_{Y,\pm} \in \mathcal{O}_{Y,Q^\pm}$ such that $(z_{Y,\pm}, b)$ maps $Y_{Q^\pm}$ biregularly onto $(\mathbb{C} \times \mathcal{B})_{(0,0)}$ and $d\hat{c}_{Y,Q^\pm} = d(z_{Y,\pm}^{-1})$.

(B1') For each $q \in X^\circ$, the germs $\hat{c}_{Y,q}$ and $\check{c}_{Y,q}$ which are induced by the deformation

$$(X^\circ, \hat{c}_q, \check{c}_q) \hookrightarrow (Y^\circ, \hat{c}_{Y^\circ,q}, \check{c}_{Y^\circ,q}) \overset{b}{\twoheadrightarrow} \mathcal{B}$$

are elements of $\mathcal{O}_{Y,q}$ such that $(\hat{c}_{Y,q}, \check{c}_{Y,q}, b)$ maps Y_q biregularly onto $V(F_q)$ with $F_q \in \mathbb{C}\{\hat{c}, \check{c}\} \hat{\otimes} \mathcal{O}_{\mathcal{B}}$ and $d\hat{c}_{Y,q} = (d\hat{c})_{Y,q}$.

(D1') The involution σ extends to an involution on Y and $\mathcal{B}$ which acts as

$$\sigma^* d\hat{c}_Y = -d\hat{c}_Y, \quad \sigma^* d\check{c}_Y = -d\check{c}_Y, \quad \sigma^* \hat{c}_{Y,q} = -\hat{c}_{Y,\sigma(q)}, \quad \sigma^* \check{c}_{Y,q} = -\check{c}_{Y,\sigma(q)},$$
$$\sigma^* F_q(\hat{c}, \check{c}, b) = (-1)^{d_l} F_{\sigma(q)}(\hat{c}, \check{c}, b)),$$

where d_l is the degree of the Weierstraß polynomial f_l describing the germ X_q. The involution σ acts trivially on $\mathcal{B}$ and commutes with the maps $X \hookrightarrow Y \twoheadrightarrow \mathcal{B}$.

(E1') If $(X, Q^+, Q^-, d\hat{c}, \sigma)$ is real, i.e. if in addition C.7 (E1) holds, then τ_2 extends to an involution of Y and $\mathcal{B}$ which acts as

$$\hat{c}_{Y,\tau_2(q)} = \tau_2^* \bar{\hat{c}}_{Y,q}, \quad \check{c}_{Y,\tau_2(q)} = \tau_2^* \bar{\check{c}}_{Y,q}, \quad z_{Y,\tau_2(q)} = \tau_2^* \bar{z}_{Y,q},$$
$$F_{\tau_2(q)}(\hat{c}, \check{c}, b) = \overline{F}_q(\bar{\hat{c}}, \bar{\check{c}}, \tau_2(b)). \tag{C.5}$$

The involution τ_2 commutes with the maps $X \hookrightarrow Y \twoheadrightarrow \mathcal{B}$.

Here, Y_p denotes for all $p \in X$ the space germ of Y at p and $d\hat{c}_{Y,p}$ denotes the germ of the 1-form $d\hat{c}_Y$ at p. Analogously, $(\mathbb{C} \times \mathcal{B})_{(0,0)}$ denotes the space germ of $\mathbb{C} \times \mathcal{B}$ at $(0,0) \in \mathbb{C} \times \mathcal{B}$.

Again, (A1') guarantees that $Q^\pm$ are smooth points of Y and $z_{Y,\pm}$ is uniquely determined by $d\hat{c}_y$. Then for $q, q' \in X^\circ$, one has $d\hat{c}_{Y,q} = d\hat{c}_Y = d\hat{c}_{Y,q'}$, and therefore one has – as in the undeformed case – that $d(\hat{c}_{Y,q} - \hat{c}_{Y,q'}) = 0$. So $\hat{c}_{Y,q} - \hat{c}_{Y,q'}$ is constant, and therefore independent from the deformation parameter $b \in \mathcal{B}$. As before, this yields that $\hat{c}_{Y,q}$ has an analytic continuation along all paths in X° starting at q and along these paths, the equality of germs of holomorphic functions

$\hat{c}_{Y,q'} = \hat{c}_{Y,q} - \hat{c}_{Y,q}(q')$ holds. Since $\hat{c}_{Y,q}$ as well as $\hat{c}_{Y,q'}$ are germs, this remains valid in a small open neighborhood of X in Y. These considerations enforce that the periods of $d\hat{c}$ are preserved along the fibers of the deformation. So condition (C) also holds on all deformed curves $X(b) \subset Y$ for $b \in \mathcal{B}$ and $\mathcal{B}$ sufficiently small. Due to (D1'), σ defines a holomorphic involution $\sigma : X(b) \to X(b)$ on all fibers over $\mathcal{B}$. So the differential $d\hat{c}_Y$ transforms as $\sigma^* d\hat{c}_Y = -d\hat{c}_Y$. Moreover, (D1') ensures that each fiber $X(b)$ of a $\hat{c}$-deformation is also endowed with a holomorphic involution σ with two fixed points $Q^{\pm}(b)$ which equal $(0, b)$ with respect to the coordinates $(z_{Y,\pm}, b)$, i.e. these points are the deformations of the points Q^+ and Q^- in $X(b)$ and also fixed points of σ on the fiber $X(b)$. Finally, (E1') implies that $d\hat{c}_Y$ transforms as $\tau_2^* d\hat{c}_Y = \overline{d\hat{c}_Y}$ and that all fibers $X(b)$ are also endowed with an antiholomorphic involution τ_2 which interchanges $Q^+(b)$ and $Q^-(b)$. So both involutions σ and τ_2 extend to involutions on Y respectively $\mathcal{B}$ which are also denoted by σ respectively τ_2. To describe the morphisms of these deformations, let

$$(X, d\hat{c}) \hookrightarrow (Y, d\hat{c}_Y) \twoheadrightarrow \mathcal{B}, \qquad (X, d\hat{c}) \hookrightarrow (\widetilde{Y}, d\hat{c}_{\widetilde{Y}}) \twoheadrightarrow \widetilde{\mathcal{B}} \tag{C.6}$$

be two such deformations and let the holomorphic maps $\varphi : \mathcal{B} \to \widetilde{\mathcal{B}}$ and $\psi : Y \to \widetilde{Y}$ define the following morphism:

$$\begin{array}{ccccc} X & \hookrightarrow & Y & \twoheadrightarrow & \mathcal{B} \\ \| & & \downarrow\psi & & \downarrow\varphi \\ X & \hookrightarrow & \widetilde{Y} & \twoheadrightarrow & \widetilde{\mathcal{B}} \end{array} \tag{C.7}$$

The deformations of the space germs at the marked points $Q^{\pm}$ and the points $q \in X^\circ$ have to be analyzed separately. This is because X is not locally planar in a small open neighborhood of $Q^{\pm}$, but locally biregular to an open set in $\mathbb{C}^2$. For $Q^{\pm}$, let $X_{Q^\pm}$, $Y_{Q^\pm}$ and $\widetilde{Y}_{Q^\pm}$ be the space germs of X, Y and $\widetilde{Y}$ at $Q^{\pm}$. The morphism (C.7) induces the following morphism of deformations of space germs:

$$\begin{array}{ccccc} X_{Q^\pm} & \hookrightarrow & Y_{Q^\pm} & \twoheadrightarrow & \mathcal{B}_0 \\ \| & & \downarrow\psi_\pm & & \downarrow\varphi_0 \\ X_{Q^\pm} & \hookrightarrow & \widetilde{Y}_{Q^\pm} & \twoheadrightarrow & \widetilde{\mathcal{B}}_0 \end{array}$$

It is imposed that this is an $\hat{c}$-morphism with respect to the morphisms $(z_\pm, z_{Y,\pm}, 0)$ and $(z_\pm, z_{\widetilde{Y},\pm}, 0)$ to the trivial deformation of space germs $\mathbb{C}_0 \hookrightarrow \mathbb{C}_0 \twoheadrightarrow \{0\}$. This is equivalent to $\psi_\pm^* z_{\widetilde{Y},\pm} = z_{Y,\pm}$. Analogously, for all other points $q \in X^\circ$, let X_q, Y_q and $\widetilde{Y}_q$ denote the space germs of X, Y and $\widetilde{Y}$ at q. The morphism (C.7) also induces a morphism of deformations of space germs:

$$\begin{array}{ccccc} X_q & \hookrightarrow & Y_q & \twoheadrightarrow & \mathcal{B}_0 \\ \| & & \downarrow\psi_q & & \downarrow\varphi_0 \\ X_q & \hookrightarrow & \widetilde{Y}_q & \twoheadrightarrow & \widetilde{\mathcal{B}}_0 \end{array}$$

This shall be a $\hat{c}$-morphism with respect to the morphisms $(\hat{c}_q, \hat{c}_{Y,q}, 0)$ and $(\hat{c}_q, \hat{c}_{\widetilde{Y},q}, 0)$ to the trivial deformation of space germs $\mathbb{C}_0 \hookrightarrow \mathbb{C}_0 \twoheadrightarrow \{0\}$. This is equivalent to $\psi_q^* \hat{c}_{\widetilde{Y},q} = \hat{c}_{Y,q}$.

Definition C.10. ([Carberry and Schmidt, 2017, Definition 3.5]) Let X have properties (A1), (B1), (D1) and optionally (E1). A *morphism from the left hand side to the right hand side of* (C.6) is a morphism (C.7) with $\psi_\pm^* z_{\tilde{Y},\pm} = z_{Y,\pm}$ for $Q^\pm$ and $\psi_q^* \hat{c}_{\tilde{Y},q} = \hat{c}_{Y,q}$ for $q \in X^\circ$ and such that σ commutes with ψ and leaves φ invariant. If in addition (E1) and (E1') hold, then τ_2 commutes with ψ and φ.

Note that for $p \in X$, one has $(\psi^* d\hat{c}_{\tilde{Y}}) = d(\hat{c}_{\tilde{Y}} \circ \psi) = d\hat{c}_Y$. We are interested in the isomorphism classes of the deformations in (C.4). Since the marked points $Q^\pm$ are smooth, the space germs of Y at these points are isomorphic to the space germs of $(\mathbb{C} \times \mathcal{B})_{(0,0)}$. Furthermore, at smooth points $q \in X$ at which $d\hat{c}_q = d(\hat{c}_q)$ does not vanish, $\hat{c}_q$ maps X_q biregularly onto $\mathbb{C}_0$ and $(\hat{c}_{Y,q}, b)$ maps Y_q biregularly onto $(\mathbb{C} \times \mathcal{B})_{(0,0)}$. From this, it is visible that any deformation should be locally trivial in the complement of some open neighborhoods of the points $q_1, \ldots, q_L$ of X which are either singularities or roots of $d\hat{c}$. In the sequel, let $F_l = F_{q_l}$ and the space germs in a neighborhood of q_l on Y is denoted as $Y_l = Y_{q_l}$. These space germs are biregular to the zero sets $X_l \simeq V(f_l)$ of $f_l \in \mathbb{C}\{\hat{c}, \check{c}\}$ and $Y_l \simeq V(F_l)$ of $F_l \in \mathbb{C}\{\hat{c}, \check{c}\} \hat{\otimes} \mathcal{O}_\mathcal{B}$. Hereby, the tie-in between the deformed and undeformed space germs is given by $F_l(\hat{c}, \check{c}, 0) = H_l(\hat{c}, \check{c}) f_l(\hat{c}, \check{c})$, where $H_l \in \mathbb{C}\{\hat{c}, \check{c}\}$ is a unit. The next Lemma shows that the isomorphism classes of deformations obeying (A1) and (B1), i.e. which provide the periods of one differential form $d\hat{c}$, are in one-to-one correspondence with the $\hat{c}$-isomorphism classes of the deformations of space germs $V(f_l) \hookrightarrow V(F_l) \twoheadrightarrow \mathcal{B}_0$ with $l \in \{1, \ldots, L\}$. This means that it is indeed sufficient to consider local deformations in the neighborhood of points which are either non smooth or the points at which $d\hat{c}$ has a root. Therefore, the number of connected components of the special fiber is preserved under the $\hat{c}$-deformations.

Lemma C.11 ([Carberry and Schmidt, 2017, Lemma 3.6])**.** *Let $(X, Q^+, Q^-, d\hat{c}, \sigma)$ be locally planar with prescribed poles and denote the points at which $\hat{c}$ is not a local coordinate by $q_1, \ldots, q_L \in X$. Moreover, suppose that for each $l = 1, \ldots, L$, there are given deformations $(X_l, \hat{c}_l, \tilde{\check{c}}_l) \hookrightarrow (\tilde{Y}_l, \hat{c}_{\tilde{Y},l}, \tilde{\check{c}}_{\tilde{Y},l}) \twoheadrightarrow \mathcal{B}_0$. Then there exists a local deformation $(X, d\hat{c}) \hookrightarrow (Y, d\hat{c}_Y) \twoheadrightarrow \mathcal{B}_0$, unique up to isomorphism, such that for each $l = 1, \ldots, L$, the induced deformation $(X_l, \hat{c}_l, \check{c}_l) \hookrightarrow (Y_l, \hat{c}_{Y,l}, \check{c}_{Y,l}) \hookrightarrow \mathcal{B}_0$ is $\hat{c}$-isomorphic to the given one.*

So the meaning of this lemma is two-fold: firstly, it is shown how to tinker a $\hat{c}$-deformation of the whole space X out of the local $\hat{c}$-deformations of the singularities of X. Secondly, one sees that the deformation of the whole space restricted to small neighborhoods of the singularities is isomorphic to the local $\hat{c}$-deformation.
The proof of this lemma is mainly based on the application of the Weierstraß Preparation Theorem [de Jong and Pfister, 2012, Theorem] in the open neighborhoods O_l of q_l, respectively $U_l \subset \mathbb{C}^2$

parametrized by $(\hat{c}, \check{c})$ on the non-deformed space X_l as in (C.2) and open neighborhoods of $(0,0) \in \mathbb{C}^2 \times \mathcal{B}$ parametrized by $(\hat{c}, \check{c}, b) \in \mathbb{C}^2 \times \mathcal{B}$ and describing Y_l locally for $l = 1, \ldots, L$. These are chosen in such a way that the branch points of the corresponding Weierstraß coverings are contained in small open balls inside of these neighborhoods and such that there is always is an annulus contained in these neighborhoods on which the Weierstraß coverings are unbranched. This and the flatness of the map from the total space to the base space is used to 'glue' the space Y together.

Corollary C.12 ([Carberry and Schmidt, 2017, Parts of Section 6]). *Let X be locally planar with prescribed poles and endowed with σ. Moreover, let $q_1, \ldots, q_L \in X$ be the points at which $\hat{c}$ is not a local coordinate and let $F_1, \ldots, F_L \in \mathbb{C}\{\hat{c}\}[\check{c}] \hat{\otimes} \mathcal{O}_{\mathcal{B}_0}$ with $F_l(\hat{c}, \check{c}, 0) = f_l(\hat{c}, \check{c})$ be the unique Weierstraß polynomial describing Y at q_l for all $l = 1, \ldots, L$. For fixed $b \in \mathcal{B}$, F_l obeys $\sigma^* F_l = (-1)^{d_l} F_{\sigma l}$ with $d_l = \deg_{\check{c}} F_l$ and Y is endowed with an involution σ.*

The next step in [Carberry and Schmidt, 2017] is the determination of the isomorphism classes of infinitesimal $\hat{c}$-deformations, compare [Greuel et al., 2007, Section II.1.4]. For the isomorphism classes of the $\hat{c}$-deformations, the following Lemma holds:

Lemma C.13 ([Carberry and Schmidt, 2017, Lemma 3.8]). *The $\hat{c}$-isomorphism classes of infinitesimal deformations of a space germ $f \in \mathbb{C}\{\hat{c}, \check{c}\}$ with $f(0,0) = 0$ are isomorphic to the elements of*

$$\mathbb{C}\{\hat{c}, \check{c}\} \Big/ \left\langle f, \frac{\partial f}{\partial \check{c}} \right\rangle . \tag{C.8}$$

In [Carberry and Schmidt, 2017], it is shown that the space in (C.8) is finite-dimensional for all $l = 1, \ldots, L$. Hereby, it is used that the Weierstraß Preparation Theorem [de Jong and Pfister, 2012, Theorem 3.2.4] yields that $\mathbb{C}\{\hat{c}, \check{c}\}/\langle f_l, \frac{\partial f_l}{\partial \check{c}} \rangle \Big|_{U_l} \simeq \mathcal{O}_{U_l} / \frac{\partial f_l}{\partial \check{c}} \mathcal{O}_{U_l}$ and that this is a coherent sheaf with finite support. Then the finite dimensionality follows due to the Noether Normalization [de Jong and Pfister, 2012, Corollary 3.3.19].

For given X obeying (A1), (B1), (D1) and optionally (E1), the next step is the construction of a particular deformation $(X, d\hat{c}) \hookrightarrow (Z, d\hat{c}_Z) \twoheadrightarrow \mathcal{T}$. Hereinafter, it will become visible that this is a universal deformation, see [Carberry and Schmidt, 2017, Theorem 3.10] with the modifications due to (D1) proposed in [Carberry and Schmidt, 2017, Section 6]. For each $l = 1, \ldots, L$, let $g_l := (g_{l,1}, \ldots, g_{l,r})$ be tuples of polynomials in $\mathbb{C}[\check{c}] \hat{\otimes} \mathcal{O}_{B_{\delta_1}(0)}$ with respect to $\check{c}$ of degree less than d_l which induce a basis of $\mathbb{C}\{\hat{c}\}[\check{c}]/\langle f_l, \frac{\partial f_l}{\partial \check{c}} \rangle \simeq \mathbb{C}[\check{c}] \hat{\otimes} \mathcal{O}_{B_{\delta_1}(0)}/\langle f_l, \frac{\partial f_l}{\partial \check{c}} \rangle$. The last isomorphy holds because in a small open neighborhood $B_{\delta_1}(0)$ of $\hat{c}$ with $\delta_1 > 0$, the coefficients of the Weierstraß polynomials $f_l \in \mathbb{C}\{\hat{c}\}[\check{c}]$ are holomorphic. Thus, the Weierstraß polynomials $f_1, \ldots, f_L$ belong to $\mathbb{C}[\check{c}] \hat{\otimes} \mathcal{O}_{B_{\delta_1}(0)}$. Due to the action of σ on f_l in (C.3), one has $\sigma^* \frac{\partial f_l}{\partial \check{c}} = (-1)^{d_l - 1} \frac{\partial f_l}{\partial \check{c}}$ since the derivative of a σ-invariant function is also invariant under σ and $\deg_{\check{c}} \left(\frac{\partial f_l}{\partial \check{c}} \right) = \deg_{\check{c}}(f_l) - 1$. Corollary C.12 yields that the deformation behavior at $\sigma(q_l)$ is determined by the deformation behavior at q_l. Therefore, also the quotient spaces determining the isomorphy classes in (C.8) have the same dimension at both of these points. Since $g_{l,1},\dots,g_{l,r}$ are polynomials and σ is an involution, the same argumentation as in the proof of Lemma C.8 yields that every basis $(g_{l,1},\dots,g_{l,r})$ can be decomposed into a direct sum of a symmetric and an antisymmetric part with respect to σ, i.e. into a part on which σ acts as $\sigma^* g_{l,i} = g_{\sigma l,i}$ and into a part on which σ acts as $\sigma^* g_{l,i} = -g_{\sigma l,i}$. To ensure that the deformed spaces $X(t)$ have the same transformation behavior under σ as X for all $t \in \mathcal{T}$, it is necessary to impose that

$$\sigma^* g_l = (-1)^{d_l} g_{\sigma l},$$

and therefore $(g_{l,1},\dots,g_{l,r})$ forms a basis of the symmetric part of $\mathbb{C}\{\hat{c},\check{c}\}/\langle f_l, \frac{\partial f_l}{\partial \check{c}}\rangle$ for d_l even and a basis of the antisymmetric part for d_l odd. This choice defines the infinitesimal $\hat{c}$-deformations of X endowed with σ, i.e. on Y acts a holomorphic involution σ which leaves $\mathcal{T}$ invariant. The number of generators r depends on l. This basis defines a $\hat{c}$-deformation on small open neighborhoods of the singularities and branch points

$$U_l(t_l) = \{(\hat{c},\check{c}) \in B_{\delta_1}(0) \times \mathbb{C} \mid G_l(\hat{c},\check{c},t_l) = 0\} \text{ with}$$
$$G_l(\hat{c},\check{c},t_l) = f_l(\hat{c},\check{c}) + t_{l,1} g_{l,1}(\hat{c},\check{c}) + \cdots + t_{l,r} g_{l,r}(\hat{c},\check{c}) =: f_l(\hat{c},\check{c}) + t_l \cdot g_l(\hat{c},\check{c})$$

of the complex analytic space germs X_l. Due to the choice of g_l, one has that $\sigma^* G_l = (-1)^{d_l} G_{\sigma l}$. One chooses δ_1 sufficiently small and small open balls $\mathcal{T}_l \subset \mathbb{C}^r$ such that the roots of the discriminant of the polynomial $f_l + t_l \cdot g_l$ with respect to $\check{c}$ belong to $\hat{c} \in B_{\delta_1/2}(0)$ for all $t_l \in \mathcal{T}_l$. With

$$Z_l = \{(\hat{c},\check{c},t_l) \in B_{\delta_1}(0) \times \mathbb{C} \times \mathcal{T}_l \mid G_l(\hat{c},\check{c},t_l) = 0\}, \tag{C.9}$$

this yields that the Weierstraß coverings $Z_l \to B_{\delta_1}(0) \times \mathcal{T}_l$ have the fixed number of d_l sheets over $(\hat{c},t_l) \in B_{\delta_1}(0) \times \mathcal{T}_l$ and are unbranched over $\left(B_{\delta_1}(0) \setminus \overline{B_{\delta_1/2}(0)}\right) \times \mathcal{T}_l$. We want to give insight into the procedure how this yields a deformation of the whole space. So as in [Carberry and Schmidt, 2017, Proof of Lemma 3.6], let

$$A_l := \{(\hat{c},\check{c}) \in U_l \mid \hat{c} \in \overline{B_{\delta_1/2}(0)}\} \quad \text{and} \quad B_l := \{(\hat{c},\check{c},t_l) \in Z_l \mid \hat{c} \in \overline{B_{\delta_1/2}(0)}\}.$$

Then for each $l = 1,\dots,L$, the spaces $(U_l \setminus A_l) \times \mathcal{T}_l$ and $Z_l \setminus B_l$ are unbranched Weierstraß coverings with an equal number of sheets over $\hat{c} \in B_{\delta_1}(0) \setminus \overline{B_{\delta_1/2}(0)})$ respectively $(\hat{c},t_l) \in (B_{\delta_1}(0) \setminus \overline{B_{\delta_1/2}(0)}) \times \mathcal{T}_l$. Therefore, $\check{c}$ is a holomorphic function on $U_l \setminus A_l$ which depends on $\hat{c} \in (B_{\delta_1}(0) \setminus \overline{B_{\delta_1/2}(0)})$ and on $Z_l \setminus B_l$, $\check{c}$ is also a holomorphic function depending on $(\hat{c},t_l) \in (B_{\delta_1}(0) \setminus \overline{B_{\delta_1/2}(0)}) \times \mathcal{T}_l$. The first function is equal to the evaluation of the second function at $t_l = 0$. Moreover, the first function has a unique global extension to all simply connected open subsets of $\hat{c} \in (B_{\delta_1}(0) \setminus \overline{B_{\delta_1/2}(0)})$ and the second function to all simply connected open subsets of $(\hat{c},t_l) \in (B_{\delta_1}(0) \setminus \overline{B\delta_1/2(0)}) \times \mathcal{T}_l$. Let

W_l be an open subset of $U_l \setminus A_l$, which is mapped by $\hat{c}$ onto a simply connected open subset of $(B_{\delta_1}(0) \setminus \overline{B_{\delta_1/2}(0)})$. The Cartesian product $W_l \times \mathcal{T}_l$ is also simply connected since $\mathcal{T}_l$ is an open ball in $\mathbb{C}^r$. Hence, there exists a unique biholomorphic map from $W_l \times \mathcal{T}_l$ onto a simply connected open subset of $Z_l \setminus B_l$ which preserves $(\hat{c}, t_l)$ and is equal to the identity for $t_l = 0$. First, the the annulus $B_{\delta_1}(0) \setminus \overline{B_{\delta_1/2}(0)}$ is covered by finitely many simply connected open subsets such that the intersection of any two of these open subsets is either empty or connected. Then $U_l \setminus A_l$ is covered by open subsets W_l which are mapped biholomorphically onto the former subsets by $\hat{c}$. Hereby, each $W_l \times \mathcal{T}_l$ is mapped to an open subset of $Z_l \setminus B_l$ as described above. Since $Z_l \setminus B_l$ is unbranched over $(\hat{c}, t_l) \in (B_{\delta_1}(0) \setminus \overline{B_{\delta_1/2}(0)}) \times \mathcal{T}_l$, the corresponding biholomorphic maps fit together to form a unique biholomorphic map $\psi_l : (U_l \setminus A_l) \times \mathcal{T}_l \to Z_l \setminus B_l$ preserving $(\hat{c}, t_l)$. For each $l = 1, \dots, L$, let C_l denote the preimage of A_l with respect to the map in (C.2). With $\mathcal{T} := \mathcal{T}_1 \times \dots \times \mathcal{T}_l$, the open subsets $(O_l \setminus C_l) \times \mathcal{T}$ of $X \setminus (C_1 \cup \dots \cup C_L) \times \mathcal{T}$ are mapped biregularly to $Z_1 \cup \dots \cup Z_L$ by $\psi_l \circ (\hat{c}, \check{c}, t_l)$. Now, $X \setminus (C_1 \cup \dots \cup C_L) \times \mathcal{T}$ and $Z_1 \cup \dots \cup Z_L$ are glued together along the maps $\psi_l \circ (\hat{c}, \check{c}, t_l)$ for $l = 1, \dots, L$. This deformation obeys the conditions in Lemma C.11 and one obtains the particular deformation

$$(X, d\hat{c}) \hookrightarrow (Z, d\hat{c}_Z) \twoheadrightarrow \mathcal{T}. \tag{C.10}$$

If in addition condition (E1) is imposed, then the maps $(\hat{c}, \check{c}) : X_l \to \mathbb{C}^2$ obey (C.1). We define an action $l \mapsto \tau_2 l$ of τ_2 on $f_1, \dots, f_L$ such that $\tau_2(q_l) = q_{\tau_2 l}$. Due to (C.1), the unique Weierstraß polynomials $f_1, \dots, f_L$ obey $f_{\tau_2 l}(\hat{c}, \check{c}) = \bar{f}_l(\bar{\hat{c}}, \bar{\check{c}})$. The basis $g_1, \dots, g_L$ is chosen in such a way that its generators also obey $g_{\tau_2 l}(\hat{c}, \check{c}) = \bar{g}_l(\bar{\hat{c}}, \bar{\check{c}})$. In particular, for $\tau_2 l = l$, the coefficients of g_l take purely imaginary values for real $\hat{c}_l \in B_{\delta_1}(0)$. Consequently, the involution τ_2 extends to a global antiholomorphic involution of (C.10) with $\tau_2(t_l) = \bar{t}_{\tau_2 l}$ which is also denoted by τ_2. The following Lemma together with Lemmata C.11 and 3.6 yields that the space of isomorphism classes of infinitesimal deformations of the fibers is a vector bundle over $\mathcal{T}$. Hereby, the next Lemma shows that the space of infinitesimal deformations of the fibers of (C.10) has a smoothly varying basis on $\mathcal{T}$:

Lemma C.14 ([Carberry and Schmidt, 2017, Lemma 3.10 and parts of Section 6])**.** *For sufficiently small $t_l \in \mathcal{T}_l$, the elements of g_l form a basis of*

$$\mathcal{O}_{U_l(t_l)} \Big/ \frac{\partial G_l}{\partial \check{c}} \mathcal{O}_{U_l(t_l)}. \tag{C.11}$$

Considering the symmetric and antisymmetric part of (C.8) *and* (C.11) *with respect to σ, the following holds: For* $\deg_{\check{c}} G_l$ *even, the elements of the symmetric part of* (C.8) *form a basis of the symmetric part of* (C.11) *and for* $\deg_{\check{c}} G_l$ *odd, the elements of the antisymmetric part of* (C.8) *form a basis of the antisymmetric part of* (C.11)*.*

The dimension of the space $\mathbb{C}\{\hat{c}, \check{c}\}/\langle f, \partial f/\partial \check{c} \rangle$ counts the number of zeros of $\partial f/\partial \check{c}$ on $V(f)$. So for smooth points $q \in X \setminus \{q_1, \dots, q_L\}$, the dimension of $\mathbb{C}\{\hat{c}, \check{c}\}/\langle f, \partial f/\partial \check{c} \rangle$ in Lemma C.13 can be interpreted as the zero order of $d\hat{c}$ over X which equals the branching order of the covering $X \to \mathbb{C}$, $(\hat{c}, \check{c}) \mapsto \hat{c}$. Because the form $(\partial f/\partial\check{c})^{-1}d\hat{c}$ is regular, the zero order of $\partial f/\partial\check{c}$ on $V(f)$ equals the zero order of $d\hat{c}$ on $V(f)$. Let us assume that $\dim(\mathbb{C}\{\hat{c}, \check{c}\}/\langle f, \partial f/\partial\check{c}\rangle)_q = 1$. Then $d\hat{c}$ has a simple zero at q. In this case, deforming a branch point by the $\hat{c}$-deformations changes the $\hat{c}$-coordinate of this branch point. If the dimension of this space is higher than one, interpreting a singularity as several branch points which coincide, these branch points are generically deformed into different directions. Since the dimension of (C.11) is locally constant, the local number of branch points stays the same in each fiber. For generic $t \in \mathcal{T}$, the zero set of f_l is deformed in such a way that all branch points are zeros of order one of $d\hat{c}$. The essential ingredient to show that the $\hat{c}$-deformation in (C.10) is universal is that the dimension of the space of infinitesimal $\hat{c}$-deformations is constant along $\mathcal{T}$. Another important ingredient is [Carberry and Schmidt, 2017, Lemma 3.11]. To formulate this, let H denote the Banach space of bounded holomorphic functions on $(\hat{c}, \check{c}) \in B_{\delta_1}(0) \times B_{\delta_2}(0)$ for positive $\delta = (\delta_1, \delta_2)$ with the uniform norm $\|\cdot\|_\infty$. Hereby, δ_2 is chosen in such a way that the corresponding set U_l is a subset of $B_{\delta_1}(0) \times B_{\delta_2}(0)$ for each $1, \ldots, L$. Furthermore, the deformation spaces $\mathcal{T}_l$ are chosen sufficiently small such that Z_l in (C.9) is contained in $B_{\delta_1}(0) \times B_{\delta_2}(0) \times \mathcal{T}_l$.

Lemma C.15 ([Carberry and Schmidt, 2017, Lemma 3.11])**.** *For $l \in \{1, \ldots, L\}$, let $t_l \in \mathcal{T}_l$ be small and let $u_l \in H$ be a polynomial with respect to $\check{c}$ of degree less than d_l such that $\|u_l - \check{c}\|_\infty$ is small. Moreover, let G_l be defined as in* (C.3)*. Then all $h_l \in H$ have a unique decomposition into triples $(\mathfrak{a}_l, \mathfrak{b}_l, \mathfrak{c}_l)$ with $\mathfrak{a}_l \in H$, $\mathfrak{b}_l \in H$ a polynomial of degree less than d_l with respect to $\check{c}$ and $\mathfrak{c}_l \in \mathbb{C}^r$ such that $h_l(\hat{c}, \check{c})$ is for all $(\hat{c}, \check{c}) \in B_{\delta_1}(0) \times B_{\delta_2}(0)$ equal to*

$$h_l(\hat{c}, \check{c}) = \mathfrak{a}_l(\hat{c}, \check{c})G_l(\hat{c}, u_l(\hat{c}, \check{c}), t_l) + \mathfrak{b}_l(\hat{c}, \check{c})\frac{\partial G_l}{\partial u_l}(\hat{c}, u_l(\hat{c}, \check{c}), t_l) + \mathfrak{c}_l \cdot g_l(\hat{c}, u_l(\hat{c}, \check{c})).$$

Furthermore $(\mathfrak{a}_l, \mathfrak{b}_l, \mathfrak{c}_l)$ depends holomorphically on $(h_l, u_l, t_l) \in H \times H \times \mathcal{T}_l$. If in addition $\sigma^* u_l = -u_{\sigma l}$ and $\sigma^* h_l = (-1)^{d_l} h_{\sigma l}$ hold, then the corresponding triples $(\mathfrak{a}_l, \mathfrak{b}_l, \mathfrak{c}_l)$ and $(\mathfrak{a}_{\sigma l}, \mathfrak{b}_{\sigma l}, \mathfrak{c}_{\sigma l})$ transform under σ as

$$\sigma^*\mathfrak{a}_l = \mathfrak{a}_{\sigma l}, \quad \sigma^*\mathfrak{b}_l = -\mathfrak{b}_{\sigma l}, \quad \sigma^*\mathfrak{c}_l = \mathfrak{c}_{\sigma l}. \tag{C.12}$$

In the proof of this lemma, the assumption that $\|u_l - \check{c}\|_\infty$ is small is essential since it allows the application of Banach's Fixed Point Theorem to $v_l \mapsto \check{c} - (u_l(\hat{c}, v_l) - v_l)$ to construct a biholomorphic map $(\hat{c}, \check{c}) \mapsto (\hat{c}, u_l(\hat{c}, \check{c}))$ with inverse map $(\hat{c}, \check{c}) \mapsto (\hat{c}, v_l(\hat{c}, \check{c}))$. So one can determine the undeformed $\check{c}$ from the deformation value $u_l(\hat{c}, \check{c})$ in the second component and vice versa. Let furthermore u_l and h_l be chosen such that $\sigma^* u_l = -u_{\sigma l}$ and $\sigma^* h_l = (-1)^{d_l} h_{\sigma l}$ holds. Since σ does not change the degree of any of the occurring polynomials in $\check{c}$ in (C.12), the action of σ on the local coordinates and on G_l as well as g_l implies that

$$\sigma^*(\mathfrak{a}_l(\hat{c},\check{c})G_l(\hat{c},u_l(\hat{c},\check{c}),t_l) + \mathfrak{b}_l(\hat{c},\check{c})\frac{\partial G_l}{\partial u_l}(\hat{c},u_l(\hat{c},\check{c}),t_l) + \mathfrak{c}_l \cdot g_l(\hat{c},u_l(\hat{c},\check{c}))) =$$
$$= \mathfrak{a}_{\sigma l}(\hat{c},\check{c})G_{\sigma l}(\hat{c},u_{\sigma l}(\hat{c},\check{c}),t_l) + \mathfrak{b}_{\sigma l}(\hat{c},\check{c})\frac{\partial G_{\sigma l}}{\partial u_l}(\hat{c},u_{\sigma l}(\hat{c},\check{c}),t_l) + \mathfrak{c}_l \cdot g_{\sigma l}(\hat{c},u_{\sigma l}(\hat{c},\check{c})).$$

Taking the transformation behavior of h_l, G_l and g_l under σ into account and comparing coefficients yields that (C.12) holds. This decomposition is used to show in [Carberry and Schmidt, 2017, Theorem 3.12] that (C.10) defines a universal $\hat{c}$-deformation:

Theorem C.16 ([Carberry and Schmidt, 2017, Theorem 3.12]). *Let $(X, Q^+, Q^-, d\hat{c}, \sigma)$ obey (A1), (B1) and (D1) with a given deformation* (C.4). *Then, after reducing the base space $\mathcal{B}$, there exists a unique holomorphic map $\varphi : \mathcal{B} \to \mathcal{T}$ such that the pullback of* (C.10) *via φ is isomorphic to* (C.4).

In the proof of this theorem, one uses that every 1-deformation is described by the local deformations around $q_1, \dots, q_L$. For an arbitrary 1-deformation $(X, d\hat{c}) \hookrightarrow (Y, d\hat{c}_Y) \twoheadrightarrow \mathcal{B}$, a map $\mathcal{B} \to \mathcal{T}$ is defined, where $\mathcal{T}$ is the candidate for the base space of the universal deformation as introduced above. A morphism between these two deformations is given by $H_l(\hat{c},\check{c},b)G_l(\hat{c},u_l(\hat{c},\check{c}),\varphi_l(b)) = F_l(\hat{c},\check{c},b)$. The trick in this proof is to apply the decomposition from Lemma C.15 to $H_l^{-1}(\hat{c},\check{c},b)\frac{\partial F_l}{\partial b}(\hat{c},\check{c},b)$. This yields a system of ordinary differential equations

$$\frac{\partial H_l}{\partial b}(\hat{c},\check{c},b) = \mathfrak{a}_l(\hat{c},\check{c})H_l(\hat{c},\check{c},b),$$
$$\frac{\partial u_l}{\partial b}(\hat{c},\check{c},b) = \mathfrak{b}_l(\hat{c},\check{c}),$$
$$\frac{\partial \varphi_l}{\partial b}(b) = \mathfrak{c}_l$$

with known start values $H_l(\hat{c},\check{c},0) = 1$, $u_l(\hat{c},\check{c},0) = \check{c}$ and $\varphi(0) = 0$. One can show that for sufficiently small base-space, all requirements for the the Picard-Lindelöf Theorem [Azad and Jost, 2013, Theorem 6.16] are fulfilled which leads to local solutions of the above differential equation. One can show that this defines the map $\phi : \mathcal{B} \to \mathcal{T}$ and that the pull back of $X \hookrightarrow Z \twoheadrightarrow \mathcal{T}$ under ϕ is isomorphic to $X \hookrightarrow Y \twoheadrightarrow \mathcal{B}$. It is explained in [Carberry and Schmidt, 2017, end of Section 3] that if condition (E1) is assumed to hold on the special fiber of the deformation (C.15) in Theorem C.16, then the assumption (E1') on the deformation space ensures that the involution τ_2 extends to involutions of (C.15) and (C.10) which are both also denoted by τ_2. In this case, Theorem C.16 implies that the morphism φ commutes with τ_2 and that φ maps the fixed point set of τ_2 in $\mathcal{B}$ to the fixed point set of τ_2 in $\mathcal{T}$. In particular, the restriction of (C.10) to the fixed point set of τ_2 in $\mathcal{T}$ is a universal deformation of real Fermi curve data $(X, Q^+, Q^-, d\hat{c}, \sigma)$.

C.4. Deformations of Curves with two differentials

The deformations $(X, Q^+, Q^-, d\hat{c}, d\check{c}, \sigma)$ obeying (A) to (D) are subfamilies of the $\hat{c}$-deformations. We will see in this section that it is possible to construct a universal deformation for the deformations which preserve the periods of both differentials $d\hat{c}$ and $d\check{c}$ as a subfamily of (C.10). Therefore, at first conditions that shall hold for the deformations preserving both differentials which are similar to the conditions (A1), (B1), (D1) and (E1) in Definition C.7 are formulated in [Carberry and Schmidt, 2017, Beginning of Section 5] as follows:
Let X be a compact one-dimensional complex analytic space together with two smooth points Q^+ and Q^-, let $\sigma : X \to X$ be a holomorphic involution and let $d\hat{c}$ and $d\check{c}$ be two meromorphic differential on X. We impose that $(X, Q^+, Q^-, d\hat{c}, d\check{c}, \sigma)$ obey the following properties:

(A2) For Q^+ and Q^-, there exist $z_\pm, w_\pm \in \mathcal{O}_{X,Q^\pm}$ which vanish at $Q^\pm$ and map $X_{Q^\pm}$ biregularly onto $\mathbb{C}_0$ such that $(d\hat{c})_{Q^\pm} = d(z_\pm^{-1})$ and $(d\check{c})_{Q^\pm} = d(w_\pm^{-1})$.

(B2) For each $q \in X^\circ$, there exist $(\hat{c}, \check{c}) \in \mathcal{O}_{X,q} \times \mathcal{O}_{X,q}$ which vanish at q and map X_q biregularly onto the zero set of some $f_q \in \mathbb{C}\{\hat{c}, \check{c}\}$ such that $(d\hat{c})_q = d(\hat{c}_q)$ and $(d\check{c})_q = d(\check{c}_q)$.

(D2) On X, there exists a holomorphic involution σ with $\sigma(Q^\pm) = Q^\pm$. The local functions $(\hat{c}_q, \check{c}_q)$ in (B2) and the differential forms $d\hat{c}$ and $d\check{c}$ transform as

$$\sigma^* d\hat{c} = -d\hat{c}, \quad \sigma^* d\check{c} = -d\check{c}, \quad \sigma^* \hat{c}_q = -\hat{c}_{\sigma(q)}, \quad \sigma^* \check{c}_q = -\check{c}_{\sigma(q)}.$$

(E2) On X, there exists an antiholomorphic involution τ_2 which acts as (C.1) and additionally $w_{Q^\pm} = \tau_2^* \bar{w}_{Q^\mp}$.

In analogy to the $\hat{c}$-deformations in Definition C.9, the deformations preserving two differentials are defined in [Carberry and Schmidt, 2017, Section 5] as follows:
For data $(X, Q^+, Q^-, d\hat{c}, d\check{c}, \sigma)$ obeying (A2), (B2), (D2) – and additionally (E2) in case of real data – the corresponding deformations are defined as

$$(X, d\hat{c}, d\check{c}) \hookrightarrow (Y, d\hat{c}_Y, d\check{c}_Y) \twoheadrightarrow \mathcal{B}. \tag{C.13}$$

This is a complex analytic space Y with two meromorphic differentials $d\hat{c}_Y$ and $d\check{c}_Y$ and base space $\mathcal{B}$ together with a deformation $X \hookrightarrow Y \twoheadrightarrow \mathcal{B}$ of complex analytic spaces obeying the following conditions:

(A2') For Q^+ and Q^-, $z_\pm$ and $w_\pm$ in (A2) extend to $z_{Y,\pm}, w_{Y,\pm} \in \mathcal{O}_{Y,Q^\pm}$ such that $(z_{Y,\pm}, b)$ and $(w_{Y,\pm}, b)$ map $Y_{Q^\pm}$ biregularly onto $(\mathbb{C} \times \mathcal{B})_{(0,0)}$ with $d\hat{c}_{Y,\pm} = d(z_{Y,\pm}^{-1})$ and $d\check{c}_{Y,Q^\pm} = d(w_{Y,\pm}^{-1})$.

(B2') For each $q \in X^\circ$, $\hat{c}_q$ and $\check{c}_q$ extend to $\hat{c}_{Y,q}, \check{c}_{Y,q} \in \mathcal{O}_{Y,q}$ such that $(\hat{c}_{Y,q}, \check{c}_{Y,q}, b)$ maps Y_q biregularly onto $V(F_q)$ with $F_q \in \mathbb{C}\{\hat{c}, \check{c}\} \hat{\otimes} \mathcal{O}_\mathcal{B}$ and $d\hat{c}_{Y,q} = d(\hat{c}_{Y,q})$ and $d\check{c}_{Y,q} = d(\check{c}_{Y,q})$.

(D2') The involution σ extends to a holomorphic involution on Y which acts as

$$\sigma^* d\hat{c}_Y = -d\check{c}_Y, \quad \sigma^* \, dy_Y = -d\check{c}_Y, \quad \sigma^* \hat{c}_{Y,q} = -\hat{c}_{Y,\sigma(q)}, \quad \sigma^* \check{c}_{Y,q} = -\check{c}_{Y,\sigma(q)}.$$

It commutes with the maps $X \hookrightarrow Y \twoheadrightarrow \mathcal{B}$ and acts trivially on $\mathcal{B}$.

For real curves X, the following condition holds additionally:

(E2') If in addition (E2) is assumed, τ_2 extends to an antiholomorphic involution of Y which acts as (C.5) and $w_{Y,\pm} = \tau_2^* \bar{w}_{Y,\mp}$. This involution commutes with the maps $X \hookrightarrow Y \twoheadrightarrow \mathcal{B}$.

Finally, the morphisms of these deformations are defined analogous to Definition C.10.

Definition C.17 ([Carberry and Schmidt, 2017, Definition 5.2])**.** Let X have properties (A2), (B2), (D2) and optionally (E2). A morphism φ from the left hand side to the right hand side of

$$(X, d\hat{c}, d\check{c}) \hookrightarrow (Y, d\hat{c}_Y, d\check{c}_Y) \twoheadrightarrow \mathcal{B}, \qquad (X, d\hat{c}, d\check{c}) \hookrightarrow (\widetilde{Y}, d\hat{c}_{\widetilde{Y}}, d\check{c}_{\widetilde{Y}}) \twoheadrightarrow \widetilde{\mathcal{B}}$$

is a morphism (C.7) with $\tilde{z}_{\widetilde{Y},\pm} \circ \psi_{Q^\pm} = z_{Y,\pm}$ as well as $\tilde{w}_{\widetilde{Y},\pm} \circ \psi_{Q^\pm} = w_{Y,\pm}$ and $\hat{c}_{\widetilde{Y},q} \circ \psi_q = \hat{c}_{Y,q}$ as well as $\check{c}_{\widetilde{Y},q} \circ \psi_q = \check{c}_{Y,q}$ for $q \in X^\circ$. The involution σ commutes with ψ and leaves φ invariant. If in addition (E2) and (E2') hold, then τ_2 commutes with ψ and φ.

To characterize the subfamily of the universal deformations in (C.13) on which $d\check{c}$ extends to a global meromorphic 1-form obeying (A2), (B2) and (D2), one starts again with a characterization of the infinitesimal deformations similar to Lemma C.13. Here, we include the proof because we think that it is necessary to understand the proof of the Lemma after that which is not contained in [Carberry and Schmidt, 2017].

Lemma C.18. *([Carberry and Schmidt, 2017, Lemma 5.3 and parts of Section 6]) The space of regular* 1*-forms ω on X° with poles of orders at most* 3 *at Q^+ and Q^- which obey $\sigma^*\omega = \omega$ parametrizes the isomorphism classes of infinitesimal deformations* (C.13)*.*

Proof. Let $F_1(\hat{c}, \check{c}, \varepsilon), \ldots, F_L(\hat{c}, \check{c}, \varepsilon) \in \mathbb{C}\{\hat{c}, \check{c}\} \hat{\otimes} \mathit{\Pi}$ with $\mathit{\Pi}$ as in Definition C.6 and chosen such that $\mathit{\Pi}$ is invariant under σ. These functions describe the infinitesimal $\hat{c}$-deformations of $f_1, \ldots, f_L$ nearby the points $q_1, \ldots, q_L$, i.e. for each $l = 1, \ldots, L$, one has $\sigma^* F_l = (-1)^{d_l} F_{\sigma l}$, where d_l is the degree of f_l and

$$F_l(\hat{c}, \check{c}, \varepsilon) = f_l(\hat{c}, \check{c}) + \varepsilon f_{l,1}(\hat{c}, \check{c}) \ \text{ with } \ f_{l,1} \in \mathbb{C}\{\hat{c}, \check{c}\}.$$

Since σ leaves $\mathit{\Pi}$ invariant, this yields that $\sigma^* f_{l,1} = (-1)^{d_l} f_{\sigma l,1}$. For all $q \in X^\circ \setminus \{q_1, \ldots, q_L\}$, $\check{c}_{Y,q}$ is a holomorphic function of $\check{c}_q$ and ε, i.e.

$$\check{c}_{Y,q} = \check{c}_q + \varepsilon \check{c}_{q,1}$$

with a germ $\check{c}_{q,1}$ of a holomorphic function on X at q. So $\omega = \check{c}_1 d\hat{c}$ is holomorphic on $X^\circ \setminus \{q_1, \ldots, q_L\}$. By comparing the Taylor coefficients up to first order, $F_l(\hat{c}_{Y,q}, \check{c}_{Y,q}, \varepsilon) = 0$ yields that

$$f_l(\hat{c}, \check{c}) = 0 \quad \text{and} \quad \varepsilon \frac{\partial f_l}{\partial \check{c}}(\hat{c}, \check{c})\check{c}_{l,1}(\hat{c}) + \varepsilon f_{l,1}(\hat{c}, \check{c}) = 0.$$

Because $f_{l,1}(\hat{c}, \check{c})$ is holomorphic, Lemma 3.6 gives that the form

$$\omega = \check{c}_{l,1} d\hat{c} = -f_{l,1}(\hat{c}, \check{c}) d\hat{c} \Big/ \frac{\partial f_l}{\partial \check{c}}(\hat{c}, \check{c}) \tag{C.14}$$

is regular at $q_1, \ldots, q_L$. At Q^+ and Q^-, the function $w_{Y,Q^\pm} = w_\pm + \varepsilon w_{\pm,1} = w_\pm(1 + \varepsilon w_\pm^{-1} w_{\pm,1})$ is a holomorphic function of $z_\pm$ and ε with $w_\pm, w_{\pm,1} \in \mathbb{C}\{z_\pm\}$ both vanishing at $z_\pm = 0$. For ε sufficiently small, one has $|\varepsilon w_\pm^{-1} w_{\pm,1}| < 1$ and since $\varepsilon \in \varPi$, it is

$$(1 + \varepsilon w_\pm^{-1} w_{\pm,1})^{-1} = \sum_{k=0}^{\infty} (\varepsilon w_\pm^{-1} w_{\pm,1})^k = 1 + \varepsilon w_\pm^{-1} w_{\pm,1}.$$

Therefore,

$$\check{c}_{Y,Q^\pm} = w_{Y,\pm}^{-1} = w_\pm^{-1} - \varepsilon (w_\pm)^{-2} w_{\pm,1} \quad \text{and} \quad \omega = -w_\pm^{-2} w_{\pm,1}\, d(z_\pm^{-1}).$$

Because $w_\pm$ has a zero of first order at $Q^\pm$, ω has poles of order at most 3 at Q^+ and Q^-. Since ω, defined like this, is holomorphic on all other open neighborhoods which cover $X^\circ \setminus \{q_1, \ldots, q_L\}$, it defines a global meromorphic 1-form on X which is regular on X° with poles of order at most 3 at Q^+ and Q^-. Furthermore, due to

$$\sigma^* d\hat{c}_Y = -d\check{c}_Y, \quad \sigma^*\, dy_Y = -d\check{c}_Y, \quad \sigma^* \hat{c}_{Y,q} = -\hat{c}_{Y,\sigma(q)}, \quad \sigma^* \check{c}_{Y,q} = -\check{c}_{Y,\sigma(q)}$$

it is $\sigma^* \check{c}_{q,1} = -\check{c}_{\sigma(q),1}$ as well as $\sigma^* d\hat{c} = -d\hat{c}$. Therefore, $\sigma^* \omega = \omega$.
Vice versa, since $\hat{c}$ as well as $\check{c}$ are known, a global meromorphic 1-form with these properties defines $\check{c}_{q,1}$ for all $q \in X^\circ$, and therefore an infinitesimal deformation obeying (A2'), (B2') and (D2'). (A2') holds since at Q^+ and Q^-, the pole order of $d\hat{c}$ is given as two and hence the pole order of $\check{c}_q$ is given as one. Since it is known how σ acts on ω, $\check{c}$ and $\hat{c}$, it is also known how σ acts on $\check{c}_{q,1}$ and (D2') holds. □

In case that (E1) holds for the given data $(X, Q^+, Q^-, d\hat{c}, d\check{c}, \sigma)$, the complex dimension of the corresponding vector bundles are halved since the vector bundles can then be obtained by considering the real part of the complex vector bundles of the deformations without an involution τ_2.
In the deformations described above, we can assume that the lattice $\varGamma$ is always normalized in such a way that $\hat{\gamma}$ is not deformed. This means that only all deformations of $\varGamma$ up to rotations and scaling are described by these deformations. However, this is no obstruction since the Fermi curves

corresponding to $c\Gamma$ with $c \neq 0$ and Fermi curves corresponding to $A \cdot \Gamma$, where A is a rotation matrix in $\mathbb{R}^2$, can always be parametrized in such a way that they are Fermi curves corresponding to Γ. It is also possible to determine the space of infinitesimal deformation which leave Γ fixed.

Lemma C.19. *The space of regular* 1*-forms* ω *on* X° *with poles of orders at most* 1 *at* Q^+ *and* Q^- *which obey* $\sigma^*\omega = \omega$ *parametrizes the isomorphism classes of infinitesimal deformations* (C.13) *such that the lattice* Γ *is not deformed up to scaling.*

Proof. Let $\hat{\gamma}, \check{\gamma} \in \Gamma$ be the generators of Γ. It is known from (4.21) that on small open neighborhoods $U_\pm \subset X$ of Q^+ and Q^- containing only smooth points of X, one has due to the action of σ on $\check{c}$ that for a Fermi curve X holds

$$\check{c}|_{U_+} = \frac{\check{\gamma}_1 - \iota\check{\gamma}_2}{z_+} + \sum_{i=0}^{\infty} a_i^+ z_+^{2i-1} \quad \text{and} \quad \check{c}|_{U_-} = \frac{\check{\gamma}_1 + \iota\check{\gamma}_2}{z_-} + \sum_{i=0}^{\infty} a_i^- z_-^{2i-1}.$$

So on Y, the first order deformations of the generators $\hat{\gamma}$ and $\check{\gamma}$ can be represented as $\tilde{\hat{\gamma}} = (\hat{\gamma}_1 + b\hat{\gamma}_{1,1}, \hat{\gamma}_2 + b\hat{\gamma}_{2,1})$, $\tilde{\check{\gamma}} = (\check{\gamma}_1 + b\check{\gamma}_{1,1}, \check{\gamma}_2 + b\check{\gamma}_{2,1})$, where $b \in \Pi$. On small open neighborhoods $U_{Y,\pm} \subset Y$ of $Q^\pm$ containing only smooth points, this yields

$$\check{c}|_{U_{Y,+}} = \frac{\tilde{\check{\gamma}}_1 - \iota\tilde{\check{\gamma}}_2}{z_{Y,+}} + \sum_{i=0}^{\infty} a_i^+(b) z_{Y,+}^{2i-1} = \frac{\check{\gamma}_1 + b\check{\gamma}_{1,1} - \iota(\check{\gamma}_2 + b\check{\gamma}_{2,1})}{z_{Y,+}} + \sum_{i=1}^{\infty} a_i^+(b) z_{Y,+}^{2i-1}$$

as well as

$$\check{c}|_{U_{Y,-}} = \frac{\tilde{\check{\gamma}}_1 + \iota\tilde{\check{\gamma}}_2}{z_{Y,-}} + \sum_{i=0}^{\infty} a_i^-(b) z_{Y,-}^{2i-1} = \frac{\check{\gamma}_1 + b\check{\gamma}_{1,1} + \iota(\check{\gamma}_2 + b\check{\gamma}_{2,1})}{z_{Y,-}} + \sum_{i=0}^{\infty} a_i^-(b) z_{Y,-}^{2i-1}.$$

Therefore,

$$\frac{\partial\check{c}}{\partial b}\bigg|_{U_{Y,+}} = \frac{\check{\gamma}_{1,1} - \iota\check{\gamma}_{2,1}}{z_{Y,+}} + \sum_{i=1}^{\infty} (a_i^+)'(b) z_{Y,+}^{2i-1} \quad \text{and} \quad \frac{\partial\check{c}}{\partial b}\bigg|_{U_{Y,-}} = \frac{\check{\gamma}_{1,1} + \iota\check{\gamma}_{2,1}}{z_{Y,-}} + \sum_{i=0}^{\infty} (a_i^-)'(b) z_{Y,-}^{2i-1}.$$

For deformations which preserve Γ, the first term of $\frac{\partial\check{c}}{\partial b}\Big|_{U_{Y,\pm}}$ on $U_\pm$. In that case $\omega = \check{c}_{Y,1} d\hat{c}$ has poles of at most of first order at Q^+ respectively Q^-. The same arguments as in the proof of Lemma C.18 ([Carberry and Schmidt, 2017, Lemma 5.3 together with the modifications in Section 6]) apply, and so the tangent space of infinitesimal deformations of X which leave Γ invariant up to shrinking and stretching is generated by a vector bundle of degree $\frac{g}{2} + 1$. □

Similar to Lemma C.13, it is shown in [Carberry and Schmidt, 2017, Lemma 5.4] that the isomorphism classes of infinitesimal deformations build a vector bundle over $\mathcal{T}$:

Lemma C.20 ([Carberry and Schmidt, 2017, Lemma 5.4 and parts of Section 6])**.** *If* $Q^\pm$ *are the only fixed points of* σ*, the* σ*-invariant, regular* 1*-forms on* X° *with poles at* Q^+ *and* Q^- *of orders* *at most* 3*, respectively* 1 *for undeformed lattices, build a complex vector bundle* $E \to \mathcal{T}$ *of rank* $\frac{g_a}{2}+3$ *respectively* $\frac{g_a}{2}+1$*, where* g_a *is the arithmetic genus of* X*. In case that (E1) holds for the Fermi curve data* $(X, Q^+, Q^-, d\hat{c}, d\check{c}, \sigma)$*, these ranks are also* $\frac{g_a}{2}+3$ *respectively* $\frac{g_a}{2}+1$ *as real vector bundles.*

Proof. Is is first shown with help of Lemma 3.6 that the regular 1-forms define a holomorphic vector bundle over $\mathcal{T}$ of rank g_a. The proof can be found in [Carberry and Schmidt, 2017]. We only show how to determine the rank of the vector bundle of σ-invariant regular 1-forms. The rank of this holomorphic vector bundle E on $\mathcal{T}$ is determined with help of the two-sheeted covering $\pi_\sigma : X \mapsto X_\sigma$ as in Proposition A.1. The invariance of ω under σ implies by Proposition A.4 that there exists a 1-form ω_σ on X_σ such that $\omega = \pi_\sigma^*\omega_\sigma$. Hence, the rank of the vector bundle describing the σ-invariant 1-forms equals the rank of the corresponding bundle on X_σ. The map π_σ is two-sheeted in a neighborhood of $Q^\pm$ since these points are fixed points of σ, compare Proposition A.4. The proof of Proposition A.1 yields that the local coordinates on X_σ centered at $\pi_\sigma(Q^\pm)$ and the local coordinates on X centered at $Q^\pm$ can be chosen such that the latter ones are the square of the former ones. A 1-form with poles of third order at $Q^\pm$ can be represented as

$$\frac{c}{z^3}\,dz = \frac{c}{z^2}\frac{dz}{z} = \frac{c}{z^2}\frac{dz^2}{2z^2}$$

with some constant $c \in \mathbb{C}$. This shows that the differential ω_σ on X_σ has a pole of second order at $\pi_\sigma(Q^\pm)$ if and only if $\omega = \pi_\sigma^*\omega_\sigma$ has a pole of third order at $Q^\pm$ and is regular on X_σ°. So the rank of this vector bundle equals the dimension of $H^0(X_\sigma, \Omega_{K_\sigma - D_\sigma})$, where K_σ is a canonical divisor of X_σ. As in the proof of Lemma C.14, the degree of the divisor D_σ of ω_σ equals $\deg(D_\sigma) = 2g_{a,\sigma} - 2 + 4 = 2g_\sigma + 2 > 2g_{a,\sigma} - 2$, where $g_{a,\sigma}$ is the arithmetic genus of X_σ. Therefore, $\dim H^1(X_\sigma, \mathcal{O}_{D_\sigma}) = 0$. Due to $H^0(X_\sigma, \Omega_{K_\sigma - D_\sigma}) \simeq H^0(X_\sigma, \mathcal{O}_{D_\sigma})$, the Riemann Roch Theorem [Forster, 1981, § 16.10] yields that

$$\dim H^0(X_\sigma, \Omega_{K_\sigma - D_\sigma}) = 1 - g_{a,\sigma} + \deg D_\sigma = 1 - g_{a,\sigma} + 2g_{a,\sigma} + 2 = g_{a,\sigma} + 3.$$

Since $2g_{a,\sigma} = g_a$, the meromorphic 1-forms on X which are invariant under σ and have poles of third order at Q^+ and Q^- form a holomorphic vector bundle E on $\mathcal{T}$ of rank $\frac{g_a}{2}+3$. The other cases follow analogously. □

As in the case of $\hat{c}$-deformations, it is shown in [Carberry and Schmidt, 2017] how to construct a particular deformation (C.13) from which can be shown that it is a universal deformation. This is again done with help of the infinitesimal deformations. To do so, note that the data $(X, Q^+, Q^-, d\hat{c}, d\check{c}, \sigma)$ obeying (A2), (B2) and (D2) are always also data $(X, Q^+, Q^-, d\hat{c}, \sigma)$ obeying (A1), (B1) and (D1). So let the initial data $(X, Q^+, Q^-, d\hat{c}, \sigma)$ of (C.10) correspond to the given data $(X, Q^+, Q^-, d\hat{c}, d\check{c}, \sigma)$ obeying (A2), (B2) and (D2). Since the deformation (C.10) is a

universal $\hat{c}$-deformation, any deformation (C.13) is endowed with unique morphisms $\varphi : \mathcal{B} \to \mathcal{T}$ and $\psi : Y \to Z$ of the deformation (C.10). Due to Lemma C.11, the maps φ and ψ are completely determined by the germs F_l describing Y_l nearby q_l for $l = 1, \dots, L$. The next Lemma [Carberry and Schmidt, 2017, Lemma 5.5] shows that the section ω defines again an ordinary differential equation for these space germs. By the construction of (C.10), the local coordinates $\hat{c}_{Y,q}$ for $q \in X^\circ$ and $z_{Y,\pm}$ at $Q^\pm$ are also given. To obtain the local coordinates $\check{c}_{Y,q}$ and $w_{y,\pm}$ one can use another differential equation than for the $\hat{c}$-deformations which depends on ω.
We choose a local trivialization of the vector bundle $E \to \mathcal{T}$ on a sufficiently small open neighborhood of $0 \in \mathcal{T}$ by linear independent sections $\omega_1, \dots, \omega_R$. Moreover, by choosing $\mathcal{T}$ small enough, one can assume that these sections trivialize E over $\mathcal{T}$ and that

$$\omega(r) := r_1\omega_1 + \cdots + r_R\omega_R \tag{C.15}$$

is a holomorphic section of $E \to \mathcal{T}$ for every $r \in \mathbb{C}^R$.

Lemma C.21 ([Carberry and Schmidt, 2017, Lemma 5.5 and parts of Section 6])**.** *Let $(X, Q^+, Q^-, d\hat{c}, d\check{c}, \sigma)$ obey (A2),(B2), (D2) and optionally (E2), let $X \hookrightarrow Z \to \mathcal{T}$ be the universal deformation in Theorem C.16 of $(X, Q^+, Q^-, d\hat{c}, \sigma)$ obeying (A1), (B1), (D1) and (E1) if (E2) holds. Every holomorphic function $r = (r_1, \dots, r_R) \in (\mathbb{C}\{b\})^R$ induces a unique deformation* (C.13) *obeying (A2),(B2), (D2) and optionally (E2) with base $\mathcal{B} = B_\varepsilon(0) \subset \mathbb{C}$ such that for all $b \in \mathcal{B}$, the corresponding infinitesimal deformation in Lemma C.18 is given by σ-invariant $\omega(r(b))$ over $\varphi(b)$. Here, $\varphi : \mathcal{B} \to \mathcal{T}$ denotes the map in Theorem C.16.*
Conversely, for any deformation (C.13) *with base $\mathcal{B} = B_\varepsilon(0) \subset \mathbb{C}$, the infinitesimal deformations in Lemma C.18 yield a holomorphic section ω of the pullback of $E \to \mathcal{T}$ with respect to the corresponding map $\varphi : \mathcal{B} \to \mathcal{T}$ in Theorem C.16. This section is equal to $\omega(r(b))$ in* (C.15) *with a unique $r \in (\mathbb{C}\{b\})^R$.*

This proof is also based on showing the existence of a solution of a differential equation, similar to the proof of Theorem C.16, whereas the directions in the tangent space of the deformations preserving two differential forms is a subspace of the tangent space of the $\hat{c}$-deformations. Only this time, the vector field on the space $(F_l, H_l, u_l, \varphi_l) \in H \times H \times H \times \mathbb{C}^r$ is for $l = 1, \dots, L$ defined by

$$\begin{aligned} \frac{\partial H_l}{\partial b}(\hat{c}, \check{c}, b) &= \mathfrak{a}_l(\hat{c}, \check{c}) H_l(\hat{c}, \check{c}, b) \left(\frac{\omega_l(b)}{d\hat{c}}\right)^{-1}, \\ \frac{\partial u_l}{\partial b}(\hat{c}, \check{c}, b) &= \mathfrak{b}_l(\hat{c}, \check{c}) \left(\frac{\omega_l(b)}{d\hat{c}}\right)^{-1}, \\ \frac{\partial \varphi_l}{\partial b}(b) &= \mathfrak{c}_l \left(\frac{\omega_l(b)}{d\hat{c}}\right)^{-1}. \end{aligned} \tag{C.16}$$

The map $t_l = \varphi_l(b)$ is used to obtain the map between the base spaces of a morphism between $(X, d\hat{c}) \hookrightarrow (Y, d\hat{c}) \twoheadrightarrow \mathcal{B}$ and the universal $\hat{c}$-deformation $(X, d\hat{c}) \hookrightarrow (Z, d\hat{c}) \twoheadrightarrow \mathcal{T}$. Hereby, it is exploited that the meromorphic function $\frac{\omega}{d\hat{c}}$ defines for each $t \in \mathcal{T}$ an isomorphism class in the even part of $\mathbb{C}\{\hat{c}, \check{c}\}/\langle f, \frac{\partial f_l}{\partial \check{c}}\rangle$ if $\deg(f_l)$ even and in the odd part of $\mathbb{C}\{\hat{c}, \check{c}\}/\langle f, \frac{\partial f_l}{\partial \check{c}}\rangle$ for $\deg(f_l)$ odd for $l = 1, \ldots, L$. Due to Lemma C.13, this isomorphism class determines the derivative of the map $\mathcal{B} \to \mathcal{T}$ which equals the map from the tangent space on $\mathcal{B}$ to the space of infinitesimal $\hat{c}$-deformations, i.e. the tangent space of $\mathcal{T}$.
To construct a particular deformation (C.13) which can then be proven to be a universal deformation for deformations obeying (A2'), (B2'), (D2') and optionally (E2'), one can define a manifold $\mathcal{R}$ for $(X, Q^+, Q^-, d\hat{c}, d\check{c}, \sigma)$ obeying (A2), (B2), (D2) and optionally (E2) and the universal $\hat{c}$-deformation (C.10) in Theorem C.16 of $(X, Q^+, Q^-, d\hat{c}, \sigma)$ together with a holomorphic map $\mathcal{R} \to \mathcal{T}$ as follows: There exists $\varepsilon > 0$ such that for all $r \in \mathcal{R} = B_\varepsilon(0) \subset \mathbb{C}^R$, the deformation family in Lemma C.21 in direction of (C.15) exists up to $\omega(r)$. Hereby, $\mathcal{R}$ is a submanifold of $\mathbb{C}^R$ of complex dimension $\frac{g_a}{2} + 3$, respectively $\frac{g_a}{2} + 1$ if the lattice Γ is not deformed. Since $\sigma^*\omega(r) = \omega(r)$, it is $\sigma^* r = r$ and let $\mathcal{R}$ be chosen as invariant under σ. It is shown in [Carberry and Schmidt, 2017] that the solutions $(F_l, H_l, u_l, \varphi_l)$ and the vector field induced by

$$\frac{\partial F_l}{\partial b}(\hat{c}, \check{c}, b) = -\frac{\partial F_l}{\partial \check{c}}(\hat{c}, \check{c}, b)\frac{\omega_l(b)}{d\hat{c}}$$

and (C.16) depends holomorphically on $r \in \mathcal{R}$. Furthermore, the local coordinates $w_{W,\pm}$ exist for $Q^\pm$ in $\mathbb{C}\{z_{W,\pm}\} \hat{\otimes} \mathcal{O}_\mathcal{R}$ and for each $q \in X^\circ \setminus (O_1 \cup \cdots \cup O_L)$, the local coordinates $\check{c}_{W,q}$ exist in $\mathbb{C}\{\hat{c}_{W,q}\} \hat{\otimes} \mathcal{O}_\mathcal{R}$. The local coordinates at $q_1, \ldots, q_L$ are obtained from F_l as in the proof of Lemma C.21. This defines a particular deformation (C.13) of the given Fermi curve data denoted by

$$(X, d\hat{c}, d\check{c}) \hookrightarrow (W, d\hat{c}_W, d\check{c}_W) \twoheadrightarrow \mathcal{R}. \tag{C.17}$$

Theorem C.16 yields a unique map $\chi : \mathcal{R} \to \mathcal{T}$ such that the deformation of $(X, Q^+, Q^-, d\hat{c}, \sigma)$ corresponding to (C.17) is isomorphic to the pullback of (C.10) with respect to χ.
If condition (E2) is imposed, the antiholomorphic involution τ_2 acts on the sections ω of E. Then $\omega_1, \ldots, \omega_R$ are chosen in such way that $\tau_2^*\omega(r) = \omega(\bar{r})$ holds. The involution τ_2 acts on r as $\tau_2^* r = \bar{r}$. Choosing $\mathcal{R}$ as invariant under the involution τ_2 ensures that τ_2 acts on (C.17), compare [Carberry and Schmidt, 2017, before Lemma 5.6]. Then the deformation in (C.17) obeys (E2').

Lemma C.22 ([Carberry and Schmidt, 2017, Lemma 5.6]). *The deformations in Lemma C.21 are equal to the pullbacks of* (C.17) *with respect to unique maps* $\xi : \mathcal{B} \to \mathcal{R}$.

With help of this lemma, the main theorem can be shown:

Theorem C.23 ([Carberry and Schmidt, 2017, Theorem 5.7]). *For* $(X, Q^+, Q^-, d\hat{c}, d\check{c}, \sigma)$ *obeying (A2), (B2), (D2) and optionally (E2), the deformation* (C.17) *is a universal deformation of the* *deformations* (C.13).

Sketch of the proof. One considers the deformations

$$\begin{array}{ccccc} X & \hookrightarrow & W & \longrightarrow & \mathcal{R} \\ & & & & \chi\downarrow \quad \nwarrow \\ X & \hookrightarrow & Z & \longrightarrow & \mathcal{T} \quad \big)\ \xi \\ & & & & \varphi\uparrow \quad \diagup \\ X & \hookrightarrow & Y & \longrightarrow & \mathcal{B} \end{array}$$

where the deformations in the first and third row are obeying conditions (A2'), (B2'), (D2') and optionally both of them (E2') and the deformation in the middle row is the corresponding universal $\hat{c}$-deformation, so it obeys (A1'), (B1'), (D1') and (E1') if (E2') holds for the other two. Due to Theorem C.16 and since the deformation in (C.13) of $(X, Q^+, Q^-, d\hat{c}, d\check{c}, \sigma)$ also induces a deformation (C.4) of $(X, Q^+, Q^-, d\hat{c}, \sigma)$, the map φ is unique. Furthermore, the deformation (C.4) is isomorphic to the pullback of (C.10) via φ. Then one shows that there exists a unique map $\xi : \mathcal{B} \to \mathcal{R}$ with $\varphi = \chi \circ \xi$ and such that the deformation (C.13) is isomorphic to the pullback of (C.17) with respect to ξ. □

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