# Essays in Microeconomic Theory

*Inauguraldissertation zur Erlangung des akademischen Grades eines Doktors der Wirtschaftswissenschaften der Universität Mannheim*

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*to my mother*

# Contents

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# Introduction

The thesis has four self-contained chapters and contributes to slightly different topics in microeconomic theory. The first three chapters have a common theme of learning, or experimentation. Learning takes place in different environments: when the decisions are made collectively, when the underlying state is restless, and when consumers deal with experience goods which are also breakable. The forth chapter is a bit of an outcast: I study a role of bundling when the monopolist faces strategic buyers.

Chapter 1 titled *Collective Experimentation with Breakdowns and Breakthroughs* complements the work already done on collective experimentation and emphasizes the importance of knowing how learning takes place. I shed light on such questions as: How incentives to implement a reform change as parties responsible for the decision—legislators, national ministers, or voters—gradually learn about it? What determines the long-term outcome, that is, whether the reform is implemented or the status quo remains? Which voting rule is socially optimal?

These questions are not trivial, because voters' incentives may be in conflict, while the decision is made collectively via voting. Indeed, because effects of the reform differ across sectors of the economy or states, there can be both winners and losers from the reform. For example, think about trade or health reforms, authorization of a certain technology on a market, and so on. Winners from the reform, of course, want it to be implemented, while losers are against it. Furthermore, the outcome is uncertain, because voters do not initially know whether they will be ultimately winners or losers.

I find that voters' incentives to experiment change as some of them learn that they lose or win from the reform. Specifically, the incentives increase with the number of losers, which I refer to as the insurance effect, and decrease with the number of winners, the anxiousness effect. That is why, for example, even if voters experiment with the reform at first, the status quo might remain not only if a majority of voters learn that they are losers, but also if unsure voters become too anxious to experiment. Furthermore, having a large voting committee might exacerbate the bias toward the status quo, though this result relies on the learning structure. Finally, the simple majority rule is not generally socially optimal. Whether there exists a qualified majority rule which is depends, once again, on how learning takes place.

Chapter 2 titled *Restless Strategic Experimentation* analyzes an abstract model of learning in restless, or changing, environments. I aim to answer the following questions: How do strategic players experiment in restless environments? More specifically, would they behave myopically, that is, care only about the current outcome, or is there an option value of learning? Is the predicted players' behavior socially optimal?

This is the first work that deals with strategic experimentation in restless environments. The predictions are strong, in the sense that there is a unique symmetric equilibrium. The equilibrium is described in closed form, that is, the relevant thresholds and value functions of players are found in closed form. Furthermore, I find that, if the environment changes quickly, then the myopic behavior is socially optimal and this is also how players choose to behave. Otherwise, if the changes are slow, then there is an option value to experiment. However, players free ride on each other and experiment too little.

The ideas and techniques presented in this chapter can be used in more applied settings. For example, consider farmers trying out a new seed. Because of changing weather and soil conditions, farmers can never be absolutely sure if the new seed will bring good yields or not. Note also that farmers can learn from each other, and so may want to free-ride on others' experimentation. Other examples may include hospitals experimenting with a new drug or decisions of drivers about whether to drink-and-drive.

Chapter 3 is titled *Gizmos*. The story behind follows these lines: Faced with the necessity to buy a new phone, should one buy, say, Google Pixie with its advanced camera or would Nexus 5X be good enough? Unless the buyer is sure that he is a photographer at heart, the answer is not clear. As a result, it is not clear what the best pricing strategy for the seller of smartphones should be.

The questions I answer in this chapter are as follows: Is it always a good idea to sell both versions of a gizmo, or other product? Does the seller find it optimal to plan obsolescence and, if so, why? Indeed, it is not a secret that phones' batteries die relatively fast. In this chapter, I consider a simple pricing mechanism, specifically, a posted-price mechanism. Therefore, the question which remains to answer and is left for future work is: What is the optimal pricing strategy for the seller of smartphones?

The seller finds it optimal to offer both basic and advanced versions of the gizmo. Furthermore, he does plan obsolescence. The seller neither wants the gizmo to break immediately nor wants it to be a durable good. The optimal breakdown rate captures the trade-off faced by the seller. On the one hand, buyers with the basic version, who have learned that they value the advanced features, upgrade their gizmo to the advanced version only upon a breakdown. That is why a higher breakdown rate increases buyers' surplus, because they get the preferred version faster, and so it also increases the seller's profit. On the other hand, with the higher breakdown rate, buyers must be quite sure that they need the advanced features, and so the seller extracts less surplus from those who would potentially value the advanced features.

In Chapter 4 titled *Bundling with Strategic Buyers*, I want to understand whether and how a seller can use bundling to increase his revenue. The seller has a finite stock of the product, which he has to sell before a deadline. He faces buyers who have multi-unit demand and different valuations for the product, and who are strategic, that is to say, who are forward-looking and possess some bargaining power. For example, think about secondary markets for planes or orders of ships, trains, or planes. As I consider a posted-price mechanism, my model is a very simplified view of these situations, but it still provides important insights into the role of bundling.

I find that bundling the products can play two roles. First, it allows the seller to discriminate among buyers with different valuations, for example, by targeting buyers with a high valuation for the product with a bundle and by selling the product unit-by-unit to buyers with a low valuation. Interestingly, the price of a two-unit bundle is more than twice the price of one unit. Second, bundling acts as a precaution when the seller chooses to target buyers with a high valuation only.

The thesis is structured as follows: The main findings of each chapter and their analysis are presented in the respective Chapters 1 to 4. The additional results and all the proofs are gathered in the respective Appendices A to D. All the references are in Bibliography.

# Chapter 1

# Collective Experimentation with Breakdowns and Breakthroughs

## 1.1 Introduction

Unanimity and simple or qualified majority rules are decision rules often used by the legislatures of democratic nations or the council of the European Union. These institutions are responsible for implementing reforms regarding, for example, trade liberalization, health care, national security, and environment. Consequences of such reforms are uncertain, and their effects vary across different sectors of the economy and across states. Fernandez and Rodrik (1991) and Rodrik (1993) argue that this uncertainty may prevent reforms that would benefit the majority, and leads to a bias toward the status quo.

This chapter sheds further light on such questions as: How incentives to implement a reform as parties responsible for the decision—legislators, national ministers, or voters—gradually learn about it? What determines the long-term outcome, that is, whether the reform is implemented or the status quo remains? Which voting rule is socially optimal?

The model is an adaptation of Strulovici (2010a) as discussed in Related literature below. The key features are as follows: First, voters' incentives may be in conflict, while the decision is made collectively via voting. As effects of the reform differ across sectors of the economy or states, there can be both winners and losers from the reform. Winners, of course, want the reform to be implemented, while losers are against it. Second, the outcome is uncertain. At the start, voters do not know whether they will be ultimately winners or losers. If voters are optimistic, then they are willing to experiment with the reform and find out whether they benefit from it. Learning is gradual and random: via bad news (or breakdowns) for losers and via good news (or breakthroughs) for winners. Third, the decision about whether to experiment is made repeatedly. That is why, for example, even if voters experiment with the reform at first, the status quo remains if a majority of them learn that they are losers. In contrast, the reform is implemented if the majority are winners.

It follows that voters always belong to one of three groups: those who have learned that they are winners or losers, or the remaining unsure voters. As long as neither winners nor losers form a majority, unsure voters are pivotal. They support experimentation with the reform if they are optimistic. The following are the main findings of this chapter.

First, two effects arise: an insurance effect and a novel anxiousness effect. In other words, unsure voters' incentives to experiment with the reform increase with the number of losers and decrease with the number of winners. Unsure voters cast their votes taking into account that choosing to experiment may lead others to learn whether they benefit from the reform. If there are many losers, then unsure voters are insured against an adverse outcome of being trapped with the reform while being a loser, because losers are likely to form a majority and to implement the status quo. In contrast, if there are many winners, then unsure voters are anxious that they turn out to be losers, while the reform is implemented forever, because winners form a majority. That is why, even if voters experiment with the reform at first, the status quo may remain not only if a majority of voters learn that they are losers, but also if unsure voters become too anxious to experiment.[1]

Second, how the size of the voting committee affects unsure voters' behavior depends on the learning structure. Unsure voters behave myopically when the number of voters grows arbitrary large and there are many winners and few losers. That is, having a large voting committee can exacerbate the bias toward the status quo. With many voters, the decision power of each is negligible, and so unsure voters are very anxious to experiment when winners are about to form a majority and thus impose the reform on everyone. However, if learning is via bad news only, then there does not have to be the bias toward the status quo. A growing group of losers insures unsure voters against being trapped with the reform and unsure voters become more optimistic in absence of news. Therefore, if relative gains from the reform are higher than relative losses, then unsure voters are always willing to experiment independently of the number of voters.

Third, the simple majority rule does not generally lead to the socially optimal outcome, but whether there exists a qualified majority rule that does depends on the learning structure. Unsure voters behave socially optimal under the unanimity rules whenever these are socially desirable. Indeed, if all voters must vote for the reform for it to be implemented, then the insurance effect is strongest, because only one loser is needed for the status quo to remain. If all voters must vote for the status quo for it to be implemented, then it must be that the relative gains from the reform are significant. This makes unsure voters willing to take the risk and experiment with the reform. This is no longer the case under other qualified majority rules if learning is via both bad and good news, because unsure voters become too anxious to experiment. However, if learning is via bad news only, then there is no anxiousness effect. As a result, there is always a qualified majority rule that leads to the socially optimal behavior in equilibrium.

The last two results emphasize the importance of knowing how learning about the reform takes place. In particular, this implies the importance of this work, which is complementary to Strulovici (2010a), whose focus is on learning via good news only.

**Related literature.** This chapter contributes to the literature on strategic experimentation and is closest to Strulovici (2010a).[2] I consider an exponential bandit model with

[1]My work supports Fernandez and Rodrik (1991) in that reforms that benefit the majority may not be implemented and that there is a bias towards the status quo because of the uncertainty about being a loser or a winner. Indeed, if voters are pessimistic, then experimentation never takes place and the status quo remains independently of whether a majority of voters are losers or winners. Moreover, even if voters experiment with the reform at first, but there happen to be a majority of losers, the status quo is implemented as soon as they learn about it. In the setting of Fernandez and Rodrik (1991), there is an infinite number of voters, and they immediately and perfectly learn whether they benefit from the reform. Therefore, there is no scope for analysis of how bad or good news received by one voter influence other voters' incentives to experiment in their paper.

[2]Bergemann and Välimäki (2008) gives the general idea of where the bandit problems come from and of main strands in economics up to 2006. Hörner and Skrzypacz (2016) surveys more recent papers of the

conclusive news, introduced in the strategic setting by Keller, Rady, and Cripps (2005) and Keller and Rady (2015a). (See also Bolton and Harris (1999) in the Brownian context.) A reform and the status quo correspond to risky and safe arms. The risky arm is either good or bad, that is, a player either wins or loses from the reform. Players are said to learn via good news if they receive news when their arm is good. They learn via bad news if news arrives when the arm is bad.

Keller, Rady, and Cripps (2005) analyzes a game with good news, while Keller and Rady (2015a) examines the bad news case. Unlike in these two papers, in which players have the risky arm of the same type and each player individually decides whether to experiment with the arm, I study a model in which the arm types are independent across players and decisions to experiment are made collectively. Strulovici (2010a) is the first and, to the best of my knowledge, only paper to look at such a setting.

This work differs from Strulovici (2010a) in two main aspects, leading to new insights into incentives for collective experimentation. First, I consider a mixed news case, in which both good and bad arms bring conclusive news. I focus on the case in which bad news arrives at a higher rate than good news. This makes it qualitatively similar to the bad news case, and thus drastically different from the good news case, which is the benchmark of Strulovici (2010a). Strulovici (2010a) gives some insights for the bad and mixed news cases, in particular, mentions the insurance effect and alludes to the failure of smooth-pasting in case of learning via bad news.[3,4] However, he refers to numerical and analytical results that are omitted from both the published and working versions of the paper (Strulovici, 2007; Strulovici, 2010a).

Keller, Rady, and Cripps (2005) and Keller and Rady (2015a) point out that the bad news case is not a mirror image of the good news case. For other examples, compare Bonatti and Hörner (2011) with Bonatti and Hörner (2017) and see Board and Meyer-ter-Vehn (2013) and Frick and Ishii (2016). The reason is rooted in dynamics of players' belief about the type of the risky arm.[5] In the good news case when no news arrives, players become more pessimistic and their belief drifts toward the cut-off and thus enters the stopping region smoothly. In contrast, in the bad news case, they become more optimistic and their belief moves away from the stopping region and enters it only upon arrival of news with a jump.

Second, I consider a model with no discounting. Bolton and Harris (2000) and Keller and Rady (2015b) analyze strategic experimentation with no discounting and the common type of the risky arm. Abstracting away from discounting is most suitable for modeling environments in which decisions are made in a very short time frame and have long-lasting effects, which is, indeed, the case of implementation of a reform. In other words, a short exploration phase is followed by a long exploitation phase. This does not

strategic experimentation literature.

[3]Keller and Rady (2015a) emphasize that the smooth-pasting condition for players' value functions does not hold at the cut-off beliefs. It does not hold here either, while players' value functions in Strulovici (2010a) are smooth at the cut-offs. However, as noted in Appendix A.1, the smooth-pasting may fail even in the good news case if there is no discounting.

[4]See Peskir and Shiryaev (2006) for a reference on smooth-pasting, referred to there as smooth fit.

[5]In absence of news, the belief of unsure voters $p_t$ about the good arm evolves as follows:

$$\dot{p}_t = (\lambda_b - \lambda_g)p_t(1 - p_t),$$

where $\lambda_g$ and $\lambda_b$ are Poisson rates at which good and bad news arrives. In the good news case of Strulovici (2010a), $\lambda_g > \lambda_b = 0$, while $\lambda_b > \lambda_g = 0$ and $\lambda_b > \lambda_g > 0$ in the bad news and mixed news cases considered here.

make the model with independent types less fruitful, because players' interests may still be in conflict with the collective decision.

No discounting also makes the model more tractable. I analyze the general case with mixed news and its special case with bad news analytically. This makes comparative statics with respect to the number of winners, losers, and voters possible. Furthermore, I compare different qualified majority rules in terms of social efficiency. Strulovici (2010a) agrues that there is no time-independent qualified majority rule that implements the utilitarian policy, but there exists a time-dependent one.[6] This is no longer the case if voters learn via bad news. There always exists a qualified majority rule that does implement the utilitarian policy. However, more complicated decision rules are required to reach social efficiency with learning via mixed news, unless the utilitarian policy prescribes the unanimity rule.

## 1.2 The Model

Time $t \in [0, \infty)$ is continuous, and the horizon is infinite. There are an odd number of players $N \geq 1$ with two arms: a safe arm $S$ (the status quo) and a risky one $R$ (a reform). Each player votes for one of the two arms over each interval of time $[t, t + \mathrm{d}t)$, and the arm chosen is determined by (simple) majority voting.[7]

Independently of any other player's risky arm, each player's risky arm can be either good or bad, that is, the player is either wins or loses from the reform. Formally, there are $N$ independent random variables $\omega^n \in \{B, G\}$, where $B$ and $G$ stand for bad and good risky arms and $n = 1, \ldots, N$.

The safe arm yields a common flow payoff $s$. An expected payoff of the risky arm per unit of time is $b := \lambda_b h_b$ if a player has the bad arm and is $g := \lambda_g h_g$ if she has the good arm, where $h_b$ and $h_g$ are lump-sums that arrive according to Poisson processes with intensities $\lambda_b > \lambda_g \geq 0$ and referred to as bad and good *news* received by the player. I refer to the case with $\lambda_g = 0$ (resp., with $\lambda_g > 0$) as the *bad news* case (resp., the *mixed news* case). For outcomes to be non-trivial, parameters are such that $g > s > b$. That is, the risky arm is preferred over the safe one if it is good, and the safe arm is preferred over the risky arm if the latter is bad.

At the beginning of the game, no player knows the type of her arm. Players share a prior $p_0 \in (0, 1)$ that the arm is good. A collective decision rule is a stochastic process $C := \{C_t\}_{t \geq 0}$ such that $C_t$ depends only on the past observations and takes values in the action space $\{S, R\}$. Formally, $C_t$ is measurable with respect to the filtration $\mathcal{F}_t$ generated by decisions made in the past and news arrived by time $t$, that is, $\mathcal{F}_t = \sigma(C_\tau, \{N^n_\tau\}_{n=1}^N, \tau \leq t)$. News is conclusive. That is, the belief of the player who receives news (and only hers) jumps to one if she receives good news, and it jumps to zero if she receives bad news. News and votes are publicly observable. Therefore, at any time $t$, players who have not received any news yet share the belief $p_t$ that the risky arm is good, where $p_t := \mathbf{P}^{p_0, C}[\omega^n = G \mid \mathcal{F}_t]$ with $n = 1, \ldots, N$.[8] The belief is determined by Bayes' rule and

[6]Strulovici (2010a) refers to the number of votes required to for the implementation of the reform as a quorum rather than a qualified majority.

[7]Qualified majority rules are considered in Section 1.4.

[8]Henceforth, I add subscript $t$ to probability and expectation functions instead of explicitly conditioning on the filtration $\mathcal{F}_t$.

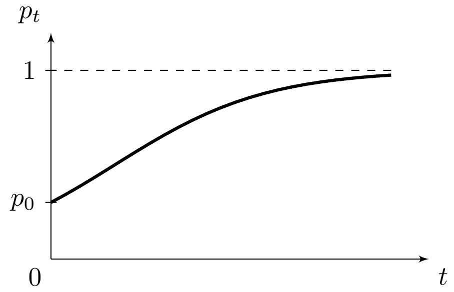


**Figure 1.1.** Evolution of the belief about having the good arm in absence of news if players experiment with the risky arm. Parameters: $(p_0, \lambda_b, \lambda_g) = (0.3, 2, 1)$.

evolves as follows in absence of news:

$$\dot{p}_t = (\lambda_b - \lambda_g) p_t (1 - p_t). \tag{1.1}$$

Note that the belief increases over time in absence of news, that is, players become more optimistic about having the good arm (see Figure 1.1).

Players have a common discount rate $r > 0$. I will focus on the limiting case of perfect patience, that is, on $r \to 0$. An expected payoff of each player $n$ is determined by a collective decision rule and is defined by

$$v_t^{n,C} := \mathbf{E}_t^{p_0,C}\left[ r \int_t^\infty e^{-r(\tau - t)} \mathrm{d}\pi_\tau^{n,C_\tau} \right],$$

where, depending on the arm chosen at time $\tau$, either $\mathrm{d}\pi_\tau^{n,R} = (b\mathbf{1}_{\omega^n = B} + g\mathbf{1}_{\omega^n = G})\mathrm{d}\tau$ or $\mathrm{d}\pi_\tau^{n,S} = s\mathrm{d}\tau$.

Because news is publicly observable and conclusive, at any time $t$, each player belongs to one of three groups: losers $I$, winners $J$, or unsure voters. Losers are those who have received bad news by time $t$ (and are *in sorrow* if the reform is implemented). Winners are those who have had good news (and are *joyful* if the reform is implemented). Let $i := |I|$ and $j := |J|$, then the remaining $N - i - j$ players (if any) are those who are still "in the dark" and thus referred to as unsure voters. I restrict attention to Markov strategies.

**Definition 1.1** (Markov Strategy)**.** *A Markov strategy for player $n$ is a function $d^n$ that maps the set of losers $I$ and winners $J$, and the common belief $p$ of unsure voters into a decision of player $n$:*

$$d^n \colon (I, J, p) \to \{S, R\}.$$

*A profile of Markov strategies is denoted by $d := (d^1, \dots, d^N)$.*

**Definition 1.2** (Markov Collective Decision Rule)**.** *Given a profile of Markov strategies $d = (d^1, \dots, d^N)$, $C$ represents a Markov collective decision rule and determined the action played,*

$$C(I, J, p) = R \quad \Leftrightarrow \quad |\{n \colon d^n(I, J, n) = R\}| > \tfrac{N}{2}.$$

*The collective rule $C$ defines the value function $v^{n,C}(I, J, p)$ of each player $n$ when the current state is $(I, J, p)$.*

Considering Markov strategies makes the game a stopping game. Indeed, if $C(I, J, p) = S$ for some $(I, J, p)$, then the safe arm is always chosen after any history that leads to

$(I, J, p)$, and each player gets $s$ thereafter, that is, $v^{n,C}(I, J, p) = s$. Keeping in mind that players follow the Markov collective decision rule, I omit the superscript $C$ from now on. Since payoffs of all players in the same group are identical, the value functions of losers, winners, and unsure voters are $l(i, j, p)$, $w(i, j, p)$, and $u(i, j, p)$, where $i$ and $j$ are the current numbers of losers and winners, and $p$ is the belief of unsure voters. The restriction to Markov strategies also allows me to characterize the optimal strategy of players in terms of solutions to the Hamilton-Jacobi-Bellman (HJB) equations.

The HJB equation for the value function of unsure voters takes the form

$$\begin{aligned} u(i,j,p) = \max \Big\{ s,\ & gp + b(1-p) \\ & + \tfrac{\lambda_b(1-p)}{r}[l(i+1,j,p) - u(i,j,p) + (N-i-j-1)(u(i+1,j,p) - u(i,j,p))] \\ & + \tfrac{\lambda_g p}{r}[w(i,j+1,p) - u(i,j,p) + (N-i-j-1)(u(i,j+1,p) - u(i,j,p))] \\ & + \tfrac{(\lambda_b-\lambda_g)p(1-p)}{r}\partial_p u(i,j,p) \Big\}. \end{aligned}$$

The first part of the maximand corresponds to the safe arm, while the second one to the risky one. Using the risky arm affects an unsure voter in six ways: (i) it yields her an expected payoff $gp+b(1-p)$, (ii) she can receive bad news at rate $\lambda_b$ with probability $1-p$ and become a loser, (iii) other unsure voter can get bad news and increase the number of losers by one, (iv) she can receive good news at rate $\lambda_g$ with probability $p$ and become a winner, (v) other unsure voter can get good news and make one more winner, and (vi) when no news arrives, she uses Bayes' rule to update her belief about having the good arm.

If the safe arm is chosen, then $u(i, j, p) = s$, and so the value function is independent of $r$. If the risky arm is chosen, then

$$\begin{aligned} u(i,j,p) = {} & gp + b(1-p) \\ & + \tfrac{\lambda_b(1-p)}{r}[l(i+1,j,p) - u(i,j,p) + (N-i-j-1)(u(i+1,j,p) - u(i,j,p))] \\ & + \tfrac{\lambda_g p}{r}[w(i,j+1,p) - u(i,j,p) + (N-i-j-1)(u(i,j+1,p) - u(i,j,p))] \\ & + \tfrac{(\lambda_b-\lambda_g)p(1-p)}{r}\partial_p u(i,j,p). \end{aligned}$$

By multiplying both sides by $r$, taking the limit $r \to 0$, and taking into account that the value function is normalized, that is, that $\lim_{r\to 0} ru(i, j, p) = 0$, I obtain

$$\begin{aligned} 0 = {} & \lambda_b(1-p)[l(i+1,j,p) - u(i,j,p) + (N-i-j-1)(u(i+1,j,p) - u(i,j,p))] \\ & + \lambda_g p[w(i,j+1,p) - u(i,j,p) + (N-i-j-1)(u(i,j+1,p) - u(i,j,p))] \\ & + (\lambda_b - \lambda_g)p(1-p)\partial_p u(i,j,p). \end{aligned}$$

Therefore, when unsure voters become infinitely patient, that is, when $r \to 0$, the HJB equation becomes

$$\begin{aligned} u(i,j,p) = \max \Big\{ s,\ & \tfrac{1}{(N-i-j)(\lambda_g p + \lambda_b(1-p))} \\ & \times \Big( \lambda_b(1-p)[l(i+1,j,p) + (N-i-j-1)u(i+1,j,p)] \\ & + \lambda_g p[w(i,j+1,p) + (N-i-j-1)u(i,j+1,p)] \end{aligned}$$

$$+ (\lambda_b - \lambda_g)p(1-p)\partial_p u(i,j,p)\Big)\Big\}.$$

Similarly, the HJB equations for the value functions of losers and winners can be found.

To avoid trivial equilibria, for example, when it is optimal to vote for the safe arm because all other players do so, I consider undominated Markov strategies only.

**Definition 1.3** (Markov Equilibrium in Undominated Strategies)**.** *The profile $d = (d^1, \ldots, d^N)$ is a Markov equilibrium in undominated strategies if, for all $(I, J, p, n)$, $d^n(I, J, p) = R$ if and only if*

$$\tfrac{1}{(N-|I|-|J|)(\lambda_g p+\lambda_b(1-p))}\Bigg(\lambda_b(1-p)\sum_{i'\notin I\cup J} v^n(I\cup\{i'\}, J, p) + \lambda_g p \sum_{j'\notin I\cup J} v^n(I, J\cup\{j'\}, p) + (\lambda_b - \lambda_g)p(1-p)\partial_p v^n(I, J, p)\Bigg) > s.$$

Given the restriction to undominated strategies, any solution to the HJB equation is optimal. The risky arm is chosen as long as it yields a strictly higher payoff than the safe arm. As mentioned above, restriction to Markov strategies implies that, if the safe arm is used once, it is going to be used thereafter. Therefore, each player votes for the risky arm until it is optimal for her.

## 1.3 The Equilibrium

Losers and winners always vote for the safe and risky arms, respectively. Unsure voters vote for the risky arm if and only if they are optimistic about the type of their arm. How optimistic they must be depends on the current number of losers and winner, because choosing the risky arm may lead others to learn the type of their arm and, if losers or winners form a majority, they impose their preferred arm on unsure voters. Note that unsure voters are pivotal until losers or winners form a majority.

The unique equilibrium is characterized by cut-offs that reflect unsure voters' incentives to experiment with the risky arm. Unsure voters choose to experiment in the presence of $i$ losers and $j$ winners if and only if their belief is above the cut-off $p(i,j)$. The lower is the cut-off $p(i,j)$, the higher are the incentives, because unsure voters are willing to experiment for a larger range of beliefs. Given the number of losers and winners, the equilibrium cut-off is the (largest) belief at which unsure voters are indifferent between the safe and risky arms. That is, $p(i,j)$ is defined by $u(i,j,p(i,j)) = s$, where $u(i,j,p)$ is the value function of unsure voters.

I start with an example that illustrates the results that lead to all the interesting findings in this chapter. Unsure voters' incentives to experiment change as some of them learn that they lose or win if the risky arm is implemented. Specifically, the incentives increase with the number of losers (an insurance effect) and decrease with the number of winners (a novel anxiousness effect). That is why, even if players experiment with the risky arm at first, the safe arm may be implemented not only if a majority of players learn that they are losers, but also if unsure voters become too anxious to experiment.

I continue with the formal statements of the results. I further show that unsure voters behave myopically when the number of players grows arbitrary large and there are many winners and few losers. This implies that having a large voting committee can exacerbate

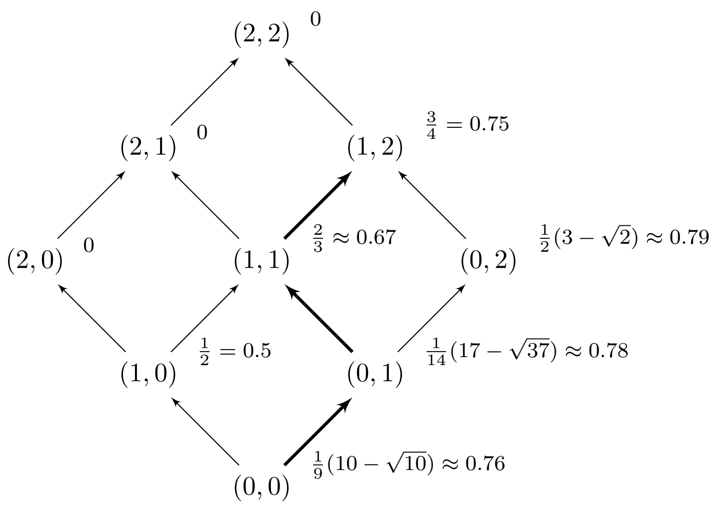


**Figure 1.2.** All possible states $(i, j)$ in terms of the number of losers $i$ and winners $j$, in which neither losers nor winners have formed a majority and hence unsure voters are decisive. Next to each state $(i, j)$ is the upper bound $\tilde{p}(i, j)$ for the cut-off $p(i, j)$ used by unsure voters. Parameters: $(N, g, s, b) = (5, 1, 0, -4)$.

the bias toward the status quo. The myopic behavior does not occur if losers are about to form a majority. Furthermore, if players learn via bad news only, that is, if $\lambda_g = 0$, then the equilibrium cut-offs are non-decreasing with the number of players. Moreover, there does not have to be the bias toward the status quo at all. If relative gains from the risky arm are higher than relative losses, that is, if $g - s \geq s - b$, then the equilibrium cut-offs are independent of the number of players and unsure voters experiment for any belief they may have. This is one of the findings which make the bad news case interesting on its own right.

### 1.3.1 An Example

As an example, I consider a game with $N = 5$ players and the risky and safe arms' payoffs equal to $g = 1$, $s = 0$, and $b = -4$. Figure 1.2 shows all possible states in terms of the number losers and winners, in which neither of these two groups has formed a majority, and so unsure voters are decisive. There are $i$ losers and $j$ winners in the state $(i, j)$.

At first, no player knows the type of her arm, that is, $(i, j) = (0, 0)$. If unsure voters are optimistic, that is, if $p_0 \geq p(0, 0)$, then they choose to experiment with the risky arm. Because unsure voters become more optimistic about the type of their arm in absence of news, their belief stays above $p(0, 0)$ and they choose the risky arm over the safe one until one of them learns whether she is a loser or a winner.

Suppose one of unsure voters receives good news and thus learns that she is a winner, that is, the state becomes $(i, j) = (0, 1)$; see the bold arrow from $(0, 0)$ to $(0, 1)$ in Figure 1.2. If $p_0 \in (p(0, 0), p(0, 1))$ and the news arrived before time[9]

$$t_{(0,1)} = \frac{1}{\lambda_b - \lambda_g} \ln \left( \frac{1-p_0}{p_0} \frac{p(0,1)}{1-p(0,1)} \right),$$

[9]The law of motion (1.1) implies that, by time $t$, unsure voters' belief reaches $p_t$ given by

$$\frac{p_t}{1-p_t} = \frac{p_0}{1-p_0} e^{(\lambda_b - \lambda_g)t},$$

where $p_0$ is the prior.

that is, before unsure voters' belief reaches $p(0,1)$, then they vote for the safe arm from the moment the news arrived on. As a result, four unsure voters vote for the safe arm, while one winner votes for the risky arm, and so the safe arm is the voting outcome. It follows that the safe arm will be implemented forever even though losers have not formed a majority.

If $p_0 \geq p(0,1)$ or the news arrived after time $t_{(0,1)}$, then unsure voters' belief is above $p(0,1)$ and experimentation continues. Suppose that now one of unsure voters receives bad news and thus learns that she is a loser; see the bold arrow from $(0,1)$ to $(1,1)$ in Figure 1.2. If experimentation goes on, which is the case if unsure voters' belief is above $p(1,1)$ at the time the news arrived, then one of unsure voters may receive good news and become a winner; see the bold arrow from $(1,1)$ to $(1,2)$ in Figure 1.2. If one more winner appears, then the risky arm will be implemented forever, because three (out of five) players who are winners will vote for it.

Figure 1.2 shows the upper bounds $\tilde{p}(i,j)$ for the cut-offs $p(i,j)$ used by unsure voters in the presence of $i$ losers and $j$ winners. These bounds bind if the prior is high, specifically, if $p_0 > \tilde{p}(0,2) = \frac{1}{2}(3-\sqrt{2}) \approx 0.79$; see Section 1.3.2 for details about the upper bounds and the cut-offs. As can be seen in Figure 1.2, $\tilde{p}(i,j)$ decrease with the number of losers $i$ and increase with the number of winners $j$. For example, $\tilde{p}(0,0) \approx 0.76 > \tilde{p}(1,0) = 0.5 > \tilde{p}(2,0) = 0$ and $\tilde{p}(0,0) \approx 0.76 < \tilde{p}(0,1) \approx 0.78 < \tilde{p}(0,2) \approx 0.79$. These display the *insurance effect* and the *anxiousness effect*. Unsure voters cast their votes taking into account that choosing the risky arm may lead others to learn the type of their arm. If there are many losers, then unsure voters are insured against an adverse outcome of being trapped with the risky arm while being losers, because losers are likely to form a majority and impose the safe arm. In contrast, if there are many winners, then unsure voters are anxious that they turn out to be losers, while the risky arm will be implemented forever, because winners form a majority.

### 1.3.2 Cut-offs

I start with two special cases. They help better understand the role of the collective decision making, in particular, that unsure voters are forward-looking and take into account that, if the risky arm is chosen, then some of them may learn the type of their risky arm.

First, the cut-off used by a *single decision maker* is equal to

$$p_{SD} = 0.$$

That is, if a voter is solemnly responsible for the decision, then she experiments with the risky arm until she learn its type. Indeed, all the decision power belongs to her, which implies she will not be trapped with the risky arm if she learns that she is a loser. Furthermore, she is patient, and thus can afford to take her time and make a fully informed decision.

Second, if voters are *myopic*, that is, if they care only about the current payoff, then unsure voters choose to experiment with the risky arm if and only if its current expected payoff is higher than that of the safe arm. Unsure voters with the belief $p$ vote for the risky arm if and only if $gp + b(1-p) > s$, that is, if and only their belief is above

$$p_M = \tfrac{s-b}{g-b},$$

the myopic cut-off.

Theorem 1.1 below describes the equilibrium *cut-offs* $p(i,j)$ and their upper bounds $\tilde{p}(i,j)$, when there are $N > 1$ strategic and forward-looking players and $i$ (resp., $j$) of them have already learned that they are losers (resp., winners). The equilibrium existence and uniqueness come from a backward induction argument on the number of losers and winners. The equilibrium cut-offs $p(i,j)$ are pinned down by the value-matching condition

$$u(i,j,p(i,j)) = s,$$

where $u(i,j,p)$ is the value function of unsure voters. The upper bounds on the cut-offs are found assuming that, if unsure voters choose to experiment for the very first time when neither of them has learned whether she is a loser or a winner, they stop experimenting if and only if losers form a majority. The upper bounds $\tilde{p}(i,j)$ are pinned down by the respective value-matching condition

$$\tilde{u}(i,j,\tilde{p}(i,j)) = s,$$

where $\tilde{u}(i,j,p)$ is the value function of unsure voters in such a case; see Section 1.3.3. Because unsure voters can always choose the safe arm rather than the risky arm, $u(i,j,p) \geq \tilde{u}(i,j,p)$ for all $p \in [0,1]$, $i \leq i_N$, and $j \leq j_N$, where $i_N := \frac{N-1}{2}$ (resp., $j_N := \frac{N-1}{2}$) is the critical number of losers (resp., winners) such that only one loser (resp., winner) is needed for them to form a majority.

**Theorem 1.1** (Equilibrium Cut-offs)**.** *There exists a unique equilibrium characterized by cut-offs $p(i,j)$. The cut-offs are such that $p(i,j) \leq \tilde{p}(i,j)$ for all $i \leq i_N$ and $j \leq j_N$, with equality if and only if $p_0 > \max_{i \leq i_N, j \leq j_N} \tilde{p}(i,j)$. The upper bounds $\tilde{p}(i,j)$ satisfy:*

- *$\tilde{p}(i+1,j) < \tilde{p}(i,j)$ and $\tilde{p}(i,j+1) > \tilde{p}(i,j)$ if $(s-b)(i_N - i) > (g-s)(j_N - j + 1)$,*
- *$\tilde{p}(i+1,j) = \tilde{p}(i,j) = 0$ and $\tilde{p}(i,j+1) \geq \tilde{p}(i,j)$ if $(s-b)(i_N - i) \leq (g-s)(j_N - j + 1)$.*

*Furthermore, $p_M > \tilde{p}(i,j) \geq p_{SD}$ for all $i \leq i_N$ and $j \leq j_N$.*

The equilibrium cut-offs are equal to their upper bounds, that is, $p(i,j) = \tilde{p}(i,j)$ for all $i \leq i_N$ and $j \leq j_N$, if and only if experimentation with the risky arm stops only if losers form a majority. Because unsure voters become more optimistic about having the good arm in absence of news, they always vote for the risky arm if and only if the prior about having the good arm is high enough, specifically, if and only if $p_0 > \max_{i \leq i_N, j \leq j_N} \tilde{p}(i,j)$. Note that $p_M > \tilde{p}(i,j) \geq p(i,j)$ for all $i \leq i_N$ and $j \leq j_N$, that is, unsure voters care not only about their current payoff. In other words, there is always an option value to learning the type of the risky arm. This implies a sufficient condition for the equilibrium cut-offs to coincide with their upper bounds, which is given in Corollary 1.1.

**Corollary 1.1.** *If $p_0 \geq p_M$, then $p(i,j) = \tilde{p}(i,j)$ for all $i \leq i_N$ and $j \leq j_N$.*

Theorem 1.1 states that the upper bounds $\tilde{p}(i,j)$ decrease with the number of losers $i$ (the *insurance effect*) and increase with the number of winners $j$ (the *anxiousness effect*). The example in Section 1.3.1 has already illustrated these effects.

The insurance effect is strongest when there is the critical number of losers, that is, $\tilde{p}(i_N,j) = p_{SD} = 0$ for all $j \leq j_N$; and it also follows that $p(i_N,j) = p_{SD} = 0$ for all $j \leq j_N$, because $p(i_N,j) \leq \tilde{p}(i_N,j)$. Indeed, similar to the single decision maker, each unsure voter can afford to make a fully informed decision. If she learns that her arm is bad, then the preferred safe arm will be implemented, because losers will form a majority. In contrast, if she learns that her arm is good, then she will continue to vote for the risky arm.

The anxiousness effect emphasizes the difference between learning with and without bad news. As I discuss in Appendix A.1, if players learn via good news only, then the equilibrium cut-offs are independent of the number of winners. If players learn via bad news as well, then the possibility of becoming one of the losers, who are in minority and whose interests are opposite to those of optimistic unsure voters and winners, makes unsure voters more anxious to experiment in the presence of many winners.

Theorem 1.1 also captures what affects unsure voters' choice between the risky and safe arms. Unsure voters take into account that, if the risky arm is a voting outcome, some of them may learn the type of the arm and that losers and winners, if they are in majority, impose their preferred arm on everyone. The fewer losers are needed for their number to be critical, the more insured are unsure voters; and the fewer winners are needed for their number to be critical, the more anxious are unsure voters to experiment. This is captured by by $i_N - i$ and $j_N - j$. Relative gains and losses from the risky arm, that is, $g - s$ and $s - b$, matter as well. If the payoff of the good arm is relatively high, alternatively, if the payoff of the bad arm is relatively small, then the risky arm becomes more attractive. These explain why $(s - b)(i_N - i) > (g - s)(j_N - j + 1)$ implies $\tilde{p}(i, j) \in (0, 1)$, while $(s - b)(i_N - i) \leq (g - s)(j_N - j + 1)$ implies $\tilde{p}(i, j) = 0$.[10] In particular, Corollary 1.2 follows.

**Corollary 1.2.** *If $(s - b)i_N \leq g - s$, then $p(i, j) = \tilde{p}(i, j) = 0$ for all $i \leq i_N$ and $j \leq j_N$.*

If learning is via bad news only, that is, if $\lambda_g = 0$, then no player ever receives good news and thus immediately learns that she is a winner; each player belong to one of two groups, either losers or unsure voters. This implies that the equilibrium cut-offs never increase over time, that is, there is no anxiousness effect. Therefore, if players choose to experiment with the risky arm when neither of them knows the type of her arm, then experimentation stops if and only if losers form a majority. It follows from Theorem 1.1 that the equilibrium cut-offs coincide with their upper bounds. Define by

$$p(i) := p(i, 0)$$

for all $i \leq i_N$ the cut-offs used by unsure voters in the bad news case. Theorem 1.2 describes the equilibrium cut-offs $p(i)$ in the presence of $i$ losers.

**Theorem 1.2** (Equilibrium Cut-offs with Learning via Bad News). *If $\lambda_g = 0$, then there exists a unique equilibrium characterized by cut-offs $p(i)$ satisfying $p(i+1) \leq p(i)$ for all $i \leq i_N$. Furthermore, $p_M > p(i) \geq p_{SD}$ for all $i \leq i_N$. Moreover, if $(s - b)\frac{N-1}{2} \leq (g - s)\frac{N+1}{2}$, then $p(i) = 0$ for all $i \leq i_N$.*

Define by $p_N(i, j)$ and $\tilde{p}_N(i, j)$ the equilibrium cut-off and its upper bound used by unsure voters when there are $i$ losers and $j$ winners in the game with $N$ players. Proposition 1.1 below states that, if winners are about to form a majority and no losers are present, unsure voters behave myopically as the number of players grows arbitrary large. Indeed, if there are many players, then the decision power of each is negligible, and so unsure voters become very anxious to experiment with the risky arm and are biased toward

[10] Technically, if $(s - b)(i_N - i) > (g - s)(j_N - j + 1)$, then the value function of unsure voters when they always experiment, that is, $\tilde{u}(i, j, p)$, is concave for $p$ close to 0; and if $(s - b)(i_N - i) \leq (g - s)(j_N - j + 1)$, then $\tilde{u}(i, j, p)$ is convex for $p$ close to 0. Because $\tilde{u}(i, j, p)$ is smooth at $p = 0$, the conditions imply that $\tilde{u}(i, j, p) < s$ and $\tilde{u}(i, j, p) > s$, respectively. These, in turn, imply that $\tilde{p}(i, j) \in (0, 1)$ and $\tilde{p}(i, j) = 0$. See Lemma A.7 and the proof of Lemma A.6 in Appendix A.2 for details.

the safe arm when winners are about to impose the risky arm on everyone. Contrary, if losers are about to form a majority, then unsure voters are insured against being trapped with the risky arm if their arm is bad. They behave as though each of them is solemnly responsible for the decision independently of the number of players and of how many of them are winners.

**Proposition 1.1** (Number of Players)**.** *For any odd number of players $N$, $\tilde{p}_N(i_N, j) = 0$ for all $j \leq j_N$. Furthermore, $\lim_{N\to\infty} \tilde{p}_N(0, j_N) = p_M$.*

Proposition 1.1 gives an idea that the learning structure is relevant for how unsure voters' incentives to experiment depend on the number of players. I further argue in Appendix A.1 that, if unsure voters learn via good news only, they are always biased toward the safe arm when the number players becomes arbitrary large. Contrary, if unsure voters learn via bad news only, there does not have to be the bias toward the safe arm at all. If relative gains from the risky arm are higher than relative losses, that is, if $g - s \geq s - b$, then the equilibrium cut-offs are independent of the number of players and unsure voters experiment for any belief they may have; see Proposition 1.2 below. In general, the equilibrium cut-offs $p_N(i)$ are non-decreasing with the number of players $N$. Given the number of losers $i$, losers are less likely to form a majority in the presence of more players, and so unsure voters are less insured against being trapped with the bad risky arm.

**Proposition 1.2** (Number of Players with Learning via Bad News)**.** *If $\lambda_g = 0$, then the equilibrium cut-offs $p_N(i)$ are non-decreasing in the number of players $N$: $p_N(i) \leq p_{N'}(i)$ for all $i \leq i_N$ and all odd $N < N'$. Furthermore, if $g - s \geq s - b$, then $p_N(i) = 0$ for all $i \leq i_N$ and all odd $N$.*

As unsure voters are unsure about the type of their risky arm, playoffs of both bad and good risky arms are relevant for their decision. The higher is the payoff of either type, the more attractive does the risky arm become. Proposition 1.3 below captures that the upper bounds for the equilibrium cut-offs are decreasing in both $b$ and $g$. In contrast, the higher is the payoff of the safe arm, the less incentives do unsure voters have to take the risk and experiment with the risky arm. In other words, the upper bounds are increasing in $s$.

**Proposition 1.3** (Payoffs of Risky and Safe Arms)**.** *The upper bounds $\tilde{p}(i, j)$ are decreasing in $b$ and $g$, and are increasing in $s$ for all $i \leq i_N$ and $j \leq j_N$.*

### 1.3.3 Value Functions

I find closed forms of bounds on the equilibrium value functions of losers, winners, and unsure voters. These bounds are tight if and only if $p_0 > \max_{i \leq i_N, j \leq j_N} \tilde{p}(i, j)$. Advantages of knowing the shape of the value functions are threefold. First, I can characterize the equilibrium cut-offs and pin down sets of parameters that correspond to different equilibrium behavior of unsure voters. Second, I can conduct extensive comparative statics on how unsure voters' incentives to experiment change with the number of losers and winners and as the number of players increases. Finally, I can analyze how unsure voters' incentives depend on the voting rule applied and can compare different voting rules in terms of efficiency; see Section 1.4.

If losers are in majority or unsure voters are pessimistic about the type of their risky arm, then the safe arm will be the voting outcome and will be implemented forever. As a result, losers, winners, and unsure voters get the same payoff $s$. If winners are in majority or unsure voters are optimistic, then the risky arm is the voting outcome. Given the number of losers $i$ and winners $j$ and the belief $p$ of unsure voters, the expected payoffs, or the value functions, of losers, winners, and unsure voters are given by $l(i,j,p)$, $w(i,j,p)$, and $u(i,j,p)$. If experimentation starts and if it stops if and only if losers form a majority, then these value functions are equal to $\tilde{l}(i,j,p)$, $\tilde{w}(i,j,p)$, and $\tilde{u}(i,j,p)$ described in detail below. Because unsure voters always have a choice to vote for the safe arm rather than the risky one, $u(i,j,p) \geq \tilde{u}(i,j,p)$. Furthermore, the value functions of losers and winners satisfy $l(i,j,p) \geq \tilde{l}(i,j,p)$ and $w(i,j,p) \leq \tilde{w}(i,j,p)$, since the safe arm is preferred by losers, while the risky arm is preferred by winners. All these are summarized in Lemma 1.1 below.

The lower bound for the equilibrium *value function of losers* is given by

$$\tilde{l}(i,j,p) = b\mathbf{P}[R \text{ is implemented}] + s\mathbf{P}[S \text{ is implemented}], \tag{1.2}$$

while the upper bound for the equilibrium *value function of winners* is given by

$$\tilde{w}(i,j,p) = g\mathbf{P}[R \text{ is implemented}] + s\mathbf{P}[S \text{ is implemented}]. \tag{1.3}$$

The only difference between these two functions is that, if the risky arm is implemented, losers have to put up with the low payoff $b$, while winners benefit from the high payoff $g$. The lower bound for the equilibrium *value function of unsure voters* is given by

$$\begin{aligned} \tilde{u}(i,j,p) = b(1-p)&\mathbf{P}[R \text{ is implemented} \mid \text{being a loser}] \\ &+ gp\mathbf{P}[R \text{ is implemented} \mid \text{being a winner}] + s\mathbf{P}[S \text{ is implemented}]. \end{aligned} \tag{1.4}$$

There are two possible outcomes for an unsure voter if the risky arm is implemented. The unsure voter has to bear the low payoff $b$ if she turns out to be a loser, and she enjoys the high payoff $g$ otherwise.

The probabilities in (1.2), (1.3), and (1.4) are defined by[11]

$$\mathbf{P}[R \text{ is implemented}] := \sum_{k=0}^{i_N - i} \tbinom{N-i-j}{k} p^{N-i-j-k}(1-p)^k,$$
$$\mathbf{P}[S \text{ is implemented}] := \sum_{m=0}^{j_N - j} \tbinom{N-i-j}{m} p^{m}(1-p)^{N-i-j-m}.$$

Each unsure voter is a winner with probability $p$ and a loser with probability $1-p$. For winners to form a majority and so for the risky arm to be implemented forever, there cannot be more that $i_N - i$ losers among the remaining $N-i-j$ unsure voters. Similarly, the safe arm is implemented forever if a majority of voters happen to be losers, that is, if no more than $j_N - j$ of the remaining unsure voters are winners. Furthermore,

$$\mathbf{P}[R \text{ is implemented} \mid \text{being a loser}] := \sum_{k=0}^{i_N - i - 1} \tbinom{N-i-j-1}{k} p^{N-i-j-1-k}(1-p)^k,$$

[11]Note that $\mathbf{P}[R \text{ is implemented}] = (1-p)\mathbf{P}[R \text{ is implemented} \mid \text{being a loser}] + p\mathbf{P}[R \text{ is implemented} \mid \text{being a winner}]$ and $\mathbf{P}[R \text{ is implemented}] + \mathbf{P}[S \text{ is implemented}] = 1$.

$$\mathbf{P}[R \text{ is implemented} \mid \text{being a winner}] := \sum_{k=0}^{i_N - i} \binom{N-i-j-1}{k} p^{N-i-j-1-k}(1-p)^k.$$

When one of $N - i - j$ unsure voters learns her type, only $N - i - j - 1$ unsure voters remain. The risky arm is implemented if no more than $i_N - i$ of unsure voters are losers, including the one who has just learned the type of her risky arm.

**Lemma 1.1** (Equilibrium Value Functions). *The equilibrium value functions of losers $l(i, j, p)$, winners $w(i, j, p)$, and unsure voters $u(i, j, p)$ satisfy:*

- *$l(i, j, p) \geq \tilde{l}(i, j, p)$, $w(i, j, p) \leq \tilde{w}(i, j, p)$, and $u(i, j, p) \geq \tilde{u}(i, j, p)$ for all $p > p(i, j)$,*
- *$l(i, j, p) = w(i, j, p) = u(i, j, p) = s$ for all $p \leq p(i, j)$,*

*for all $i \leq i_N$ and $j \leq j_N$.*

## 1.4 Qualified Majority and Socially Optimal Rules

I introduce a larger class of voting rules, namely, qualified majority rules or *$Q$-rules* for short, to discuss efficiency. Under the $Q$-rule, $Q$ votes are required for the safe arm to be implemented.[12] In particular, the safe arm is implemented if there are at least $Q$ losers. I denote by $i_N^Q := Q - 1$ (resp., $j_N^Q := N - Q$) the critical number of losers (resp., winners) such that only one loser (resp., winner) is needed for them to form a majority. The majority rule obtains for $Q = Q_N := \frac{N+1}{2}$.

I show that unsure voters behave socially optimal under the unanimity rules whenever these are socially desirable. This is no longer the case under other qualified majority rules if learning is via both bad and good news. However, if learning is via bad news only, then there is always a qualified majority rule that leads to the socially optimal outcome. This again emphasizes the importance of the bad news case on its own right.

### 1.4.1 Qualified Majority Rules

All the results in Section 1.3 extend to any game with qualified majority voting, because any game with the $Q$-rule can be described as a game with the majority rule by adding *phantom players* with a certain type of their risky arm. This makes it straightforward to find how unsure voters' incentives to experiment depend on the voting rule applied and that there may exist a qualified majority rule that leads to the socially optimal outcome. I give an idea of the procedure next. The reader interested in the implication of the optimal voting rules may skip this without loss of continuity.

Any game with the $Q$-rule is solved by backward induction on the number of losers and winners. To distinguish between games with different $Q$-rules, I denote by $p^Q(i, j)$ the cut-off used by unsure voters in the presence of $i$ losers and $j$ winners. What actually drives the choice of the cut-offs $p^Q(i, j)$ is the number of unsure voters $N - i - j$, because how the game evolves depends on the type of the risky arms of currently unsure voters and on their belief $p$. This observation makes it possible to describe the game with qualified majority voting as a game with simple majority voting.

Suppose $i$ among $N$ players have learned that they are losers and $j$ players have learned that they are winners (and both groups are still in qualified minority); the current belief of the remaining $N - i - j$ unsure voters is $p$; and players use the $Q$-rule to

[12] Strulovici (2010a) refers to the number of votes required for the risky arm to be implemented as a "quorum." That is, $N - Q + 1$ defines the quorum in his paper.

decide about the voting outcome. If players use a submajority rule, that is, if $Q < Q_N$, then incentives of unsure voters are the same as incentives of unsure voters in a game with $N + i^Q_{ph}$ players, the majority rule, $i + i^Q_{ph}$ losers, $j$ winners, and the belief $p$ of unsure voters, where

$$i^Q_{ph} := 2(Q_N - Q)$$

is the number of *phantom losers*. If players use a supermajority rule, that is, if $Q > Q_N$, then incentives of unsure voters are the same as incentives of unsure voters in a game with $N + j^Q_{ph}$ players, the majority rule, $i$ losers, $j + j^Q_{ph}$ winners, and the belief $p$ of unsure voters, where

$$j^Q_{ph} := 2(Q - Q_N)$$

is the number of *phantom winners*. I refer to two states in two games as "equivalent" if unsure voters' incentives at these states are the same. Lemmata 1.2 and 1.3 follow.

**Lemma 1.2** (Submajority Rules)**.** *A state $(i, j, p)$ in the game with $N$ players and the $Q$-rule with $Q < Q_N$ is equivalent to a state $(i + i^Q_{ph}, j, p)$ in the game with $N + i^Q_{ph}$ players and the majority rule.*

**Lemma 1.3** (Supermajority Rules)**.** *A state $(i, j, p)$ in the game with $N$ players and the $Q$-rule with $Q > Q_N$ is equivalent to a state $(i, j + j^Q_{ph}, p)$ in the game with $N + j^Q_{ph}$ players and the majority rule.*

Theorems 1.3 and 1.4 are immediate consequences of Theorems 1.1 and 1.2 and Lemmata 1.2 and 1.3. Define by $p^Q(i) := p^Q(i, 0)$ for all $i \le i^Q_N$ the cut-offs used by unsure voters in the bad news case.

**Theorem 1.3** (Equilibrium Cut-offs with the $Q$-Rule)**.** *There exists a unique equilibrium characterized by cut-offs $p^Q(i, j)$. The cut-offs are such that $p^Q(i, j) \le \tilde{p}^Q(i, j)$ for all $i \le i^Q_N$ and $j \le j^Q_N$, with equality if and only if $p_0 > \max_{i \le i^Q_N, j \le j^Q_N} \tilde{p}^Q(i, j)$. The upper bounds $\tilde{p}^Q(i, j)$ satisfy:*

- *$\tilde{p}^Q(i+1, j) < \tilde{p}^Q(i, j)$ and $\tilde{p}^Q(i, j+1) > \tilde{p}^Q(i, j)$ if $(s-b)(i^Q_N - i) > (g-s)(j^Q_N - j + 1)$,*
- *$\tilde{p}^Q(i+1, j) = \tilde{p}^Q(i, j) = 0$ and $\tilde{p}^Q(i, j+1) \ge \tilde{p}^Q(i, j)$ if $(s-b)(i^Q_N - i) \le (g-s)(j^Q_N - j + 1)$.*

*Furthermore, $p_M > \tilde{p}^Q(i, j) \ge p_{SD}$ for all $i \le i^Q_N$ and $j \le j^Q_N$. Moreover, $\tilde{p}^Q(i, j)$ are non-decreasing in the $Q$-rule: $\tilde{p}^Q(i, j) \le \tilde{p}^{Q'}(i, j)$ for all $Q < Q'$, $i \le i^Q_N$, and $j \le j^{Q'}_N$.*

**Theorem 1.4** (Equilibrium Cut-offs with the $Q$-Rule and with Learning via Bad News)**.** *If $\lambda_g = 0$, then there exists a unique equilibrium characterized by cut-offs $p^Q(i)$ satisfying $p^Q(i + 1) \le p^Q(i)$ for all $i \le i^Q_N$. Furthermore, $p_M > p^Q(i) \ge p_{SD}$ for all $i \le i^Q_N$ and, if $(s-b)(Q-1) \le (g-s)(N-Q+1)$, then $p^Q(i) = 0$ for all $i \le i^Q_N$. Moreover, $p^Q(i)$ are non-decreasing in the $Q$-rule: $p^Q(i) \le p^{Q'}(i)$ for all $Q < Q'$ and $i \le i^Q_N$.*

Theorems 1.3 and 1.4 also state that the equilibrium upper bounds $\tilde{p}^Q(i, j)$ in the mixed news case and the equilibrium cut-offs $p^Q(i)$ in the bad news case are non-decreasing with the $Q$-rule applied. Given the same number of losers and winners, the more losers are needed for the safe arm to be implemented forever, the less insured are unsure voters against being trapped with the bad risky arm and the more anxious are they to experiment.

### 1.4.2 The Socially Optimal Rule

The utilitarian social planner who is interested in maximizing the sum of players' expected payoffs makes a fully informed decision, because players are patient, and so there is time to make that decision. The social planner chooses the risky arm over the safe one unless he learns that the safe arm is socially desirable, that is, unless at least $i^*$ players receive bad news, where $i^*$ is the smallest $i$ such that $Ns \geq (N-i)g + ib$. That is, $i^*$ is defined by[13]

$$i^* := \left\lceil \tfrac{g-s}{g-b} N \right\rceil = \lceil (1-p_M)N \rceil .$$

Therefore, the *socially optimal rule* is the $Q^*$-rule, where $Q^* := i^*$. This is summarized by Lemma 1.4 below. Note that, by definition of the $Q^*$-rule,

$$p_M \geq 1 - \tfrac{Q^*}{N}.$$

It follows that the higher are the relative gains form the risky arm $g - s$ or the lower are the relative losses $s - b$, that is, the lower is the myopic cut-off $p_M$, the more players must turn out to be losers for the safe arm to be socially desirable.

**Lemma 1.4** (Social Planner)**.** *The utilitarian social planner chooses the risky arm over the safe arm for any belief of unsure voters unless $Q^*$ players learn that they are losers.*

For the equilibrium to be socially optimal, two conditions must be satisfied. First, the $Q^*$-rule must be applied. Second, unsure voters must vote the risky arm for any belief they may have, that is, the cut-offs must be $p^{Q^*}(i,j) = 0$ for all $i \leq i_N^{Q^*}$ and $j \leq j_N^{Q^*}$, which is guaranteed if the upper bounds are $\tilde{p}^{Q^*}(i,j) = 0$ for all $i \leq i_N^{Q^*}$ and $j \leq j_N^{Q^*}$.

Theorem 1.3 implies that, with learning via both bad and good news, $\tilde{p}^{Q^*}(i,j) = 0$ for all $i \leq i_N^{Q^*}$ and $j \leq j_N^{Q^*}$ if and only if the highest upper bound is $\tilde{p}^{Q^*}(0, j_N^{Q^*}) = 0$, that is, if and only if $(s-b)(Q^*-1) \leq g - s$. This restricts the set of qualified majority rules under which unsure voters experiment with the risky arm for any belief they may have and thus behave socially optimal. Theorem 1.5 below states that they do so in equilibrium if one of the unanimity rules leads to the socially desirable outcome, that is, if $Q^* = 1$ or $Q^* = N$. If $Q^* = 1$, then the insurance effect is strongest, because the safe arm is implemented if there is at least one loser. If $Q^* = N$, then the relative gains from the risky arm are very high or the relative losses are very low, and so unsure voters are willing to take the risk and experiment with the risky arm. Under other qualified majority rules, unsure voters may be too anxious to experiment for low beliefs, that is, $\tilde{p}^{Q^*}(0, j_N^{Q^*}) > 0$ for some $i \leq i_N^{Q^*}$ and $j \leq j_N^{Q^*}$.

**Theorem 1.5** (Socially Optimal Rule with Learning via Mixed News)**.** *If $\lambda_g > 0$, then the equilibrium of the game with the $Q^*$-rule is socially optimal if $Q^* = 1$ or $Q^* = N$.*

If learning is via bad news only, then there is no anxiousness effect and the $Q^*$-rule ensures that unsure voters are always willing to experiment with the risky arm. Theorem 1.4 implies that the highest cut-off is $p^{Q^*}(0) = 0$ if and only if $(s-b)(Q^*-1) \leq (g-s)(N - Q^* + 1)$, which holds strictly by definition of the $Q^*$-rule. Theorem 1.6 follows.

**Theorem 1.6** (Socially Optimal Rule with Learning via Bad News)**.** *If $\lambda_g = 0$, then the equilibrium of the game with the $Q^*$-rule is socially optimal.*

[13]The ceiling function $\lceil x \rceil$ gives the smallest integer not less than $x$.

## 1.5 Conclusion

The chapter has studied a dynamic problem of collective decision making via voting and thus has contributed to understanding of collective experimentation, the work started by Strulovici (2010a). Considering an undiscounted model buys tractability and does not make it less fruitful, because players' interests may be in conflict, while the decision has to be made collectively.

The first extension one may consider is the case of inconclusive news, which is bound to have richer dynamics.[14] For example, consider learning via bad news only but with inconclusive bad news, that is, with bad news arriving not only if a voter is a loser, but also at a lower rate if she is a winner. Then upon receiving news the unsure voter becomes a loser, but she may not be the loser forever for two reasons. On the one hand, if experimentation continues, then her belief about being a winner increases in absence of other bad news and may go above the cut-off used by unsure voters. This makes her willing to experiment with the reform again. On the other hand, because of the insurance effect, the appearance of another loser increases unsure voters' incentives to experiment. As a result, the previous loser may turn out to be optimistic enough now to vote for the reform.

Another natural extension is to look at players with correlated types of their risky arms. Indeed, there is often some common ground between voters regarding the reform. That is why it is plausible that voters learn about whether they benefit from the reform after observing news received by others. Considering correlated types also makes a bridge between the work of Keller, Rady, and Cripps (2005) and Keller and Rady (2015a) and that of Strulovici (2010a) and mine. Working with an undiscounted model may allow to go beyond a two-player case and a unanimity rule like in Strulovici (2010b), and see what else can be said besides that players seem to have higher incentives to experiment and efficiency is improved.

[14]Keller and Rady (2010) analyzes an exponential bandit model with inconclusive good news and the same type of the risky arm among players. Keller and Rady (2015a) studies inconclusive bad news.

# Chapter 2

# Restless Strategic Experimentation

## 2.1 Introduction

Success of many decisions crucially depends on whether it is taken into account that environments change over time. A decision about whether to adopt a new seed is of vital interest, especially in developing countries. There, even a modest improvement of yields can provide food security to millions of people. Suitability of the new seed depends on weather and soil conditions, but these conditions change over time. Knowing this and also being able to learn from experimentation by neighbors, how would and should farmers behave? Alternatively, think about experimentation with a new drug which is believed to be effective against a certain virus. Given that viruses mutate over time, even though the drug may succeed at the moment, it does not mean that it will always perform better than placebo. How should the experimentation be carried out?

Questions I would like to answer are: How do players experiment in changing, or *restless*, environments? Is it always socially optimal to learn the current state? Can players behave efficiently on their own?

I provide a model of how strategic players experiment in restless environments under free flow of information among players and uncertainty regarding the current state. Specifically, there is a finite number of players endowed with one unit of perfectly divisible resource each. Each player continuously chooses how to split her resource between two arms: safe and risky. The risky arm is preferred over the safe one if and only if it is good. The state of the arm, good or bad, can be learned through experimentation with it. Learning is either via good news or via bad news. For example, with learning via good news, players receive so-called news if and only if the state is good. The arrival of news is publicly observable. The state of the arm changes, or "reboots," exogenously at times unobservable to players. Therefore, the arrival of news reveals the current state only. Allowing for the restless state is the key contribution of this chapter.

The first finding is the social optimality of myopic behavior. Surprisingly, because players are forward-looking, myopic behavior turns out to be socially optimal when the state is restless and learning is relatively slow. The learning can be slow, because the state changes quickly or because there are few players to learn from. Furthermore, this result holds even when players are patient.

I look for symmetric Markov perfect equilibria in pure strategies. That is, in equilibrium, players allocate the same fraction of their resource to the risky arm given the belief about the state being good. I solve in closed form for such an equilibrium and establish its existence and uniqueness analytically in some cases and numerically in others. The second finding is the efficiency of symmetric equilibrium. I find that the equilibrium is efficient only if it is myopic when learning is via good news, while the equilibrium does not have to be myopic for it to be efficient when learning is via bad news.

**Related literature.** This chapter contributes to the literature on strategic experimentation with Bolton and Harris (1999) the founding paper. In Bolton and Harris (1999), news arrives according to a Brownian process rather than a Poisson process.[1] It builds on Keller, Rady, and Cripps (2005) and Keller and Rady (2015a) allowing the underlying state to change exogenously over time.

Belief dynamics when the state changes over time are qualitatively different from those with the rested state. The belief about the state of the risky arm evolves even if players do not experiment. Furthermore, players are only certain about the state upon arrival of news, that is, they learn about the current state only. In absence of news, players belief drifts to a certain stationary belief. Inconclusiveness of news about the state in general is reminiscent of models with inconclusive or imperfect news, in which, for example, good news arrives not only if the state is good, but also at a slower rate if it is bad. However, with inconclusive good news, which is analyzed in Keller and Rady (2010), players are unsure about the state even after the arrival of news. Furthermore, they also always become more pessimistic about the state in absence of news. Keller and Rady (2015a) considers a model with inconclusive bad news.

Vasama (2017) extends the model of Bolton and Harris (1999) and allows the state to change endogenously over time. Specifically, if players experiment, they not only learn the current state, but also make it more likely that the state is good in the next instant. It is game of both information and payoff externalities. Vasama (2017) states a system of differential equations which each player' value function satisfies and concludes existence of possibly multiple symmetric Markov equilibria.

Fryer and Harms (2015) studies a general two-armed bandit model in which the expected return from the risky arm increases if the arm is chosen and decreases otherwise. They show that the optimal strategy can be described by Gittins index (Gittins, 1979). Bose and Makris (2016) analyzes a decision problem of whether to undertake a project which is comprised of several tasks and, if so, whether to put an effort or slack on each task. Effort is costly, but slacking may lead to more difficult tasks ahead.

Board and Meyer-ter-Vehn (2013) considers a model of firm reputation in which the reputation is restless from the point of view of consumers and depends on the firm's investments in the quality. Board and Meyer-ter-Vehn (2014) analyzes an extension with the firm not observing the underlying quality either and being allowed to exit the market. Halac and Prat (2016) studies a two-sided moral hazard model of managerial attention and agent effort which has similar dynamics.

Keller and Rady (1999) analyzes a model of a monopoly who faces changing demand curve. Keller and Rady (2003) considers an extension to a duopoly.

Quite a few papers in the literature on operations research and engineering study restless bandits and mostly focus on the optimality of the Whittle index (Whittle, 1988), a generalization of the Gittins index. Whittle (1988) considers a model for optimizing the allocation of effort among $m$ out of $n$ arms and describes how to use the Lagrangian multiplier approach to assign indices to arms. He conjectures that allocating effort to $m$ arms with the highest indices is optimal as $m$ and $n$ go to infinity. Weber and Weiss (1990) provide sufficient conditions for the conjecture to hold. Bertsimas and Niño-Mora

[1]Hörner and Skrzypacz (2016) surveys the strategic experimentation literature.

(1996) and Niño-Mora (2001) also provide sufficient conditions for indexability. Gittins, Glazebrook, and Weber (2011) gathers together different models of multi-armed bandits and discusses optimality of allocation indices.

## 2.2 The Model

**Players, actions, and states.** There are $I \geq 1$ players, each endowed with one unit of perfectly divisible resource per unit of time. Time $t \in [0, \infty)$ is continuous, and the horizon is infinite. At each time $t$, players choose how to split their resource between two arms: a safe arm $S$ and a risky arm $R$. Specifically, player $i$ chooses which fraction $x_{i,t} \in [0, 1]$ to allocate to $R$ in the interval $[t, t + \mathrm{d}t)$. She then allocates $1 - x_{i,t}$ to $S$. I say that player $i$ experiments if $x_{i,t} > 0$, where $i = 1, \ldots, I$.

At time $t$, the state of the risky arm is $\omega_t \in \{0, 1\}$. The arm is said to be good if $\omega_t = 1$, and it is bad otherwise. The initial state $\omega_0$ is exogenously specified. The state is *restless*: it changes, or "reboots," over time. The state reboot is modeled as follows: First, independently of the current state and actions taken by players, times at which the state may change are determined by a Poisson process with intensity $\phi > 0$. These are referred to as the reboot times. Second, if such a time arrives, the arm is good in the next instant with probability $\pi \in [0, 1]$, referred to as the post-reboot probability. Formally, the state $\omega_t$ reboots in the interval $[t, t + \mathrm{d}t)$ with probability $\phi \mathrm{d}t$, and then $\omega_{t+\mathrm{d}t} = 1$ with probability $\pi$:[2,3]

| | $\omega_{t+\mathrm{d}t} = 1$ | $\omega_{t+\mathrm{d}t} = 0$ |
|---|---|---|
| $\omega_t = 1$ | $1 - (1 - \pi)\phi \mathrm{d}t$ | $(1 - \pi)\phi \mathrm{d}t$ |
| $\omega_t = 0$ | $\pi \phi \mathrm{d}t$ | $1 - \pi \phi \mathrm{d}t$ |

I say that the state is *absorbing* after the reboot if $\pi$ is equal to $0$ or $1$.

**Information.** Actions are publicly observable. In contrast, players observe neither the initial state $\omega_0$ nor the reboot times. Instead, they learn about the current state through public news. Given the state $\omega_t$ and the aggregate resource allocation $X_t := \sum_{i=1}^{I} x_{i,t}$ to $R$ at time $t$, news is generated according to a Poisson process with intensity $\lambda_{\omega_t} X_t$, where $\lambda_\omega \geq 0$. I say that learning is via *good news* if $\lambda := \lambda_1 > \lambda_0 = 0$, and it is via *bad news* if $\lambda := \lambda_0 > \lambda_1 = 0$. Without loss of generality, $\lambda$ can be normalized to $1$.

**Payoffs.** The safe arm yields a constant flow payoff normalized to $0$. In the good news case, the risky arm has a flow payoff $-b$. If player $i$ allocates $x_{i,t} > 0$ to $R$ at time $t$,

[2]The state follows a Markov chain. In discrete time with period length $1$, the transition matrix takes the following form:

| | $\omega_{t+1} = 1$ | $\omega_{t+1} = 0$ |
|---|---|---|
| $\omega_t = 1$ | $p_{11}$ | $1 - p_{11}$ |
| $\omega_t = 0$ | $1 - p_{00}$ | $p_{00}$ |

Here, $p_{11} := 1 - (1 - \pi)\phi$ (resp., $p_{00} := 1 - \pi\phi$) is the probability that, if the state is $1$ (resp., $0$) in period $t$, it will be $1$ (resp., $0$) in the next period $t + 1$ as well. The reboot rate $\phi$ and the post-reboot probability $\pi$ pin down the probabilities $p_{11}$ and $p_{00}$.

[3]Keller, Rady, and Cripps (2005) and Keller and Rady (2015a) are special cases with $\phi = 0$.

the risky arm also yields a lump-sum $g + b$ at Poisson times with intensity $x_{i,t}$ only if the current state $\omega_t$ is 1. In the bad news case, the risky arm has a flow payoff $g$. If player $i$ allocates $x_{i,t} > 0$ to $R$, the risky arm also yields a lump-sum $-(g + b)$ at Poisson times with intensity $x_{i,t}$ only if $\omega_t = 0$. Normalization of payoffs is without loss and for notational convenience. In both good news and bad news cases, if player $i$ allocates the whole resource to $R$ at time $t$, that is, if $x_{i,t} = 1$, the expected flow payoff in the interval $[t, t + \mathrm{d}t)$ is $g$ if $\omega_t = 1$ and $-b$ if $\omega_t = 0$. Parameters are such that $g > 0 > -b$. Players prefer the good risky arm over the safe one, but the safe arm over the bad risky one.

Conditional on the state, arrival of lump-sums is independent across players. These lump-sums are interpreted as news received by players.

Players are risk-neutral and discount future at (common) rate $r > 0$. Given player $i$'s actions $\{x_{i,t}\}_{t\geq 0}$ and the number $N_{i,t}$ of lump-sums she receives up to time $t$, her realized payoff in the good news case is

$$\int_0^\infty re^{-rt}(-bx_{i,t}\mathrm{d}t + (g + b)\mathrm{d}N_{i,t}).$$

Player $i$'s realized payoff in the bad news case is

$$\int_0^\infty re^{-rt}(gx_{i,t}\mathrm{d}t - (g + b)\mathrm{d}N_{i,t}).$$

It is a game of informational externalities, there is *no payoff externality*. There is no common interest, in the sense that each player would like others to experiment on her behalf, not to bear the cost herself.

**Belief dynamics, strategies, and equilibrium.** Players have a (common) prior belief $p_0 \in [0, 1]$ that the initial state $\omega_0$ is 1. As they share the same information, it is natural to assume they have a (common) belief $p_t$ that $\omega_t = 1$ at any time $t$.

I assume that players' actions depend on time $t$ and public history only via the left-sided limit of the belief $p_{t-} = \lim_{\varepsilon\to 0} p_{t-\varepsilon}$. That is, I consider *Markov strategies*. A pure Markov strategy of player $i$ is a measurable function $x_i\colon [0, 1] \to [0, 1]$ such that $x_{i,t} = x_i(p_{t-})$ for each $i = 1, \ldots , I$. To analyze the belief trajectory, suppose each $x_i$ is continuous at $p \in [0, 1]$ and define a function $X\colon [0, 1] \to [0, I]$ by $X(p) := \sum_{i=1}^{I} x_i(p)$.

If there is no news in the interval $[t, t + \mathrm{d}t)$, it follows from Bayes' rule that

$$p_{t+\mathrm{d}t} = \phi\mathrm{d}t \cdot \pi + (1 - \phi\mathrm{d}t) \cdot \tfrac{p_t(1-\lambda_1 X(p_t)\mathrm{d}t)}{p_t(1-\lambda_1 X(p_t)\mathrm{d}t)+(1-p_t)(1-\lambda_0 X(p_t)\mathrm{d}t)} + o(\mathrm{d}t).$$

The term $\phi\mathrm{d}t \cdot \pi$ reflects the possibility that the state reboots in $[t, t + \mathrm{d}t)$ and $\omega_{t+\mathrm{d}t} = 1$ with probability $\pi$. The second term reflects players learning about the state if no news arrives in $[t, t + \mathrm{d}t)$. Taking $\mathrm{d}t \to 0$, the ordinary differential equation that governs the belief is as follows:

$$\dot{p}_t = \phi(\pi - p_t) - (\lambda_1 - \lambda_0)X(p_t)p_t(1 - p_t). \tag{2.1}$$

The second term is standard for experimentation with the rested, or unchanging, state. It captures that, in absence of news, players become more pessimistic about the state if learning is via good news, $1 = \lambda_1 > \lambda_0 = 0$, and they become more optimistic if learning is via bad news, $1 = \lambda_0 > \lambda_1 = 0$. The term $\phi(\pi - p_t)$ captures restlessness of the state. Even if players do not experiment at time $t$, that is, even if $X(p_t) = 0$, and unless $p_t = \pi$, the belief is not stationary and drifts toward $\pi$.

Let $f$ denote *drift* in the belief, that is,

$$f(p) := \phi(\pi - p) - (\lambda_1 - \lambda_0)X(p)p(1-p).$$

To ensure that $\dot{p} = f(p)$ has a unique solution, I impose some regularity conditions on player $i$'s strategy $x_i$, similar to Klein and Rady (2011). Specifically, $x_i$ is *admissible at* $p^* \in [0,1]$ if one of the following three conditions for $x_i$ (and so for $X$ and $f$) is satisfied:

(i) $f(p^*) = 0$,
(ii) $f(p^*) > 0$ and $x_i$ (and so $X$ and $f$) is right-continuous at $p^*$, or
(iii) $f(p^*) < 0$ and $x_i$ (and so $X$ and $f$) is left-continuous at $p^*$.

The action $x_i$ is called *admissible* if there is a finite number $n$ of beliefs $p_j^*$, where $j = 1, \ldots, n$, with $0 \leq p_1^* < \cdots < p_n^* \leq 1$ such that $x_i$ is Lipschitz continuous on each interval $[0, p_1^*)$, ..., $(p_j^*, p_{j+1}^*)$, ..., $(p_{n-1}^*, p_n^*]$ and $x_i$ is admissible at each $p_j^*$.

If news arrives at time $t$, the belief jumps from $p_{t-}$ to $p_t$ defined by Bayes' rule as follows:

$$p_t = \frac{\lambda_1 p_{t-}}{\lambda_1 p_{t-} + \lambda_0(1-p_{t-})}.$$

In the good news case, the belief jumps to 1, and so players learn immediately that the current state $\omega_t$ is 1. Unless the post-reboot probability $\pi$ is equal to 1, the belief does not stay at 1, but drifts down according to (2.1) in the next instant. In the bad news case, the belief jumps to 0 and players learn that $\omega_t = 0$. Unless $\pi = 0$, the belief does not stay at 0, but drifts up according to (2.1) in the next instant. In other words, because the state is restless, the arrived news is conclusive, but only about the current state.

I look for symmetric *Markov perfect equilibria* in pure strategies. That is, in equilibrium, players allocate the same fraction of their resource to the risky arm given the belief.

## 2.3 Learning with the Restless State

To understand how learning with a restless state affects the social optimum and equilibrium, it turns out to be enough to allow for one potential change of the state only. In other words, the socially optimal and equilibrium behavior with $\pi \in (0,1)$ is qualitatively the same as with $\pi = 1$ or $\pi = 0$. The argument is presented for the social planner's problem and is summarized by Lemmata 2.1 and 2.2. A similar reasoning applies for the strategic problem.

### 2.3.1 Learning via Good News

Suppose $\pi \in (0,1)$ and learning is via good news, $1 = \lambda_1 > \lambda_0 = 0$. The equation (2.1) that governs evolution of the belief in absence of news takes the form:

$$\dot{p} = \phi(\pi - p) - X(p)p(1-p). \tag{2.2}$$

If players do not experiment at the belief $p$, that is, if $X(p) = 0$, then their belief still evolves and drifts toward $\pi$ according to

$$\dot{p} = \phi(\pi - p). \tag{2.3}$$

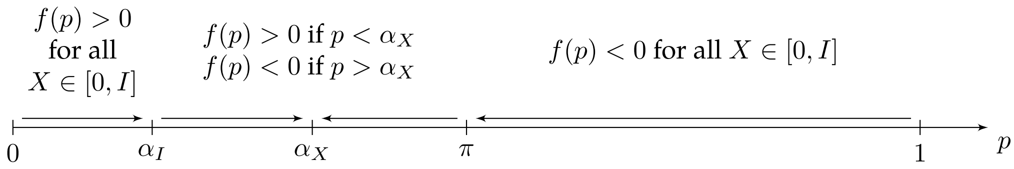


**Figure 2.1.** Dependence of the drift in the belief $f(p)$ on the belief $p$ and the aggregate allocation $X$ if learning is via good news. Parameters: $(I, X, \phi, \pi) = (2, 0.57, 0.75, 0.5)$.

If players allocate $X(p) > 0$ to $R$ altogether, then their belief evolves according to (2.2), which can be rewritten as follows:

$$\dot{p} = -X(p)(p - \alpha_{X(p)}) \underbrace{(\beta_{X(p)} - p)}_{>0}, \tag{2.4}$$

where $\alpha_X$ and $\beta_X$ are defined in Appendix B.1.1 and are such that $\alpha_X \in (0, \pi)$ and $\beta_X > 1$ for all $X \in (0, I]$, $\alpha_X$ is decreasing with $X$, and $\lim_{X \to 0} \alpha_X = \pi$. It is convenient to define $\alpha_0 := \pi$.

The belief $\alpha_X$ is the *stationary belief* for the aggregate allocation $X$. This is the belief at which two forces, described next, balance each other out. The first force makes players more optimistic about the state of $R$ when their belief is below the post-reboot probability $\pi$, because the state may reboot and then it is good with probability $\pi$. The second force makes players more pessimistic in absence of news. The larger is the fraction $X$ allocated to $R$, the higher is the rate at which players expect news to arrive. If news does not arrive, then players become more pessimistic about the state. Hence, the stationary belief is lower for larger $X$.

The stationary belief $\alpha_I$ and the post-reboot probability $\pi$ divide the unit interval into three regions (see Figure 2.1). The drift in the belief $p$ depends on the aggregate allocation $X$ differently across these regions, as immediately follows from (2.3) and (2.4). If $p < \alpha_I$, then $f(p) > 0$ for all $X \in [0, I]$. If $p > \pi$, then $f(p) < 0$ for all $X \in [0, I]$. That is, the drift is independent of players' actions. No matter what players do, they become more optimistic about the state of $R$ in absence of news if $p < \alpha_I$ and more pessimistic if $p > \pi$. If $p \in [\alpha_I, \pi]$, then $f(p) > 0$ if $p < \alpha_{X(p)}$, $f(p) < 0$ if $\alpha_{X(p)}$, and $f(p) = 0$ otherwise. That is, the drift depends on players' actions.

The socially optimal resource allocation has the bang-bang property, as stated in Section 2.4 below and Appendix B.1.1. That is, it is optimal to allocated all resources to $R$ for beliefs above a certain cut-off $p^*$ and not to experiment for beliefs below $p^*$. Belief dynamics around the cut-off and so the socially optimal behavior depend on the position of $p^*$ relative to $\alpha_I$ and $\pi$. The behavior is qualitatively different for $p^* < \alpha_I$, $p^* \in (\alpha_I, \pi)$, and $p^* > \pi$.

**Lemma 2.1** (Socially Optimal Experimentation in the Good News Case)**.**

(*i*) *If $p^* < \alpha_I$, there exists $t^*$ such that $p_t \in [\alpha_I, 1]$ and $X^*(p_t) = I$ for all $t \geq t^*$, and*
  - *if $p_0 < \alpha_I$, then $t^*$ is the first time news arrives;*
  - *if $p_0 > \alpha_I$, then $t^* = 0$.*

(*ii*) *If $p^* \in (\alpha_I, \pi)$, there exists $t^*$ such that $p_t \in [p^*, 1]$ and $X^*(p_t) \in \left\{\frac{\phi(\pi - p^*)}{p^*(1-p^*)}, I\right\}$ for all $t \geq t^*$, and*
  - *if $p_0 < p^*$, then $t^*$ is the time $p^*$ is reached;*
  - *if $p_0 > p^*$, then $t^* = 0$.*

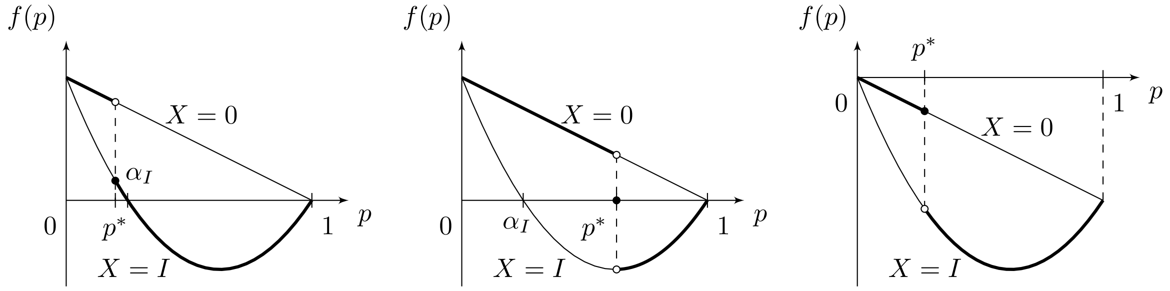


**Figure 2.2.** The drift in the belief $f(p)$ as a function of the belief $p$ given the aggregate resource allocation $X$ and with learning via good news. Thick curves in the left and middle panels (resp., in the right panel) show the socially optimal allocation when $\pi = 1$ (resp., when $\pi = 0$). Parameters: $(I, \phi, r, g, b) = (2, 0.5, 0.1, 1, b)$ with $b = 0.25$ (left), $b = 4$ (middle), and $b = 1$ (right).

(*iii*) *If* $p^* > \pi$, *there exists* $t^*$ *such that* $p_t \in [0, \pi]$ *or* $p_t \in [\pi, p^*]$ *and* $X^*(p_t) = 0$ *for all* $t \geq t^*$, *and*
- *if* $p_0 < p^*$, *then* $t^* = 0$;
- *if* $p_0 > p^*$, *then* $t^*$ *is the time* $p^*$ *is reached.*

With $p^* < \alpha_I$ and independently of the prior, experimentation always starts, and all resources are allocated to $R$ from that point onward. With $p^* \in (\alpha_I, \pi)$, experimentation also starts independently of the prior, but now only a fraction of resources is allocated to $R$ at $p^*$. If the state is good from the first reboot onward, that is, if $\pi = 1$, one of these two cases arises. In contrast, with $p^* > \pi$, experimentation never starts or ceases in finite time. This is what is observed if the state is bad after the first reboot onward, that is, if $\pi = 0$.

Suppose $\pi = 1$, that is, the state is absorbing and it is good after the reboot onward. Given the aggregate allocation $X$ to $R$, the stationary belief $\alpha_X$ takes the simple form

$$\alpha_X := \min\left\{\tfrac{\phi}{X}, 1\right\}.$$

If $p^* < \alpha_I$, then $f(p^*_-) > 0$ and $f(p^*_+) > 0$, where $f(p^*_-) := \lim_{\varepsilon \to 0} f(p^* - \varepsilon)$ and $f(p^*_+) := \lim_{\varepsilon \to 0} f(p^* + \varepsilon)$ are the left and right limits of drift in the belief. This cut-off is referred to as permeable.[4] If $p < p^*$, then the belief drifts up toward $p^*$ due to the possible state reboot and reaches it in finite time. If $p > p^*$, then $p^*$ is never reached. Because the state may reboot and in absence of news, the belief drifts toward $\alpha_I$. It jumps to 1 and stays there if news arrives. At $p^*$, admissibility requires

$$X(p^*) = X(p^*_+) = I,$$

and so the belief drifts through $p^*$. It follow that the belief is eventually trapped in $[\alpha_I, 1]$. The left panel in Figure 2.2 illustrates this case.

If $p^* > \alpha_I$, then $f(p^*_-) > 0$ and $f(p^*_+) < 0$. This cut-off is convergent. If $p < p^*$, then the belief drifts up toward $p^*$ due to the possible state reboot and reaches it in finite time. If $p > p^*$, then absence of news reverses the drift direction. The belief drifts down toward $p^*$ and reaches it in finite time if no news arrives. If news does arrive, then the belief jumps to 1 and stays there. The positive drift below $p^*$ eliminates $f(p^*) < 0$, while the

[4] I adopt terminology of Board and Meyer-ter-Vehn (2013). I say that cut-off $p^* \in (0, 1)$ is (i) *permeable* if both $f(p^*_-) > 0$ and $f(p^*_+) > 0$, or both $f(p^*_-) < 0$ and $f(p^*_+) < 0$; (ii) *convergent* if $f(p^*_-) \geq 0 \geq f(p^*_+)$, with at most one equality; and (iii) *divergent* if $f(p^*_-) \leq 0 \leq f(p^*_+)$, with at most one equality.

negative drift above $p^*$ eliminates $f(p^*) > 0$. Therefore, admissibility implies $f(p^*) = 0$, pinning down

$$X(p^*) = \tfrac{\phi}{p^*}.$$

That is, the belief drift vanishes at $p^*$. It follows that the belief stays in $[p^*, 1]$. This case is illustrated in the middle panel in Figure 2.2.

Suppose $\pi = 0$, that is, the state is absorbing and is bad after the reboot. For any $X \in [0, I]$, the stationary belief is equal to $0$. For any $p^* \in (0, 1)$, $f(p^*_-) < 0$ and $f(p^*_+) < 0$, and so $p^*$ is permeable. If $p < p^*$, then $p^*$ is never reached. The belief drifts toward $0$ due to the possible state reboot. If $p > p^*$, then both the possibility of the reboot and absence of news push the belief toward $p^*$. If no news arrives, then the belief reaches $p^*$ in finite time. If news arrives, then the belief jumps to $1$, whereupon it starts drifting back toward $p^*$. At $p^*$, admissibility requires

$$X(p^*) = X(p^*_-) = 0,$$

and so the belief drifts through $p^*$. The right panel in Figure 2.2 illustrates this case.

### 2.3.2 Learning via Bad News

Suppose $\pi \in (0, 1)$ and learning is via bad news, $1 = \lambda_0 > \lambda_1 = 0$. The equation (2.1) that governs evolution of the belief in absence of news is as follows:

$$\dot{p} = \phi(\pi - p) + X(p)p(1 - p). \tag{2.5}$$

If players do not experiment at $p$, that is, if $X(p) = 0$, then the belief drifts toward $\pi$ according to

$$\dot{p} = \phi(\pi - p).$$

If players allocate $X(p) > 0$ to $R$ altogether, then the belief evolves according to (2.5), which can be rewritten as follows:

$$\dot{p} = -X(p)(p - \alpha_{X(p)}) \underbrace{(p - \beta_{X(p)})}_{>0}, \tag{2.6}$$

where $\alpha_X$ and $\beta_X$ are defined in Appendix B.1.2 and are such that $\alpha_X \in (\pi, 1)$ and $\beta_X < 0$ for all $X \in (0, I]$, $\alpha_X$ is increasing with $X$, and $\lim_{X \to 0} \alpha_X = \pi$. Define $\alpha_0 := \pi$.

The belief $\alpha_X$ is the *stationary belief* when fraction $X$ is allocated to $R$. This is the belief at which two forces, described next, balance each other out. The first force makes players more pessimistic about the state of $R$, because the state may reboot and then it becomes good with only probability $\pi$. The second force makes players more optimistic in absence of news. The larger is the fraction $X$ allocated to $R$, the higher is the rate players expect news to arrive. If news does not arrive, then players become more optimistic that the state is good. Hence, the stationary belief is higher for larger $X$.

The post-reboot probability $\pi$ and the stationary belief $\alpha_I$ divide the unit interval into three regions (see Figure 2.3). The drift in the belief depends on $X$ differently across these regions. If $p < \pi$, then $f(p) > 0$ for all $X \in [0, I]$. If $p > \alpha_I$, then $f(p) < 0$ for all $X \in [0, I]$. That is, the drift is independent of players' actions. Whether players experiment or not, they become more optimistic if $p < \pi$ and more pessimistic if $p > \alpha_I$. If $p \in [\pi, \alpha_I]$, then $f(p) > 0$ if $p < \alpha_{X(p)}$, $f(p) < 0$ if $p > \alpha_{X(p)}$, and $f(p) = 0$ otherwise.

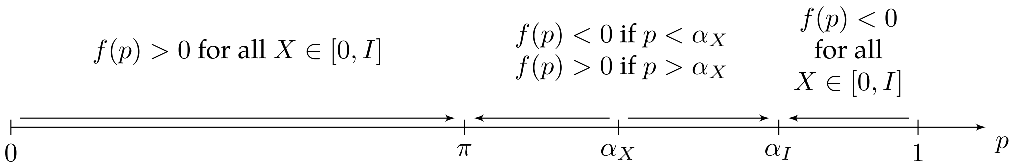


**Figure 2.3.** Dependence of the drift in the belief $f(p)$ on the belief $p$ and the aggregate allocation $X$ if learning is via bad news. Parameters: $(I, X, \phi, \pi) = (2, 0.57, 0.75, 0.5)$.

That is, the drift depends on players actions.

As in the good news case, the socially optimal resource allocation has the bang-bang property; see Section 2.5 below and Appendix B.1.2. Let $p^*$ be the socially optimal cut-off. Belief dynamics around the cut-off and so the socially optimal behavior depend on the position of $p^*$ relative to $\pi$ and $\alpha_I$. This behavior is qualitatively different for $p^* < \pi$, $p^* \in (\pi, \alpha_I)$, and $p^* > \alpha_I$.

**Lemma 2.2** (Socially Optimal Experimentation in the Bad News Case)**.**
(*i*) *If $p^* < \pi$, there exist $t_1^* < t_1^{**} < t_2^* < t_2^{**} < t_3^* < t_3^{**} < \ldots$ such that $p_t \in [p^*, \alpha_I]$ or $p_t \in [\alpha_I, 1]$ for all $t \in [t_1^*, t_1^{**}]$, $p_t \in [p^*, \alpha_I]$ for all $t \in [t_n^*, t_n^{**}]$, $n = 2, 3, \ldots$, $X^*(p_t) = I$ for all $t \in [t_n^*, t_n^{**}]$ and $X^*(p_t) = 0$ for all $t \in (t_n^{**}, t_{n+1}^*)$, $n = 1, 2, 3, \ldots$, and*
  – *if $p_0 < p^*$, then $t_1^* = 0$;*
  – *if $p_0 > p^*$, then $t_1^*$ is the first time $p^*$ is reached;*
  – *$t_n^{**}$ is the first time news arrives after $t_n^*$;*
  – *$t_{n+1}^*$ is the first time $p^*$ is reached after $t_n^{**}$.*

(*ii*) *If $p^* \in (\pi, \alpha_I)$, there exists $t^*$ such that $p_t \in [0, \pi]$ or $p_t \in [\pi, p^*]$ and $X^*(p_t) = 0$ for all $t \geq t^*$, and*
  – *if $p_0 < p^*$, then $t^* = 0$;*
  – *if $p_0 > p^*$, then $t^*$ is the first time news arrives.*

(*iii*) *If $p^* > \alpha_I$, there exists $t^*$ such that $p_t \in [0, \pi]$ or $p_t \in [\pi, p^*]$ and $X^*(p_t) = 0$ for all $t \geq t^*$, and*
  – *if $p_0 < p^*$, then $t^* = 0$;*
  – *if $p_0 > p^*$, then $t^*$ is the first time news arrives or the time $p^*$ is reached.*

With $p^* < \pi$ and independently of the prior, experimentation always start and, upon arrival of news, it ceases only for a while. This behavior is observed if the state is good after the first reboot, that is, if $\pi = 1$. In contrast, with $p^* \in (\pi, \alpha_I)$, experimentation never starts or ceases as soon as first news arrives. With $p^* > \alpha_I$, experimentation also never starts or ceases in finite time. If the state is bad after the first reboot onward, that is, if $\pi = 0$, then one of these two cases arises.

Suppose $\pi = 1$. For any $X \in [0, I]$, the stationary belief is equal to 1. For any $p^* \in (0, 1)$, $f(p_-^*) > 0$ and $f(p_+^*) > 0$, and so $p^*$ is permeable. If $p < p^*$, then the belief drifts up toward $p^*$ due to the possible state reboot and reaches it in finite time. If $p > p^*$, then both the possibility of the reboot and absence of news push the belief toward 1. If news arrives, then the belief jumps to 0, whereupon it start drifting toward $p^*$. At $p^*$, admissibility requires

$$X(p^*) = X(p_+^*) = I,$$

and so the belief drifts through $p^*$. This case is illustrated in the left panel in Figure 2.4.

Suppose $\pi = 0$. Given $X$, the stationary belief $\alpha_X$ takes the simple form

$$\alpha_X := \max\left\{1 - \tfrac{\phi}{X}, 0\right\}.$$

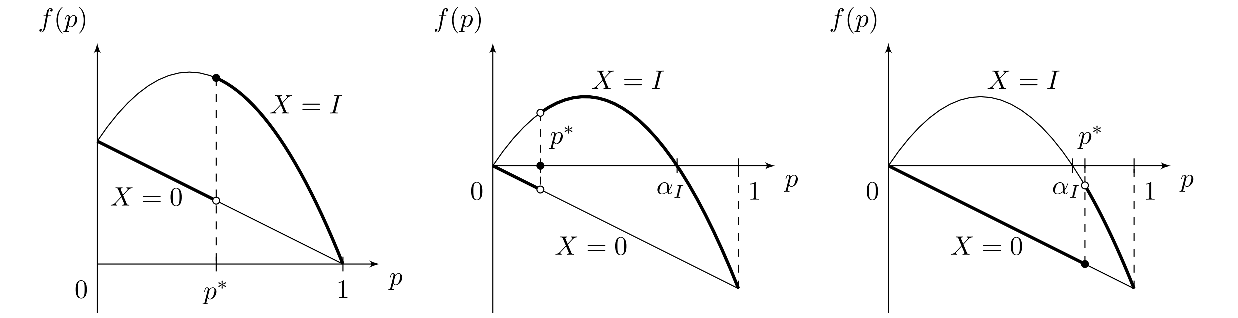


**Figure 2.4.** The drift in the belief $f(p)$ as a function of the belief $p$ given the aggregate resource allocation $X$ and with learning via bad news. Thick curves in the left panel (resp., in the middle and right panels) show the socially optimal allocation when $\pi = 1$ (resp., when $\pi = 0$). Parameters: $(I, \phi, r, g, b) = (2, 0.5, 0.1, 1, b)$ with $b = 2$ (left), $b = 0.7$ (middle), and $b = 4$ (right).

If $p^* < \alpha_I$, then $f(p^*_-) < 0$ and $f(p^*_+) > 0$. This cut-off is referred to as divergent. If $p < p^*$, then $p^*$ is never reached. The belief drifts toward $0$ due to the possible state reboot. If $p > p^*$, then the belief does not reach $p^*$ either. In absence of news, the belief drifts to $\alpha_I$. If news arrives, then the belief jumps to $0$ and stays there. At $p^*$, multiple values of $X(p^*)$ are admitted by admissibility. Specifically, it requires

$$X(p^*) \in \left\{0, \tfrac{\phi}{1-p^*}, I\right\}.$$

If $X(p^*) = 0$ or $X(p^*) = I$, then the belief drifts from $p^*$ to the region with no or full experimentation, respectively. If $X(p^*) = \frac{\phi}{1-p^*}$, then belief drift vanishes at $p^*$. The middle panel in Figure 2.4 illustrates this case with $X(p^*) = \frac{\phi}{1-p^*}$.

If $p^* > \alpha_I$, then $f(p^*_-) < 0$ and $f(p^*_+) < 0$. This cut-off is permeable. If $p < p^*$, then $p^*$ is never reached. The belief drifts toward $0$ because of the possible state reboot. If $p > p^*$, then belief drift stays negative even in absence of news. The belief drifts toward $p^*$ and reaches it in finite time if no news arrives. The belief jumps to $0$ and stays there upon the arrival of news. At $p^*$, admissibility requires

$$X(p^*) = X(p^*_-) = 0,$$

and so the belief drifts through $p^*$. This case is illustrated in the right panel in Figure 2.4.

## 2.4 Good News

Surprisingly, because players are forward-looking and learn from others' experimentation, I find that behaving myopically can be socially optimal when the state is restless. Furthermore, the symmetric equilibrium can be efficient. Whenever the myopic behavior is socially optimal, the equilibrium is myopic, and hence it is efficient. With learning via good news, the equilibrium is efficient only if it is myopic. However, with learning via bad news and as discussed in Section 2.5, the equilibrium does not have to be myopic for it to be efficient.

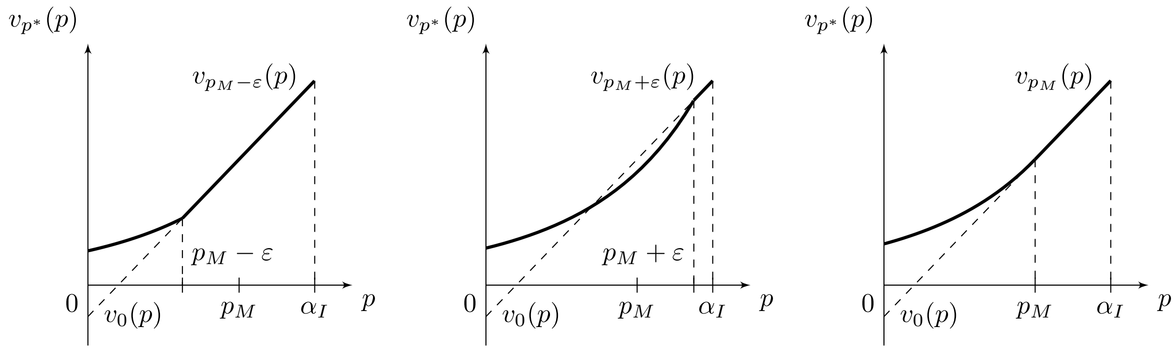


**Figure 2.5.** Heuristic explanation of social optimality of the myopic behavior if learning is via good news. The function $v_{p^*}(p)$ is the average value function when the cut-off $p^*$ is used, where $p^* = p_M - \varepsilon$ (left), $p^* = p_M + \varepsilon$ (middle), and $p^* = p_M$ (right). Parameters: $(I, \phi, r, g, b, \varepsilon) = (2, 0.6, 3, 1, 0.25, 0.075)$.

### 2.4.1 Social Planner's Problem

The myopic cut-off is a belief at which a myopic player, the one who cares about the immediate payoff only, would stop experimenting. This is the belief $p_M$ at which the expected payoff of $R$ is equal to the flow payoff of $S$, that is, $p_M g - (1 - p_M)b = 0$, and so

$$p_M = \tfrac{b}{g+b}.$$

The socially optimal behavior has the bang-bang property, which captures the trade-off between exploration and exploitation. Let $p^*$ denote the socially optimal cut-off. It is optimal not to experiment for beliefs below $p^*$ and to allocate all resources to $R$ for beliefs above $p^*$. Proposition 2.1 below describes the socially optimal behavior with $\pi = 1$. In particular, it states the first interesting result which restlessness brings: it can be optimal to forgo exploration and behave myopically. That is to say, there is a range of parameters for which $p^* = p_M$.

**Proposition 2.1** (Social Optimal in the Good News Case with $\pi = 1$)**.** *The optimal strategy of the social planner is (essentially) unique. It is bang-bang with $X^*(p) = I$ for $p > p^*$ and $X^*(p) = 0$ for $p < p^*$, where $p^*$ and $X^*(p^*)$ are as follows:*
*(i) if $p_M < \frac{\phi}{I}$, then $p^* = p_M$ and $X^*(p^*) = I$;*
*(ii) if $\frac{\phi}{I} < p_M$, then $p^* < p_M$ and $X^*(p^*) = \frac{\phi}{p^*}$.*

The myopic behavior is socially optimal when learning is relatively slow compared to how fast the state reboots. This is the case when the reboot rate $\phi$ is high or there are few players $I$ to learn from, and so $\frac{\phi}{I} > p_M$. Note that optimality of the myopic behavior is independent of the discount rate $r$ and hence of players' patience.

To understand the result heuristically, consider varying $p^*$ and its effect on players' average value function $v_{p^*}(p)$ (see Figure 2.5). Recall that the stationary belief for the aggregate allocation $I$ is $\alpha_I := \min\left\{\frac{\phi}{I}, 1\right\}$. The value function $v_{p^*}(p)$ is linear for $p > p^*$ whenever $p^* < \frac{\phi}{I}$. If players experiment for all beliefs, that is, if $p^* = 0$, then the value function is $v_0(p)$, the dashed line in Figure 2.5. Because players' belief drifts up no matter what they do for $p < \frac{\phi}{I}$, the value function $v_0(p)$ is a continuation value for starting experimentation at $p$. Hence, its slope captures the marginal benefit of waiting at a given belief.

Suppose $p^* = p_M - \varepsilon$ for some small $\varepsilon > 0$. The value function $v_{p_M-\varepsilon}(p)$ is given by the thick curve in the left panel in Figure 2.5. The kink at $p_M - \varepsilon$ captures that the marginal benefit of starting experimentation at $p_M - \varepsilon$ is still below the marginal benefit of waiting. Therefore, if players waited for their belief to increase and allocate resources to $S$ for beliefs slightly above $p_M - \varepsilon$, then the average value function would increase. Suppose $p^* = p_M + \varepsilon$. The value function $v_{p_M+\varepsilon}(p)$ is shown in the middle panel in Figure 2.5. Players would be better off if they allocated resources to $R$ for beliefs slightly below $p_M + \varepsilon$. At $p^* = p_M$, the marginal benefit of waiting is equal to the marginal benefit of starting experimentation, as the right panel in Figure 2.5 shows. Therefore, it is optimal to experiment if and only if the belief is above the myopic cut-off.

When the state reboots slowly, that is, when the reboot rate $\phi$ is low, or when there are quite a few players $I$ to learn from, learning is relatively fast. In such a case, there is an option value to experiment, in the sense that $p^* < p_M$. The optimal cut-off is given by

$$p^* = \tfrac{-b(\phi-r)+\sqrt{\Delta}}{2[br+g(I+r)]}, \tag{2.7}$$

where

$$\Delta := b^2(\phi - r)^2 + 4b\phi[br + g(I + r)].$$

Because learning is faster with more players, it is socially optimal to experiment for a larger range of beliefs. In contrast, a higher reboot rate makes learning optimal for a smaller range of beliefs. Corollaries 2.1 and 2.2 follow.

**Corollary 2.1** (Number of Players)**.** *If $\frac{\phi}{I} < p_M$, then the socially optimal cut-off $p^*$ decreases in the number of players $I$.*

**Corollary 2.2** (Reboot Rate)**.** *If $\frac{\phi}{I} < p_M$, then the socially optimal cut-off $p^*$ increases in the reboot rate $\phi$.*

Proposition 2.2 below describes the socially optimal behavior with $\pi = 0$. There is always an option value to experiment.

**Proposition 2.2** (Social Optimal in the Good News Case with $\pi = 0$)**.** *The optimal strategy of the social planner is (essentially) unique. It is bang-bang with $X^*(p) = I$ for $p > p^*$ and $X^*(p) = 0$ for $p < p^*$, where $p^* < p_M$ and $X^*(p^*) = 0$.*

If the state is bad after the first reboot onward, then the only chance players get to benefit from the good arm is when the initial state is good. That is why, $p^* < p_M$. The socially optimal cut-off $p^*$ is defined implicitly by (B.2) in Appendix B.2.1.

### 2.4.2 Strategic Problem

The symmetric equilibrium either has the bang-bang property or is characterized by two cut-offs $\underline{p}$ and $\bar{p}$ such that $\underline{p} < \bar{p}$. With two cut-offs, neither player experiments for beliefs below $\underline{p}$, each players allocates the whole resource to $R$ for beliefs above $\bar{p}$ and gradually increases the fraction allocated $R$ for beliefs in between.

Proposition 2.3 below states that the symmetric equilibrium exists and is unique when $\pi = 1$. The second interesting result which restlessness bring from Proposition 2.3: the symmetric equilibrium can be efficient. That is, there is a range of parameters for which the equilibrium is bang-bang and the cut-off players use coincides with the socially optimal cut-off $p^*$.

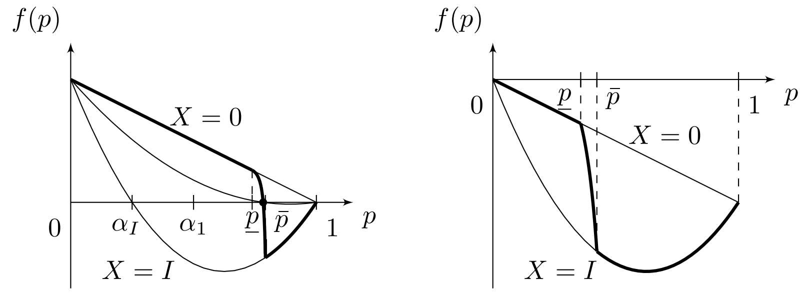


**Figure 2.6.** The drift in the belief $f(p)$ as a function of the belief $p$ given the aggregate resource allocation $X$ and with learning via good news. Thick curve in the left panel (resp., in the right panel) shows the equilibrium allocation when $\pi = 1$ and $\alpha_1 < p_M$ (resp., when $\pi = 0$). Parameters: $(I, \phi, r, g, b) = (2, 0.5, 0.1, 1, b)$ with $b = 4$ (left) and $b = 1$ (right).

**Proposition 2.3** (Symmetric Equilibrium in the Good News Case with $\pi = 1$). *There exists the (essentially) unique equilibrium such that $x^e(p) = 1$ for $p > \bar{p}$ and $x^e(p) = 0$ for $p < \underline{p}$, where $\underline{p}$, $\bar{p}$, and $x^e(p)$ for $p \in [\underline{p}, \bar{p}]$ are as follows:*

(*i*) *if $p_M < \frac{\phi}{I}$, then $\underline{p} = \bar{p} = p_M$ and $x^e(\bar{p}) = 1$;*
(*ii*) *if $\frac{\phi}{I} < p_M < \phi$, then $\underline{p} = \bar{p} = p_M$ and $x^e(\bar{p}) = \frac{\phi}{I\bar{p}}$;*
(*iii*) *if $\phi < p_M$, then $\underline{p} < \bar{p} < p_M$ and $x^e(p)$ increases in $p$.*

Players behave myopically when the reboot rate $\phi$ is high, that is, when $p_M < \phi$. Note that players disregard how many of them are present and behave myopically whenever they would do so if they were alone. Indeed, Proposition 2.1 implies that the socially optimal cut-off with $I = 1$, and so the single player's cut-off, is $p^* = p_M$ when $p_M < \phi$. Recall that the stationary belief for the aggregate allocation 1 is $\alpha_1 := \min\{\phi, 1\}$. It also follows from Proposition 2.1 that myopic behavior is optimal when $p_M < \frac{\phi}{I}$. Corollary 2.3 follows.

**Corollary 2.3** (Efficiency). *If $p_M < \frac{\phi}{I}$, then the symmetric equilibrium is efficient.*

When the state reboots slowly, the equilibrium is characterized by two cut-offs $\underline{p}$ and $\bar{p}$. The lower cut-off is given by

$$\underline{p} = \tfrac{-b(\phi-r)+\sqrt{\Delta}}{2[br+g(1+r)]}, \tag{2.8}$$

where

$$\Delta := b^2(\phi - r)^2 + 4b\phi[br + g(1 + r)],$$

while the upper cut-off $\bar{p}$ is defined implicitly by (B.18) in Appendix B.2.2. The fraction of the resource each player allocates to $R$ for beliefs in $(\underline{p}, \bar{p})$ is $x^e(p)$ given by (B.19) in Appendix B.2.2 and gradually increases from 0 to 1. The equilibrium resource allocation and the corresponding drift in the belief are shown in the left panel in Figure 2.6.

With the low reboot rate $\phi$, there is an option value to experiment, in the sense that $\bar{p} < p_M$. However, players experiment too little. Let $p_I^*$ denote the socially optimal cut-off with $I$ players, defined by (2.7). Comparing $\underline{p}$ defined by (2.8) with $p_1^*$ gives $\underline{p} = p_1^*$. Because the socially optimal cut-off $p_I^*$ is decreasing with $I$ by Corollary 2.1, players not only allocate only a fraction of their resource to $R$ when it is optimal to allocate the whole resource, but also do not experiment for beliefs in $(p_I^*, p_1^*)$. Corollary 2.4 follows.

**Corollary 2.4** (Free-Riding). *If $\frac{\phi}{I} < p_M$, then $\underline{p} = p_1^* > p_I^*$.*

Whenever exists, the symmetric equilibrium with $\pi = 1$ takes the form described in Proposition 2.4 below. I observe existence of the equilibrium numerically, but it is left to show this analytically.

**Proposition 2.4** (Symmetric Equilibrium in the Good News Case with $\pi = 0$)**.** *The equilibrium is such that $x^e(p) = 1$ for $p > \bar{p}$ and $x^e(p) = 0$ for $p < \underline{p}$, where $\underline{p} < \bar{p}$ and $x^e(p)$ increases in $p$ for $p \in [\underline{p}, \bar{p}]$.*

The equilibrium is always characterized by two cut-offs $\underline{p}$ and $\bar{p}$, which are implicitly defined by (B.22) and (B.23). The fraction of the resource each player allocates to $R$ for beliefs in $(\underline{p}, \bar{p})$ is $x^e(p)$ given by (B.24) in Appendix B.2.2 and gradually increases from $0$ to $1$. The right panel in Figure 2.6 depicts this case.

## 2.5 Bad News

With learning via bad news, behaving myopically can also be socially optimal because of the restless state. Furthermore, the symmetric equilibrium can be efficient not only when it is myopic.

### 2.5.1 Social Planner's Problem

The socially optimal resource allocation is bang-bang. Proposition 2.5 below characterized the socially optimal behavior with $\pi = 0$. In particular, similar to the good news case, behaving myopically is optimal when learning is relatively slow compared to how fast the state reboots, and this is independent of players' patience.

**Proposition 2.5** (Social Optimal in the Bad News Case with $\pi = 0$)**.** *The optimal strategy of the social planner is (essentially) unique. It is bang-bang with $X^*(p) = I$ for $p > p^*$ and $X^*(p) = 0$ for $p < p^*$, where $p^*$ and $X(p^*)$ are as follows:*

*(i) if $p_M > 1 - \frac{\phi}{I}$, then $p^* = p_M$ and $X^*(p^*) = 0$;*

*(ii) if $1 - \frac{\phi}{I} > p_M$, then $p^* < p_M$ and $X^*(p^*) \in \left\{0, \frac{\phi}{1-p^*}, I\right\}$.*

To understand the optimality of the myopic behavior heuristically, consider varying $p^*$ and its effect on players' average value function $v_{p^*}(p)$ (see Figure 2.7). Recall that the stationary belief for the aggregate allocation $I$ is $\alpha_I := \max\{1 - \frac{\phi}{I}, 0\}$, and so players' belief drifts down no matter what they do for $p > 1 - \frac{\phi}{I}$. Furthermore, if players stop experimenting at any such belief, their belief continues drifting down and experimentation never resumes. Because the continuation value at $p^*$ is equal to $0$ and so is the marginal benefit of stopping, the slope of $v_{p^*}(p^*_+)$ captures the marginal benefit of experimenting at $p^*$.

Suppose $p^* = p_M - \varepsilon$ for some small $\varepsilon > 0$. The value function $v_{p_M-\varepsilon}(p)$ is given by the thick curve in the left panel in Figure 2.7. The slope $v_{p_M-\varepsilon}(p^*_+)$ is negative. Therefore, players would be better off if they stopped experimenting for beliefs slightly above $p_M - \varepsilon$. Suppose $p^* = p_M + \varepsilon$. The value function $v_{p_M+\varepsilon}(p)$ is the thick curve in the middle panel in Figure 2.7. The kink at $p_M + \varepsilon$ captures that the marginal benefit of experimenting at $p_M + \varepsilon$ is above the marginal benefit of stopping. Therefore, if players experimented for beliefs slightly below $p_M + \varepsilon$, the average value function would increase. At $p^* = p_M$, the marginal benefit of experimenting is equal to the marginal benefit of stopping, as the

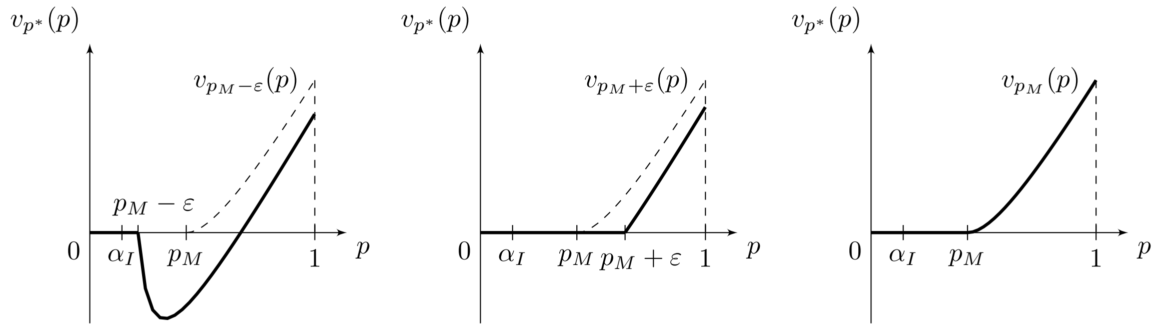


**Figure 2.7.** Heuristic explanation of social optimality of the myopic behavior if learning is via bad news. The function $v_{p^*}(p)$ is the average value function when the cut-off $p^*$ is used, where $p^* = p_M - \varepsilon$ (left), $p^* = p_M + \varepsilon$ (middle), and $p^* = p_M$ (right). Parameters: $(I, \phi, r, g, b, \varepsilon) = (2, 0.6, 1, 1, 4, 0.075)$.

right panel in Figure 2.7 shows. Therefore, it is optimal to experiment if and only if the belief is above the myopic cut-off.

When the reboot rate $\phi$ is low or there are quite a few players $I$ to learn from, there is an option value to experiment, in the sense that $p^* < p_M$. The optimal cut-off is given by

$$p^* = \frac{b(r+\phi)}{br+g(I+r)}. \tag{2.9}$$

It is socially optimal to experiment for a larger range of beliefs when more players are present, and so learning is faster, or when players are more patient. In contrast, a higher reboot rate makes learning optimal for a smaller range of beliefs. Corollaries 2.5 to 2.7 follow.

**Corollary 2.5** (Number of Players). *If $1-\frac{\phi}{I} > p_M$, then the socially optimal cut-off $p^*$ decreases in the number of players $I$.*

**Corollary 2.6** (Reboot Rate). *If $1 - \frac{\phi}{I} > p_M$, then the socially optimal cut-off $p^*$ increases in the reboot rate $\phi$.*

**Corollary 2.7** (Patience of Players). *If $1-\frac{\phi}{I} > p_M$, then the socially optimal cut-off $p^*$ increases in the discount rate $r$.*

Proposition 2.6 below describes the socially optimal cut-off with $\pi = 1$. There is always an option to experiment.

**Proposition 2.6** (Social Optimal in the Bad News Case with $\pi = 1$). *The optimal strategy of the social planner is (essentially) unique. It is bang-bang with $X^*(p) = I$ for $p > p^*$ and $X^*(p) = 0$ for $p < p^*$, where $p^* < p_M$ and $X(p^*) = I$.*

If the state is good after the first reboot onward, it pays off start experimentation even after arrival of news. Therefore, $p^* < p_M$. The socially optimal cut-off $p^*$ is defined implicitly by (B.10) in Appendix B.2.1.

### 2.5.2 Strategic Problem

The symmetric equilibrium either has the bang-bang property or is characterized by two cut-offs $\underline{p}$ and $\bar{p}$ such that $\underline{p} < \bar{p}$. Proposition 2.7 below states existence and uniqueness of the symmetric equilibrium and describes this equilibrium. Similar to the good news case, the symmetric equilibrium can be efficient, but now it does not have to be myopic.

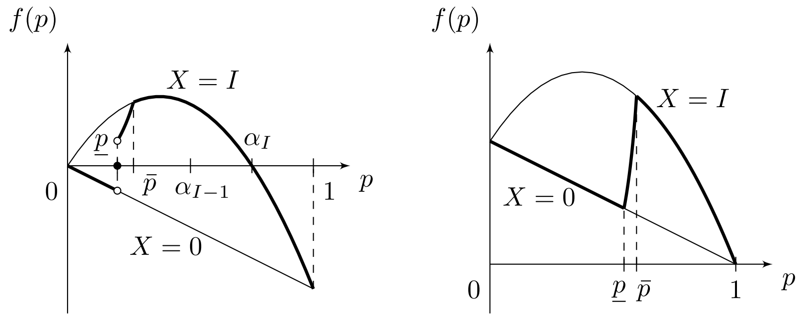


**Figure 2.8.** The drift in the belief $f(p)$ as a function of the belief $p$ given the aggregate resource allocation $X$ and with learning via bad news. Thick curves in the left panel (resp., in the right panel) show the equilibrium allocation with $x^e(\underline{p}) = \frac{\phi}{I(1-\bar{p})}$ when $\pi = 0$ and $\frac{(I-1-\phi)(I+r)}{(I-1)(I+r)-\phi} > p_M$ (resp., when $\pi = 1$). Parameters: $(I, \phi, r, g, b) = (2, 0.5, 0.1, 1, b)$ with $b = 0.7$ (left) and $b = 2$ (right).

**Proposition 2.7** (Symmetric Equilibrium in the Bad News Case with $\pi = 0$)**.** *There exists the (essentially) unique equilibrium such that $x^e(p) = 1$ for $p > \bar{p}$ and $x^e(p) = 0$ for $p < \underline{p}$, where $\underline{p}$, $\bar{p}$, and $x^e(p)$ for $p \in [\underline{p}, \bar{p}]$ are as follows:*

(*i*) *if $p_M > 1 - \frac{\phi}{I}$, then $\underline{p} = \bar{p} = p_M$ and $x^e(\bar{p}) = 0$;*

(*ii*) *if $1 - \frac{\phi}{I} > p_M > \frac{(I-1-\phi)(I+r)}{(I-1)(I+r)-\phi}$, then $\underline{p} = \bar{p} < p_M$ and $x^e(\underline{p}) \in \left\{0, \frac{\phi}{I(1-\underline{p})}, \frac{\phi}{(I-1)(1-\underline{p})}\right\}$;*

(*iii*) *if $\frac{(I-1-\phi)(I+r)}{(I-1)(I+r)-\phi} > p_M$, then $\underline{p} < \bar{p} < p_M$ and $x^e(p)$ increases in $p$.*

Players behave myopically when it is socially optimal to do so according to Proposition 2.5, that is, when learning is relatively slow. The equilibrium is also efficient when it is not myopic. Indeed, if $1 - \frac{\phi}{I} > p_M > \frac{(I-1-\phi)(I+r)}{(I-1)(I+r)-\phi}$, then

$$\underline{p} = \bar{p} = \tfrac{b(\phi+r)}{br+g(I+r)},$$

which is equal to the socially optimal cut-off $p^*$ given by (2.9). It follows that $\underline{p} = \bar{p} \in \left(1 - \frac{\phi}{I-1}, 1 - \frac{\phi}{I}\right)$. Recall that the stationary beliefs for the aggregate allocations $I - 1$ and $I$ are $\alpha_{I-1} := \max\left\{1 - \frac{\phi}{I-1}, 0\right\}$ and $\alpha_I := \max\left\{1 - \frac{\phi}{I}, 0\right\}$. Efficiency is observed, because experimentation by the $I$-th player is crucial, in the sense that it reverts the drift in the belief. If there were $I - 1$ players, then they would become more pessimistic over time no matter what they did. If all $I$ players experiment for beliefs in $\left(1 - \frac{\phi}{I-1}, 1 - \frac{\phi}{I}\right)$, then they become more optimistic in absence of news. Corollary 2.8 follows.

**Corollary 2.8** (Efficiency)**.** *If $p_M > \frac{(I-1-\phi)(I+r)}{(I-1)(I+r)-\phi}$, then the symmetric equilibrium is efficient.*

Otherwise, the equilibrium is characterized by two cut-offs $\underline{p}$ and $\bar{p}$. The upper cut-off is given by

$$\bar{p} = \tfrac{b(I+r-1)}{(g+b)(I+r)-b}, \tag{2.10}$$

while the lower cut-off $\underline{p}$ is defined implicitly by (B.31) in Appendix B.2.2. In this case, there is always an option value to experiment, in the sense that $\bar{p} < p_M$. The fraction of the resource each player allocates to $R$ for beliefs in $(\underline{p}, \bar{p})$ is $x^e(p)$ given by (B.32) in Appendix B.2.2 and gradually increases from $\frac{\phi}{(I-1)(1-\underline{p})}$ to 1. The equilibrium resource allocation with $x^e(\underline{p}) = \frac{\phi}{I(1-\bar{p})}$ and the corresponding drift in the belief are presented in the left panel in Figure 2.8.

Whenever exists, the symmetric equilibrium with $\pi = 1$ takes the form characterized in Proposition 2.8 below. I observe existence of the equilibrium numerically, but it is left to show this analytically.

**Proposition 2.8** (Symmetric Equilibrium in the Bad News Case with $\pi = 1$)**.** *The equilibrium is such that $x^e(p) = 1$ for $p > \bar{p}$ and $x^e(p) = 0$ for $p < \underline{p}$, where $\underline{p} < \bar{p}$ and $x^e(p)$ increases in $p$ for $p \in [\underline{p}, \bar{p}]$.*

The equilibrium is always characterized by two cut-offs $\underline{p}$ and $\bar{p}$, which are implicitly defined by (B.35) and (B.36). The fraction of the resource each player allocates to $R$ for beliefs in $(\underline{p}, \bar{p})$ is $x^e(p)$ given by (B.37) in Appendix B.2.2 and gradually increases from $0$ to $1$. The right panel in Figure 2.8 shows this case.

## 2.6 Conclusion

A natural variation is to allow news to arrive continuously but with some noise, that is, according to a Brownian process. I have not solved this variant of the model, though I suspect that similar results would arise. Specifically, if the state is restless, then the change in the belief will not be distributed normally with zero mean, as is the case in Bolton and Harris (1999). If the drift is large, then, independently of players' actions and depending on the drift sign, players will become either more optimistic or pessimistic about the state. As a result, myopic behavior can be socially optimal.

# Chapter 3

# Gizmos

## 3.1 Introduction

Consider a recently graduated PhD student who landed at the top university or got a nice job at the private sector. Should she buy Google Pixie with its advanced camera now that she can afford it? It may turn out though that there is no time to discover a latent photographer, and so Nexus 5X is good enough. Alternatively, think about a middle age researcher who realizes that she should live a more active lifestyle. Suppose the choice fell on mountain biking. Should she buy an advanced bike like Giant Trance Advanced right away or start with a basic version of Giant Stance? If she is a quick learner and an adventurer at heart, then having the advanced bike seems to be the right decision. In contrast, if she does not venture into the difficult slopes, then the basic version may be a more reasonable choice.

In turn, what is the optimal pricing strategy for the producer of smartphones or mountain bikes? Is it always a good idea to sell both versions of the product? If so, should the price of the advanced version be low enough to tempt unsure buyers? The phone producers appear to post prices which are high, but reasonably so. No high-ranked employee at the private sector would think twice about buying the advanced phone on a whim. In contrast, the sky seems to be the limit for bike prices, which may make a buyer ponder about getting the advanced version. It is also not a secret that phones' batteries die relatively fast and that bikes require a lot of maintenance. The question which arises is whether the producer resorts to planned obsolescence and, if so, why?

In this chapter, I analyze the optimal behavior of buyers who enter the market unsure about their own needs and the optimal pricing strategy of a monopolist seller who faces these buyers. I develop a model in which the seller offers advanced and basic versions of a product, or a *gizmo*, to a population of buyers and posts prices she wants to charge for each version. When a buyer enters the market, he is not sure whether he values the advanced features the advanced version provides. The buyer learns about his needs as he experiences either version of the gizmo. The gizmos break, and so the buyer may have to choose again which version to buy given the acquired experience.

Given the prices of the advanced and basic versions, the optimal strategy for a buyer takes the simple form of a cut-off. That is, a buyer without the gizmo buys the advanced version if his belief about needing the advanced features is above the cut-off, and he buys the basic version otherwise. The seller finds it optimal to offer both versions of the gizmo. The optimal prices are such that a buyer buys a new gizmo if and only if the one he has breaks. If the buyer has learned whether he values the advanced features by the time his gizmo breaks, then he may choose to buy a different version, which can be an upgrade or a downgrade over the old one. Depending on parameters, it can be only those who have learned that they value the advanced features who buy the advanced version.

If the seller were to choose the breakdown rate of the gizmos as well as the prices, then she would plan obsolescence. The seller neither wants the gizmos to break immediately nor wants them to be a durable good. The optimal breakdown rate captures the trade-off faced by the seller. On the one hand, because buyers with the basic version who have learned that they need the advanced features upgrade their gizmo to the advanced version only upon a breakdown, a higher breakdown rate increases buyers' surplus, for they get the preferred version faster. As a result, the higher breakdown rate also increases the seller's profit. On the other hand, the cut-off used by buyers increases with the breakdown rate, and so fewer buyers go for the advanced version. Therefore, the seller extracts less surplus from those who would value the advanced features.

**Related literature.** This chapter contributes to several strands of literature: experience goods, planned obsolescence, and learning. In the experience goods literature, the closest paper is Bergemann and Välimäki (2006). Bergemann and Välimäki (2006) considers a monopoly pricing model, in which buyers are uncertain about their valuations when the product is introduced. There is only one version of the product and the product is a durable good. As buyers experience with the product, they learn their valuation stochastically via perfectly revealing signals. Bergemann and Välimäki (2006) looks for the optimal dynamic pricing pattern and argues that all markets can be split into niche markets and mass markets, in which the optimal price patterns are qualitatively different. The monopolist finds it optimal to use penetration pricing in the niche markets and skimming pricing in the mass markets.

Bonatti (2011) also analyzes a dynamic model of monopoly pricing, in which the monopolist offers a menu of contracts to a population of buyers. Buyers privately know about their willingness to pay for the product, but are uncertain about its quality. Information about the product quality is generated through experimentation with the product and is increasing with the total quantity sold. The monopolist is interested in both generating information and screening buyers. Bonatti (2011) shows that the monopolist finds it optimal to increase the sales and lower marginal prices compared to the myopic benchmark. Learning of the product quality is similar to learning of the demand curve in Keller and Rady (1999).[1]

Bulow (1986) studies planned obsolescence and argues that, by making the product less durable, the monopolist can alleviate the commitment problem put forward in Coase (1972).[2] Bulow (1986) uses durability as a proxy for obsolescence, but points out that the matter of obsolescence is often about introduction of a new product and compatibility of the new product with its older versions. Waldman (1993) and Choi (1994) analize the monopolist's incentives to introduce new versions of the product that are incompatible with the old versions. Waldman (1996) considers the monopolist who makes the old versions obsolete by investing in R&D and thus introducing products of superior quality. Pesendorfer (1995) develops a model of fashion cycles. See also Fudenberg and Tirole (1998), Lee and Lee (1998), and Ellison and Fudenberg (2000). Fishman and Rob (2000) considers both technological and physical obsolescence and shows that the monopolist,

[1] Keller and Rady (2003) considers a duopoly that faces the changing demand curve.

[2] For formal proofs of the Coase conjecture, see Stokey (1981), Gul, Sonnenschein, and Wilson (1986), and Ausubel and Deneckere (1989).

who can shorten the product's life, introduces technologically advanced versions at the socially optimal pace.

Deneckere and Liang (2008) considers a monopolist who sells a durable good that depreciates stochastically over time. Deneckere and Liang (2008) characterizes a complete set of stationary equilibria and finds that, if the durability is sufficiently low, then the monopoly equilibrium is the unique equilibrium. That is, through planned obsolescence, the monopolist overcomes the commitment problem introduced in Coase (1972) and does not lose any monopoly power. The following papers suggest other ways to mitigate the commitment problem. Kühn and Padilla (1996) shows that the Coase conjecture fails if the monopolist sells both durable and non-durable goods. Hahn (2006) argues that it does not hold either if the monopolist introduces a damaged good.[3] Board and Pycia (2014) shows that the conjecture fails if buyers can exercise an outside option. Nava and Schiraldi (2018) argues that these examples are not failures of classical insights on Coasian dynamics, but rather the conjecture itself must be revisited. Nava and Schiraldi (2018) considers a monopolist who sells different varieties of a durable good and establishes a revisited Coase conjecture. The conjecture states that the force behind any Coasian equilibrium is market clearing rather than competition or efficiency.

I consider the exponential learning structure via conclusive news, introduced in the strategic setting by Keller, Rady, and Cripps (2005) and Keller and Rady (2015a). (See also Bolton and Harris (1999) in the Brownian context.) As buyers experiment with the product, they receive news which reveal whether they value the advanced features or not. The arrival rate of both types of news is the same, and so buyers' belief stays unchanged in its absence.

## 3.2 The Model

**Players and actions.** Time $t \in [0, \infty)$ is continuous, and the horizon is infinite. A monopolist seller (she) has two versions of a gizmo for sale, $A$ and $B$. The seller can produce either version of the gizmo immediately upon demand and *at no cost*. The $A$-gizmo is an advanced version with extra features, while the $B$-gizmo is a basic version without them. The seller posts the respective gizmo prices $p_A$ and $p_B$ once and for all at the beginning of time. I assume that the prices must be such that *the market is covered*, that is, all buyers are willing to buy one of the versions.

A unit mass of buyers (each he) continuously enters the market at rate $\delta > 0$. Buyers leave the market at rate $\delta$ as well. Each buyer buys a gizmo, if he wants to, as he enters the market. Independent of the version, the gizmo breaks at *exogenous* Poisson rate $\beta > 0$. (I endogenize the breakdown rate in Section 3.4.) If the current gizmo breaks, then the buyer decides whether to buy a new one. The buyer can also replace the gizmo he has and buy a new one at any other time.

Each buyer is of one of two types, $L$ or $H$. Types are independent across buyers. The $L$-type does not care for the extra features the $A$-gizmo provides, while the $H$-type values them, as reflected by the payoffs below.

**Information.** A buyer who enters the market does not know his type. His prior belief $x$ that he is the $H$-type is distributed according to the distribution function $F(x)$ on $[0, 1]$. I

[3] Deneckere and McAfee (1996) is the first to analyze how introduction of the damaged good helps the monopolist price discriminate.

assume that $F$ admits a positive and differentiable density function $f$ satisfying $2f(x) + xf'(x) > 0$ for all $x \in (0, 1)$. The buyer can learn the type through news only if he owns a gizmo.

Independent of the gizmo's version, both the $H$-type and the $L$-type receive news at Poisson rate $\lambda > 0$. News is independent across buyers. It follows from Bayes' rule that the buyer's belief remains constant in absence of news. If news does arrive, then the buyer learns his type perfectly. That is, if the $H$-type (resp., the $L$-type) receives news, then his belief jumps to $1$ (resp., $0$). Therefore, at each moment in time, the belief of a buyer who enters the market with the prior $x$ belongs to the set $\{0, x, 1\}$.

**Payoffs.** The seller and buyers are risk-neutral and discount future at (common) rate $r > 0$. Independent of his type, if a buyer does not buy a gizmo, then he receives a constant flow payoff normalized to $0$. Either version of the gizmo yields a flow payoff $w > 0$ to either type of the buyer. If the $H$-type has the $A$-gizmo, then news is accompanied by a lump-sum $W_A > 0$. Hence, the lump-sums arrive at rate $\lambda$ for such a buyer. (Clearly, this is not a second process.) Define $w_A := \lambda W_A$. I *assume* that $w > w_A$. One interpretation of this assumption is that the gizmo has been on the market for a while, and even though there are technological advances, they are incremental rather than radical innovations. The $L$-type with the $A$-gizmo does not receive any lump-sum. No lump-sum arrives if the buyer has the $B$-gizmo.

Suppose that a buyer buys the $A$-gizmo at time $t$ and that the gizmo breaks at time $t + \tau_A$. Let $N_{A,s}$ denote the number of lump-sums the buyer receives from $t$ to $s$, where $s \in (t, t + \tau_A)$. (If the buyer is the $L$-type, then $N_{A,s} = 0$ for all $s$.) The buyer's realized payoff is

$$\int_t^{t+\tau_A} e^{-(r+\delta)s}[w\mathrm{d}s + w_A \mathrm{d}N_{A,s}] - p_A.$$

Similarly, if the buyer buys the $B$-gizmo at time $t$ and if the gizmo breaks at time $t + \tau_B$, then his realized payoff is

$$\int_t^{t+\tau_B} e^{-(r+\delta)s} w\mathrm{d}s - p_B.$$

Given the structure of the buyer's payoffs, the buyer may want to buy a new gizmo, for example, when he enters the market, when his gizmo breaks, or when he learns his type, in particular, when he learns that he is the $H$-type. Let $M_{A,t}$ (resp., $M_{B,t}$) denote the aggregate mass of buyers who buy the $A$-gizmo (resp., the $B$-gizmo) by time $t$. The seller's realized payoff is

$$\int_0^{\infty} e^{-rt}[p_A \mathrm{d}M_{A,t} + p_B \mathrm{d}M_{B,t}].$$

**Measures of buyers in stationary environment.** Let $m(x, t)$ denote the measure of buyers with the belief $x$ at time $t$. Let $M_H(t)$ (resp., $M_L(t)$) denote the mass of the $H$-types (resp., the $L$-types) who have learned their type by time $t$. Arrival of news determines the evolution of $m(x, t)$, $M_H(t)$, and $M_L(t)$. I focus on the *stationary* environment, in which the distributions of $m(x, t)$, $M_H(t)$, and $M_L(t)$ are independent of $t$. This requires specifying the "right" exogenous distributions at time 0; see Lemma 3.1 below. This also implies that the distributions of $M_{A,t}$ and $M_{B,t}$ are stationary as well.

Now I derive the stationary distributions of $m(x, t)$, $M_H(t)$, and $M_L(t)$. The balance equation for the measure $m(x,t)$ of buyers with the belief $x$ at time $t$ takes the form

$$\frac{\mathrm{d}m(x,t)}{\mathrm{d}t} = \underbrace{\delta f(x)}_{\substack{\text{inflow}\\\text{of buyers}\\\text{with the}\\\text{belief } x\\\text{who enter}\\\text{the market}}} - \underbrace{\delta m(x,t)}_{\substack{\text{outflow of}\\\text{buyers}\\\text{who leave}\\\text{the market}}} - \underbrace{\lambda x m(x,t)}_{\substack{\text{outflow of buyers}\\\text{who learn that}\\\text{they are}\\\text{the } H\text{-types}}} - \underbrace{\lambda(1-x)m(x,t)}_{\substack{\text{outflow of buyers}\\\text{who learn that}\\\text{they are}\\\text{the } L\text{-types}}}.$$

In a stationary environment, $\frac{\mathrm{d}m(x,t)}{\mathrm{d}t} = 0$. Therefore,

$$m(x) \coloneqq m(x,t) = \tfrac{\delta}{\delta+\lambda} f(x).$$

The balance equations for the mass $M_H(t)$ of the $H$-types and the mass $M_L(t)$ of the $L$-types who have learned their type are as follows:

$$\frac{\mathrm{d}M_H(t)}{\mathrm{d}t} = \underbrace{\lambda \int_0^1 x m(x,t)\mathrm{d}x}_{\substack{\text{inflow of buyers}\\\text{who learn that}\\\text{they are}\\\text{the } H\text{-types}}} - \underbrace{\delta M_H(t)}_{\substack{\text{outflow of}\\\text{the } H\text{-types}\\\text{who leave}\\\text{the market}}}, \qquad \frac{\mathrm{d}M_L(t)}{\mathrm{d}t} = \underbrace{\lambda \int_0^1 (1-x) m(x,t)\mathrm{d}x}_{\substack{\text{inflow of buyers}\\\text{who learn that}\\\text{they are}\\\text{the } L\text{-types}}} - \underbrace{\delta M_L(t)}_{\substack{\text{outflow of}\\\text{the } L\text{-types}\\\text{who leave}\\\text{the market}}}.$$

In a stationary environment, $\frac{\mathrm{d}M_H(t)}{\mathrm{d}t} = 0$ and $\frac{\mathrm{d}M_L(t)}{\mathrm{d}t} = 0$. Therefore,

$$M_H \coloneqq M_H(t) = \tfrac{\lambda}{\delta} \int_0^1 x m(x,t)\mathrm{d}x = \tfrac{\lambda}{\delta+\lambda} \int_0^1 x f(x)\mathrm{d}x,$$
$$M_L \coloneqq M_L(t) = \tfrac{\lambda}{\delta} \int_0^1 (1-x) m(x,t)\mathrm{d}x = \tfrac{\lambda}{\delta+\lambda} \int_0^1 (1-x) f(x)\mathrm{d}x.$$

Lemma 3.1 follows.

**Lemma 3.1.** *In stationary environment,*
- *the measure of buyers with the belief $x$ is $m(x) = \frac{\delta}{\delta+\lambda} f(x)$ for all $x \in [0,1]$;*
- *the mass of $H$-types who have learned their type is $M_H = \frac{\lambda}{\delta+\lambda} \int_0^1 x f(x)\mathrm{d}x$;*
- *the mass of $L$-types who have learned their type is $M_L = \frac{\lambda}{\delta+\lambda} \int_0^1 (1-x) f(x)\mathrm{d}x$.*

*Furthermore, $\int_0^1 m(x)\mathrm{d}x + M_H + M_L = 1$.*

**Strategies and optimum.** A strategy of the seller prescribes which prices $p_A$ and $p_B$ she posts at the beginning of time. That is, it is an element $\sigma^S \in \mathbf{R} \times \mathbf{R}$.

A buyer best-responds to the prices $p_A$ and $p_B$ and chooses whether to buy a gizmo and, if so, which gizmo to buy. Given $p_A$ and $p_B$, the best response is a *Markov strategy* that maps the buyer's belief $x$ and whether the buyer has a gizmo and, if so, its version into his action. In other words, it is a measurable function

$$\sigma^B\colon [0,1] \times \{A, B, \text{no gizmo}\} \to \{A, B, \text{no gizmo}\}.$$

I look for the profit maximizing prices and focus on the *seller-preferred* optimum.

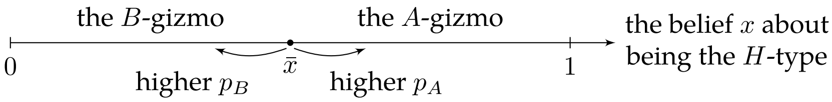


**Figure 3.1.** If the prices $p_A$ and $p_B$ posted by the seller are such that $\frac{1}{r+\delta+\beta}w_A \geq p_A - p_B \geq 0$, then there exists a cut-off belief $\bar{x} \in [0,1]$ such that a buyer prefers the advanced version over the basic version if his belief $x \geq \bar{x}$ and the basic version over the advanced version if $x \leq \bar{x}$. The cut-off $\bar{x}$ increases in $p_A$ and decreases in $p_B$.

# 3.3 The Seller-Preferred Optimum

I start with the description of the buyers' optimal strategy. It takes the simple form of a cut-off. I proceed with the characterization of the seller's preferred optimum. I find that both versions of the gizmo are sold at the optimum. The optimal prices are such that buyers buy a new gizmo if and only if the gizmo they have breaks. Furthermore, depending on parameters, it may be that only the $H$-types, who have already learned their type, go for the advanced version.

## 3.3.1 What Is Optimal for the Buyers?

Given the prices $p_A$ and $p_B$ posted by the seller, the optimal strategy for a buyer takes the simple form of a *cut-off* (see Figure 3.1). Indeed, buyers are small and so their decisions affect others only through the prices. It follows that a buyer without a gizmo buys the advanced version if his belief is above the cut-off and buys the basic version otherwise. The optimal cut-off is given in Proposition 3.1.

**Proposition 3.1.** *For given $p_A$ and $p_B$, a buyer without the gizmo behaves as follows:*

- *if $0 \geq p_A - p_B$, then the buyer prefers the $A$-gizmo over the $B$-gizmo for all $x \in [0,1]$;*
- *if $\frac{1}{r+\delta+\beta}w_A \geq p_A - p_B \geq 0$, then there exists $\bar{x} \in [0,1]$ given by*

$$\bar{x} = \frac{p_A - p_B}{\frac{1}{r+\delta+\beta}w_A - \frac{\lambda}{r+\delta+\beta+\lambda}\max\left\{\frac{1}{r+\delta+\beta}w_A - p_A, 0\right\}}$$

  *such that the buyer prefers the $A$-gizmo over the $B$-gizmo if $x \geq \bar{x}$ and the $B$-gizmo over the $A$-gizmo if $x \leq \bar{x}$;*
- *if $p_A - p_B \geq \frac{1}{r+\delta+\beta}w_A$, then the buyer prefers the $B$-gizmo over the $A$-gizmo for all $x \in [0,1]$.*

The optimal cut-off $\bar{x}$ is the belief at which the buyer is indifferent between the two versions. The cut-off belongs to a unit interval if and only if the difference between prices of the two versions does not exceed the "discounted" value of the advanced features to the $H$-types and the $L$-types prefer the basic version. That is, $\bar{x} \in [0,1]$ if and only if $\frac{1}{r+\delta+\beta}w_A \geq p_A - p_B \geq 0$. The boundaries on the price difference are determined by preferences of buyers who have already learned their type. The $H$-types prefer the advanced version over the basic one if and only if $\frac{1}{r+\delta+\beta}w_A \geq p_A - p_B$. In contrast, the $L$-types choose the basic version over the advanced one if and only if $p_A - p_B \geq 0$.

Given that buyers' beliefs do not change in absence of news and gizmos do not depreciate gradually but break at random times, buyers choose to buy a new gizmo only if the gizmo they have breaks or if they learn their type. If buyers are still unsure about their type and their gizmo breaks, then they buy the same gizmo as before. Upon learning that he is the $L$-type, the buyer does not buy a new gizmo unless the one he has breaks, because he gets the same payoff from either version of the gizmo. If his gizmo breaks, then the $L$-type buys the basic version. In contrast, upon learning that he is the $H$-type, the buyer may immediately replace the basic version with the advanced one if the price of the latter is relatively low, specifically, if $p_A \leq \frac{1}{r+\delta+\beta} w_A$.

**Lemma 3.2.** *The optimal cut-off $\bar{x}$ increases in $p_A$ and decreases in $p_B$.*

Dependence of the optimal cut-off $\bar{x}$ on the prices $p_A$ and $p_B$ is intuitive. Indeed, the higher is the price $p_A$ of the advanced version, the more optimistic about their type must buyers be to be willing to pay for it, and so the higher is the cut-off. On the contrary, the higher is the price $p_B$ of the basic version, the more attractive does the advanced version become, because it gives a higher payoff in case the buyer turns out to be the $H$-type, and so the lower is the cut-off.

Lemma 3.3 below states partial effects of the discount rate $r$, the entry/exit rate $\delta$, the breakdown rate $\beta$, and the learning rate $\lambda$ on the optimal cut-off $\bar{x}$. That is, it shows how the optimal cut-off varies with these rates if the prices posted by the seller stay fixed. The full effects in case of the uniform distribution of the prior beliefs are discussed in Section 3.3.2 below.

**Lemma 3.3.** *Given $p_A$ and $p_B$ that satisfy $\frac{1}{r+\delta+\beta} w_A \geq p_A - p_B \geq 0$, the optimal cut-off $\bar{x}$*
- *increases in $r$;*
- *increases in $\delta$;*
- *increases in $\beta$;*
- *increases in $\lambda$ if $p_A < \frac{1}{r+\delta+\beta} w_A$ and is independent of $\lambda$ if $p_A \geq \frac{1}{r+\delta+\beta} w_A$.*

Fix the prices $p_A$ and $p_B$. The more impatient are buyers, that is, the higher is the discount rate $r$, the less value do buyers put on what happens after they learn their type. Buyers without the gizmo go for the advanced version if they are optimistic about being the $H$-types and thus about needing the advanced features. The more impatient are buyers, the more optimistic they must be to buy the advanced version, that is, the higher is the optimal cut-off $\bar{x}$.

The only role the entry/exit rate $\delta$ plays, when it has no effect on prices, is that of additional discounting. As a result, it has the same effect on $\bar{x}$ as $r$.

With fixed prices, the only role of the breakdown rate $\beta$ is that of how often buyers must buy a new gizmo. That is, it "discounts" the value of each version, in particular, it "discounts" the value of the advanced features the advanced version provides. As a result, buyers must be more optimistic to choose it over the basic version. The optimal cut-off $\bar{x}$ increases with $\beta$.

The learning rate $\lambda$ affects the optimal cut-off only if the price of the advanced version is low enough and buyers with the basic version replace it with the advanced one immediately upon learning that they are the $H$-types, that is, only if $p_A < \frac{1}{r+\delta+\beta} w_A$. This result relies on the fact that both advanced and basic versions break at the same rate. Indeed, if the price of the advanced version is high, that is, if $p_A \geq \frac{1}{r+\delta+\beta} w_A$, then buyers buy a new gizmo if the one they have breaks and not if they learn their type. That is why, with the same breakdown rate and fixed prices, $\lambda$ has no effect on $\bar{x}$. If the price of the advanced version is low, that is, if $p_A < \frac{1}{r+\delta+\beta} w_A$, then $\bar{x}$ increases with $\lambda$. If buyers learn their type quickly and the price of the advanced version is low, then buyers might as well wait until they learn that they are, indeed, the $H$-types and buy the advanced version then, rather than buy it when they are not that sure about their type.

### 3.3.2 What Is Optimal for the Seller?

The seller is always better off by making both versions of the gizmo acceptable to buyers. Indeed, if only one of the versions is offered, then, independent of the version, the most the seller can get from each buyer is the value of the gizmo to the $L$-types. The seller would be strictly better off by selling the basic version at that price and the advanced version at a slightly higher price so that the $H$-types go for it.

In the optimum, the seller posts a too high price of the advanced version for a buyer with the basic version to replace it immediately with the advanced one upon learning that he is the $H$-type. The buyer always waits until his basic version breaks and only then buys the advanced version. For the immediate replacement to take place, the price of the advanced version should not exceed the value its advanced features provide. However, this value is too low for the seller to be willing to post such a price.

Whether the price of the advanced version is such that only the $H$-types who have learned their type buy it depends on parameters of the model, in particular, on how large the mass of these buyers is. This is summarized in Theorem 3.1.

**Theorem 3.1.** *There exists a unique seller-preferred optimum. The optimum is characterized by a unique set of prices $p_A^*$ and $p_B^*$ given by $p_B^* = \frac{1}{r+\delta+\beta}w$ and*

- *if $\frac{\beta}{r+\beta}M_H \geq \frac{\delta}{r+\delta}f(1) + \frac{\beta}{r+\beta}m(1)$, then $p_A^* = \frac{1}{r+\delta+\beta}(w + w_A)$ and $\bar{x}^* = 1$;*
- *if $\frac{\beta}{r+\beta}M_H < \frac{\delta}{r+\delta}f(1) + \frac{\beta}{r+\beta}m(1)$, then $p_A^* = \frac{1}{r+\delta+\beta}(w + w_A\bar{x}^*)$ and $\bar{x}^* \in (0,1)$.*

*In the optimum, a buyer buys a new gizmo if and only if the one he has breaks.*

If the mass of the $H$-types who have learned their type is above a certain cut-off, that is, if $M_H \geq \bar{M}_H$, where

$$\bar{M}_H := \frac{\frac{\delta}{r+\delta} + \frac{\beta}{r+\beta}\frac{\delta}{\delta+\lambda}}{\frac{\beta}{r+\beta}} f(1),$$

then it is optimal for the seller to post a high price for the advanced version so that only the $H$-types buy it. This way she extracts full surplus from the $H$-types. This occurs when the learning rate $\lambda$ or the breakdown rate $\beta$ is high or when the entry/exit rate $\delta$ is low. Indeed, if buyers learn their type quickly, then many buyers on the market know their type and, in particular, there are many $H$-types. With a high breakdown rate, buyers often have to buy a new gizmo. That is why the mass of the $H$-types per se does not have to be large for the seller to find it optimal to charge a high price for the advanced version. If the market "refreshes" slowly, that is, if the entry/exit rate is low, then buyers who know their type accumulate on the marker, and so the mass of the $H$-types is large.

If $M_H < \bar{M}_H$, then the seller finds it optimal to make the advanced version also acceptable by buyers who are still unsure about their type. The larger is the mass of the $H$-types who have learned their type, the higher is the price of the advanced version. As a result, unsure buyers must be more optimistic to go for the advanced version.

If the prior beliefs are distributed uniformly, that is, if $f(x) = 1$ for all $x \in [0,1]$, then $\bar{x}^* = 1$ if $\frac{1}{2}\frac{\beta}{r+\beta}\frac{\lambda}{\delta+\lambda} \geq \frac{\delta}{r+\delta} + \frac{\beta}{r+\beta}\frac{\delta}{\delta+\lambda}$. In contrast, if $\frac{1}{2}\frac{\beta}{r+\beta}\frac{\lambda}{\delta+\lambda} < \frac{\delta}{r+\delta} + \frac{\beta}{r+\beta}\frac{\delta}{\delta+\lambda}$, then $\bar{x}^* \in (0,1)$ and is equal to

$$\bar{x}^* = \frac{1}{2} + \frac{1}{4}\frac{\frac{\beta}{r+\beta}\frac{\lambda}{\delta+\lambda}}{\frac{\delta}{r+\delta} + \frac{\beta}{r+\beta}\frac{\delta}{\delta+\lambda}}.$$

The first observation is that $\bar{x}^* > \frac{1}{2}$, which emphasizes that the $H$-types who have learned their type are the primary focus of the seller when choosing the price of the advanced version. Dependences of $\bar{x}^*$ on the discount rate $r$, the entry/exit rate $\delta$, the breakdown rate $\beta$, and the learning rate $\lambda$ are stated in the next lemma.

**Lemma 3.4.** *Suppose the distribution of the prior beliefs is uniform, that is, $f(x) = 1$ for all $x \in [0,1]$, and that $\frac{1}{2}\frac{\beta}{r+\beta}\frac{\lambda}{\delta+\lambda} < \frac{\delta}{r+\delta} + \frac{\beta}{r+\beta}\frac{\delta}{\delta+\lambda}$. The optimal cut-off $\bar{x}^*$*

- *decreases in $r$ if $\beta \leq \delta$ and increases in $r$ if $\beta \geq \delta$;*
- *decreases in $\delta$;*
- *increases in $\beta$;*
- *increases in $\lambda$.*

If buyers are likely to leave the market before their gizmo breaks, that is, if $\beta \leq \delta$, then it is likely that the only way they buy the advanced version is upon entering the market. The more impatient is the seller, that is, the higher is the discount rate, the lower is the price of the advanced version and hence the lower is the difference between the two prices. That is to say, $\bar{x}^*$ is decreasing in $r$. If $\beta \geq \delta$, then the full effect of $r$ coincides with its partial effect given in Lemma 3.3, and $\bar{x}^*$ is increasing in $r$, even though the price of the advanced version may decrease.

When the dependence of the difference between the optimal prices on the entry/exit rate $\delta$ is taken into account, its effect on the optimal cut-off is reversed. The cut-off $\bar{x}^*$ is decreasing in $\delta$; see Lemma 3.3. If the market "refreshes" quickly, that is, if $\delta$ is high, then the advanced version is likely to be bought only by buyers who enter the market. That is why the seller is willing to lower the price of the advanced version and thus to lower the difference between the two prices.

Even though the price of the advanced version may decrease with the breakdown rate $\beta$, the full effect of $\beta$ on the optimal cut-off coincides with its partial effect given in Lemma 3.3, and $\bar{x}^*$ is increasing in $\beta$. A higher breakdown rate induces the seller to lower the price of the advanced version for buyers to be willing to buy the gizmo. However, higher $\beta$ also implies that buyers often have an opportunity to replace the basic version with the advanced one if they want to, and so they do not have to buy the advanced version right away.

The optimal cut-off depends on the learning rate $\lambda$ only via the prices; recall that Lemma 3.3 states $\bar{x}^*$ does not depend on $\lambda$ directly. Specifically, $\bar{x}^*$ increases with $\lambda$. If buyers learn their type quickly, then many $H$-types who have learned their type are present on the market. As a result, while choosing the price of the advanced version, the seller focuses on the $H$-types and buyers who are more optimistic about their type. The price of advanced version increases with $\lambda$, and so does the difference between the prices of the two versions.

## 3.4 Planned Obsolescence

As a motivation for planned obsolescence, the seller's profit as a function of the breakdown rate $\beta$ is depicted in Figure 3.2. It is single-peaked and is maximized at a certain rate $\beta^* \in (0, \infty)$. Therefore, in a game in which the seller chooses $\beta$ at the beginning of time and commits to it, she picks this $\beta^*$ or, in other words, she plans obsolescence. This is stated in Theorem 3.2.

**Theorem 3.2.** *In the seller-preferred optimum, the seller plans obsolescence, that is, she chooses the breakdown rate $\beta^* \in (0, \infty)$.*

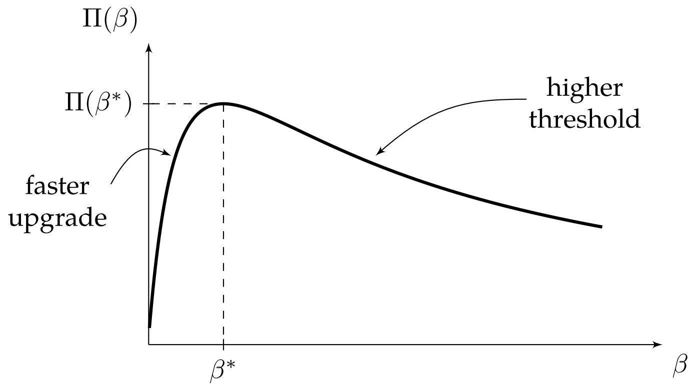


**Figure 3.2.** The seller's profit $\Pi(\beta)$ as a function of the breakdown rate $\beta$ if the prices she posts are $p_A^*$ and $p_B^*$. In the seller-preferred optimum, the seller chooses $\beta^* \in (0, \infty)$. Parameters: $(r, \delta, \lambda, w, w_A) = (1, 0.01, 10, 2, 1)$ and $f(x) = 1$ for all $x \in [0, 1]$, the uniform distribution of the prior beliefs.

The seller does not want the gizmos to break immediately, but she also does not want them to be a durable good. The optimal breakdown rate $\beta^*$ captures the trade-off faced by the seller. On the one hand, because buyers with the basic version who have learned that they are the $H$-types upgrade their gizmo to the advanced version only upon a breakdown, a higher breakdown rate increases buyers' surplus, for they get the preferred version faster. As a result, the higher breakdown rate also increases the seller's profit. On the other hand, the cut-off used by buyers increases with the breakdown rate, and so fewer buyers go for the advanced version. Therefore, the seller extracts less surplus from those who would appreciate the advanced features.

## 3.5 Conclusion

I want to conclude by pointing out two questions which naturally arise from the analysis and are to be answered. First, what is the socially optimal breakdown rate of the gizmos? The trade-off faced by the social planner is as non-trivial as the one faced by the seller, and so it is not clear a priori if the breakdown rate chosen by the seller is inefficiently high or low. Second, what is the optimal dynamic pricing? The answer to the second question is of particular interest in the non-stationary environment.

# Chapter 4

# Bundling with Strategic Buyers

## 4.1 Introduction

I aim to answer the following question: How can a monopolist seller use *bundling* to increase his revenue when he faces strategic buyers? The seller has a finite stock of the product, which he has to sell before a deadline. Buyers have multi-unit demand and different valuations for the product. They are forward-looking and posses some bargaining power. For example, think about secondary markets for planes or orders of ships, trains, or planes.

The property which helps answer the question is the multi-unit single-crossing property. I find that, for given unit and bundle prices, buyers with a high valuation for the product choose to purchase weakly more than buyers with a low valuation. For example, if buyers with the low valuation accept the offer of the unit, then buyers with the high valuation either also accept the offer of the unit or accept the offer of the bundle.

I find that bundling players *two roles* which may allow the seller to increase his revenue. First, it can be used a tool to discriminate among buyers with high and low valuations by either screening buyers with the high valuation in the first period or by posting the same-period unit and bundle prices such that buyers with different valuations accept different offers. Interestingly, the bundle price is higher than double of the unit price. Buyers are willing to pay such a high bundle price, because they fear that they will not get any product if they do not accept this offer. Second, bundling can act as a precaution when the seller chooses to target buyers with the high valuation only and does so by selling two units as a bundle.

**Related literature.** This chapter contributes to the revenue management literature and analyzes the roles bundling can play to increase the seller's revenue. This is an adaptation of Hörner and Samuelson (2011). Hörner and Samuelson (2011) considers the seller with a fixed quantity of the product which he has to sell before a deadline. The seller in that paper faces buyers who have unit demand for the product, and so there is no scope for bundling.

Stigler (1963) is the first paper to point out that bundling can be profitable for a monopolist seller by giving an example of block booking of feature films. Adams and Yellen (1976) gives a series of examples to illustrate that (mixed) bundling can be a useful price discrimination strategy.[1] They do not give a general condition for optimality of bundling.

[1]In Adams and Yellen (1976), "pure bundling" implies that the seller commits to selling units of the

McAfee, McMillan, and Whinston (1989) provides a sufficient condition on the distribution of buyers' valuations for bundling to dominate unbundled sales. Chu, Leslie, and Sorensen (2011) gives numerical examples illustrating that bundling is profitable in a series of special cases, in particular, shows profitability of bundle-size pricing. Chen and Riordan (2013) establishes new general conditions for profitability of bundling.

Bundling can be thought of as non-linear pricing. Armstrong (1996) finds an optimal multi-product non-linear tariff in a special case that puts restrictions on the distribution of buyers' valuations. Rochet and Choné (1998) gives a special example of an optimal non-linear tariff and also points out non-robustness of examples in Armstrong (1996). Armstrong (1999) is a complementary paper to Armstrong (1996), which shows that almost optimal non-linear tariffs can be found in a more general setting if the number of products is arbitrary large. This is a more general but similar result to Bakos and Brynjolfsson (1999); see also Bakos and Brynjolfsson (2000). In contrast, Fang and Norman (2006) focuses on a finite number of products and looks for conditions under which bundling is an attractive pricing strategy. Nocke, Peitz, and Rosar (2011) analyzes the optimality of advance-purchase discounts.

## 4.2 The Model

I consider a two-period dynamic game between a single seller (he), with two units of the same product for sale, and two *strategic* buyers (both she) who have *multi-unit* demand. Both units are sold and consumed at the end of the second period, and have no value afterward. The seller has two periods to agree with one or two buyers on the purchase of one or two units.

In each period, the seller posts unit and bundle prices; he posts a unit price only when there is just one unit for sale in a given period.[2] The seller has *no commitment* power. That is, prices must be sequentially rational given his beliefs. After observing the prices, buyers simultaneously and independently decide whether they accept or not an offer of one or two units at the corresponding posted unit or bundle price. I use the following tie-breaking assumptions.

**Assumption 4.1.** *If both buyers accept the offer of a bundle (resp., a unit) when two units (resp., one unit) are for sale, then the seller randomly selects the buyer who gets the bundle (resp., the unit).*

**Assumption 4.2.** *If one buyer accepts the offer of the unit, while another buyer accepts the offer of the bundle, then the seller sells both units as a bundle.*

The seller's behavior under Assumption 4.1 is sequentially rational, because he has to give the bundle or the remaining unit to at least one of the buyers and he is indifferent among them. Regarding Assumption 4.2, favoring the buyer, who prefers the offer of one unit, does not bring much to the analysis and allows for an arguably unnatural behavioral pattern; see Appendix D.1 for details.

Prices and sales are observed by everyone. The game ends if either offers for both units have been accepted or the second period is over.

product as a bundle only. What they call "mixed bundling" is a strategy of the seller when he does not make such a commitment and may sell them either separately or together. Because the seller lacks commitment in my model, it is always the case of mixed bundling here.

[2]The model can be thought of as a one with non-linear pricing.

Buyers' valuations for one unit, or buyers' types, $v_i$ and $v_{-i}$ are privately known, and independently and identically distributed. They do not change over time. A buyer has a high valuation, normalized to $1$ without loss of generality, for one unit of the product with probability $\alpha$, and a low valuation $v \in (0,1)$ with probability $1-\alpha$, where $\alpha \in (0,1)$. A buyer $i$'s valuation for consuming both units is scaled down by a parameter $\gamma \in \left(\frac{1}{2}, 1\right)$, that is, it is $2\gamma v_i$, where $i = 1, 2$.

If buyer $i$ ends up with one unit, then her payoff is her valuation $v_i$ minus the unit price she pays for it. If she gets two units, then her payoff is $2\gamma v_i$ minus either the bundle price or the sum of unit prices if she accepts the offers for each unit in different periods. Recall that there is no utility flow, and units are consumed in the end of the second period. If the buyer does not buy anything, then her payoff is $0$. The seller has a zero reservation value for both units, and his revenue is the price or sum of prices at which he sells them. Hereafter, by buyers' "payoff" and the seller's "revenue," I mean their *expected* payoff and revenue. There is no discounting.

Formally, pure strategies of the seller and buyers are as follows: In period $t$, where $t = 1, 2$, the seller posts the unit and bundle prices $p_t \in \mathbf{R}$ and $q_t \in \mathbf{R}$. After observing the offers, each buyer picks an action from the set $\{B, U, N\}$, where $B$ (resp., $U$) means that the buyer accepts the offer of the bundle (resp., the offer of the unit), and $N$ means that she rejects the offers. Let $h^t \in H^t$ be a non-trivial history when the game is not effectively over. That is, $H^1 = \{\varnothing\}$ and $H^2$ contains prices posted by the seller and an allocation of a unit (if the offer for one unit has been accepted) in the first period. It follows that, in period $t$, a pure strategy of the seller is $\sigma_t^S\colon H^t \to \mathbf{R} \times \mathbf{R}$ if two units are for sale; his strategy is $\sigma_t^S\colon H^t \to \mathbf{R}$ if only one unit remains for sale. A pure strategy of buyer $i$ is $\sigma_t^{B_i}\colon \{v, 1\} \times H^t \times \mathbf{R} \times \mathbf{R} \to \{B, U, N\}$, where $i = 1, 2$.

At the end of the first period, either both units remain for sale, only one unit remains for sale, or the game is over. To keep notation simple, I denote by $p$ and $q$ the unit and bundle prices posted by the seller in the first period, by $p''$ and $q''$ the prices posted in the second period if two units remain, and by $p'$ the unit price posted in the second period if only one unit remains for sale.

I look for perfect Bayesian equilibria in pure strategies and in which buyers use symmetric strategies, that is, buyers of a given type behave identically.[3] Therefore, in each period $t$, it suffices to look separately at the strategies of buyers with the high and low valuations only. I denote them by $\bar{\sigma}_t^B$ and $\underline{\sigma}_t^B$. I also make the following tie-breaking assumptions on the behavior of buyers.

**Assumption 4.3** (Intra-Period Indifference)**.** *In any period, buyers accept one of the offers if they are indifferent between accepting and rejecting.*

**Assumption 4.4** (Inter-Period Indifference)**.** *Buyers accept the offer in the first period if they are indifferent between accepting it and waiting for the second period.*

If the two indifference assumptions are relaxed, the number of equilibria increases, but the number of equilibria in terms of revenue stays the same. As the seller's revenue is of interest in this chapter, making such assumptions is without loss.

[3] The equilibrium beliefs of the seller about buyers' types and of buyer $i$ about buyer $-i$'s type have to be as follows: (i) Even off path, posterior beliefs of the seller are independent, and both types of buyer $i$ have the same belief. (ii) Bayes' rule is used to update beliefs whenever possible. (iii) There is "no signaling what you don't know." (iv) The seller and buyer $i$ have the same belief about the type of buyer $-i$. The assumptions (i)–(iv) correspond to assumptions B(i)–B(iv) of Fudenberg and Tirole (1991) (Section 8.2.3, pages 331–332).

## 4.3 One Period to Make a Deal

In this section, I show that bundling can make the seller better off even if he has only one period to make a deal with one or two buyers. Specifically, I show that bundling can allow the seller to discriminate among buyers with different valuations and that it can act as a precaution if the seller targets buyers with the high valuation only. It is also helpful to note that buyers' equilibrium behavior satisfies the multi-unit single-crossing property.

### 4.3.1 Equilibria in the Game between Buyers

A pair of prices $(p, q)$ defines a game between buyers, which is the game I solve here. The equilibrium behavior of buyers for given unit and bundle prices is summarized by Lemma 4.1 below. Their reaction depends on parameters of the model: the low valuation $v$, the scaling parameter $\gamma$, and the probability $\alpha$ that a buyer has the high valuation.

**Lemma 4.1** (Multi-Unit Single-Crossing Property)**.** *For any pair of prices, buyers with the high valuation for the product choose to purchase weakly more than buyers with the low valuation.*

The lemma is the multi-unit analog of the familiar single-crossing property saying that higher types are more willing to buy. In other words, for a given pair of prices, if buyers with the low valuation accept the offer of the bundle, then so do buyers with the high valuation. If buyers with the low valuation choose the offer of the unit, then buyers with the high valuation find it optimal to buy the unit or the bundle. Rejection of both offers by buyers with the low valuation implies that buyers with the high valuation can accept any of the offers or reject both of them.

Lemma 4.1 implies that, across parameters, there are six kinds of equilibria, that is, six different pairs $(\bar{\sigma}^B, \underline{\sigma}^B)$ of equilibrium strategies of buyers with high and low valuations; see Figure 4.1.[4] Furthermore, for a given pair of prices $(p, q)$, there are at most three equilibria among buyers.

Each kind of equilibria has its own pattern or color in Figure 4.1. For example, the light gray region represents an equilibrium with $(U, U)$, which is the equilibrium in which buyers with both high and low valuations given the unit and bundle prices. The gray region corresponds to an equilibrium with $(B, U)$: buyers with the high valuation accept the offer of the bundle, while buyers with the low valuation accept the offer of the unit. Regions in which patterns or colors intersect correspond to the prices that support multiple equilibria. For example, the striped region with light gray and gray lines corresponds to pairs of the unit and bundle prices which support both $(U, U)$ and $(B, U)$ as equilibria.[5]

[4] In Figure 4.1, parameters are such that $2\gamma - 1 > 2\gamma v$. If $2\gamma - 1 \leq 2\gamma v$, then one more case arises. Specifically, there are unit and bundle prices $(p, q)$ such that $(U, U)$, $(B, U)$, and $(B, B)$ can be supported as equilibria.

[5] To understand how the regions are found, consider, for example, the case when $(B, U)$ is an equilibrium. Given $(p, q)$, buyers with the high valuation prefer the offer of the bundle over the offer of the unit if and only if

$$\alpha \cdot \tfrac{1}{2}(2\gamma - q) + (1 - \alpha) \cdot (2\gamma - q) \geq \alpha \cdot 0 + (1 - \alpha) \cdot (1 - p) \quad \Leftrightarrow \quad q \leq 2\gamma - \tfrac{2(1-\alpha)}{2-\alpha}(1 - p).$$

Furthermore, buyers with the high valuation accept the offer of the bundle rather than reject it if and only if

$$\alpha \cdot \tfrac{1}{2}(2\gamma - q) + (1 - \alpha) \cdot (2\gamma - q) \geq 0 \quad \Leftrightarrow \quad q \leq 2\gamma.$$

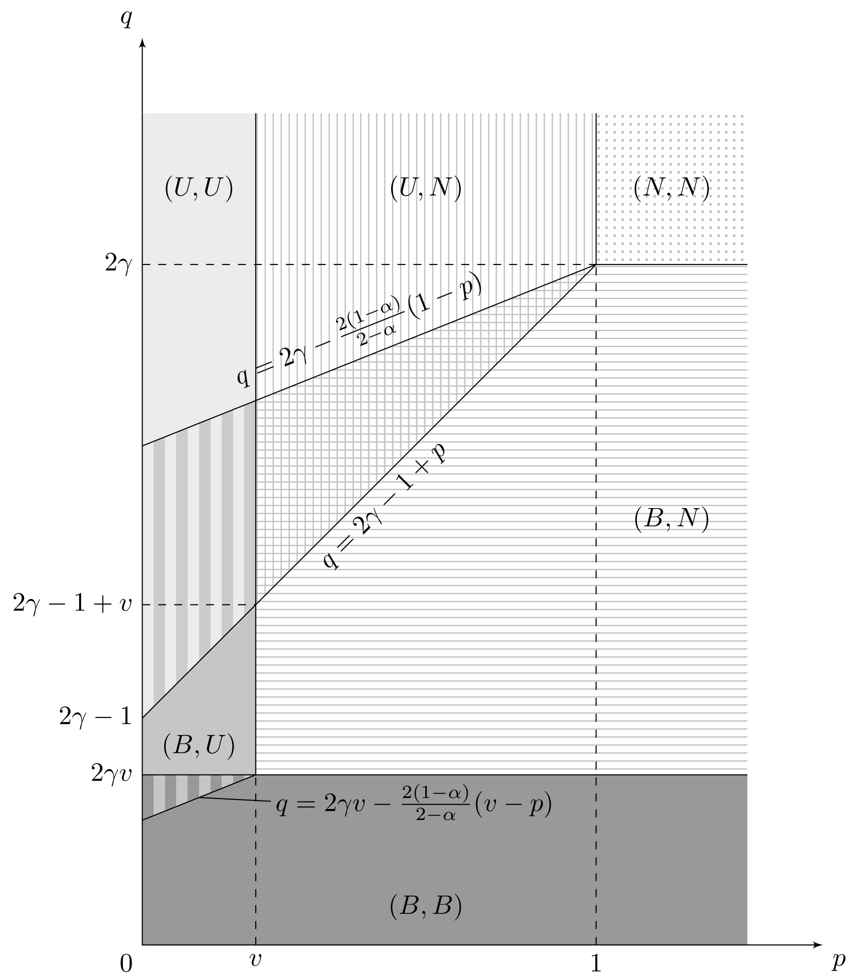


**Figure 4.1.** The equilibrium strategies of buyers $(\bar{\sigma}^B, \underline{\sigma}^B)$ given unit and bundle prices $(p, q)$. Parameters: $(v, \gamma, \alpha) = \left(\frac{1}{4}, \frac{3}{4}, \frac{3}{4}\right)$.

It follows that, by choosing the pair of prices carefully, the seller can induce the buyers' reaction which brings him the largest revenue. This is explained next.

### 4.3.2 Equilibria with One Period

Now I analyze the game between all players: the seller and the two buyers. The seller moves first and posts unit and bundle prices. After observing the prices, buyers simultaneously and independently decide whether to accept one of the offers.

The seller's equilibrium revenue $\mathcal{R}$ is is necessarily one of the following four revenues:

$$\mathcal{R}_{UU} = 2v,$$
$$\mathcal{R}_{BU} = 2\alpha(2-\alpha)\gamma + 2(1-\alpha)(v-\alpha),$$

Similarly, buyers with the low valuation prefer the offer of the unit over the offer of the bundle and over not buying anything if and only if

$$\alpha \cdot 0 + (1-\alpha) \cdot (v-p) > \alpha \cdot \tfrac{1}{2}(2\gamma v - q) + (1-\alpha) \cdot (2\gamma v - q) \quad \Leftrightarrow \quad q > 2\gamma v - \tfrac{2(1-\alpha)}{2-\alpha}(v-p)$$

and

$$\alpha \cdot 0 + (1-\alpha) \cdot (v-p) \geq 0 \quad \Leftrightarrow \quad p \leq v.$$

The inequalities are written taking into account Assumptions 4.1 to 4.4.

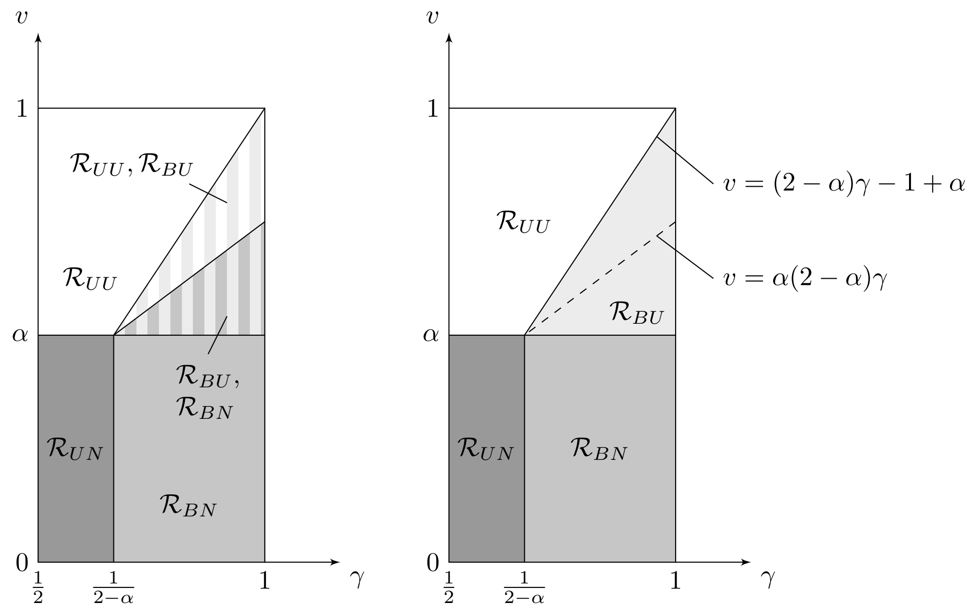


**Figure 4.2.** The seller's equilibrium revenue (left) and his largest equilibrium revenue (right) in the game with one period. Parameters: $\alpha = \frac{1}{2}$.

$$\mathcal{R}_{UN} = 2\alpha,$$
$$\mathcal{R}_{BN} = 2\alpha(2-\alpha)\gamma.$$

That is, in any equilibrium, the revenue the seller gets is $\mathcal{R}_{UU}$, $\mathcal{R}_{BU}$, $\mathcal{R}_{UN}$, or $\mathcal{R}_{BN}$, which one depends on the parameters of the model and the equilibrium behavior of buyers. This is summarized in Proposition 4.1 below and is illustrated in the left panel in Figure 4.2. For some parameters, there are multiple equilibria with distinct revenues; see the striped regions in the left panel in Figure 4.2. Therefore, the seller cannot necessarily secure the largest of the four possible revenues. Which of the four revenues is the largest depends on the parameters and is illustrated in right panel in Figure 4.2.

**Proposition 4.1** (Equilibria with One Period). *In any equilibrium of the one-period game, the seller's revenue is as follows:*

– (Fig. 4.2, "$\mathcal{R}_{UU}$") *if $v \geq \alpha$ and $v \geq (2-\alpha)\gamma - 1 + \alpha$, then $\mathcal{R} = \mathcal{R}_{UU}$;*
– (Fig. 4.2, "$\mathcal{R}_{UU}, \mathcal{R}_{BU}$") *if $(2-\alpha)\gamma - 1 + \alpha \geq v \geq \alpha(2-\alpha)\gamma$, then $\mathcal{R} \in \{\mathcal{R}_{UU}, \mathcal{R}_{BU}\}$;*[6,7]
– (Fig. 4.2, "$\mathcal{R}_{BU}, \mathcal{R}_{BN}$") *if $\alpha(2-\alpha)\gamma \geq v \geq \alpha$, then $\mathcal{R} \in \{\mathcal{R}_{BU}, \mathcal{R}_{BN}\}$;*
– (Fig. 4.2, "$\mathcal{R}_{UN}$") *if $\alpha \geq v$ and $\frac{1}{2-\alpha} \geq \gamma$, then $\mathcal{R} = \mathcal{R}_{UN}$;*
– (Fig. 4.2, "$\mathcal{R}_{BN}$") *if $\alpha \geq v$ and $\gamma \geq \frac{1}{2-\alpha}$, then $\mathcal{R} = \mathcal{R}_{BN}$.*

If the valuation $v$ is high, then the best the seller can do is to sell both units unit-by-unit at the price which is acceptable by buyers with both high and low valuations; see the "$\mathcal{R}_{UU}$"-region in Figure 4.2. The seller get $\mathcal{R}_{UU}$ with the unit price $p = v$ and by making sure that buyers prefer the offer of the unit over the offer of the bundle, that is, by setting a high bundle price, specifically, $q > 2\gamma - 1 + v$.

If the valuation $v$ is high and if buyers value having both units, that is, if the scaling parameter $\gamma$ is high, then there are multiple equilibria in terms of revenue; see the

[6]Hereafter, by the set of revenues, I mean that all revenues in the set occur for some equilibria.
[7]The presence of two (not three) equilibria should not come as a surprise, because I am concerned with equilibria in pure strategies only.

"$\mathcal{R}_{BU}$"-region in the right panel in Figure 4.2. The seller gets the largest revenue $\mathcal{R}_{BU}$ in equilibrium in which he uses bundling to *discriminate* among buyers with different valuations. In such an equilibrium, he posts the unit and bundle prices equal to $p = v$ and $q = 2\gamma - \frac{2(1-\alpha)}{2-\alpha}(1-v)$, which makes buyers with the high valuation willing to accept the offer of the bundle, while buyers with the low valuation accept the offer of the unit.

If it is likely that buyers have the high valuation, that is, if the probability $\alpha$ is such that $\alpha > v$, then the seller is better of by targeting buyers with the high valuation only. If buyers do not value having both units that much, that is, if the scaling parameter $\gamma$ is low, then the seller sells both units unit-by-unit; see the "$\mathcal{R}_{UN}$"-region in Figure 4.2. The seller post the unit price equal to $p = 1$ and the bundle price $q > 2\gamma$ to ensure that buyers prefer the offer of the unit over the offer of the bundle. This gives him the revenue $\mathcal{R}_{UN}$. In contrast, if buyers value having both units, that is, if $\gamma$ is high, then the seller takes advantage of it and uses bundling as a *precaution* to increase the likelihood of selling both units; see the "$\mathcal{R}_{BN}$"-region in Figure 4.2. This brings him the revenue $\mathcal{R}_{BN}$.

## 4.4 Two Periods to Make a Deal

If the seller has two periods to make a deal with one or two buyers, then he may find it optimal to screen buyers with the high valuation in the first period. This is another way of how the seller can use bundling to *discriminate* among buyers with different valuations.

### 4.4.1 Equilibria with and without Screening

To screen buyers, the seller can post the first-period unit and bundle prices such that buyers with different valuations accept different offers; I refer to this as a *screening* strategy. Given the multi-unit single-crossing property, if a screening strategy is applied, then the equilibrium reaction of buyers in the first period $(\bar{\sigma}_1^B, \underline{\sigma}_1^B)$ is one of the following: $(B, U)$, $(B, N)$, or $(U, N)$. Furthermore, buyers' valuations become fully and publicly known.

Equilibria with screening exist for some but not all parameters. Lemma 4.2 below gives not only the equilibrium revenue when such an equilibrium exists, but also a candidate revenue if the seller tries to screen buyers.

**Lemma 4.2** (Screening of Buyers with a High Valuation)**.** *In any equilibrium in which the seller applies a screening strategy, his revenue is $\mathcal{R} = \mathcal{R}_{BU}$.*

If the seller does not screen buyers, that is, if he posts unit and bundle prices such that buyers with different valuations accept the same offer or reject both offers, then the following two observations apply. First, the seller never posts such prices that buyers with the high and low valuations accept the offer of the bundle, because he would be better off by selling the product unit-by-unit. Therefore, the equilibrium reaction of buyers in the first period $(\bar{\sigma}_1^B, \underline{\sigma}_1^B)$ is either $(U, U)$ or $(N, N)$. Second, if buyers reject the first-period offers, that is, if $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (N, N)$, then the continuation game is identical to the one-period game. Indeed, there is no learning, and so the seller has the same belief $\alpha$ that a given buyer has the high valuation. Similarly, each buyer's belief that another buyer has the high valuation stays the same and is equal to $\alpha$. Lemma 4.3 follows.

**Lemma 4.3** (No Screening)**.** *In any equilibrium in which the seller does not apply a screening strategy, his revenue is as in the one-period game.*

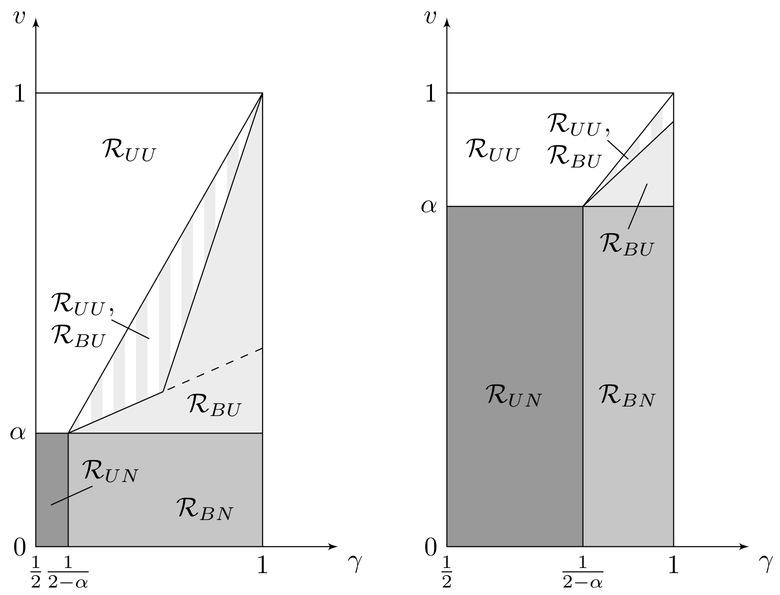


**Figure 4.3.** The seller's equilibrium revenue in the game with two periods if $\alpha < \frac{1}{2}$ (left) and $\alpha \geq \frac{1}{2}$ (right). Parameters: $\alpha = \frac{1}{4}$ (left) and $\alpha = \frac{3}{4}$ (right).

### 4.4.2 Equilibria with Two Periods

Lemmata 4.2 and 4.3 imply that the seller's revenue in any equilibrium is necessary one of the same four revenues an in the one-period game: $\mathcal{R}_{UU}$, $\mathcal{R}_{BU}$, $\mathcal{R}_{UN}$, or $\mathcal{R}_{BN}$. However, multiplicity of equilibria in terms of revenue, which is a feature of that game for all parameters of the model for which $\mathcal{R}_{BU}$ is the largest revenue, disappears for some parameters. This is captured in Proposition 4.2 below and is illustrated in Figure 4.3.

**Proposition 4.2** (Equilibria in the Dynamic Game)**.** *In any equilibrium of the two-period game, the seller's revenue is as in the one-period game, except for:*

– (left Fig. 4.3, "$\mathcal{R}_{BU}$") *if* $\alpha < \frac{1}{2}$ *and either* $\frac{2(1-\alpha)}{1-2\alpha}\gamma - \frac{1}{1-2\alpha} \geq v \geq \alpha(2-\alpha)\gamma$ *or* $\alpha(2-\alpha)\gamma \geq v \geq \alpha$*, then* $\mathcal{R} = \mathcal{R}_{BU}$*;*
– (right Fig. 4.3, "$\mathcal{R}_{BU}$") *if* $\alpha \geq \frac{1}{2}$ *and* $\alpha(2-\alpha)\gamma \geq v \geq \alpha$*, then* $\mathcal{R} = \mathcal{R}_{BU}$*.*

Proposition 4.2 implies that, as in the game with one period, if the valuation $v$ is high, then the best the seller can do is to sell both units unit-by-unit at the price which is acceptable by buyers with both high and low valuations; see "$\mathcal{R}_{UU}$"-region in Figure 4.3. The seller does it in the first period with the unit price $p = v$ or posts unacceptable first-period prices and sell both units in the second period with the unit price $p'' = v$.

If it is likely that buyers have the high valuation, that is, if the probability $\alpha$ is such that $\alpha > v$, then the seller finds it optimal to target buyers with the high valuation only; see "$\mathcal{R}_{UN}$"-region or "$\mathcal{R}_{BN}$"-region in Figure 4.3. However, because the seller lacks commitment power, he has to wait until the second period to do so. That is, the seller posts unacceptable first-period unit and bundle prices, and the prices he posts in the second period are identical to those in the one-period game.

Having two periods to make a deal with one or two buyers can be advantageous for the seller when $\mathcal{R}_{BU}$ is the largest revenue for given parameters. For a subset of parameters, there is now a unique equilibrium in terms of revenue in which the seller gets $\mathcal{R}_{BU}$; see "$\mathcal{R}_{BU}$"-region in Figure 4.3.

The seller can use bundling to get the desired $\mathcal{R}_{BU}$. Specifically, he posts the first-period unit and bundle prices equal to $p = v$ and $q = 2\gamma - \frac{2(1-\alpha)}{2-\alpha}(1-v)$, and buyers with the high valuation accept the offer of the bundle, while buyers with the low valuation accept the offer of the unit. Note that $q > 2p$ for parameters for which $\mathcal{R}_{BU}$ is the equilibrium revenue. Buyers with the high valuation are willing to pay such a high price for the bundle, because they fear that they would end up without the product otherwise. The seller can also get $\mathcal{R}_{BU}$ by posting the unit price $p = \alpha\gamma + (1-\alpha)v$ and the bundle price such that buyers with the high valuation accept the offer of the unit in the first period, while buyers with the low valuation reject both offers. If the offer of the unit is accepted by one of the buyers, and so if there is only one buyer with the high valuation, then the seller sells the second unit to the same buyer in the second period. The seller uses this screening strategy only if buyers value having both units of the product, that is, only if the scaling parameter $\gamma$ is such that $2\gamma - 1 \geq v$.

## 4.5 To Bundle or not to Bundle

Which seller's strategy yields him the largest revenue depends on the parameters of the model: the low valuation $v$, the scaling parameter $\gamma$, and the probability $\alpha$ that a buyer has the high valuation. In this section, I discuss these dependences in detail.

Lemma 4.4 below states the dependence of the seller's equilibrium revenue on the parameters. If buyers with the low valuation are in the picture, then the equilibrium revenue of the seller is $\mathcal{R}_{UU}$ or $\mathcal{R}_{BU}$, and it increases in the valuation $v$. If the seller targets buyers with the high valuation only, then his revenue is $\mathcal{R}_{UN}$ or $\mathcal{R}_{BN}$, and it is independent of $v$, because buyers with the low valuation always reject the offers, and so their valuation is irrelevant. Whenever the seller takes advantage of buyers' appreciation for having both units, his equilibrium revenue is $\mathcal{R}_{BU}$ or $\mathcal{R}_{BN}$, and it increases in the scaling parameter $\gamma$. The higher is $\gamma$, the higher is, for example, the bundling price the seller can post, and thus the larger is his revenue. If the seller targets buyers with different valuations with different offers, then his equilibrium revenue is $\mathcal{R}_{BU}$, $\mathcal{R}_{UN}$, or $\mathcal{R}_{BN}$, and it increases in the probability $\alpha$ of having buyers with the high valuation. Indeed, the seller behaves this way, because he wants to take advantage of buyers' high valuation for the product, and so he is better off if he is like to face such buyers.

**Lemma 4.4.** *The seller's equilibrium revenue depends on the parameters as follows:*
– *$\mathcal{R}_{UN}$, $\mathcal{R}_{BN}$ are independent of, while $\mathcal{R}_{UU}$, $\mathcal{R}_{BU}$ increase in $v$;*
– *$\mathcal{R}_{UU}$, $\mathcal{R}_{UN}$ are independent of, while $\mathcal{R}_{BU}$, $\mathcal{R}_{BN}$ increase in $\gamma$;*
– *$\mathcal{R}_{UU}$ is independent of, while $\mathcal{R}_{BU}$, $\mathcal{R}_{UN}$, $\mathcal{R}_{BN}$ increase in $\alpha$.*

It follows from Lemma 4.4 that he interplay between the low valuation $v$ and the probability $\alpha$ of having buyers with the high valuation determines whether the seller targets buyers with the high valuation only or posts such unit and bundle prices that buyers with both high and low valuations accept one of the offers. Furthermore, the scaling parameter $\gamma$ determines whether the seller sells both units as a bundle. Hereafter, I focus on the strategies which bring the seller the largest revenue for given parameters.

If the *valuation* $v$ is low, then the seller targets buyers with the high valuation only; see the left panel in Figure 4.4. He sells the product unit-by-unit and gets $\mathcal{R}_{UN}$ if the probability of having buyers with the high valuation is high. The seller sells the product as a bundle and gets $\mathcal{R}_{BN}$ if buyers value having both units. In contrast, if the valuation $v$ is high, then the seller is better off by selling the two units unit-by-unit at the price acceptable by buyers with both high and low valuations. This brings him $\mathcal{R}_{UU}$; see the right panel in Figure 4.4.

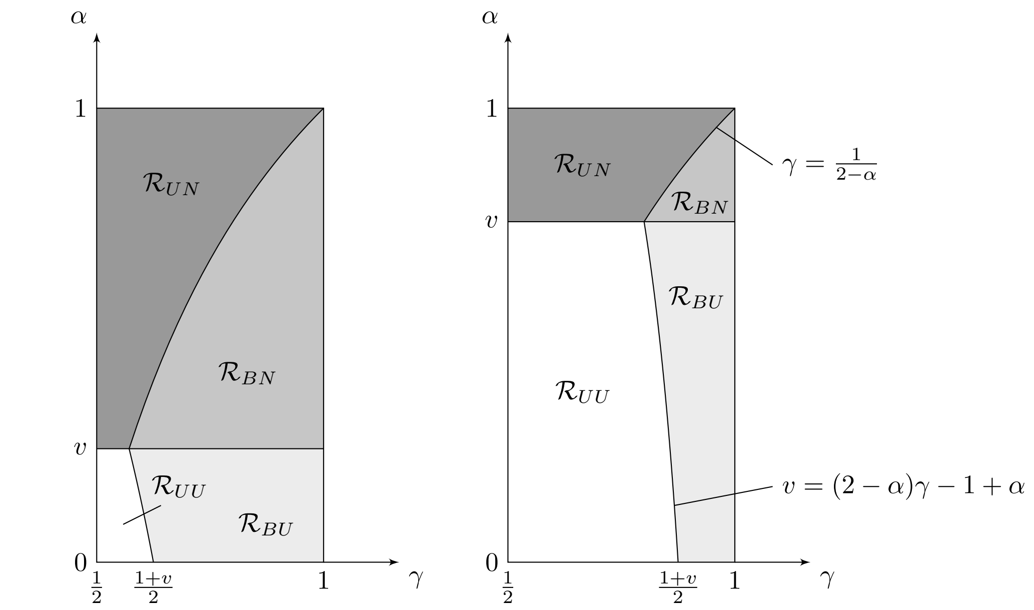


**Figure 4.4.** The largest equilibrium revenue. Parameters: $v = \frac{1}{4}$ (left) and $v = \frac{3}{4}$ (right).

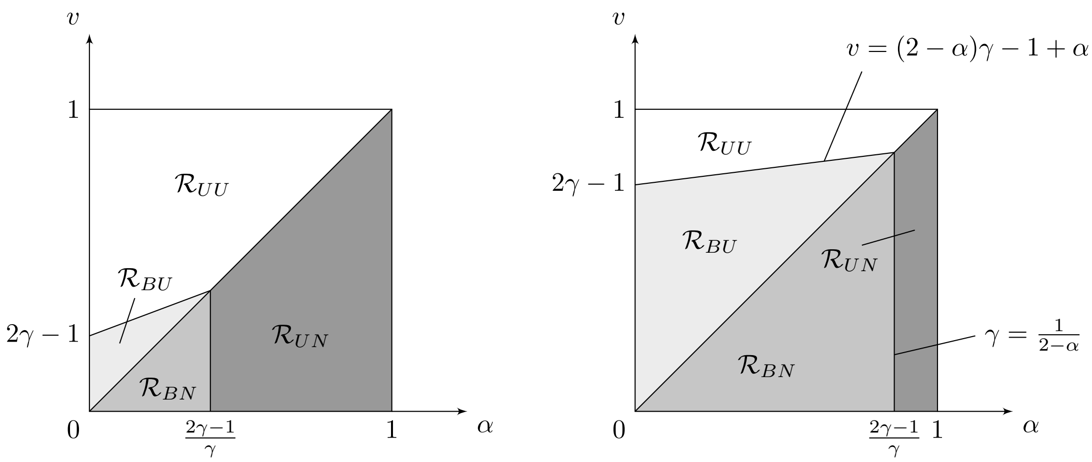


**Figure 4.5.** The largest equilibrium revenue. Parameters: $\gamma = \frac{5}{8}$ (left) and $\gamma = \frac{7}{8}$ (right).

If buyers do not care for having both units of the product, that is, if the *scaling parameter* $\gamma$ is low, then the seller is better off by selling the product unit-by-unit; see the left panel in Figure 4.5. Whether he posts the unit price which is acceptable by buyers with both high and low valuations or by buyers with the high valuation only depends on how high is the low valuation and how likely it is that buyers have the high valuation. If the low valuation is high, then the seller sells to buyers with both high and low valuations and gets $\mathcal{R}_{UU}$. If the probability of having buyers with the high valuation is high, then he sells to buyers with the high valuation only, which brings him $\mathcal{R}_{UN}$. In contrast, if buyers appreciate having both units, that is, if the scaling parameter $\gamma$ is high, then the seller either uses bundling to discriminate among buyers with different valuations or targets buyers with the high valuation with the bundle; see the right panel in Figure 4.5. This yields him $\mathcal{R}_{BU}$ or $\mathcal{R}_{BN}$ and again depends on the interplay between the low valuation and the probability of having buyers with the high valuation.

If it is unlikely that buyers have the high valuation, that is, if the *probability* $\alpha$ is low, then the seller posts such unit and bundle prices that buyers with both high and low

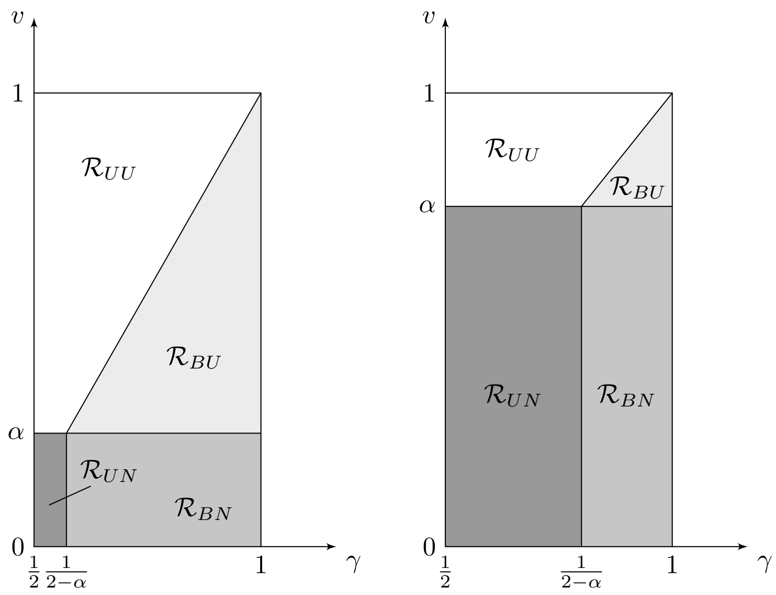


**Figure 4.6.** The largest equilibrium revenue. Parameters: $\alpha = \frac{1}{4}$ (left) and $\alpha = \frac{3}{4}$ (right).

valuations accept one of the offer; see the left panel in Figure 4.6. This brings him $\mathcal{R}_{UU}$ or $\mathcal{R}_{BU}$. The seller chooses to target buyers with the high valuation with the offer of the bundle if buyers care for having both units. In contrast, if the probability $\alpha$ of having buyers with the high valuation is high, then the seller sells the product only to these buyers; see the right panel in Figure 4.6. Whether he sells the two units unit-by-unit which yields $\mathcal{R}_{UN}$ or as a bundle which yields $\mathcal{R}_{BN}$ depends on how much buyers value having both units.

## 4.6 Conclusion

A natural extension is to consider buyers with positively correlated valuations. I expect the seller would more often find it optimal to use bundling to discriminate among buyers with different valuations, in particular, to screen buyers with the high valuation in the first period. In contrast, the bundling would be used as a precaution less. Indeed, if the buyers' valuations are correlated and it is likely that buyers have the high valuation, then the seller is better off by selling the product to these buyers unit-by-unit rather than as a bundle.

# Appendix A

# Addendum to Chapter 1

## A.1 Undiscounted Version of Strulovici (2010a)

The model with learning via good news only is as in Section 1.2 with only few exceptions. No news arrives if the risky arm is bad, that is, if $\lambda_b = 0$. It follows that the expected payoff of the bad risky arm is $b = 0$, and so $g > s > 0$. Furthermore, there are two groups of players at any time $t$: winners and unsure voters. The belief of unsure voters decreases over time according to $\dot{p}_t = -\lambda_g p_t(1 - p_t)$, and so they become more pessimistic about the type of their risky arm in absence of news. It follows that, if winners do not form a majority before the belief reaches the cut-off used by unsure voters, experimentation would cease. I denote by $p(j)$ the cut-off used by unsure voters in the presence of $j$ winners. Finally, I denote by $w(j,p)$ and $u(j,p)$ the value functions of winners and unsure voters when there are $j$ winners and unsure voters' belief is $p$.

### A.1.1 Cut-offs

Theorem A.1 below describes the equilibrium *cut-offs* $p(j)$, when there are $N > 1$ strategic and forward-looking players and $j$ of them have already leaned that they are winners. The equilibrium existence and uniqueness come from the backward induction argument on the number of winners. The equilibrium cut-offs $p(j)$ are pinned down by the value matching condition $u(j, p(j)) = s$, where $u(j,p)$ is the value function of unsure voters.[1]

**Theorem A.1** (Equilibrium Cut-offs with Learning via Good News)**.** *If $\lambda_b = 0$, then there exists a unique equilibrium characterized by cut-offs $p(j)$. The cut-offs are such that $p(j) = \bar{p}$ for all $j \leq j_N$, where*

- $\bar{p} = 0$ *if* $g \geq s\frac{N+1}{2}$,
- $\bar{p} = 1 - \frac{N+1}{N-1}\left(1 - \frac{s}{g}\right)$ *if* $g < s\frac{N+1}{2}$.

*Furthermore,* $p_M > p(j) \geq p_{SD}$ *for all* $j \leq j_N$.

Theorem A.1 states that the equilibrium cut-offs $p(j)$ are independent of the number of winners $j$. The foremost reason why unsure voters choose to experiment with the risky arm is that they hope to benefit from the high payoff of the good arm. Because this is

[1] The value function $u(j,p)$ and the cut-off $p(j)$ are found by solving the respective ordinary differential equation subject to the value-matching constraint $u(j,p(j)) = s$ and, in case $p(j) > 0$, the smooth-pasting constraint $\partial_p u(j, p(j)) = 0$. Note that the smooth-pasting constraint does not hold if $p(j) = 0$. See the proof of Lemma A.16 in Appendix A.2 for details.

also the reason why winners prefer the risky arm and because players are patient, what determines unsure voters' choice of the cut-offs is the worst case scenario when they are about to lose power over decision making, that is, when there is the critical number of winners $j_N$. In Strulovici (2010a), players discount future payoffs. As a result, the equilibrium cut-offs are above $\bar{p}$ and depend on the number of winners. Note though that, if players learn the type of their risky arm immediately, then $p(j_N) = \bar{p}$; see Corollary 1 in Strulovici (2010a).

Theorem A.1 also states that, if the payoff of the good arm is relatively high, then even pessimistic unsure voters find it worthwhile to take the risk and experiment with the risky arm, that is, $\bar{p} = 0$. If the payoff of the good arm is not high, then unsure voters must be optimistic enough to choose the risky arm over the safe one. Specifically, $\bar{p}$ satisfies

$$1 - \bar{p} = \tfrac{N+1}{N-1}(1 - p_M).$$

Recall that the myopic threshold is $p_M = \frac{s}{g}$. It follows that $\bar{p} < p_M$, that is, unsure voters still care not only about the current payoff. However, if the number of players grows arbitrary large, then they start behaving myopically; see Proposition A.1 below. In general, the equilibrium cut-off $\bar{p}$ increases with the number of players $N$. Indeed, if there are many players, then unsure voters are more anxious to experiment with the risky arm and become more biased toward the safe arm.

**Proposition A.1** (Number of Players with Learning via Good News)**.** *If $\lambda_b = 0$, then the equilibrium cut-offs are non-decreasing in the number of players $N$. Furthermore, $\lim_{N\to\infty} \bar{p} = p_M$.*

## A.1.2 Value Functions

Let $q$ denote a normalized belief of unsure voters, which is defined by

$$q := \tfrac{p-\bar{p}}{1-\bar{p}},$$

where $p$ is the belief of unsure voters. The normalized belief $q$ can be interpreted as a probability that an unsure voter who actually has the good arm learns about it before her belief falls to $\bar{p}$. It is convenient to work with normalized beliefs, because, unless winners form a majority, experimentation goes on as long as the belief $p$ of unsure voters is above $\bar{p}$, that is, as long as $q$ is above $0$.

If winners are not in majority when the belief of unsure voters falls to $\bar{p}$, then the safe arm is implemented. As a result, winners and unsure voters get the same payoff $s$. If winners are in majority or unsure voters are optimistic, then the risky arm is the voting outcome. Given the number of winners $j$ and the belief $p$ of unsure voters, the expected payoffs, or the *value functions*, of winners and unsure voters are given by $w(j,p)$ and $u(j,p)$. With a slight abuse of notation regarding the dependence on the belief $p$ and on the normalized belief $q$, $w(j,p)$ and $u(j,p)$ are equal to $\tilde{w}(j,q)$ and $\tilde{u}(j,q)$ described in detail below. All these are summarized in Lemma A.1 below.

The *value function of winners* is given by

$$\tilde{w}(j,q) = g\mathbf{P}[R \text{ is implemented}] + s\mathbf{P}[S \text{ is implemented}], \tag{A.1}$$

where the probabilities are defined by

$$\mathbf{P}[R \text{ is implemented}] := \sum_{k=0}^{\frac{N-1}{2}} \binom{N-j}{k} q^{N-j-k}(1-q)^k,$$
$$\mathbf{P}[S \text{ is implemented}] := \sum_{m=0}^{j_N-j} \binom{N-j}{m} q^m(1-q)^{N-j-m}.$$

The *value function of unsure voters* is given by

$$\begin{aligned}\tilde{u}(j,q) &= gq\mathbf{P}[R \text{ is implemented} \mid \text{being a winner}] \\ &\quad + \bar{p}g(1-q)\mathbf{P}[R \text{ is implemented} \mid \text{being a loser}] + s\mathbf{P}[S \text{ is implemented}], \quad \text{(A.2)}\end{aligned}$$

where the probabilities are defined by

$$\mathbf{P}[R \text{ is implemented} \mid \text{being a winner}] := \sum_{k=0}^{\frac{N-1}{2}} \binom{N-j-1}{k} q^{N-j-1-k}(1-q)^k,$$
$$\mathbf{P}[R \text{ is implemented} \mid \text{being a loser}] := \sum_{k=0}^{\frac{N-1}{2}-1} \binom{N-j-1}{k} q^{N-j-1-k}(1-q)^k$$

The interpretation of the value functions and the probabilities is as in Section 1.3.3.

**Lemma A.1** (Equilibrium Value Functions with Learning via Good News)**.** *The equilibrium value functions of winners $w(j,p)$ and unsure voters $u(j,p)$ satisfy:*
- $w(j,p) = \tilde{w}(j,q)$ *and* $u(j,p) = \tilde{u}(j,q)$ *for all* $p > \bar{p}$, *where* $q := \frac{p-\bar{p}}{1-\bar{p}}$,
- $w(j,p) = u(j,p) = s$ *for all* $p \leq \bar{p}$,

*for all* $j \leq j_N$.

## A.2 Proofs

Identities defined in Claims A.1 and A.2 are used in the proofs below.

**Claim A.1.** *For* $a_1 \in \mathbb{N}_0$, $a_2 \in \mathbb{N}_0$, *and* $q \in (0,1)$,[2]

$$\Phi_q(a_1,a_2) := \sum_{k=0}^{a_2} \binom{a_1+a_2}{k} \left(1 - \tfrac{a_1+a_2+1}{a_1+a_2+1-k} q\right) q^{a_2-k}(1-q)^k = \binom{a_1+a_2}{a_2}(1-q)^{a_2+1}.$$

**Proof.** I prove the claim using the induction argument on $a_2$. Let $a_2 = 0$, then both sides are equal to $1-q$. I assume that the equality holds for $a_2 - 1$, and I show next that it is also holds for $a_2$. I rewrite $\Phi_q(a_1,a_2)$ as follows:

$$\Phi_q(a_1,a_2) = q \sum_{k=0}^{a_2-1} \binom{a_1+a_2}{k} \left(1 - \tfrac{a_1+a_2+1}{a_1+a_2+1-k} q\right) q^{a_2-1-k}(1-q)^k$$

[2] The set $\mathbb{N}_0$ is a set of natural numbers including zero, that is, $\mathbb{N}_0 := \{0,1,2,3,\ldots\}$.

$$+ \binom{a_1+a_2}{a_2}\left(1 - \tfrac{a_1+a_2+1}{a_1+1}q\right)(1-q)^{a_2}.$$

The sum in the last expression is equal to $\Phi_q(a_1+1, a_2-1)$. Applying the assumption for $a_2 - 1$ yields

$$\begin{aligned}\Phi_q(a_1,a_2) = \tbinom{a_1+a_2}{a_2-1}q(1-q)^{a_2} + \tbinom{a_1+a_2}{a_2}(1-q)^{a_2+1} - \tbinom{a_1+a_2}{a_2-1}q(1-q)^{a_2}\\ = \tbinom{a_1+a_2}{a_2}(1-q)^{a_2+1},\end{aligned}$$

which gives the result. ■

**Claim A.2.** *For $a_1 \in \mathbb{N}_0$, $a_2 \in \mathbb{N}_0$, and $q \in (0,1)$,*

$$\begin{aligned}\Psi_q(a_1,a_2) := \sum_{k=0}^{a_2} \tbinom{a_1+a_2}{k}\left[1 - \tfrac{(a_1+a_2+1)(a_1+a_2+2)}{(a_1+a_2+1-k)(a_1+a_2+2-k)}q^2\right] q^{a_2-k}(1-q)^k\\ = \tbinom{a_1+a_2+1}{a_2}q(1-q)^{a_2+1} + \tbinom{a_1+a_2}{a_2}(1-q)^{a_2+1}.\end{aligned}$$

**Proof.** I prove the claim using the induction argument on $a_2$. Let $a_2 = 0$, then both sides are equal to $(1+q)(1-q)$. I assume that the equality holds for $a_2 - 1$, and I show next that it also holds for $a_2$. I rewrite $\Psi_q(a_1, a_2)$ as follows:

$$\begin{aligned}\Psi_q(a_1,a_2) = q\sum_{k=0}^{a_2-1} \tbinom{a_1+a_2}{k}\left[1 - \tfrac{(a_1+a_2+1)(a_1+a_2+2)}{(a_1+a_2+1-k)(a_1+a_2+2-k)}q^2\right] q^{a_2-1-k}(1-q)^k\\ + \tbinom{a_1+a_2}{a_2}\left[1 - \tfrac{(a_1+a_2+1)(a_1+a_2+2)}{(a_1+1)(a_1+2)}q^2\right](1-q)^{a_2}.\end{aligned}$$

The sum in the last expression is equal to $\Psi_q(a_1+1, a_2-1)$. Applying the assumption for $a_2 - 1$ yields

$$\begin{aligned}\Psi_q(a_1,a_2) = q\left[\tbinom{a_1+a_2+1}{a_2-1}q(1-q)^{a_2} + \tbinom{a_1+a_2}{a_2-1}(1-q)^{a_2}\right]\\ + \tbinom{a_1+a_2}{a_2}\left[1 - \tfrac{(a_1+a_2+1)(a_1+a_2+2)}{(a_1+1)(a_1+2)}q^2\right](1-q)^{a_2}.\end{aligned}$$

Note the following: $\binom{a_1+a_2+1}{a_2-1} = \frac{(a_1+a_2+1)a_2}{(a_1+1)(a_1+2)}\binom{a_1+a_2}{a_2}$, $\binom{a_1+a_2}{a_2-1} = \binom{a_1+a_2+1}{a_2} - \binom{a_1+a_2}{a_2}$, and $\binom{a_1+a_2+1}{a_2} = \frac{a_1+a_2+1}{a_1+1}\binom{a_1+a_2}{a_2}$. Therefore,

$$\Psi_q(a_1,a_2) = \left[\tbinom{a_1+a_2+1}{a_2} - \tbinom{a_1+a_2}{a_2}\right]q(1-q)^{a_2} + \tbinom{a_1+a_2}{a_2}q^2(1-q)^{a_2} - \tbinom{a_1+a_2+1}{a_2}q^2(1-q)^{a_2},$$

which gives the result. ■

For the proofs that follow, it is convenient to define $x := i_N - i$ and $y := j_N - j$. The functions $\tilde{l}(i,j,p)$, $\tilde{w}(i,j,p)$, and $\tilde{u}(i,j,p)$ in (1.2), (1.3), and (1.4) can be rewritten as follows:

$$\begin{aligned}\tilde{l}(i_N - x, j_N - y, p) = bp^{y+1}\sum_{k=0}^{x}\tbinom{x+y+1}{k}p^{x-k}(1-p)^k\\ + s(1-p)^{x+1}\sum_{m=0}^{y}\tbinom{x+y+1}{m}p^m(1-p)^{y-m}, \quad \text{(A.3)}\end{aligned}$$

$$\tilde{w}(i_N - x, j_N - y, p) = gp^{y+1} \sum_{k=0}^{x} \binom{x+y+1}{k} p^{x-k}(1-p)^k + s(1-p)^{x+1} \sum_{m=0}^{y} \binom{x+y+1}{m} p^m (1-p)^{y-m}, \quad \text{(A.4)}$$

and

$$\tilde{u}(i_N - x, j_N - y, p) = bp^{y+1}(1-p) \sum_{k=0}^{x-1} \binom{x+y}{k} p^{x-1-k}(1-p)^k + gp^{y+1} \sum_{k=0}^{x} \binom{x+y}{k} p^{x-k}(1-p)^k + s(1-p)^{x+1} \sum_{m=0}^{y} \binom{x+y+1}{m} p^m (1-p)^{y-m}. \quad \text{(A.5)}$$

Similarly, the functions $\tilde{w}(j, q)$ and $\tilde{u}(j, q)$ in (A.1) and (A.2) can be rewritten as follows:

$$\tilde{w}(j_N - y, q) = gq^{y+1} \sum_{k=0}^{\frac{N-1}{2}} \binom{\frac{N+1}{2}+y}{k} q^{\frac{N-1}{2}-k}(1-q)^k + s(1-q)^{\frac{N+1}{2}} \sum_{m=0}^{y} \binom{\frac{N+1}{2}+y}{m} q^m (1-q)^{y-m}, \quad \text{(A.6)}$$

$$\tilde{u}(j_N - y, q) = gq^{y+1} \sum_{k=0}^{\frac{N-1}{2}} \binom{\frac{N+1}{2}+y-1}{k} q^{\frac{N-1}{2}-k}(1-q)^k + \bar{p}gq^{y+1}(1-q) \sum_{k=0}^{\frac{N-1}{2}-1} \binom{\frac{N+1}{2}+y-1}{k} q^{\frac{N-1}{2}-1-k}(1-q)^k + s(1-q)^{\frac{N+1}{2}} \sum_{m=0}^{y} \binom{\frac{N+1}{2}+y}{m} q^m (1-q)^{y-m}. \quad \text{(A.7)}$$

### A.2.1 Proof of Theorems 1.1 and 1.2, Corollaries 1.1 and 1.2, and Lemma 1.1

Theorem 1.1 and Lemma 1.1 follow from the next lemmata. Theorem 1.2 is a special case of Theorem 1.1. Corollaries 1.1 and 1.2 are direct consequences of Theorem 1.1.

**Lemma A.2.** *The equilibrium exists and is unique.*

**Proof.** Similar to Strulovici (2010a), the equilibrium existence and uniqueness come from a backward induction argument on the number of losers and winners. ■

**Lemma A.3.** *The value functions of losers, winners, and unsure voters satisfy $l(i, j, p) \geq \tilde{l}(i, j, p)$, $w(i, j, p) \leq \tilde{w}(i, j, p)$, and $u(i, j, p) \geq \tilde{u}(i, j, p)$ for all $p \in [0, 1]$, $i \leq i_N$, and $j \leq j_N$, with equalities if $p_0 > \max_{i \leq i_N, j \leq j_N} p(i, j)$.*

**Proof.** Assume that unsure voters start experimenting with the risky arm and stop experimentation if and only if losers form a majority. In particular, this is the case if $p_0 >$

$\max_{i\leq i_N, j\leq j_N} p(i,j)$, because the belief of unsure voters is always above the cut-off they use given the current number of losers and winner, i.e., $p \geq p(i,j)$ for all $i \leq i_N$ and $j \leq j_N$.

Because unsure voters can become either losers or winners, but not other way around, the sets of losers $I$ and winners $J$ can only grow over time. I proceed with an induction argument based on the number of losers $i$ and winners $j$.

If $i > i_N$, then losers form a majority, and so the safe arm is implemented. In particular, it follows that $l(i_N+1, j_N, p) = w(i_N+1, j_N, p) = s$. If $j > j_N$, then winners form a majority, and so the risky arm is implemented. In particular, it follows that $l(i_N, j_N+1, p) = b$ and $w(i_N, j_N+1, p) = g$.

If $x = 0$ and $y = 0$, i.e., if $i = i_N$ and $j = j_N$, then $N - i_N - j_N = 1$. The value function of unsure voters, when they experiment, satisfies the following ordinary differential equation (ODE):

$$u(i_N, j_N, p) = \tfrac{1}{\lambda_g p + \lambda_b(1-p)} \left[\lambda_b(1-p)s + \lambda_g pg + (\lambda_b - \lambda_g)p(1-p)\partial_p u(i_N, j_N, p)\right].$$

The solution to the ODE takes the form:[3]

$$u(i_N, j_N, p) = pg + (1-p)s + C(1-p)\left(\tfrac{p}{1-p}\right)^{\frac{\lambda_b}{\lambda_b - \lambda_g}},$$

where $C$ is a constant of integration. The value function of unsure voters must satisfy the boundary condition $u(i_N, j_N, 1) = g$. Therefore, $C = 0$, and so

$$u(i_N, j_N, p) = pg + (1-p)s.$$

Note that $u(i_N, j_N, p) = \tilde{u}(i_N, j_N, p)$. Similarly, the value functions of losers and winners must solve:

$$l(i_N, j_N, p) = \tfrac{1}{\lambda_g p + \lambda_b(1-p)} \left[\lambda_g pb + \lambda_b(1-p)s + (\lambda_b - \lambda_g)p(1-p)\partial_p l(i_N, j_N, p)\right]$$

subject to $l(i_N, j_N, 1) = b$, and

$$w(i_N, j_N, p) = \tfrac{1}{\lambda_g p + \lambda_b(1-p)} \left[\lambda_g pg + \lambda_b(1-p)s + (\lambda_b - \lambda_g)p(1-p)\partial_p w(i_N, j_N, p)\right]$$

subject to $w(i_N, j_N, 1) = g$. It follows that $l(i_N, j_N, p) = \tilde{l}(i_N, j_N, p)$ and $w(i_N, j_N, p) = \tilde{w}(i_N, j_N, p)$.

Assume that the lemma holds for all $x' \in \{0, \ldots, x\}$ (i.e., for all $i' \in \{i, \ldots, i_N\}$) and all $y' \in \{0, \ldots, y\}$ (i.e., for all $j' \in \{j, \ldots, j_N\}$), but not when both $x' = x$ and $y' = y$ at the same time. Next I show that it also holds for $x$ and $y$ (i.e., for $i$ and $j$).

The value function of unsure voters, when they experiment, satisfies the following ODE:

$$u(i,j,p) = \tfrac{1}{(N-i-j)(\lambda_g p + \lambda_b(1-p))}\Big(\lambda_b(1-p)[l(i+1,j,p) + (N-i-j-1)u(i+1,j,p)]$$

[3]The solution to the ODE of the form $f'(x) + a(x)f(x) = b(x)$ is given by

$$f(x) = e^{-A(x)}\left(\int b(x)e^{A(x)}\mathrm{d}x + C\right),$$

where $A(x) = \int a(x)\mathrm{d}x$ and $C$ is a constant of integration.

$$+ \lambda_g p[w(i,j+1,p) + (N-i-j-1)u(i,j+1,p)] + (\lambda_b - \lambda_g)p(1-p)\partial_p u(i,j,p)\Big),$$

where $u(i+1,j,p) = \tilde{u}(i+1,j,p)$, $l(i+1,j,p) = \tilde{l}(i+1,j,p)$, $u(i,j+1,p) = \tilde{u}(i,j+1,p)$, and $w(i,j+1,p) = \tilde{w}(i,j+1,p)$ by assumption. Solving the ODE subject to $u(i,j,1) = g$ yields $u(i,j,p) = \tilde{u}(i,j,p)$. Similarly, the value functions of losers and winners must solve:

$$\begin{aligned} l(i,j,p) = \tfrac{1}{(N-i-j)(\lambda_g p + \lambda_b(1-p))}[&\lambda_b(1-p)(N-i-j)l(i+1,j,p) \\ &+ \lambda_g p(N-i-j)l(i,j+1,p) + (\lambda_b - \lambda_g)p(1-p)\partial_p l(i,j,p)] \end{aligned}$$

subject to $l(i,j,1) = b$, and

$$\begin{aligned} w(i,j,p) = \tfrac{1}{(N-i-j)(\lambda_g p + \lambda_b(1-p))}[&\lambda_b(1-p)(N-i-j)w(i+1,j,p) \\ &+ \lambda_g p(N-i-j)w(i,j+1,p) + (\lambda_b - \lambda_g)p(1-p)\partial_p w(i,j,p)] \end{aligned}$$

subject to $w(i,j,1) = g$, where $l(i+1,j,p) = \tilde{l}(i+1,j,p)$, $l(i,j+1,p) = \tilde{l}(i,j+1,p)$, $w(i+1,j,p) = \tilde{w}(i+1,j,p)$, and $w(i,j+1,p) = \tilde{w}(i,j+1,p)$ by assumption. It follows that $l(i,j,p) = \tilde{l}(i,j,p)$ and $w(i,j,p) = \tilde{w}(i,j,p)$.

The function $\tilde{u}(i,j,p)$ is found assuming that, if unsure voters choose to experiment with the risky arm for the very first time when neither of them has learned whether she is a loser or a winner, they stop if and only if losers form a majority. Because unsure voters can always vote for the safe arm instead, their value function satisfies $u(i,j,p) \geq \tilde{u}(i,j,p)$. Furthermore, the value functions of losers and winners satisfy $l(i,j,p) \geq \tilde{l}(i,j,p)$ and $w(i,j,p) \leq \tilde{w}(i,j,p)$, since the safe arm is preferred by losers, while the risky arm is preferred by winners. ■

Let $\tilde{p}(i,j)$ be such that $\tilde{u}(i,j,\tilde{p}(i,j)) = s$ and $\tilde{u}(i,j,p) > s$ for all $p > \tilde{p}(i,j)$. That is, $\tilde{p}(i,j)$ is the largest $p$ for which the value-matching condition holds.

**Lemma A.4.** *$\tilde{p}(i,j) \geq p(i,j)$ for all $i \leq i_N$ and $j \leq j_N$.*

**Proof.** Because the value function of unsure voters satisfies $u(i,j,p) \geq \tilde{u}(i,j,p)$ for all $p \in [0,1]$, $i \leq i_N$, and $j \leq j_N$, $\tilde{p}(i,j)$ are upper bounds for the cut-offs $p(i,j)$. ■

**Lemma A.5.** *The function $\tilde{l}(i,j,p)$ decreases in $p$, while the function $\tilde{w}(i,j,p)$ increases in $p$ for all $p \in [0,1]$, $i \leq i_N$, and $j \leq j_N$.*

**Proof.** I work with $\tilde{l}(i,j,p)$ and $\tilde{w}(i,j,p)$ written as (A.3) and (A.4) with $x \coloneqq i_N - i$ and $y \coloneqq j_N - j$. The partial derivative with respect to $p$ of $\tilde{l}(i_N - x, j_N - y, p)$ is as follows:

$$\begin{aligned} \partial_p \tilde{l}(i_N - x, j_N - y, p) = b \sum_{k=0}^{x} \tbinom{x+y+1}{k} \left[(x+y+1-k)p^{x+y-k}(1-p)^k - kp^{x+y+1-k}(1-p)^{k-1}\right] \\ + s \sum_{m=0}^{y} \tbinom{x+y+1}{m} \left[mp^{m-1}(1-p)^{x+y+1-m} - (x+y+1-m)p^m(1-p)^{x+y-m}\right]. \end{aligned}$$

Note that

$$\sum_{k=0}^{x} \tbinom{x+y+1}{k} kp^{x+y+1-k}(1-p)^{k-1} = \sum_{k=1}^{x} \tbinom{x+y+1}{k} kp^{x+y+1-k}(1-p)^{k-1}$$

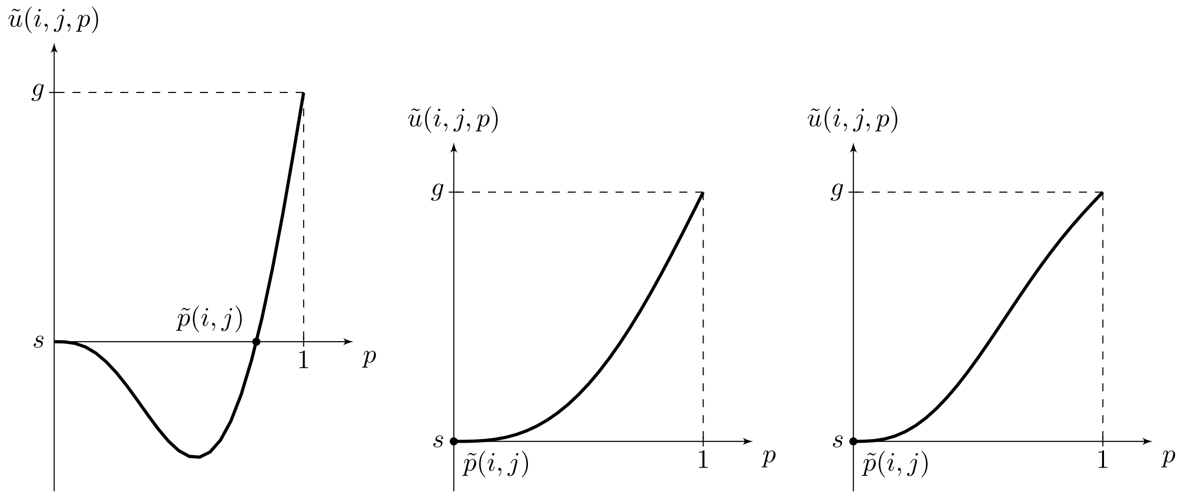


**Figure A.1.** The shape of $\tilde{u}(i,j,p)$ as a function of $p$. Case 1 (left): $\tilde{u}(i,j,p)$ is concave and then convex, and $\tilde{p}(i,j) > 0$. Case 2 (middle): $\tilde{u}(i,j,p)$ is convex, and $\tilde{p}(i,j) = 0$. Case 3 (right): $\tilde{u}(i,j,p)$ is convex and then concave, and $\tilde{p}(i,j) = 0$. Parameters: $(N,g,s,b,i,j) = (5,1,0,b,0,0)$ with $b = -5$ (left), $b = -1$ (middle), and $b = -0.05$ (right).

$$= \sum_{k=0}^{x-1} \binom{x+y+1}{k}(x+y+1-k)p^{x+y-k}(1-p)^k.$$

Similarly,

$$\sum_{m=0}^{y} \binom{x+y+1}{m} m p^{m-1}(1-p)^{x+y+1-m} = \sum_{m=0}^{y-1} \binom{x+y+1}{m}(x+y+1-m)p^m(1-p)^{x+y-m}.$$

Therefore,

$$\partial_p \tilde{l}(i_N - x, j_N - y, p) = -(s-b)(x+y+1)\binom{x+y}{x} p^y (1-p)^x < 0,$$

for all $p \in (0,1)$.

The function $\tilde{w}(i,j,p)$ is the same as $\tilde{l}(i,j,p)$, except $b$ in the latter should be replaced by $g$. Thus, the partial derivative with respect to $p$ of $\tilde{w}(i_N - x, j_N - y, p)$ takes the form

$$\partial_p \tilde{w}(i_N - x, j_N - y, p) = (g-s)(x+y+1)\binom{x+y}{x} p^y (1-p)^x > 0,$$

for all $p \in (0,1)$. ■

**Lemma A.6.** *The function $\tilde{u}(i,j,p)$ increases in $p$ for all $p \geq \tilde{p}(i,j)$, $i \leq i_N$, and $j \leq j_N$.*

**Proof.** The lemma is proved by exploiting convexity of $\tilde{u}(i,j,p)$ as a function of $p$. Depending on parameters, $\tilde{u}(i,j,p)$ can take only one of three shapes (see Figure A.1), which correspond to three cases analyzed in the end of the proof. In particular, $\tilde{u}(i,j,p)$ has at most one inflection point.

I work with $\tilde{u}(i,j,p)$ written as (A.5) with $x := i_N - i$ and $y := j_N - j$. As it is useful later on, note that

$$\tilde{u}(i_N - x, j_N - y, 0) = s,$$
$$\tilde{u}(i_N - x, j_N - y, 1) = g,$$

for all $x \leq i_N$ and $y \leq j_N$.

I start with the analysis of particular cases, namely, $i = i_N$ (i.e., $x = 0$) and $j = j_N$ (i.e., $y = 0$), because they are slightly different from the others (as will be emphasized in footnote 4). When $i = i_N$, i.e., when $x = 0$, $\tilde{u}(i_N, j, p)$ takes the form

$$\tilde{u}(i_N, j_N - y, p) = s \sum_{m=0}^{y} \tbinom{y+1}{m} p^m (1-p)^{y+1-m} + gp^{y+1}$$
$$= s\left(1 - p^{y+1}\right) + gp^{y+1} = s + (g-s)p^{y+1}.$$

Its partial derivative with respect to $p$ is as follows:

$$\partial_p \tilde{u}(i_N, j_N - y, p) = (g-s)(y+1)(1-p)^y > 0,$$

for all $p \in (0,1)$, i.e., $\tilde{u}(i_N, j, p)$ is increasing in $p$ for all $p \in (0,1)$. Together with $\tilde{u}(i_N, j, 0) = s$, it implies that $\tilde{p}(i_N, j) = 0$ for all $j \leq j_N$.

When $j = j_N$, i.e., when $y = 0$, $\tilde{u}(i, j_N, p)$ takes the form

$$\tilde{u}(i_N - x, j_N, p) = b(1-p) \sum_{k=0}^{x-1} \tbinom{x}{k} p^{x-k}(1-p)^k + s(1-p)^{x+1} + gp \sum_{k=0}^{x} \tbinom{x}{k} p^{x-k}(1-p)^x$$
$$= b(1-p)\left(1-(1-p)^x\right) + s(1-p)^{x+1} + gp = g - (g-b)(1-p) + (s-b)(1-p)^{x+1}.$$

Its partial derivative with respect to $p$ is as follows:

$$\partial_p \tilde{u}(i_N - x, j_N, p) = (g-b) - (s-b)(x+1)(1-p)^x.$$

In particular, $\partial_p \tilde{u}(i_N, j_N, 1) = g - b > 0$. The second order partial derivative with respect to $p$ yields

$$\partial^2_{pp} \tilde{u}(i_N - x, j_N, p) = (s-b)x(x+1)(1-p)^{x-1} > 0,$$

i.e., $\tilde{u}(i, j_N, p)$ is convex (or linear if $i = i_N$, i.e., if $x = 0$) for all $p \in (0,1)$. Therefore, since $\partial_p \tilde{u}(i, j_N, 1) > 0$ as well as $\tilde{u}(i, j_N, 0) = s$, $\tilde{u}(i, j_N, p)$ is increasing for all $p \geq \tilde{p}(i, j_N)$. Moreover, $\tilde{p}(i, j_N) = 0$ whenever

$$g - b > (s-b)(x+1) \quad \Leftrightarrow \quad g - s > (s-b)x,$$

i.e., whenever $\tilde{u}(i, j_N, p)$ is decreasing for all $p \in (0,1)$. Otherwise, $\tilde{p}(i, j_N) \in (0,1)$.

Next I look at the case with arbitrary $x \in \{1, \ldots, i_N\}$ and $y \in \{1, \ldots, j_N\}$. I consider partial derivatives with respect to $p$ of each term of $\tilde{u}(i, j, p)$ separately. The partial derivative of the term corresponding to $b$ is as follows:

$$\partial_p \left( \sum_{k=0}^{x-1} \tbinom{x+y}{k} p^{x+y-k}(1-p)^{k+1} \right)$$
$$= \sum_{k=0}^{x-1} \tbinom{x+y}{k} \left[ (x+y-k)p^{x+y-1-k}(1-p)^{k+1} - (k+1)p^{x+y-k}(1-p)^k \right],$$

where

$$\sum_{k=0}^{x-1} \binom{x+y}{k} k p^{x+y-k}(1-p)^k = \sum_{k=0}^{x-2} \binom{x+y}{k}(x+y-k)p^{x+y-1-k}(1-p)^{k+1}.$$

Hence,

$$\partial_p \left( \sum_{k=0}^{x-1} \binom{x+y}{k} p^{x+y-k}(1-p)^{k+1} \right) = -\sum_{k=0}^{x-1} \binom{x+y}{k} p^{x+y-k}(1-p)^k + x\binom{x+y}{x} p^y (1-p)^x.$$

The partial derivative of the term corresponding to $g$ is as follows:

$$\begin{aligned}\partial_p &\left( \sum_{k=0}^{x} \binom{x+y}{k} p^{x+y+1-k}(1-p)^k \right) \\ &= \sum_{k=0}^{x} \binom{x+y}{k} \left[ (x+y+1-k)p^{x+y-k}(1-p)^k - k p^{x+y+1-k}(1-p)^{k-1} \right],\end{aligned}$$

where

$$\sum_{k=0}^{x} \binom{x+y}{k} k p^{x+y+1-k}(1-p)^{k-1} = \sum_{k=0}^{x-1} \binom{x+y}{k}(x+y-k)p^{x+y-k}(1-p)^k.$$

Hence,

$$\partial_p \left( \sum_{k=0}^{x} \binom{x+y}{k} p^{x+y+1-k}(1-p)^k \right) = \sum_{k=0}^{x-1} \binom{x+y}{k} p^{x+y-k}(1-p)^k + (y+1)\binom{x+y}{x} p^y (1-p)^x.$$

As shown in the proof of Lemma A.5, the partial derivative of the term corresponding to $s$ is as follows:

$$\partial_p \left( \sum_{m=0}^{y} \binom{x+y+1}{m} p^m (1-p)^{x+y+1-m} \right) = -s(x+y+1)\binom{x+y}{x} p^y (1-p)^x.$$

All in all, the partial derivative with respect to $p$ of $\tilde{u}(i,j,p)$ takes the form

$$\begin{aligned}\partial_p \tilde{u}(i_N - x, j_N - y, p) = (g-b) &\sum_{k=0}^{x-1} \binom{x+y}{k} p^{x+y-k}(1-p)^k \\ &- \left[ -bx + s(x+y+1) - g(y+1) \right] \binom{x+y}{x} p^y (1-p)^x.\end{aligned}$$

Note for future reference that[4]

$$\begin{aligned}\partial_p \tilde{u}(i_N - x, j_N - y, 0) &= 0, \\ \partial_p \tilde{u}(i_N - x, j_N - y, 1) &= g - b > 0.\end{aligned}$$

To prove the lemma, I exploit convexity of $\tilde{u}(i,j,p)$ as a function of $p$. For this reason,

[4]Note that $\partial_p \tilde{u}(i_N, j_N - y, 1) = (g-s)(y+1)$ when $x = 0$, and $\partial_p \tilde{u}(i_N - x, j_N, 0) = (g-b) - (s-b)(x+1)$ when $y = 0$, which makes these cases slightly different.

I look at its second order partial derivative with respect to $p$ next. Note that

$$\begin{aligned}
&\partial_p\left(\sum_{k=0}^{x-1}\tbinom{x+y}{k}p^{x+y-k}(1-p)^k\right)\\
&\quad=\sum_{k=0}^{x-1}\tbinom{x+y}{k}\left[(x+y-k)p^{x+y-1-k}(1-p)^k-kp^{x+y-k}(1-p)^{k-1}\right]\\
&\quad=\sum_{k=0}^{x-1}\tbinom{x+y}{k}(x+y-k)p^{x+y-1-k}(1-p)^k-\sum_{k=0}^{x-2}\tbinom{x+y}{k}(x+y-k)p^{x+y-1-k}(1-p)^k\\
&\quad=x\tbinom{x+y}{x}p^y(1-p)^{x-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\partial^2_{pp}\tilde{u}(i_N-x,j_N-y,p)&=(g-b)x\tbinom{x+y}{x}p^y(1-p)^{x-1}\\
&\quad-[-bx+s(x+y+1)-g(y+1)]\tbinom{x+y}{x}\left[yp^{y-1}(1-p)^x-xp^y(1-p)^{x-1}\right],
\end{aligned}$$

which simplifies to

$$\partial^2_{pp}\tilde{u}(i_N-x,j_N-y,p)=\tbinom{x+y}{x}p^{y-1}(1-p)^{x-1}D(x,y,p),$$

where $D(x,y,p):=[A(x,y)+B(x,y)]\,p-B(x,y)$ with

$$\begin{aligned}
A(x,y)&:=x\left[-b(x+1)+s(x+y+1)-gy\right],\\
B(x,y)&:=y\left[-bx+s(x+y+1)-g(y+1)\right].
\end{aligned}$$

Therefore, the sign of $\partial^2_{pp}\tilde{u}(i_N-x,j_N-y,p)$ for $p\in(0,1)$ coincides with the sign of $D(x,y,p)$. Note also that

$$\begin{aligned}
A(x,y)>0\quad&\Leftrightarrow\quad(s-b)(x+1)>(g-s)y,\\
B(x,y)>0\quad&\Leftrightarrow\quad(s-b)x>(g-s)(y+1),
\end{aligned}$$

and let $\bar{p}:=\frac{B(x,y)}{A(x,y)+B(x,y)}$. There are three cases to consider.

Case 1: If $(s-b)x>(g-s)(y+1)$, i.e., if $A(x,y)>0$ and $B(x,y)>0$, then

$$\partial^2_{pp}\tilde{u}(i_N-x,j_N-y,p)>0\quad\Leftrightarrow\quad[A(x,y)+B(x,y)]\,p-B(x,y)>0\quad\Leftrightarrow\quad p>\bar{p}\in(0,1).$$

Hence, $\tilde{u}(i_N-x,j_N-y,p)$ is concave for $p\in(0,\bar{p})$ and convex for $p\in(\bar{p},1)$.

Case 2: If $(s-b)x<(g-s)(y+1)$, but $(s-b)(x+1)>(g-s)y$, i.e., if $A(x,y)>0$ and $B(x,y)<0$, then

$$\partial^2_{pp}\tilde{u}(i_N-x,j_N-y,p)>0.$$

Hence, $\tilde{u}(i_N-x,j_N-y,p)$ is convex for all $p\in(0,1)$.

Case 3: If $(s-b)(x+1)>(g-s)y$, i.e., if $A(x,y)<0$ and $B(x,y)<0$, then

$$\partial^2_{pp}\tilde{u}(i_N-x,j_N-y,p)>0\quad\Leftrightarrow\quad[A(x,y)+B(x,y)]\,p-B(x,y)>0\quad\Leftrightarrow\quad p<\bar{p}\in(0,1).$$

Hence, $\tilde{u}(i_N-x,j_N-y,p)$ is convex for $p\in(0,\bar{p})$ and concave for $p\in(\bar{p},1)$.

Now it is easy to see that the lemma also holds for $x\in\{1,\dots,i_N\}$ and $y\in\{1,\dots,j_N\}$, i.e., for $i \le i_N - 1$ and $j \le j_N - 1$. First, $\tilde{u}(i,j,1) = g$ and $\partial_p \tilde{u}(i,j,1) = g - b > 0$ imply that $\tilde{u}(i,j,p)$ is increasing for large $p$, and is larger than $s$. Furthermore, $\tilde{u}(i,j,0) = s$ and $\partial_p \tilde{u}(i,j,0) = 0$ imply that $\tilde{u}(i,j,p)$ reaches $s$ smoothly. Therefore, in Case 1, $\tilde{u}(i,j,p)$ must reach $s$ at $p = 0$ from below. Hence, in such a case, the cut-off is $\tilde{p}(i,j) > 0$, and $\tilde{u}(i,j,p)$ is increasing for all $p \ge \tilde{p}(i,j)$. In Cases 2 and 3, $\tilde{u}(i,j,p)$ must reach $s$ at $p = 0$ from above. It follows that $\tilde{p}(i,j) = 0$ and $\tilde{u}(i,j,p)$ is increasing for all $p \in (0,1)$. ■

**Lemma A.7.** *The upper bounds $\tilde{p}(i,j)$ satisfy:*
- *$\tilde{p}(i,j) \in (0,1)$ if $(s-b)(i_N - i) > (g-s)(j_N - j + 1)$,*
- *$\tilde{p}(i,j) = 0$ if $(s-b)(i_N - i) \le (g-s)(j_N - j + 1)$,*

*for all $i \le i_N$ and $j \le j_N$.*

**Proof.** The lemma follows immediately from the proof of Lemma A.6. ■

**Lemma A.8.** *The functions $\tilde{l}(i,j,p)$ and $\tilde{w}(i,j,p)$ depend on $i$ and $j$ as follows:*
- *$\tilde{l}(i+1,j,p) > \tilde{l}(i,j,p)$ and $\tilde{w}(i+1,j,p) < \tilde{w}(i,j,p)$ for all $i \le i_N - 1$ and $j \le j_N$,*
- *$\tilde{l}(i,j+1,p) < \tilde{l}(i,j,p)$ and $\tilde{w}(i,j+1,p) > \tilde{w}(i,j,p)$ for all $i \le i_N$ and $j \le j_N - 1$,*

*for all $p \in (0,1)$.*

**Proof.** I work with $\tilde{l}(i,j,p)$ and $\tilde{w}(i,j,p)$ written as (A.3) and (A.4) with $x \coloneqq i_N - i$ and $y \coloneqq j_N - j$. I begin with dependence of $\tilde{l}(i,j,p)$ on $i$:

$$\begin{aligned}\tilde{l}(i+1,j,p) - \tilde{l}(i,j,p) &= \tilde{l}(i_N - (x-1), j_N - y, p) - \tilde{l}(i_N - x, j_N - y, p) \\ &= b \sum_{k=0}^{x-1} \tbinom{x+y}{k} p^{x+y-k}(1-p)^k + s \sum_{m=0}^{y} \tbinom{x+y}{m} p^m (1-p)^{x+y-m} \\ &\quad - b \sum_{k=0}^{x} \tbinom{x+y+1}{k} p^{x+y+1-k}(1-p)^k - s \sum_{m=0}^{y} \tbinom{x+y+1}{m} p^m (1-p)^{x+y+1-m},\end{aligned}$$

which can be rewritten as

$$\tilde{l}(i+1,j,p) - \tilde{l}(i,j,p) = bp^y \Phi_p(y,x) - b\tbinom{x+y}{x} p^y (1-p)^x + s(1-p)^x \Phi_{1-p}(x,y),$$

where $\Phi_q(a_1,a_2)$ is defined in Claim A.1. It follows that

$$\begin{aligned}\tilde{l}(i+1,j,p) - \tilde{l}(i,j,p) &= bp^y \tbinom{x+y}{x}(1-p)^{x+1} - b\tbinom{x+y}{x} p^y (1-p)^x + s(1-p)^x \tbinom{x+y}{y} p^{y+1} \\ &= (s-b)\tbinom{x+y}{x} p^{y+1}(1-p)^x > 0,\end{aligned}$$

for all $p \in (0,1)$.

The function $\tilde{w}(i,j,p)$ is the same as $\tilde{l}(i,j,p)$, except $b$ in the latter should be replaced by $g$. Hence,

$$\tilde{w}(i+1,j,p) - \tilde{w}(i,j,p) = -(g-s)\tbinom{x+y}{x} p^{y+1}(1-p)^x < 0,$$

for all $p \in (0,1)$.

Next I examine the dependence of $\tilde{l}(i,j,p)$ on $j$:

$$\begin{aligned}\tilde{l}(i,j+1,p) - \tilde{l}(i,j,p) &= \tilde{l}(i_N - x, j_N - (y-1), p) - \tilde{l}(i_N - x, j_N - y, p) \\ &= b \sum_{k=0}^{x} \tbinom{x+y}{k} p^{x+y-k}(1-p)^k + s \sum_{m=0}^{y-1} \tbinom{x+y}{m} p^m (1-p)^{x+y-m}\end{aligned}$$

$$- b \sum_{k=0}^{x} \binom{x+y+1}{k} p^{x+y+1-k}(1-p)^k - s \sum_{m=0}^{y} \binom{x+y+1}{m} p^m (1-p)^{x+y+1-m},$$

which can be rewritten as

$$\tilde{l}(i, j+1, p) - \tilde{l}(i, j, p) = bp^y \Phi_p(y, x) + s(1-p)^x \Phi_{1-p}(x, y) - s\binom{x+y}{y} p^y (1-p)^x,$$

where $\Phi_q(a_1, a_2)$ is defined in Claim A.1. It follows that

$$\begin{aligned}\tilde{l}(i, j+1, p) - \tilde{l}(i, j, p) &= bp^y \binom{x+y}{x}(1-p)^{x+1} + s(1-p)^x \binom{x+y}{y} p^{y+1} - s\binom{x+y}{y} p^y (1-p)^x \\ &= -(s-b)\binom{x+y}{x} p^y (1-p)^{x+1} < 0,\end{aligned}$$

for all $p \in (0, 1)$.

Applying the same argument as above, yields

$$\tilde{w}(i, j+1, p) - \tilde{w}(i, j, p) = (g-s)\binom{x+y}{x} p^y (1-p)^{x+1} > 0,$$

for all $p \in (0, 1)$. ■

**Lemma A.9.** *The function $\tilde{u}(i, j, p)$ depends on $i$ and $j$ as follows:*

- *if $(s-b)(i_N - i) > (g-s)(j_N - j)$, then*
  - *$\tilde{u}(i+1, j, p) > \tilde{u}(i, j, p)$ for all $i \le i_N - 1$ and $j \le j_N$, and*
  - *$\tilde{u}(i, j+1, p) < \tilde{u}(i, j, p)$ for all $i \le i_N$ and $j \le j_N - 1$,*
- *if $(s-b)(i_N - i) < (g-s)(j_N - j)$, then*
  - *$\tilde{u}(i+1, j, p) < \tilde{u}(i, j, p)$ for all $i \le i_N - 1$ and $j \le j_N$, and*
  - *$\tilde{u}(i, j+1, p) > \tilde{u}(i, j, p)$ for all $i \le i_N$ and $j \le j_N - 1$,*

*for all $p \in (0, 1)$.*

**Proof.** I work with $\tilde{u}(i, j, p)$ written as (A.5) with $x := i_N - i$ and $y := j_N - j$. I begin with the dependence on $i$:

$$\begin{aligned}\tilde{u}(i+1, j, p) - \tilde{u}(i, j, p) &= \tilde{u}(i_N - (x-1), j_N - y, p) - \tilde{u}(i_N - x, j_N - y, p) \\ &= b\sum_{k=0}^{x-2} \binom{x+y-1}{k} p^{x+y-1-k}(1-p)^{k+1} + s\sum_{m=0}^{y} \binom{x+y}{m} p^m (1-p)^{x+y-m} \\ &\quad + g\sum_{k=0}^{x-1} \binom{x+y-1}{k} p^{x+y-k}(1-p)^k - b\sum_{k=0}^{x-1} \binom{x+y}{k} p^{x+y-k}(1-p)^{k+1} \\ &\quad - s\sum_{m=0}^{y} \binom{x+y+1}{m} p^m (1-p)^{x+y+1-m} - g\sum_{k=0}^{x} \binom{x+y}{k} p^{x+y+1-k}(1-p)^k,\end{aligned}$$

which can be rewritten as follows:

$$\begin{aligned}\tilde{u}(i+1, j, p) - \tilde{u}(i, j, p) &= bp^y(1-p)\Phi_p(y, x-1) - b\binom{x+y-1}{x-1} p^y (1-p)^x \\ &\quad + s(1-p)^x \Phi_{1-p}(x, y) + gp^y \Phi_p(y, x) - g\binom{x+y-1}{x} p^y (1-p)^x,\end{aligned}$$

where $\Phi_q(a_1, a_2)$ is defined in Claim A.1. It follows that

$$\begin{aligned}\tilde{u}(i+1, j, p) - \tilde{u}(i, j, p) &= bp^y(1-p)\binom{x+y-1}{x-1}(1-p)^x - b\binom{x+y-1}{x-1} p^y (1-p)^x \\ &\quad + s(1-p)^x \binom{x+y}{y} p^{y+1} + gp^y \binom{x+y-1}{x}(1-p)^{x+1} - g\binom{x+y-1}{x} p^y (1-p)^x\end{aligned}$$

$$= \tfrac{1}{x+y}\left[(s-b)x-(g-s)y\right]\tbinom{x+y}{x}p^{y+1}(1-p)^x,$$

i.e., the sign of $\tilde{u}(i+1,j,p)-\tilde{u}(i,j,p)$ coincides with the sign of $[(s-b)x-(g-s)y]$.

Next I examine the dependence on $j$:

$$\begin{aligned}\tilde{u}(i,j+1,p)-\tilde{u}(i,j,p) &= \tilde{u}(i_N-x,j_N-(y-1),p)-\tilde{u}(i_N-x,j_N-y,p)\\ &= b\sum_{k=0}^{x-1}\tbinom{x+y-1}{k}p^{x+y-1-k}(1-p)^{k+1}+s\sum_{m=0}^{y-1}\tbinom{x+y}{m}p^m(1-p)^{x+y-m}\\ &\quad+g\sum_{k=0}^{x}\tbinom{x+y-1}{k}p^{x+y-k}(1-p)^k-b\sum_{k=0}^{x-1}\tbinom{x+y}{k}p^{x+y-k}(1-p)^{k+1}\\ &\quad-s\sum_{m=0}^{y}\tbinom{x+y+1}{m}p^m(1-p)^{x+y+1-m}-g\sum_{k=0}^{x}\tbinom{x+y}{k}p^{x+y+1-k}(1-p)^k,\end{aligned}$$

which can be rewritten as follows:

$$\begin{aligned}\tilde{u}(i,j+1,p)-\tilde{u}(i,j,p) &= bp^y(1-p)\Phi_p(y,x-1)\\ &\quad+s(1-p)^x\Phi_{1-p}(x,y)-s\tbinom{x+y}{y}p^y(1-p)^x+gp^y\Phi_p(y,x),\end{aligned}$$

where $\Phi_q(a_1,a_2)$ is defined in Claim A.1. It follows that

$$\begin{aligned}\tilde{u}(i,j+1,p)-\tilde{u}(i,j,p) &= bp^y(1-p)\tbinom{x+y-1}{x-1}(1-p)^x+s(1-p)^x\tbinom{x+y}{y}p^{y+1}\\ &\quad-s\tbinom{x+y}{y}p^y(1-p)^x+gp^y\tbinom{x+y-1}{x}(1-p)^{x+1}\\ &= \tfrac{1}{x+y}\left[-(s-b)x+(g-s)y\right]\tbinom{x+y}{x}p^y(1-p)^{x+1},\end{aligned}$$

i.e., the sign of $\tilde{u}(i,j+1,p)-\tilde{u}(i,j,p)$ coincides with the sign of $[-(s-b)x+(g-s)y]$. ■

**Lemma A.10.** *The upper bounds $\tilde{p}(i,j)$ depend on $i$ and $j$ as follows:*

- *if $(s-b)(i_N-i)>(g-s)(j_N-j+1)$, then*
  - *$\tilde{p}(i+1,j)<\tilde{p}(i,j)$ for all $i\le i_N-1$ and $j\le j_N$, and*
  - *$\tilde{p}(i,j+1)>\tilde{p}(i,j)$ for all $i\le i_N$ and $j\le j_N-1$,*
- *if $(s-b)(i_N-i)\le(g-s)(j_N-j+1)$, then*
  - *$\tilde{p}(i+1,j)=\tilde{p}(i,j)$ for all $i\le i_N-1$ and $j\le j_N$, and*
  - *$\tilde{p}(i,j+1)\ge\tilde{p}(i,j)$ for all $i\le i_N$ and $j\le j_N-1$.*

**Proof.** If $(g-s)(j_N-j+1)<(s-b)(i_N-i)$, then $\tilde{p}(i,j)\in(0,1)$ by Lemma A.7. It also follows that $(g-s)(j_N-j)<(s-b)(i_N-i)$. Therefore, $\tilde{p}(i+1,j)<\tilde{p}(i,j)$ and $\tilde{p}(i,j+1)>\tilde{p}(i,j)$ by Lemmata A.6 and A.9 and the definition of $\tilde{p}(i,j)$, in particular, $\tilde{u}(i,j,\tilde{p}(i,j))=s$.

If $(g-s)(j_N-j+1)\ge(s-b)(i_N-i)$, then $\tilde{p}(i,j)=0$ by Lemma A.7. It also follows that $(g-s)(j_N-j+1)\ge(s-b)(i_N-i-1)$. Therefore, $\tilde{p}(i+1,j)=0$ by Lemma A.7. However, $(g-s)(j_N-(j+1)-1)$ may be greater or smaller than $(s-b)(i_N-i)$, and so $\tilde{p}(i,j+1)=0$ or $\tilde{p}(i,j+1)\in(0,1)$ by Lemma A.7. ■

**Lemma A.11.** *The upper bounds $\tilde{p}(i,j)$ are such that $\tilde{p}(i,j)<p_M$ for all $i\le i_N$ and $j\le j_N$.*

**Proof.** Lemma A.10 implies that the upper bounds $\tilde{p}(i,j)$ are decreasing in $i$ and increasing in $j$. Therefore, it suffices to compare the highest upper bound $\tilde{p}(0,j_N)$ with the myopic cut-off $p_M$.

The function $\tilde{u}(0, j_N, p)$ is as follows:

$$\tilde{u}(0, j_N, p) = b(1-p) \sum_{k=0}^{\frac{N-1}{2}-1} \binom{\frac{N-1}{2}}{k} p^{\frac{N-1}{2}-k}(1-p)^k \\ + gp \sum_{k=0}^{\frac{N-1}{2}} \binom{\frac{N-1}{2}}{k} p^{\frac{N-1}{2}-k}(1-p)^k + s(1-p)^{\frac{N+1}{2}}.$$

It can be rewritten in the following way

$$\tilde{u}(0, j_N, p) = b(1-p)\left(1-(1-p)^{\frac{N-1}{2}}\right) + gp + s(1-p)^{\frac{N+1}{2}} \\ = gp + b(1-p) + (s-b)(1-p)^{\frac{N+1}{2}}.$$

Applying the definition of the myopic cut-off, i.e., $s = gp_M + b(1-p_M)$, yields

$$\tilde{u}(0, j_N, p_M) = s + (s-b)(1-p_M)^{\frac{N+1}{2}} > s.$$

It follows from $\tilde{u}(0, j_N, \tilde{p}(0, j_N)) = s$ and Lemma A.6 that $\tilde{p}(0, j_N) < p_M$. ■

### A.2.2 Proof of Proposition 1.1

The first part of Proposition 1.1 is an immediate consequence of Theorem 1.1. As for the second part, it follows from Theorem 1.1 that the largest upper bound is $\tilde{p}(0, j_N)$. The function $\tilde{u}(0, j_N, p)$ is as follows:

$$\tilde{u}(0, j_N, p) = b(1-p) \sum_{k=0}^{\frac{N-1}{2}-1} \binom{\frac{N-1}{2}}{k} p^{\frac{N-1}{2}-k}(1-p)^k \\ + gp \sum_{k=0}^{\frac{N-1}{2}} \binom{\frac{N-1}{2}}{k} p^{\frac{N-1}{2}-k}(1-p)^k + s(1-p)^{\frac{N+1}{2}}.$$

It can be rewritten in the following way

$$\tilde{u}(0, j_N, p) = b(1-p)\left(1-(1-p)^{\frac{N-1}{2}}\right) + gp + s(1-p)^{\frac{N+1}{2}} \\ = gp + b(1-p) + (s-b)(1-p)^{\frac{N+1}{2}}.$$

Therefore,

$$\lim_{N\to\infty} \tilde{u}(0, j_N, p) = gp + b(1-p)$$

for all $p \in (0,1)$. The definition of the upper bound $\tilde{p}(0, j_N)$, specifically, $\tilde{u}(0, j_N, \tilde{p}(0, j_N)) = s$, together with the definition of the myopic cut-off, i.e., $gp_M + b(1-p_M) = s$, imply that $\lim_{N\to\infty} \tilde{p}(0, j_N) = p_M$.

### A.2.3 Proof of Proposition 1.2

The exposition is simpler for the general setting. I go back to learning via bad news only in the end of the proof.

Define by $p_N(i,j)$ and $\tilde{p}_N(i,j)$ the cut-off and its upper bound used by unsure voters in the presence of $i$ losers and $j$ winners in the game with $N$ players. It suffices to compare the upper bounds $\tilde{p}_N(i,j)$ and $\tilde{p}_{N'}(i,j)$ in two games with $N$ and $N' = N+2$ players and the same number of losers and winners, $i \le i_N$ and $j \le j_N$. Define by $\tilde{u}_N(i,j,p)$ the function $\tilde{u}(i,j,p)$ in the game with $N$ players.

Define $\Lambda(x,y)$ by

$$\Lambda(x,y) \coloneqq \tfrac{(s-b)(x+1)-(g-s)y}{(s-b)x-(g-s)(y+1)}, \tag{A.8}$$

where $x \coloneqq i_N - i$ and $y \coloneqq j_N - j$. The proof makes use of the following lemma.

**Lemma A.12.** *Let $N$ be an odd positive integer and $N' = N+2$. The function $\tilde{u}_N(i,j,p)$ depends on $N$ as follows:*

– *$\tilde{u}_{N'}(i,j,p) > \tilde{u}_N(i,j,p)$ for all $\frac{1-p}{p} > \frac{y+1}{x+1}\Lambda(x,y)$ if $(s-b)x > (g-s)(y+1)$,*

– *$\tilde{u}_{N'}(i,j,p) < \tilde{u}_N(i,j,p)$ for all $\frac{1-p}{p} > \frac{y+1}{x+1}\Lambda(x,y)$ if $(s-b)x < (g-s)(y+1)$,*

*for all $i \le i_N$ and $j \le j_N$.*

**Proof.** I work with the function $\tilde{u}_N(i,j,p)$ written as (A.5). It follows that

$$\begin{aligned}\tilde{u}_N(i,j,p) = bp\sum_{k=0}^{x-1}\tbinom{x+y}{k}p^{x+y-k}(1-p)^k + g(1-p)\sum_{k=0}^{x}\tbinom{x+y}{k}p^{x+y-k}(1-p)^k \\ + sp\sum_{m=0}^{y}\tbinom{x+y+1}{m}p^m(1-p)^{x+y-m},\end{aligned}$$

and

$$\begin{aligned}\tilde{u}_{N'}(i,j,p) = bp\sum_{k=0}^{x}\tbinom{x+y+2}{k}p^{x+y+2-k}(1-p)^k + g(1-p)\sum_{k=0}^{x+1}\tbinom{x+y+2}{k}p^{x+y+2-k}(1-p)^k \\ + sp\sum_{m=0}^{y+1}\tbinom{x+y+3}{m}p^m(1-p)^{x+y+2-m}\end{aligned}$$

for $N' = N+2$. Therefore,

$$\begin{aligned}\tilde{u}_{N'}(i,j,p) - \tilde{u}_N(i,j,p) = &- bp^{y+1}(1-p)\Psi_p(y+1,x-1) + b\tbinom{x+y+2}{x}p^{y+2}(1-p)^{x+1} \\ &- s(1-p)^{x+1}\Psi_{1-p}(x+1,y) + s\tbinom{x+y+3}{y+1}p^{y+1}(1-p)^{x+2} \\ &- gp^{y+1}\Psi_p(y,x) + g\tbinom{x+y+2}{x+1}p^{y+2}(1-p)^{x+1},\end{aligned}$$

where $\Psi_q(a_1,a_2)$ is defined in Claim A.2. It follows that

$$\begin{aligned}\tilde{u}_{N'}(i,j,p) - \tilde{u}_N(i,j,p) = &\, bp^{y+1}(1-p)^{x+1}\left[\tbinom{x+y+2}{x}p - \tbinom{x+y+1}{x-1}p - \tbinom{x+y}{x-1}\right] \\ &+ sp^{y+1}(1-p)^{x+1}\left[\tbinom{x+y+3}{y+1}(1-p) - \tbinom{x+y+2}{y}(1-p) - \tbinom{x+y+1}{y}\right] \\ &+ gp^{y+1}(1-p)^{x+1}\left[\tbinom{x+y+2}{x+1}p - \tbinom{x+y+1}{x}p - \tbinom{x+y}{x}\right].\end{aligned}$$

Note the following: $\binom{x+y+1}{x-1} = \frac{x(x+1)}{(y+2)(x+y+2)}\binom{x+y+2}{x+1}$, $\binom{x+y}{x-1} = \frac{x(x+1)}{(x+y+1)(x+y+2)}\binom{x+y+2}{x+1}$, and $\binom{x+y}{x} = \frac{(x+1)(y+1)}{(x+y+1)(x+y+2)}\binom{x+y+2}{x+1}$. Hence,

$$\begin{aligned}\tilde{u}_{N'}(i,j,p) - \tilde{u}_N(i,j,p) &= b\tfrac{x+1}{x+y+2}\tbinom{x+y+2}{x+1}p^{y+1}(1-p)^{x+1}\left(p - \tfrac{x}{x+y+1}\right) \\ &+ s\tbinom{x+y+2}{x+1}p^{y+1}(1-p)^{x+1}\left(\tfrac{x+1}{x+y+2} - p\right) + g\tfrac{y+1}{x+y+2}\tbinom{x+y+2}{x+1}p^{y+1}(1-p)^{x+1}\left(p - \tfrac{x+1}{x+y+1}\right),\end{aligned}$$

which can be rearranged as follows:

$$\begin{aligned}\tilde{u}_{N'}(i,j,p) - \tilde{u}_N(i,j,p) &= \tbinom{x+y+2}{x+1}p^{y+1}(1-p)^{x+1} \\ &\times \left[\left(b\tfrac{x+1}{x+y+2} - s + g\tfrac{y+1}{x+y+2}\right)p - \tfrac{x+1}{x+y+2}\left(b\tfrac{x}{x+y+1} - s + g\tfrac{y+1}{x+y+1}\right)\right].\end{aligned}$$

The sign of $\tilde{u}_{N'}(i,j,p) - \tilde{u}_N(i,j,p)$ coincides with the sign of the expression in brackets, which, if multiplied by $(x+y+1)(x+y+2)$, can be written in the following way

$$-(x+y+1)\left[(s-b)(x+1) - (g-s)(y+1)\right]p + (x+1)\left[(s-b)x - (g-s)(y+1)\right],$$

which, in turn, can be rewritten as follows:

$$(x+1)\left[(s-b)x - (g-s)(y+1)\right](1-p) - (y+1)\left[(s-b)(x+1) - (g-s)y\right]p.$$

Therefore, the sign of $\tilde{u}_{N'}(i,j,p) - \tilde{u}_N(i,j,p)$ coincides with the sign of the last expression. The statement of the lemma follows. ∎

If $(s-b)x \le (g-s)(y+1)$, then $\tilde{p}_N(i,j) = 0$ by Theorem 1.1 and $\tilde{u}_{N'}(i,j,p) \le \tilde{u}_N(i,j,p)$ for all $p \le \hat{p}$ by Lemma A.12, where $\hat{p}$ is defined by

$$\tfrac{1-\hat{p}}{\hat{p}} = \tfrac{y+1}{x+1}\Lambda(x,y).$$

It follows that $\tilde{p}_{N'}(i,j) \ge \tilde{p}_N(i,j)$.

If $(s-b)x > (g-s)(y+1)$, then $\tilde{p}_N(i,j) \in (0,1)$ by Theorem 1.1 and $\tilde{u}_{N'}(i,j,p) \le \tilde{u}_N(i,j,p)$ for all $p \ge \hat{p}$ by Lemma A.12. When $y \neq 0$, it follows from the proof of Lemma A.6 that, for $\tilde{p}_{N'}(i,j) \ge \tilde{p}_N(i,j)$, it is sufficient to show that $\hat{p} \le \bar{p}$, where $\bar{p}$ is the inflection point of $\tilde{u}_N(i,j,p)$, defined in the proof of Lemma A.6 by

$$\bar{p} \coloneqq \tfrac{B(x,y)}{A(x,y)+B(x,y)}$$

where $A(x,y)$ and $B(x,y)$ are as follows:

$$\begin{aligned}A(x,y) &\coloneqq x[(s-b)(x+1) - (g-s)y], \\ B(x,y) &\coloneqq y[(s-b)x - (g-s)(y+1)].\end{aligned}$$

Observe that $\Lambda(x,y)$ given by (A.8) can be rewritten as

$$\Lambda(x,y) = \tfrac{y}{x}\tfrac{A(x,y)}{B(x,y)},$$

and hence

$$\hat{p} = \tfrac{x(x+1)B(x,y)}{y(y+1)A(x,y)+x(x+1)B(x,y)}.$$

Therefore, $\tilde{p}_{N'}(i,j) \geq \tilde{p}_N(i,j)$ if

$$\hat{p} \leq \bar{p} \quad \Leftrightarrow \quad \tfrac{x(x+1)B(x,y)}{y(y+1)A(x,y)+x(x+1)B(x,y)} \leq \tfrac{B(x,y)}{A(x,y)+B(x,y)} \quad \Leftrightarrow \quad y(y+1) \geq x(x+1).$$

With learning via bad news only, there are no winners, i.e., $j = 0$, and thus $y = j_N$. Furthermore, the uppers bounds coincide with the cut-offs used by unsure voters, i.e., $p_N(i) \coloneqq p_N(i,0) = \tilde{p}_N(i,0)$ for all $i \leq i_N$, and so are the value functions, i.e., $u_N(i,p) \coloneqq u_N(i,0,p) = \tilde{u}_N(i,0,p)$. It follows from the argument above that $p_{N'}(i) \geq p_N(i)$ for $N$ and $N' = N+2$ and for all $i \leq i_N$. Finally, Theorem 1.2 implies that, if $g - s \geq s - b$, then $p_N(i)$ is independent of the number of players $N$ and $p_N(i) = 0$ for all $i \leq i_N$.

### A.2.4 Proof of Proposition 1.3

The upper bounds $\tilde{p}(i,j)$ are determined by the value-matching condition $\tilde{u}(i,j,\tilde{p}(i,j)) = s$ for all $i \leq i_N$ and $j \leq j_N$. It follows that

$$\begin{aligned} \tfrac{\mathrm{d}\tilde{u}(i,j,\tilde{p}(i,j))}{\mathrm{d}b} &= \partial_b \tilde{u}(i,j,\tilde{p}(i,j)) + \partial_p \tilde{u}(i,j,\tilde{p}(i,j)) \tfrac{\mathrm{d}\tilde{p}(i,j)}{\mathrm{d}b} \\ &= (1-\tilde{p}(i,j)) \sum_{k=0}^{i_N-i-1} \tbinom{N-i-j-1}{k} \tilde{p}(i,j)^{N-i-j-1-k}(1-\tilde{p}(i,j))^k + \partial_p \tilde{u}(i,j,\tilde{p}(i,j)) \tfrac{\mathrm{d}\tilde{p}(i,j)}{\mathrm{d}b} = 0, \end{aligned}$$

where $\partial_p \tilde{u}(i,j,\tilde{p}(i,j)) \geq 0$ by Lemma A.6. Therefore, $\tilde{p}(i,j)$ is decreasing in $b$. Similarly,

$$\begin{aligned} \tfrac{\mathrm{d}\tilde{u}(i,j,\tilde{p}(i,j))}{\mathrm{d}g} &= \partial_g \tilde{u}(i,j,\tilde{p}(i,j)) + \partial_p \tilde{u}(i,j,\tilde{p}(i,j)) \tfrac{\mathrm{d}\tilde{p}(i,j)}{\mathrm{d}g} \\ &= \tilde{p}(i,j) \sum_{k=0}^{i_N-i} \tbinom{N-i-j-1}{k} \tilde{p}(i,j)^{N-i-j-1-k}(1-\tilde{p}(i,j))^k + \partial_p \tilde{u}(i,j,\tilde{p}(i,j)) \tfrac{\mathrm{d}\tilde{p}(i,j)}{\mathrm{d}g} = 0, \end{aligned}$$

and so $\tilde{p}(i,j)$ is decreasing in $g$. Finally,

$$\begin{aligned} \tfrac{\mathrm{d}\tilde{u}(i,j,\tilde{p}(i,j))}{\mathrm{d}s} &= \partial_s \tilde{u}(i,j,\tilde{p}(i,j)) + \partial_p \tilde{u}(i,j,\tilde{p}(i,j)) \tfrac{\mathrm{d}\tilde{p}(i,j)}{\mathrm{d}s} \\ &= \sum_{m=0}^{j_N-j} \tbinom{N-i-j}{m} \tilde{p}(i,j)^m (1-\tilde{p}(i,j))^{N-i-j-m} + \partial_p \tilde{u}(i,j,\tilde{p}(i,j)) \tfrac{\mathrm{d}\tilde{p}(i,j)}{\mathrm{d}s} = 1. \end{aligned}$$

Because $\partial_p \tilde{u}(i,j,\tilde{p}(i,j)) \geq 0$ by Lemma A.6 and

$$\sum_{m=0}^{j_N-j} \tbinom{N-i-j}{m} \tilde{p}(i,j)^m (1-\tilde{p}(i,j))^{N-i-j-m} \leq 1,$$

it follows that $\tilde{p}(i,j)$ is increasing in $s$.

### A.2.5 Proof of Lemma 1.2

The evolution of the game with $N$ players and the $Q$-rule such that $Q < Q_N$ from the state $(i,j,p)$ on is equivalent to the evolution of the game with $N + i_{ph}^Q$ players and the majority rule from the state $(i + i_{ph}^Q, j, p)$ on. Recall that $i_{ph}^Q \coloneqq 2(Q_N - Q)$ is the number of phantom losers. Indeed, the number of unsure voters is the same $(N + i_{ph}^Q) - (i + i_{ph}^Q) - j = N - i - j$

and so is their belief $p$. Furthermore, if

$$\tfrac{(N+i^Q_{ph})+1}{2} - (i + i^Q_{ph}) = \tfrac{N+1}{2} - \tfrac{i^Q_{ph}}{2} - i = Q_N - (Q_N - Q) - i = Q - i$$

more players become losers, then the safe arm will be implemented.

### A.2.6 Proof of Lemma 1.3

The evolution of the game with $N$ players and the $Q$-rule such that $Q > Q_N$ from the state $(i, j, p)$ on is equivalent to the evolution of the game with $N + j^Q_{ph}$ players and the majority rule from the state $(i, j + j^Q_{ph}, p)$ on. Recall that $j^Q_{ph} := 2(Q - Q_N)$ is the number of phantom winners. Indeed, the number of unsure voters is the same $(N + j^Q_{ph}) - i - (j + j^Q_{ph}) = N - i - j$ and so is their belief $p$. Furthermore, if

$$\tfrac{(N+j^Q_{ph})+1}{2} - i = \tfrac{N+1}{2} + \tfrac{j^Q_{ph}}{2} - i = Q_N + (Q - Q_N) - i = Q - i$$

more players become losers, then the safe arm will be implemented.

### A.2.7 Proof of Theorems 1.3 and 1.4

It suffices to compare the upper bounds for the equilibrium cut-offs $\tilde{p}_Q(i, j)$ and $\tilde{p}_{Q'}(i, j)$ in two games with $Q$ and $Q'$ such that $Q' = Q + 1$. The lower bound for the value function of unsure voters in the game with the $Q$-rule is given by

$$\begin{aligned}\tilde{u}^Q(i, j, p) &= b(1-p) \sum_{k=0}^{Q-2-i} \tbinom{N-i-j-1}{k} p^{N-i-j-1-k}(1-p)^k \\ &\quad + gp \sum_{k=0}^{Q-1-i} \tbinom{N-i-j-1}{k} p^{N-i-j-1-k}(1-p)^k + s \sum_{m=0}^{N-Q-j} \tbinom{N-i-j}{m} p^m (1-p)^{N-i-j-m}\end{aligned}$$

for all $p \geq \tilde{p}^Q(i, j)$, $i \leq i^Q_N$, and $j \leq j^Q_N$. Therefore,

$$\begin{aligned}\tilde{u}^{Q'}(i, j, p) - \tilde{u}^Q(i, j, p) &= b(1-p)\tbinom{N-i-j-1}{Q-1-i} p^{N-Q-j}(1-p)^{Q-1-i} \\ &\quad + gp\tbinom{N-i-j-1}{Q-i} p^{N-Q-1-j}(1-p)^{Q-i} - s\tbinom{N-i-j}{N-Q-j} p^{N-Q-j}(1-p)^{Q-i}\end{aligned}$$

for all $i \leq i^Q_N$ and all $j \leq j^{Q'}_N$. It follows that

$$\tilde{u}^{Q'}(i, j, p) - \tilde{u}^Q(i, j, p) = \left(b\tfrac{Q-i}{N-i-j} + g\tfrac{N-Q-j}{N-i-j} - s\right) \tbinom{N-i-j}{N-Q-j} p^{N-Q-j}(1-p)^{Q-i}.$$

The sign of $\tilde{u}^{Q'}(i, j, p) - \tilde{u}^Q(i, j, p)$ coincides with the sign of

$$-(s-b)(Q-i) + (g-s)(N-Q-j) = -(s-b)(Q'-1-i) + (g-s)(N-Q'+1-j).$$

If $(s-b)(i^{Q'}_N - i) > (g-s)(j^{Q'}_N - j + 1)$, then $\tilde{p}_{Q'}(i, j) \in (0, 1)$ and so $\tilde{p}^Q(i, j) < \tilde{p}^{Q'}(i, j)$. If $(s-b)(i^{Q'}_N - i) \leq (g-s)(j^{Q'}_N - j + 1)$, then $\tilde{p}^Q(i, j) = \tilde{p}^{Q'}(i, j) = 0$.

### A.2.8 Proof of Lemma 1.4

If the social value function is greater than $s$ for all beliefs of unsure voters, then the utilitarian social planner chooses the risky arm over the safe arm unless $Q^*$ players receive bad news. It follows from Lemmata 1.2 and 1.3 that, in the game with the $Q$-rule, the bounds on the equilibrium value functions of losers, winners, and unsure voters are given by

$$\tilde{l}^Q(i,j,p) = b \sum_{k=0}^{Q-1-i} \binom{N-i-j}{k} p^{N-i-j-k}(1-p)^k + s \sum_{m=0}^{N-Q-j} \binom{N-i-j}{m} p^m (1-p)^{N-i-j-m}, \quad \text{(A.9)}$$

$$\tilde{w}^Q(i,j,p) = g \sum_{k=0}^{Q-1-i} \binom{N-i-j}{k} p^{N-i-j-k}(1-p)^k + s \sum_{m=0}^{N-Q-j} \binom{N-i-j}{m} p^m (1-p)^{N-i-j-m}, \quad \text{(A.10)}$$

$$\begin{aligned}\tilde{u}_{N,Q}(i,j,p) &= b(1-p) \sum_{k=0}^{Q-2-i} \binom{N-i-j-1}{k} p^{N-i-j-1-k}(1-p)^k \\ &+ gp \sum_{k=0}^{Q-1-i} \binom{N-i-j-1}{k} p^{N-i-j-1-k}(1-p)^k + s \sum_{m=0}^{N-Q-j} \binom{N-i-j}{m} p^m (1-p)^{N-i-j-m}, \quad \text{(A.11)}\end{aligned}$$

for all $i \le i_N^Q$ and $j \le j_N^Q$. Define $V^{Q^*}(i,j,p)$ by

$$V^{Q^*}(i,j,p) := i\tilde{l}^{Q^*}(i,j,p) + j\tilde{w}^{Q^*}(i,j,p) + (N-i-j)\tilde{u}^{Q^*}(i,j,p) \quad \text{(A.12)}$$

for all $p \in [0,1]$, $i \le i_N^{Q^*}$, and $j \le j_N^{Q^*}$. The statement of the lemma follows from Lemmata A.13 and A.14 below and the observation that $\tilde{l}^{Q^*}(i,j,0) = \tilde{w}^{Q^*}(i,j,0) = \tilde{u}^{Q^*}(i,j,0) = s$ for all $i \le i_N^{Q^*}$ and $j \le j_N^{Q^*}$.

**Lemma A.13.** *The social value function is given by $V^{Q^*}(i,j,p)$ for all $p \in [0,1]$, $i \le i_N^{Q^*}$, and $j \le j_N^{Q^*}$.*

**Proof.** It follows from the proof of Lemmata 1.2 and 1.3 and the proof of Lemma 1.1 that, if unsure voters experiment for all beliefs in a game with the $Q^*$-rule, then the value functions of losers, winners, and unsure voters are $l^{Q^*}(i,j,p) = \tilde{l}^{Q^*}(i,j,p)$, $w^{Q^*}(i,j,p) = \tilde{w}^{Q^*}(i,j,p)$, and $u^{Q^*}(i,j,p) = \tilde{u}^{Q^*}(i,j,p)$ for all $p \in [0,1]$, $i \le i_N^{Q^*}$, and $j \le j_N^{Q^*}$. Therefore, if the utilitarian social planner experiments with the risky arm for all beliefs of unsure voters, then the social value function is given by $V^{Q^*}(i,j,p)$. ■

**Lemma A.14.** *The function $V^{Q^*}(i,j,p)$ increases in $p$ for all $p \in [0,1]$, $i \le i_N^{Q^*}$, and $j \le j_N^{Q^*}$.*

**Proof.** The functions $\tilde{l}^{Q^*}(i,j,p)$, $\tilde{w}^{Q^*}(i,j,p)$, and $\tilde{u}^{Q^*}(i,j,p)$ are given by (A.9), (A.10), and (A.11). It follows that

$$\begin{aligned}\tilde{l}^{Q^*}(i,j,p) - \tilde{u}^{Q^*}(i,j,p) &= -(g-b) \sum_{k=0}^{Q^*-1-i} \tfrac{N-i-j-k}{N-i-j} \binom{N-i-j}{k} p^{N-i-j-k}(1-p)^k \\ &= -(g-b)p^{y+1} \sum_{k=0}^{x} \binom{x+y}{k} p^{x-k}(1-p)^k\end{aligned}$$

and

$$\tilde{w}^{Q^*}(i,j,p) - \tilde{u}^{Q^*}(i,j,p) = (g-b) \sum_{k=0}^{Q^*-1-i} \tfrac{k}{N-i-j} \tbinom{N-i-j}{k} p^{N-i-j-k}(1-p)^k$$
$$= (g-b)p^{y+1}(1-p) \sum_{k=0}^{x-1} \tbinom{x+y}{k} p^{x-1-k}(1-p)^k,$$

where $x := Q^* - 1 - i = i_N^{Q^*} - i$ and $y := N - Q^* - j = j_N^{Q^*} - j$. Hence, $V^{Q^*}(i,j,p)$ can be written as follows:

$$V^{Q^*}(i,j,p) = -i(g-b)p^{y+1} \sum_{k=0}^{x} \tbinom{x+y}{k} p^{x-k}(1-p)^k$$
$$+ j(g-b)p^{y+1}(1-p) \sum_{k=0}^{x-1} \tbinom{x+y}{k} p^{x-1-k}(1-p)^k + N\tilde{u}^{Q^*}(i,j,p).$$

The argument from now on follows the steps of the proof of Lemma A.6. If $x = 0$, i.e., if $i = i_N^{Q^*}$, then

$$V^{Q^*}(i_N^{Q^*}, j, p) = -i_N^{Q^*}(g-b)p^{y+1} + N\left[gp^{y+1} + s\left(1 - p^{y+1}\right)\right]$$
$$= [i_N^{Q^*} b + (N - i_N^{Q^*})g - Ns]p^{y+1} + Ns,$$

where $i_N^{Q^*} b + (N - i_N^{Q^*})g > Ns$ by definition of $Q^*$. Therefore, the partial derivative with respect to $p$ is as follows:

$$\partial_p V^{Q^*}(i_N^{Q^*}, j, p) = (y+1)[i_N^{Q^*} b + (N - i_N^{Q^*})g - Ns](1-p)^y > 0$$

for all $p \in (0,1)$.

If $y = 0$, i.e., if $j = j_N^{Q^*}$, then

$$V^{Q^*}(i, j_N^{Q^*}, p)$$
$$= i(b-g)p - j_N^{Q^*}(b-g)(1-p)\left(1 - (1-p)^x\right) + N[b(1-p)(1-(1-p)^x) + gp + s(1-p)^{x+1}]$$
$$= ib + (N-i)g - (N - i - j_N^{Q^*})(g-b)(1-p) - [(N - j_N^{Q^*})b + j_N^{Q^*} g - Ns](1-p)^{x+1},$$

where $(N - j_N^{Q^*})b + j_N^{Q^*} g < Ns$ by definition of $Q^*$. Therefore, the partial derivative with respect to $p$ is as follows:

$$\partial_p V^{Q^*}(i, j_N^{Q^*}, p) = (N - i - j_N^{Q^*})(g-b) + (x+1)[(N - j_N^{Q^*})b + j_N^{Q^*} g - Ns](1-p)^x,$$

and the second order partial derivative takes the form

$$\partial_{pp}^2 V^{Q^*}(i, j_N^{Q^*}, p) = -x(x+1)[(N - j_N^{Q^*})b + j_N^{Q^*} g - Ns]p^{x-1} \geq 0,$$

with strict inequality unless $x = 0$ and for all $p \in (0,1)$. Furthermore, $\partial_p V^{Q^*}(i, j_N^{Q^*}, 1) > 0$ and $V^{Q^*}(i, j_N^{Q^*}, 0) = s$. It follows that $V^{Q^*}(i, j_N^{Q^*}, p)$ increases in $p$.

If neither $x = 0$ nor $y = 0$, then the partial derivative with respect to $p$ is as follows:

$$\partial_p V^{Q^*}(i,j,p) = -i(g-b)\left[\sum_{k=0}^{x-1}\tbinom{x+y}{k}p^{x+y-k}(1-p)^k + (y+1)\tbinom{x+y}{k}p^y(1-p)^x\right]$$
$$+ j(g-b)\left[-\sum_{k=0}^{x-1}\tbinom{x+y}{k}p^{x+y-k}(1-p)^k + x\tbinom{x+y}{k}p^y(1-p)^x\right] + N\partial_p\tilde{u}^{Q^*}(i,j,p),$$

where I use calculations made in the proof of Lemma A.6. As a result,

$$\partial_p V^{Q^*}(i,j,p) = (N-i-j)(g-b)\sum_{k=0}^{x-1}\tbinom{x+y}{k}p^{x+y-k}(1-p)^k$$
$$-\left[i(g-b)(y+1) - j(g-b)x + N\left(-bx + s(x+y+1) - g(y+1)\right)\right]\tbinom{x+y}{x}p^y(1-p)^x.$$

As found in the proof of Lemma A.6,

$$\partial_p\left(\sum_{k=0}^{x-1}\tbinom{x+y}{k}p^{x+y-k}(1-p)^k\right) = x\tbinom{x+y}{x}p^y(1-p)^{x-1}.$$

Therefore,

$$\partial^2_{pp}V^{Q^*}(i,j,p) = \tbinom{x+y}{x}p^{y-1}(1-p)^{x-1}D(x,y),$$

where $D(x,y,p) \coloneqq [A(x,y)+B(x,y)]p - B(x,y)$ with

$$A(x,y) \coloneqq x[\alpha(i)y - \beta(j)(x+1)],$$
$$B(x,y) \coloneqq y[\alpha(i)(y+1) - \beta(j)x],$$

where $\alpha(i) \coloneqq Ns - [ib + (N-i)g] < 0$ and $\beta(j) \coloneqq [(N-j)b + jg] - Ns < 0$ by definition of $Q^*$. Up to the definition of $A(x,y)$ and $B(x,y)$, there are the same three cases to consider as in the proof of Lemma A.6. However, as shown next, $B(x,y) < 0$ for all $x$ and $y$, and so Cases 2 and 3 are relevant only. Indeed,

$$\begin{aligned}\alpha(i)(y+1) - \beta(j)x &= -[N(g-s) - i(g-b)](y+1) + [(N-j)(g-b) - N(g-s)]x\\ &= -N(x+y+1)(g-s) + [i(y+1) + (N-j)x](g-b)\\ &= -N(x+y+1)(g-s) + [i(x+y+1) + (N-i-j)x](g-b).\end{aligned}$$

Observe that $N-i-j = x+y+1$ and $x+i = i_N^{Q^*}$. Therefore,

$$\begin{aligned}\alpha(i)(y+1) - \beta(j)x &= (x+y+1)[-N(g-s) + i_N^{Q^*}(g-b)]\\ &= (x+y+1)[Ns - i_N^{Q^*}b - (N - i_N^{Q^*})g] < 0\end{aligned}$$

by definition of $Q^*$. ■

### A.2.9 Proof of Theorem 1.5

By definition of the $Q^*$-rule,

$$[N-(Q^*-1)]g + (Q^*-1)b > Ns \quad \Leftrightarrow \quad 1 - p_M > \tfrac{Q^*-1}{N},$$
$$(N-Q^*)g + Q^*b \le Ns \quad \Leftrightarrow \quad 1 - p_M \le \tfrac{Q^*}{N}.$$

By Theorem 1.3, the upper bounds on the equilibrium cut-offs decrease in $i$ and increase in $j$. Therefore, a sufficient condition for the equilibrium in the game with the $Q^*$-rule to be socially optimal is $p^{Q^*}(0, j_N^{Q^*}) = \tilde{p}^{Q^*}(0, j_N^{Q^*}) = 0$, that is,

$$(s-b)(Q^*-1) \leq g - s \quad \Leftrightarrow \quad 1 - p_M \geq \tfrac{Q^*-1}{Q^*}.$$

The last inequality is satisfied for $Q^* = 1$ and $Q^* = N$.

### A.2.10 Proof of Theorem A.1 and Lemma A.1

Theorem A.1 and Lemma A.1 follow from the following lemmata.

**Lemma A.15.** *The equilibrium exists and is unique.*

**Proof.** Similar to Strulovici (2010a), the equilibrium existence and uniqueness come from a backward induction argument on the number of winners. ■

**Lemma A.16.** *The cut-offs used by unsure voters are $p(j) = \bar{p}$ for all $j \leq j_N$, where*
- $\bar{p} = 0$ *if* $g \geq s\frac{N+1}{2}$,
- $\bar{p} = 1 - \frac{N+1}{N-1}\left(1 - \frac{s}{g}\right)$ *if* $g < s\frac{N+1}{2}$.

*The value functions of winners and unsure voters are $w(j,p) = \tilde{w}(j,q)$ and $u(j,p) = \tilde{u}(j,q)$ for all $p \geq \bar{p}$ and $j \leq j_N$, where $q := \frac{p-\bar{p}}{1-\bar{p}}$.*

**Proof.** Because unsure voters can become winners, but not other way around, the set of winners $J$ can only grow over time. I proceed with an induction argument based on the number of winners $j$. Note that, if $j > j_N$, then winners form a majority, and so the risky arm is implemented. In particular, it follows that $w(j_N+1, p) = g$ and $u(j_N+1, p) = pg$.

If $y = 0$, i.e., if $j = j_N$, then $N - j_N = \frac{N+1}{2}$. The value function of unsure voters, when they experiment, satisfies the following ODE:

$$u(j_N, p) = \frac{1}{\frac{N+1}{2}}\left[g + \tfrac{N-1}{2}pg - (1-p)\partial_p u(j_N, p)\right].$$

The solution to the ODE takes the form:

$$u(j_N, p) = pg + C(1-p)^{\frac{N+1}{2}},$$

where $C$ is a constant of integration. The value function of unsure voters must satisfy the value-matching condition $u(j_N, p(j_N)) = s$ and, if $p(j_N) > 0$, the smooth-pasting condition $\partial_p u(j_N, p(j_N)) = 0$. It follows that

$$u(j_N, p(j_N)) = p(j_N)g + C(1 - p(j_N))^{\frac{N+1}{2}} = s,$$
$$\partial_p u(j_N, p(j_N)) = g - \tfrac{N+1}{2}C(1 - p(j_N))^{\frac{N-1}{2}} = 0,$$

and so

$$p(j_N) = 1 - \tfrac{N+1}{N-1}\left(1 - \tfrac{s}{g}\right).$$

Note that $p(j_N) > 0$ if and only if $g < s\frac{N+1}{2}$; $p(j_N) = 0$ otherwise. Therefore, $p(j_N) = \bar{p}$. The constant of integration follows from the value-matching condition and is equal to

$$C = (s - \bar{p}g)(1-\bar{p})^{-\frac{N+1}{2}}.$$

Therefore,

$$u(j_N, p) = pg + (s - \bar{p}g)\left(\frac{1-p}{1-\bar{p}}\right)^{\frac{N+1}{2}},$$

which is equal to $\tilde{u}(j_N, q)$, where $q := \frac{p-\bar{p}}{1-\bar{p}}$ is the normalized belief of unsure voters. Similarly, the value function of winners must solve:

$$w(j_N, p) = \frac{1}{\frac{N+1}{2}}\left[\frac{N+1}{2}g - (1-p)\partial_p w(j_N, p)\right].$$

subject to the value-matching condition $w(j_N, p(j_N)) = s$. It follows that $w(j_N, p) = \tilde{w}(j_N, q)$ with $q = \frac{p-\bar{p}}{1-\bar{p}}$.

Assume that the lemma holds for all $y' \in \{0, \ldots, y-1\}$ (i.e., for all $j' \in \{j+1, \ldots, j_N\}$). Next I show that it also holds for $y$ (i.e., for $j$).

The value function of unsure voters, when they experiment, satisfies the following ODE:

$$u(j, p) = \frac{1}{N-j}\left[w(j+1, p) + (N-j-1)u(j+1, p) - (1-p)\partial_p u(j, p)\right],$$

where $u(j+1, p) = \tilde{u}(j+1, q)$ and $w(j+1, p) = \tilde{w}(j+1, q)$ with $q = \frac{p-\bar{p}}{1-\bar{p}}$. Solving the ODE subject to the value-matching condition $u(j, p(j)) = s$ and, if $p(j) > 0$, the smooth-pasting condition $\partial_p u(j, p(j)) = 0$ yields $p(j) = \bar{p}$ and $u(j, p) = \tilde{u}(j, q)$ with $q = \frac{p-\bar{p}}{1-\bar{p}}$. Similarly, the value function of winners must solve:

$$w(j, p) = \frac{1}{N-j}\left[(N-j)w(j+1, p) - (1-p)\partial_p w(j, p)\right]$$

subject to the value-matching condition $w(j, p(j)) = s$. It follows that $w(j, p) = \tilde{w}(j, q)$ with $q = \frac{p-\bar{p}}{1-\bar{p}}$. ■

**Lemma A.17.** *The function $w(j, p)$ increases in $p$ for all $p \in [\bar{p}, 1]$ and $j \leq j_N$.*

**Proof.** The function $w(j, p)$ increases in $p$ for all $p \in [\bar{p}, 1]$ and $j \leq j_N$ if and only if $\tilde{w}(j, q)$ increases in $q$ for all $q \in [0, 1]$ and $j \leq j_N$, where $q := \frac{p-\bar{p}}{1-\bar{p}}$. I work with the function $\tilde{w}(j, q)$ written as (A.6). Because it coincides with $\tilde{w}(0, j, q)$ written as (A.4), following the steps of the proof of Lemma A.5 yields

$$\partial_q \tilde{w}(j_N - y, q) = (g - s)\left(\frac{N+1}{2} + y\right)\binom{\frac{N-1}{2} + y}{y} q^y (1-q)^{\frac{N-1}{2}} > 0$$

for all $q \in (0, 1)$, where $y := j_N - j$. ■

**Lemma A.18.** *The function $u(j, p)$ increases in $p$ for all $p \in [\bar{p}, 1]$ and $j \leq j_N$.*

**Proof.** The function $u(j, p)$ increases in $p$ for all $p \in [\bar{p}, 1]$ and $j \leq j_N$ if and only if $\tilde{u}(j, q)$ increases in $q$ for all $q \in [0, 1]$ and $j \leq j_N$, where $q := \frac{p-\bar{p}}{1-\bar{p}}$. I work with the function $\tilde{u}(j, q)$ written as (A.7). It coincides with $\tilde{w}(0, j, q)$ written as (A.5), except $b$ should be replaced by $\bar{p}g$ in the latter. Therefore, following the steps of the proof of Lemma A.6 while keeping in mind that $g > s > \bar{p}g$ yields $\partial_q \tilde{u}(j_N - y, q) > 0$ for all $q \in (0, 1)$, where $y := j_N - j$. ■

### A.2.11 Proof of Proposition A.1

It follows from Theorem A.1 that, if $g \geq s\frac{N+1}{2}$, then the equilibrium cut-offs are equal to $\bar{p} = 0$, and so are independent of the number of players $N$. If $g < s\frac{N+1}{2}$, then the equilibrium cut-offs are equal to

$$\bar{p} = 1 - \tfrac{N+1}{N-1}(1 - p_M),$$

where $p_M = \frac{s}{g}$ is the myopic cut-off. Therefore, $\bar{p}$ increases in $N$. Furthermore,

$$\lim_{N\to\infty} \bar{p} = 1 - (1 - p_M) = p_M.$$

# Appendix B

# Addendum to Chapter 2

## B.1 General Case

The results in the general case with the post-reboot probability $\pi \in (0,1)$ are gathered in this appendix. Social optimality of the myopic behavior and efficiency of the symmetric equilibrium are observed here as well. Closed forms of players' value function and of the socially optimal and equilibrium cut-offs or equations which implicitly define the cut-offs can be found in the respective part of Appendix B.2.

### B.1.1 Good News

With learning via good news, if players allocate $X(p) > 0$ to $R$ altogether, then their belief evolves according to (2.4), which takes the following form

$$\dot{p} = -X(p)(p - \alpha_{X(p)})(\beta_{X(p)} - p),$$

where

$$\alpha_X := \tfrac{1}{2X}\left(X + \phi - \sqrt{(X+\phi)^2 - 4X\phi\pi}\right),$$
$$\beta_X := \tfrac{1}{2X}\left(X + \phi + \sqrt{(X+\phi)^2 - 4X\phi\pi}\right).$$

Parameters $\alpha_X$ and $\beta_X$ are such that $\alpha_X \in (0,\pi)$ and $\beta_X > 1$ for all $X \in (0, I]$, $\alpha_X$ and $\beta_X$ are decreasing with $X$, and $\lim_{X\to 0}\alpha_X = \pi$.[1]

#### Social Planner's Problem

The socially optimal behavior is bang-bang. Proposition B.1 below describes the socially optimal behavior with $\pi \in (0,1)$. The left panel in Figure B.1 shows parameter regions which correspond to different kinds of the social optima.

[1] Note that

$$(\alpha_X)'_X = \frac{\phi\left(-2X\pi + X + \phi - \sqrt{(X+\phi)^2 - 4X\phi\pi}\right)}{2X^2\sqrt{(X+\phi)^2-4X\phi\pi}} = \frac{\phi(\alpha_X - \pi)}{X\sqrt{(X+\phi)^2-4X\phi\pi}} < 0,$$
$$(\beta_X)'_X = \frac{\phi\left(2X\pi - (X+\phi) - \sqrt{(X+\phi)^2 - 4X\phi\pi}\right)}{2X^2\sqrt{(X+\phi)^2-4X\phi\pi}} = \frac{\phi(\pi - \beta_X)}{X\sqrt{(X+\phi)^2-4X\phi\pi}} < 0.$$

The limit $\lim_{X\to 0}\alpha_X = \pi$ is obtained applying l'Hôpital rule.

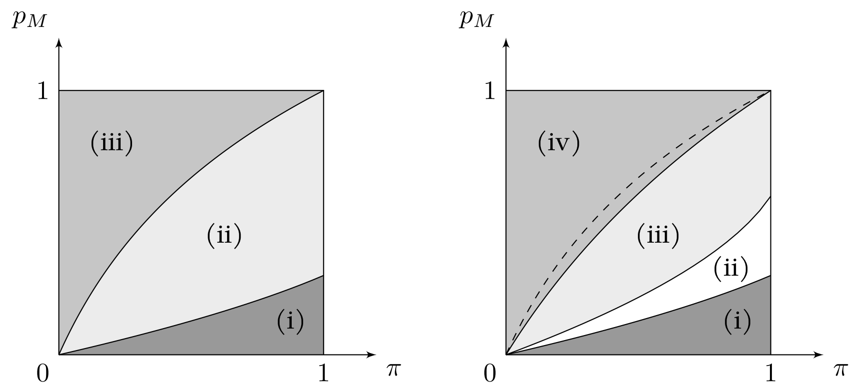


**Figure B.1.** Parameter regions of different kinds of the social optima (left) and of the equilibria (right) if learning is via good news. Regions (i)–(iii) in the left panel correspond to (i)–(iii) in Proposition B.1. Regions (i)–(iv) in the right panel correspond to (i)–(iv) in Proposition B.2. Parameters: $(I, \phi, r) = (2, 0.6, 1)$.

**Proposition B.1** (Social Optimal in the Good News Case with $\pi \in (0,1)$)**.** *The optimal strategy of the social planner is (essentially) unique. It is bang-bang with $X^*(p) = I$ for $p > p^*$ and $X^*(p) = 0$ for $p < p^*$, where $p^*$ and $X^*(p^*)$ are as follows:*

(*i*) *if $p_M < \alpha_I$, then $p^* \in (0, \alpha_I)$, $p^* = p_M$, and $X^*(p^*) = I$;*
(*ii*) *if $\alpha_I < p_M < \tilde{p}$, then $p^* \in (\alpha_I, \pi)$, $p^* < p_M$, and $X^*(p^*) = \frac{\phi(\pi - p^*)}{p^*(1-p^*)}$;*
(*iii*) *if $\tilde{p} < p_M$, then $p^* \in (\pi, 1)$, $p^* < p_M$, and $X^*(p^*) = 0$.*

**Strategic Problem**

The symmetric equilibrium either has the bang-bang property or is characterized by two cut-offs $\underline{p}$ and $\bar{p}$ such that $\underline{p} < \bar{p}$. Whenever exists, the symmetric equilibrium with $\pi \in (0,1)$ takes the form characterized in Proposition B.2 below. I observe existence of the equilibrium numerically, but it is left to show this analytically. The right panel in Figure B.1 shows parameter regions which correspond to different kinds of the equilibria. The dashed line is the line that separates regions (ii) and (iii) in the left panel in Figure B.1.

**Proposition B.2** (Symmetric Equilibrium in the Good News Case with $\pi \in (0,1)$)**.** *The equilibrium is such that $x^e(p) = 1$ for $p > \bar{p}$ and $x^e(p) = 0$ for $p < \underline{p}$, where $\underline{p}$, $\bar{p}$, and $x^e(p)$ for $p \in [\underline{p}, \bar{p}]$ are as follows:*

(*i*) *if $p_M < \alpha_I$, then $\underline{p} \in (0, \alpha_I)$, $\underline{p} = \bar{p} = p_M$ and $x^e(\bar{p}) = 1$;*
(*ii*) *if $\alpha_I < p_M < \alpha_1$, then $\underline{p} \in (\alpha_I, \alpha_1)$, $\underline{p} = \bar{p} = p_M$, and $x^e(\bar{p}) = \frac{\phi(\pi - \bar{p})}{I\bar{p}(1-\bar{p})}$;*
(*iii*) *if $\alpha_1 < p_M < \ldots$, then $\underline{p} \in (\alpha_1, \pi)$, $\underline{p} < \bar{p} < p_M$, and $x^e(p)$ is increasing in $p$;*
(*iv*) *if $\ldots < p_M$, then $\underline{p} \in (\pi, 1)$, $\underline{p} < \bar{p} < p_M$, and $x^e(p)$ increases in $p$.*

## B.1.2 Bad News

With learning via bad news, if players allocate $X(p) > 0$ to $R$ altogether, then their belief evolves according to (2.6), which takes the following form

$$\dot{p} = -X(p)(p - \alpha_{X(p)})(p - \beta_{X(p)}),$$

where

$$\alpha_X := \frac{1}{2X}\left(X - \phi + \sqrt{(X-\phi)^2 + 4X\phi\pi}\right),$$

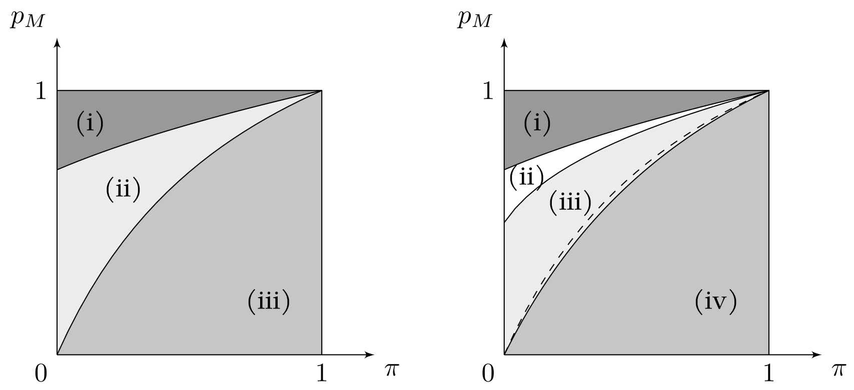


**Figure B.2.** Parameter regions of different kinds of the social optima (left) and of the equilibria (right) if learning is via bad news. Regions (i)–(iii) in the left panel correspond to (i)–(iii) in Proposition B.3. Regions (i)–(iv) in the right panel correspond to (i)–(iv) in Proposition B.4. Parameters: $(I, \phi, r) = (2, 0.6, 1)$.

$$\beta_X := \tfrac{1}{2X}\left(X - \phi - \sqrt{(X-\phi)^2 + 4X\phi\pi}\right).$$

Parameters $\alpha_X$ and $\beta_X$ are such that $\alpha_X \in (\pi, 1)$ and $\beta_X < 0$ for all $X \in (0, I]$, $\alpha_X$ and $\beta_X$ are increasing in $X$, and $\lim_{X\to 0} \alpha_X = \pi$.[2]

**Social Planner's Problem**

The socially optimal behavior is bang-bang. Proposition B.3 below describes the socially optimal behavior with $\pi \in (0,1)$. The left panel in Figure B.2 shows parameter regions which correspond to different kinds of the social optima.

**Proposition B.3** (Social Optimal in the Bad News Case with $\pi \in (0,1)$)**.** *The optimal strategy of the social planner is (essentially) unique. It is bang-bang with $X^*(p) = I$ for $p > p^*$ and $X^*(p) = 0$ for $p < p^*$, where $p^*$ and $X^*(p^*)$ are as follows:*

(*i*) *if $p_M > \alpha_I$, then $p^* \in (\alpha_I, 1)$, $p^* = p_M$, and $X^*(p^*) = I$;*

(*ii*) *if $\alpha_I > p_M > \frac{\pi(I+r+\phi)}{\pi I + r + \phi}$, then $p^* \in (\pi, \alpha_I)$, $p^* < p_M$, and $X^*(p^*) \in \left\{0, \frac{\phi(p^*-\pi)}{p^*(1-p^*)}, I\right\}$;*

(*iii*) *if $\frac{\pi(I+r+\phi)}{\pi I + r + \phi} > p_M$, then $p^* \in (0, \pi)$, $p^* < p_M$, and $X^*(p^*) = I$.*

**Strategic Problem**

The symmetric equilibrium either has the bang-bang property or is characterized by two cut-offs $\underline{p}$ and $\bar{p}$ such that $\underline{p} < \bar{p}$. Whenever exists, the symmetric equilibrium with $\pi \in (0,1)$ takes the form characterized in Proposition B.4 below. I observe existence of the equilibrium numerically, but it is left to show this analytically. The right panel in Figure B.2 shows parameter regions which correspond to different kinds of equilibria. The dashed line is the line which separates regions (ii) and (iii) in the left panel in Figure B.2.

[2] Note that

$$(\alpha_X)'_X = \frac{\phi\left(-2X\pi + X - \phi + \sqrt{(X-\phi)^2 + 4X\phi\pi}\right)}{2X^2\sqrt{(X-\phi)^2+4X\phi\pi}} = \frac{\phi(\alpha_X - \pi)}{X\sqrt{(X-\phi)^2+4X\phi\pi}} > 0,$$

$$(\beta_X)'_X = \frac{\phi\left(2X\pi - (X-\phi) - \sqrt{(X-\phi)^2 + 4X\phi\pi}\right)}{2X^2\sqrt{(X-\phi)^2+4X\phi\pi}} = \frac{\phi(\pi - \beta_X)}{X\sqrt{(X-\phi)^2+4X\phi\pi}} > 0.$$

The limit $\lim_{X\to 0} \alpha_X = \pi$ is obtained applying l'Hôpital rule.

**Proposition B.4** (Symmetric Equilibrium in the Bad News Case with $\pi \in (0,1)$). *The equilibrium is such that $x^e(p) = 1$ for $p > \bar{p}$ and $x^e(p) = 0$ for $p < \underline{p}$, where $\underline{p}$, $\bar{p}$, and $x^e(p)$ for $p \in [\underline{p}, \bar{p}]$ are as follows:*

(*i*) *if $p_M > \alpha_I$, then $\underline{p} \in (\alpha_I, 1)$, $\underline{p} = \bar{p} = p_M$, and $x^e(\bar{p}) = 0$;*
(*ii*) *if $\alpha_I > p_M > \frac{(I+r)\alpha_{I-1}+\phi\pi}{I\alpha_{I-1}+r+\phi}$, then $\underline{p} \in (\alpha_{I-1}, \alpha_I)$, $\underline{p} = \bar{p} < p_M$, and $x^e(\bar{p}) \in \left\{0, \frac{\phi(\bar{p}-\pi)}{I\bar{p}(1-\bar{p})}, \frac{\phi(\underline{p}-\pi)}{(I-1)\underline{p}(1-\underline{p})}\right\}$;*
(*iii*) *if $\frac{(I+r)\alpha_{I-1}+\phi\pi}{I\alpha_{I-1}+r+\phi} > p_M > \ldots$, then $\underline{p} \in (\pi, \alpha_1)$, $\underline{p} < \bar{p} < p_M$, and $x^e(p)$ is increasing in $p$;*
(*iv*) *if $\ldots > p_M$, then $\underline{p} \in (0, \pi)$, $\underline{p} < \bar{p} < p_M$, and $x^e(p)$ is increasing in $p$.*

# B.2 Proofs

## B.2.1 Proofs for the Social Planner's Problem

The social planner chooses which fraction of all resources $X(p) \in [0, I]$ to allocate to $R$ given the belief $p$ and in order to maximize the sum of players' expected payoffs, or equivalently the average expected payoff. By the principle of optimality, the social planner's problem can written as the solution to the Hamilton-Jacobi-Bellman (HJB) equation:

$$v(p) = \max_{X \in [0,I]} \left\{ r\mathrm{d}t \cdot \tfrac{X}{I}[pg - (1-p)b] + (1 - r\mathrm{d}t) \cdot \mathbf{E}[v(p + \mathrm{d}p) \mid p] \right\} + o(\mathrm{d}t),$$

where $v(p)$ is the average value function. To find $\mathbf{E}[v(p + \mathrm{d}p) \mid p]$, note that

- with probability $\lambda_1 X p \mathrm{d}t$, good news arrives: the value function jumps to $v(1)$;
- with probability $\lambda_0 X(1-p)\mathrm{d}t$, bad news arrives: the value function jumps to $v(0)$;
- with probability $p(1 - \lambda_1 X \mathrm{d}t) + (1-p)(1 - \lambda_0 X \mathrm{d}t) = 1 - \lambda_1 X p \mathrm{d}t - \lambda_0 X(1-p)\mathrm{d}t$, there is no news: assuming differentiability, the value function becomes

$$v(p) + v'(p)\mathrm{d}p = v(p) + [\phi(\pi - p) - (\lambda_1 - \lambda_0)Xp(1-p)]v'(p)\mathrm{d}t.$$

Therefore, the social planner's problem takes the form:

$$v(p) = \tfrac{\phi}{r}(\pi - p)v'(p) + \max_{X \in [0,I]} \left\{ X \left( b_v(p) - \tfrac{c(p)}{I} \right) \right\}, \tag{B.1}$$

where

$$\begin{aligned} b_v(p) &:= \tfrac{\lambda_1}{r} p[v(1) - v(p) - (1-p)v'(p)] + \tfrac{\lambda_0}{r}(1-p)[v(0) - v(p) + pv'(p)], \\ c(p) &:= (1-p)b - pg. \end{aligned}$$

The function $b_v(p)$ stands for the normalized expected benefit from $R$. It captures jumps in the value function upon arrival of good and bad news, $v(1) - v(p)$ and $v(0) - v(p)$, and the gradual change in value $v'(p)$ in absence of news. The function $c(p)$ is the opportunity cost of using $R$.

Propositions 2.1, 2.2, 2.5, 2.6, B.1, and B.3 are proved by applying the verification argument.

**Proof of Proposition 2.1**

Case (i): If $p_M < \frac{\phi}{I}$ , then $p^* = p_M$. If players use the specified strategy, then the average value function is equal to

$$v^*(p) = \begin{cases} (g+b)\frac{\phi+rp}{\phi+r} - b & \text{if } p \geq p^*, \\ g\frac{\phi}{\phi+r}\left(\frac{1-p^*}{1-p}\right)^{\frac{r}{\phi}} & \text{if } p < p^*. \end{cases}$$

Case (ii): If $\frac{\phi}{I} < p_M$, then $p^*$ is given by

$$p^* = \frac{-b(\phi-r)+\sqrt{\Delta}}{2[br+g(I+r)]},$$

where

$$\Delta := b^2(\phi-r)^2 + 4b\phi[br+g(I+r)].$$

Furthermore, $p^* < p_M$. If players use the specified strategy, then the average value function is equal to

$$v^*(p) = \begin{cases} (g+b)\frac{\phi+rp}{\phi+r} - b + C_1(1-p)\left(\frac{1-p}{p-\phi_I}\right)^{\frac{r+\phi}{I-\phi}} & \text{if } p \geq p^*, \\ b\frac{\phi^2(1-p^*)}{I(\phi+r)(p^*)^2}\left(\frac{1-p^*}{1-p}\right)^{\frac{r}{\phi}} & \text{if } p < p^*, \end{cases}$$

where

$$C_1(1-p^*)\left(\frac{1-p^*}{p^*-\phi_I}\right)^{\frac{r+\phi}{I-\phi}} = \frac{Ip^*-\phi}{r+\phi}\frac{r}{I}\frac{b-(g+b)p^*}{p^*}.$$

In each case, $b_{v^*}(p) > \frac{c(p)}{I}$ for $p > p^*$, $b_{v^*}(p) < \frac{c(p)}{I}$ for $p < p^*$, and $b_{v^*}(p^*) = \frac{c(p^*)}{I}$ for the specified $v^*$. Therefore, $v^*$ solves the HJB equation (B.1), and so it is the value function for the social planner's problem. Because the strategy specified in the proposition achieves the maximum in the HJB equation, this strategy is optimal.

**Details.** Suppose no resource is allocated to $R$, i.e., $X = 0$. The function $v_0(p)$ that solves

$$v(p) = \frac{\phi}{r}(1-p)v'(p)$$

is given by

$$v_0(p) = \frac{C_0}{(1-p)^{\frac{r}{\phi}}},$$

where $C_0$ is a constant of integration. The constant $C_0$ is pinned down by appropriate boundary conditions. It follows that

$$v_0'(p) = \frac{r}{\phi}\frac{C_0}{(1-p)^{\frac{r}{\phi}+1}} = \frac{r}{\phi(1-p)}v_0(p),$$

$$v_0''(p) = \frac{r(r+\phi)}{\phi^2}\frac{C_0}{(1-p)^{\frac{r}{\phi}+2}} = \frac{r+\phi}{\phi(1-p)}v_0'(p) = \frac{r(r+\phi)}{\phi^2(1-p)^2}v_0(p).$$

If news arrives, then it means that the initial state is good or the reboot has taken place. Either way the state is good and players allocated all resources to $R$ thereafter. Therefore, $v(1) = v_1(1) = g$, where $v_1(p)$ is defined next. Suppose all resources are allocated to $R$, i.e., $X = I$. The function $v_1(p)$ that solves

$$v(p) = \tfrac{\phi}{r}(1-p)v'(p) + \tfrac{I}{r}p[g - v(p) - (1-p)v'(p)] - [(1-p)b - pg]$$

is given by

$$v_1(p) = \underbrace{(g+b)\tfrac{\phi+rp}{r+\phi} - b}_{=:w_1(p)} + C_1(1-p)\left(\tfrac{1-p}{p-\phi_I}\right)^{\frac{r+\phi}{I-\phi}},$$

where $C_1$ is a constant of integration. The constant $C_1$ is pinned down by appropriate boundary conditions. In particular, if $X = I$ for $p \leq \phi_I := \frac{\phi}{I}$, then $C_1 = 0$, and so $v_1(p) = w_1(p)$. Furthermore,

$$v_1'(p) = (g+b)\tfrac{r}{r+\phi} - C_1\tfrac{Ip+r}{Ip-\phi}\left(\tfrac{1-p}{p-\phi_I}\right)^{\frac{r+\phi}{I-\phi}},$$
$$v_1''(p) = C_1\tfrac{(I+r)(r+\phi)}{(1-p)(Ip-\phi)^2}\left(\tfrac{1-p}{p-\phi_I}\right)^{\frac{r+\phi}{I-\phi}}.$$

Note that $g - w_1(p) - (1-p)w_1'(p) = 0$. Therefore,

$$g - v_1(p) - (1-p)v_1'(p) = C_1(1-p)\left(\tfrac{1-p}{p-\phi_I}\right)^{\frac{r+\phi}{I-\phi}}\tfrac{r+\phi}{Ip-\phi}.$$

Case (i): Because $b_{w_1}(p) = 0$ for all $p$, $b_{w_1}(p) \geq c(p)$ if and only if $p \geq p_M$, with equality when $p = p_M$. Therefore, $p^* = p_M$ if $p_M \leq \frac{\phi}{I}$. The value-matching at $p^*$ implies $v_0(p^*) = v_1(p^*)$, and so

$$\tfrac{C_0}{(1-p^*)^{\frac{r}{\phi}}} = (g+b)\tfrac{\phi+rp^*}{r+\phi} - b = g\tfrac{\phi}{r+\phi}.$$

Case (ii): It follows from $b_{v_1}(p^*) = \frac{c(p^*)}{I}$ that

$$C_1\left(\tfrac{1-p^*}{p^*-\phi_I}\right)^{\frac{r+\phi}{I-\phi}} = \tfrac{Ip^*-\phi}{(r+\phi)(1-p^*)}\tfrac{r}{I}\tfrac{b-(g+b)p^*}{p^*}$$

and

$$v_1''(p^*) = b\tfrac{r}{I(p^*)^2(1-p^*)}.$$

Therefore,

$$b\tfrac{r}{I(p^*)^2(1-p^*)} = \tfrac{(I+r)(r+\phi)}{(1-p^*)(Ip^*-\phi)^2}\tfrac{Ip^*-\phi}{(r+\phi)(1-p^*)}\tfrac{r}{I}\tfrac{b-(g+b)p^*}{p^*},$$

and so

$$[br + g(I+r)](p^*)^2 + b(\phi - r)p^* - b\phi = 0.$$

The discriminant is equal to

$$\Delta := b^2(\phi-r)^2 + 4b\phi[br + g(I+r)].$$

Therefore,

$$p^* = \tfrac{-b(\phi-r)+\sqrt{\Delta}}{2[br+g(I+r)]}.$$

Note that

$$(I + r - Ip_M)p_M^2 + (\phi - r)p_M^2 - \phi p_M = Ip_M(1-p_M)\left(p_M - \tfrac{\phi}{I}\right) > 0,$$

whenever $p_M > \frac{\phi}{I}$. This implies that, if $p_M > \frac{\phi}{I}$, then $p^* < p_M$. Furthermore, the second-order smooth-pasting at $p^*$ implies $v_0''(p^*) = v_1''(p^*)$, and so

$$C_0 = b\tfrac{\phi^2(1-p^*)}{I(r+\phi)(p^*)^2}(1-p^*)^{\frac{r}{\phi}}.$$

**Proof of Proposition 2.2**

The cut-off $p^*$ solves

$$b\tfrac{r+Ip^*}{Ip^*} = (g+b)\tfrac{r(I+r+\phi)}{I(r+\phi)} + b\tfrac{\phi}{r+\phi}\left(\tfrac{\phi p^*}{I+\phi-Ip^*}\right)^{\frac{r}{I+\phi}}. \tag{B.2}$$

Such $p^*$ exists and is unique. Furthermore, $p^* < p_M$. See details below. If players use the specified strategy, then the average value function is equal to

$$v^*(p) = \begin{cases} \left(v^*(1) + (g+b)\frac{r}{I} + b\right)\frac{Ip}{I+r+\phi} - b + C_1(1+\phi_I - p)\left(\frac{1+\phi_I-p}{p}\right)^{\frac{r}{I+\phi}} & \text{if } p > p^*, \\ 0 & \text{if } p \le p^*, \end{cases}$$

where

$$C_1(1+\phi_I-p^*)\left(\tfrac{1+\phi_I-p^*}{p^*}\right)^{\frac{r}{I+\phi}} = b\tfrac{I+\phi-Ip^*}{I+r+\phi}.$$

Furthermore, $b_{v^*}(p) > \frac{c(p)}{I}$ for $p > p^*$, $b_{v^*}(p) < \frac{c(p)}{I}$ for $p < p^*$, and $b_{v^*}(p^*) = \frac{c(p^*)}{I}$ for the specified $v^*$. Therefore, $v^*$ solves the HJB equation (B.1), and so it is the value function for the social planner's problem. Because the strategy specified in the proposition achieves the maximum in the HJB equation, this strategy is optimal.

**Details.** Suppose no resource is allocated to $R$, i.e., $X = 0$. The function $v_0(p)$ that solves

$$v(p) = -\tfrac{\phi}{r}pv'(p)$$

is given by

$$v_0(p) = \tfrac{C_0}{p^{\frac{r}{\phi}}},$$

where $C_0$ is a constant of integration. If players do not experiment, then their belief drifts down toward 0. It follows that $C_0 = 0$, and so $v_0(p) = 0$.

If the belief is 1, then players allocate all resources to $R$. Therefore, $v(1) = v_1(1)$, where $v_1(p)$ is defined next. Suppose all resources are allocated to $R$, i.e., $X = I$. The function $v_1(p)$ that solves

$$v(p) = -\tfrac{\phi}{r}pv'(p) + \tfrac{I}{r}p[v(1) - v(p) - (1-p)v'(p)] - [(1-p)b - pg]$$

is given by

$$v_1(p) = \underbrace{\left(v_1(1) + (g+b)\tfrac{r}{I} + b\right)\tfrac{Ip}{I+r+\phi} - b}_{=:w_1(p)} + C_1(1+\phi_I-p)\left(\tfrac{1+\phi_I-p}{p}\right)^{\frac{r}{I+\phi}},$$

where $C_1$ is a constant of integration. The constant $C_1$ is pinned down by appropriate boundary conditions. It follows that

$$v_1(1) = -b + (g+b)\tfrac{r}{r+\phi} + \tfrac{I+r+\phi}{r+\phi} C_1 \phi_I^{\frac{I+r+\phi}{I+\phi}},$$

or equivalently that

$$\left(v_1(1) + (g+b)\tfrac{r}{I} + b\right)\tfrac{I}{I+r+\phi} = (g+b)\tfrac{r}{r+\phi} + \tfrac{\phi}{r+\phi} C_1 \phi_I^{\frac{r}{I+\phi}}.$$

Furthermore,

$$\begin{aligned} v_1'(p) &= \left(v_1(1) + (g+b)\tfrac{r}{I} + b\right)\tfrac{I}{I+r+\phi} - C_1 \tfrac{r+Ip}{Ip}\left(\tfrac{1+\phi_I-p}{p}\right)^{\frac{r}{I+\phi}}, \\ v_1''(p) &= C_1 \tfrac{r(I+r+\phi)}{Ip^2(I+\phi-Ip)}\left(\tfrac{1+\phi_I-p}{p}\right)^{\frac{r}{I+\phi}}. \end{aligned}$$

Note that

$$v_1(1) - w_1(p) - (1-p)w_1'(p) = (v_1(1)+b)\tfrac{\phi}{I+r+\phi} + (v_1(1)-g)\tfrac{r}{I+r+\phi} = C_1 \phi_I^{\frac{I+r+\phi}{I+\phi}}.$$

The value-matching and the smooth-pasting at $p^*$ imply $v_1(p^*) = v_0(p^*) = 0$ and $v_1'(p^*) = v_0'(p^*) = 0$. It follows from $b_{v_1}(p^*) = \frac{c(p^*)}{I}$ that

$$v_1(1) = \tfrac{r}{I}\tfrac{b-(g+b)p^*}{p^*},$$

or equivalently that

$$p^* = \tfrac{br}{Iv_1(1)+(g+b)r}.$$

Furthermore, $v_1(p^*) = 0$ and $v_1'(p^*) = 0$ imply

$$\begin{aligned} \left(v_1(1) + (g+b)\tfrac{r}{I} + b\right)\tfrac{Ip^*}{I+r+\phi} - b + C_1(1+\phi_I-p^*)\left(\tfrac{1+\phi_I-p^*}{p^*}\right)^{\frac{r}{I+\phi}} &= 0, \\ \left(v_1(1) + (g+b)\tfrac{r}{I} + b\right)\tfrac{I}{I+r+\phi} - C_1\tfrac{r+Ip^*}{Ip^*}\left(\tfrac{1+\phi_I-p^*}{p^*}\right)^{\frac{r}{I+\phi}} &= 0. \end{aligned}$$

Subtracting from the first equality the second one multiplied by $p^*$ yields

$$C_1\left(\tfrac{1+\phi_I-p^*}{p^*}\right)^{\frac{r}{I+\phi}} = b\tfrac{I}{I+r+\phi},$$

and so

$$\left(v_1(1) + (g+b)\tfrac{r}{I} + b\right)\tfrac{I}{I+r+\phi} = b\tfrac{r+Ip^*}{(I+r+\phi)p^*}.$$

All in all, it follows that

$$b\tfrac{r+Ip^*}{(I+r+\phi)p^*} = (g+b)\tfrac{r}{r+\phi} + b\tfrac{\phi}{r+\phi}\tfrac{I}{I+r+\phi}\left(\tfrac{\phi p^*}{I+\phi-Ip^*}\right)^{\frac{r}{I+\phi}},$$

which can be rewritten as follows:[3]

$$b\frac{r+Ip^*}{Ip^*} = (g+b)\frac{r(I+r+\phi)}{I(r+\phi)} + b\frac{\phi}{r+\phi}\left(\frac{\phi p^*}{I+\phi-Ip^*}\right)^{\frac{r}{I+\phi}}. \tag{B.3}$$

Define

$$h_L(p) := b\frac{r+Ip}{Ip},$$
$$h_R(p) := (g+b)\frac{r(I+r+\phi)}{I(r+\phi)} + b\frac{\phi}{r+\phi}\left(\frac{\phi p}{I+\phi-Ip}\right)^{\frac{r}{I+\phi}}.$$

There exists unique $p^* \in [0,1]$ that solves (B.3) if and only if $h_L(p)$ and $h_R(p)$ intersect once on $[0,1]$. The function $h_L(p)$ decreases in $p$. The function $h_R(p)$ increases in $p$, because its first derivative is equal to

$$h_R'(p) = b\frac{\phi}{r+\phi}\frac{r}{p(I+\phi-Ip)}\left(\frac{\phi p}{I+\phi-Ip}\right)^{\frac{r}{I+\phi}} > 0.$$

Therefore, if there exists $p^*$ that solves (B.3), then it is unique. Such $p^*$ exists. Indeed, $\lim_{p\to 0} h_L(p) = \infty$, while

$$h_R(0) = (g+b)\frac{r(I+r+\phi)}{I(r+\phi)},$$

and so $h_L(p)$ is above $h_R(p)$ as $p$ goes to 0. Furthermore, $h_L(1) = b\frac{I+r}{I}$, while

$$h_R(1) = (g+b)\frac{r(I+r+\phi)}{I(r+\phi)} + b\frac{\phi}{r+\phi} = g\frac{r(I+r+\phi)}{I(r+\phi)} + b\frac{I+r}{I} > b\frac{I+r}{I}.$$

Furthermore, $p^* < p_M$ if and only if $h_L(p_M) < h_R(p_M)$. Equivalently, this is the case if and only if

$$(r+\phi)p_M - r < \phi p_M\left(\frac{\phi p_M}{I+\phi-Ip_M}\right)^{\frac{r}{I+\phi}}. \tag{B.4}$$

Define

$$g_L(p) := (r+\phi)p - r,$$
$$g_R(p) := \phi p\left(\frac{\phi p}{I+\phi-Ip}\right)^{\frac{r}{I+\phi}}.$$

The function $g_L(p)$ is an increasing, linear function of $p$. The function $g_R(p)$ is increasing and convex in $p$. Indeed, its first derivative is equal to

$$g_R'(p) = \frac{\phi(r+\phi+I(1-p))}{\phi+I(1-p)}\left(\frac{\phi p}{I+\phi-Ip}\right)^{\frac{r}{I+\phi}} > 0,$$

and its second derivative is equal to

$$g_R''(p) = \frac{r\phi(I+r+\phi)}{p[\phi+I(1-p)]^2}\left(\frac{\phi p}{I+\phi-Ip}\right)^{\frac{r}{I+\phi}} > 0.$$

Note that $g_L(1) = g_R(1) = \phi$ and $g_R'(1) = g_L'(1) = r+\phi$. Therefore, $g_L(p)$ lies below $g_R(p)$ for all $p \in (0,1)$, and so the condition (B.4) holds.

[3]Note that (B.3) is a special case with $\pi = 0$ of (B.8) below.

**Proof of Proposition B.1**

Case (i): If $p_M < \alpha_I$, then $p^* = p_M$. If players use the specified strategy, then the average value function if equal to

$$v^*(p) = \begin{cases} (g+b)\frac{\phi\pi + rp}{r+\phi} - b & \text{if } p \geq p^*, \\ [g\pi - b(1-\pi)]\frac{\phi}{r+\phi}\left(\frac{\pi - p^*}{\pi - p}\right)^{\frac{\phi}{r}} & \text{if } p < p^*. \end{cases}$$

Case (ii): If $\alpha_I < p_M$ and

$$p_M < \frac{[r^2 + r(I+\phi) + I\phi\pi]\pi}{(r+\phi)(r+I\pi) - I^2\pi(1-\pi)\frac{\beta_I - 1}{\beta_I - \pi}\left(\frac{\beta_I - 1}{1-\alpha_I}\frac{\pi - \alpha_I}{\beta_I - \pi}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I - \alpha_I}}},$$

where the right side belongs to $(\pi, 1)$, then $p^* \in (\alpha_I, \pi)$ and $p^*$ solves

$$b\frac{[\phi(\pi - p^*) - Ip^*(1-p^*)]^2}{p^*(1-p^*)}\frac{\beta_I - 1}{\beta_I - p^*}\left(\frac{\beta_I - 1}{1-\alpha_I}\frac{p^* - \alpha_I}{\beta_I - p^*}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I - \alpha_I}}$$
$$= b\frac{[\phi(\pi - p^*) + r(1-p^*)][\phi(\pi - p^*) - Ip^*(1-p^*)]}{p^*(1-p^*)} + [r^2 + r(I+\phi) + I\phi\pi][b - (g+b)p^*].$$

Such $p^*$ exists and is unique. Furthermore, $p^* < p_M$. See details below. If players use the specified strategy, then the average value function is equal to

$$v^*(p) = \begin{cases} \left(v^*(1) + (g+b)\frac{r}{I} + b\right)\frac{I(\phi\pi + rp)}{r^2 + r(I+\phi) + I\phi\pi} - b + C_1(\beta_I - p)\left(\frac{\beta_I - p}{p - \alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I - \alpha_I}} & \text{if } p \geq p^*, \\ b\frac{\phi^2(\pi - p^*)^2}{(r+\phi)(p^*)^2(1-p^*)}\left(\frac{\pi - p^*}{\pi - p}\right)^{\frac{\phi}{r}} & \text{if } p < p^*, \end{cases}$$

where

$$C_1(\beta_I - p^*)\left(\frac{\beta_I - p^*}{p^* - \alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I - \alpha_I}} = \frac{r}{I}\frac{b - (g+b)p^*}{p^*\left(\frac{\beta_I - 1}{\beta_I - p^*}\left(\frac{\beta_I - 1}{1-\alpha_I}\frac{p^* - \alpha_I}{\beta_I - p^*}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I - \alpha_I}} + \frac{\phi(\pi - p^*) + r(1-p^*)}{I(p^* - \alpha_I)(\beta_I - p^*)}\right)}.$$

Case (iii): If

$$p_M > \frac{[r^2 + r(I+\phi) + I\phi\pi]\pi}{(r+\phi)(r+I\pi) - I^2\pi(1-\pi)\frac{\beta_I - 1}{\beta_I - \pi}\left(\frac{\beta_I - 1}{1-\alpha_I}\frac{\pi - \alpha_I}{\beta_I - \pi}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I - \alpha_I}}},$$

then $p^* > \pi$ and $p^*$ solves

$$b(r+\phi)(r+Ip^*) - (g+b)[r^2 + r(I+\phi) + I\phi\pi]p^*$$
$$= -bI[\phi(\pi - p^*) - Ip^*(1-p^*)]\frac{\beta_I - 1}{\beta_I - p^*}\left(\frac{\beta_I - 1}{1-\alpha_I}\frac{p^* - \alpha_I}{\beta_I - p^*}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I - \alpha_I}}.$$

Such $p^*$ exists and is unique. Furthermore, $p^* < p_M$. See details below. If players use the

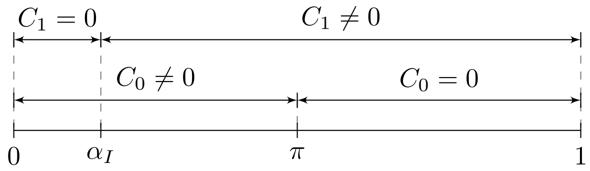


**Figure B.3.** Constants $C_0$ and $C_1$ depending on which region $p^*$ belongs to. Parameters: $(I, \phi, \pi) = (2, 0.75, 0.5)$.

specified strategy, then the average value function if equal to

$$v^*(p) = \begin{cases} \left(v^*(1) + (g+b)\frac{r}{I} + b\right) \frac{I(\phi\pi + rp)}{r^2 + r(I+\phi) + I\phi\pi} - b + C_1(\beta_I - p)\left(\frac{\beta_I - p}{p - \alpha_I}\right)^{\frac{\frac{r}{I} + \alpha_I}{\beta_I - \alpha_I}} & \text{if } p \geq p^*, \\ 0 & \text{if } p < p^*, \end{cases}$$

where

$$C_1(\beta_I - p^*)\left(\frac{\beta_I - p^*}{p^* - \alpha_I}\right)^{\frac{\frac{r}{I} + \alpha_I}{\beta_I - \alpha_I}} = b\frac{r}{r^2 + r(I+\phi) + I\phi\pi}\frac{I(p^* - \alpha_I)(\beta_I - p^*)}{p^*}.$$

In each case, $b_{v^*}(p) > \frac{c(p)}{I}$ for $p > p^*$, $b_{v^*}(p) < \frac{c(p)}{I}$ for $p < p^*$, and $b_{v^*}(p^*) = \frac{c(p^*)}{I}$ for the specified $v^*$. Therefore, $v^*$ solves the HJB equation (B.1), and so it is the value function for the social planner's problem. Because the strategy specified in the proposition achieves the maximum in the HJB equation, this strategy is optimal.

**Details.** Suppose no resource is allocated to $R$, i.e., $X = 0$. The function $v_0(p)$ that solves

$$v(p) = \tfrac{\phi}{r}(\pi - p)v'(p)$$

is given by

$$v_0(p) = \frac{C_0}{(\pi - p)^{\frac{r}{\phi}}},$$

where $C_0$ is a constant of integration. The constant $C_0$ is pinned down by appropriate boundary conditions. In particular, if $X = 0$ for $p \geq \pi$, then $C_0 = 0$ (see Figure B.3), and so $v_0(p) = 0$. It follows that

$$\begin{aligned} v_0'(p) &= \frac{r}{\phi}\frac{C_0}{(\pi - p)^{\frac{r}{\phi}+1}} = \frac{r}{\phi(\pi - p)}v_0(p), \\ v_0''(p) &= \frac{r(r+\phi)}{\phi^2}\frac{C_0}{(\pi - p)^{\frac{r}{\phi}+2}} = \frac{r+\phi}{\phi(\pi - p)}v_0'(p) = \frac{r(r+\phi)}{\phi^2(\pi - p)^2}v_0(p). \end{aligned}$$

If the belief is $1$, players allocate all resources to $R$. Therefore, $v(1) = v_1(1)$, where $v_1(p)$ is defined next. Suppose all resources are allocated to $R$, i.e., $X = I$. The function $v_1(p)$ that solves

$$v(p) = \tfrac{\phi}{r}(\pi - p)v'(p) + \tfrac{I}{r}p[v(1) - v(p) - (1-p)v'(p)] - [(1-p)b - pg]$$

is given by

$$v_1(p) = \underbrace{\left(v_1(1) + (g+b)\tfrac{r}{I} + b\right) \tfrac{I(\phi\pi + rp)}{r^2 + r(I+\phi) + I\phi\pi} - b}_{=:w_1(p)} + C_1(\beta_I - p)\left(\tfrac{\beta_I - p}{p - \alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I - \alpha_I}},$$

where $C_1$ is a constant of integration. The constant $C_1$ is pinned down by appropriate boundary conditions. In particular, if $X = I$ for $p \leq \alpha_I$, then $C_1 = 0$ (see Figure B.3), and so $v_1(p) = w_1(p)$. It follows that

$$v_1(1) = -b + (g+b)\tfrac{r+\phi\pi}{r+\phi} + C_1 \tfrac{r^2 + r(I+\phi) + I\phi\pi}{r(r+\phi)}(\beta_I - 1)\left(\tfrac{\beta_I - 1}{1 - \alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I - \alpha_I}},$$

or equivalently that

$$\left(v_1(1) + (g+b)\tfrac{r}{I} + b\right) \tfrac{I}{r^2 + r(I+\phi) + I\phi\pi} = (g+b)\tfrac{1}{r+\phi} + C_1 \tfrac{I}{r(r+\phi)}(\beta_I - 1)\left(\tfrac{\beta_I - 1}{1 - \alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I - \alpha_I}}.$$

Furthermore,

$$\begin{aligned} v_1'(p) &= \left(v_1(1) + (g+b)\tfrac{r}{I} + b\right) \tfrac{Ir}{r^2 + r(I+\phi) + I\phi\pi} - C_1 \tfrac{Ip + r}{I(p - \alpha_I)}\left(\tfrac{\beta_I - p}{p - \alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I - \alpha_I}}, \\ v_1''(p) &= C_1 \tfrac{(r + I\alpha_I)(r + I\beta_I)}{I^2 (p - \alpha_I)^2 (\beta_I - p)}\left(\tfrac{\beta_I - p}{p - \alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I - \alpha_I}} = C_1 \tfrac{r^2 + r(I+\phi) + I\phi\pi}{I^2 (p - \alpha_I)^2 (\beta_I - p)}\left(\tfrac{\beta_I - p}{p - \alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I - \alpha_I}}. \end{aligned}$$

Note that

$$\begin{aligned} v_1(1) - w_1(p) - (1-p)w_1'(p) &= (v_1(1) + b)\tfrac{r\phi(1-\pi)}{r^2 + r(I+\phi) + I\phi\pi} + (v_1(1) - g)\tfrac{r(r+\phi\pi)}{r^2 + r(I+\phi) + I\phi\pi} \\ &= C_1(\beta_I - 1)\left(\tfrac{\beta_I - 1}{1 - \alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I - \alpha_I}}. \end{aligned}$$

Therefore,

$$\begin{aligned} &v_1(1) - v_1(p) - (1-p)v_1'(p) \\ &\quad = C_1(\beta_I - 1)\left(\tfrac{\beta_I - 1}{1 - \alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I - \alpha_I}} - C_1(\beta_I - p)\left(\tfrac{\beta_I - p}{p - \alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I - \alpha_I}} + C_1 \tfrac{(1-p)(Ip + r)}{I(p - \alpha_I)}\left(\tfrac{\beta_I - p}{p - \alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I - \alpha_I}}. \end{aligned}$$

Case (i): Because $b_{w_1}(p) = 0$ for all $p$ whenever $C_1 = 0$, $b_{w_1}(p) \geq c(p)$ if and only if $p \geq p_M$, with equality when $p = p_M$. Therefore, $p^* = p_M$ if $p_M \leq \alpha_I$. Furthermore,

$$v_1(p) = (g+b)\tfrac{\phi\pi + rp}{r+\phi} - b.$$

The value-matching at $p^*$ implies $v_0(p^*) = v_1(p^*)$, and so

$$\tfrac{C_0}{(1-p^*)^{\frac{r}{\phi}}} = (g+b)\tfrac{\phi\pi + rp^*}{r+\phi} - b = [g\pi - b(1-\pi)]\tfrac{\phi}{r+\phi}.$$

Case (ii): It follows from $b_{v_1}(p^*) = \frac{c(p^*)}{I}$ that

$$C_1(\beta_I - 1)\left(\frac{\beta_I-1}{1-\alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}} + C_1\frac{\phi(\pi-p^*)+r(1-p^*)}{I(p^*-\alpha_I)}\left(\frac{\beta_I-p^*}{p^*-\alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}} = \frac{r}{I}\frac{b-(g+b)p^*}{p^*},$$

and so

$$C_1(\beta_I - p^*)\left(\frac{\beta_I-p^*}{p^*-\alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}} = \frac{r}{I}\frac{b-(g+b)p^*}{p^*\left(\frac{\beta_I-1}{\beta_I-p^*}\left(\frac{\beta_I-1}{1-\alpha_I}\frac{p^*-\alpha_I}{\beta_I-p^*}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}}+\frac{\phi(\pi-p^*)+r(1-p^*)}{I(p^*-\alpha_I)(\beta_I-p^*)}\right)}.$$

Furthermore,

$$v_1''(p^*) = b\frac{r}{I(p^*)^2(1-p^*)}.$$

Therefore, the second-order smooth-pasting at $p^*$ implies that $v_0''(p^*) = v_1''(p^*)$, and so

$$C_0 = b\frac{\phi^2(\pi-p^*)^2}{(r+\phi)(p^*)^2(1-p^*)}(\pi-p^*)^{\frac{r}{\phi}}$$

and

$$b\frac{r}{I(p^*)^2(1-p^*)} = \frac{r^2+r(I+\phi)+I\phi\pi}{I^2(p^*-\alpha_I)^2(\beta_I-p^*)^2}\frac{r}{I}\frac{b-(g+b)p^*}{p^*\left(\frac{\beta_I-1}{\beta_I-p^*}\left(\frac{\beta_I-1}{1-\alpha_I}\frac{p^*-\alpha_I}{\beta_I-p^*}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}}+\frac{\phi(\pi-p^*)+r(1-p^*)}{I(p^*-\alpha_I)(\beta_I-p^*)}\right)}.$$

The last equality can be rewritten as follows:

$$\begin{aligned} &b\frac{[\phi(\pi-p^*)-Ip^*(1-p^*)]^2}{p^*(1-p^*)}\frac{\beta_I-1}{\beta_I-p^*}\left(\frac{\beta_I-1}{1-\alpha_I}\frac{p^*-\alpha_I}{\beta_I-p^*}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}} \\ &\quad = b\frac{[\phi(\pi-p^*)+r(1-p^*)][\phi(\pi-p^*)-Ip^*(1-p^*)]}{p^*(1-p^*)} + [r^2+r(I+\phi)+I\phi\pi][b-(g+b)p^*]. \end{aligned} \tag{B.5}$$

Define

$$\begin{aligned} h_L(p) &:= b\frac{[\phi(\pi-p)-Ip(1-p)]^2}{p(1-p)}\frac{\beta_I-1}{\beta_I-p}\left(\frac{\beta_I-1}{1-\alpha_I}\frac{p-\alpha_I}{\beta_I-p}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}}, \\ h_R(p) &:= b\frac{[\phi(\pi-p)+r(1-p)][\phi(\pi-p)-Ip(1-p)]}{p(1-p)} + [r^2+r(I+\phi)+I\phi\pi][b-(g+b)p]. \end{aligned}$$

There exists unique $p^* \in [\alpha_I, \pi]$ that solves (B.5) if and only if $h_L(p)$ and $h_R(p)$ intersect once on $[\alpha_I, \pi]$. The function $h_L(p)$ is positive for all $p > \alpha_I$. Furthermore, it increases in $p$ for $p \in [\alpha_I, \pi]$. Indeed, its first derivative is equal to

$$\begin{aligned} h_L'(p) = &-b\frac{\phi(\pi-p)-Ip(1-p)}{p^2(1-p)^2}\frac{\beta_I-1}{\beta_I-p}\left(\frac{\beta_I-1}{1-\alpha_I}\frac{p-\alpha_I}{\beta_I-p}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}} \\ &\times[\phi p(1-p)+\phi(\pi-p)(1-p)+\phi(1-\pi)p+Ip(1-p)^2+rp(1-p)] > 0 \end{aligned}$$

for $p \in (\alpha_I, \pi]$, where the following identity $-I(p-\alpha_I)(\beta_I-p) = \phi(\pi-p) - Ip(1-p)$ is used. The function $h_L(p)$ is convex in $p$ on $[\alpha_I, \pi]$. Indeed, its second derivative is equal to

$$h_L''(p) = b\tfrac{2[\phi(\pi-p)(1-p)+\phi(1-\pi)p][\phi(\pi-p)(1-p)+\phi(1-\pi)p+r(1-p)^2]}{p^3(1-p)^3} + b\tfrac{2\phi(\pi-p)[\phi(\pi-p)+r(1-p)]}{p^2(1-p)^2} > 0.$$

The first derivative of the first term of $h_R(p)$ is as follows:

$$-b\tfrac{(r+\phi)[\phi(\pi-p)-Ip(1-p)]}{p(1-p)} - b\tfrac{[\phi(\pi-p)+r(1-p)][\phi(\pi-p)(1-p)+\phi(1-\pi)p]}{p^2(1-p)^2} < 0$$

for $p \in [\alpha_I, \tilde{p})$ for some $\tilde{p} > \alpha_I$. Because the second term of $h_R(p)$ decreases in $p$, it follows that the function $h_R(p)$ decreases in $p$ for $p \in [\alpha_I, \tilde{p})$. Furthermore, $h_L(\alpha_I) = 0$, while

$$h_R(\alpha_I) = [r^2 + r(I+\phi) + I\phi\pi][b - (g+b)\alpha_I] \geq 0$$

whenever $p_M \geq \alpha_I$, with equality when $p_M = \alpha_I$. Therefore, there exists unique $p^* \in [\alpha_I, \pi]$ that solves (B.5) if only if $h_L(\pi) \geq h_R(\pi)$. That is, this is the case if and only if the following condition is satisfied

$$bI^2\pi(1-\pi)\tfrac{\beta_I-1}{\beta_I-\pi}\left(\tfrac{\beta_I-1}{1-\alpha_I}\tfrac{\pi-\alpha_I}{\beta_I-\pi}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}} \geq b(r+\phi)(r+I\pi) - (g+b)[r^2 + r(I+\phi) + I\phi\pi]\pi,$$

or equivalently if and only if

$$p_M \leq \frac{[r^2+r(I+\phi)+I\phi\pi]\pi}{(r+\phi)(r+I\pi)-I^2\pi(1-\pi)\frac{\beta_I-1}{\beta_I-\pi}\left(\frac{\beta_I-1}{1-\alpha_I}\frac{\pi-\alpha_I}{\beta_I-\pi}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}}}. \tag{B.6}$$

Note that

$$I\phi\pi - I^2\pi(1-\pi)\tfrac{\beta_I-1}{\beta_I-\pi}\left(\tfrac{\beta_I-1}{1-\alpha_I}\tfrac{\pi-\alpha_I}{\beta_I-\pi}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}} = I^2\alpha_I\left(\beta_I - (1-\pi)\left(\tfrac{\alpha_I}{\beta_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}}\right) > 0,$$

because $\alpha_I \in (0, \pi)$ and $\beta_I > 0$, and so the denominator of

$$\tilde{p} := \frac{[r^2+r(I+\phi)+I\phi\pi]\pi}{(r+\phi)(r+I\pi)-I^2\pi(1-\pi)\frac{\beta_I-1}{\beta_I-\pi}\left(\frac{\beta_I-1}{1-\alpha_I}\frac{\pi-\alpha_I}{\beta_I-\pi}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}}},$$

which features in (B.6) and (B.9) below, is positive. Moreover, $\tilde{p} \in (\pi, 1)$. Furthermore, $p^* < p_M$ if and only if $h_L(p_M) > h_R(p_M)$. Equivalently, this is the case if and only if

$$[\phi(\pi-p_M) - Ip_M(1-p_M)]\tfrac{\beta_I-1}{\beta_I-p_M}\left(\tfrac{\beta_I-1}{1-\alpha_I}\tfrac{p_M-\alpha_I}{\beta_I-p_M}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}} < \phi(\pi-p_M) + r(1-p_M), \tag{B.7}$$

whenever $p_M > \alpha_I$. Define

$$\begin{aligned} g_L(p) &:= [\phi(\pi-p) - Ip(1-p)]\tfrac{\beta_I-1}{\beta_I-p}\left(\tfrac{\beta_I-1}{1-\alpha_I}\tfrac{p-\alpha_I}{\beta_I-p}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}}, \\ g_R(p) &:= \phi(\pi-p) + r(1-p). \end{aligned}$$

The function $g_R(p)$ is a decreasing, linear function of $p$. The function $g_L(p)$ is decreasing and concave in $p$ for $p > \alpha_I$. Indeed, its first derivative is equal to

$$g_L'(p) = -[r+\phi+I(1-p)]\frac{\beta_I-1}{\beta_I-p}\left(\frac{\beta_I-1}{1-\alpha_I}\frac{p-\alpha_I}{\beta_I-p}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}} < 0,$$

and its second derivative is equal to

$$g_L''(p) = -\frac{r^2+r(I+\phi)+I\phi\pi}{I(p-\alpha_I)(\beta_I-p)}\frac{\beta_I-1}{\beta_I-p}\left(\frac{\beta_I-1}{1-\alpha_I}\frac{p-\alpha_I}{\beta_I-p}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}} < 0.$$

Note that $g_L(1) = g_R(1) = -\phi(1-\pi)$ and $g_L'(1) = g_R'(1) = -(r+\phi)$. Therefore, $g_L(p) < g_R(p)$ for all $p \in (\alpha_I, 1)$, and so the condition (B.7) holds.

Case (iii): The value-matching and the smooth-pasting at $p^*$ imply $v_1(p^*) = v_0(p^*) = 0$ and $v_1'(p^*) = v_0'(p^*) = 0$. It follows from $b_{v_1}(p^*) = \frac{c(p^*)}{I}$ that

$$v_1(1) = \frac{r}{I}\frac{b-(g+b)p^*}{p^*},$$

and so

$$v_1(1) + (g+b)\frac{r}{I} + b = b\frac{Ip^*+r}{Ip^*}.$$

Equivalently,

$$p^* = \frac{br}{Iv_1(1)+(g+b)r}.$$

Furthermore, $v_1(p^*) = 0$ and $v_1'(p^*) = 0$ imply

$$\left(v_1(1) + (g+b)\frac{r}{I} + b\right)\frac{I(\phi\pi+rp^*)}{r^2+r(I+\phi)+I\phi\pi} - b + C_1(\beta_I - p^*)\left(\frac{\beta_I-p^*}{p^*-\alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}} = 0,$$
$$\left(v_1(1) + (g+b)\frac{r}{I} + b\right)\frac{Ir}{r^2+r(I+\phi)+I\phi\pi} - C_1\frac{Ip^*+r}{I(p^*-\alpha_I)}\left(\frac{\beta_I-p^*}{p^*-\alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}} = 0.$$

Subtracting from the first equality the second one multiplied by $\frac{\phi\pi+rp^*}{r}$ yields

$$C_1\left(\frac{\beta_I-p^*}{p^*-\alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}} = b\frac{Ir}{r^2+r(I+\phi)+I\phi\pi}\frac{p^*-\alpha_I}{p^*},$$

and so

$$C_1(\beta_I - p^*)\left(\frac{\beta_I-p^*}{p^*-\alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}} = -b\frac{r}{r^2+r(I+\phi)+I\phi\pi}\frac{\phi(\pi-p^*)-Ip^*(1-p^*)}{p^*}.$$

All in all, it follows that

$$b\frac{Ip^*+r}{Ip^*} = (g+b)\frac{r^2+r(I+\phi)+I\phi\pi}{I(r+\phi)} - b\frac{\phi(\pi-p^*)-Ip^*(1-p^*)}{(r+\phi)p^*}\frac{\beta_I-1}{\beta_I-p^*}\left(\frac{\beta_I-1}{1-\alpha_I}\frac{p^*-\alpha_I}{\beta_I-p^*}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}}.$$

The last equality can be rewritten as follows:

$$b(r+\phi)(r+Ip^*) - (g+b)[r^2+r(I+\phi)+I\phi\pi]p^*$$

$$= -bI[\phi(\pi - p^*) - Ip^*(1-p^*)]\frac{\beta_I - 1}{\beta_I - p^*}\left(\frac{\beta_I - 1}{1-\alpha_I}\frac{p^* - \alpha_I}{\beta_I - p^*}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I - \alpha_I}}. \quad \text{(B.8)}$$

Define

$$h_L(p) := b(r+\phi)(r+Ip) - (g+b)[r^2 + r(I+\phi) + I\phi\pi]p,$$

$$h_R(p) := -bI[\phi(\pi - p) - Ip(1-p)]\frac{\beta_I - 1}{\beta_I - p}\left(\frac{\beta_I - 1}{1-\alpha_I}\frac{p - \alpha_I}{\beta_I - p}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I - \alpha_I}}.$$

There exists unique $p^* \in [\pi, 1]$ that solves (B.8) if and only if $h_L(p)$ and $h_R(p)$ intersect once on $[\pi, 1]$. The function $h_L(p)$ is linear in $p$. The function $h_R(p)$ is positive for $p > \alpha_I$, because

$$\phi(\pi - p) - Ip(1-p) = -I(p - \alpha_I)(\beta_I - p) < 0.$$

Furthermore, it increases in $p$ for $p > \alpha_I$. Indeed, its first derivative is equal to

$$h_R'(p) = bI[r + \phi + I(1-p)]\frac{\beta_I - 1}{\beta_I - p}\left(\frac{\beta_I - 1}{1-\alpha_I}\frac{p - \alpha_I}{\beta_I - p}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I - \alpha_I}} > 0.$$

The function $h_R(p)$ is convex in $p$ for $p > \alpha_I$. Indeed, its second derivative is equal to

$$h_R''(p) = bI\frac{r^2 + r(I+\phi) + I\phi\pi}{I(p-\alpha_I)(\beta_I - p)}\frac{\beta_I - 1}{\beta_I - p}\left(\frac{\beta_I - 1}{1-\alpha_I}\frac{p - \alpha_I}{\beta_I - p}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I - \alpha_I}} > 0.$$

Furthermore, $h_R(1) = bI\phi(1-\pi)$, while

$$h_L(1) = bI\phi(1-\pi) - g[r^2 + r(I+\phi) + I\phi\pi] < bI\phi(1-\pi).$$

Therefore, there exists unique $p^* \geq \pi$ that solves (B.8) if and only if $h_L(\pi) > h_R(\pi)$. That is, this is the case if and only if

$$b(r+\phi)(r+I\pi) - (g+b)[r^2 + r(I+\phi) + I\phi\pi]\pi \geq bI^2\pi(1-\pi)\frac{\beta_I - 1}{\beta_I - \pi}\left(\frac{\beta_I - 1}{1-\alpha_I}\frac{\pi - \alpha_I}{\beta_I - \pi}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I - \alpha_I}},$$

or equivalently if and only if

$$p_M \geq \frac{[r^2 + r(I+\phi) + I\phi\pi]\pi}{(r+\phi)(r+I\pi) - I^2\pi(1-\pi)\frac{\beta_I - 1}{\beta_I - \pi}\left(\frac{\beta_I - 1}{1-\alpha_I}\frac{\pi - \alpha_I}{\beta_I - \pi}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I - \alpha_I}}}. \quad \text{(B.9)}$$

Compare with the condition (B.6). Furthermore, $p^* < p_M$ if and only if $h_R(p_M) > h_L(p_M)$. Divide both sides of (B.8) by $(g+b)$. Then $p^* < p_M$ if and only if

$$-Ip_M[\phi(\pi - p_M) + r(1-p_M)] < -Ip_M[\phi(\pi - p_M) - Ip_M(1-p_M)]\frac{\beta_I - 1}{\beta_I - p_M}\left(\frac{\beta_I - 1}{1-\alpha_I}\frac{p_M - \alpha_I}{\beta_I - p_M}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I - \alpha_I}},$$

or equivalently if and only if (B.7) holds, whenever $p_M > \alpha_I$. As shown above, (B.7) holds for all $p_M \in (\alpha_I, 1)$.

**Proof of Proposition 2.5**

Case (i): If $p_M > 1 - \frac{\phi}{I}$, then $p^* = p_M$. If players use the specified strategy, then the average value function is equal to

$$v^*(p) = \begin{cases} g - \left((g+b)\frac{r}{I} + g\right)\frac{I(\phi+r(1-p))}{(I+r)(\phi+r)} + C_1 p \left(\frac{p}{p-(1-\phi_I)}\right)^{\frac{r+\phi}{I-\phi}} & \text{if } p > p^*, \\ 0 & \text{if } p \leq p^*, \end{cases}$$

where

$$C_1 p^* \left(\frac{p^*}{p^*-(1-\phi_I)}\right)^{\frac{r+\phi}{I-\phi}} = (g+b)\frac{rp^*(Ip^*-I+\phi)}{(I+r)(\phi+r)}.$$

Case (ii): If $1 - \frac{\phi}{I} > p_M$, then $p^* < 1 - \frac{\phi}{I}$ and $p^*$ is given by

$$p^* = \frac{b(r+\phi)}{br+g(I+r)}.$$

Furthermore, $p^* < p_M$. If players use the specified strategy, then the average value function is equal to

$$v^*(p) = \begin{cases} g - \left((g+b)\frac{r}{I} + g\right)\frac{I(\phi+r(1-p))}{(I+r)(\phi+r)} & \text{if } p > p^*, \\ 0 & \text{if } p \leq p^*. \end{cases}$$

In each case, $b_{v^*}(p) > \frac{c(p)}{I}$ for $p > p^*$, $b_{v^*}(p) < \frac{c(p)}{I}$ for $p < p^*$, and $b_{v^*}(p^*) = \frac{c(p^*)}{I}$ for the specified $v^*$. Therefore, $v^*$ solves the HJB equation (B.1), and so it is the value function for the social planner's problem. Because the strategy specified in the proposition achieves the maximum in the HJB equation, this strategy is optimal.

**Details.** Suppose no resource is allocated to $R$, i.e., $X = 0$. The function $v_0(p)$ that solves

$$v(p) = -\frac{\phi}{r} p v'(p)$$

is given by

$$v_0(p) = \frac{C_0}{p^{\frac{r}{\phi}}},$$

where $C_0$ is a constant of integration. If players do not experiment, then their belief drifts down toward $0$. It follows that $C_0 = 0$, and so $v_0(p) = 0$. If the belief is $0$, then players allocate no resource to $R$. Therefore, $v(0) = v_0(0) = 0$.

Suppose all resources are allocated to $R$, i.e., $X = I$. The function $v_1(p)$ that solves

$$v(p) = -\frac{\phi}{r} p v'(p) + \frac{I}{r}(1-p)[-v(p) + pv'(p)] - [(1-p)b - pg]$$

is given by

$$v_1(p) = \underbrace{g - \left((g+b)\frac{r}{I} + g\right)\frac{I(\phi+r(1-p))}{(I+r)(\phi+r)}}_{=:w_1(p)} + C_1 p \left(\frac{p}{p-(1-\phi_I)}\right)^{\frac{r+\phi}{I-\phi}},$$

where $\phi_I := \frac{\phi}{I}$ and $C_1$ is a constant of integration. The constant $C_1$ is pinned down by appropriate boundary conditions. In particular, if $X = I$ for $p \leq 1 - \frac{\phi}{I}$, then $C_1 = 0$, and so $v_1(p) = w_1(p)$. Furthermore,

$$v_1'(p) = \left((g+b)\tfrac{r}{I} + g\right) \tfrac{Ir}{(I+r)(r+\phi)} - C_1 \tfrac{I+r-Ip}{Ip-I+\phi} \left(\tfrac{p}{p-(1-\phi_I)}\right)^{\frac{r+\phi}{I-\phi}},$$
$$v_1''(p) = C_1 \tfrac{(I+r)(\phi+r)}{p(Ip-I+\phi)^2} \left(\tfrac{p}{p-(1-\phi_I)}\right)^{\frac{r+\phi}{I-\phi}}.$$

Note that

$$-w_1(p) + pw_1'(p) = b\tfrac{r}{I+r}.$$

Case (i): The value-matching and the smooth-pasting at $p^*$ imply $v_1(p^*) = v_0(p^*) = 0$ and $v_1'(p^*) = v_0'(p^*) = 0$. Therefore, $b_{v_1}(p^*) = 0$. It follows from $b_{v_1}(p^*) = \frac{c(p^*)}{I}$ that $p^* = p_M$. Furthermore, $v_1(p^*) = 0$ implies

$$C_1 p^* \left(\tfrac{p^*}{p^*-(1-\phi_I)}\right)^{\frac{r+\phi}{I-\phi}} = (g+b)\tfrac{rp^*(Ip^*-I+\phi)}{(I+r)(\phi+r)}.$$

Case (ii): The value-matching at $p^*$ implies $w_1(p^*) = v_0(p^*) = 0$, and so

$$p^* = \tfrac{b(\phi+r)}{br+g(I+r)} = \tfrac{p_M(\phi+r)}{I(1-p_M)+r}.$$

Note that $p^* < p_M$ if and only if $p_M < 1 - \frac{\phi}{I}$. Furthermore,

$$b_{w_1}(p^*) - \tfrac{c(p^*)}{I} = \tfrac{1}{r}(1-p^*)b\tfrac{r}{I+r} - \tfrac{b(1-p^*)-gp^*}{I} = b\tfrac{\phi}{I(I+r)} > 0.$$

**Proof of Proposition 2.6**

The cut-off $p^*$ solves

$$g\phi(1-p^*)^{\frac{r}{\phi}+1} = (g\phi - br)(1-p^*) + \tfrac{r(I+r+\phi)}{I}[(g+b)(1-p^*) - g]. \tag{B.10}$$

Such $p^*$ exists and is unique. Furthermore, $p^* < p_M$. See details below. If players use the specified strategy, then the average value function is equal to

$$v^*(p) = \begin{cases} g - \left(-C_0 + (g+b)\frac{r}{I} + g\right)\frac{I(1-p)}{I+r+\phi} & \text{if } p \geq p^*, \\ \frac{C_0}{(1-p)^{\frac{r}{\phi}}} & \text{if } p < p^*, \end{cases}$$

where

$$C_0 = g\tfrac{\phi}{\phi+r}(1-p^*)^{\frac{r}{\phi}}.$$

Furthermore, $b_{v^*}(p) > \frac{c(p)}{I}$ for $p > p^*$, $b_{v^*}(p) < \frac{c(p)}{I}$ for $p < p^*$, and $b_{v^*}(p^*) = \frac{c(p^*)}{I}$ for the specified $v^*$. Therefore, $v^*$ solves the HJB equation (B.1), and so it is the value function for the social planner's problem. Because the strategy specified in the proposition achieves the maximum in the HJB equation, this strategy is optimal.

**Details.** Suppose no resource is allocated to $R$, i.e., $X = 0$. The function $v_0(p)$ that solves

$$v(p) = \tfrac{\phi}{r}(1-p)v'(p)$$

is given by
$$v_0(p) = \frac{C_0}{(1-p)^{\frac{r}{\phi}}},$$
where $C_0$ is a constant of integration. The constant $C_0$ is pinned down by appropriate boundary conditions. It follows that
$$\begin{aligned} v_0'(p) &= \tfrac{r}{\phi}\frac{C_0}{(1-p)^{\frac{r}{\phi}+1}} = \tfrac{r}{\phi(1-p)}v_0(p), \\ v_0''(p) &= \tfrac{r(r+\phi)}{\phi^2}\frac{C_0}{(1-p)^{\frac{r}{\phi}+2}} = \tfrac{r+\phi}{\phi(1-p)}v_0'(p) = \tfrac{r(r+\phi)}{\phi^2(1-p)^2}v_0(p). \end{aligned}$$

Furthermore, if the belief is $0$, then players allocate no resource to $R$. Therefore, $v(0) = v_0(0) = C_0$.

Suppose all resources are allocated to $R$, i.e., $X = I$. The function $v_1(p)$ that solves
$$v(p) = \tfrac{\phi}{r}(1-p)v'(p) + \tfrac{I}{r}(1-p)[C_0 - v(p) + pv'(p)] - [(1-p)b - pg]$$
is given by
$$v_1(p) = \underbrace{g - \left(-C_0 + (g+b)\tfrac{r}{I} + g\right)\tfrac{I(1-p)}{I+r+\phi}}_{=:w_1(p)} + C_1(\phi_I + p)\left(\tfrac{\phi_I+p}{1-p}\right)^{\frac{r}{I+\phi}},$$
where $C_1$ is a constant of integration. If players experiment, then their belief drifts toward $1$ in absence of news. It follows that $C_1 = 0$, and so $v_1(p) = w_1(p)$. Note that
$$C_0 - w_1(p) + pw_1'(p) = (C_0 - g)\tfrac{\phi}{I+r+\phi} + (C_0 + b)\tfrac{r}{I+r+\phi}.$$

The value-matching and the smooth-pasting at $p^*$ imply $v_1(p^*) = v_0(p^*)$ and $v_1'(p^*) = v_0'(p)$, and so
$$\begin{aligned} g - \left(-C_0 + (g+b)\tfrac{r}{I} + g\right)\tfrac{I(1-p^*)}{I+r+\phi} &= \frac{C_0}{(1-p^*)^{\frac{r}{\phi}}}, \\ \left(-C_0 + (g+b)\tfrac{r}{I} + g\right)\tfrac{I}{I+r+\phi} &= \tfrac{r}{\phi}\frac{C_0}{(1-p^*)^{\frac{r}{\phi}+1}}. \end{aligned}$$
It follows that
$$g - \tfrac{r}{\phi}\frac{C_0}{(1-p^*)^{\frac{r}{\phi}+1}}(1-p^*) = \frac{C_0}{(1-p^*)^{\frac{r}{\phi}}},$$
and so
$$C_0 = g\tfrac{\phi}{r+\phi}(1-p^*)^{\frac{r}{\phi}}.$$

All in all, it follows from $b_{v_1}(p^*) = \frac{c(p^*)}{I}$ that
$$(C_0 - g)\tfrac{\phi}{I+r+\phi} + (C_0 + b)\tfrac{r}{I+r+\phi} = \tfrac{r}{I}\tfrac{b(1-p^*)-gp^*}{1-p^*},$$
that is,
$$g\tfrac{\phi}{r+\phi}(1-p^*)^{\frac{r}{\phi}}\tfrac{r+\phi}{I+r+\phi} - g\tfrac{\phi}{I+r+\phi} + b\tfrac{r}{I+r+\phi} = \tfrac{r}{I}\tfrac{b(1-p^*)-gp^*}{1-p^*}.$$

The last equality can be rewritten as follows:[4]

$$g\phi(1-p^*)^{\frac{r}{\phi}+1} = (g\phi - br)(1-p^*) + \tfrac{r(I+r+\phi)}{I}[(g+b)(1-p^*) - g]. \tag{B.11}$$

Define

$$\begin{aligned} h_L(p) &:= g\phi(1-p)^{\frac{r}{\phi}+1}, \\ h_R(p) &:= (g\phi - br)(1-p) + \tfrac{r(I+r+\phi)}{I}[(g+b)(1-p) - g]. \end{aligned}$$

There exists unique $p^* \in [0,1]$ that solves (B.11) if and only if $h_L(p)$ and $h_R(p)$ intersect once on $[0,1]$. The function $h_L(p)$ decreases in $p$. Indeed, its first derivative is equal to

$$h_L'(p) = -g(r+\phi)(1-p^*)^{\frac{r}{\phi}} < 0$$

for $p < 1$. The function $h_L(p)$ is convex in $p$. Indeed, its second derivative is equal to

$$h_L''(p) = g\tfrac{r(r+\phi)}{\phi}(1-p)^{\frac{r}{\phi}-1} > 0$$

for $p < 1$. The $h_R(p)$ is a decreasing, linear function of $p$. Furthermore, $h_L(0) = g\phi$, while

$$h_R(0) = g\phi + b\tfrac{r(r+\phi)}{I} > g\phi.$$

Moreover, $h_L(1) = 0$, while

$$h_R(1) = -g\tfrac{r(I+r+\phi)}{I} < 0.$$

Therefore, there exists and is unique $p^* \in (0,1)$ that solves (B.11).

It is left to show that $p^* < p_M$, which is the case if and only if $h_L(p_M) > h_R(p_M)$. Equivalently, when divided by $(g+b)(1-p_M)$, this is the case if and only if

$$\phi(1-p_M)^{\frac{r}{\phi}+1} > \phi - (r+\phi)p_M. \tag{B.12}$$

Define

$$\begin{aligned} g_L(p) &:= \phi(1-p)^{\frac{r}{\phi}+1}, \\ g_R(p) &:= \phi - (r+\phi)p. \end{aligned}$$

The function $g_L(p)$ is a decreasing, convex function of $p$ with the first derivative equal to

$$g_L'(p) = -(r+\phi)(1-p)^{\frac{r}{\phi}}.$$

The function $g_R(p)$ is a decreasing, linear function of $p$ with the first derivative equal to

$$g_R'(p) = -(r+\phi) \leq -(r+\phi)(1-p)^{\frac{r}{\phi}},$$

with equality at $p = 0$. That is, $g_R(p)$ decreases in $p$ faster than $g_L(p)$. Note that $g_L(0) = g_R(0) = \phi$ and $g_L'(0) = g_R'(0) = -(r+\phi)$. Therefore, $g_L(p) > g_R(p)$ for all $p \in (0,1)$, and so the condition (B.12) holds.

[4] Note that (B.11) is a special case with $\pi = 1$ of (B.14) below.

**Proof of Proposition B.3**

Case (i): If $p_M > \alpha_I$, then $p^* = p_M$. If players use the specified strategy, then the average value function is equal to

$$v^*(p) = \begin{cases} g - \left((g+b)\frac{r}{I} + g\right) \frac{I(\phi(1-\pi)+r(1-p))}{r^2+r(I+\phi)+I\phi(1-\pi)} + C_1(p-\beta_I)\left(\frac{p-\beta_I}{p-\alpha_I}\right)^{\frac{\frac{r}{I}+1-\alpha_I}{\alpha_I-\beta_I}} & \text{if } p > p^*, \\ 0 & \text{if } p \leq p^*, \end{cases}$$

where

$$C_1(p^*-\beta_I)\left(\frac{p^*-\beta_I}{p^*-\alpha_I}\right)^{\frac{\frac{r}{I}+1-\alpha_I}{\alpha_I-\beta_I}} = (b+g)\frac{rp^*(Ip^*-I+\phi)}{r^2+r(I+\phi)+I\phi(1-\pi)}.$$

Case (ii): If $\alpha_I > p_M > \frac{\pi(I+r+\phi)}{\pi I+r+\phi}$, then $p^* \in (\pi, \alpha_I)$ and $p^*$ is given by

$$p^* = \frac{b[r+\phi(1-\pi)]-g\phi\pi}{Ig+(g+b)r}.$$

Furthermore, $p^* < p_M$. If players use the specified strategy, then the average value function is equal to

$$v^*(p) = \begin{cases} g - \left((g+b)\frac{r}{I} + g\right) \frac{I(\phi(1-\pi)+r(1-p))}{r^2+r(I+\phi)+I\phi(1-\pi)} & \text{if } p > p^*, \\ 0 & \text{if } p \leq p^*. \end{cases}$$

Case (iii): If $\frac{\pi(I+r+\phi)}{\pi I+r+\phi} > p_M$, then $p^* < \pi$ and $p^*$ solves

$$\begin{aligned} g\phi\pi\left(\frac{\pi-p^*}{\pi}\right)^{\frac{r}{\phi}+1} &= g\phi\pi(1-p^*) - b[r+\phi(1-\pi)](1-p^*) \\ &\quad + \frac{r^2+r(I+\phi)+I\phi(1-\pi)}{I}[(g+b)(1-p^*)-g]. \end{aligned}$$

Such $p^*$ exists and is unique. Furthermore, $p^* < p_M$. See details below. If players use the specified strategy, then the average value function is equal to

$$v^*(p) = \begin{cases} g - \left(-\frac{C_0}{\pi^{\frac{r}{\phi}}} + (g+b)\frac{r}{I} + g\right) \frac{I(\phi(1-\pi)+r(1-p))}{r^2+r(I+\phi)+I\phi(1-\pi)} & \text{if } p > p^*, \\ \frac{C_0}{(\pi-p)^{\frac{r}{\phi}}} & \text{if } p \leq p^*, \end{cases}$$

where

$$C_0 = g\frac{\phi(\pi-p^*)}{(r+\phi)(1-p^*)}(\pi-p^*)^{\frac{r}{\phi}}.$$

In each case, $b_{v^*}(p) > \frac{c(p)}{I}$ for $p > p^*$, $b_{v^*}(p) < \frac{c(p)}{I}$ for $p < p^*$, and $b_{v^*}(p^*) = \frac{c(p^*)}{I}$ for the specified $v^*$. Therefore, $v^*$ solves the HJB equation (B.1), and so it is the value function for the social planner's problem. Because the strategy specified in the proposition achieves the maximum in the HJB equation, this strategy is optimal.

**Details.** Suppose no resource is allocated to $R$, i.e., $X = 0$. The function $v_0(p)$ that solves

$$v(p) = \frac{\phi}{r}(\pi - p)v'(p)$$

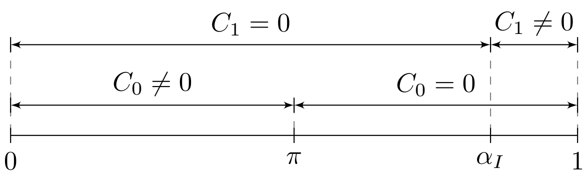


**Figure B.4.** Constants $C_0$ and $C_1$ depending on which region $p^*$ belongs to. Parameters: $(I, \phi, \pi) = (2, 0.75, 0.5)$.

is given by

$$v_0(p) = \frac{C_0}{(\pi-p)^{\frac{r}{\phi}}},$$

where $C_0$ is a constant of integration. The constant $C_0$ is pinned down by appropriate boundary conditions. In particular, if $X = 0$ for $p \geq \pi$, then $C_0 = 0$ (see Figure B.4), and so $v_0(p) = 0$. Furthermore,

$$v_0'(p) = \frac{r}{\phi} \frac{C_0}{(\pi-p)^{\frac{r}{\phi}+1}} = \frac{r}{\phi(\pi-p)} v_0(p),$$

$$v_0''(p) = \frac{r(r+\phi)}{\phi^2} \frac{C_0}{(\pi-p)^{\frac{r}{\phi}+2}} = \frac{r+\phi}{\phi(\pi-p)} v_0'(p) = \frac{r(r+\phi)}{\phi^2(\pi-p)^2} v_0(p).$$

Note that, if the belief is 0, then players allocate no resource to $R$. Therefore, $v(0) = v_0(0) = \frac{C_0}{\pi^{\frac{r}{\phi}}}$.

Suppose all resources are allocated to $R$, i.e., $X = I$. The function $v_1(p)$ that solves

$$v(p) = \tfrac{\phi}{r}(\pi - p)v'(p) + \tfrac{I}{r}(1-p)[v(0) - v(p) + pv'(p)] - [(1-p)b - pg]$$

is given by

$$v_1(p) = \underbrace{g - \left(-v_0(0) + (g+b)\tfrac{r}{I} + g\right) \frac{I(\phi(1-\pi)+r(1-p))}{r^2+r(I+\phi)+I\phi(1-\pi)}}_{=:w_1(p)} + C_1(p - \beta_I)\left(\frac{p-\beta_I}{p-\alpha_I}\right)^{\frac{\frac{r}{I}+1-\alpha_I}{\alpha_I-\beta_I}},$$

where $C_1$ is a constant of integration. The constant $C_1$ is pinned down by appropriate boundary conditions. In particular, if $X = I$ for $p \leq \alpha_I$, then $C_1$ (see Figure B.4), and so $v_1(p) = w_1(p)$. Furthermore,

$$v_1'(p) = \left(-v_0(0) + (g+b)\tfrac{r}{I} + g\right) \frac{Ir}{r^2+r(I+\phi)+I\phi(1-\pi)} - C_1 \frac{I(1-p)+r}{I(p-\alpha_I)} \left(\frac{p-\beta_I}{p-\alpha_I}\right)^{\frac{\frac{r}{I}+1-\alpha_I}{\alpha_I-\beta_I}},$$

$$v_1''(p) = C_1 \frac{(r+I(1-\alpha_I))(r+I(1-\beta_I))}{I^2(p-\alpha_I)^2(p-\beta_I)} \left(\frac{p-\beta_I}{p-\alpha_I}\right)^{\frac{\frac{r}{I}+1-\alpha_I}{\alpha_I-\beta_I}} = C_1 \frac{r^2+r(I+\phi)+I\phi(1-\pi)}{I^2(p-\alpha_I)^2(p-\beta_I)} \left(\frac{p-\beta_I}{p-\alpha_I}\right)^{\frac{\frac{r}{I}+1-\alpha_I}{\alpha_I-\beta_I}}.$$

Note that

$$v_0(0) - w_1(p) + pw_1'(p) = (v_0(0) + b)\frac{r(r+\phi(1-\pi))}{r^2+r(I+\phi)+I\phi(1-\pi)} + (v_0(0) - g)\frac{r\phi\pi}{r^2+r(I+\phi)+I\phi(1-\pi)}.$$

Case (i): The value-matching and the smooth-pasting at $p^*$ imply $v_1(p^*) = v_0(p^*) = 0$ and $v_1'(p^*) = v_0'(p^*) = 0$. Therefore, $b_{v_1}(p^*) = 0$. It follows from $b_{v_1}(p^*) = \frac{c(p^*)}{I}$ that

$p^* = p_M$. Furthermore, $v_1(p^*) = 0$ implies

$$C_1(p^* - \beta_I)\left(\frac{p^*-\beta_I}{p^*-\alpha_I}\right)^{\frac{\frac{r}{I}+1-\alpha_I}{\alpha_I-\beta_I}} = (b+g)\frac{rp^*(Ip^*-I+\phi)}{r^2+r(I+\phi)+I\phi(1-\pi)}.$$

Case (ii): The value-matching at $p^*$ implies $w_1(p^*) = v_0(p^*) = 0$, and so

$$p^* = \frac{-Iv_0(0)(r+\phi(1-\pi))+br(r+\phi(1-\pi))-gr\phi\pi}{Ir\left(-v_0(0)+(g+b)\frac{r}{I}+g\right)}.$$

Because $v_0(0) = 0$,

$$p^* = \frac{b[r+\phi(1-\pi)]-g\phi\pi}{Ig+(g+b)r} = \frac{p_M[r+\phi(1-\pi)]-(1-p_M)\phi\pi}{I(1-p_M)+r} = \frac{p_M(r+\phi)-\phi\pi}{I(1-p_M)+r}.$$

It follows that $p^* \leq p_M$ if and only if

$$0 \leq \phi(\pi - p_M) + Ip_M(1-p_M) = -I(p_M-\alpha_I)(p_M-\beta_I).$$

That is, $p^* \leq p_M$ if and only if $p_M \leq \alpha_I$, with equality when $p_M = \alpha_I$. Note that $p^* \geq \pi$ if and only if

$$p_M \geq \frac{\pi(I+r+\phi)}{\pi I+r+\phi}. \tag{B.13}$$

Furthermore,

$$\begin{aligned} b_{w_1}(p^*) - \frac{c(p^*)}{I} &= \frac{1}{r}(1-p^*)\left(b\frac{r(r+\phi(1-\pi))}{r^2+r(I+\phi)+I\phi(1-\pi)} - g\frac{r\phi\pi}{r^2+r(I+\phi)+I\phi(1-\pi)}\right) - \frac{b(1-p^*)-gp^*}{I} \\ &= (g+b)\frac{\phi[p_M(I\pi+r+\phi)-\pi(I+r+\phi)]}{I[r^2+r(I+\phi)+I\phi(1-\pi)]}, \end{aligned}$$

and $b_{w_1}(p^*) - \frac{c(p^*)}{I} \geq 0$ if and only if

$$p_M \geq \frac{\pi(I+r+\phi)}{\pi I+r+\phi}.$$

Case (iii): The value-matching and the smooth-pasting at $p^*$ imply $w_1(p^*) = v_0(p^*)$ and $w_1'(p^*) = v_0'(p^*)$, and so

$$\begin{aligned} g - \left(-\frac{C_0}{\pi^{\frac{r}{\phi}}} + (g+b)\frac{r}{I} + g\right)\frac{I(\phi(1-\pi)+r(1-p^*))}{r^2+r(I+\phi)+I\phi(1-\pi)} &= \frac{C_0}{(\pi-p^*)^{\frac{r}{\phi}}}, \\ \left(-\frac{C_0}{\pi^{\frac{r}{\phi}}} + (g+b)\frac{r}{I} + g\right)\frac{Ir}{r^2+r(I+\phi)+I\phi(1-\pi)} &= \frac{r}{\phi}\frac{C_0}{(\pi-p^*)^{\frac{r}{\phi}+1}}. \end{aligned}$$

It follows that

$$g - \frac{C_0[\phi(1-\pi)+r(1-p^*)]}{\phi(\pi-p^*)^{\frac{r}{\phi}+1}} = \frac{C_0}{(\pi-p^*)^{\frac{r}{\phi}}},$$

and so

$$C_0 = g\frac{\phi(\pi-p^*)}{(r+\phi)(1-p^*)}(\pi-p^*)^{\frac{r}{\phi}}.$$

It follows from $b_{w_1}(p^*) = \frac{c(p^*)}{I}$ that

$$(v_0(0)+b)\frac{r[r+\phi(1-\pi)]}{r^2+r(I+\phi)+I\phi(1-\pi)} + (v_0(0)-g)\frac{r\phi\pi}{r^2+r(I+\phi)+I\phi(1-\pi)} = \frac{r}{I}\frac{b(1-p^*)-gp^*}{1-p^*},$$

that is,

$$g\tfrac{\phi(\pi-p^*)}{(r+\phi)(1-p^*)}\left(\tfrac{\pi-p^*}{\pi}\right)^{\frac{r}{\phi}}\tfrac{r+\phi}{r^2+r(I+\phi)+I\phi(1-\pi)} \\ + b\tfrac{r+\phi(1-\pi)}{r^2+r(I+\phi)+I\phi(1-\pi)} - g\tfrac{\phi\pi}{r^2+r(I+\phi)+I\phi(1-\pi)} = \tfrac{1}{I}\tfrac{b(1-p^*)-gp^*}{1-p^*}.$$

The last equality can be rewritten as follows:

$$g\phi\pi\left(\tfrac{\pi-p^*}{\pi}\right)^{\frac{r}{\phi}+1} \\ = g\phi\pi(1-p^*) - b[r+\phi(1-\pi)](1-p^*) + \tfrac{r^2+r(I+\phi)+I\phi(1-\pi)}{I}[(g+b)(1-p^*)-g]. \quad \text{(B.14)}$$

Define

$$h_L(p) := g\phi\pi\left(\tfrac{\pi-p}{\pi}\right)^{\frac{r}{\phi}+1},$$
$$h_R(p) := g\phi\pi(1-p) - b[r+\phi(1-\pi)](1-p) + \tfrac{r^2+r(I+\phi)+I\phi(1-\pi)}{I}[(g+b)(1-p)-g].$$

There exists unique $p^* \in [0,\pi]$ that solves (B.14) if and only if $h_L(p)$ and $h_R(p)$ intersect once on $[0,\pi]$. The function $h_L(p)$ decreases in $p$ for $p \leq \pi$. Indeed, its first derivative is equal to

$$h_L'(p) = -g(r+\phi)\left(\tfrac{\pi-p}{\pi}\right)^{\frac{r}{\phi}} < 0$$

for $p < \pi$. The function $h_L(p)$ is convex in $p$ for $p \leq \pi$. Indeed, its second derivative is equal to

$$h_L''(p) = g\tfrac{r(r+\phi)}{\phi\pi}\left(\tfrac{\pi-p}{\pi}\right)^{\frac{r}{\phi}-1} > 0$$

for $p < \pi$. The function $h_R(p)$ is a decreasing, linear function of $p$. Furthermore, $h_L(0) = g\phi\pi$, while

$$h_R(0) = g\phi\pi + b\tfrac{r(r+\phi)}{I} > g\phi\pi.$$

Therefore, there exists $p^* \in [0,\pi]$ that solves (B.14) if and only if $h_L(\pi) \geq h_R(\pi)$. That is, this is the case when the following condition is satisfied

$$0 \geq g\phi\pi(1-\pi) - b[r+\phi(1-\pi)](1-\pi) + \tfrac{r^2+r(I+\phi)+I\phi(1-\pi)}{I}[(g+b)(1-\pi)-g],$$

or equivalently if and only if

$$p_M \leq \tfrac{\pi(I+r+\phi)}{\pi I+r+\phi}. \quad \text{(B.15)}$$

Compare with the condition (B.13). It is left to show that such $p^*$ is below $p_M$. If $p_M \geq \pi$, then it is done, because $p^* \leq \pi$. If $p_M < \pi$, then this is the case if and only if $h_L(p_M) > h_R(p_M)$. Equivalently, when divided by $(g+b)(1-p_M)$, this is the case if and only if

$$\phi\pi\left(\tfrac{\pi-p_M}{\pi}\right)^{\frac{r}{\phi}+1} > \phi\pi - (r+\phi)p_M. \quad \text{(B.16)}$$

Define

$$g_L(p) := \phi\pi\left(\tfrac{\pi-p}{\pi}\right)^{\frac{r}{\phi}+1},$$
$$g_R(p) := \phi\pi - (r+\phi)p.$$

The function $g_L(p)$ is a decreasing, convex function of $p$ with the first derivative equal to

$$g_L'(p) = -(r+\phi)\left(\tfrac{\pi-p}{\pi}\right)^{\frac{r}{\phi}}.$$

The function $g_R(p)$ is a decreasing, linear function of $p$ with the first derivative equal to

$$g_R'(p) = -(r+\phi) \leq -(r+\phi)\left(\tfrac{\pi-p}{\pi}\right)^{\frac{r}{\phi}},$$

with equality at $p=0$. That is, $g_R(p)$ decreases in $p$ faster than $g_L(p)$. Note that $g_L(0) = g_R(0) = \phi\pi$ and $g_L'(0) = g_R'(0) = -(r+\phi)$. Therefore, $g_L(p) > g_R(p)$ for all $p \in (0,1)$, and so the condition (B.16) holds.

Note that

$$\tilde{p} := \tfrac{\pi(I+r+\phi)}{\pi I+r+\phi},$$

which features in conditions (B.13) and (B.15), is below $\alpha_I$. Indeed, $\tilde{p} \leq \alpha_I$ if and only if

$$\tfrac{\pi(I+r+\phi)}{\pi I+r+\phi} \leq \tfrac{I-\phi+\sqrt{(I-\phi)^2+4I\phi\pi}}{2I},$$

i.e., if and only if

$$(I-\phi)^2 + 4I\phi\pi - \left(2I\tfrac{\pi(I+r+\phi)}{\pi I+r+\phi} - (I-\phi)\right)^2 = \tfrac{4I^2\pi(1-\pi)[r^2+r(I+\phi)+I\phi(1-\pi)]}{(\pi I+r+\phi)^2} \geq 0.$$

### B.2.2 Proofs for the Strategic Problem

Player $i$ chooses which fraction of her resource $x_i(p) \in [0,1]$ to allocate to $R$ given the belief $p$ and in order to maximize her expected payoff $v_i(p)$, where $i = 1,\dots,I$. By the principle of optimality, her problem can be written as the solution to the HJB equation:

$$v_i(p) = \max_{x_i\in[0,1]}\{r\mathrm{d}t \cdot x_i[pg-(1-p)b] + (1-r\mathrm{d}t)\cdot \mathbf{E}[p+\mathrm{d}p \mid p]\} + o(\mathrm{d}t).$$

Let $X_{-i} := \sum_{j\neq i} x_j$ be the aggregate resource allocation to $R$ by all players but player $i$. To find $\mathbf{E}[v(p+\mathrm{d}p) \mid p]$, note that

- with probability $\lambda_1(X_{-i}+x_i)p\mathrm{d}t$, good news arrives: the value function jumps to $v_i(1)$;
- with probability $\lambda_0(X_{-i}+x_i)(1-p)\mathrm{d}t$, bad news arrives: the value function jumps to $v_i(0)$;
- with probability

$$\begin{aligned}p(1-\lambda_1(X_{-i}+x_i)\mathrm{d}t) + (1-p)(1-\lambda_0(X_{-i}+x_i)\mathrm{d}t)\\ = 1-\lambda_1(X_{-i}+x_i)p\mathrm{d}t - \lambda_0(X_{-i}+x_i)(1-p)\mathrm{d}t,\end{aligned}$$

  there is no news: assuming differentiability, the value function becomes

$$v(p) + v'(p)\mathrm{d}p = v(p) + [\phi(\pi-p) - (\lambda_1-\lambda_0)(X_{-i}+x_i)p(1-p)]v'(p)\mathrm{d}t.$$

Therefore, player $i$'s problem takes the form:

$$v_i(p) = \tfrac{\phi}{r}(\pi-p)v_i'(p) + X_{-i}b_{v_i}(p) + \max_{x_i\in[0,1]}\{x_i\left(b_{v_i}(p)-c(p)\right)\}, \tag{B.17}$$

where

$$\begin{aligned} b_{v_i}(p) &:= \tfrac{\lambda_1}{r} p[v_i(1) - v_i(p) - (1-p)v_i'(p)] + \tfrac{\lambda_0}{r}(1-p)[v_i(0) - v_i(p) + p v_i'(p)], \\ c(p) &:= (1-p)b - pg. \end{aligned}$$

The function $b_{v_i}(p)$ stands for the normalized expected benefit from $R$. It captures jumps in the value function upon arrival of good and bad news, $v_i(1)-v_i(p)$ and $v_i(0)-v_i(p)$, and the gradual change in value $v_i'(p)$ in absence of news. The function $c(p)$ is the opportunity cost of using $R$.

As I look for a symmetric equilibrium, the subscript $i$ is omitted hereafter. Propositions 2.3, 2.4, 2.7, 2.8, B.2, and B.4 are proved by applying the verification argument.

**Proof of Proposition 2.3**

Cases (i) and (ii): If $p_M < \phi$, then $\underline{p} = \bar{p} = p_M$. If players use the specified strategy, then each player's value function is equal to

$$v^e(p) = \begin{cases} (g+b)\frac{\phi + rp}{\phi + r} - b & \text{if } p \geq \bar{p}, \\ g\frac{\phi}{\phi + r}\left(\frac{1-\bar{p}}{1-p}\right)^{\frac{r}{\phi}} & \text{if } p < \bar{p}. \end{cases}$$

Case (iii): If $\phi < p_M$, then $\underline{p} < \bar{p}$ and $\underline{p}$ is given by

$$\underline{p} = \tfrac{-b(\phi - r) + \sqrt{\Delta}}{2[br + g(1+r)]},$$

where

$$\Delta := b^2(\phi - r)^2 + 4b\phi[br + g(1+r)],$$

while $\bar{p}$ solves

$$\begin{aligned} g + gr - b\frac{r\underline{p} + \phi}{\underline{p}^2}(1-\bar{p}) - b\phi\frac{\bar{p} - \underline{p}}{\underline{p}\bar{p}}(1-\bar{p}) - b(r+\phi)(1-\bar{p})\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{\bar{p}}{1-\bar{p}}\right) \\ = (I-1)[b(1-\bar{p}) - g\bar{p}]. \quad \text{(B.18)} \end{aligned}$$

Such $\bar{p}$ exists and is unique. Furthermore, $\bar{p} < p_M$. See details below. If players use the specified strategy with

$$x^e(p) = \frac{g + gr - b\frac{r\underline{p}+\phi}{\underline{p}^2}(1-p) - b\phi\frac{p - \underline{p}}{\underline{p}p}(1-p) - b(r+\phi)(1-p)\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right)}{(I-1)[b(1-p) - gp]}. \quad \text{(B.19)}$$

for $p \in [\underline{p}, \bar{p}]$, then each player's value function is equal to

$$v^e(p) = \begin{cases} (g+b)\frac{\phi + rp}{\phi + r} - b + C_1(1-p)\left(\frac{1-p}{p - \phi_I}\right)^{\frac{r+\phi}{I-\phi}} & \text{if } p > \bar{p}, \\ g + gr - br\frac{(r+\phi)\underline{p} + \phi}{(r+\phi)\underline{p}^2}(1-p) - br(1-p)\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right) & \text{if } p \in [\underline{p}, \bar{p}], \\ b\frac{\phi^2(1-\underline{p})}{(\phi + r)\underline{p}^2}\left(\frac{1-\underline{p}}{1-p}\right)^{\frac{r}{\phi}} & \text{if } p < \underline{p}, \end{cases}$$

where

$$C_1(1-\bar{p})\left(\frac{1-\bar{p}}{\bar{p}-\phi_I}\right)^{\frac{r+\phi}{I-\phi}} = \frac{I\bar{p}-\phi}{r+\phi}r\frac{b-(g+b)\bar{p}}{\bar{p}}.$$

In each case, $b_{v^e}(p) > c(p)$ for $p > \bar{p}$, $b_{v^e}(p) < c(p)$ for $p < \underline{p}$, and $b_{v^e}(p) = c(p)$ for $p \in [\underline{p}, \bar{p}]$ for the specified $v^e$. Therefore, $v^e$ solves the HJB equation (B.17), and so it is the value function in the strategic problem. Because the strategy specified in the proposition achieves the maximum in the HJB equation, this strategy is optimal.

**Details.** Suppose no resource is allocated to $R$, i.e., $x = 0$. The function $v_0(p)$ that solves

$$v(p) = \tfrac{\phi}{r}(1-p)v'(p)$$

is given by

$$v_0(p) = \frac{C_0}{(1-p)^{\frac{r}{\phi}}},$$

where $C_0$ is a constant of integration. The constant $C_0$ is pinned down by appropriate boundary conditions. It follows that

$$v_0'(p) = \frac{r}{\phi}\frac{C_0}{(1-p)^{\frac{r}{\phi}+1}} = \frac{r}{\phi(1-p)}v_0(p),$$

$$v_0''(p) = \frac{r(r+\phi)}{\phi^2}\frac{C_0}{(1-p)^{\frac{r}{\phi}+2}} = \frac{r+\phi}{\phi(1-p)}v_0'(p) = \frac{r(r+\phi)}{\phi^2(1-p)^2}v_0(p).$$

If news arrives, then it means that the initial state is good or the reboot has taken place. Either way the state is good and players allocate the whole resource to $R$ thereafter. Therefore, $v(1) = v_1(1) = g$, where $v_1(p)$ is defined next. Suppose each player allocates the whole resource to $R$, i.e., $x = 1$. The function $v_1(p)$ that solves

$$v(p) = \tfrac{\phi}{r}(1-p)v'(p) + \tfrac{I}{r}p[g - v(p) - (1-p)v'(p)] - [(1-p)b - pg]$$

is given by

$$v_1(p) = \underbrace{(g+b)\frac{\phi+rp}{\phi+r} - b}_{=:w_1(p)} + C_1(1-p)\left(\frac{1-p}{p-\phi_I}\right)^{\frac{r+\phi}{I-\phi}},$$

where $C_1$ is a constant of integration. The constant $C_1$ is pinned down by appropriate boundary conditions. In particular, if $x = 1$ for $p \leq \phi_I := \frac{\phi}{I}$, then $C_1 = 0$, and so $v_1(p) = w_1(p)$. Furthermore,

$$v_1'(p) = (g+b)\frac{r}{\phi+r} - C_1\frac{Ip+r}{Ip-\phi}\left(\frac{1-p}{p-\phi_I}\right)^{\frac{r+\phi}{I-\phi}},$$

$$v_1''(p) = C_1\frac{(I+r)(r+\phi)}{(1-p)(Ip-\phi)^2}\left(\frac{1-p}{p-\phi_I}\right)^{\frac{r+\phi}{I-\phi}}.$$

Note that $g - w_1(p) - (1-p)w_1'(p) = 0$. Therefore,

$$g - v_1(p) - (1-p)v_1'(p) = C_1(1-p)\left(\frac{1-p}{p-\phi_I}\right)^{\frac{r+\phi}{I-\phi}}\frac{r+\phi}{Ip-\phi}.$$

Suppose each player allocates only a fraction of her resource to $R$, i.e., $x \in (0,1)$. The function $v_x(p)$ that solves $b_v(p) = c(p)$, i.e.,

$$\tfrac{1}{r}p[g - v(p) - (1-p)v'(p)] = (1-p)b - pg,$$

is given by

$$v_x(p) = g + gr + br(1-p)\ln\left(\tfrac{1-p}{p}\right) + C(1-p),$$

where $C$ is a constant of integration. The constant $C$ is pinned down by appropriate boundary conditions. It follows that

$$\begin{aligned} v_x'(p) &= -br\ln\left(\tfrac{1-p}{p}\right) - br\tfrac{1}{p} - C, \\ v_x''(p) &= b\tfrac{r}{p^2(1-p)}. \end{aligned}$$

Cases (i) and (ii): Because $b_{w_1}(p) = 0$ for all $p$, $b_{w_1}(p) \geq c(p)$ if and only if $p \geq p_M$, with equality when $p = p_M$. Therefore, $\underline{p} = \bar{p} = p_M$ if $p_M \leq \phi$. The value-matching condition at $\bar{p}$ implies $v_0(\bar{p}) = w_1(\bar{p})$, and so

$$\tfrac{C_0}{(1-\bar{p})^{\frac{r}{\phi}}} = (g+b)\tfrac{\phi + r\bar{p}}{r+\phi} - b = g\tfrac{\phi}{r+\phi}.$$

Case (iii): It follows from $b_{v_1}(\bar{p}) = c(\bar{p})$ that

$$C_1\left(\tfrac{1-\bar{p}}{\bar{p}-\phi_I}\right)^{\frac{r+\phi}{I-\phi}} = \tfrac{I\bar{p}-\phi}{(r+\phi)(1-\bar{p})} r\tfrac{b-(g+b)\bar{p}}{\bar{p}}.$$

The second-order smooth-pasting condition at $\underline{p}$ implies $v_0''(\underline{p}) = v_x''(\underline{p})$, and so

$$\tfrac{r(r+\phi)}{\phi^2}\tfrac{C_0}{(1-\underline{p})^{\frac{r}{\phi}+2}} = b\tfrac{r}{\underline{p}^2(1-\underline{p})},$$

and so

$$C_0 = b\tfrac{\phi^2}{r+\phi}\tfrac{(1-\underline{p})^{\frac{r}{\phi}+1}}{\underline{p}^2}.$$

It follows from $b_{v_0}(\underline{p}) = c(\underline{p})$ that

$$\begin{aligned} g - \tfrac{C_0}{(1-\underline{p})^{\frac{r}{\phi}}} - (1-\underline{p})\tfrac{r}{\phi}\tfrac{C_0}{(1-\underline{p})^{\frac{r}{\phi}+1}} &= r\tfrac{b-(g+b)\underline{p}}{\underline{p}}, \\ g - \tfrac{r+\phi}{\phi}\tfrac{C_0}{(1-\underline{p})^{\frac{r}{\phi}}} &= r\tfrac{b-(g+b)\underline{p}}{\underline{p}}. \end{aligned}$$

Therefore,

$$[g + r(g+b)]\underline{p}^2 + b(\phi - r)\underline{p} - b\phi = 0.$$

The discriminant is equal to

$$\Delta := b^2(\phi - r)^2 + 4b\phi[g + r(g+b)].$$

Therefore,

$$\underline{p} = \tfrac{-b(\phi - r) + \sqrt{\Delta}}{2[g + r(g+b)]}. \tag{B.20}$$

Note that

$$(1-p_M+r)p_M^2+(\phi-r)p_M^2-\phi p_M = p_M(1-p_M)(p_M-\phi)>0,$$

whenever $p_M > \phi$. This implies that, if $p_M > \phi$, then $\underline{p} < p_M$. The smooth-pasting condition at $\underline{p}$ implies $v_x'(\underline{p}) = v_0'(\underline{p})$, that is,

$$-br\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\right) - br\tfrac{1}{\underline{p}} - C = \tfrac{r}{\phi}\tfrac{C_0}{(1-\underline{p})^{\frac{r}{\phi}+1}},$$

and so

$$C = -br\tfrac{1}{\underline{p}} - br\tfrac{\phi}{r+\phi}\tfrac{1}{\underline{p}^2} - br\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\right).$$

Therefore,

$$v_x(p) = g+gr-br\tfrac{(r+\phi)\underline{p}+\phi}{(r+\phi)\underline{p}^2}(1-p) - br(1-p)\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{p}{1-p}\right),$$

and so

$$\begin{aligned} &v_x(p) - \tfrac{\phi}{r}(1-p)v_x'(p) \\ &\qquad = g+gr-b\tfrac{r\underline{p}+\phi}{\underline{p}^2}(1-p) - b\phi\tfrac{p-\underline{p}}{\underline{p}p}(1-p) - b(r+\phi)(1-p)\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{p}{1-p}\right). \end{aligned}$$

It follows from $b_{v_x}(p) = c(p)$ that

$$v_x(p) - \tfrac{\phi}{r}(1-p)v_x'(p) = (I-1)x(p)c(p),$$

and so

$$x(p) = \frac{g+gr-b\frac{r\underline{p}+\phi}{\underline{p}^2}(1-p)-b\phi\frac{p-\underline{p}}{\underline{p}p}(1-p)-b(r+\phi)(1-p)\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right)}{(I-1)[b(1-p)-gp]}.$$

Note that $x(\underline{p}) = 0$. The allocation $x(p)$ increases in $p$ on $[\underline{p}, \bar{p}]$. Indeed, its first derivative is equal to

$$x'(p) = \frac{b\frac{r\underline{p}+\phi}{\underline{p}^2} - b\frac{rp+\phi}{p^2} + b\phi\frac{p-\underline{p}}{\underline{p}p} + b(r+\phi)\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right)}{(I-1)[b(1-p)-gp]} + \frac{g+b}{[b(1-p)-gp]}x(p) \geq 0,$$

with equality when $p = \underline{p}$. Furthermore, $x(\bar{p}) = 1$ implies

$$\begin{aligned} g+gr-b\tfrac{r\underline{p}+\phi}{\underline{p}^2}(1-\bar{p}) - b\phi\tfrac{\bar{p}-\underline{p}}{\underline{p}\bar{p}}(1-\bar{p}) - b(r+\phi)(1-\bar{p})\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{\bar{p}}{1-\bar{p}}\right) \\ = (I-1)[b(1-\bar{p})-g\bar{p}]. \end{aligned} \tag{B.21}$$

Fix $\underline{p}$. Define

$$\begin{aligned} h_L(p) &:= g+gr-b\tfrac{r\underline{p}+\phi}{\underline{p}^2}(1-p) - b\phi\tfrac{p-\underline{p}}{\underline{p}p}(1-p) - b(r+\phi)(1-p)\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{p}{1-p}\right), \\ h_R(p) &:= (I-1)[b(1-p)-gp]. \end{aligned}$$

Given $\underline{p}$, there exists unique $\bar{p} \in [\underline{p}, 1]$ that solves (B.21) if and only if $h_L(p)$ and $h_R(p)$ intersect once on $[\underline{p}, 1]$. The function $h_R(p)$ decreases in $p$. The function $h_L(p)$ increases in $p$, because its first derivative is equal to

$$h_L'(p) = b\frac{rp+\phi}{\underline{p}^2} - b\frac{rp+\phi}{p^2} + b\phi\frac{p-\underline{p}}{\underline{p}p} + b(r+\phi)\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right) > 0$$

for $p > \underline{p}$. Therefore, if there exists $\bar{p}$ that solves (B.21), then it is unique. Such $\bar{p}$ exists. Indeed, $h_L(\underline{p}) = 0$, while

$$h_R(\underline{p}) = (I-1)[b(1-\underline{p}) - g\underline{p}] = (I-1)(g+b)(p_M - \underline{p}) > 0,$$

because $\underline{p} < p_M$. Furthermore, $h_L(1) = g(1+r) > 0$, while $h_R(1) = -(I-1)g < 0$. It is left to show that $\bar{p} < p_M$. Because $h_L(p)$ increases in $p$, $h_L(\underline{p}) = 0$, and $\underline{p} < p_M$, it follows that $h_L(p_M) > 0$. In contrast, $h_R(p_M) = 0$.

**Proof of Proposition 2.4**

The cut-offs are $\underline{p} < \bar{p}$, where $\underline{p}$ and $\bar{p}$ solve

$$\left(r\frac{b-(g+b)\underline{p}}{\underline{p}} + (g+b)\frac{r}{I} + b\right)\frac{r+\phi}{I+r+\phi} - (g+b)\frac{r}{I} = \frac{r[b(1-\bar{p})-g\bar{p}]}{\bar{p}\left(1+\frac{r-(r+\phi)\bar{p}}{\phi\bar{p}}\left(\frac{1+\phi_I-\bar{p}}{\phi_I\bar{p}}\right)^{\frac{r}{I+\phi}}\right)} \tag{B.22}$$

and

$$b(r+\phi)\frac{\bar{p}-\underline{p}}{\underline{p}} - b[r-(r+\phi)\bar{p}]\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{\bar{p}}{1-\bar{p}}\right) = (I-1)[b(1-\bar{p}) - g\bar{p}]. \tag{B.23}$$

Such $\underline{p}$ and $\bar{p}$ exists and are unique. Furthermore, $\bar{p} < p_M$. See details below. If players use the specified strategy with

$$x^e(p) = \frac{b(r+\phi)\frac{p-\underline{p}}{\underline{p}} - b[r-(r+\phi)p]\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right)}{(I-1)(b(1-p)-gp)} \tag{B.24}$$

for $p \in [\underline{p}, \bar{p}]$, then each player's value function is equal to

$$v^e(p) = \begin{cases} \left(v^e(1) + (g+b)\frac{r}{I} + b\right)\frac{Ip}{I+r+\phi} - b + C_1(1+\phi_I - p)\left(\frac{1+\phi_I-p}{p}\right)^{\frac{r}{I+\phi}} & \text{if } p > \bar{p}, \\ br\frac{p-\underline{p}}{\underline{p}} - br(1-p)\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right) & \text{if } p \in [\underline{p}, \bar{p}], \\ 0 & \text{if } p < \underline{p}, \end{cases}$$

where

$$C_1\phi_I^{1+\frac{r}{I+\phi}} = \left(r\frac{b-(g+b)\underline{p}}{\underline{p}} + (g+b)\frac{r}{I} + b\right)\frac{r+\phi}{I+r+\phi} - (g+b)\frac{r}{I}.$$

Furthermore, $b_{v^e}(p) > c(p)$ for $p > \bar{p}$, $b_{v^e}(p) < c(p)$ for $p < \underline{p}$, and $b_{v^e}(p) = c(p)$ for $p \in [\underline{p}, \bar{p}]$ for the specified $v^e$. Therefore, $v^e$ solves the HJB equation (B.17), and so it is the value function in the strategic problem. Because the strategy specified in the proposition achieves the maximum in the HJB equation, this strategy is optimal.

**Details.** Suppose no resource is allocated to $R$, i.e., $x = 0$. The function $v_0(p)$ that solves

$$v(p) = -\frac{\phi}{r}pv'(p)$$

is given by

$$v_0(p) = \frac{C_0}{p^{\frac{r}{\phi}}},$$

where $C_0$ is a constant of integration. If players do not experiment, then their belief drifts down toward $0$. It follows that $C_0 = 0$, and so $v_0(p) = 0$.

If the belief is $1$, then players allocate the whole resource to $R$. Therefore, $v(1) = v_1(1)$, where $v_1(p)$ is defined next. Suppose each player allocates the whole resource to $R$, i.e., $x = 1$. The function $v_1(p)$ that solves

$$v(p) = -\tfrac{\phi}{r} p v'(p) + \tfrac{I}{r} p[v(1) - v(p) - (1-p)v'(p)] - [(1-p)b - pg]$$

is given by

$$v_1(p) = \underbrace{\left(v_1(1) + (g+b)\tfrac{r}{I} + b\right) \tfrac{Ip}{I+r+\phi} - b}_{=:w_1(p)} + C_1(1+\phi_I - p)\left(\tfrac{1+\phi_I - p}{p}\right)^{\frac{r}{I+\phi}},$$

where $C_1$ is a constant of integration. The constant $C_1$ is pinned down by appropriate boundary conditions. It follows that

$$v_1(1) = -b + (g+b)\tfrac{r}{r+\phi} + \tfrac{I+r+\phi}{r+\phi} C_1 \phi_I^{1+\frac{r}{I+\phi}},$$

or equivalently

$$\left(v_1(1) + (g+b)\tfrac{r}{I} + b\right) \tfrac{I}{I+r+\phi} = (g+b)\tfrac{r}{r+\phi} + \tfrac{I}{r+\phi} C_1 \phi_I^{1+\frac{r}{I+\phi}}.$$

Therefore,

$$C_1 \phi_I^{1+\frac{r}{I+\phi}} = \left(v_1(1) + (g+b)\tfrac{r}{I} + b\right) \tfrac{r+\phi}{I+r+\phi} - (g+b)\tfrac{r}{I}.$$

Furthermore,

$$v_1'(p) = \left(v(1) + (g+b)\tfrac{r}{I} + b\right) \tfrac{I}{I+r+\phi} - C_1 \tfrac{r+Ip}{Ip} \left(\tfrac{1+\phi_I - p}{p}\right)^{\frac{r}{I+\phi}},$$
$$v_1''(p) = C_1 \tfrac{r(I+r+\phi)}{Ip^2(I+\phi-Ip)} \left(\tfrac{1+\phi_I - p}{p}\right)^{\frac{r}{I+\phi}}.$$

Note that

$$v_1(1) - w_1(p) - (1-p)w_1'(p) = (v_1(1)+b)\tfrac{\phi}{I+r+\phi} + (v_1(1) - g)\tfrac{r}{I+r+\phi} = C_1 \phi_I^{1+\frac{r}{I+\phi}}.$$

Therefore,

$$v_1(1) - v_1(p) - (1-p)v_1'(p) = C_1 \phi_I^{1+\frac{r}{I+\phi}} + C_1 \tfrac{r-(r+\phi)p}{Ip} \left(\tfrac{1+\phi_I - p}{p}\right)^{\frac{r}{I+\phi}}.$$

Suppose each player allocates only a fraction of her resource to $R$, i.e., $x \in (0,1)$. The function $v_x(p)$ that solves $b_v(p) = c(p)$, i.e.,

$$\tfrac{1}{r} p[v(1) - v(p) - (1-p)v'(p)] = (1-p)b - pg,$$

is given by

$$v_x(p) = v_1(1) + gr + br(1-p)\ln\left(\tfrac{1-p}{p}\right) + C(1-p),$$

where $C$ is a constant of integration. The constant $C$ is pinned down by appropriate boundary conditions. It follows that

$$\begin{aligned} v_x'(p) &= -br\ln\left(\tfrac{1-p}{p}\right) - br\tfrac{1}{p} - C, \\ v_x''(p) &= b\tfrac{r}{p^2(1-p)}. \end{aligned}$$

It follows from $b_{v_1}(\bar{p}) = c(\bar{p})$ that

$$C_1\phi_I^{1+\frac{r}{I+\phi}} = \frac{r[b(1-\bar{p})-g\bar{p}]}{\bar{p}\left(1+\frac{r-(r+\phi)\bar{p}}{\phi\bar{p}}\left(\frac{1+\phi_I-\bar{p}}{\phi_I\bar{p}}\right)^{\frac{r}{I+\phi}}\right)}.$$

The value-matching and the smooth-pasting at $\underline{p}$ imply $v_x(\underline{p}) = v_0(\underline{p}) = 0$ and $v_x'(\underline{p}) = v_0'(\underline{p}) = 0$. It follows from $b_{v_0}(\underline{p}) = c(\underline{p})$ that

$$v_1(1) = r\frac{b-(g+b)\underline{p}}{\underline{p}},$$

or equivalently that

$$\underline{p} = \tfrac{br}{v_1(1)+r(g+b)}.$$

Therefore,

$$v_1(1) + (g+b)\tfrac{r}{I} + b = r\frac{b-(g+b)\underline{p}}{\underline{p}} + (g+b)\tfrac{r}{I} + b.$$

It follows that

$$C_1\phi_I^{1+\frac{r}{I+\phi}} = \left(r\frac{b-(g+b)\underline{p}}{\underline{p}} + (g+b)\tfrac{r}{I} + b\right)\tfrac{r+\phi}{I+r+\phi} - (g+b)\tfrac{r}{I}.$$

Therefore,

$$\left(r\frac{b-(g+b)\underline{p}}{\underline{p}} + (g+b)\tfrac{r}{I} + b\right)\tfrac{r+\phi}{I+r+\phi} - (g+b)\tfrac{r}{I} = \frac{r[b(1-\bar{p})-g\bar{p}]}{\bar{p}\left(1+\frac{r-(r+\phi)\bar{p}}{\phi\bar{p}}\left(\frac{1+\phi_I-\bar{p}}{\phi_I\bar{p}}\right)^{\frac{r}{I+\phi}}\right)}. \tag{B.25}$$

Furthermore, $v_x'(\underline{p}) = v_0'(\underline{p}) = 0$ implies

$$C = -br\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\right) - br\tfrac{1}{\underline{p}}.$$

Therefore,

$$v_x(p) = v_1(1) + gr - br\tfrac{1-p}{\underline{p}} - br(1-p)\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{p}{1-p}\right) = br\tfrac{p-\underline{p}}{\underline{p}} - br(1-p)\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{p}{1-p}\right),$$

and so

$$v_x(p) + \tfrac{\phi}{r}pv_x'(p) = b(r+\phi)\tfrac{p-\underline{p}}{\underline{p}} - b[r-(r+\phi)p]\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{p}{1-p}\right).$$

It follows from $b_{v_x}(p) = c(p)$ that

$$v_x(p) + \tfrac{\phi}{r} p v_x'(p) = (I-1)x(p)c(p),$$

and so that

$$x(p) = \frac{b(r+\phi)\frac{p-\underline{p}}{\underline{p}} - b[r-(r+\phi)p]\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right)}{(I-1)[b(1-p)-gp]}.$$

Note that $x(\underline{p}) = 0$. The allocation $x(p)$ increases in $p$ on $[\underline{p}, \bar{p}]$. Indeed, its first derivative is equal to

$$x'(p) = \frac{b(r+\phi)\left(\frac{1}{\underline{p}} - \frac{1}{p}\right) + b\phi\frac{1}{p(1-p)} + b(r+\phi)\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right)}{(I-1)[b(1-p)-gp]} + \frac{g+b}{[b(1-p)-gp]}x(p) \geq 0,$$

with equality when $p = \underline{p}$. Furthermore, $x(\bar{p}) = 1$ implies

$$b(r+\phi)\frac{\bar{p}-\underline{p}}{\underline{p}} - b[r-(r+\phi)\bar{p}]\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{\bar{p}}{1-\bar{p}}\right) = (I-1)[b(1-\bar{p}) - g\bar{p}]. \tag{B.26}$$

Equations (B.25) and (B.26) pin down $\underline{p}$ and $\bar{p}$. It is left to argue that the solution to (B.25) and (B.26) exists and is unique. It is also left to argue that $\bar{p} < p_M$.

**Proof of Proposition B.2**

Cases (i) and (ii): If $p_M < \alpha_1$, then $\underline{p} = \bar{p} = p_M$. If players use the specified strategy, then each player's value function is equal to

$$v^e(p) = \begin{cases} (g+b)\frac{\phi\pi + rp}{r+\phi} - b & \text{if } p \geq \bar{p}, \\ [g\pi - b(1-\pi)]\frac{\phi}{r+\phi}\left(\frac{\pi-\bar{p}}{\pi-p}\right)^{\frac{r}{\phi}} & \text{if } p < \bar{p}. \end{cases}$$

Case (iii): If $\alpha_1 < p_M < \ldots$, then $\underline{p} < \bar{p}$, where $\underline{p} \in (\alpha_1, \pi)$ and $\underline{p}$ with $\bar{p}$ solve[5]

$$\left(b\frac{\phi(\pi-\underline{p})[\phi(\pi-\underline{p})+r(1-\underline{p})]}{(r+\phi)\underline{p}^2(1-\underline{p})} + r\frac{b-(g+b)\underline{p}}{\underline{p}} + (g+b)\frac{r}{I} + b\right)\frac{r(r+\phi)}{r^2+r(I+\phi)+I\phi\pi} - (g+b)\frac{r}{I}$$
$$= r\frac{b-(g+b)\bar{p}}{\bar{p}\left(1 - \frac{\phi(\pi-\bar{p})+r(1-\bar{p})}{\phi(\pi-\bar{p})-I\bar{p}(1-\bar{p})}\frac{\beta_I-\bar{p}}{\beta_I-1}\left(\frac{\beta_I-\bar{p}}{\bar{p}-\alpha_I}\frac{1-\alpha_I}{\beta_I-1}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}}\right)}.$$

and

$$b\frac{[r\bar{p}-\phi(\pi-\bar{p})](\bar{p}-\underline{p})}{\underline{p}\bar{p}} + b\frac{\phi(\pi-\underline{p})(\bar{p}-\underline{p})}{\underline{p}^2(1-\underline{p})} - b[\phi(\pi-\bar{p})+r(1-\bar{p})]\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{\bar{p}}{1-\bar{p}}\right)$$
$$= (I-1)[b(1-\bar{p})-g\bar{p}].$$

Such $\underline{p}$ with $\bar{p}$ exist and are unique. Furthermore, $\bar{p} < p_M$. See details below. If players use the specified strategy with

$$x^e(p) = \frac{b\frac{[rp-\phi(\pi-p)](p-\underline{p})}{\underline{p}p} + b\frac{\phi(\pi-\underline{p})(p-\underline{p})}{\underline{p}^2(1-\underline{p})} - b[\phi(\pi-p)+r(1-p)]\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right)}{(I-1)[b(1-p)-gp]}$$

[5] The upper bound for $p_M$, i.e., the precise range of parameters for this case, is to be found.

for $p \in [\underline{p}, \bar{p}]$, then each player's value function is equal to

$$v^e(p) = \begin{cases} \left(v^e(1) + (g+b)\frac{r}{I} + b\right) \frac{I(\phi\pi + rp)}{r^2 + r(I+\phi) + I\phi\pi} - b + C_1(\beta_I - p)\left(\frac{\beta_I - p}{p - \alpha_I}\right)^{\frac{\frac{r}{I} + \alpha_I}{\beta_I - \alpha_I}} & \text{if } p > \bar{p}, \\ b\frac{r(p-\underline{p})}{\underline{p}} + b\frac{\phi(\pi-\underline{p})[\phi(\pi-\underline{p}) + r(p-\underline{p})]}{(r+\phi)\underline{p}^2(1-\underline{p})} - br(1-p)\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right) & \text{if } p \in [\underline{p}, \bar{p}], \\ b\frac{\phi^2(\pi-\underline{p})^2}{(r+\phi)\underline{p}^2(1-\underline{p})}\left(\frac{\pi-\bar{p}}{\pi-p}\right)^{\frac{\bar{r}}{\phi}} & \text{if } p < \underline{p}, \end{cases}$$

where

$$\begin{aligned} &C_1(\beta_I - 1)\left(\frac{\beta_I - 1}{1-\alpha_I}\right)^{\frac{\frac{r}{I} + \alpha_I}{\beta_I - \alpha_I}} \\ &\quad = \left(b\frac{\phi(\pi-\underline{p})[\phi(\pi-\underline{p}) + r(1-\underline{p})]}{(r+\phi)\underline{p}^2(1-\underline{p})} + r\frac{b-(g+b)\underline{p}}{\underline{p}} + (g+b)\frac{r}{I} + b\right)\frac{r(r+\phi)}{r^2 + r(I+\phi) + I\phi\pi} - (g+b)\frac{r}{I}. \end{aligned}$$

Case (iv): If $\ldots < p_M$, then $\underline{p} < \bar{p}$, where $\underline{p} > \pi$ and $\underline{p}$ with $\bar{p}$ solve[6]

$$\begin{aligned} &\left(r\frac{b-(g+b)\underline{p}}{\underline{p}} + (g+b)\frac{r}{I} + b\right)\frac{r(r+\phi)}{r^2 + r(I+\phi) + I\phi\pi} - (g+b)\frac{r}{I} \\ &\quad = r\frac{b-(g+b)\bar{p}}{\bar{p}\left(1 - \frac{\phi(\pi-\bar{p}) + r(1-\bar{p})}{\phi(\pi-\bar{p}) - I\bar{p}(1-\bar{p})}\frac{\beta_I - \bar{p}}{\beta_I - 1}\left(\frac{\beta_I - \bar{p}}{\bar{p} - \alpha_I}\frac{1-\alpha_I}{\beta_I - 1}\right)^{\frac{\frac{r}{I} + \alpha_I}{\beta_I - \alpha_I}}\right)}. \end{aligned}$$

and

$$b\frac{[r\bar{p} - \phi(\pi-\bar{p})](\bar{p}-\underline{p})}{\underline{p}\bar{p}} - b[\phi(\pi-\bar{p}) + r(1-\bar{p})]\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{\bar{p}}{1-\bar{p}}\right) = (I-1)[b(1-\bar{p}) - g\bar{p}].$$

Such $\underline{p}$ with $\bar{p}$ exist and are unique. Furthermore, $\bar{p} < p_M$. See details below. If players use the specified strategy with

$$x^e(p) = \frac{b\frac{[rp - \phi(\pi-p)](p-\underline{p})}{\underline{p}p} - b[\phi(\pi-p) + r(1-p)]\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right)}{(I-1)[b(1-p) - gp]}$$

for $p \in [\underline{p}, \bar{p}]$, then each player's value function is equal to

$$v^e(p) = \begin{cases} \left(v^e(1) + (g+b)\frac{r}{I} + b\right) \frac{I(\phi\pi + rp)}{r^2 + r(I+\phi) + I\phi\pi} - b + C_1(\beta_I - p)\left(\frac{\beta_I - p}{p - \alpha_I}\right)^{\frac{\frac{r}{I} + \alpha_I}{\beta_I - \alpha_I}} & \text{if } p > \bar{p}, \\ b\frac{r(p-\underline{p})}{\underline{p}} - br(1-p)\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right) & \text{if } p \in [\underline{p}, \bar{p}], \\ 0 & \text{if } p < \underline{p}, \end{cases}$$

where

$$C_1(\beta_I - 1)\left(\frac{\beta_I - 1}{1-\alpha_I}\right)^{\frac{\frac{r}{I} + \alpha_I}{\beta_I - \alpha_I}} = \left(r\frac{b-(g+b)\underline{p}}{\underline{p}} + (g+b)\frac{r}{I} + b\right)\frac{r(r+\phi)}{r^2 + r(I+\phi) + I\phi\pi} - (g+b)\frac{r}{I}.$$

In each case, $b_{v^e}(p) > c(p)$ for $p > \bar{p}$, $b_{v^e}(p) < c(p)$ for $p < \underline{p}$, and $b_{v^e}(p) = c(p)$ for

[6] The lower bound for $p_M$, i.e., the precise range of parameters for this case, is to be found.

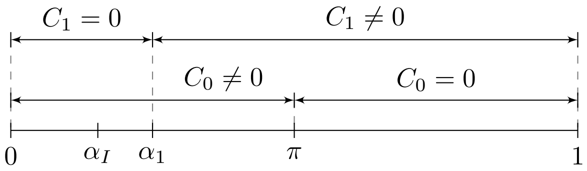


**Figure B.5.** Constants $C_0$ and $C_1$ depending on which region $\underline{p}$ belongs to. Parameters: $(I, \phi, \pi) = (2, 0.75, 0.5)$.

$p \in [\underline{p}, \bar{p}]$ for the specified $v^e$. Therefore, $v^e$ solves the HJB equation (B.17), and so it is the value function in the strategic problem. Because the strategy specified in the proposition achieves the maximum in the HJB equation, this strategy is optimal.

**Details.** Suppose no resource is allocated to $R$, i.e., $x = 0$. The function $v_0(p)$ that solves

$$v(p) = \tfrac{\phi}{r}(\pi - p)v'(p)$$

is given by

$$v_0(p) = \frac{C_0}{(\pi-p)^{\frac{r}{\phi}}},$$

where $C_0$ is a constant of integration. The constant is pinned down by appropriate boundary conditions. In particular, if $x = 0$ for $p \geq \pi$, then $C_0 = 0$ (see Figure B.5), and so $v_0(p) = 0$. Furthermore,

$$\begin{aligned} v_0'(p) &= \tfrac{r}{\phi}\frac{C_0}{(\pi-p)^{\frac{r}{\phi}+1}} = \tfrac{r}{\phi(\pi-p)}v_0(p), \\ v_0''(p) &= \tfrac{r(r+\phi)}{\phi^2}\frac{C_0}{(\pi-p)^{\frac{r}{\phi}+2}} = \tfrac{r+\phi}{\phi(\pi-p)}v_0'(p) = \tfrac{r(r+\phi)}{\phi^2(\pi-p)^2}v_0(p). \end{aligned}$$

If the belief is 1, then players allocate the whole resource to $R$. Therefore, $v(1) = v_1(1)$, where $v_1(p)$ is defined next. Suppose each player allocates the whole resource to $R$, i.e., $x = 1$. The function $v_1(p)$ that solves

$$v(p) = \tfrac{\phi}{r}(\pi - p)v'(p) + \tfrac{I}{r}p[v(1) - v(p) - (1-p)v'(p)] - [(1-p)b - pg]$$

is given by

$$v_1(p) = \underbrace{\left(v_1(1) + (g+b)\tfrac{r}{I} + b\right)\tfrac{I(\phi\pi+rp)}{r^2+r(I+\phi)+I\phi\pi} - b}_{=:w_1(p)} + C_1(\beta_I - p)\left(\tfrac{\beta_I-p}{p-\alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}},$$

where $C_1$ is a constant of integration. The constant $C_1$ is pinned down by appropriate boundary conditions. In particular, if $x = 1$ for $p \leq \alpha_I$, then $C_1 = 0$ (see Figure B.5), and so $v_1(p) = w_1(p)$. Furthermore,

$$v_1(1) = -b + (g+b)\tfrac{r+\phi\pi}{r+\phi} + C_1\tfrac{r^2+r(I+\phi)+I\phi\pi}{r(r+\phi)}(\beta_I - 1)\left(\tfrac{\beta_I-1}{1-\alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}},$$

or equivalently

$$\left(v_1(1)+(g+b)\tfrac{r}{I}+b\right)\tfrac{I}{r^2+r(I+\phi)+I\phi\pi}=(g+b)\tfrac{1}{r+\phi}+C_1\tfrac{I}{r(r+\phi)}(\beta_I-1)\left(\tfrac{\beta_I-1}{1-\alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}}.$$

Therefore,

$$C_1(\beta_I-1)\left(\tfrac{\beta_I-1}{1-\alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}}=\left(v_1(1)+(g+b)\tfrac{r}{I}+b\right)\tfrac{r(r+\phi)}{r^2+r(I+\phi)+I\phi\pi}-(g+b)\tfrac{r}{I}.$$

It also follows that

$$\begin{aligned}v_1'(p)&=\left(v_1(1)+(g+b)\tfrac{r}{I}+b\right)\tfrac{Ir}{r^2+r(I+\phi)+I\phi\pi}-C_1\tfrac{Ip+r}{I(p-\alpha_I)}\left(\tfrac{\beta_I-p}{p-\alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}},\\ v_1''(p)&=C_1\tfrac{(r+I\alpha_I)(r+I\beta_I)}{I^2(p-\alpha_I)^2(\beta_I-p)}\left(\tfrac{\beta_I-p}{p-\alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}}.\end{aligned}$$

Note that

$$\begin{aligned}v_1(1)-w_1(p)-(1-p)w_1'(p)&=(v_1(1)+b)\tfrac{r\phi(1-\pi)}{r^2+r(I+\phi)+I\phi\pi}+(v_1(1)-g)\tfrac{r(r+\phi\pi)}{r^2+r(I+\phi)+I\phi\pi}\\&=C_1(\beta_I-1)\left(\tfrac{\beta_I-1}{1-\alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}}.\end{aligned}$$

Therefore,

$$v_1(1)-v_1(p)-(1-p)v_1'(p)=C_1(\beta_I-1)\left(\tfrac{\beta_I-1}{1-\alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}}+C_1\tfrac{\phi(\pi-p)+r(1-p)}{I(p-\alpha_I)}\left(\tfrac{\beta_I-p}{p-\alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}}.$$

Suppose each player allocates only a fraction of her resource to $R$, i.e., $x\in(0,1)$. The function $v_x(p)$ that solves $b_v(p)=c(p)$, i.e.,

$$\tfrac{1}{r}p[v(1)-v(p)-(1-p)v'(p)]=(1-p)b-pg,$$

is given by

$$v_x(p)=v_1(1)+gr+br(1-p)\ln\left(\tfrac{1-p}{p}\right)+C(1-p),$$

where $C$ is a constant of integration. The constant $C$ is pinned down by appropriate boundary conditions. It follows that

$$\begin{aligned}v_x'(p)&=-br\ln\left(\tfrac{1-p}{p}\right)-br\tfrac{1}{p}-C,\\ v_x''(p)&=b\tfrac{r}{p^2(1-p)}.\end{aligned}$$

Cases (i) and (ii): Because $b_{w_1}(p)=0$ for all $p$ whenever $C_1=0$, $b_{w_1}(p)\geq c(p)$ if and only if $p\geq p_M$, with equality when $p=p_M$. Therefore, $\underline{p}=\bar{p}=p_M$ if $p_M\leq\alpha_1$. Furthermore,

$$v_1(p)=(g+b)\tfrac{\phi\pi+rp}{r+\phi}-b.$$

The value-matching condition at $\bar{p}$ implies $v_0(\bar{p}) = v_1(\bar{p})$, and so

$$\frac{C_0}{(\pi-\bar{p})^{\frac{r}{\phi}}} = (g+b)\tfrac{\phi\pi+r\bar{p}}{r+\phi} - b = [g\pi - b(1-\pi)]\tfrac{\phi}{r+\phi}.$$

Case (iii): It follows from $b_{v_1}(\bar{p}) = c(\bar{p})$ that

$$C_1(\beta_I - 1)\left(\tfrac{\beta_I-1}{1-\alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}} = r\frac{b-(g+b)\bar{p}}{\bar{p}\left(1-\frac{\phi(\pi-\bar{p})+r(1-\bar{p})}{\phi(\pi-\bar{p})-I\bar{p}(1-\bar{p})}\frac{\beta_I-\bar{p}}{\beta_I-1}\left(\frac{\beta_I-\bar{p}}{\bar{p}-\alpha_I}\frac{1-\alpha_I}{\beta_I-1}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}}\right)}.$$

The second-order smooth-pasting at $\underline{p}$ implies $v_0''(\underline{p}) = v_x''(\underline{p})$, i.e.,

$$\tfrac{r(r+\phi)}{\phi^2}\frac{C_0}{(\pi-\underline{p})^{\frac{r}{\phi}+2}} = b\tfrac{r}{\underline{p}^2(1-\underline{p})},$$

and so

$$C_0 = b\tfrac{\phi^2(\pi-\underline{p})^2}{(r+\phi)\underline{p}^2(1-\underline{p})}(\pi-\underline{p})^{\frac{r}{\phi}}.$$

It follows from $b_{v_0}(\underline{p}) = c(\underline{p})$ that

$$\begin{aligned} v_1(1) - \frac{C_0}{(\pi-\underline{p})^{\frac{r}{\phi}}} - (1-\underline{p})\tfrac{r}{\phi}\frac{C_0}{(\pi-\underline{p})^{\frac{r}{\phi}+1}} &= r\tfrac{b-(g+b)\underline{p}}{\underline{p}}, \\ v_1(1) - \frac{\phi(\pi-\underline{p})+r(1-\underline{p})}{\phi(\pi-\underline{p})^{\frac{r}{\phi}+1}}C_0 &= r\tfrac{b-(g+b)\underline{p}}{\underline{p}}. \end{aligned}$$

Therefore,

$$v_1(1) = b\tfrac{\phi(\pi-\underline{p})[\phi(\pi-\underline{p})+r(1-\underline{p})]}{(r+\phi)\underline{p}^2(1-\underline{p})} + r\tfrac{b-(g+b)\underline{p}}{\underline{p}}.$$

It follows that

$$\begin{aligned} &\left(b\tfrac{\phi(\pi-\underline{p})[\phi(\pi-\underline{p})+r(1-\underline{p})]}{(r+\phi)\underline{p}^2(1-\underline{p})} + r\tfrac{b-(g+b)\underline{p}}{\underline{p}} + (g+b)\tfrac{r}{I} + b\right)\tfrac{r(r+\phi)}{r^2+r(I+\phi)+I\phi\pi} - (g+b)\tfrac{r}{I} \\ &\qquad = r\frac{b-(g+b)\bar{p}}{\bar{p}\left(1-\frac{\phi(\pi-\bar{p})+r(1-\bar{p})}{\phi(\pi-\bar{p})-I\bar{p}(1-\bar{p})}\frac{\beta_I-\bar{p}}{\beta_I-1}\left(\frac{\beta_I-\bar{p}}{\bar{p}-\alpha_I}\frac{1-\alpha_I}{\beta_I-1}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}}\right)}. \end{aligned} \tag{B.27}$$

The smooth-pasting at $\underline{p}$ implies $v_x'(\underline{p}) = v_0'(\underline{p})$, that is,

$$-br\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\right) - br\tfrac{1}{\underline{p}} - C = \tfrac{r}{\phi}\frac{C_0}{(\pi-\underline{p})^{\frac{r}{\phi}+1}},$$

and so

$$C = -br\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\right) - br\tfrac{1}{\underline{p}} - b\tfrac{r\phi(\pi-\underline{p})}{(r+\phi)\underline{p}^2(1-\underline{p})}.$$

Therefore,

$$v_x(p) = v_1(1) + gr - br\tfrac{1-p}{\underline{p}} - br(1-p)\tfrac{\phi(\pi-\underline{p})}{(r+\phi)\underline{p}^2(1-\underline{p})} - br(1-p)\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{p}{1-p}\right)$$

$$= b\tfrac{r(p-\underline{p})}{\underline{p}} + b\tfrac{\phi(\pi-\underline{p})[\phi(\pi-\underline{p})+r(p-\underline{p})]}{(r+\phi)\underline{p}^2(1-\underline{p})} - br(1-p)\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{p}{1-p}\right)$$

and

$$v_x'(p) = b\tfrac{r(p-\underline{p})}{\underline{p}p} + b\tfrac{r\phi(\pi-\underline{p})}{(r+\phi)\underline{p}^2(1-\underline{p})} + br\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{p}{1-p}\right).$$

It follows that

$$\begin{aligned}&v_x(p) - \tfrac{\phi}{r}(\pi-p)v_x'(p)\\ &\quad= b\tfrac{[rp-\phi(\pi-p)](p-\underline{p})}{\underline{p}p} + b\tfrac{\phi(\pi-\underline{p})(p-\underline{p})}{\underline{p}^2(1-\underline{p})} - b[\phi(\pi-p)+r(1-p)]\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{p}{1-p}\right).\end{aligned}$$

It follows from $b_{v_x}(p) = c(p)$ that

$$v_x(p) - \tfrac{\phi}{r}(\pi-p)v_x'(p) = (I-1)x(p)c(p),$$

and so that

$$x(p) = \frac{b\tfrac{[rp-\phi(\pi-p)](p-\underline{p})}{\underline{p}p} + b\tfrac{\phi(\pi-\underline{p})(p-\underline{p})}{\underline{p}^2(1-\underline{p})} - b[\phi(\pi-p)+r(1-p)]\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{p}{1-p}\right)}{(I-1)[b(1-p)-gp]}.$$

Note that $x(\underline{p}) = 0$. It is left to argue that $x(p)$ increases in $p$ on $[\underline{p}, \bar{p}]$. Its first derivative is equal to

$$x'(p) = \frac{b(r+\phi)\left(\tfrac{1}{\underline{p}}-\tfrac{1}{p}\right) + b\tfrac{\phi(\pi-\underline{p})}{\underline{p}^2(1-\underline{p})} - b\tfrac{\phi(\pi-p)}{p^2(1-p)} + b(r+\phi)\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{p}{1-p}\right)}{(I-1)[b(1-p)-gp]} + \tfrac{g+b}{[b(1-p)-gp]}x(p).$$

Furthermore, $x(\bar{p}) = 1$ implies

$$\begin{aligned}&b\tfrac{[r\bar{p}-\phi(\pi-\bar{p})](\bar{p}-\underline{p})}{\underline{p}\bar{p}} + b\tfrac{\phi(\pi-\underline{p})(\bar{p}-\underline{p})}{\underline{p}^2(1-\underline{p})} - b[\phi(\pi-\bar{p})+r(1-\bar{p})]\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{\bar{p}}{1-\bar{p}}\right)\\ &\qquad= (I-1)[b(1-\bar{p})-g\bar{p}]. \quad \text{(B.28)}\end{aligned}$$

Equation (B.27) and (B.28) pin down $\underline{p}$ and $\bar{p}$ in this case. In this case, $\underline{p} \in (\alpha_1, \pi)$. It is left to argue that the solution to (B.27) and (B.28) exists and is unique. It is also left to show that $\bar{p} < p_M$.

Case (iv): The value-matching and the smooth-pasting at $\underline{p}$ imply $v_x(\underline{p}) = v_0(\underline{p}) = 0$ and $v_x'(\underline{p}) = v_0'(\underline{p}) = 0$. It follows from $b_{v_0}(\underline{p}) = c(\underline{p})$ that

$$v_1(1) = r\tfrac{b-(g+b)\underline{p}}{\underline{p}},$$

or equivalently that

$$\underline{p} = \tfrac{br}{v_1(1)+r(g+b)}.$$

Therefore,

$$C_1(\beta_I - 1)\left(\tfrac{\beta_I-1}{1-\alpha_I}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}} = \left(r\tfrac{b-(g+b)\underline{p}}{\underline{p}} + (g+b)\tfrac{r}{I} + b\right)\tfrac{r(r+\phi)}{r^2+r(I+\phi)+I\phi\pi} - (g+b)\tfrac{r}{I}.$$

It follows that

$$\left(r\frac{b-(g+b)\underline{p}}{\underline{p}}+(g+b)\frac{r}{I}+b\right)\frac{r(r+\phi)}{r^2+r(I+\phi)+I\phi\pi}-(g+b)\frac{r}{I}$$
$$=r\frac{b-(g+b)\bar{p}}{\bar{p}\left(1-\frac{\phi(\pi-\bar{p})+r(1-\bar{p})}{\phi(\pi-\bar{p})-I\bar{p}(1-\bar{p})}\frac{\beta_I-\bar{p}}{\beta_I-1}\left(\frac{\beta_I-\bar{p}}{\bar{p}-\alpha_I}\frac{1-\alpha_I}{\beta_I-1}\right)^{\frac{\frac{r}{I}+\alpha_I}{\beta_I-\alpha_I}}\right)}. \quad \text{(B.29)}$$

The smooth-pasting at $\underline{p}$ implies $v_x'(\underline{p})=v_0'(\underline{p})=0$, and so

$$C=-br\ln\left(\frac{1-\underline{p}}{\underline{p}}\right)-br\frac{1}{\underline{p}}.$$

Therefore,

$$v_x(p)=v_1(1)+gr-br\frac{1-\underline{p}}{\underline{p}}-br(1-p)\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right)=b\frac{r(p-\underline{p})}{\underline{p}}-br(1-p)\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right),$$
$$v_x'(p)=b\frac{r(p-\underline{p})}{\underline{p}p}+br\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right),$$

and so

$$v_x(p)-\frac{\phi}{r}(\pi-p)v_x'(p)=b\frac{[rp-\phi(\pi-p)](p-\underline{p})}{\underline{p}p}-b[\phi(\pi-p)+r(1-p)]\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right).$$

It follows from $b_{v_x}(p)=c(p)$ that

$$v_x(p)-\frac{\phi}{r}(\pi-p)v_x'(p)=(I-1)x(p)c(p),$$

and so that

$$x(p)=\frac{b\frac{[rp-\phi(\pi-p)](p-\underline{p})}{\underline{p}p}-b[\phi(\pi-p)+r(1-p)]\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right)}{(I-1)[b(1-p)-gp]}.$$

Note that $x(\underline{p})=0$. The allocation $x(p)$ increases in $p$ on $[\underline{p},\bar{p}]$. Indeed, its first derivative is equal to

$$x'(p)=\frac{b(r+\phi)\left(\frac{1}{\underline{p}}-\frac{1}{p}\right)-b\frac{\phi(\pi-p)}{p^2(1-p)}+b(r+\phi)\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right)}{(I-1)[b(1-p)-gp]}+\frac{g+b}{[b(1-p)-gp]}x(p)\geq 0,$$

because $\underline{p}>\pi$, and with equality when $p=\underline{p}$. Furthermore, $x(\bar{p})=1$ implies

$$b\frac{[r\bar{p}-\phi(\pi-\bar{p})](\bar{p}-\underline{p})}{\underline{p}\bar{p}}-b[\phi(\pi-\bar{p})+r(1-\bar{p})]\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{\bar{p}}{1-\bar{p}}\right)=(I-1)[b(1-\bar{p})-g\bar{p}]. \quad \text{(B.30)}$$

Equations (B.29) and (B.30) pin down $\underline{p}$ and $\bar{p}$ in this case. Note that (B.27) and (B.28) coincide with (B.29) and (B.30) when $\underline{p}=\pi$. In this case, $\underline{p}>\pi$. It is left to argue that the solution to (B.29) and (B.30) exists and is unique. It is also left to show that $\bar{p}<p_M$.

**Proof of Proposition 2.7**

Case (i): If $p_M>1-\frac{\phi}{I}$, then $\underline{p}=\bar{p}=p_M$. If players use the specified strategy, then each player's value function is equal to

$$v^e(p)=\begin{cases} g-\left((g+b)\frac{r}{I}+g\right)\frac{I(\phi+r(1-p))}{(I+r)(\phi+r)}+C_1p\left(\frac{p}{p-(1-\phi I)}\right)^{\frac{r+\phi}{I-\phi}} & \text{if } p>\bar{p},\\ 0 & \text{if } p\leq\bar{p},\end{cases}$$

where

$$C_1\bar{p}\left(\frac{\bar{p}}{\bar{p}-(1-\phi_I)}\right)^{\frac{r+\phi}{I-\phi}} = (g+b)\frac{r\bar{p}(I\bar{p}-I+\phi)}{(I+r)(\phi+r)}.$$

Case (ii): If $1-\frac{\phi}{I} > p_M > \frac{(I-1-\phi)(I+r)}{(I-1)(I+r)-\phi}$, then $\underline{p} = \bar{p} \in \left(1-\frac{\phi}{I-1}, 1-\frac{\phi}{I}\right)$ and

$$\underline{p} = \bar{p} = \frac{b(\phi+r)}{br+g(I+r)}.$$

Furthermore, $\bar{p} < p_M$. If players use the specified strategy, then each player's value function is equal to

$$v^e(p) = \begin{cases} g - \left((g+b)\frac{r}{I}+g\right)\frac{I(\phi+r(1-p))}{(I+r)(\phi+r)} & \text{if } p > \bar{p}, \\ 0 & \text{if } p \leq \bar{p}. \end{cases}$$

Case (iii): If $\frac{(I-1-\phi)(I+r)}{(I-1)(I+r)-\phi} > p_M$, then $\underline{p} < \bar{p}$, where $\underline{p} < 1-\frac{\phi}{I-1}$ and $\bar{p}$ is given by

$$\bar{p} = \frac{b(I+r-1)}{(g+b)(I+r)-b},$$

while $\underline{p}$ solves

$$-br + b(r+\phi)\frac{\bar{p}}{\underline{p}} - g\phi\frac{\bar{p}}{1-\bar{p}} - g(r+\phi)\bar{p}\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{\bar{p}}{1-\bar{p}}\right) = (I-1)[b(1-\bar{p}) - g\bar{p}]. \qquad \text{(B.31)}$$

Such $\underline{p}$ exists and is unique. Furthermore, $\bar{p} < p_M$. See details below. If players use the specified strategy with

$$x^e(p) = \frac{-br+b(r+\phi)\frac{p}{\underline{p}} - g\phi\frac{p}{1-p} - g(r+\phi)p\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right)}{(I-1)[b(1-p)-gp]} \qquad \text{(B.32)}$$

for $p \in [\underline{p}, \bar{p}]$, then each player's value function is equal to

$$v^e(p) = \begin{cases} g - \left((g+b)\frac{r}{I}+g\right)\frac{I[\phi+r(1-p)]}{(I+r)(\phi+r)} & \text{if } p > \bar{p}, \\ br\frac{p-\underline{p}}{\underline{p}} - grp\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right) & \text{if } p \in [\underline{p}, \bar{p}], \\ 0 & \text{if } p < \underline{p}. \end{cases}$$

In each case, $b_{v^e}(p) > c(p)$ for $p > \bar{p}$, $b_{v^e}(p) < c(p)$ for $p < \underline{p}$, and $b_{v^e}(p) = c(p)$ for $p \in [\underline{p}, \bar{p}]$ for the specified $v^e$. Therefore, $v^e$ solves the HJB equation (B.17), and so it is the value function in the strategic problem. Because the strategy specified in the proposition achieves the maximum in the HJB equation, this strategy is optimal.

**Details.** Suppose no resource is allocated to $R$, i.e., $x = 0$. The function $v_0(p)$ that solves

$$v(p) = -\frac{\phi}{r}pv'(p)$$

is given by

$$v_0(p) = \frac{C_0}{p^{\frac{r}{\phi}}},$$

where $C_0$ is a constant of integration. If players do not experiment, then their belief drifts down toward $0$. It follows that $C_0 = 0$, and so $v_0(p) = 0$. If the belief is $0$, then players allocate no resource to $R$. Therefore, $v(0) = v_0(0) = 0$.

Suppose each player allocates the whole resource to $R$, i.e., $x = 1$. The function $v_1(p)$ that solves

$$v(p) = -\tfrac{\phi}{r} p v'(p) + \tfrac{I}{r}(1-p)[-v(p) + p v'(p)] - [(1-p)b - pg]$$

is given by

$$v_1(p) = \underbrace{g - \left((g+b)\tfrac{r}{I} + g\right) \tfrac{I(\phi + r(1-p))}{(I+r)(\phi+r)}}_{=:w_1(p)} + C_1 p \left(\tfrac{p}{p-(1-\phi_I)}\right)^{\frac{r+\phi}{I-\phi}},$$

where $\phi_I := \frac{\phi}{I}$ and $C_1$ is a constant of integration. The constant $C_1$ is pinned down by appropriate boundary conditions. In particular, if $x = 1$ for $p \leq 1 - \frac{\phi}{I}$, then $C_1 = 0$, and so $v_1(p) = w_1(p)$. Furthermore,

$$v_1'(p) = \left((g+b)\tfrac{r}{I} + g\right) \tfrac{Ir}{(I+r)(r+\phi)} - C_1 \tfrac{I+r-Ip}{Ip-I+\phi} \left(\tfrac{p}{p-(1-\phi_I)}\right)^{\frac{r+\phi}{I-\phi}},$$

$$v_1''(p) = C_1 \tfrac{(I+r)(\phi+r)}{p(Ip-I+\phi)^2} \left(\tfrac{p}{p-(1-\phi_I)}\right)^{\frac{r+\phi}{I-\phi}}.$$

Note that

$$-w_1(p) + p w_1'(p) = b\tfrac{r}{I+r}.$$

Suppose each player allocates only a fraction of her resource to $R$, i.e., $x \in (0,1)$. The function $v_x(p)$ that solves $b_v(p) = c(p)$, that is,

$$\tfrac{1}{r}(1-p)[-v(p) + p v'(p)] = (1-p)b - pg,$$

is given by

$$v_x(p) = -br + grp \ln\left(\tfrac{1-p}{p}\right) + Cp,$$

where $C$ is a constant of integration. The constant $C$ is pinned down by appropriate boundary conditions. It follows that

$$v_x'(p) = gr \ln\left(\tfrac{1-p}{p}\right) - gr\tfrac{1}{1-p} + C,$$

$$v_x''(p) = -g\tfrac{r}{p(1-p)^2}.$$

Case (i): The value-matching and the smooth-pasting at $\underline{p} = \bar{p}$ imply $v_1(\bar{p}) = v_0(\bar{p}) = 0$ and $v_1'(\bar{p}) = v_1'(\bar{p}) = 0$. Therefore, $b_{v_1}(\bar{p}) = 0$. It follows from $b_{v_1}(\bar{p}) = c(\bar{p})$ that $\bar{p} = p_M$. Furthermore, $v_1(\bar{p}) = 0$ implies

$$C_1 \bar{p} \left(\tfrac{\bar{p}}{\bar{p}-(1-\phi_I)}\right)^{\frac{r+\phi}{I-\phi}} = (g+b)\tfrac{r\bar{p}(I\bar{p}-I+\phi)}{(I+r)(\phi+r)}.$$

Case (ii): The value-matching at $\underline{p} = \bar{p}$ implies $w_1(\bar{p}) = v_0(\bar{p}) = 0$, and so

$$\bar{p} = \tfrac{b(\phi+r)}{br+g(I+r)} = \tfrac{p_M(\phi+r)}{I(1-p_M)+r}.$$

Note that $\bar{p} \le p_M$ if and only if $p_M \le 1 - \frac{\phi}{I}$. It follows that

$$b_{w_1}(\bar{p}) - c(\bar{p}) = \tfrac{1}{r}(1-\bar{p})b\tfrac{r}{I+r} - [b(1-\bar{p}) - g\bar{p}] = b^2\tfrac{\phi(I+r-1)}{(I+r)(br+g(I+r))} - gb\tfrac{I-1-\phi}{br+g(I+r)}$$

Therefore, $b_{w_1}(\bar{p}) - c(\bar{p}) \ge 0$ if and only if

$$b\phi(I+r-1) - g(I-1-\phi)(I+r) \ge 0,$$

or equivalently

$$[(I-1)(I+r) - \phi]p_M \ge (I-1-\phi)(I+r).$$

In this case, $1 - \frac{\phi}{I} \ge \bar{p} \ge 1 - \frac{\phi}{I-1}$. Therefore, it must be the case that $I - \phi \ge 0$, and so $(I-1)(I+r) - \phi > 0$. It follows that $b_w(\bar{p}) - c(\bar{p}) \ge 0$ if and only if

$$p_M \ge \tfrac{(I-1-\phi)(I+r)}{(I-1)(I+r)-\phi}. \tag{B.33}$$

Case (iii): It follows from $b_{w_1}(\bar{p}) = c(\bar{p})$ that

$$\tfrac{1}{r}(1-\bar{p})b\tfrac{r}{I+r} = (1-\bar{p})b + \bar{p}g,$$

and so that

$$1 - \bar{p} = \tfrac{g(I+r)}{(g+b)(I+r)-b}$$

or

$$\bar{p} = \tfrac{b(I+r-1)}{(g+b)(I+r)-b} = \tfrac{p_M(I+r-1)}{I+r-p_M}.$$

It follows that $\bar{p} \le p_M$. Note that $w_1(\bar{p}) \ge 0$ if and only if

$$g - \left((g+b)\tfrac{r}{I} + g\right)\frac{I\left(\phi + r\frac{g(I+r)}{(g+b)(I+r)-b}\right)}{(I+r)(\phi+r)} \ge 0.$$

Multiplying both side by $(I+r)(\phi+r)[(g+b)(I+r)-b] > 0$ and simplifying the left side yield

$$br(b\phi + (I+r)[g(I-1) - (g+b)\phi]) \ge 0.$$

Therefore,

$$b\phi + (I+r)[g(I-1) - (g+b)\phi] \ge 0,$$

or equivalently

$$(I-1-\phi)(I+r) \ge [(I-1)(I+r) - \phi]p_M.$$

In this case, $\bar{p} \le 1 - \frac{\phi}{I-1}$. Therefore, it must be that $I-1-\phi \ge 0$, and so $(I-1)(I+r) - \phi > 0$. It follow that $w(\bar{p}) \ge 0$ if and only if

$$p_M \le \tfrac{(I-1-\phi)(I+r)}{(I-1)(I+r)-\phi}.$$

Compare with the condition (B.33). The value-matching at $\underline{p}$ implies $v_x(\underline{p}) = v_0(\underline{p}) = 0$, and so

$$C = br\tfrac{1}{\underline{p}} - gr\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\right).$$

Therefore,

$$v_x(p) = br\tfrac{p-\underline{p}}{\underline{p}} - grp\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{p}{1-p}\right),$$

and so

$$v_x(p) + \tfrac{\phi}{r} p v_x'(p) = -br + b(r+\phi)\tfrac{p}{\underline{p}} - g\phi\tfrac{p}{1-p} - g(r+\phi)p\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{p}{1-p}\right).$$

It follows from $b_{v_x}(p) = c(p)$ that

$$v_x(p) + \tfrac{\phi}{r} p v_x'(p) = (I-1)x(p)c(p),$$

and so that

$$x(p) = \frac{-br + b(r+\phi)\frac{p}{\underline{p}} - g\phi\frac{p}{1-p} - g(r+\phi)p\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right)}{(I-1)[b(1-p)-gp]}.$$

It is left to show that $x(p)$ increases in $p$ for $p \in [\underline{p}, \bar{p}]$. Note that

$$x(\underline{p}) = \tfrac{\phi}{(I-1)(1-\underline{p})} > 0.$$

At $\bar{p}$, $x(\bar{p}) = 1$, i.e.,[7]

$$-br + b(r+\phi)\tfrac{\bar{p}}{\underline{p}} - g\phi\tfrac{\bar{p}}{1-\bar{p}} - g(r+\phi)\bar{p}\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{\bar{p}}{1-\bar{p}}\right) = (I-1)[b(1-\bar{p}) - g\bar{p}]. \tag{B.34}$$

Fix $\bar{p}$. Define

$$\begin{aligned} h_L(p) &:= -br + b(r+\phi)\tfrac{\bar{p}}{p} - g\phi\tfrac{\bar{p}}{1-\bar{p}} - g(r+\phi)\bar{p}\ln\left(\tfrac{1-p}{p}\tfrac{\bar{p}}{1-\bar{p}}\right), \\ h_R(p) &:= (I-1)[b(1-\bar{p}) - g\bar{p}]. \end{aligned}$$

Given $\bar{p}$, there exists unique $\underline{p} \in [0, \bar{p}]$ that solves (B.34) if and only if $h_L(p)$ and $h_R(p)$ intersect once on $[0, \bar{p}]$. The function $h_R(p)$ is constant for all $p$. The function $h_L(p)$ decreases in $p$ for $p \le \bar{p} \le p_M$. Indeed, its first derivative is equal to

$$h_L'(p) = -(r+\phi)\bar{p}\tfrac{b(1-p)-gp}{p^2(1-p)} < 0$$

for $p < \bar{p} \le p_M$. Furthermore, $\lim_{p\to 0} h_L(p) = \infty$, and so $h_L(p)$ is above $h_R(p)$ as $p$ goes to $0$. Moreover,

$$h_L(\bar{p}) = \phi\tfrac{b(1-\bar{p})-g\bar{p}}{1-\bar{p}} < h_R(\bar{p})$$

for $\bar{p} \le p_M$ and $\bar{p} \le 1 - \frac{\phi}{I-1}$. Therefore, there exists and is unique $\underline{p} \in (0, \bar{p})$ that solves (B.34).

**Proof of Proposition 2.8**

The cut-offs are $\underline{p} < \bar{p}$, where $\underline{p}$ and $\bar{p}$ solve

$$1 - \bar{p} = \frac{gr(I+r+\phi)}{(g+b)r(I+r+\phi) + (g+b)\phi - (C_0+b)(r+\phi)} \tag{B.35}$$

and

$$b\frac{r(\bar{p}-\underline{p}) - \phi(1-\bar{p})}{\underline{p}} + g\phi - g[r\bar{p} - \phi(1-\bar{p})]\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{\bar{p}}{1-\bar{p}}\right)$$

[7]Note that (B.34) is a special case with $\pi = 0$ of (B.44) below.

$$+ C_0 + C_0 \frac{r\bar{p}-\phi(1-\bar{p})}{r\underline{p}} \left( -1 + \frac{1}{(1-\underline{p})^{\frac{r}{\phi}}} \right) = (I-1)[b(1-\bar{p}) - g\bar{p}], \quad \text{(B.36)}$$

where

$$C_0 = \frac{b(1-\underline{p})-g\underline{p}}{1-\underline{p}} \frac{r}{1+\frac{r\underline{p}-\phi(1-\underline{p})}{\phi(1-\underline{p})^{\frac{r}{\phi}+1}}}.$$

Such $\underline{p}$ and $\bar{p}$ exist and are unique. Furthermore, $\bar{p} < p_M$. See details below. If players use the specified strategy with

$$x^e(p) = \frac{b\frac{r(p-\underline{p})-\phi(1-p)}{\underline{p}} + g\phi - g[rp-\phi(1-p)] \ln\left(\frac{1-\underline{p}}{\underline{p}} \frac{p}{1-p}\right) + C_0 + C_0 \frac{rp-\phi(1-p)}{r\underline{p}} \left( -1 + \frac{1}{(1-\underline{p})^{\frac{r}{\phi}}} \right)}{(I-1)[b(1-p)-gp]} \quad \text{(B.37)}$$

for $p \in [\underline{p}, \bar{p}]$, then each player's value function is equal to

$$v^e(p) = \begin{cases} g - \left(-C_0 + (g+b)\frac{r}{I} + g\right) \frac{I(1-p)}{I+r+\phi} & \text{if } p > \bar{p}, \\ C_0 + br\frac{p-\underline{p}}{\underline{p}} - grp \ln\left(\frac{1-\underline{p}}{\underline{p}} \frac{p}{1-p}\right) - C_0 \frac{p}{\underline{p}} + \frac{C_0}{(1-\underline{p})^{\frac{r}{\phi}}} \frac{p}{\underline{p}} & \text{if } p \in [\underline{p}, \bar{p}], \\ \frac{C_0}{(1-p)^{\frac{r}{\phi}}} & \text{if } p < \underline{p}. \end{cases}$$

Furthermore, $b_{v^e}(p) > c(p)$ for $p > \bar{p}$, $b_{v^e}(p) < c(p)$ for $p < \underline{p}$, and $b_{v^e}(p) = c(p)$ for $p \in [\underline{p}, \bar{p}]$ for the specified $v^e$. Therefore, $v^e$ solves the HJB equation (B.17), and so it is the value function in the strategic problem. Because the strategy specified in the proposition achieves the maximum in the HJB equation, this strategy is optimal.

**Details.** Suppose no resource is allocated to $R$, i.e., $x = 0$. The function $v_0(p)$ that solves

$$v(p) = \tfrac{\phi}{r}(1-p)v'(p)$$

is given by

$$v_0(p) = \frac{C_0}{(1-p)^{\frac{r}{\phi}}},$$

where $C_0$ is a constant of integration. The constant $C_0$ is pinned down by appropriate boundary conditions. It follows that

$$v_0'(p) = \frac{r}{\phi} \frac{C_0}{(1-p)^{\frac{r}{\phi}+1}} = \frac{r}{\phi(1-p)} v_0(p),$$

$$v_0''(p) = \frac{r(r+\phi)}{\phi^2} \frac{C_0}{(1-p)^{\frac{r}{\phi}+2}} = \frac{r+\phi}{\phi(1-p)} v_0'(p) = \frac{r(r+\phi)}{\phi^2(1-p)^2} v_0(p).$$

Furthermore, if the belief is 0, then players allocate no resource to $R$. Therefore, $v(0) = v_0(0) = C_0$.

Suppose each player allocates the whole resource to $R$, i.e., $x = 1$. The function $v_1(p)$ that solves

$$v(p) = \tfrac{\phi}{r}(1-p)v'(p) + \tfrac{I}{r}(1-p)[C_0 - v(p) + pv'(p)] - [(1-p)b - pg]$$

is given by

$$v_1(p) = \underbrace{g - \left(-C_0 + (g+b)\tfrac{r}{I} + g\right) \tfrac{I(1-p)}{I+r+\phi}}_{=:w_1(p)} + C_1(\phi_I + p)\left(\tfrac{\phi_I+p}{1-p}\right)^{\frac{r}{I+\phi}},$$

where $C_1$ is a constant of integration. If players experiment, then their belief drifts toward $1$ in absence of news. It follows that $C_1 = 0$, and so $v_1(p) = w_1(p)$. Note that

$$C_0 - w_1(p) + pw_1'(p) = (C_0 - g)\tfrac{\phi}{I+r+\phi} + (C_0 + b)\tfrac{r}{I+r+\phi}.$$

Suppose each player allocates only a fraction of her resource to $R$, i.e., $x \in (0,1)$. The function $v_x(p)$ that solves $b_v(p) = c(p)$, i.e.,

$$\tfrac{1}{r}(1-p)[C_0 - v(p) + pv'(p)] = (1-p)b - pg,$$

is given by

$$v_x(p) = C_0 - br + grp\ln\left(\tfrac{1-p}{p}\right) + Cp,$$

where $C$ is a constant of integration. The constant $C$ is pinned down by appropriate boundary conditions. It follows that

$$\begin{aligned} v_x'(p) &= gr\ln\left(\tfrac{1-p}{p}\right) - gr\tfrac{1}{1-p} + C, \\ v_x''(p) &= -g\tfrac{r}{p(1-p)^2}. \end{aligned}$$

It follows from $b_{w_1}(\bar{p}) = c(\bar{p})$ that

$$\tfrac{1}{r}(1-\bar{p})\left(C_0\tfrac{r+\phi}{I+r+\phi} + b\tfrac{r}{I+r+\phi} - g\tfrac{\phi}{I+r+\phi}\right) = (1-\bar{p})b - g\bar{p},$$

and so that

$$1 - \bar{p} = \tfrac{gr(I+r+\phi)}{(g+b)r(I+r+\phi)+(g+b)\phi-(C_0+b)(r+\phi)},$$

i.e.,

$$\bar{p} = \tfrac{br(I+r+\phi)+(g+b)\phi-(C_0+b)(r+\phi)}{(g+b)r(I+r+\phi)+(g+b)\phi-(C_0+b)(r+\phi)} = \tfrac{p_M r(I+r+\phi)+\phi-\left(\frac{C_0}{g+b}+p_M\right)(r+\phi)}{r(I+r+\phi)+\phi-\left(\frac{C_0}{g+b}+p_M\right)(r+\phi)}. \tag{B.38}$$

Note that the right side of (B.38) decreases in $C_0$. The value-matching at $\underline{p}$ implies $v_0(\underline{p}) = v_x(\underline{p})$, i.e.,

$$\tfrac{C_0}{(1-\underline{p})^{\frac{r}{\phi}}} = C_0 - br + gr\underline{p}\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\right) + C\underline{p},$$

and so

$$C = br\tfrac{1}{\underline{p}} - gr\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\right) - C_0\tfrac{1}{\underline{p}} + \tfrac{C_0}{(1-\underline{p})^{\frac{r}{\phi}}}\tfrac{1}{\underline{p}}.$$

The smooth-pasting at $\underline{p}$ implies $v_0'(\underline{p}) = v_x'(\underline{p})$, i.e.,

$$\tfrac{r}{\phi}\tfrac{C_0}{(1-\underline{p})^{\frac{r}{\phi}+1}} = gr\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\right) - gr\tfrac{1}{1-\underline{p}} + C,$$

and so

$$C = -gr\ln\left(\frac{1-\underline{p}}{\underline{p}}\right) + gr\frac{1}{1-\underline{p}} + \frac{r}{\phi}\frac{C_0}{(1-\underline{p})^{\frac{r}{\phi}+1}}.$$

It follows that

$$C_0 = \frac{b(1-\underline{p})-g\underline{p}}{1-\underline{p}}\frac{r}{1+\frac{r\underline{p}-\phi(1-\underline{p})}{\phi(1-\underline{p})^{\frac{r}{\phi}+1}}} = (g+b)\frac{p_M-\underline{p}}{1-\underline{p}}\frac{r}{1+\frac{r\underline{p}-\phi(1-\underline{p})}{\phi(1-\underline{p})^{\frac{r}{\phi}+1}}}. \tag{B.39}$$

Note that $C_0$ decreases in $\underline{p}$. Indeed, the first derivative of $\frac{C_0}{g+b}$ with respect to $\underline{p}$ is equal to

$$-\frac{1-p_M}{(1-\underline{p})^2}\frac{r}{1+\frac{r\underline{p}-\phi(1-\underline{p})}{\phi(1-\underline{p})^{\frac{r}{\phi}+1}}} - \frac{p_M-\underline{p}}{1-\underline{p}}\frac{r}{\left(1+\frac{r\underline{p}-\phi(1-\underline{p})}{\phi(1-\underline{p})^{\frac{r}{\phi}+1}}\right)^2}\frac{r(r+\phi)\underline{p}}{\phi^2(1-\underline{p})^{\frac{r}{\phi}+2}} < 0$$

for $\underline{p} < p_M$. Furthermore,

$$v_x(p) = C_0 + br\frac{p-\underline{p}}{\underline{p}} - grp\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right) - C_0\frac{p}{\underline{p}} + \frac{C_0}{(1-\underline{p})^{\frac{r}{\phi}}}\frac{p}{\underline{p}},$$

$$v_x'(p) = br\frac{1}{\underline{p}} - gr\frac{1}{1-p} - gr\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right) - C_0\frac{1}{\underline{p}} + \frac{C_0}{(1-\underline{p})^{\frac{r}{\phi}}}\frac{1}{\underline{p}},$$

and so

$$v_x(p) - \frac{\phi}{r}(1-p)v_x'(p) = b\frac{r(p-\underline{p})-\phi(1-p)}{\underline{p}} + g\phi - g[rp-\phi(1-p)]\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right) + C_0 + C_0\frac{rp-\phi(1-p)}{r\underline{p}}\left(-1+\frac{1}{(1-\underline{p})^{\frac{r}{\phi}}}\right).$$

It follows from $b_{v_x}(p) = c(p)$ that

$$v_x(p) - \frac{\phi}{r}(1-p)v_x'(p) = (I-1)x(p)c(p),$$

so that

$$x(p) = \frac{b\frac{r(p-\underline{p})-\phi(1-p)}{\underline{p}} + g\phi - g[rp-\phi(1-p)]\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right) + C_0 + C_0\frac{rp-\phi(1-p)}{r\underline{p}}\left(-1+\frac{1}{(1-\underline{p})^{\frac{r}{\phi}}}\right)}{(I-1)[b(1-p)-gp]}.$$

It is left to show that $x(p)$ is increasing in $p$ for $p \in [\underline{p},\bar{p}]$. Note that $x(\underline{p}) = 0$. At $\bar{p}$, $x(\bar{p}) = 1$, i.e.,

$$b\frac{r(\bar{p}-\underline{p})-\phi(1-\bar{p})}{\underline{p}} + g\phi - g[r\bar{p}-\phi(1-\bar{p})]\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{\bar{p}}{1-\bar{p}}\right) + C_0 + C_0\frac{r\bar{p}-\phi(1-\bar{p})}{r\underline{p}}\left(-1+\frac{1}{(1-\underline{p})^{\frac{r}{\phi}}}\right) = (I-1)[b(1-\bar{p})-g\bar{p}],$$

which can be rewritten as follows:

$$p_M\frac{r(\bar{p}-\underline{p})-\phi(1-\bar{p})}{\underline{p}} + (1-p_M)\phi - (1-p_M)[r\bar{p}-\phi(1-\bar{p})]\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{\bar{p}}{1-\bar{p}}\right)$$

$$+\tfrac{C_0}{g+b}+\tfrac{C_0}{g+b}\tfrac{r\bar{p}-\phi(1-\bar{p})}{r\underline{p}}\left(-1+\frac{1}{(1-\underline{p})^{\frac{r}{\phi}}}\right)=(I-1)(p_M-\bar{p}), \quad \text{(B.40)}$$

Given $C_0$ defined by (B.39), equations (B.38) and (B.40) pin down $\underline{p}$ and $\bar{p}$. It is left to argue that the solution to (B.38) and (B.40) exists and is unique.

**Proof of Proposition B.4**

Case (i): If $p_M > \alpha_I$, then $\underline{p} = \bar{p} = p_M$. If players use the specified strategy, then each player's value function is equal to

$$v^e(p) = \begin{cases} g - \left((g+b)\frac{r}{I}+g\right)\frac{I(\phi(1-\pi)+r(1-p))}{r^2+r(I+\phi)+I\phi(1-\pi)} + C_1(p-\beta_I)\left(\frac{p-\beta_I}{p-\alpha_I}\right)^{\frac{\frac{r}{I}+1-\alpha_I}{\alpha_I-\beta_I}} & \text{if } p > \bar{p}, \\ 0 & \text{if } p \le \bar{p}, \end{cases}$$

where

$$C_1(\bar{p}-\beta_I)\left(\tfrac{\bar{p}-\beta_I}{\bar{p}-\alpha_I}\right)^{\frac{\frac{r}{I}+1-\alpha_I}{\alpha_I-\beta_I}} = (b+g)\tfrac{r\bar{p}(I\bar{p}-I+\phi)}{r^2+r(I+\phi)+I\phi(1-\pi)}.$$

Case (ii): If $\alpha_I > p_M > \frac{(I+r)\alpha_{I-1}+\phi\pi}{I\alpha_{I-1}+r+\phi}$, then $\underline{p} = \bar{p} \in (\alpha_{I-1}, \alpha_I)$ and

$$\underline{p} = \bar{p} = \tfrac{b[r+\phi(1-\pi)]-g\phi\pi}{Ig+(g+b)r}.$$

Furthermore, $\bar{p} < p_M$. If players use the specified strategy, then each player's value function is equal to

$$v^e(p) = \begin{cases} g - \left((g+b)\frac{r}{I}+g\right)\frac{I(\phi(1-\pi)+r(1-p))}{r^2+r(I+\phi)+I\phi(1-\pi)} & \text{if } p > \bar{p}, \\ 0 & \text{if } p \le \bar{p}. \end{cases}$$

Case (iii): If $\frac{(I+r)\alpha_{I-1}+\phi\pi}{I\alpha_{I-1}+r+\phi} > p_M$ and

$$p_M\tfrac{\phi(\pi-\bar{p})^2}{\pi\bar{p}} - p_M\tfrac{r(\pi-\bar{p})}{\pi} - (1-p_M)[r\bar{p}-\phi(\pi-\bar{p})]\ln\left(\tfrac{1-\pi}{\pi}\tfrac{\bar{p}}{1-\bar{p}}\right)$$
$$> (p_M-\bar{p})\tfrac{\phi(\pi-\bar{p})+(I-1)\bar{p}(1-\bar{p})}{\bar{p}(1-\bar{p})},$$

which gives the lower bound on $p_M$, then $\underline{p} < \bar{p}$, where $\underline{p} \in (\pi, \alpha_{I-1})$ and $\bar{p}$ is given by

$$\bar{p} = \tfrac{b[r^2+r(I+\phi)+I\phi(1-\pi)]+(g+b)\phi\pi-b(r+\phi)}{(g+b)[r^2+r(I+\phi)+I\phi(1-\pi)]+(g+b)\phi\pi-b(r+\phi)} = \tfrac{p_M[r^2+r(I+\phi)+I\phi(1-\pi)]+\phi\pi-p_M(r+\phi)}{r^2+r(I+\phi)+I\phi(1-\pi)+\phi\pi-p_M(r+\phi)},$$

while $\underline{p}$ solves

$$b\tfrac{r(\bar{p}-\underline{p})-\phi(\pi-\bar{p})}{\underline{p}} + g\tfrac{\phi(\pi-\bar{p})}{1-\bar{p}} - g[r\bar{p}-\phi(\pi-\bar{p})]\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{\bar{p}}{1-\bar{p}}\right) = (I-1)[b(1-\bar{p})-g\bar{p}].$$

Such $\underline{p}$ exists and is unique. Furthermore, $\bar{p} < p_M$. See details below. If players use the specified strategy with

$$x^e(p) = \frac{b\frac{r(p-\underline{p})-\phi(\pi-p)}{\underline{p}}+g\frac{\phi(\pi-p)}{1-p}-g[rp-\phi(\pi-p)]\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right)}{(I-1)[b(1-p)-gp]}$$

for $p \in [\underline{p}, \bar{p}]$, then each player's value function is equal to

$$v^e(p) = \begin{cases} g - \left((g+b)\frac{r}{I} + g\right) \frac{I(\phi(1-\pi)+r(1-p))}{r^2+r(I+\phi)+I\phi(1-\pi)} & \text{if } p > \bar{p}, \\ br\frac{p-\underline{p}}{\underline{p}} - grp\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right) & \text{if } p \in [\underline{p}, \bar{p}], \\ 0 & \text{if } p < \underline{p}. \end{cases}$$

Note that $x^e(\underline{p}) > 0$.

Case (iv): Otherwise, $\underline{p} < \bar{p}$, where $\underline{p} < \pi$ and $\underline{p}$ with $\bar{p}$ solve

$$1 - \bar{p} = \frac{g[r^2+r(I+\phi)+I\phi(1-\pi)]}{(g+b)[r^2+r(I+\phi)+I\phi(1-\pi)]+(g+b)\phi\pi-\left(\frac{C_0}{\pi^{\frac{r}{\phi}}}+b\right)(r+\phi)}$$

and

$$b\frac{r(\bar{p}-\underline{p})-\phi(\pi-\bar{p})}{\underline{p}} + g\frac{\phi(\pi-\bar{p})}{1-\bar{p}} - g[r\bar{p}-\phi(\pi-\bar{p})]\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{\bar{p}}{1-\bar{p}}\right)$$
$$+ \frac{C_0}{\pi^{\frac{r}{\phi}}} + \frac{C_0}{\pi^{\frac{r}{\phi}}}\frac{r\bar{p}-\phi(\pi-\bar{p})}{r\underline{p}}\left(-1+\left(\frac{\pi}{\pi-\underline{p}}\right)^{\frac{r}{\phi}}\right) = (I-1)[b(1-\bar{p})-g\bar{p}],$$

where

$$\frac{C_0}{\pi^{\frac{r}{\phi}}} = \frac{b(1-\underline{p})-g\underline{p}}{1-\underline{p}}\frac{r}{1+\frac{r\underline{p}-\phi(\pi-\underline{p})}{\phi(\pi-\underline{p})}\left(\frac{\pi}{\pi-\underline{p}}\right)^{\frac{r}{\phi}}}.$$

Such $\underline{p}$ and $\bar{p}$ exist and are unique. Furthermore, $\bar{p} < p_M$. See details below. If players use the specified strategy with

$$x^e(p) = \frac{1}{(I-1)[b(1-p)-gp]}\left[b\frac{r(p-\underline{p})-\phi(\pi-p)}{\underline{p}} + g\frac{\phi(\pi-p)}{1-p} - g[rp-\phi(\pi-p)]\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right)\right.$$
$$\left. + \frac{C_0}{\pi^{\frac{r}{\phi}}} + \frac{C_0}{\pi^{\frac{r}{\phi}}}\frac{rp-\phi(\pi-p)}{r\underline{p}}\left(-1+\left(\frac{\pi}{\pi-\underline{p}}\right)^{\frac{r}{\phi}}\right)\right]$$

for $p \in [\underline{p}, \bar{p}]$, then each player's value function is equal to

$$v^e(p) = \begin{cases} g - \left(-\frac{C_0}{\pi^{\frac{r}{\phi}}} + (g+b)\frac{r}{I} + g\right)\frac{I(\phi(1-\pi)+r(1-p))}{r^2+r(I+\phi)+I\phi(1-\pi)} & \text{if } p > \bar{p}, \\ \frac{C_0}{\pi^{\frac{r}{\phi}}} + br\frac{p-\underline{p}}{\underline{p}} - grp\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right) - \frac{C_0}{\pi^{\frac{r}{\phi}}}\frac{p}{\underline{p}} + \frac{C_0}{\pi^{\frac{r}{\phi}}}\left(\frac{\pi}{\pi-\underline{p}}\right)^{\frac{r}{\phi}}\frac{p}{\underline{p}} & \text{if } p \in [\underline{p}, \bar{p}], \\ \frac{C_0}{(\pi-p)^{\frac{r}{\phi}}} & \text{if } p < \underline{p}. \end{cases}$$

In each case, $b_{v^e}(p) > c(p)$ for $p > \bar{p}$, $b_{v^e}(p) < c(p)$ for $p < \underline{p}$, and $b_{v^e}(p) = c(p)$ for $p \in [\underline{p}, \bar{p}]$ for the specified $v^e$. Therefore, $v^e$ solves the HJB equation (B.17), and so it is the value function in the strategic problem. Because the strategy specified in the proposition achieves the maximum in the HJB equation, this strategy is optimal.

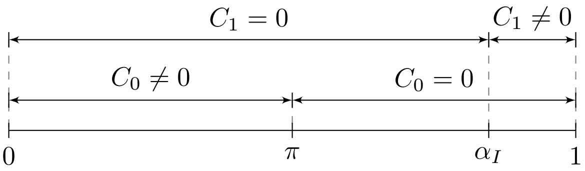


**Figure B.6.** Constants $C_0$ and $C_1$ depending on which region $\underline{p}$ belongs to. Parameters: $(I, \phi, \pi) = (2, 0.75, 0.5)$.

**Details.** Suppose no resource is allocated to $R$, i.e., $x = 0$. The function $v_0(p)$ that solves

$$v(p) = \tfrac{\phi}{r}(\pi - p)v'(p)$$

is given by

$$v_0(p) = \frac{C_0}{(\pi-p)^{\frac{r}{\phi}}},$$

where $C_0$ is a constant of integration. The constant $C_0$ is pinned down by appropriate boundary conditions. In particular, if $x = 0$ for $p \geq \pi$, then $C_0 = 0$ (see Figure B.6), and so $v_0(p) = 0$. Furthermore,

$$\begin{aligned} v_0'(p) &= \tfrac{r}{\phi}\frac{C_0}{(\pi-p)^{\frac{r}{\phi}+1}} = \tfrac{r}{\phi(\pi-p)}v_0(p), \\ v_0''(p) &= \tfrac{r(r+\phi)}{\phi^2}\frac{C_0}{(\pi-p)^{\frac{r}{\phi}+2}} = \tfrac{r+\phi}{\phi(\pi-p)}v_0'(p) = \tfrac{r(r+\phi)}{\phi^2(\pi-p)^2}v_0(p). \end{aligned}$$

Note that, if the belief is 0, then players allocate no resource to $R$. Therefore, $v(0) = v_0(0) = \frac{C_0}{\pi^{\frac{r}{\phi}}}$.

Suppose each players allocates the whole resource to $R$, i.e., $x = 1$. The function $v_1(p)$ that solves

$$v(p) = \tfrac{\phi}{r}(\pi - p)v'(p) + \tfrac{I}{r}(1-p)[v(0) - v(p) + pv'(p)] - [(1-p)b - pg]$$

is given by

$$v_1(p) = \underbrace{g - \left(-v_0(0) + (g+b)\tfrac{r}{I} + g\right)\tfrac{I(\phi(1-\pi)+r(1-p))}{r^2+r(I+\phi)+I\phi(1-\pi)}}_{=:w_1(p)} + C_1(p-\beta_I)\left(\tfrac{p-\beta_I}{p-\alpha_I}\right)^{\frac{\frac{r}{I}+1-\alpha_I}{\alpha_I-\beta_I}},$$

where $C_1$ is a constant of integration. The constant $C_1$ is pinned down by appropriate boundary conditions. In particular, if $x = 1$ for $p \leq \alpha_I$, then $C_1 = 0$ (see Figure B.6), and so $v_1(p) = w_1(p)$. Furthermore,

$$\begin{aligned} v_1'(p) &= \left(-v_0(0) + (g+b)\tfrac{r}{I} + g\right)\tfrac{Ir}{r^2+r(I+\phi)+I\phi(1-\pi)} - C_1\tfrac{I(1-p)+r}{I(p-\alpha_I)}\left(\tfrac{p-\beta_I}{p-\alpha_I}\right)^{\frac{\frac{r}{I}+1-\alpha_I}{\alpha_I-\beta_I}}, \\ v_1''(p) &= C_1\tfrac{(r+I(1-\alpha_I))(r+I(1-\beta_I))}{I^2(p-\alpha_I)^2(p-\beta_I)}\left(\tfrac{p-\beta_I}{p-\alpha_I}\right)^{\frac{\frac{r}{I}+1-\alpha_I}{\alpha_I-\beta_I}} = C_1\tfrac{r^2+r(I+\phi)+I\phi(1-\pi)}{I^2(p-\alpha_I)^2(p-\beta_I)}\left(\tfrac{p-\beta_I}{p-\alpha_I}\right)^{\frac{\frac{r}{I}+1-\alpha_I}{\alpha_I-\beta_I}}. \end{aligned}$$

Note that

$$v_0(0) - w_1(p) + pw_1'(p) = (v_0(0) + b)\tfrac{r(r+\phi(1-\pi))}{r^2+r(I+\phi)+I\phi(1-\pi)} + (v_0(0) - g)\tfrac{r\phi\pi}{r^2+r(I+\phi)+I\phi(1-\pi)}.$$

Suppose each player allocates only a fraction of her resource to $R$, i.e., $x \in (0, 1)$. The function $v_x(p)$ that solves $b_v(p) = c(p)$, i.e.,

$$\tfrac{1}{r}(1-p)[v(0) - v(p) + pv'(p)] = (1-p)b - pg,$$

is given by

$$v_x(p) = v_0(0) - br + grp\ln\left(\tfrac{1-p}{p}\right) + Cp,$$

where $C$ is a constant of integration. The constant $C$ is pinned down by appropriate boundary conditions. It follows that

$$\begin{aligned} v_x'(p) &= gr\ln\left(\tfrac{1-p}{p}\right) - gr\tfrac{1}{1-p} + C,\\ v_x''(p) &= -g\tfrac{r}{p(1-p)^2}. \end{aligned}$$

Case (i): The value-matching and the smooth-pasting at $\underline{p} = \bar{p}$ imply $v_1(\bar{p}) = v_0(\bar{p}) = 0$ and $v_1'(\bar{p}) = v_0'(\bar{p}) = 0$. Therefore, $b_{v_1}(\bar{p}) = 0$. It follows from $b_{v_1}(\bar{p}) = c(\bar{p})$ that $\bar{p} = p_M$. Furthermore, $v_1(\bar{p}) = 0$ implies

$$C_1(\bar{p} - \beta_I)\left(\tfrac{\bar{p}-\beta_I}{\bar{p}-\alpha_I}\right)^{\frac{\frac{r}{I}+1-\alpha_I}{\alpha_I-\beta_I}} = (b+g)\tfrac{r\bar{p}(I\bar{p}-I+\phi)}{r^2+r(I+\phi)+I\phi(1-\pi)}.$$

Case (ii): The value-matching at $\underline{p} = \bar{p}$ implies $w_1(\bar{p}) = v_0(\bar{p}) = 0$, and so

$$\bar{p} = \tfrac{-Iv_0(0)(r+\phi(1-\pi))+br(r+\phi(1-\pi))-gr\phi\pi}{Ir\left(-v_0(0)+(g+b)\frac{r}{I}+g\right)}.$$

Because $v_0(0) = 0$,

$$\bar{p} = \tfrac{b[r+\phi(1-\pi)]-g\phi\pi}{Ig+(g+b)r} = \tfrac{p_M(r+\phi)-\phi\pi}{I(1-p_M)+r}.$$

It follows that $\bar{p} \le p_M$ if and only if

$$0 \le \phi(\pi - p_M) + Ip_M(1-p_M) = -I(p_M - \alpha_I)(p_M - \beta_I).$$

That is, $\bar{p} \le p_M$ if and only if $p_M \le \alpha_I$, with equality when $p_M = \alpha_I$. Note that $\bar{p} \ge \pi$ if and only if

$$p_M \ge \tfrac{\pi(I+r+\phi)}{\pi I+r+\phi}.$$

Note that $\bar{p} \ge \alpha_{I-1}$ if and only if

$$p_M \ge \tfrac{(I+r)\alpha_{I-1}+\phi\pi}{I\alpha_{I-1}+r+\phi}. \tag{B.41}$$

Furthermore,

$$\begin{aligned} b_{w_1}(\bar{p}) - c(\bar{p}) &= \tfrac{1}{r}(1-\bar{p})\left(b\tfrac{r(r+\phi(1-\pi))}{r^2+r(I+\phi)+I\phi(1-\pi)} - g\tfrac{r\phi\pi}{r^2+r(I+\phi)+I\phi(1-\pi)}\right) - [b(1-\bar{p}) - g\bar{p}]\\ &= (g+b)\tfrac{\phi[p_M(I\pi+r+\phi)-\pi(I+r+\phi)]}{I[r^2+r(I+\phi)+I\phi(1-\pi)]} - (g+b)\tfrac{I-1}{I}\tfrac{\phi(\pi-p_M)+Ip_M(1-p_M)}{I(1-p_M)+r}, \end{aligned}$$

and $b_{w_1}(\bar{p}) - c(\bar{p}) \geq 0$ if and only if

$$\begin{aligned} &p_M^2[-I\phi(I\pi + r + \phi) + I(I-1)[r^2 + r(I+\phi) + I\phi(1-\pi)]] \\ &+ p_M[(I+r)\phi(I\pi + r + \phi) + I\phi\pi(I + r + \phi) + (I-1)(\phi - I)[r^2 + r(I+\phi) + I\phi(1-\pi)]] \\ &\qquad - \phi\pi(I+r)(I+r+\phi) - (I-1)\phi\pi[r^2 + r(I+\phi) + I\phi(1-\pi)] \geq 0. \end{aligned}$$

Divided by

$$-I\phi(I\pi + r + \phi) + I(I-1)[r^2 + r(I+\phi) + I\phi(1-\pi)] > 0, \tag{B.42}$$

where the sign must be shown, the last inequality can be rewritten as follows:[8]

$$\left(p_M - \tfrac{(I+r)\alpha_{I-1} + \phi\pi}{I\alpha_{I-1} + r + \phi}\right)\left(p_M - \tfrac{(I+r)\beta_{I-1} + \phi\pi}{I\beta_{I-1} + r + \phi}\right) \geq 0.$$

It follows from footnote 8 that

$$\tfrac{(I+r)\alpha_{I-1} + \phi\pi}{I\alpha_{I-1} + r + \phi} \in (\alpha_{I-1}, \alpha_I), \quad \tfrac{(I+r)\beta_{I-1} + \phi\pi}{I\beta_{I-1} + r + \phi} < 0.$$

Therefore, $b_{w_1}(\bar{p}) - c(\bar{p}) \geq 0$ if and only if

$$p_M \geq \tfrac{(I+r)\alpha_{I-1} + \phi\pi}{I\alpha_{I-1} + r + \phi}.$$

Case (iii): It follows from $b_{w_1}(\bar{p}) = c(\bar{p})$ that

$$\tfrac{1}{r}(1-\bar{p})\left(b\tfrac{r(r+\phi(1-\pi))}{r^2 + r(I+\phi) + I\phi(1-\pi)} - g\tfrac{r\phi\pi}{r^2 + r(I+\phi) + I\phi(1-\pi)}\right) = (1-\bar{p})b - g\bar{p},$$

and so that

$$1 - \bar{p} = \tfrac{g[r^2 + r(I+\phi) + I\phi(1-\pi)]}{(g+b)[r^2 + r(I+\phi) + I\phi(1-\pi)] + (g+b)\phi\pi - b(r+\phi)},$$

[8] Define

$$h(x) := \tfrac{(I+r)x + \phi\pi}{Ix + r + \phi}.$$

The function $h(x)$ increases in $x$. Indeed, its first derivative is equal to

$$h'(x) = \tfrac{r(r+\phi) + I[r + \phi(1-\pi)]}{(Ix + r + \phi)^2} > 0.$$

Furthermore, $h(x) \geq x$ if and only if $\phi(\pi - x) + Ix(1-x) = -I(x - \alpha_I)(x - \beta_I) \geq 0$, or equivalently if and only if $x \in [\beta_I, \alpha_I]$. Because $\alpha_X \in [\pi, \alpha_I]$ for all $X \in [0, I]$, it follows that $h(\alpha_X) \in [\pi, \alpha_I]$. Note that $h(x) \leq 0$ if and only if

$$x \leq x_1 := -\tfrac{\phi\pi}{I+r},$$

where $x_1 \geq \beta_I$. Indeed, $x_1 \geq \beta_I$ if and only if

$$-\tfrac{\phi\pi}{I+r} \geq \tfrac{I - \phi - \sqrt{(I-\phi)^2 + 4I\phi\pi}}{2I},$$

i.e., if and only if

$$(I-\phi)^2 + 4I\phi\pi - \left(I - \phi + 2I\tfrac{\phi\pi}{I+r}\right)^2 = \tfrac{4I\phi\pi[r^2 + r(I+\phi) + I\phi(1-\pi)]}{(I+r)^2} \geq 0.$$

Because $\beta_X$ is increasing in $X$, it follows that $h(\beta_X) \leq 0$ for all $X \in [0, I]$.

i.e.,

$$\bar{p} = \frac{b[r^2+r(I+\phi)+I\phi(1-\pi)]+(g+b)\phi\pi-b(r+\phi)}{(g+b)[r^2+r(I+\phi)+I\phi(1-\pi)]+(g+b)\phi\pi-b(r+\phi)} = \frac{p_M[r^2+r(I+\phi)+I\phi(1-\pi)]+\phi\pi-p_M(r+\phi)}{r^2+r(I+\phi)+I\phi(1-\pi)+\phi\pi-p_M(r+\phi)}. \tag{B.43}$$

It follows that $\bar{p} \leq p_M$ if and only if

$$(1-p_M)\left(\frac{\phi\pi}{r+\phi} - p_M\right) \leq 0.$$

That is, $\bar{p} \leq p_M$ if and only if $p_M \in \left[\frac{\phi\pi}{r+\phi}, 1\right]$. Note that $\frac{\phi\pi}{r+\phi} < \pi$. Furthermore, $\bar{p} \leq \alpha_{I-1}$ if and only if

$$p_M \leq \frac{\alpha_{I-1}[r^2+r(I+\phi)+I\phi(1-\pi)]-(1-\alpha_{I-1})\phi\pi}{r^2+r(I+\phi)+I\phi(1-\pi)-(1-\alpha_{I-1})(r+\phi)} = \frac{(I+r)\alpha_{I-1}+\phi\pi}{I\alpha_{I-1}+r+\phi}.$$

Compare with the condition (B.41). Note that $w_1(\bar{p}) \geq 0$ if and only if

$$g - \left((g+b)\tfrac{r}{I}+g\right)\frac{I\left(\phi(1-\pi)+r\frac{g[r^2+r(I+\phi)+I\phi(1-\pi)]}{(g+b)[r^2+r(I+\phi)+I\phi(1-\pi)]+(g+b)\phi\pi-b(r+\phi)}\right)}{r^2+r(I+\phi)+I\phi(1-\pi)} \geq 0.$$

The last inequality can be rewritten as follows:

$$g - [Ig+(g+b)r]\left(\frac{\phi(1-\pi)}{r^2+r(I+\phi)+I\phi(1-\pi)} + \frac{gr}{(g+b)[r^2+r(I+\phi)+I\phi(1-\pi)]+(g+b)\phi\pi-b(r+\phi)}\right) \geq 0,$$

or equivalently

$$(1-p_M) - [I(1-p_M)+r]\left(\frac{\phi(1-\pi)}{r^2+r(I+\phi)+I\phi(1-\pi)} + \frac{r(1-p_M)}{r^2+r(I+\phi)+I\phi(1-\pi)+\phi\pi-p_M(r+\phi)}\right) \geq 0.$$

It follows that $w_1(\bar{p}) \geq 0$ if and only if

$$\begin{aligned} p_M^2&[(I+r+\phi)(r+\phi) - I[r^2+r(I+\phi)+I\phi(1-\pi)]] \\ &+ p_M[-(I+r+\phi\pi)(r+\phi) - (I+r+\phi)\phi\pi + (I-\phi)[r^2+r(I+\phi)+I\phi(1-\pi)]] \\ &\qquad + [r^2+r(I+\phi)+I\phi(1-\pi)+I+r+\phi\pi]\phi\pi \geq 0. \end{aligned}$$

Divide the last inequality by

$$(I+r+\phi)(r+\phi) - I[r^2+r(I+\phi)+I\phi(1-\pi)] < 0,$$

which can be rewritten as follows:

$$-\phi(I\pi+r+\phi) + (I-1)[r^2+r(I+\phi)+I\phi(1-\pi)] > 0.$$

Compare with (B.42). The inequality is to be shown. It follows that $w_1(\bar{p}) \geq 0$ if and only if

$$\left(p_M - \frac{(I+r)\alpha_{I-1}+\phi\pi}{I\alpha_{I-1}+r+\phi}\right)\left(p_M - \frac{(I+r)\beta_{I-1}+\phi\pi}{I\beta_{I-1}+r+\phi}\right) \leq 0.$$

Therefore, taking into account footnote 8, $w_1(\bar{p}) \geq 0$ if and only if

$$p_M \leq \frac{(I+r)\alpha_{I-1}+\phi\pi}{I\alpha_{I-1}+r+\phi}.$$

The value-matching at $\underline{p}$ implies $v_x(\underline{p}) = v_0(\underline{p}) = 0$, and so

$$C = br\tfrac{1}{\underline{p}} - gr\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\right).$$

Therefore,

$$\begin{aligned} v_x(p) &= br\tfrac{p-\underline{p}}{\underline{p}} - grp\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{p}{1-p}\right), \\ v_x'(p) &= br\tfrac{1}{\underline{p}} - gr\tfrac{1}{1-p} - gr\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{p}{1-p}\right), \end{aligned}$$

and so

$$v_x(p) - \tfrac{\phi}{r}(\pi - p)v_x'(p) = b\tfrac{r(p-\underline{p})-\phi(\pi-p)}{\underline{p}} + g\tfrac{\phi(\pi-p)}{1-p} - g[rp - \phi(\pi-p)]\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{p}{1-p}\right).$$

It follows from $b_{v_x}(p) = c(p)$ that

$$v_x(p) - \tfrac{\phi}{r}(\pi - p)v_x'(p) = (I-1)x(p)c(p),$$

so that

$$x(p) = \frac{b\frac{r(p-\underline{p})-\phi(\pi-p)}{\underline{p}} + g\frac{\phi(\pi-p)}{1-p} - g[rp-\phi(\pi-p)]\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right)}{(I-1)[b(1-p)-gp]}.$$

It is left to show that $x(p)$ increases in $p$ for $p \in [\underline{p}, \bar{p}]$. Note that

$$x(\underline{p}) = -\tfrac{\phi(\pi-\underline{p})}{(I-1)\underline{p}(1-\underline{p})} \geq 0$$

for $\underline{p} \geq \pi$, with equality when $\underline{p} = \pi$. At $\bar{p}$, $x(\bar{p}) = 1$, i.e.,

$$b\tfrac{r(\bar{p}-\underline{p})-\phi(\pi-\bar{p})}{\underline{p}} + g\tfrac{\phi(\pi-\bar{p})}{1-\bar{p}} - g[r\bar{p} - \phi(\pi-\bar{p})]\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{\bar{p}}{1-\bar{p}}\right) = (I-1)[b(1-\bar{p}) - g\bar{p}]. \tag{B.44}$$

Fix $\bar{p}$. Define

$$\begin{aligned} h_L(p) &:= b\tfrac{r(\bar{p}-p)-\phi(\pi-\bar{p})}{p} + g\tfrac{\phi(\pi-\bar{p})}{1-\bar{p}} - g[r\bar{p} - \phi(\pi-\bar{p})]\ln\left(\tfrac{1-p}{p}\tfrac{\bar{p}}{1-\bar{p}}\right), \\ h_R(p) &:= (I-1)[b(1-\bar{p}) - g\bar{p}]. \end{aligned}$$

Given $\bar{p}$, there exists unique $\underline{p} \in [\pi, \bar{p}]$ that solves (B.44) if and only if $h_L(p)$ and $h_R(p)$ intersect once on $[\pi, \bar{p}]$. The function $h_R(p)$ is constant for all $p$. The function $h_L(p)$ decreases in $p$ for $p \leq \bar{p} \leq p_M$. Indeed, its first derivative is equal to

$$h_L'(p) = -[r\bar{p} - \phi(\pi-\bar{p})]\tfrac{b(1-p)-gp}{p^2(1-p)} < 0$$

for $p < \bar{p} \leq p_M$ and $\bar{p} > \frac{\phi\pi}{r+\phi}$, in particular for $\bar{p} \geq \pi$. Furthermore,

$$h_L(\bar{p}) = -\tfrac{\phi(\pi-\bar{p})}{\bar{p}(1-\bar{p})}[b(1-\bar{p}) - g\bar{p}] < h_R(\bar{p})$$

$\bar{p} \leq p_M$ and $\bar{p} \leq \alpha_{I-1}$, because

$$0 \leq \phi(\pi - \bar{p}) - (I-1)\bar{p}(1-\bar{p}) = -I(\bar{p} - \alpha_{I-1})(\bar{p} - \beta_{I-1}).$$

Therefore, there exists $\underline{p} \in [\pi, \bar{p}]$ that solves (B.44) if and only $h_L(\pi) > h_R(\pi)$. That is, this is the case if and only if the following condition is satisfied

$$b\tfrac{\phi(\pi-\bar{p})^2}{\pi\bar{p}} - b\tfrac{r(\pi-\bar{p})}{\pi} - g[r\bar{p}-\phi(\pi-\bar{p})]\ln\left(\tfrac{1-\pi}{\pi}\tfrac{\bar{p}}{1-\bar{p}}\right) \geq [b(1-\bar{p})-g\bar{p}]\tfrac{\phi(\pi-\bar{p})+(I-1)\bar{p}(1-\bar{p})}{\bar{p}(1-\bar{p})},$$

which can be rewritten as follows:

$$\begin{aligned} p_M\tfrac{\phi(\pi-\bar{p})^2}{\pi\bar{p}} - p_M\tfrac{r(\pi-\bar{p})}{\pi} - (1-p_M)[r\bar{p}-\phi(\pi-\bar{p})]\ln\left(\tfrac{1-\pi}{\pi}\tfrac{\bar{p}}{1-\bar{p}}\right)& \\ \geq (p_M-\bar{p})\tfrac{\phi(\pi-\bar{p})+(I-1)\bar{p}(1-\bar{p})}{\bar{p}(1-\bar{p})},& \quad \text{(B.45)}\end{aligned}$$

where $\bar{p}$ is given by (B.43). It is left to argue that (B.45) is the condition that bounds $p_M$ below for this case.

Case (iv): It follows from $b_{w_1}(\bar{p}) = c(\bar{p})$ that

$$\begin{aligned}\tfrac{1}{r}(1-\bar{p})\left(v_0(0)\tfrac{r(r+\phi)}{r^2+r(I+\phi)+I\phi(1-\pi)} + b\tfrac{r(r+\phi(1-\pi))}{r^2+r(I+\phi)+I\phi(1-\pi)} - g\tfrac{r\phi\pi}{r^2+r(I+\phi)+I\phi(1-\pi)}\right)& \\ = (1-\bar{p})b - g\bar{p},&\end{aligned}$$

and so that

$$1-\bar{p} = \tfrac{g[r^2+r(I+\phi)+I\phi(1-\pi)]}{(g+b)[r^2+r(I+\phi)+I\phi(1-\pi)]+(g+b)\phi\pi-(v_0(0)+b)(r+\phi)},$$

that is,

$$\begin{aligned}\bar{p} &= \tfrac{b[r^2+r(I+\phi)+I\phi(1-\pi)]+(g+b)\phi\pi-(v_0(0)+b)(r+\phi)}{(g+b)[r^2+r(I+\phi)+I\phi(1-\pi)]+(g+b)\phi\pi-(v_0(0)+b)(r+\phi)} \\ &= \frac{p_M[r^2+r(I+\phi)+I\phi(1-\pi)]+\phi\pi-\left(\frac{v_0(0)}{g+b}+p_M\right)(r+\phi)}{r^2+r(I+\phi)+I\phi(1-\pi)+\phi\pi-\left(\frac{v_0(0)}{g+b}+p_M\right)(r+\phi)}. \quad \text{(B.46)}\end{aligned}$$

Compare with (B.43). Note that the right side of (B.46) decreases in $v_0(0)$. The value-matching at $\underline{p}$ implies $v_0(\underline{p}) = v_x(\underline{p})$, that is,

$$v_0(0)\left(\tfrac{\pi}{\pi-\underline{p}}\right)^{\frac{r}{\phi}} = v_0(0) - br + gr\underline{p}\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\right) + C\underline{p},$$

and so

$$C = br\tfrac{1}{\underline{p}} - gr\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\right) - v_0(0)\tfrac{1}{\underline{p}} + v_0(0)\left(\tfrac{\pi}{\pi-\underline{p}}\right)^{\frac{r}{\phi}}\tfrac{1}{\underline{p}}.$$

The smooth-pasting at $\underline{p}$ implies $v_0'(\underline{p}) = v_x'(\underline{p})$, i.e.,

$$v_0(0)\tfrac{r}{\phi\pi}\left(\tfrac{\pi}{\pi-\underline{p}}\right)^{\frac{r}{\phi}+1} = gr\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\right) - gr\tfrac{1}{1-\underline{p}} + C,$$

and so

$$C = -gr\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\right) + gr\tfrac{1}{1-\underline{p}} + v_0(0)\tfrac{r}{\phi\pi}\left(\tfrac{\pi}{\pi-\underline{p}}\right)^{\frac{r}{\phi}+1}.$$

It follows that

$$v_0(0) = \tfrac{b(1-\underline{p})-g\underline{p}}{1-\underline{p}}\frac{r}{1+\frac{r\underline{p}-\phi(\pi-\underline{p})}{\phi(\pi-\underline{p})}\left(\frac{\pi}{\pi-\underline{p}}\right)^{\frac{r}{\phi}}} = (g+b)\tfrac{p_M-\underline{p}}{1-\underline{p}}\frac{r}{1+\frac{r\underline{p}-\phi(\pi-\underline{p})}{\phi(\pi-\underline{p})}\left(\frac{\pi}{\pi-\underline{p}}\right)^{\frac{r}{\phi}}}. \quad \text{(B.47)}$$

Note that $v_0(0)$ decreases in $\underline{p}$. Indeed, the first derivative of $\frac{v_0(0)}{g+b}$ with respect to $\underline{p}$ is equal to

$$-\frac{1-p_M}{(1-\underline{p})^2}\frac{r}{1+\frac{r\underline{p}-\phi(\pi-\underline{p})}{\phi(\pi-\underline{p})}\left(\frac{\pi}{\pi-\underline{p}}\right)^{\frac{r}{\phi}}}-\frac{p_M-\underline{p}}{1-\underline{p}}\frac{r}{\left(1+\frac{r\underline{p}-\phi(\pi-\underline{p})}{\phi(\pi-\underline{p})}\left(\frac{\pi}{\pi-\underline{p}}\right)^{\frac{r}{\phi}}\right)^2}\frac{r(r+\phi)\underline{p}}{\phi^2(\pi-\underline{p})^2}\left(\frac{\pi}{\pi-\underline{p}}\right)^{\frac{r}{\phi}}<0$$

for $\underline{p}<p_M$. Furthermore,

$$\begin{aligned}
v_x(p) &= v_0(0)+br\tfrac{p-\underline{p}}{\underline{p}}-grp\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{p}{1-p}\right)-v_0(0)\tfrac{p}{\underline{p}}+v_0(0)\left(\tfrac{\pi}{\pi-\underline{p}}\right)^{\frac{r}{\phi}}\tfrac{p}{\underline{p}},\\
v_x'(p) &= br\tfrac{1}{\underline{p}}-gr\tfrac{1}{1-p}-gr\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{p}{1-p}\right)-v_0(0)\tfrac{1}{\underline{p}}+v_0(0)\left(\tfrac{\pi}{\pi-\underline{p}}\right)^{\frac{r}{\phi}}\tfrac{1}{\underline{p}},
\end{aligned}$$

and so

$$\begin{aligned}
v_x(p)-\tfrac{\phi}{r}(\pi-p)v_x'(p) = b\tfrac{r(p-\underline{p})-\phi(\pi-p)}{\underline{p}}+g\tfrac{\phi(\pi-p)}{1-p}-g[rp-\phi(\pi-p)]\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{p}{1-p}\right)\\
+v_0(0)+v_0(0)\tfrac{rp-\phi(\pi-p)}{r\underline{p}}\left(-1+\left(\tfrac{\pi}{\pi-\underline{p}}\right)^{\frac{r}{\phi}}\right).
\end{aligned}$$

It follows from $b_{v_x}(p)=c(p)$ that

$$v_x(p)-\tfrac{\phi}{r}(\pi-p)v_x'(p)=(I-1)x(p)c(p),$$

so that

$$x(p)=\frac{b\frac{r(p-\underline{p})-\phi(\pi-p)}{\underline{p}}+g\frac{\phi(\pi-p)}{1-p}-g[rp-\phi(\pi-p)]\ln\left(\frac{1-\underline{p}}{\underline{p}}\frac{p}{1-p}\right)+v_0(0)+v_0(0)\frac{rp-\phi(\pi-p)}{r\underline{p}}\left(-1+\left(\frac{\pi}{\pi-\underline{p}}\right)^{\frac{r}{\phi}}\right)}{(I-1)[b(1-p)-gp]}.$$

It is left to show that $x(p)$ increases in $p$ for $p\in[\underline{p},\bar{p}]$. Note that $x(\underline{p})=0$. At $\bar{p}$, $x(\bar{p})=1$, i.e.,

$$\begin{aligned}
b\tfrac{r(\bar{p}-\underline{p})-\phi(\pi-\bar{p})}{\underline{p}}+g\tfrac{\phi(\pi-\bar{p})}{1-\bar{p}}-g[r\bar{p}-\phi(\pi-\bar{p})]\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{\bar{p}}{1-\bar{p}}\right)\\
+v_0(0)+v_0(0)\tfrac{r\bar{p}-\phi(\pi-\bar{p})}{r\underline{p}}\left(-1+\left(\tfrac{\pi}{\pi-\underline{p}}\right)^{\frac{r}{\phi}}\right)=(I-1)[b(1-\bar{p})-g\bar{p}],
\end{aligned}$$

which can be rewritten as follows:

$$\begin{aligned}
p_M\tfrac{r(\bar{p}-\underline{p})-\phi(\pi-\bar{p})}{\underline{p}}+(1-p_M)\tfrac{\phi(\pi-\bar{p})}{1-\bar{p}}-(1-p_M)[r\bar{p}-\phi(\pi-\bar{p})]\ln\left(\tfrac{1-\underline{p}}{\underline{p}}\tfrac{\bar{p}}{1-\bar{p}}\right)\\
+\tfrac{v_0(0)}{g+b}+\tfrac{v_0(0)}{g+b}\tfrac{r\bar{p}-\phi(\pi-\bar{p})}{r\underline{p}}\left(-1+\left(\tfrac{\pi}{\pi-\underline{p}}\right)^{\frac{r}{\phi}}\right)=(I-1)(p_M-\bar{p}),
\end{aligned}\tag{B.48}$$

Given $v_0(0)$ defined by (B.47), equations (B.46) and (B.48) pin down $\underline{p}$ and $\bar{p}$. It is left to argue that the solution to (B.46) and (B.48) exists and is unique, and to show that the condition on parameter is the opposite to (B.45).

# Appendix C

# Addendum to Chapter 3

## C.1 A Buyer's Problem

Proposition 3.1 and Lemmata 3.2 and 3.3 follow from the following analysis. Suppose the buyer with the belief $x$ has the $A$-gizmo. Because the gizmo breaks at rate $\beta$ and the buyer learns that he is the $H$-type at rate $\lambda x$ and that he is the $L$-type at rate $\lambda(1-x)$, the buyer's value function satisfies the following Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned}(r+\delta)V_A(x,p_A,p_B) &= w + w_A x\\ &\quad + \beta[\max\{V_A(x,p_A,p_B)-p_A, V_B(x,p_A,p_B)-p_B\} - V_A(x,p_A,p_B)]\\ &\quad + \lambda x[\max\{V_A(1,p_A,p_B), V_B(1,p_A,p_B)-p_B\} - V_A(x,p_A,p_B)]\\ &\quad + \lambda(1-x)[\max\{V_A(0,p_A,p_B), V_B(0,p_A,p_B)-p_B\} - V_A(x,p_A,p_B)],\end{aligned}$$

$$\begin{aligned}(r+\delta+\beta+\lambda)V_A(x,p_A,p_B) &= w + w_A x + \beta\max\{V_A(x,p_A,p_B)-p_A, V_B(x,p_A,p_B)-p_B\}\\ &\quad + \lambda x\max\{V_A(1,p_A,p_B), V_B(1,p_A,p_B)-p_B\}\\ &\quad + \lambda(1-x)\max\{V_A(0,p_A,p_B), V_B(0,p_A,p_B)-p_B\}.\end{aligned}$$

Similarly, for the buyer with the belief $x$ and the $B$-gizmo, the HJB equation is as follows:

$$\begin{aligned}(r+\delta)V_B(x,p_A,p_B) &= w\\ &\quad + \beta[\max\{V_A(x,p_A,p_B)-p_A, V_B(x,p_A,p_B)-p_B\} - V_B(x,p_A,p_B)]\\ &\quad + \lambda x[\max\{V_A(1,p_A,p_B)-p_A, V_B(1,p_A,p_B)\} - V_B(x,p_A,p_B)]\\ &\quad + \lambda(1-x)[\max\{V_A(0,p_A,p_B)-p_A, V_B(0,p_A,p_B)\} - V_B(x,p_A,p_B)],\end{aligned}$$

$$\begin{aligned}(r+\delta+\beta+\lambda)V_B(x,p_A,p_B) &= w + \beta\max\{V_A(x,p_A,p_B)-p_A, V_B(x,p_A,p_B)-p_B\}\\ &\quad + \lambda x\max\{V_A(1,p_A,p_B)-p_A, V_B(1,p_A,p_B)\}\\ &\quad + \lambda(1-x)\max\{V_A(0,p_A,p_B)-p_A, V_B(0,p_A,p_B)\}.\end{aligned}$$

Consider the $H$-type (who has learned his type) with the $A$-gizmo. Because the gizmo breaks at rate $\beta$, the buyer's value function satisfies

$$(r+\delta)V_A(1,p_A,p_B)$$

$$= w + w_A + \beta[\max\{V_A(1,p_A,p_B) - p_A, V_B(1,p_A,p_B) - p_B\} - V_A(1,p_A,p_B)],$$

$$(r+\delta+\beta)V_A(1,p_A,p_B) = w + w_A + \beta\max\{V_A(1,p_A,p_B) - p_A, V_B(1,p_A,p_B) - p_B\}.$$

The value function of the $L$-type (who has learned his type) with the $A$-gizmo satisfies

$$(r+\delta)V_B(1,p_A,p_B)$$
$$= w + \beta[\max\{V_A(1,p_A,p_B) - p_A, V_B(1,p_A,p_B) - p_B\} - V_B(1,p_A,p_B)],$$

$$(r+\delta+\beta)V_B(1,p_A,p_B) = w + \beta\max\{V_A(1,p_A,p_B) - p_A, V_B(1,p_A,p_B) - p_B\}.$$

It follows that $V_A(1,p_A,p_B) > V_B(1,p_A,p_B)$, and so $\max\{V_A(1,p_A,p_B), V_B(1,p_A,p_B) - p_B\} = V_A(1,p_A,p_B)$. Note that the $H$-types prefers the $A$-gizmo over the $B$-gizmo if and only if $V_A(1,p_A,p_B) - p_A \geq V_B(1,p_A,p_B) - p_B$, i.e., if and only if $\frac{1}{r+\delta+\beta}w_A \geq p_A - p_B$. In such a case,

$$\begin{aligned} V_A(1,p_A,p_B) &= \tfrac{1}{r+\delta}(w+w_A) - \tfrac{\beta}{r+\delta}p_A, \\ V_B(1,p_A,p_B) &= \tfrac{1}{r+\delta}w + \tfrac{\beta}{(r+\delta)(r+\delta+\beta)}w_A - \tfrac{\beta}{r+\delta}p_A. \end{aligned}$$

Note that $V_B(1,p_A,p_B) \geq V_A(1,p_A,p_B) - p_A$ if and only if $p_A \geq \frac{1}{r+\delta+\beta}w_A$.

Consider the $H$-type (who has learned his type) with the $B$-gizmo. Because the gizmo breaks at rate $\beta$, the buyer's value function satisfies

$$(r+\delta)V_A(0,p_A,p_B)$$
$$= w + \beta[\max\{V_A(0,p_A,p_B) - p_A, V_B(0,p_A,p_B) - p_B\} - V_A(0,p_A,p_B)],$$

$$(r+\delta+\beta)V_A(0,p_A,p_B) = w + \beta\max\{V_A(0,p_A,p_B) - p_A, V_B(0,p_A,p_B) - p_B\}.$$

The value function of the $L$-type (who has learned his type) with the $B$-gizmo satisfies

$$(r+\delta)V_B(0,p_A,p_B)$$
$$= w + \beta[\max\{V_A(0,p_A,p_B) - p_A, V_B(0,p_A,p_B) - p_B\} - V_B(0,p_A,p_B)],$$

$$(r+\delta+\beta)V_B(0,p_A,p_B) = w + \beta\max\{V_A(0,p_A,p_B) - p_A, V_B(0,p_A,p_B) - p_B\}.$$

It follows that $V_A(0,p_A,p_B) = V_B(0,p_A,p_B)$, and so $\max\{V_A(0,p_A,p_B), V_B(0,p_A,p_B) - p_B\} = V_A(0,p_A,p_B)$ and $\max\{V_A(0,p_A,p_B) - p_A, V_B(0,p_A,p_B)\} = V_B(0,p_A,p_B)$. Note that the $L$-types prefers the $B$-gizmo over the $A$-gizmo if and only if $V_B(0,p_A,p_B) - p_B \geq V_A(0,p_A,p_B) - p_A$, i.e., if and only if $p_A - p_B \geq 0$. In such a case,

$$V_A(0,p_A,p_B) = V_B(0,p_A,p_B) = \tfrac{1}{r+\delta}w - \tfrac{\beta}{r+\delta}p_B.$$

The optimal cut-off $\bar{x}$ is the belief at which a buyer without the gizmo is indifferent between the two versions given the prices $p_A$ and $p_B$. If $\frac{1}{r+\delta+\beta}w_A \geq p_A - p_B \geq 0$, then $V_A(\bar{x},p_A,p_B) - p_A = V_B(\bar{x},p_A,p_B) - p_B$ at

$$\begin{aligned} \bar{x} &= \frac{(r+\delta+\beta+\lambda)(p_A-p_B)}{w_A+\lambda[V_A(1,p_A,p_B)-\max\{V_A(1,p_A,p_B)-p_A,V_B(1,p_A,p_B)\}]} \\ &= \frac{(p_A-p_B)(r+\delta+\beta+\lambda)}{w_A+\lambda[V_A(1,p_A,p_B)-V_B(1,p_A,p_B)-\max\{V_A(1,p_A,p_B)-V_B(1,p_A,p_B)-p_A,0\}]} \\ &= \frac{p_A-p_B}{\frac{1}{r+\delta+\beta}w_A-\frac{\lambda}{r+\delta+\beta+\lambda}\max\left\{\frac{1}{r+\delta+\beta}w_A-p_A,0\right\}}. \end{aligned}$$

Suppose $x \geq \bar{x}$, and so $V_A(x, p_A, p_B) - p_A \geq V_B(x, p_A, p_B) - p_B$. The value function of a buyer with the belief $x$ who has the $A$-gizmo is given by

$$V_A(x, p_A, p_B) = \tfrac{1}{r+\delta}(w + w_A x) - \tfrac{\beta}{r+\delta+\lambda} p_A - \tfrac{\beta\lambda}{(r+\delta)(r+\delta+\lambda)}[x p_A + (1-x) p_B]. \tag{C.1}$$

Note that

$$\tfrac{\mathrm{d}V_A(x,p_A,p_B)}{\mathrm{d}x} = \tfrac{r+\delta+\beta+\lambda}{(r+\delta+\beta)(r+\delta+\lambda)} w_A + \tfrac{\beta\lambda}{(r+\delta)(r+\delta+\lambda)} \left[\tfrac{1}{r+\delta+\beta} w_A - (p_A - p_B)\right] > 0$$

if $\frac{1}{r+\delta+\beta} w_A \geq p_A - p_B$. The value function of a buyer with the belief $x$ who has the $B$-gizmo is given by

$$\begin{aligned} V_B(x, p_A, p_B) = \tfrac{1}{r+\delta} w + \tfrac{\beta}{(r+\delta)(r+\delta+\beta)} w_A x - \tfrac{\beta}{r+\delta+\lambda} p_A - \tfrac{\beta\lambda}{(r+\delta)(r+\delta+\lambda)}[x p_A + (1-x) p_B] \\ + \tfrac{\lambda}{r+\delta+\beta+\lambda} x \max\left\{\tfrac{1}{r+\delta+\beta} w_A - p_A, 0\right\}. \end{aligned}$$

Note that

$$\begin{aligned} \tfrac{\mathrm{d}V_B(x,p_A,p_B)}{\mathrm{d}x} = \tfrac{\beta}{(r+\delta+\beta)(r+\delta+\lambda)} w_A + \tfrac{\beta\lambda}{(r+\delta)(r+\delta+\lambda)} \left[\tfrac{1}{r+\delta+\beta} w_A - (p_A - p_B)\right] \\ + \tfrac{\lambda}{r+\delta+\beta+\lambda} \max\left\{\tfrac{1}{r+\delta+\beta} w_A - p_A, 0\right\} > 0 \end{aligned}$$

if $\frac{1}{r+\delta+\beta} w_A \geq p_A - p_B$. Furthermore,

$$V_A(x, p_A, p_B) - V_B(x, p_A, p_B) = \tfrac{1}{r+\delta+\beta} w_A x - \tfrac{\lambda}{r+\delta+\beta+\lambda} x \max\left\{\tfrac{1}{r+\delta+\beta} w_A - p_A, 0\right\} \geq 0,$$

with equality if and only if $x = 0$.

Suppose $x \leq \bar{x}$, and so $V_B(x, p_A, p_B) - p_B \geq V_A(x, p_A, p_B) - p_A$. The value function of a buyer with the belief $x$ who has the $A$-gizmo is given by

$$\begin{aligned} V_A(x, p_A, p_B) = \tfrac{1}{r+\delta}(w + w_A x) - \tfrac{\beta}{(r+\delta+\beta)(r+\delta+\lambda)} w_A x - \tfrac{\beta}{r+\delta+\lambda} p_B \\ - \tfrac{\beta\lambda}{(r+\delta)(r+\delta+\lambda)}[x p_A + (1-x) p_B] + \tfrac{\beta\lambda}{(r+\delta+\lambda)(r+\delta+\beta+\lambda)} x \max\left\{\tfrac{1}{r+\delta+\beta} w_A - p_A, 0\right\}. \end{aligned}$$

Note that

$$\begin{aligned} \tfrac{\mathrm{d}V_A(x,p_A,p_B)}{\mathrm{d}x} = \tfrac{1}{r+\delta+\beta} w_A + \tfrac{\beta\lambda}{(r+\delta)(r+\delta+\lambda)} \left[\tfrac{1}{r+\delta+\beta} w_A - (p_A - p_B)\right] \\ + \tfrac{\beta\lambda}{(r+\delta+\lambda)(r+\delta+\beta+\lambda)} \max\left\{\tfrac{1}{r+\delta+\beta} w_A - p_A, 0\right\} > 0 \end{aligned}$$

if $\frac{1}{r+\delta+\beta} w_A \geq p_A - p_B$. The value function of a buyer with the belief $x$ who has the $B$-gizmo is given by

$$\begin{aligned} V_B(x, p_A, p_B) = \tfrac{1}{r+\delta} w + \tfrac{\beta\lambda}{(r+\delta)(r+\delta+\beta)(r+\delta+\lambda)} w_A x \\ - \tfrac{\beta}{r+\delta+\lambda} p_B - \tfrac{\beta\lambda}{(r+\delta)(r+\delta+\lambda)}[x p_A + (1-x) p_B] \\ + \tfrac{\lambda}{r+\delta+\lambda} x \max\left\{\tfrac{1}{r+\delta+\beta} w_A - p_A, 0\right\}. \end{aligned} \tag{C.2}$$

Note that

$$\frac{\mathrm{d}V_B(x,p_A,p_B)}{\mathrm{d}x} = \frac{\beta\lambda}{(r+\delta)(r+\delta+\lambda)}\left[\frac{1}{r+\delta+\beta}w_A - (p_A - p_B)\right] + \frac{\beta\lambda}{(r+\delta+\lambda)(r+\delta+\beta+\lambda)}\max\left\{\frac{1}{r+\delta+\beta}w_A - p_A, 0\right\} \geq 0$$

if $\frac{1}{r+\delta+\beta}w_A \geq p_A - p_B$. Furthermore,

$$V_A(x,p_A,p_B) - V_B(x,p_A,p_B) = \frac{1}{r+\delta+\beta}w_A x - \frac{\lambda}{r+\delta+\beta+\lambda}x\max\left\{\frac{1}{r+\delta+\beta}w_A - p_A, 0\right\} \geq 0,$$

with equality if and only if $x = 0$.

## C.2 The Seller's Problem

Theorems 3.1 and 3.2 and Lemma 3.4 follow from the following analysis. Let $\Pi(p_A, p_B)$ define the seller's profit when she posts prices $p_A$ and $p_B$. The seller's problem can be divide into two subproblems depending on whether, given the prices $p_A$ and $p_B$, buyers with the $B$-gizmo replace it immediately with the $A$-gizmo upon learning that they are the $H$-types. There is no immediate replacement if $p_A \geq \frac{1}{r+\delta+\beta}w_A$.

### C.2.1 No Replacement

The seller's subproblem takes the form

$$\max_{p_A,p_B}\Pi(p_A,p_B) = \left[\frac{\delta}{r+\delta}\int_{\bar{x}}^{1} f(x)\mathrm{d}x + \frac{\beta}{r+\beta}\left(\int_{\bar{x}}^{1} m(x)\mathrm{d}x + M_H\right)\right]p_A + \left[\frac{\delta}{r+\delta}\int_{0}^{\bar{x}} f(x)\mathrm{d}x + \frac{\beta}{r+\beta}\left(\int_{0}^{\bar{x}} m(x)\mathrm{d}x + M_L\right)\right]p_B$$

$$\begin{aligned}
\text{sub. to}\quad & p_A - p_B \leq \tfrac{1}{r+\delta+\beta}w_A, && (\mathrm{IC}_1^A)\\
& p_A - p_B \geq 0, && (\mathrm{IC}_0^B)\\
& p_A \geq \tfrac{1}{r+\delta+\beta}w_A, && (\mathrm{NR})\\
& V_A(\bar{x},p_A,p_B) - p_A \geq 0, && (\mathrm{IR}_{\bar{x}}^A)\\
& V_B(0,p_A,p_B) - p_B \geq 0, && (\mathrm{IR}_0^B)
\end{aligned}$$

where $\bar{x} := \bar{x} = \frac{p_A - p_B}{\frac{1}{r+\delta+\beta}w_A}$ by Proposition 3.1. The incentive constraint ($\mathrm{IC}_1^A$) (resp., ($\mathrm{IC}_0^B$)) says that the $H$-types (resp., the $L$-types) prefer the $A$-gizmo (resp., the $B$-gizmo) over the $B$-gizmo (resp., the $A$-gizmo). The constraint (NR) captures that the $A$-gizmo price is too high for buyers with the $B$-gizmo to replace it immediately with the $A$-gizmo upon learning that they are the $H$-types. Given the monotonicity of the value functions, the individual rationality constraint ($\mathrm{IR}_{\bar{x}}^A$) (resp., ($\mathrm{IR}_0^B$)) says that buyers with the beliefs above $\bar{x}$ (resp., with any belief, including the $L$-types) prefer to have the $A$-gizmo (resp., the $B$-gizmo) over not having a gizmo at all.

Given the value function of the buyer with the $A$-gizmo in (C.1), the constraint ($\mathrm{IR}_{\bar{x}}^A$)

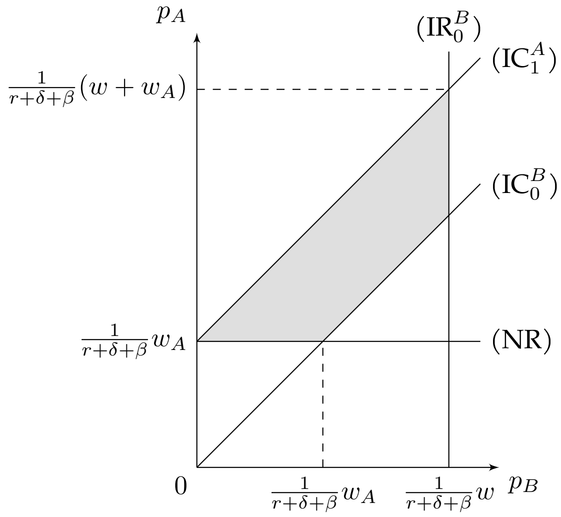


**Figure C.1.** The feasibility set of the seller's subproblem if buyers do not replace the $B$-gizmo with the $A$-gizmo immediately upon learning that they are the $H$-types. Parameters: $(r, \delta, \beta, w, w_A) = (1, 1, 1, 2, 1)$.

is equivalent to

$$p_A \le \tfrac{1}{r+\delta+\beta}(w + w_A\bar{x}) + \tfrac{\beta\lambda}{(r+\delta+\beta)(r+\delta+\lambda)}(1-\bar{x})(p_A - p_B).$$

Given the value function of the buyer with the $B$-gizmo in (C.2), the constraint $(\mathrm{IR}_0^B)$ is equivalent to

$$p_B \le \tfrac{1}{r+\delta+\beta}w \quad \Leftrightarrow \quad p_A \le \tfrac{1}{r+\delta+\beta}(w + w_A\bar{x}),$$

taking into account the definition of $\bar{x}$. Therefore, $(\mathrm{IR}_{\bar{x}}^A)$ is redundant. The feasibility set is the shaded area in Figure C.1.

The Lagrangian takes the following form

$$\begin{aligned}\mathcal{L}(p_A, p_B, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = &\left[\tfrac{\delta}{r+\delta}\int_{\bar{x}}^1 f(x)\mathrm{d}x + \tfrac{\beta}{r+\beta}\left(\int_{\bar{x}}^1 m(x)\mathrm{d}x + M_H\right)\right]p_A \\ &+ \left[\tfrac{\delta}{r+\delta}\int_0^{\bar{x}} f(x)\mathrm{d}x + \tfrac{\beta}{r+\beta}\left(\int_0^{\bar{x}} m(x)\mathrm{d}x + M_L\right)\right]p_B \\ + \alpha_1\left(\tfrac{1}{r+\delta+\beta}w_A - (p_A - p_B)\right) &+ \alpha_2(p_A - p_B) + \alpha_3\left(p_A - \tfrac{1}{r+\delta+\beta}w_A\right) + \alpha_4\left(\tfrac{1}{r+\delta+\beta}w - p_B\right).\end{aligned}$$

The Kuhn-Tucker conditions are necessary for the optimum and are as follows:

$$\begin{aligned}\mathcal{L}'_{p_A} = \tfrac{\delta}{r+\delta}\int_{\bar{x}}^1 f(x)\mathrm{d}x + \tfrac{\beta}{r+\beta}\left(\int_{\bar{x}}^1 m(x)\mathrm{d}x + M_H\right)& \\ - \left(\tfrac{\delta}{r+\delta}f(\bar{x}) + \tfrac{\beta}{r+\beta}m(\bar{x})\right)\bar{x} - \alpha_1 + \alpha_2 + \alpha_3 = 0,&\end{aligned}$$

$$\begin{aligned}\mathcal{L}'_{p_B} = \tfrac{\delta}{r+\delta}\int_0^{\bar{x}} f(x)\mathrm{d}x + \tfrac{\beta}{r+\beta}\left(\int_0^{\bar{x}} m(x)\mathrm{d}x + M_L\right)& \\ + \left(\tfrac{\delta}{r+\delta}f(\bar{x}) + \tfrac{\beta}{r+\beta}m(\bar{x})\right)\bar{x} + \alpha_1 - \alpha_2 - \alpha_4 = 0,&\end{aligned}$$

$$\alpha_1\left(\tfrac{1}{r+\delta+\beta}w_A-(p_A-p_B)\right)=0,\quad \alpha_1\geq 0,$$
$$\alpha_2(p_A-p_B)=0,\quad \alpha_2\geq 0,$$
$$\alpha_3\left(p_A-\tfrac{1}{r+\delta+\beta}w_A\right)=0,\quad \alpha_3\geq 0,$$
$$\alpha_4\left(\tfrac{1}{r+\delta+\beta}w-p_B\right)=0,\quad \alpha_4\geq 0.$$

It follows from $\mathcal{L}'_{p_B}=0$ and $\alpha_i\geq 0$ for $i=1,\ldots,4$ that $\alpha_2>0$ or $\alpha_4>0$. If $\alpha_2>0$, then $p_A=p_B$, and so $\bar{x}=0$ and $\alpha_1=0$. This implies that $\mathcal{L}'_{p_A}>0$, a contradiction. Therefore, $\alpha_2^*=0$ and $\alpha_4^*>0$, and so $p_B^*=\frac{1}{r+\delta+\beta}w$. That is, the constraint $(\mathrm{IR}_0^B)$ binds.

Given $p_B^*=\frac{1}{r+\delta+\beta}w$ and the definition of $\bar{x}$, the seller's profit is equal to

$$\begin{aligned}&\left[\tfrac{\delta}{r+\delta}\int_{\bar{x}}^1 f(x)\mathrm{d}x+\tfrac{\beta}{r+\beta}\left(\int_{\bar{x}}^1 m(x)\mathrm{d}x+M_H\right)\right]\tfrac{1}{r+\delta+\beta}(w+w_A\bar{x})\\&\qquad+\left[\tfrac{\delta}{r+\delta}\int_0^{\bar{x}} f(x)\mathrm{d}x+\tfrac{\beta}{r+\beta}\left(\int_0^{\bar{x}} m(x)\mathrm{d}x+M_L\right)\right]\tfrac{1}{r+\delta+\beta}w\\&=\left[\tfrac{\delta}{r+\delta}\int_{\bar{x}}^1 f(x)\mathrm{d}x+\tfrac{\beta}{r+\beta}\left(\int_{\bar{x}}^1 m(x)\mathrm{d}x+M_H\right)\right]\tfrac{1}{r+\delta+\beta}w_A\bar{x}+\left(\tfrac{\delta}{r+\delta}+\tfrac{\beta}{r+\beta}\right)\tfrac{1}{r+\delta+\beta}w.\end{aligned}$$

The seller's subproblem can be rewritten as follows:

$$\max_{\bar{x}\in[0,1]}\left[\tfrac{\delta}{r+\delta}\int_{\bar{x}}^1 f(x)\mathrm{d}x+\tfrac{\beta}{r+\beta}\left(\int_{\bar{x}}^1 m(x)\mathrm{d}x+M_H\right)\right]\tfrac{1}{r+\delta+\beta}w_A\bar{x}+\left(\tfrac{\delta}{r+\delta}+\tfrac{\beta}{r+\beta}\right)\tfrac{1}{r+\delta+\beta}w.$$

The first-order derivative with respect to $\bar{x}$ and divided by $\frac{1}{r+\delta+\beta}w_A$ is equal to

$$\tfrac{\delta}{r+\delta}\int_{\bar{x}}^1 f(x)\mathrm{d}x+\tfrac{\beta}{r+\beta}\left(\int_{\bar{x}}^1 m(x)\mathrm{d}x+M_H\right)-\left(\tfrac{\delta}{r+\delta}f(\bar{x})+\tfrac{\beta}{r+\beta}m(\bar{x})\right)\bar{x}.$$

Define

$$H(y):=\tfrac{\delta}{r+\delta}\int_y^1 f(x)\mathrm{d}x+\tfrac{\beta}{r+\beta}\left(\int_y^1 m(x)\mathrm{d}x+M_H\right)-\left(\tfrac{\delta}{r+\delta}f(y)+\tfrac{\beta}{r+\beta}m(y)\right)y.$$

Note that $H(0)>0$ and

$$\begin{aligned}H'(y)&=-2\left(\tfrac{\delta}{r+\delta}f(y)+\tfrac{\beta}{r+\beta}m(y)\right)-\left(\tfrac{\delta}{r+\delta}f'(y)+\tfrac{\beta}{r+\beta}m'(y)\right)y\\&=-\left(\tfrac{\delta}{r+\delta}+\tfrac{\beta}{r+\beta}\tfrac{\delta}{\delta+\lambda}\right)[2f(y)+yf'(y)]<0\end{aligned}$$

if $2f(y)+yf'(y)>0$. Assume $2f(y)+yf'(y)>0$ for all $y\in[0,1]$. If $H(1)\geq 0$, i.e., if

$$\tfrac{\beta}{r+\beta}M_H\geq\tfrac{\delta}{r+\delta}f(1)+\tfrac{\beta}{r+\beta}m(1),$$

then $\bar{x}^*=1$ solves the seller's subproblem. That is, $p_A^*=\frac{1}{r+\delta+\beta}(w+w_A)$. If $H(1)<0$, then there exists a unique $\bar{x}^*\in(0,1)$ that solves $H(\bar{x}^*)=0$. This $\bar{x}^*$ solves the seller's subproblem and pins down $p_A^*=\frac{1}{r+\delta+\beta}(w+w_A\bar{x}^*)\in\left(\frac{1}{r+\delta+\beta}w,\frac{1}{r+\delta+\beta}(w+w_A)\right)$.

If $\frac{\beta}{r+\beta}M_H \geq \frac{\delta}{r+\delta}f(1) + \frac{\beta}{r+\beta}m(1)$, then the seller's profit is equal to

$$\Pi_{(\mathrm{NR})} := \Pi(p_A^*, p_B^*) = \frac{\beta}{r+\beta}\frac{\lambda}{\delta+\lambda}\int_0^1 xf(x)\mathrm{d}x \cdot \frac{1}{r+\delta+\beta}w_A + \left(\frac{\delta}{r+\delta} + \frac{\beta}{r+\beta}\right)\frac{1}{r+\delta+\beta}w.$$

If $\frac{\beta}{r+\beta}M_H < \frac{\delta}{r+\delta}f(1) + \frac{\beta}{r+\beta}m(1)$, then the profit is

$$\Pi_{(\mathrm{NR})} := \Pi(p_A^*, p_B^*) = \left(\frac{\delta}{r+\delta} + \frac{\beta}{r+\beta}\frac{\delta}{\delta+\lambda}\right)(\bar{x}^*)^2 f(\bar{x}^*)\frac{1}{r+\delta+\beta}w_A + \left(\frac{\delta}{r+\delta} + \frac{\beta}{r+\beta}\right)\frac{1}{r+\delta+\beta}w.$$

**Uniform distribution of priors.** Suppose the prior belief of buyers who enter the market is uniformly distributed on $[0,1]$, i.e., $f(x) = 1$ for all $x \in [0,1]$. If $\frac{1}{2}\frac{\beta}{r+\beta}\frac{\lambda}{\delta+\lambda} \geq \frac{\delta}{r+\delta} + \frac{\beta}{r+\beta}\frac{\delta}{\delta+\lambda}$, then $\bar{x}^* = 1$. If $\frac{1}{2}\frac{\beta}{r+\beta}\frac{\lambda}{\delta+\lambda} < \frac{\delta}{r+\delta} + \frac{\beta}{r+\beta}\frac{\delta}{\delta+\lambda}$, then

$$\bar{x}^* = \frac{1}{2} + \frac{1}{4}\frac{\frac{\beta}{r+\beta}\frac{\lambda}{\delta+\lambda}}{\frac{\delta}{r+\delta}+\frac{\beta}{r+\beta}\frac{\delta}{\delta+\lambda}}.$$

The optimal cut-off $\bar{x}^*$ depends on parameters $\beta$, $\lambda$, $\delta$, and $r$ as follows:

$$\bar{x}^{*\prime}_\beta = \frac{1}{4}\frac{\frac{\delta}{r+\delta}\frac{\lambda}{\delta+\lambda}}{\left(\frac{\delta}{r+\delta}+\frac{\beta}{r+\beta}\frac{\delta}{\delta+\lambda}\right)^2}\frac{r}{(r+\beta)^2} > 0,$$

$$\bar{x}^{*\prime}_\lambda = \frac{1}{4}\frac{\frac{\beta}{r+\beta}\left(\frac{\delta}{r+\delta}+\frac{\beta}{r+\beta}\right)}{\left(\frac{\delta}{r+\delta}+\frac{\beta}{r+\beta}\frac{\delta}{\delta+\lambda}\right)^2}\frac{\delta}{(r+\delta)^2} > 0,$$

$$\bar{x}^{*\prime}_\delta = -\frac{1}{4}\frac{\frac{\beta}{r+\beta}\frac{\lambda}{\delta+\lambda}}{\left(\frac{\delta}{r+\delta}+\frac{\beta}{r+\beta}\frac{\delta}{\delta+\lambda}\right)^2}\left[\frac{1}{\delta+\lambda}\left(\frac{\delta}{r+\delta} + \frac{\beta}{r+\beta}\frac{\delta}{\delta+\lambda}\right) + \frac{r}{(r+\delta)^2} + \frac{\beta}{r+\beta}\frac{\lambda}{(\delta+\lambda)^2}\right] < 0,$$

$$\bar{x}^{*\prime}_r = \frac{1}{4}\frac{\frac{\beta}{r+\beta}\frac{\lambda}{\delta+\lambda}}{\left(\frac{\delta}{r+\delta}+\frac{\beta}{r+\beta}\frac{\delta}{\delta+\lambda}\right)^2}\frac{\delta(\beta-\delta)}{(r+\delta)^2(r+\beta)},$$

where $\bar{x}^{*\prime}_r > 0$ if $\beta > \delta$ and $\bar{x}^{*\prime}_r < 0$ if $\beta < \delta$.

### C.2.2 Replacement

The seller's subproblem takes the form

$$\begin{aligned}
\max_{p_A, p_B} \Pi(p_A, p_B) \\
&= \left[\frac{\delta}{r+\delta}\int_{\bar{x}}^1 f(x)\mathrm{d}x + \frac{\beta}{r+\beta}\left(\int_{\bar{x}}^1 m(x)\mathrm{d}x + M_H\right) + \frac{\lambda}{r+\lambda}\int_0^{\bar{x}} xm(x)\mathrm{d}x\right]p_A \\
&\quad + \left[\frac{\delta}{r+\delta}\int_0^{\bar{x}} f(x)\mathrm{d}x + \frac{\beta}{r+\beta}\left(\int_0^{\bar{x}} m(x)\mathrm{d}x + M_L\right)\right]p_B
\end{aligned}$$

sub. to

$$p_A - p_B \leq \frac{1}{r+\delta+\beta}w_A, \tag{$\mathrm{IC}_1^A$}$$

$$p_A - p_B \geq 0, \tag{$\mathrm{IC}_0^B$}$$

$$p_A \leq \frac{1}{r+\delta+\beta}w_A, \tag{R}$$

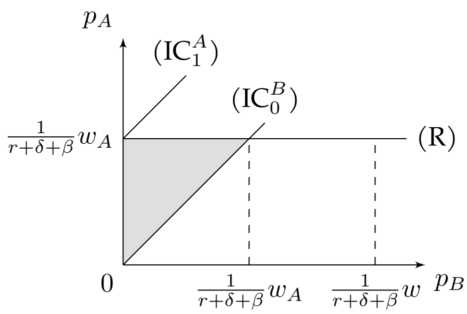


**Figure C.2.** The feasibility set of the seller's subproblem if buyers replace the $B$-gizmo with the $A$-gizmo immediately upon learning that they are the $H$-types. Parameters: $(r,\delta,\beta,w,w_A)=(1,1,1,2,1)$.

$$V_A(\bar{x},p_A,p_B)-p_A\geq 0, \qquad (\mathrm{IR}_{\bar{x}}^A)$$
$$V_B(0,p_A,p_B)-p_B\geq 0, \qquad (\mathrm{IR}_0^B)$$

where $\bar{x}:=\bar{x}=\frac{p_A-p_B}{\frac{1}{r+\delta+\beta}w_A-\frac{\lambda}{r+\delta+\beta+\lambda}\left(\frac{1}{r+\delta+\beta}w_A-p_A\right)}$ by Proposition 3.1. Note that

$$\bar{x}'_{p_A}=\frac{\frac{1}{r+\delta+\beta}w_A-\frac{\lambda}{r+\delta+\beta+\lambda}\left(\frac{1}{r+\delta+\beta}w_A-p_B\right)}{\left[\frac{1}{r+\delta+\beta}w_A-\frac{\lambda}{r+\delta+\beta+\lambda}\left(\frac{1}{r+\delta+\beta}w_A-p_A\right)\right]^2}=\frac{1-\frac{\lambda}{r+\delta+\beta+\lambda}\bar{x}}{\frac{1}{r+\delta+\beta}w_A-\frac{\lambda}{r+\delta+\beta+\lambda}\left(\frac{1}{r+\delta+\beta}w_A-p_A\right)},$$

$$\bar{x}'_{p_B}=-\frac{1}{\frac{1}{r+\delta+\beta}w_A-\frac{\lambda}{r+\delta+\beta+\lambda}\left(\frac{1}{r+\delta+\beta}w_A-p_A\right)}.$$

The seller gets $\frac{\lambda}{r+\lambda}\int_0^{\bar{x}}xm(x)\mathrm{d}x\cdot p_A$ from buyers with the $B$-gizmo who learn that they are the $H$-types and immediately replace their gizmo with the $A$-gizmo. The constraint (R) captures that the price of the $A$-gizmo is relatively low for these buyers to be willing to replace their $B$-gizmo immediately. The remaining constraints are similar to those of the previous subproblem.

Given the value function of the buyer with the $A$-gizmo in (C.1), the constraint $(\mathrm{IR}_{\bar{x}}^A)$ is equivalent to

$$p_A\leq\frac{1}{r+\delta+\beta}(w+w_A\bar{x})+\frac{\beta\lambda}{(r+\delta+\beta)(r+\delta+\lambda)}(1-\bar{x})(p_A-p_B).$$

Given the value function of the buyer with the $B$-gizmo in (C.2), the constraint $(\mathrm{IR}_0^B)$ is equivalent to

$$p_B\leq\frac{1}{r+\delta+\beta}w \quad\Leftrightarrow\quad p_A\leq\frac{1}{r+\delta+\beta}w+\bar{x}\left[\frac{1}{r+\delta+\beta}w_A-\frac{\lambda}{r+\delta+\beta+\lambda}\left(\frac{1}{r+\delta+\beta}w_A-p_A\right)\right],$$

taking into account the definition of $\bar{x}$. It follows from $(\mathrm{IR}_0^B)$ and (R) that $(\mathrm{IR}_{\bar{x}}^A)$ is redundant. The constraint (R) also implies $(\mathrm{IC}_1^A)$. Furthermore, (R) implies $(\mathrm{IR}_0^B)$ when $w>w_A$. The feasibility set is the shaded area in Figure C.2.

The Lagrangian takes the following form

$$\mathcal{L}(p_A,p_B,\alpha_1,\alpha_2)$$
$$=\left[\frac{\delta}{r+\delta}\int_{\bar{x}}^1 f(x)\mathrm{d}x+\frac{\beta}{r+\beta}\left(\int_{\bar{x}}^1 m(x)\mathrm{d}x+M_H\right)+\frac{\lambda}{r+\lambda}\int_0^{\bar{x}}xm(x)\mathrm{d}x\right]p_A$$

$$+ \left[ \tfrac{\delta}{r+\delta} \int_0^{\bar{x}} f(x)\mathrm{d}x + \tfrac{\beta}{r+\beta} \left( \int_0^{\bar{x}} m(x)\mathrm{d}x + M_L \right) \right] p_B$$
$$+ \alpha_1(p_A - p_B) + \alpha_2 \left( \tfrac{1}{r+\delta+\beta} w_A - p_A \right).$$

The Kuhn-Tucker conditions are necessary for the optimum and are as follows:

$$\mathcal{L}'_{p_A} = \tfrac{\delta}{r+\delta} \int_{\bar{x}}^1 f(x)\mathrm{d}x + \tfrac{\beta}{r+\beta} \left( \int_{\bar{x}}^1 m(x)\mathrm{d}x + M_H \right) + \tfrac{\lambda}{r+\lambda} \int_0^{\bar{x}} xm(x)\mathrm{d}x$$
$$+ \tfrac{\lambda}{r+\lambda} \bar{x} m(\bar{x}) \frac{1 - \frac{\lambda}{r+\delta+\beta+\lambda}\bar{x}}{\frac{1}{r+\delta+\beta}w_A - \frac{\lambda}{r+\delta+\beta+\lambda}\left(\frac{1}{r+\delta+\beta}w_A - p_A\right)} p_A$$
$$- \left( \tfrac{\delta}{r+\delta} f(\bar{x}) + \tfrac{\beta}{r+\beta} m(\bar{x}) \right) \bar{x} \left( 1 - \tfrac{\lambda}{r+\delta+\beta+\lambda}\bar{x} \right) + \alpha_1 - \alpha_2 = 0,$$

$$\mathcal{L}'_{p_B} = \tfrac{\delta}{r+\delta} \int_0^{\bar{x}} f(x)\mathrm{d}x + \tfrac{\beta}{r+\beta} \left( \int_0^{\bar{x}} m(x)\mathrm{d}x + M_L \right)$$
$$- \tfrac{\lambda}{r+\lambda} \bar{x} m(\bar{x}) \frac{1}{\frac{1}{r+\delta+\beta}w_A - \frac{\lambda}{r+\delta+\beta+\lambda}\left(\frac{1}{r+\delta+\beta}w_A - p_A\right)} p_A + \left( \tfrac{\delta}{r+\delta} f(\bar{x}) + \tfrac{\beta}{r+\beta} m(\bar{x}) \right) \bar{x} - \alpha_1 = 0,$$

$$\alpha_1(p_A - p_B) = 0, \quad \alpha_1 \geq 0,$$
$$\alpha_2 \left( \tfrac{1}{r+\delta+\beta} w_A - p_A \right) = 0, \quad \alpha_2 \geq 0.$$

Suppose $p_A < \tfrac{1}{r+\delta+\beta} w_A$, and so $\alpha_2 = 0$. It follows that $\mathcal{L}'_{p_A} + \mathcal{L}'_{p_B} = 0$ is equivalent to

$$\tfrac{\delta}{r+\delta} + \tfrac{\beta}{r+\beta} + \tfrac{\lambda}{r+\lambda} \int_0^{\bar{x}} xm(x)\mathrm{d}x - \tfrac{\lambda}{r+\delta+\beta+\lambda} \tfrac{\lambda}{r+\lambda} \bar{x}^2 m(\bar{x}) \frac{1}{\frac{1}{r+\delta+\beta}w_A - \frac{\lambda}{r+\delta+\beta+\lambda}\left(\frac{1}{r+\delta+\beta}w_A - p_A\right)} p_A$$
$$+ \tfrac{\lambda}{r+\delta+\beta+\lambda} \left( \tfrac{\delta}{r+\delta} f(\bar{x}) + \tfrac{\beta}{r+\beta} m(\bar{x}) \right) \bar{x}^2 = 0,$$

which implies

$$\tfrac{\lambda}{r+\delta+\beta+\lambda} \bar{x} \mathcal{L}'_{p_B} = \tfrac{\lambda}{r+\delta+\beta+\lambda} \bar{x} \left[ \tfrac{\delta}{r+\delta} \int_0^{\bar{x}} f(x)\mathrm{d}x + \tfrac{\beta}{r+\beta} \left( \int_0^{\bar{x}} m(x)\mathrm{d}x + M_L \right) \right]$$
$$- \tfrac{\delta}{r+\delta} - \tfrac{\beta}{r+\beta} - \tfrac{\lambda}{r+\lambda} \int_0^{\bar{x}} xm(x)\mathrm{d}x - \tfrac{\lambda}{r+\delta+\beta+\lambda} \bar{x} \alpha_1 < 0,$$

a contradiction. Therefore, $p_A^* = \tfrac{1}{r+\delta+\beta} w_A$. That is, the constraint (R) binds.

Given $p_A^* = \tfrac{1}{r+\delta+\beta} w_A$ and the definition of $\bar{x}$, the seller's subproblem can be rewritten as follows:

$$\max_{\bar{x} \in [0,1]} \left[ \tfrac{\delta}{r+\delta} \int_{\bar{x}}^1 f(x)\mathrm{d}x + \tfrac{\beta}{r+\beta} \left( \int_{\bar{x}}^1 m(x)\mathrm{d}x + M_H \right) + \tfrac{\lambda}{r+\lambda} \int_0^{\bar{x}} xm(x)\mathrm{d}x \right] \tfrac{1}{r+\delta+\beta} w_A$$
$$+ \left[ \tfrac{\delta}{r+\delta} \int_0^{\bar{x}} f(x)\mathrm{d}x + \tfrac{\beta}{r+\beta} \left( \int_0^{\bar{x}} m(x)\mathrm{d}x + M_L \right) \right] \tfrac{1}{r+\delta+\beta} w_A (1 - \bar{x}).$$

The first-order derivative with respect to $\bar{x}$ divided by $\frac{1}{r+\delta+\beta} w_A$ is equal to

$$\tfrac{\lambda}{r+\lambda}\bar{x}m(\bar{x}) - \left(\tfrac{\delta}{r+\delta}f(\bar{x}) + \tfrac{\beta}{r+\beta}m(\bar{x})\right)\bar{x} \\ - \left[\tfrac{\delta}{r+\delta}\int_0^{\bar{x}} f(x)\mathrm{d}x + \tfrac{\beta}{r+\beta}\left(\int_0^{\bar{x}} m(x)\mathrm{d}x + M_L\right)\right] < 0,$$

because

$$\tfrac{\lambda}{r+\lambda}m(\bar{x}) - \tfrac{\delta}{r+\delta}f(\bar{x}) = \left(\tfrac{\lambda}{r+\lambda}\tfrac{\delta}{\delta+\lambda} - \tfrac{\delta}{r+\delta}\right)f(\bar{x}) = -\tfrac{\delta(r\delta+\lambda^2)}{(r+\lambda)(\delta+\lambda)(r+\delta)}f(\bar{x}).$$

It follows that $\bar{x}^* = 0$, and so $p_B^* = \frac{1}{r+\delta+\beta}w_A$. In particular, this implies that only the $L$-types who have learned their type buy the $B$-gizmo.

The seller's profit is equal to

$$\Pi_{(\mathrm{R})} \coloneqq \Pi(p_A^*, p_B^*) = \left[\tfrac{\delta}{r+\delta} + \tfrac{\beta}{r+\beta}\tfrac{\delta}{\delta+\lambda}\left(1 + \tfrac{\lambda}{\delta}\int_0^1 xf(x)\mathrm{d}x\right)\right]\tfrac{1}{r+\delta+\beta}w_A.$$

Note that $\Pi_{(\mathrm{R})} < \Pi_{(\mathrm{NR})}$ when $w > w_A$.

### C.2.3 Planned Obsolescence

**The optimal cut-off $\bar{x}^* = 1$.** If $\frac{\beta}{r+\beta}M_H \geq \frac{\delta}{r+\delta}f(1) + \frac{\beta}{r+\beta}m(1)$, then the seller's profit as a function of the breakdown rate $\beta$ is equal to

$$\Pi(\beta) \coloneqq \tfrac{1}{r+\delta+\beta}\left[\tfrac{\beta}{r+\beta}M_H w_A + \left(\tfrac{\delta}{r+\delta} + \tfrac{\beta}{r+\beta}\right)w\right].$$

The first-order derivative with respect to $\beta$ is

$$\Pi_\beta'(\beta) = \tfrac{1}{(r+\beta)^2(r+\delta+\beta)^2}\left[\left(r(r+\delta) - \beta^2\right)(M_H w_A + w) - \tfrac{\delta}{r+\delta}(r+\beta)^2 w\right].$$

Note that $\Pi_\beta' > 0$ for $\beta \in [0, \beta^*)$ and $\Pi_\beta' < 0$ for $\beta \in (\beta^*, \infty)$, where

$$\beta^* \coloneqq \tfrac{\sqrt{rC[rC+\delta(C+D)]}-rD}{C+D}$$

with $C \coloneqq M_H w_A + w$ and $D \coloneqq \frac{\delta}{r+\delta}w$.[1] It follows that the seller's profit $\Pi(\beta)$ is maximized at $\beta = \beta^*$.

[1]Consider a quadratic equation in $\beta$

$$[r(r+\delta) - \beta^2]C - (r+\beta)^2 D = r(r+\delta)C - r^2 D - 2rD\beta - (C+D)\beta^2 = 0.$$

The determinant is equal to $\frac{\Delta}{4} = r^2D^2 + (C+D)[r(r+\delta)C - r^2D] = rC[rC + \delta(C+D)]$. The solution to the equation above is as follows:

$$\beta_1 = \tfrac{rD+\sqrt{rC[rC+\delta(C+D)]}}{-(C+D)}, \quad \beta_2 = \tfrac{rD-\sqrt{rC[rC+\delta(C+D)]}}{-(C+D)}.$$

For $C \coloneqq M_H w_A + w$ and $D \coloneqq \frac{\delta}{r+\delta}w$, note that $\beta_1 < 0$, while $\beta_2 > 0$ because $C > D$.

**The optimal cut-off $\bar{x}^* \in (0,1)$.** If $\frac{\beta}{r+\beta}M_H < \frac{\delta}{r+\delta}f(1) + \frac{\beta}{r+\beta}m(1)$, then the seller's profit as a function of the breakdown rate $\beta$ is equal to

$$\Pi(\beta) := \tfrac{1}{r+\delta+\beta}\left[\left[\tfrac{\delta}{r+\delta}\int_{\bar{x}^*}^{1} f(x)\mathrm{d}x + \tfrac{\beta}{r+\beta}\left(\int_{\bar{x}^*}^{1} m(x)\mathrm{d}x + M_H\right)\right] w_A\bar{x}^* + \left(\tfrac{\delta}{r+\delta} + \tfrac{\beta}{r+\beta}\right) w\right].$$

Applying the Envelope theorem, the first-order derivative with respect to $\beta$ is

$$\begin{aligned}\Pi'_\beta(\beta) = \tfrac{1}{(r+\beta)^2(r+\delta+\beta)^2}\Bigg[&(r(r+\delta)-\beta^2)\left[\left(\int_{\bar{x}^*}^{1} m(x)\mathrm{d}x + M_H\right) w_A\bar{x}^* + w\right] \\ &- \tfrac{\delta}{r+\delta}(r+\beta)^2\left(\int_{\bar{x}^*}^{1} f(x)\mathrm{d}x \cdot w_A\bar{x}^* + w\right)\Bigg].\end{aligned}$$

Note that $\Pi'_\beta > 0$ for $\beta \in [0,\beta^*)$ and $\Pi'_\beta < 0$ for $\beta \in (\beta^*,\infty)$, where

$$\beta^* := \tfrac{\sqrt{rC[rC+\delta(C+D)]}-rD}{C+D}$$

with $C := \left(\int_{\bar{x}^*}^{1} m(x)\mathrm{d}x + M_H\right) w_A\bar{x}^* + w$ and $D := \frac{\delta}{r+\delta}\left(\int_{\bar{x}^*}^{1} f(x)\mathrm{d}x \cdot w_A\bar{x}^* + w\right)$.[2] It follows that the seller's profit $\Pi(\beta)$ is maximized at $\beta = \beta^*$.

[2]See footnote 1. For $C := \left(\int_{\bar{x}^*}^{1} m(x)\mathrm{d}x + M_H\right) w_A\bar{x}^* + w$ and $D := \frac{\delta}{r+\delta}\left(\int_{\bar{x}^*}^{1} f(x)\mathrm{d}x \cdot w_A\bar{x}^* + w\right)$, note that $\beta_2 > \beta_1$ and $\beta_1 < 0$, while $\beta_2 > 0$ when $w > w_A$. Indeed, if $w > w_A$, then $0 \in (\beta_1, \beta_2)$ because

$$\begin{aligned}r(r+\delta)C - r^2D &> r\left[(r+\delta)w - \tfrac{r\delta}{r+\delta}\left(\int_{\bar{x}^*}^{1} f(x)\mathrm{d}x \cdot w_A\bar{x}^* + w\right)\right] \\ &> r\left[(r+\delta)w - \tfrac{r\delta}{r+\delta}\cdot 2w\right] = \tfrac{r(r-\delta)^2}{r+\delta}w \geq 0.\end{aligned}$$

# Appendix D

# Addendum to Chapter 4

## D.1 Another Tie-Breaking Rule

I show that, if the seller uses an alternative tie-breaking assumption, then the equilibrium behavior of buyers for given unit and bundle prices seems unnatural. In particular, the multi-unit single-crossing property does not hold anymore.

**Assumption D.1.** *If one buyer accepts the offer of the unit, while another buyer accepts the offer of the bundle, then the seller sells one unit.*

**Lemma D.1** (No Single-Crossing Property)**.** *Under Assumptions 4.1, D.1, and 4.3, there are unit and bundle prices such that buyers with the high valuation accept the offer of the unit, while buyers with the low valuation accept the offer of the bundle.*

Figure D.1, which is a counterpart of Figure 4.1, shows the equilibrium strategies of buyers with high and low valuations for given unit and bundle prices under Assumption D.1 rather than under Assumption 4.2. If $1 - \frac{1}{2}(2\gamma - q) \geq p > v$ and $q \leq 2\gamma v$, then there is an equilibrium in the game between buyers, in which buyers with the high valuation accept the offer of the unit, while buyers with the low valuation accept the offer of the bundle, that is, $(\bar{\sigma}^B, \underline{\sigma}^B) = (U, B)$.

## D.2 Proofs

### D.2.1 Proof of Lemma 4.1

Individual rationality of the seller implies that $p \geq 0$ and $q \geq 0$. Furthermore, because only equilibria in pure strategies are of interest here, there are nine equilibrium candidates $(\bar{\sigma}^B, \underline{\sigma}^B)$ for given unit and bundle prices: $(N, N)$, $(U, N)$, $(N, U)$, $(U, U)$, $(B, N)$, $(N, B)$, $(B, U)$, $(U, B)$, or $(B, B)$. I consider each pair $(\bar{\sigma}^B, \underline{\sigma}^B)$ separately, and look for prices (if any), for which the pair of strategies forms an equilibrium. While writing the inequalities which capture buyers' preferences over the offers, I take into account that the seller uses the tie-breaking rules given by Assumptions 4.1 and 4.2, and buyers behave according to Assumption 4.3 in case of indifference.

**Case with $(\bar{\sigma}^B, \underline{\sigma}^B) = (N, N)$.** Buyers with both high and low valuations reject both offers.

– Preferences of buyers with the high valuation:

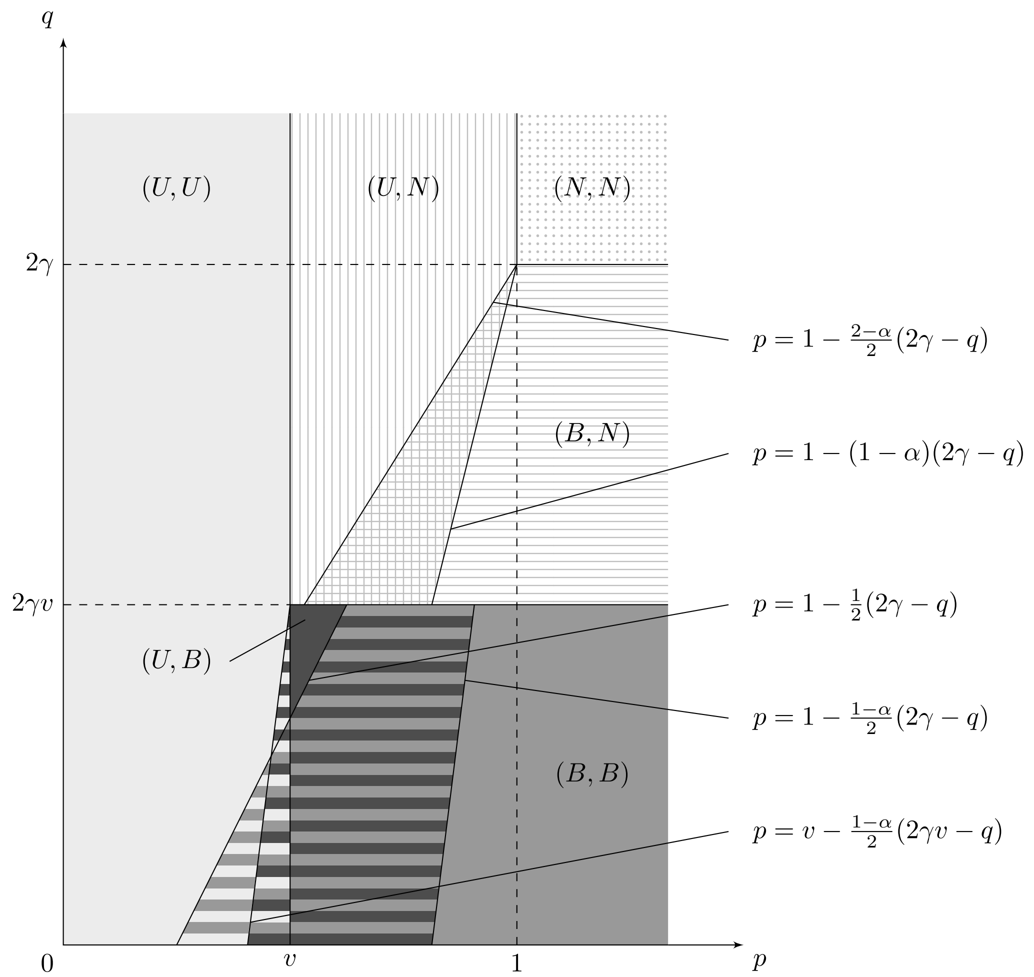


**Figure D.1.** The equilibrium strategies of buyers $(\bar{\sigma}^B, \underline{\sigma}^B)$ given unit and bundle prices $(p, q)$ and under Assumption D.1 rather than under Assumption 4.2. Parameters: $(v, \gamma, \alpha) = \left(\frac{1}{2}, \frac{3}{4}, \frac{3}{4}\right)$.

- $N \succ B$:

$$0 > 2\gamma - q \quad \Leftrightarrow \quad q > 2\gamma;$$

- $N \succ U$:

$$0 > 1 - p \quad \Leftrightarrow \quad p > 1.$$

– Preferences of buyers with the low valuation:
- $N \succ B$:

$$0 > 2\gamma v - q \quad \Leftrightarrow \quad q > 2\gamma v;$$

- $N \succ U$:

$$0 > v - p \quad \Leftrightarrow \quad p > v.$$

Therefore, this is an equilibrium when the unit and bundle prices satisfy $p > 1$ and $q > 2\gamma$.

**Case with $(\bar{\sigma}^B, \underline{\sigma}^B) = (U, N)$.** Buyers with the high valuation accept the offer of the unit, while buyers with the low valuation reject both offers.

– Preferences of buyers with the high valuation:
- $U \succ B$:

$$1 - p > 2\gamma - q \quad \Leftrightarrow \quad q > 2\gamma - 1 + p;$$

– $U \succ N$:

$$1 - p \geq 0 \quad \Leftrightarrow \quad p \leq 1.$$

– Preferences of buyers with the low valuation:

– $N \succ B$:

$$0 > 2\gamma v - q \quad \Leftrightarrow \quad q > 2\gamma v;$$

– $N \succ U$:

$$0 > v - p \quad \Leftrightarrow \quad p > v.$$

Note that $2\gamma - 1 + p > 2\gamma v$ if and only if $2\gamma(1 - v) > 1 - p$, which is always the case for $\gamma \in \left(\frac{1}{2}, 1\right)$ and $p > v$. Therefore, this is an equilibrium when the unit and bundle prices satisfy $1 \geq p > v$ and $q > 2\gamma - 1 + p$.

**Case with $(\bar{\sigma}^B, \underline{\sigma}^B) = (N, U)$.** Buyers with the high valuation reject both offers, which buyers with the low valuation accept the offer of the unit.

– Preferences of buyers with the high valuation:

– $N \succ U$:

$$0 > 1 - p \quad \Leftrightarrow \quad p > 1.$$

– Preferences of buyers with the low valuation:

– $U \succ N$:

$$v - p \geq 0 \quad \Leftrightarrow \quad p \leq v.$$

Because $v \in (0, 1)$, this gives a contradiction. Therefore, this is never an equilibrium.

**Case with $(\bar{\sigma}^B, \underline{\sigma}^B) = (U, U)$.** Buyers with both high and low valuations accept the offer of the unit.

– Preferences of buyers with the high valuation:

– $U \succ B$:

$$1 - p > 2\gamma - q \quad \Leftrightarrow \quad q > 2\gamma - 1 + p;$$

– $U \succ N$:

$$1 - p \geq 0 \quad \Leftrightarrow \quad p \leq 1.$$

– Preferences of buyers with the low valuation:

– $U \succ B$:

$$v - p > 2\gamma v - q \quad \Leftrightarrow \quad q > (2\gamma - 1)v + p;$$

– $U \succ N$:

$$v - p \geq 0 \quad \Leftrightarrow \quad p \leq v.$$

Observe that $2\gamma - 1 + p > (2\gamma - 1)v + p$ if and only if $2\gamma - 1 > (2\gamma - 1)v$, which is always a case for $\gamma \in \left(\frac{1}{2}, 1\right)$ and $v \in (0, 1)$. Therefore, this is an equilibrium when the unit and bundle prices satisfy $p \leq v$ and $q > 2\gamma - 1 + p$.

**Case with $(\bar{\sigma}^B, \underline{\sigma}^B) = (B, N)$.** Buyers with the high valuation accept the offer of the bundle, while buyers with the low valuation reject both offers.

– Preferences of buyers with the high valuation:

– $B \succ U$:

$$\alpha \cdot \tfrac{1}{2}(2\gamma - q) + (1 - \alpha) \cdot (2\gamma - q) \geq \alpha \cdot 0 + (1 - \alpha) \cdot (1 - p),$$

$$q \leq 2\gamma - \tfrac{2(1-\alpha)}{2-\alpha}(1-p);$$

- $B \succ N$:

$$\alpha \cdot \tfrac{1}{2}(2\gamma - q) + (1-\alpha) \cdot (2\gamma - q) \geq 0 \quad \Leftrightarrow \quad q \leq 2\gamma.$$

- Preferences of buyers with the low valuations:
  - $N \succ B$:

$$0 > \alpha \cdot \tfrac{1}{2}(2\gamma v - q) + (1-\alpha) \cdot (2\gamma v - q) \quad \Leftrightarrow \quad q > 2\gamma v;$$

  - $N \succ U$:

$$0 > \alpha \cdot 0 + (1-\alpha) \cdot (v - p) \quad \Leftrightarrow \quad p > v.$$

Therefore, this is an equilibrium when the unit and bundle prices satisfy the four inequalities above.

**Case with** $(\bar{\sigma}^B, \underline{\sigma}^B) = (N, B)$**.** Buyers with the high valuation reject both offers, while buyers with the low valuation accept the offer of the bundle.

- Preferences of buyers with the high valuation:
  - $N \succ B$:

$$0 > \alpha \cdot (2\gamma - q) + (1-\alpha) \cdot \tfrac{1}{2}(2\gamma - q) \quad \Leftrightarrow \quad q > 2\gamma.$$

- Preferences of buyers with the low valuation:
  - $B \succ N$:

$$\alpha \cdot (2\gamma v - q) + (1-\alpha) \cdot \tfrac{1}{2}(2\gamma v - q) \geq 0 \quad \Leftrightarrow \quad q \leq 2\gamma v.$$

Because $v \in (0,1)$, this gives a contradiction. Therefore, this is never an equilibrium.

**Case with** $(\bar{\sigma}^B, \underline{\sigma}^B) = (B, U)$**.** Buyers with the high valuation accept the offer of the bundle, while buyers with the low valuation accept the offer of the unit.

- Preferences of buyers with the high valuation:
  - $B \succ U$:

$$\alpha \cdot \tfrac{1}{2}(2\gamma - q) + (1-\alpha) \cdot (2\gamma - q) \geq \alpha \cdot 0 + (1-\alpha) \cdot (1-p),$$
$$q \leq 2\gamma - \tfrac{2(1-\alpha)}{2-\alpha}(1-p);$$

  - $B \succ N$:

$$\alpha \cdot \tfrac{1}{2}(2\gamma - q) + (1-\alpha) \cdot (2\gamma - q) \geq 0 \quad \Leftrightarrow \quad q \leq 2\gamma.$$

- Preferences of buyers with the low valuation:
  - $U \succ B$:

$$\alpha \cdot 0 + (1-\alpha) \cdot (v - p) > \alpha \cdot \tfrac{1}{2}(2\gamma v - q) + (1-\alpha) \cdot (2\gamma v - q),$$
$$q > 2\gamma v - \tfrac{2(1-\alpha)}{2-\alpha}(v-p);$$

  - $U \succ N$:

$$\alpha \cdot 0 + (1-\alpha) \cdot (v - p) \geq 0 \quad \Leftrightarrow \quad p \leq v.$$

Therefore, this is an equilibrium when the unit and bundle prices satisfy $p \leq v$ and $2\gamma - \frac{2(1-\alpha)}{2-\alpha}(1-p) \geq q > 2\gamma v - \frac{2(1-\alpha)}{2-\alpha}(v-p)$.

**Case with** $(\bar{\sigma}^B, \underline{\sigma}^B) = (U, B)$**.** Buyers with the high valuation accept the offer of the unit, while buyers with the low valuation accept the offer of the bundle.

- Preferences of buyers with the high valuation:
  - $U \succ B$:

$$\alpha \cdot (1-p) + (1-\alpha) \cdot 0 > \alpha \cdot (2\gamma - q) + (1-\alpha) \cdot \tfrac{1}{2}(2\gamma - q),$$
$$q > 2\gamma - \tfrac{2\alpha}{1+\alpha}(1-p).$$

- Preferences of buyers with the low valuation:
  - $B \succ U$:

$$\alpha \cdot (2\gamma v - q) + (1-\alpha) \cdot \tfrac{1}{2}(2\gamma v - q) \geq \alpha \cdot (v-p) + (1-\alpha) \cdot 0,$$
$$q \leq 2\gamma v - \tfrac{2\alpha}{1+\alpha}(v-p).$$

Note that $2\gamma - \frac{2\alpha}{1+\alpha}(1-p) > 2\gamma v - \frac{2\alpha}{1+\alpha}(v-p)$ if and only if $2\left(\gamma - \frac{\alpha}{1+\alpha}\right) > 2\left(\gamma - \frac{\alpha}{1+\alpha}\right) v$, which is always the case for $v \in (0,1)$, $\gamma \in \left(\frac{1}{2}, 1\right)$, and $\alpha \in (0,1)$ because $\frac{\alpha}{1+\alpha} \in \left(0, \frac{1}{2}\right)$. It is a contradiction. Therefore, this is never an equilibrium.

**Case with** $(\bar{\sigma}^B, \underline{\sigma}^B) = (B, B)$**.** Buyers with both high and low valuations accept the offer of the bundle.

- Preferences of buyers with the high valuation:
  - $B \succ U$ and $B \succ N$:

$$\tfrac{1}{2}(2\gamma - q) \geq 0 \quad \Leftrightarrow \quad q \leq 2\gamma.$$

- Preferences of buyers with the low valuation:
  - $B \succ U$ and $B \succ N$:

$$\tfrac{1}{2}(2\gamma v - q) \geq 0 \quad \Leftrightarrow \quad q \leq 2\gamma v.$$

Therefore, this is an equilibrium for any unit price $p$ and the bundle price that satisfies $q \leq 2\gamma v$.

### D.2.2 Proof of Proposition 4.1

It follows from Lemma 4.1 that, for given unit and bundle prices, the equilibrium pair of strategies $(\bar{\sigma}^B, \underline{\sigma}^B)$ is necessary one of the following six: $(N, N)$, $(U, N)$, $(U, U)$, $(B, N)$, $(B, U)$, and $(B, B)$.

#### Candidate Revenue

I consider each pair of the six strategies separately. I look for the largest revenue the seller can get by posting prices that support such a pair of strategies in an equilibrium.

**Case with** $(\bar{\sigma}^B, \underline{\sigma}^B) = (N, N)$**.** For $p > 1$ and $q > 2\gamma$,

$$\mathcal{R} = 0.$$

**Case with** $(\bar{\sigma}^B, \underline{\sigma}^B) = (U, N)$**.** For $1 \geq p > v$ and $q > 2\gamma - 1 + p$,

$$\mathcal{R} = \alpha^2 \cdot 2p + 2\alpha(1-\alpha) \cdot p = 2\alpha p.$$

The revenue is the largest for $p = 1$ and $q > 2\gamma$, and is equal to

$$\mathcal{R}_{UN} = 2\alpha.$$

**Case with** $(\bar{\sigma}^B, \underline{\sigma}^B) = (U, U)$**.** For $p \leq v$ and $q > 2\gamma - 1 + p$,

$$\mathcal{R} = 2p.$$

The revenue is the largest for $p = v$ and $q > 2\gamma - 1 + v$, and is equal to

$$\mathcal{R}_{UU} = 2v.$$

**Case with** $(\bar{\sigma}^B, \underline{\sigma}^B) = (B, N)$**.** For $p > v$, $q \leq 2\gamma - \frac{2(1-\alpha)}{1-\alpha}(1-p)$, $q \leq 2\gamma$, and $q > 2\gamma v$,

$$\mathcal{R} = [\alpha^2 + 2\alpha(1-\alpha)] \cdot q = \alpha(2-\alpha)q.$$

The revenue is the largest for $p \geq 1$ and $q = 2\gamma$, and is equal to

$$\mathcal{R}_{BN} = 2\alpha(2-\alpha)\gamma.$$

**Case with** $(\bar{\sigma}^B, \underline{\sigma}^B) = (B, U)$**.** For $p \leq v$ and $2\gamma - \frac{2(1-\alpha)}{2-\alpha}(1-p) \geq q > 2\gamma v - \frac{2(1-\alpha)}{2-\alpha}(v-p)$,

$$\mathcal{R} = [\alpha^2 + 2\alpha(1-\alpha)] \cdot q + (1-\alpha)^2 \cdot 2p = \alpha(2-\alpha)q + 2(1-\alpha)^2 p.$$

The revenue is the largest for $p = v$ and $q = 2\gamma - \frac{2(1-\alpha)}{2-\alpha}(1-v)$, and is equal to

$$\mathcal{R}_{BU} = 2\alpha(2-\alpha)\gamma + 2(1-\alpha)(v-\alpha).$$

**Case with** $(\bar{\sigma}^B, \underline{\sigma}^B) = (B, B)$**.** For any $p \geq 0$ and $q \leq 2\gamma v$,

$$\mathcal{R} = q < \mathcal{R}_{UU} = 2v.$$

Therefore, the pair of strategies $(\bar{\sigma}^B, \underline{\sigma}^B) = (B, B)$ never occurs in equilibrium.

**Equilibrium Revenue**

It follows that the equilibrium revenue is necessary one of the following four revenues: $\mathcal{R}_{UU}$, $\mathcal{R}_{BU}$, $\mathcal{R}_{UN}$, and $\mathcal{R}_{BN}$. If the seller can post unit and bundle prices that result in a unique reaction of buyers and give him the largest revenue for given parameters, then this is the unique equilibrium in terms of revenue. This is the case whenever $\mathcal{R}_{UU}$, $\mathcal{R}_{UN}$, or $\mathcal{R}_{BN}$ is the largest for given parameters. There are multiple equilibria in terms of revenue, specifically, whenever $\mathcal{R}_{BU}$ the largest revenue for given parameters.

$\mathcal{R}_{UU}$ **vs.** $\mathcal{R}_{BU}$**.** The seller is better off by selling the product unit-by-unit to buyers with both high and low valuations rather than selling the bundle to buyers with the high valuation and the unit to buyers with the low valuation if and only if

$$\begin{aligned} \mathcal{R}_{UU} &> \mathcal{R}_{BU}, \\ 2v &> 2\alpha(2-\alpha)\gamma + 2(1-\alpha)(v-\alpha), \\ \alpha v &> \alpha(2-\alpha)\gamma - \alpha(1-\alpha), \\ v &> (2-\alpha)\gamma - 1 + \alpha. \end{aligned}$$

**$\mathcal{R}_{UU}$ vs. $\mathcal{R}_{UN}$.** The seller is better off by selling the product unit-by-unit to buyers with both high and low valuations rather than only to buyers with the high valuation if and only if

$$\mathcal{R}_{UU} > \mathcal{R}_{UN} \quad \Leftrightarrow \quad 2v > 2\alpha \quad \Leftrightarrow \quad v > \alpha.$$

**$\mathcal{R}_{UU}$ vs. $\mathcal{R}_{BN}$.** The seller is better off by selling the product unit-by-unit to buyers with both high and low valuations rather than selling the bundle to buyers with the high valuation only if and only if

$$\mathcal{R}_{UU} > \mathcal{R}_{BN} \quad \Leftrightarrow \quad 2v > 2\alpha(2-\alpha)\gamma \quad \Leftrightarrow \quad v > \alpha(2-\alpha)\gamma.$$

**$\mathcal{R}_{BU}$ vs. $\mathcal{R}_{UN}$.** The seller is better off by selling the bundle to buyers with the high valuation and the unit to buyers with the low valuation rather than selling the product unit-by-unit to buyers with the high valuation only if and only if

$$\begin{aligned} \mathcal{R}_{BU} &> \mathcal{R}_{UN}, \\ 2\alpha(2-\alpha)\gamma + 2(1-\alpha)(v-\alpha) &> 2\alpha, \\ (1-\alpha)v &> -\alpha(2-\alpha)\gamma + \alpha(2-\alpha), \\ v &> \tfrac{\alpha}{1-\alpha}(2-\alpha)(1-\gamma). \end{aligned}$$

**$\mathcal{R}_{BU}$ vs. $\mathcal{R}_{BN}$.** The seller is better off by selling the bundle to buyers with the high valuation and the unit to buyers with the low valuation rather than selling the bundle to buyers with the high valuation only if and only if

$$\begin{aligned} \mathcal{R}_{BU} &> \mathcal{R}_{BN}, \\ 2\alpha(2-\alpha)\gamma + 2(1-\alpha)(v-\alpha) &> 2\alpha(2-\alpha)\gamma, \\ (1-\alpha)(v-\alpha) &> 0, \\ v &> \alpha. \end{aligned}$$

**$\mathcal{R}_{UN}$ vs. $\mathcal{R}_{BN}$.** The seller is better off by selling the product unit-by-unit to buyers with the high valuation only rather than selling them the bundle if and only if

$$\mathcal{R}_{UN} > \mathcal{R}_{BN} \quad \Leftrightarrow \quad 2\alpha > 2\alpha(2-\alpha)\gamma \quad \Leftrightarrow \quad \tfrac{1}{2-\alpha} > \gamma.$$

### D.2.3 Proof of Lemma 4.2

If the seller wants to screen buyers, then he must post the first-period prices such that buyers with different valuations react differently. Because I focus on equilibria in pure strategies only, there are six equilibrium candidates $(\bar{\sigma}_1^B, \underline{\sigma}_1^B)$ given the unit and bundle prices: $(U, N)$, $(N, U)$, $(B, N)$, $(N, B)$, $(B, U)$, or $(U, B)$. I consider each pair $(\bar{\sigma}_1^B, \underline{\sigma}_1^B)$ separately, and look for prices (if any), for which the pair of strategies forms an equilibrium. While writing the inequalities that capture buyers' preferences over the offers, I take into account that the seller uses the tie-breaking rules given by Assumptions 4.1 and 4.2, and that buyers behave according to Assumptions 4.3 and 4.4 in case of indifference. I find that the first-period equilibrium behavior of buyers $(\bar{\sigma}_1^B, \underline{\sigma}_1^B)$ is necessary one of the following three: $(U, N)$, $(B, N)$, and $(B, U)$. Furthermore, the seller's revenue is always $\mathcal{R}_{BU}$.

**Case with $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (U, N)$.** Buyers with the high valuation accept the offer of the unit in the first period, while buyers with the low valuation reject both offers. If the second period is reached, then beliefs of the seller (and of buyers) are as follows:

(i) if one unit is left for sale, then the buyer with the unit has the high valuation, while the buyer without the unit has the low valuation;

(ii) if two units remain for sale, then both buyers have the low valuation.

In (i), if the buyer with the high valuation rejects the second period offer, then her payoff is $1-p$, where $p$ is the price she paid for one unit of the product in the first period. If she accepts the offer, while another buyer does not, then she gets $2\gamma - (p + p') = 1 - p + (2\gamma - 1) - p'$. If she accept the offer and so does another buyers, then she gets $\frac{1}{2}(1 - p) + \frac{1}{2}(2\gamma - (p + p')) = 1 - p + \frac{1}{2}((2\gamma - 1) - p')$. Therefore, she accept the offer if and only if $2\gamma - 1 \geq p'$. If only the buyer with the low valuation accepts the second-period offer, then her payoff is $v - p'$. If both buyers accept the offer, then the buyer with the low valuation gets $\frac{1}{2}(v - p')$. Therefore, she does accept the offer if and only if $v \geq p'$. To maximize his revenue, the seller posts $p' = \max\{v, 2\gamma - 1\}$. It follows that the continuation payoff of the deviating buyer with the high valuation (and without the unit) is equal to $1 - v$ if $v > 2\gamma - 1$ (note that $p' = v$) and to $1 - \gamma$ if $2\gamma - 1 \geq v$ (note that $p' = 2\gamma - 1$); indeed, she gets the remaining unit and so gets $1 - (2\gamma - 1)$ with probability $\frac{1}{2}$. The payoff of non-deviating buyers is always $0$, because either they reject the offers or the seller extracts full surplus from them.

In (ii), the seller can gain $2v$ at most. For this to happen, buyers with the low valuation have to prefer the offer of the unit over the offer of the bundle when the unit price is $p'' = v$. This is the case whenever $v - p'' > 2\gamma v - q''$ and $p'' = v$. Therefore, the bundle price must satisfy $q'' > 2\gamma v$. It follows that, for $(p'', q'')$ to be a part of an equilibrium, the following must hold. In case with $(\bar{\sigma}_2^B, \underline{\sigma}_2^B) = (U, U)$, preferences of the deviating buyer with the high valuation are

– $U \succ B$:
$$1 - v > 2\gamma - q'' \quad \Leftrightarrow \quad q'' > 2\gamma - 1 + v;$$

– $U \succ N$:
$$1 - v \geq 0.$$

Therefore, $q'' > 2\gamma - 1 + v$ and the seller's revenue $\mathcal{R} = 2v$ independently of whether there is a deviating buyer or not. In case with $(\bar{\sigma}_2^B, \underline{\sigma}_2^B) = (B, U)$, preferences of the deviating buyer with the high valuations are

– $B \succ U$:
$$2\gamma - q'' \geq 1 - v \quad \Leftrightarrow \quad q'' \leq 2\gamma - 1 + v;$$

– $B \succ N$:

$$2\gamma - q'' \geq 0 \quad \Leftrightarrow \quad q'' \leq 2\gamma.$$

To maximize his revenue

$$\mathcal{R} = [\alpha^2 + 2\alpha(1-\alpha)] \cdot q'' + (1-\alpha)^2 \cdot 2v,$$

the seller posts $q'' = 2\gamma - 1 + v$, which gives him $\mathcal{R} = 2v + \alpha(2-\alpha)(2\gamma - 1 - v)$. All in all, if $v \geq 2\gamma - 1$, then the seller posts $p'' = v$ and $q'' > 2\gamma - 1 + v$, which lead to $(\bar{\sigma}_2^B, \underline{\sigma}_2^B) = (U, U)$ in the continuation game. If $v \leq 2\gamma - 1$, then he posts $p'' = v$ and $q'' = 2\gamma - 1 + v$, which lead to $(\bar{\sigma}_2^B, \underline{\sigma}_2^B) = (B, U)$. Therefore, the deviating buyer with the high valuation always gets $1 - v$ in the second period with two units.

Next I look for the first-period unit and bundle prices that support $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (U, N)$ in equilibrium.

– Preferences of buyers with the high valuation:
  – $U \succ B$:

$$(1-\alpha) \cdot (1-p) + (1-\alpha) \cdot (1-p) > \alpha \cdot (2\gamma - q) + (1-\alpha) \cdot (2\gamma - q),$$
$$q > 2\gamma - 1 + v;$$

  – $U \succ N$ if $v > 2\gamma - 1$ (i.e., if $p' = v$):

$$1 - p \geq \alpha \cdot (1-v) + (1-\alpha) \cdot (1-v),$$
$$p \leq v;$$

  – $U \succ N$ if $2\gamma - 1 \geq v$ (i.e., if $p' = 2\gamma - 1$):

$$1 - p \geq \alpha \cdot \tfrac{1}{2}(1 - (2\gamma - 1)) + (1-\alpha) \cdot (1-v) = \alpha(1-\gamma) + (1-\alpha)(1-v),$$
$$p \leq \alpha\gamma + (1-\alpha)v.$$

– Preferences of buyers with the low valuation:
  – $N \succ B$:

$$0 > \alpha \cdot (2\gamma v - q) + (1-\alpha) \cdot (2\gamma v - q),$$
$$q > 2\gamma v;$$

  – $N \succ U$:

$$0 > \alpha \cdot (v - p) + (1-\alpha) \cdot (v - p),$$
$$p > v.$$

Therefore, $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (U, N)$ can occur in equilibrium only if $2\gamma - 1 \geq v$. In such a case, $\gamma > \frac{1+v}{2} > v$ and $\alpha\gamma + (1-\alpha)v > v$.

The seller wants to maximize his revenue

$$\mathcal{R} = \alpha^2 \cdot 2p + 2\alpha(1-\alpha) \cdot (p + p') + (1-\alpha)^2 \cdot 2p''.$$

Therefore, the equilibrium unit and bundle prices are as follows:

$$p = \alpha\gamma + (1-\alpha)v \text{ and } q > 2\gamma - 1 + \alpha\gamma + (1-\alpha)v,$$

$$
\begin{gathered}
p' = 2\gamma - 1, \\
p'' = v \text{ and } q'' > 2\gamma - 1 + v, \text{ if } 2\gamma - 1 = v, \\
p'' = v \text{ and } q'' = 2\gamma - 1 + v, \text{ if } 2\gamma - 1 \geq v.
\end{gathered}
$$

The equilibrium revenue is equal to

$$\mathcal{R}_{BU} = 2\alpha(2-\alpha)\gamma + 2(1-\alpha)(v-\alpha).$$

**Case with $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (N, U)$.** Buyers with the high valuation rejects both offers in the first period, while buyers with the low valuation accept the offer of the unit. If the second period is reached, then beliefs of the seller (and of buyers) are as follows:

(i) if one unit is left for sale, then the buyer with the unit has the low valuation, while the buyer without the unit has the high valuation;

(ii) if two units remain for sale, then both buyers have the high valuation.

In (i), the buyer with the low valuation is of no interest to the seller, because he can get at most $(2\gamma-1)v$ from her. Indeed, if the buyer buys the second unit, then her payoff is $2\gamma v - (p+p') = v - p + (2\gamma-1)v - p'$. In contrast, the seller can get $1 > (2\gamma-1)v$ from the high-type buyer by posting $p' = 1$.

In (ii), the seller does best by selling the product unit-by-unit at the unit price $p'' = 1$ and the bundle price $q''$ such that buyers with the high valuation prefer the offer of the unit over the offer of a bundle, i.e.,

$$1 - p'' = 0 > 2\gamma - q'' \quad \Leftrightarrow \quad q'' > 2\gamma.$$

It follows that the seller always extracts full surplus from buyers with the high valuation in the second period. Therefore, their continuation payoff is $0$. The continuation payoff of the deviating buyer with whichever valuation is also $0$, because she does not buy anything.

For $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (N, U)$ to occur in equilibrium, the first-period unit and bundle prices must satisfy the following.

- Preferences of buyers with the high valuation:
  - $N \succ U$:

$$0 > 1 - p \quad \Leftrightarrow \quad p > 1.$$

- Preferences of buyers with the low valuation:
  - $U \succ N$:

$$v - p \geq 0 \quad \Leftrightarrow \quad p \leq v.$$

Because $v \in (0,1)$, this gives a contradiction. Therefore, $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (N, U)$ can never occur in equilibrium.

**Case with $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (B, N)$.** Buyers with the high valuation accept the offer of the bundle in the first period, while buyers with the low valuation reject both offers. If the second period is reached, then beliefs of the seller (and of buyers) are as follows:

(i) if one unit is left for sale (which occurs off path only), then the buyer without the unit has the low valuation, while the deviating buyer with the unit has the high valuation with probability $\alpha$;

(ii) if two units remain for sale, then both buyers have the low valuation.

In (i), if the deviating buyer with the unit has the high valuation, then she is willing to pay at most $2\gamma - 1$ for the second unit. If she has the low valuation, then she does not pay more than $(2\gamma - 1)v$. The buyer without the unit has the low valuation, and so she will pay at most $v$ for the remaining unit. Therefore, the seller can get $v$ with probability $1$, and he can get $2\gamma - 1$ with probability $\alpha$ if $2\gamma - 1 > v$ or with probability $1$ if $2\gamma - 1 \leq v$. The seller posts $p'$ equal to $v$ or $2\gamma - 1$ depending on which price yields him a higher revenue, i.e., on whether $v$ or $\alpha(2\gamma - 1)$ is greater. Therefore, the continuation payoff of the deviating buyer with the high valuation (and with the unit) is as follows:
- $0$ if $v \geq 2\gamma - 1$ (i.e., if $p' = v$) or if $\alpha(2\gamma - 1) \geq v$ (i.e., if $p' = 2\gamma - 1$);
- $\frac{1}{2}(2\gamma - 1 - v)$ if $2\gamma - 1 > v \geq \alpha(2\gamma - 1)$ (i.e., if $p' = v$).

The continuation payoff of the deviating buyer with the low valuation (and with the unit) is $0$.

In (ii), the seller posts the same unit and bundle prices which he would post if $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (U, N)$ were expected in the first period. That is, he posts $p'' = v$ and $q'' > 2\gamma - 1 + v$ if $v \geq 2\gamma - 1$ and $p'' = v$ and $q'' = 2\gamma - 1 + v$ if $v \leq 2\gamma - 1$, which lead to $(\bar{\sigma}_2^B, \underline{\sigma}_2^B) = (U, U)$ and $(\bar{\sigma}_2^B, \underline{\sigma}_2^B) = (B, U)$, respectively. Therefore, the continuation payoff of buyers with the low valuation is $0$, while the deviating buyer with the high valuation obtains $1 - v$.

Next I look for the first-period unit and bundle prices that support $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (B, N)$ in equilibrium.

- Preferences of buyers with the high valuation:
  - $B \succ U$ if their off-path continuation payoff is $0$:

$$\alpha \cdot \tfrac{1}{2}(2\gamma - q) + (1 - \alpha) \cdot (2\gamma - q) \geq \alpha \cdot 0 + (1 - \alpha) \cdot (1 - p),$$
$$\tfrac{2-\alpha}{2}(2\gamma - 1) \geq (1 - \alpha)(1 - p),$$
$$q \leq 2\gamma - \tfrac{2(1-\alpha)}{2-\alpha}(1 - p);$$

  - $B \succ U$ if their off-path continuation payoff is $\frac{1}{2}(2\gamma - 1 - v)$:

$$\alpha \cdot \tfrac{1}{2}(2\gamma - q) + (1 - \alpha) \cdot (2\gamma - q) \geq \alpha \cdot 0 + (1 - \alpha) \cdot \left(1 - p + \tfrac{1}{2}(2\gamma - 1 - v)\right),$$
$$q \leq 2\gamma - \tfrac{2(1-\alpha)}{2-\alpha}(1 - p) - \tfrac{1-\alpha}{2-\alpha}(2\gamma - 1 - v),$$
$$q \leq 2\gamma - \tfrac{2(1-\alpha)}{2-\alpha}(1 - v) + \tfrac{2(1-\alpha)}{2-\alpha}\left(p - \tfrac{1}{2}(2\gamma - 1 + v)\right);$$

  - $B \succ N$:

$$\alpha \cdot \tfrac{1}{2}(2\gamma - q) + (1 - \alpha) \cdot (2\gamma - q) \geq \alpha \cdot 0 + (1 - \alpha) \cdot (1 - v),$$
$$q \leq 2\gamma - \tfrac{2(1-\alpha)}{2-\alpha}(1 - v).$$

- Preferences of buyers with the low valuation:
  - $N \succ B$:

$$0 > \alpha \cdot \tfrac{1}{2}(2\gamma v - q) + (1 - \alpha) \cdot (2\gamma v - q),$$
$$q > 2\gamma v;$$

  - $N \succ U$:

$$0 > \alpha \cdot 0 + (1 - \alpha) \cdot (v - p),$$

$$p > v.$$

Note that $2\gamma - \frac{2(1-\alpha)}{2-\alpha}(1-v) > 2\gamma v$ if and only if $2\gamma(1-v) > \frac{2(1-\alpha)}{2-\alpha}(1-v)$, which is always the case for $\gamma \in \left(\frac{1}{2}, 1\right)$, $v \in (0,1)$, and $\alpha \in (0,1)$.

The seller wants to maximize his revenue

$$\mathcal{R} = [\alpha^2 + 2\alpha(1-\alpha)] \cdot q + (1-\alpha)^2 \cdot 2p''.$$

Therefore, the equilibrium unit and bundle prices are as follows:

$$\begin{gathered}
p > v \text{ and } q = 2\gamma - \tfrac{2(1-\alpha)}{2-\alpha}(1-v), \text{ if } v \geq 2\gamma - 1 \text{ or } \alpha(2\gamma-1) \geq v, \\
p \geq \tfrac{1}{2}(2\gamma - 1 + v) \text{ and } q = 2\gamma - \tfrac{2(1-\alpha)}{2-\alpha}(1-v), \text{ if } 2\gamma - 1 > v \geq \alpha(2\gamma - 1), \\
p' = v, \text{ if } v \geq \alpha(2\gamma-1), \\
p' = 2\gamma - 1, \text{ if } \alpha(2\gamma - 1) \geq v, \\
p'' = v \text{ and } q'' > 2\gamma - 1 + v, \text{ if } v \geq 2\gamma - 1, \\
p'' = v \text{ and } q'' = 2\gamma - 1 + v, \text{ if } 2\gamma - 1 \geq v.
\end{gathered}$$

The equilibrium revenue is equal to

$$\mathcal{R}_{BU} = 2\alpha(2-\alpha)\gamma + 2(1-\alpha)(v-\alpha).$$

**Case with $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (N, B)$.** Buyers with the high valuation rejects both offers in the first period, while buyers with the low valuation accept the offer of the bundle. If the second period is reached, then beliefs of the seller (and of buyers) are as follows:

(i) if one unit is left for sale (which occurs off path only), then the buyer without the unit has the high valuation, while the deviating buyer with the unit has the high valuation with probability $\alpha$;

(ii) if two units remain for sale, then both buyers have the high valuation.

In (i), if the deviating buyer with the unit has the high valuation, then she is willing to pay at most $2\gamma - 1$ for the second unit. If she has the low valuation, then she does not pay more than $(2\gamma - 1)v$. The buyer without the unit has the high valuation, and so she will pay at most $1$ for the remaining unit. Therefore, the seller does best by targeting the buyer without the unit and by posting $p' = 1$. It follows that the continuation payoff of the deviating buyer (with the unit) is $0$, because she does not buy the second unit independently of which valuation she has.

In (ii), the seller is better off by selling the product unit-by-unit at the unit price equal to $p'' = 1$, just as she does in the case with $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (N, U)$. For buyers to prefer the offer of the unit over the offer of the bundle, the seller must post a high enough bundle price, i.e.,

$$1 - p'' = 0 > 2\gamma - q'' \quad \Leftrightarrow \quad q'' > 2\gamma.$$

Therefore, he extracts full surplus form buyers with the high valuation in the second period. Furthermore, the continuation payoff of the deviating buyer with the low valuation is $0$, because she does not buy anything at such prices.

For $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (N, B)$ to occur in equilibrium, the first-period unit and bundle prices must satisfy the following.

– Preferences of buyers with the high valuation:
  – $N \succ B$:

$$\begin{aligned} 0 > \alpha \cdot (2\gamma - q) + (1-\alpha) \cdot \tfrac{1}{2}(2\gamma - q), \\ q > 2\gamma. \end{aligned}$$

– Preferences of buyers with the low valuation:
  – $B \succ N$:

$$\begin{aligned} \alpha \cdot (2\gamma v - q) + (1-\alpha) \cdot \tfrac{1}{2}(2\gamma v - q) \geq 0, \\ q \leq 2\gamma v. \end{aligned}$$

Because $v \in (0,1)$, this gives a contradiction. Therefore, $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (N, B)$ can never occur in equilibrium.

**Case with $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (B, U)$.** Buyers with the high valuation accept the offer of the bundle in the first period, while buyers with the low valuation accept the offer of the unit. Note that it is never the case that two units remain for sale in the second period, neither on nor off path. Furthermore, the second period with one unit remaining for sale is reached off path only. If the second period is reached, then beliefs of the seller (and of buyers) is as follows: the buyer with the unit has the low valuation, while the deviating buyer without the unit has the high valuation with probability $\alpha$.

The buyer with the low valuation and with the unit is of no interest to the seller, because he can get at most $(2\gamma - 1)v$ from her, while he can always get $v > (2\gamma - 1)v$ from the deviating buyer. If the buyer without the unit has the high valuation, then the seller can get $1$ by posting $p' = 1$. Therefore, the seller posts $p' = v$ if $v \geq \alpha$ and $p' = 1$ if $v \leq \alpha$. It follows that $p' \geq v$ and the continuation payoff of the deviating buyer with the low valuation is $0$, while the continuation payoff of the deviating buyer with the high valuation is at most $1 - v$.

Next I look for the first-period unit and bundle prices that support $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (B, U)$ in equilibrium.

– Preferences of buyers with the high valuation:
  – $B \succ U$:

$$\begin{aligned} \alpha \cdot \tfrac{1}{2}(2\gamma - q) + (1-\alpha) \cdot (2\gamma - q) \geq \alpha \cdot 0 + (1-\alpha) \cdot (1-p), \\ q \leq 2\gamma - \tfrac{2(1-\alpha)}{2-\alpha}(1-p); \end{aligned}$$

  – $B \succ N$:

$$\begin{aligned} \alpha \cdot \tfrac{1}{2}(2\gamma - q) + (1-\alpha) \cdot (2\gamma - q) \geq \alpha \cdot 0 + (1-\alpha) \cdot (1-p'), \\ q \leq 2\gamma - \tfrac{2(1-\alpha)}{2-\alpha}(1-p'). \end{aligned}$$

– Preferences of buyers with the low valuation:
  – $U \succ B$:

$$\begin{aligned} \alpha \cdot 0 + (1-\alpha) \cdot (v-p) > \alpha \cdot \tfrac{1}{2}(2\gamma v - q) + (1-\alpha) \cdot (2\gamma v - q), \\ q > 2\gamma v - \tfrac{2(1-\alpha)}{2-\alpha}(v-p); \end{aligned}$$

– $U \succ N$:

$$\alpha \cdot 0 + (1-\alpha) \cdot (v-p) \geq 0,$$
$$p \leq v.$$

Because $p' \geq v$ and $p \leq v$, what matters is that the bundle price is satisfies $2\gamma - \frac{2(1-\alpha)}{2-\alpha}(1-p) \geq q > 2\gamma v - \frac{2(1-\alpha)}{2-\alpha}(v-p)$. Note that, if $p = v$, then the inequalities take form: $2\gamma - \frac{2(1-\alpha)}{2-\alpha}(1-v) \geq q > 2\gamma v$. Because $2\gamma - \frac{2(1-\alpha)}{2-\alpha}(1-v) > 2\gamma v$, there does exist such $q$.

The seller wants to maximize his revenue

$$\mathcal{R} = [\alpha^2 + 2\alpha(1-\alpha)] \cdot q + (1-\alpha)^2 \cdot 2p.$$

Therefore, the equilibrium unit and bundle prices are as follows:

$$p = v \text{ and } q = 2\gamma - \tfrac{2(1-\alpha)}{2-\alpha}(1-v),$$
$$p' = v, \text{ if } v \geq \alpha,$$
$$p' = 1, \text{ if } \alpha \geq v,$$
$$p'' \geq 0 \text{ and } q'' \geq 0.$$

The equilibrium revenue is equal to

$$\mathcal{R}_{BU} = 2\alpha(2-\alpha)\gamma + 2(1-\alpha)(v-\alpha).$$

**Case with $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (U, B)$.** Buyers with the high valuation accept the offer of the unit in the first period, while buyers with the low valuation accept the offer of the bundle. Note that it is never the case that two units remain for sale in the second period, neither on nor off path. Furthermore, the second period with one unit remaining for sale is reached off path only. If the second period is reached, then beliefs of the seller (and of buyers) is as follows: the buyer with the unit has the high valuation, while the deviating buyer without the unit has the high valuation with probability $\alpha$.

The buyer with the high valuation and the unit is willing to pay at most $2\gamma - 1$ for the second unit. If the deviating buyer has the high valuation, then she pays at most $1$ for the remaining unit. If she has the low valuation, then she pays at most $v$. Therefore, the seller can get $2\gamma - 1$ or $v$ with probability $1$, and he can get $1$ with probability $\alpha$. The seller posts $p' = \max\{2\gamma-1, v\}$ if $\max\{2\gamma-1, v\} \geq \alpha$ and $p' = 1$ if $\max\{2\gamma-1, v\} \leq \alpha$. It follows that the continuation payoff of the deviating buyer with the low valuation is $0$, while the continuation payoff of the deviating buyer with the high valuation is at most $1-v$.

For $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (U, B)$ to occur in equilibrium, the first-period unit and bundle prices must satisfy the following.

– Preferences of buyers with the high valuation:

  – $U \succ B$:

$$\alpha \cdot (1-p) + (1-\alpha) \cdot 0 > \alpha \cdot (2\gamma - q) + (1-\alpha) \cdot \tfrac{1}{2}(2\gamma - q),$$
$$q > 2\gamma - \tfrac{2\alpha}{1+\alpha}(1-p).$$

– Preferences of buyers with the low valuation:

– $B \succ U$:

$$\alpha \cdot (2\gamma v - q) + (1-\alpha) \cdot \tfrac{1}{2}(2\gamma v - q) \geq \alpha \cdot (v - p) + (1-\alpha) \cdot 0,$$
$$q \leq 2\gamma v - \tfrac{2\alpha}{1+\alpha}(v-p).$$

Note that $2\gamma - \frac{2\alpha}{1+\alpha}(1-p) > 2\gamma v - \frac{2\alpha}{1+\alpha}(v-p)$ if and only if $2\gamma - \frac{2\alpha}{1+\alpha} > \left(2\gamma - \frac{2\alpha}{1+\alpha}\right)v$, which is always the case for $v \in (0,1)$, $\gamma \in \left(\frac{1}{2}, 1\right)$, and $\alpha \in (0,1)$. Therefore, this gives a contradiction. It follows that $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (U, B)$ is never a part of an equilibrium.

### D.2.4 Proof of Lemma 4.3

If the seller does not screen buyers, then he must post the first-period prices such that buyers with different valuations react the same. Because I focus on equilibria in pure strategies only, there are three equilibrium candidates $(\bar{\sigma}_1^B, \underline{\sigma}_1^B)$ given the unit and bundle prices: $(N,N)$, $(U,U)$, or $(B,B)$. I consider each pair $(\bar{\sigma}_1^B, \underline{\sigma}_1^B)$ separately, and look for prices (if any), for which the pair of strategies forms an equilibrium. While writing the inequalities that capture buyers' preferences over the offers, I take into account that the seller uses the tie-breaking rules given by Assumptions 4.1 and 4.2, and that buyers behave according to Assumptions 4.3 and 4.4 in case of indifference.

**Case with $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (N,N)$.** Buyers with both high and low valuations reject the offers in the first period. It follows that one unit remains for sale in the second period only off path. Furthermore, buyers' valuations stay unknown. That is, the seller (and buyers) believes that either buyer has the high valuation with probability $\alpha$.

If the deviating buyer with the unit has the high valuation, then she is willing to pay at most $2\gamma - 1$ for the second unit. If she has the low valuation, then she does not pay more than $(2\gamma-1)v$. If the buyer without the unit has the high valuation, then she is willing to pay $1$ for the remaining unit. If she has the low valuation, then she does not pay more than $v$. Therefore, the seller expects to get $v$ with probability $1$, $1$ with probability $\alpha$, and $2\gamma - 1$ with probability $\alpha^2 + 2\alpha(1-\alpha) = \alpha(2-\alpha)$ if $2\gamma - 1 > v$ or with probability $1$ if $v \geq 2\gamma - 1$. The seller posts $p'$ equal to $v$, $1$, or $2\gamma - 1$ depending on which price yields him a higher revenue, i.e., on whether $v$, $\alpha$, or $\alpha(2-\alpha)(2\gamma-1)$ is greater; see Figure D.2. It follows that the continuation payoff of the deviating buyer with the high valuation (and with the unit) is as follows:

- $\frac{1}{2}(2\gamma - 1 - v)$ if $2\gamma - 1 \geq v \geq \alpha(2-\alpha)(2\gamma-1)$ and $v \geq \alpha$ (i.e., if $p' = v$); indeed, the deviating buyer has to compete with another buyer (whichever valuation she has) for the remaining unit;
- $0$ if $v \geq 2\gamma - 1$ and $v \geq \alpha$ (i.e., if $p' = v$), if $\alpha(2-\alpha)(2\gamma-1) \geq v$ and $\alpha(2-\alpha)(2\gamma-1) \geq \alpha$ (i.e., if $p' = 2\gamma - 1$), or if $\alpha \geq v$ and $\alpha \geq \alpha(2-\alpha)(2\gamma-1)$ (i.e., if $p' = 1$).

The continuation payoff of the deviating buyer with the low valuation (and with the unit) is $0$.

The second period with two units remaining for sale is reached on path. The continuation game is identical to the one-period game. Therefore, as follows from Proposition 4.1, the continuation payoff of buyers with the low valuation is $0$, while the continuation payoff of buyers with the high valuation is as follows:

- $1 - v$ if $(\bar{\sigma}_2^B, \underline{\sigma}_2^B) = (U,U)$;
- $(1-\alpha)(1-v)$ if $(\bar{\sigma}_2^B, \underline{\sigma}_2^B) = (B,U)$; indeed, it is $\frac{2(1-\alpha)}{2-\alpha}(1-v)$ with probability $\frac{1}{2}\alpha + (1-\alpha)$ and $0$ with probability $\frac{1}{2}\alpha$;

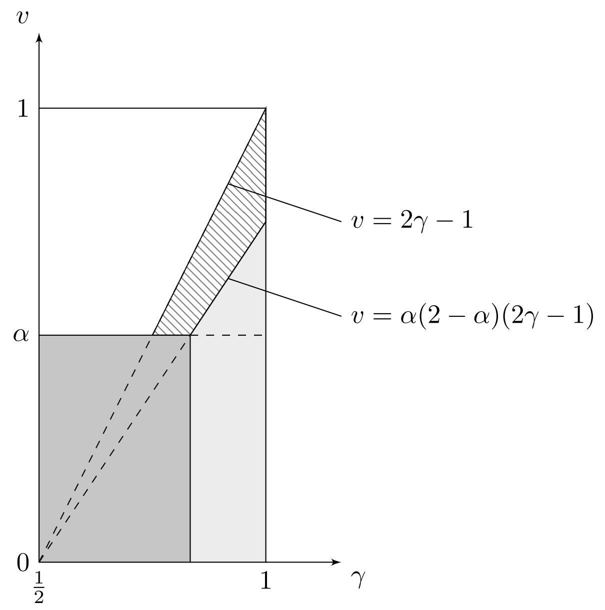


**Figure D.2.** The unit price in the off-path second period with one unit left for sale if the first-period expected behavior is $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (N, N)$: $p' = v$ (white), $p' = 2\gamma - 1$ (light gray), and $p' = 1$ (gray). The off-path continuation payoff of the deviating buyer with the high valuation is different from 0 and is equal to $\frac{1}{2}(2\gamma - 1 - v)$ in the hatched region. Parameters: $\alpha = \frac{1}{2}$.

– 0 if $(\bar{\sigma}_2^B, \underline{\sigma}_2^B) = (U, N)$ or $(\bar{\sigma}_2^B, \underline{\sigma}_2^B) = (B, N)$.

Next I look for the first-period unit and bundle prices that support $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (N, N)$ in equilibrium. Figure D.3 shows the largest equilibrium revenue of the continuation game. The hatched region in Figure D.3 is the same as in Figure D.2 and corresponds to the set of parameters for which the off-path payoff of the deviating buyer with the high valuation is different from 0 and is equal to $\frac{1}{2}(2\gamma - 1 - v)$.

– Preferences of buyers with the high valuation if their off-path continuation payoff is $\frac{1}{2}(2\gamma - 1 - v)$ and their on-path continuation payoff is $1 - v$:
  – $N \succ B$:
$$1 - v > 2\gamma - q \quad \Leftrightarrow \quad q > 2\gamma - 1 + v;$$
  – $N \succ U$:
$$1 - v > 1 - p + \tfrac{1}{2}(2\gamma - 1 - v),$$
$$p > \tfrac{1}{2}(2\gamma - 1 + v).$$

– Preferences of buyers with the high valuation if their off-path continuation payoff is $\frac{1}{2}(2\gamma - 1 - v)$ and their on-path continuation payoff is $(1 - \alpha)(1 - v)$:
  – $N \succ B$:
$$(1-\alpha)(1-v) > 2\gamma - q \quad \Leftrightarrow \quad q > 2\gamma - 1 + v + \alpha(1 - v);$$
  – $B \succ U$:
$$(1-\alpha)(1-v) > 1 - p + \tfrac{1}{2}(2\gamma - 1 - v),$$
$$p > \alpha(1 - v) + \tfrac{1}{2}(2\gamma - 1 + v).$$

– Preferences of buyers with the high valuation if their off-path continuation payoff

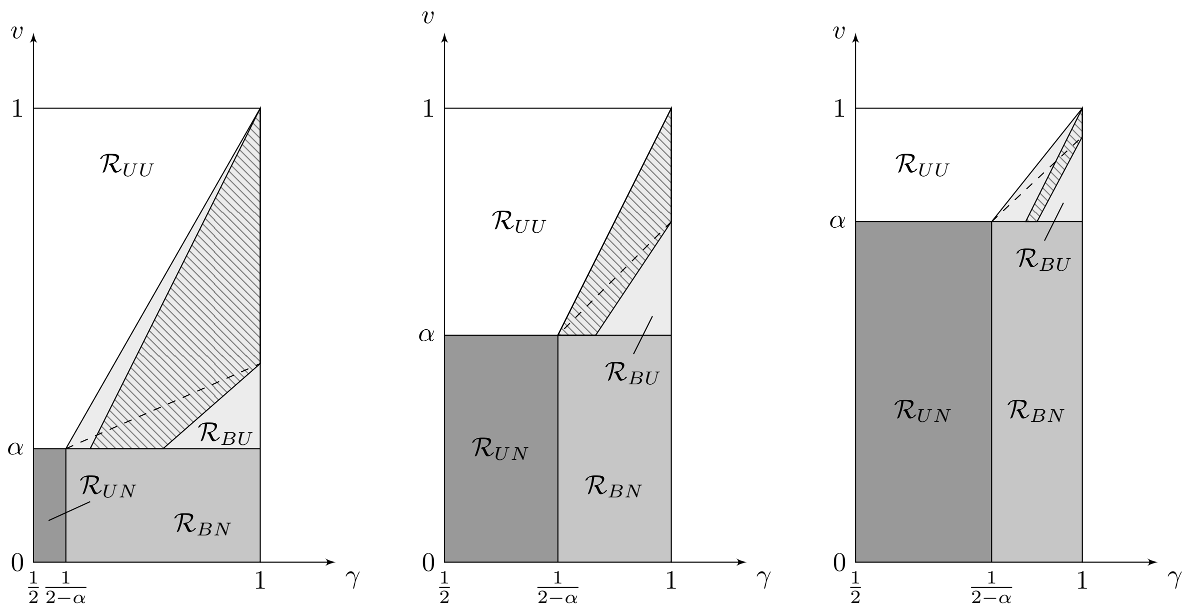


**Figure D.3.** The largest equilibrium revenue of the seller in the continuation game if the first-period expected behavior is $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (N, N)$ and if $\alpha < \frac{1}{2}$ (left), $\alpha = \frac{1}{2}$, and $\alpha > \frac{1}{2}$ (right). The off-path continuation payoff of the deviating buyer with the high valuation is different from 0 and is equal to $\frac{1}{2}(2\gamma - 1 - v)$ in the hatched region. Parameters: $\alpha = \frac{1}{4}$ (left), $\alpha = \frac{1}{2}$ (middle), and $\alpha = \frac{3}{4}$ (right).

is $\frac{1}{2}(2\gamma - 1 - v)$ and their on-path continuation payoff is $0$:
  - $N \succ B$:

$$0 > 2\gamma - q \quad \Leftrightarrow \quad q > 2\gamma;$$

  - $N \succ U$:

$$0 > 1 - p + \tfrac{1}{2}(2\gamma - 1 - v),$$
$$p > 1 - v + \tfrac{1}{2}(2\gamma - 1 + v).$$

- Preferences of buyers with the high valuation if their off-path continuation payoff is $0$ and their on-path continuation payoff is $1 - v$:
  - $N \succ B$:

$$1 - v > 2\gamma - q \quad \Leftrightarrow \quad q > 2\gamma - 1 + v;$$

  - $N \succ U$:

$$1 - v > 1 - p \quad \Leftrightarrow \quad p > v.$$

- Preferences of buyers with the high valuation if their off-path continuation payoff is $0$ and their on-path continuation payoff is $(1 - \alpha)(1 - v)$:
  - $N \succ B$:

$$(1 - \alpha)(1 - p) > 2\gamma - q \quad \Leftrightarrow \quad q > 2\gamma - 1 + v + \alpha(1 - v);$$

  - $N \succ U$:

$$(1 - \alpha)(1 - v) > 1 - p \quad \Leftrightarrow \quad p > v + \alpha(1 - v).$$

- Preferences of buyers with the high valuation if their off-path and on-path continuation payoffs are $0$:

  - $N \succ B$:

$$0 > 2\gamma - q \quad \Leftrightarrow \quad q > 2\gamma;$$

  - $N \succ U$:

$$0 > 1 - p \quad \Leftrightarrow \quad p > 1.$$

- Preferences of buyers with the low valuation:
  - $N \succ B$:

$$0 > 2\gamma v - q \quad \Leftrightarrow \quad q > 2\gamma v;$$

  - $N \succ U$:

$$0 > v - p \quad \Leftrightarrow \quad p > v.$$

Therefore, the equilibrium first-period prices are determined by preferences of buyers with the high valuation, and the seller's revenue is as in the one-period game.

**Case with $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (U, U)$.** Buyers with both high and low valuations accept the offer of the unit. Note that the second period with two units remaining for sale is never reached, neither on nor off path. One unit remains for sale in the second period only off path. Furthermore, buyers' valuations stay unknown. That is, the seller (and buyers) believes that either buyer has the high valuation with probability $\alpha$. As in the case with $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (N, N)$, the seller expects to get $v$ with probability $1$, $1$ with probability $\alpha$, and $2\gamma - 1$ with probability $\alpha(2-\alpha)$ if $2\gamma - 1 > v$ or with probability $1$ if $v \geq 2\gamma - 1$. The seller posts $p'$ equal to $v$, $1$, or $2\gamma - 1$ depending on which price yields him a higher revenue, i.e., on whether $v$, $\alpha$, or $\alpha(2-\alpha)(2\gamma-1)$ is greater; see Figure D.2. Therefore, the continuation payoff of the deviating buyer with the high valuation (and with or without the unit) is as follows:
- $1 - v$ if $v > 2\gamma - 1$ and $v \geq \alpha$ (i.e., if $p' = v$);
- $\frac{2-\alpha}{2}(1 - v)$ if $2\gamma - 1 \geq v \geq \alpha(2-\alpha)(2\gamma - 1)$ and $v \geq \alpha$ (i.e., if $p' = v$); indeed, she gets the remaining unit, and so gets $1 - v$ with probability $\frac{1}{2}\alpha + (1-\alpha)$ and $0$ with probability $\frac{1}{2}\alpha$;
- $(2-\alpha)(1-\gamma)$ if $\alpha(2-\alpha)(2\gamma - 1) \geq v$ and $\alpha(2-\alpha)(2\gamma-1) \geq \alpha$ (i.e., if $p' = 2\gamma - 1$); indeed, she gets the remaining unit, and so gets $1 - (2\gamma - 1)$ with probability $\frac{1}{2}\alpha + (1-\alpha)$ and $0$ with probability $\frac{1}{2}\alpha$;
- $0$ if $\alpha \geq v$ and $\alpha \geq \alpha(2-\alpha)(2\gamma - 1)$ (i.e., if $p' = 1$).

The continuation payoff of the deviating buyer with the low valuation is always $0$, because $p' \geq v$.

For $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (U, U)$ to occur in equilibrium, the first-period unit and bundle prices must satisfy the following.
- Preferences of buyers with the high valuation if their off-path continuation payoff is $1 - v$:
  - $U \succ B$:

$$1 - p > 2\gamma - q \quad \Leftrightarrow \quad q > 2\gamma - 1 + p;$$

  - $U \succ N$:

$$1 - p \geq 1 - v \quad \Leftrightarrow \quad p \leq v.$$

- Preferences of buyers with the high valuation if their off-path continuation payoff is $\frac{2-\alpha}{2}(1 - v)$:
  - $U \succ B$:

$$1 - p > 2\gamma - q \quad \Leftrightarrow \quad q > 2\gamma - 1 + p;$$

- $U \succ N$:
$$1 - p \geq \tfrac{2-\alpha}{2}(1 - v) \quad \Leftrightarrow \quad p \leq v + \tfrac{\alpha}{2}(1 - v).$$

– Preferences of buyers with the high valuation if their off-path continuation payoff is $(2 - \alpha)(1 - \gamma)$:
- $U \succ B$:
$$1 - p > 2\gamma - q \quad \Leftrightarrow \quad q > 2\gamma - 1 + p;$$
- $U \succ N$:
$$1 - p \geq (2 - \alpha)(1 - \gamma) \quad \Leftrightarrow \quad p \leq 1 - (2 - \alpha)(1 - \gamma).$$

– Preferences of buyers with the high valuation if their off-path continuation payoff is 0:
- $U \succ B$:
$$1 - p > 2\gamma - q \quad \Leftrightarrow \quad q > 2\gamma - 1 + p;$$
- $U \succ N$:
$$1 - p \geq 0 \quad \Leftrightarrow \quad p \leq 1.$$

– Preferences of buyers with the low valuation:
- $U \succ B$:
$$v - p > 2\gamma v - q \quad \Leftrightarrow \quad q > (2\gamma - 1)v + p;$$
- $U \succ N$:
$$v - p \geq 0 \quad \Leftrightarrow \quad p \leq v.$$

Note that $1 - (2 - \alpha)(1 - \gamma) > \alpha(2 - \alpha)(2\gamma - 1)$. Therefore, the unit price must satisfy $p \leq v$.

The seller wants to maximize his revenue
$$\mathcal{R} = 2p.$$

Therefore, the equilibrium unit and bundle prices are as follows:
$$\begin{gathered} p = v \text{ and } q > 2\gamma - 1 + v, \\ p' = v, \text{ if } v \geq \alpha(2 - \alpha)(2\gamma - 1) \text{ and } v \geq \alpha, \\ p' = 2\gamma - 1, \text{ if } \alpha(2 - \alpha)(2\gamma - 1) \geq v \text{ and } \alpha(2 - \alpha)(2\gamma - 1) \geq \alpha, \\ p' = 1, \text{ if } \alpha \geq v \text{ and } \alpha \geq \alpha(2 - \alpha)(2\gamma - 1), \\ p'' \geq 0 \text{ and } q'' \geq 0, \end{gathered}$$

The equilibrium revenue is equal to
$$\mathcal{R}_{UU} = 2v.$$

**Case with $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (B, B)$.** Buyers with both high and low valuations accept the offer of the bundle. Note that the second period is never reached, neither on nor off path.

For $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (B, B)$ to occur in equilibrium, the first-period unit and bundle prices must satisfy the following.

– Preferences of buyers with the high valuation:
- $B \succ U$ and $B \succ N$:
$$\tfrac{1}{2}(2\gamma - q) \geq 0 \quad \Leftrightarrow \quad q \leq 2\gamma.$$

– Preferences of buyers with the low valuation:

– $B \succ U$ and $B \succ N$:

$$\tfrac{1}{2}(2\gamma v - q) \geq 0 \quad \Leftrightarrow \quad q \leq 2\gamma v.$$

Therefore, the seller's revenue is as follows:

$$\mathcal{R} = q < \mathcal{R}_{UU} = 2v.$$

Because the seller can always post unacceptable first-period unit and bundle prices (e.g., $p > 1$ and $q > 2\gamma$) and get $\mathcal{R}_{UU}$ in the second period, $(\bar{\sigma}_1^B, \underline{\sigma}_1^B) = (B, B)$ is never a part of an equilibrium.

### D.2.5 Proof of Proposition 4.2

Proposition 4.2 follows from Lemmata 4.2 and 4.3.

### D.2.6 Proof of Lemma 4.4

Dependence of $\mathcal{R}_{UU}$ and $\mathcal{R}_{UN}$ on the parameters of the model is trivial. As for $\mathcal{R}_{BU}$ and $\mathcal{R}_{BN}$, they both increase in $\gamma$, $\mathcal{R}_{BU}$ increases in $v$, while $\mathcal{R}_{BN}$ is independent of $v$. Furthermore,

$$\tfrac{\partial \mathcal{R}_{BU}}{\partial \alpha} = 2[(2-\alpha)\gamma - 1 + \alpha - v + (1-\gamma)\alpha] > 0$$

if $\mathcal{R}_{BU}$ is the equilibrium revenue, i.e., if $(2-\alpha)\gamma - 1 + \alpha > v$. Finally,

$$\tfrac{\partial \mathcal{R}_{BN}}{\partial \alpha} = 4(1-\alpha)\gamma > 0,$$

because $\alpha \in (0, 1)$.

### D.2.7 Proof of Lemma D.1

Individual rationality of the seller implies that $p \geq 0$ and $q \geq 0$. Furthermore, because only equilibria in pure strategies are of interest here, there are nine equilibrium candidates $(\bar{\sigma}^B, \underline{\sigma}^B)$ for given unit and bundle prices: $(N, N)$, $(U, N)$, $(N, U)$, $(U, U)$, $(B, N)$, $(N, B)$, $(B, U)$, $(U, B)$, or $(B, B)$. I consider each pair $(\bar{\sigma}^B, \underline{\sigma}^B)$ separately, and look for prices (if any), for which the pair of strategies forms an equilibrium. While writing the inequalities which capture buyers' preferences over the offers, I take into account that the seller uses the tie-breaking rules given by Assumptions 4.1 and D.1, and buyers behave according to Assumption 4.3 in case of indifference.

**Case with $(\bar{\sigma}^B, \underline{\sigma}^B) = (N, N)$.** Buyers with both high and low valuations reject both offers.

– Preferences of buyers with the high valuation:

– $N \succ B$:

$$0 > 2\gamma - q \quad \Leftrightarrow \quad q > 2\gamma;$$

– $N \succ U$:

$$0 > 1 - p \quad \Leftrightarrow \quad p > 1.$$

– Preferences of buyers with the low valuation:
 – $N \succ B$:

$$0 > 2\gamma v - q \quad \Leftrightarrow \quad q > 2\gamma v;$$

 – $N \succ U$:

$$0 > v - p \quad \Leftrightarrow \quad p > v.$$

Therefore, this is an equilibrium when the unit and bundle prices satisfy $p > 1$ and $q > 2\gamma$.

**Case with** $(\bar{\sigma}^B, \underline{\sigma}^B) = (U, N)$**.** Buyers with the high valuation accept the offer of the unit, while buyers with the low valuation reject both offers.
– Preferences of buyers with the high valuation:
 – $U \succ B$:

$$1 - p \geq \alpha \cdot 0 + (1-\alpha) \cdot (2\gamma - q),$$
$$p \leq 1 - (1-\alpha)(2\gamma - q);$$

 – $U \succ N$:

$$1 - p \geq 0 \quad \Leftrightarrow \quad p \leq 1.$$

– Preferences of buyers with the low valuation:
 – $N \succ B$:

$$0 > \alpha \cdot 0 + (1-\alpha) \cdot (2\gamma v - q),$$
$$q > 2\gamma v;$$

 – $N \succ U$:

$$0 > v - p \quad \Leftrightarrow \quad p > v.$$

Therefore, this is an equilibrium when the unit and bundle prices satisfy the four inequalities above.

**Case with** $(\bar{\sigma}^B, \underline{\sigma}^B) = (N, U)$**.** Buyers with the high valuation reject both offers, while buyers with the low valuation accept the offer of the unit.
– Preferences of buyers with the high valuation:
 – $N \succ U$:

$$0 > 1 - p \quad \Leftrightarrow \quad p > 1.$$

– Preferences of buyers with the low valuation:
 – $U \succ N$:

$$v - p \geq 0 \quad \Leftrightarrow \quad p \leq v.$$

Because $v \in (0, 1)$, this gives a contradiction. Therefore, this is never an equilibrium.

**Case with** $(\bar{\sigma}^B, \underline{\sigma}^B) = (U, U)$**.** Buyers with both high and low valuations accept the offer of the unit.
– Preferences of buyers with the high valuation:
 – $U \succ B$ and $U \succ N$:

$$1 - p \geq 0 \quad \Leftrightarrow \quad p \leq 1.$$

– Preferences of buyers with the low valuation:

– $U \succ B$ and $U \succ N$:

$$v - p \geq 0 \quad \Leftrightarrow \quad p \leq v.$$

Therefore, this is an equilibrium for the unit price that satisfies $p \leq v$ and for any bundle price $q$.

**Case with** $(\bar{\sigma}^B, \underline{\sigma}^B) = (B, N)$**.** Buyers with the high valuation accept the offer of the bundle, while buyers with the low valuation reject both offers.

– Preferences of buyers with the high valuation:
  – $B \succ U$:

$$\alpha \cdot \tfrac{1}{2}(2\gamma - q) + (1 - \alpha) \cdot (2\gamma - q),$$
$$p > 1 - \tfrac{2-\alpha}{2}(2\gamma - q);$$

  – $B \succ N$:

$$\alpha \cdot \tfrac{1}{2}(2\gamma - q) + (1 - \alpha) \cdot (2\gamma - q) \geq 0,$$
$$q \leq 2\gamma.$$

– Preferences of buyers with the low valuation:
  – $N \succ B$:

$$0 > \alpha \cdot \tfrac{1}{2}(2\gamma v - q) + (1 - \alpha) \cdot (2\gamma v - q),$$
$$q > 2\gamma v;$$

  – $N \succ U$:

$$0 > v - p \quad \Leftrightarrow \quad p > v.$$

Therefore, this is an equilibrium when the unit and bundle prices satisfy the four inequalities above.

**Case with** $(\bar{\sigma}^B, \underline{\sigma}^B) = (N, B)$**.** Buyers with the high valuation reject both offers, while buyers with the low valuation accept the offer of the bundle.

– Preferences of buyers with the high valuation:
  – $N \succ B$:

$$0 > \alpha \cdot (2\gamma - q) + (1 - \alpha) \cdot \tfrac{1}{2}(2\gamma - q),$$
$$q > 2\gamma.$$

– Preferences of buyers with the low valuation:
  – $B \succ N$:

$$\alpha \cdot (2\gamma v - q) + (1 - \alpha) \cdot \tfrac{1}{2}(2\gamma v - q) \geq 0,$$
$$q \leq 2\gamma v.$$

Because $v \in (0, 1)$, this gives a contradiction. Therefore, this is never an equilibrium.

**Case with** $(\bar{\sigma}^B, \underline{\sigma}^B) = (B, U)$**.** Buyers with the high valuation accept the offer of the bundle, while buyers with the low valuation accept the offer of the unit.

– Preferences of buyers with the high valuation:
  – $B \succ U$:

$$\alpha \cdot \tfrac{1}{2}(2\gamma - q) + (1-\alpha)\cdot 0 > 1 - p,$$
$$p > 1 - \tfrac{\alpha}{2}(2\gamma - q).$$

– Preferences of buyers with the low valuation:
  – $U \succ B$:

$$v - p \geq \alpha \cdot \tfrac{1}{2}(2\gamma v - q) + (1-\alpha)\cdot 0,$$
$$p \leq v - \tfrac{\alpha}{2}(2\gamma v - q).$$

Note that $1 - \frac{\alpha}{2}(2\gamma - q) > v - \frac{\alpha}{2}(2\gamma v - q)$ if and only if $1 - \alpha\gamma > (1-\alpha\gamma)v$, which is always the case for $\gamma \in \left(\frac{1}{2}, 1\right)$, $v \in (0,1)$, and $\alpha \in (0,1)$. This gives a contradiction. Therefore, this is never an equilibrium.

**Case with $(\bar{\sigma}^B, \underline{\sigma}^B) = (U, B)$.** Buyers with the high valuation accept the offer of the unit, while buyers with the low valuation accept the offer of the bundle.
– Preferences of buyers with the high valuation:
  – $U \succ B$:

$$1 - p \geq \alpha \cdot 0 + (1-\alpha)\cdot \tfrac{1}{2}(2\gamma - q),$$
$$p \leq 1 - \tfrac{1-\alpha}{2}(2\gamma - q);$$

  – $U \succ N$:

$$1 - p \geq 0 \quad \Leftrightarrow \quad p \leq 1.$$

– Preferences of buyers with the low valuation:
  – $B \succ U$:

$$\alpha \cdot 0 + (1-\alpha)\cdot \tfrac{1}{2}(2\gamma v - q) > v - p,$$
$$p > v - \tfrac{1-\alpha}{2}(2\gamma v - q);$$

  – $B \succ N$:

$$\alpha \cdot 0 + (1-\alpha)\cdot \tfrac{1}{2}(2\gamma v - q) \geq 0,$$
$$q \leq 2\gamma v.$$

Therefore, this is an equilibrium when the unit and bundle prices satisfy $v - \frac{1-\alpha}{2}(2\gamma v - q) < p \leq 1 - \frac{1-\alpha}{2}(2\gamma - q)$ and $q \leq 2\gamma v$.

**Case with $(\bar{\sigma}^B, \underline{\sigma}^B) = (B, B)$.** Buyers with both high and low valuations accept the offer of the bundle.
– Preferences of buyers with the high valuation:
  – $B \succ U$:

$$\tfrac{1}{2}(2\gamma - q) > 1 - p \quad \Leftrightarrow \quad p > 1 - \tfrac{1}{2}(2\gamma - q),$$

  – $B \succ N$:

$$\tfrac{1}{2}(2\gamma - q) \geq 0 \quad \Leftrightarrow \quad q \leq 2\gamma.$$

- Preferences of buyers with the low valuation:
  - $B \succ U$:
$$\tfrac{1}{2}(2\gamma v - q) > v - p \quad \Leftrightarrow \quad p > v - \tfrac{1}{2}(2\gamma v - q),$$
  - $B \succ N$:
$$\tfrac{1}{2}(2\gamma v - q) \geq 0 \quad \Leftrightarrow \quad q \leq 2\gamma v.$$

Note that $1 - \frac{1}{2}(2\gamma - q) > v - \frac{1}{2}(2\gamma v - q)$ if and only if $1 - \gamma > (1 - \gamma)v$, which is always the case for $\gamma \in \left(\frac{1}{2}, 1\right)$, and $v \in (0, 1)$. Therefore, this is an equilibrium when the unit and bundle prices satisfy $p > 1 - \frac{1}{2}(2\gamma - q)$ and $q \leq 2\gamma v$.

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# Eidesstattliche Erklärung

Hiermit erkläre ich, dass ich die vorliegende Dissertation selbstständig angefertigt und die benutzten Hilfsmittel vollständig und deutlich angegeben habe.

Mannheim, 13.09.2018 Daria Khromenkova