Lévy processes conditioned to avoid and hit intervals

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Abstract

In this thesis we consider two classical problems in the theory of Markov processes for the special case of a Lévy process: Conditioning the process to avoid a Borel set B and conditioning the process to hit a Borel set B continuously from the outside. The aim in both settings is to characterise the conditioned process as a Doob-h-transform of the process killed on entering the set B.
In the first setting we use an invariant function to transform the killed process to a non-killed process whose state space is the complement of B. Moreover, this transformed process is connected to the process conditioned to avoid B and the long-time behaviour of the transformed process is analysed. We tackle this problem when B is an interval for two classes of Lévy processes. The first one consists of Lévy processes with finite variance and the second one consists of stable processes. Different techniques are needed to handle both cases.
The second problem is kind of a counterpart of the first one. Here, the aim is to transform the process killed on hitting B to a process which hits the boundary of B continuously before hitting the inner of B. For this we use a harmonic function and show that the underlying h-transformed process indeed hits the boundary of B continuously. Furthermore, we connect the h-transformed process to the process conditioned to be absorbed by B in a meaningful way. This problem is tackled when B is an interval and the Lévy process is a stable process.

Zusammenfassung

In dieser Arbeit betrachten wir zwei klassische Probleme der Markov-Prozess-Theorie für den Spezialfall eines Lévy-Prozesses: Einerseits soll der Prozess darauf bedingt werden, eine Borelmenge B zu vermeiden und andererseits soll er darauf bedingt werden, eine Borelmenge B stetig von außerhalb zu treffen. Das Ziel in beiden Situationen ist es, den bedingten Prozess als eine Doob-h-Transformation des Prozesses, getötet bei Eintritt in B, darzustellen.
In der ersten Situation nutzen wir eine invariante Funktion, um den getöteten Prozess in einen nicht-getöteten Prozess, dessen Zustandsraum das Komplement von B ist, zu transformieren. Außerdem wird dieser transformierte Prozess mit dem Prozess bedingt B zu vermeiden verbunden und das Langzeitverhalten des transformierten Prozesses wird analysiert. Wir gehen das Problem im Fall wenn B ein Intervall ist für zwei Klassen von Lévy-Prozessen an: Die erste besteht aus Lévy-Prozessen mit endlicher Varianz und die zweite besteht aus stabilen Prozessen. Um beide Klassen zu behandeln, werden verschiedene Techniken benötigt.
Das zweite Problem ist eine Art Gegenstück des ersten Problems. Hier ist das Ziel, den Prozess getötet in B zu einem anderen Prozess zu transformieren, der den Rand von B stetig trifft bevor er das Innere von B trifft. Dazu nutzen wir eine harmonische Funktion und zeigen, dass der zugehörige h-transformierte Prozess den Rand von B stetig trifft. Außerdem verbinden wir den h-transformierten Prozess zum Prozess bedingt von B absorbiert zu werden in einer aussagekräftigen Art und Weise. Dieses Problem wird für die Situation angegangen, falls B ein Intervall ist und der Lévy-Prozess ein stabiler Prozess ist.

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Contents

1 Introduction

1.1 Markov processes conditioned to avoid a set

Conditioning Markov processes to avoid sets is a classical problem and has been studied in many settings. Suppose $(\mathbb{P}^x)_{x\in\mathbb{R}}$ is a family of Markov probabilities on the state space $\mathbb{R}$, and that

$$T_B := \inf\{t \geq 0 : \xi_t \in B\}$$

is the first hitting time of a fixed Borel set $B \subseteq \mathbb{R}$. When T_B is infinite with positive probability the conditioning is trivial since one can just condition on the event $\{T_B = \infty\}$. In the case that T_B is almost surely finite, it is non-trivial to construct and characterise the conditioned process through the natural limiting procedure

$$\lim_{s\to\infty} \mathbb{P}^x(\Lambda \,|\, s + t < T_B) \tag{1.1}$$

or the randomized version

$$\lim_{q\to 0} \mathbb{P}^x(\Lambda, t < e_q \,|\, e_q < T_B), \tag{1.2}$$

for $\Lambda \in \mathcal{F}_t$ and $x \in \mathbb{R} \setminus B$. Here, $(\mathcal{F}_t)_{t\geq 0}$ denotes the natural filtration of the underlying Markov process and e_q are independent exponentially distributed random variables with parameter $q > 0$. The limiting procedures (1.1) translates to conditioning the process not to hit B until time $s + t$ and then s tends to ∞ and (1.2) translates to conditioning the process not to hit B until some exponential distributed time and then its parameter tends to 0.

The classical aim concerning these limits is to show that they lead to a so-called Doob-h-transform of the process killed on entering B using some invariant function. This yields on the one hand that the process conditioned to avoid B in the sense of (1.1) or (1.2) is again a (strong) Markov process. On the other hand the invariant function sometimes has analytic or probabilistic properties which can be used to analyse the conditioned process concerning questions like recurrence and transience or the long-time behaviour.

A classical example is Brownian motion conditioned to avoid the negative half-line. In this case, the limits (1.1) and (1.2) lead to the Doob-h-transform of the Brownian motion killed on entering the negative half-line, by the positive invariant function $h(x) = x$ on $(0, \infty)$. This Doob-h-transform turns out (see e.g. [47] or Chapter VI.3 of [50]) to be the Bessel process of dimension 3, which is transient. This example is typical, in that a conditioning procedure leads to a new process which is transient where the original process was recurrent.

Extensions of this result have been obtained to condition Lévy processes to stay positive, see Chaumont and Doney [16]. In that case, the associated invariant function that plays the role of h in the Brownian example above is given by the potential function of the dual ladder height process. In a similar spirit Bertoin and Doney [5] have shown how to condition random walks conditioned to stay non-negative. Other examples of Markov processes conditioned to avoid domains via a limiting procedure, and thus characterised as a Doob-h-transform of the original process killed on exiting the specified domain, include random walks conditioned to stay in cones (Denisov and Wachtel [20]), random walks with finite second moments conditioned to avoid an interval (Vysotsky [57]), spectrally negative Lévy processes conditioned to stay in an interval (Lambert [43]), subordinators conditioned to stay in an interval (Kyprianou et al. [39]), Lévy processes conditioned to avoid the origin (Pantí [45] and Yano [58]), self-similar Markov processes conditioned to avoid the origin (Kyprianou et al. [37]) or stable processes conditioned to stay in cones (Kyprianou et al. [42]) .

The purpose of a part of this work is to take advantage of the path discontinuities of a Lévy process to condition it to avoid a bounded interval. In contrast to Brownian motion it is possible that a Lévy process reaches the area above and below the interval without hitting it. Hence, the process conditioned to avoid the interval in the spirit of (1.1) or (1.2) is not just the process conditioned to stay below the interval when the initial value is below the interval and above the interval when the initial value is above the interval whereas for the Brownian motion exactly this is the case.

Typically, there are two ingredients needed to connect an h-transformed process with the process conditioned to avoid B in the sense of (1.1) or (1.2):

(i) An invariant function h for the process killed on entering B.

(ii) Tail asymptotics for the distribution of T_B of the form

$$\lim_{s\to\infty} f(s)\mathbb{P}^x(s < T_B) = h(x)$$

or in the randomized version

$$\lim_{q\to 0} f(q)\mathbb{P}^x(e_q < T_B) = h(x).$$

If one can solve these two problems, it is usually a standard procedure to reach our aim.

1.2 Markov processes conditioned to be absorbed by a set

The contrary problem of conditioning a Markov process to avoid a set is to condition a process to hit a Borel set $B \subseteq \mathbb{R}$ continuously in finite time or more precisely, condition it to hit the boundary of a set continuously without hitting the inner of the set. Similar to the problem before the interesting case is when $T_{\partial B}$ is infinite almost surely. We define $B_\delta = (\bigcup_{x\in\partial B} D_\delta(x)) \setminus B$ where $D_\delta(x)$ is the open δ-ball around x. The usual aim for our problem is to characterise the limit

$$\lim_{\varepsilon\searrow 0} \mathbb{P}^x(\Lambda, t < T_{B\cup B_\delta} \,|\, T_{B_\varepsilon} < T_B), \quad \Lambda \in \mathcal{F}_t, x \in \mathbb{R}\setminus B \tag{1.3}$$

as the h-transform using a harmonic function for the process killed on entering B. The benefit of a harmonic function is that the h-transformed process leaves all compact subsets of $\mathbb{R}\setminus B$ before it is killed. Sometimes this can be used to show that the h-transformed process is killed in finite time and has indeed a limit at its killing time which is in ∂B.

A classical example is a one-dimensional Lévy process conditioned to be absorbed by 0 from above, i.e. without hitting the negative half-line (Chaumont [15], Silverstein [55]). Silverstein showed that for a Lévy process which does not drift to $-\infty$ and fulfils some mild assumptions under which the potential of the dual ladder height process has a density u_-, this density is harmonic for the process killed on entering $(-\infty, 0]$. The limit (1.3) in this setting translates to

$$\lim_{\varepsilon \searrow 0} \mathbb{P}^x(\Lambda, t < T_{(-\infty,\delta)} \,|\, T_{(0,\varepsilon)} < T_{(-\infty,0]}), \quad \Lambda \in \mathcal{F}_t, x > 0$$

and Chaumont identified it as the h-transformed process using the harmonic function found by Silverstein. Furthermore, Chaumont used the harmonicity of u_- to show that the conditioned process is indeed absorbed by 0 in finite time without hitting the negative half-line.

More recent is the one-dimensional stable process with self-similarity index $\alpha < 1$ conditioned to be absorbed by 0 (Kyprianou et al. [37]). In this setting the harmonic function is $u(-x)$ where u is the potential density which is known to be

$$u(x) = \begin{cases} \sin(\pi\alpha\rho)x^{\alpha-1} & \text{if } x > 0 \\ \sin(\pi\alpha\hat{\rho})|x|^{\alpha-1} & \text{if } x < 0 \end{cases}.$$

The limiting procedure (1.3) translates to

$$\lim_{\varepsilon \searrow 0} \mathbb{P}^x(\Lambda, t < T_{(-\delta,\delta)} \,|\, T_{(-\varepsilon,\varepsilon)} < \infty), \quad \Lambda \in \mathcal{F}_t, x \neq 0.$$

Moreover, the authors showed again that the conditioned process hits 0 continuously in finite time.

Similarly to Section 1.1 parts of this work consider the described problem of conditioning a Markov process to be absorbed by a Borel set B when the Markov process is a one-dimensional Lévy process and B is a bounded interval. Usually these problems splits up in two subproblems, too. Namely, we need the following

(i) A harmonic function h for the process killed on entering B.

(ii) Asymptotics for the probability of T_{B_ε} being finite of the form

$$\lim_{\varepsilon \searrow 0} f(\varepsilon)\mathbb{P}^x(T_{B_\varepsilon} < T_B) = h(x).$$

Similarly to the problem of Section 1.1 the achievement of (i) and (ii) leads to the aim.

1.3 Outline

Chapter 2

We start this work by introducing the main tools we need in the later chapters. We insert the topics of Lévy processes, their special class of stable processes, self-similar Markov processes, excessive functions, Doob-h-transforms and some applications of Doob-h-transforms to Lévy processes in the spirit of our upcoming results.

Chapter 3

Chapter 3 is a warm up to become familiar with overshoots of one-dimensional Lévy processes which play a crucial role in Chapter 4. An overshoot over the level $a \in \mathbb{R}$ is the value $\xi_{T_{[a,\infty)}} - a$. A classical question is if the overshoot distribution converges weakly when the starting value of the Lévy process tends to $-\infty$. This was already answered with a sufficient and necessary condition. We change the perspective of this limiting distribution and consider it as an invariant measure for the Markov process $(\xi_{T_{[a,\infty)}} - a)_{a \geq 0}$. We prove that there exists an invariant measure for this Markov process also in the case when the weak limit from above does not exist and give an explicit form of this measure.

Chapter 4

In Chapter 4 we consider the problem of Section 1.1 for one-dimensional Lévy processes which have finite variance when $B = [a, b]$ is an interval. The limit (1.2) is split up in the sense that the event we condition on is separated by $\xi_{e_q} > b$ and $\xi_{e_q} < a$. We prove that both limits correspond to an h-transform using two different invariant functions. Furthermore, we show that a particular linear combination of these two invariant functions corresponds to the process conditioned to avoid $[a, b]$ in the sense of (1.2). Finally, we analyse all of these conditioned processes concerning the questions of transience and long-time behaviour. For that the presented form of the invariant function is used.
The analogous problem was considered by Vysotsky [57] for random walks and it turns out that the invariant function we extract is the analogous one to Vysotsky's one. But the techniques of proving invariance are different and are rather based on ideas of Chaumont and Doney [16].
This chapter is based on joint work with Leif Döring and Alexander R. Watson. The corresponding paper *Lévy processes with finite variance conditioned to avoid an interval* forms [28].

Chapter 5

Here, we consider again the problem of conditioning a one-dimensional Lévy process to avoid an interval. But this time the role of the Lévy process is played by an α-stable process, $\alpha \in (0, 2)$ which is not included in the class of Lévy process with finite variance. Thanks to the scaling property we can reduce our analysis to the interval $[-1, 1]$. The methods of extracting the right invariant function for the conditioning (1.1) are based on the deep factorisation of the stable process (see Kyprianou [35] and Kyprianou et al. [39]). Moreover, the methods differ for $\alpha < 1$, $\alpha = 1$ and $\alpha > 1$. For the first range of α the conditioning is trivial since the process is transient. We use recent results to calculate the probability of $T_{[-1,1]}$ being infinite which then forms an invariant function. For $\alpha = 1$ we use the Lamperti-Kiu transform to reduce the problem to conditioning a Markov additive process (MAP) to stay positive. The third case when $\alpha > 1$ is the most involved one. First, we condition the process to avoid the origin via an h-transform for the process killed on hitting 0. Then we condition this h-transformed process to avoid an interval via an h-transform using an invariant function for the h-transformed process killed on entering the interval.
The tail-asymptotics for the distribution of $T_{[-1,1]}$ which are needed to connect the h-transform to the conditioning (1.1) are based on classical works of Blumenthal et al. [10] and Port [48].

This chapter is based on joint work with Leif Döring and Andreas E. Kyprianou and contains the results of the paper *Stable processes conditioned to avoid an interval* which forms [27].

Chapter 6

In this chapter we focus on the problem described in Section 1.2 when the Markov process is again a one-dimensional stable Lévy process and B is the interval $[-1,1]$. Roughly speaking an h-transform of a Markov process using a harmonic function which has a pole corresponds to the process conditioned to be absorbed by this pole. Here, the role of the Markov process is played by the stable process killed on entering $[-1,1]$. To pursue our goal we would like to find a harmonic function for this killed process which has a pole in ± 1. Recent work of Profeta and Simon [49] establishes explicit formulas for the potential densities of the killed process (so-called Green's functions). From these it can be seen that the potential densities can have poles in the initial value. Moreover, classic theory of Kunita and Watanabe [33] states that these potential densities are harmonic functions. As a consequence our approach to extract harmonic functions is to send the initial value of the (scaled) potential densities to the boundary points of the interval. For the two different boundary points we get two harmonic functions which belong to generalised versions of the conditioned process (1.3).
The asymptotic probabilities of $T_{(-1-\varepsilon,1+\varepsilon)} < T_{[-1,1]}$ for $\varepsilon \searrow 0$ which are needed for the connection between the h-transformed process and the conditioned process are based on the deep factorisation of the stable process.
This chapter is based on joint work with Leif Döring and forms the paper *Stable processes conditioned to hit an interval continuously from the outside* [26].

2 Preliminaries

To be prepared for tackling the actual issues of this work we introduce the tools used in the later chapters. The main topics which are touched here are Lévy processes, especially fluctuation theory and Lévy processes from the Markov process point of view, a detailed view on their subclass of stable processes, self-similar Markov processes and their relations to Lévy processes. We end this section by considering some applications which we will need in the remaining parts of this work.

2.1 Definitions and basic results on Lévy processes

The first part contains definitions and elementary properties of Lévy processes. For more details see e.g. Applebaum [2], Bertoin [4], Kyprianou [34] and Sato [53]. Roughly speaking a Lévy process is a generalisation of a Brownian motion and a Poisson process, i.e. it has just the properties which are fulfilled by both of these classes of processes.

Definition 2.1.1. *An $\mathbb{R}$-valued stochastic process $X = (X_t)_{t\geq 0}$ on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$ is called a (one-dimensional) Lévy process if the following holds:*

(i) $X_0 = 0$ *almost surely.*

(ii) For all $t \geq s \geq 0$ the random variable X_{t-s} has the same distribution as $X_t - X_s$ (stationary increments).

(iii) For all $t \geq s \geq 0$ the random variable X_{t-s} is independent of $(X_u, u \leq s)$ (independent increments).

(iv) The map $t \mapsto X_t$ is almost surely càdlàg (right-continuous with existing left-limits).

A few of the most common examples are linear Brownian motions, Poisson processes and compound Poisson processes. Later we will introduce the class of stable Lévy processes which are Lévy processes whose marginals have stable distributions.

Let us fix a Lévy process $X = (X_t)_{t\geq 0}$. By the stationary and independent increments it is not hard to show that X_t has an infinitely divisible distribution for all $t \geq 0$ in the sense that for all $n \in \mathbb{N}$ there are independent and identically distributed random variables $X_t^{(1,n)}, \dots, X_t^{(n,n)}$ such that

$$X_t \overset{(d)}{=} \sum_{i=1}^{n} X_t^{(i,n)},$$

where $\overset{(d)}{=}$ stands for equality in distribution. One just has to choose $X_t^{(i,n)} = X_{\frac{it}{n}} - X_{\frac{(i-1)t}{n}}$ for $i = 1, \dots, n$. Hence, one can apply the Lévy-Khintchine formula for infinitely divisible distributions to X_t. To speak about the Lévy-Khintchine formula we define the characteristic exponent of X_t,

$$\Psi_t(\theta) := -\log \mathbf{E}\left[\exp(\mathrm{i}\theta X_t)\right], \quad \theta \in \mathbb{R},$$

which uniquely determines the distribution of X_t by classic results on the characteristic function. One can use the stationary and independent increments and the right-continuity of the paths to show that $\Psi_t = t\Psi_1$, i.e. for all $t \geq 0$ the characteristic exponent of X_t is characterised by the characteristic exponent of X_1. In other words for all $t \geq 0$ the distribution of X_t is uniquely determined by the distribution of X_1. As a consequence let $\Psi := \Psi_1$ and consider from now on just Ψ which is called the characteristic exponent of the Lévy process X.

Now we state the classic Lévy-Khintchine formula for infinitely divisible distributions.

Theorem 2.1.2. *(Lévy-Khintchine formula)*
Let Z be a $\mathbb{R}$-valued random variable which has an infinitely divisible distribution. Then there exists a unique triple (γ, σ^2, Π), where $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$ and Π is a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ satisfying $\Pi(\{0\}) = 0$ and $\int_{\mathbb{R}} \min(1, |x|^2)\, \Pi(\mathrm{d}x) < \infty$ such that the corresponding characteristic exponent Ψ_Z has the following form:

$$\Psi_Z(\theta) = \mathrm{i}\theta\gamma + \frac{1}{2}\theta^2\sigma^2 + \int_{\mathbb{R}} \left(1 - \exp(\mathrm{i}\theta x) + \mathrm{i}\theta x \mathbb{1}_{\{|x|<1\}}\right) \Pi(\mathrm{d}x), \quad \theta \in \mathbb{R}.$$

In this case (γ, σ^2, Π) is called the characteristic triplet of Z.

Since X_1 has an infinitely divisible distribution we obtain that Ψ has the form given in the Lévy-Khintchine formula as well. With our explanations above we obtain that a Lévy process is uniquely determined by the corresponding characteristic triplet of X_1 which we call from now on the characteristic triplet of the Lévy process X. We should give the characteristic triplet in the mentioned examples:

(i) If X is a linear Brownian motion, i.e. $X_t = \gamma t + \sigma B_t$ with $\gamma, \sigma \in \mathbb{R}$ and a Brownian motion B, then the characteristic triplet is $(-\gamma, \sigma^2, 0)$.

(ii) If X is a Poisson process with rate $\lambda > 0$, then the characteristic triplet is $(0, 0, \lambda\delta_1)$ where δ_1 is the dirac measure in 1.

(iii) If X is a compound Poisson process with rate $\lambda > 0$ and jump distribution F, then the characteristic triplet is $(-\lambda \int_{(-1,1)\setminus\{0\}} x\, F(\mathrm{d}x), 0, \lambda F)$.

We also note that the sum of (finitely many) independent Lévy processes is again a Lévy process and the new characteristic exponent is just the sum of the original characteristic exponents as well as the new triplet consists just of the componentwise sum of the original triplets.

The natural question concerning this theorem is if there is a converse to the Lévy-Khintchine formula in some sense, i.e. given a triplet (γ, σ^2, Π) satisfying the conditions in Theorem 2.1.2, is there a Lévy process X with characteristic triple (γ, σ^2, Π). The answer can be obtained by the following.

Theorem 2.1.3. *(Lévy-Itô decomposition)*
Let $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$ and Π a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ which satisfies $\Pi(\{0\}) = 0$ and $\int_{\mathbb{R}} \min(1, |x|^2)\, \mathrm{d}x < \infty$. Then there exist three independent Lévy processes $X^{(1)}$, $X^{(2)}$ and $X^{(3)}$ on a common probability space $(\Omega, \mathcal{A}, \mathbf{P})$ such that:

(i) $X^{(1)}$ *is a linear Brownian motion with drift* $-\gamma$ *and volatility* σ^2.

(ii) $X^{(2)}$ *is a compound Poisson process with rate* $\Pi(\{x \in \mathbb{R} : |x| > 1\})$ *and jump measure*

$$\frac{\mathbb{1}_{\{|x|>1\}}\Pi(\mathrm{d}x)}{\Pi(\{x \in \mathbb{R} : |x| > 1\})}.$$

(iii) $X^{(3)}$ *is a square integrable martingale which almost surely makes a countable number of jumps in every finite interval and all jumps have values in* $\{x \in \mathbb{R} : |x| \leq 1\}$. *Moreover,* $X^{(3)}$ *has characteristic exponent*

$$\Psi^{(3)}(\theta) = \int\limits_{(-1,1)} \big(1 - \exp(\mathrm{i}\theta x) + \mathrm{i}\theta x\big)\,\Pi(\mathrm{d}x), \quad \theta \in \mathbb{R}.$$

By our examples it holds that $X^{(1)}$ has characteristic exponent $\Psi^{(1)} = \mathrm{i}\theta\gamma + \frac{1}{2}\theta\sigma^2\theta$ and $X^{(2)}$ has characteristic exponent $\Psi^{(2)} = \int_{\{x\in\mathbb{R}:|x|>1\}} \big(1 - \exp(\mathrm{i}\theta x)\big)\,\Pi(\mathrm{d}x)$. Hence the sum $X = X^{(1)} + X^{(2)} + X^{(3)}$ is a Lévy process with characteristic exponent

$$\Psi(\theta) = \mathrm{i}\theta\gamma + \frac{1}{2}\theta^2\sigma^2 + \int\limits_{\mathbb{R}} \big(1 - \exp(\mathrm{i}\theta x) + \mathrm{i}\theta x\mathbb{1}_{\{|x|<1\}}\big)\,\Pi(\mathrm{d}x), \quad \theta \in \mathbb{R},$$

i.e. the Lévy process X has characteristic triplet (γ, σ^2, Π). To sum up, the Lévy-Khintchine formula and the Lévy-Itô decomposition tell us that there is a one-to-one connection between Lévy processes and triplets (γ, σ^2, Π) which fulfil the conditions of Theorem 2.1.2. As a last remark of this section we introduce $\psi(\theta) := -\Psi(-i\theta) = \log \mathbb{E}[\exp(\theta X_1)]$ which is called Laplace exponent if it exists. Sometimes it is more practicable to work with the Laplace exponent instead of the characteristic exponent.

2.2 Lévy processes as Markov processes

Very often it is useful to consider a Lévy process as a so-called universal Markov process (also called right-continuous realisation of a Markov process), that is roughly speaking a collection of probability measures $(\mathbb{P}^x)_{x\in\mathbb{R}}$ on the space of càdlàg paths such that the canonical process is a Markov process and starts from x almost surely under $\mathbb{P}^x$ for all $x \in \mathbb{R}$. Furthermore, we will model killing the Lévy process when it enters a given Borel set $B \subseteq \mathbb{R}$. As a last part of this section we introduce excessive functions for killed Lévy processes which we can use to transform the killed Lévy process via a so-called Doob-h-transform.

For general theory on universal Markov processes, see e.g. Blumenthal and Getoor [9], Chung and Walsh [18], Dellacherie and Meyer [19], Doob [24], Kallenberg [31], Rogers and Williams [51] or Sharpe [54]. We will focus on the case when the Markov process is a Lévy process apart from the definition of sub-Markov semigroups.

The path measure

To introduce the mentioned measures $\mathbb{P}^x$ we fix a Lévy process $X = (X_t)_{t\geq 0}$ on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$ and we assume that $\mathcal{A}$ is complete with respect to null-sets, i.e

$$\mathcal{A} = \sigma(\mathcal{A} \cup \mathcal{N}),$$

where $\mathcal{N} := \{S \subseteq \Omega \,|\, \exists N \in \sigma(X_s, s \in [0,\infty)) : S \subseteq N, \mathbf{P}(N) = 0\}$. Furthermore, we define $\mathcal{A}_t = \sigma(\sigma(X_s, s \leq t) \cup \mathcal{N})$ which is known to be right-continuous, i.e. $\mathcal{A}_t = \bigcap_{\varepsilon>0} \mathcal{A}_{t+\varepsilon}$. The filtration $(\mathcal{A}_t)_{t\geq 0}$ is called the natural enlargement of the filtration induced by X.

It is well known that one can use the stationary and independent increments of X to prove that X is a $(\mathcal{A}_t)_{t\geq 0}$-Markov process, in the sense that

$$\mathbf{P}(B \,|\, \mathcal{A}_t) = \mathbf{P}(B \,|\, X_t) := \mathbf{P}(B \,|\, \sigma(X_t)), \tag{2.1}$$

for all $B \in \sigma(X_u, u \in [t,\infty))$. There are lots of equivalent characterisations of a Markov process. One of them which is important for us leads us to Markov semigroups which will be defined now.

Definition 2.2.1. *Let $(E,\mathcal{E})$ be a measurable space. A family $(p_t)_{t\geq 0}$ of functions $p_t : E \times \mathcal{E} \to [0,1]$ is called sub-Markov semigroup on $(E,\mathcal{E})$ if the following is satisfied:*

(i) For all $t \geq 0, x \in E$ the map $A \mapsto p_t(x,A)$ is a measure on $(E,\mathcal{E})$.

(ii) For all $t \geq 0, A \in \mathcal{E}$ the map $x \mapsto p_t(x,A)$ is measurable.

(iii) For all $s,t \geq 0, x \in E, A \in \mathcal{E}$ it holds

$$p_{s+t}(x,A) = \int_E p_s(y,A)\, p_t(x,\mathrm{d}y). \tag{2.2}$$

If for all $t \geq 0, x \in E$ it holds $p_t(x,E) = 1$, we call $(p_t)_{t\geq 0}$ a Markov semigroup.

Usually (2.2) is called Chapman-Kolmogorov equation. With a Markov semigroup we associate a family of operators on $b\mathcal{E} := \{f : E \to \mathbb{R},\ f \text{ is measurable and bounded}\}$ for a measurable space $(E,\mathcal{E})$ via

$$p_t f(x) = \int_E f(y)\, p_t(x,\mathrm{d}y), \quad x \in E, t \geq 0.$$

From the Chapman-Kolmogorov equation we get

$$p_t(p_s f)(x) = p_{t+s} f(x),$$

for all $s,t \geq 0$ and $x \in E$. This leads to the semigroup property of a family of operators in the functional analysis view which is the reason why $(p_t)_{t\geq 0}$ is called Markov semigroup. We remark that one can define $p_t f$ in the same way also for non-bounded measurable functions $f : E \to \mathbb{R}$ but in this case it is not clear if $p_t f(x)$ exists.

For the fixed Lévy process X it is possible to show by stationary and independent increments that

$$p_t(x,\mathrm{d}y) := \mathbf{P}(X_t + x \in \mathrm{d}y), \quad x,y \in \mathbb{R}, t \geq 0,$$

defines a Markov semigroup on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$. With this one can show that

$$\mathbf{E}\left[f(X_{t+s}) \,|\, \mathcal{A}_s\right] = p_t f(X_s), \tag{2.3}$$

for all $s,t \geq 0$ and $f \in b\mathcal{B}(\mathbb{R})$. One says that $(p_t)_{t\geq 0}$ is a Markov semigroup for the process X. A nice consequence of the Chapman-Kolmogorov equation is that the finite dimensional distributions of X can be characterised via

$$\mathbf{P}(X_{t_1} \in \mathrm{d}x_1, ..., X_{t_n} \in \mathrm{d}x_n)$$

$$= p_{t_n - t_{n-1}}(x_{n-1}, \mathrm{d}x_n) \dots p_{t_2 - t_1}(x_1, \mathrm{d}x_2)\, p_{t_1}(0, \mathrm{d}x_1), \tag{2.4}$$

for $t_n \geq \ldots \geq t_1 \geq 0$ and $x_1, \ldots, x_n \in \mathbb{R}$. Finally we characterise the Lévy process X via a universal Markov process. For this fix the following:

(i) Some point $\Delta \notin \mathbb{R}$.

(ii) The path space

$$D = \{\omega : [0,\infty) \to \mathbb{R} \cup \{\Delta\} \mid \exists \zeta(\omega) \in [0,\infty] : \omega \text{ is càdlàg on } [0,\zeta(\omega)),$$
$$\omega_t \in \mathbb{R} \text{ for } t < \zeta(\omega), \omega_t = \Delta \text{ for } t \geq \zeta(\omega)\},$$

and $\xi_t : D \to \mathbb{R}, \xi_t(\omega) = \omega_t$ for $t \geq 0$ (the canonical process).

(iii) The σ-algebra $\mathcal{F}$ on D induced by the Skorohod topology (see e.g. Billingsley [8] or Jacod and Shiryaev [30]) and the natural completion of the filtration induced by ξ which we denote by $(\mathcal{F}_t)_{t\geq 0}$ (see e.g. Sato [53], Section 40 for this notion).

(iv) For all $t \geq 0$ the mapping $\theta_t : D \to D, \omega \mapsto (s \mapsto \omega_{t+s})$. We remark that it holds

$$\theta_t \circ \theta_s = \theta_{t+s} \quad \text{and} \quad \xi_t \circ \theta_s = \xi_{t+s},$$

for all $s \geq 0$.

Classic theory on Markov processes implies that for all $x \in \mathbb{R}$ there is a probability measure $\mathbb{P}^x$ on $(D, \mathcal{F})$ such that $\mathbb{P}^x(\xi_0 = x) = 1$ and

$$\mathbb{P}^x(f(\xi_{t+s}) \,|\, \mathcal{F}_s) = p_t f(\xi_s),$$

for all $s, t \geq 0$, $f \in b\mathcal{B}(\mathbb{R})$. This is known to be equivalent to ξ being an $(\mathcal{F}_t)_{t\geq 0}$-Markov process under $\mathbb{P}^x$ and the finite dimensional distributions can again be characterised by

$$\mathbb{P}^x(\xi_{t_1} \in \mathrm{d}x_1, ..., \xi_{t_n} \in \mathrm{d}x_n)$$
$$= p_{t_n - t_{n-1}}(x_{n-1}, \mathrm{d}x_n) \dots p_{t_2 - t_1}(x_1, \mathrm{d}x_2)\, p_{t_1}(x, \mathrm{d}x_1), \tag{2.5}$$

for $t_n \geq \ldots \geq t_1 \geq 0$ and $x, x_1, \ldots, x_n \in \mathbb{R}$. In particular it holds

$$\mathbb{P}^x(\zeta = \infty) = \lim_{t\to\infty} \mathbb{P}^x(\xi_t \in \mathbb{R}) = \lim_{t\to\infty} p_t(x, \mathbb{R}) = 1,$$

for $x \in \mathbb{R}$. Furthermore, the Markov property can be translated to

$$\mathbb{E}^x\big[f(\xi_{t+s}) \,|\, \mathcal{F}_t\big] = \mathbb{E}^{\xi_t}\big[f(\xi_s)\big], \tag{2.6}$$

for $f \in b\mathcal{B}(\mathbb{R})$ or more general to

$$\mathbb{E}^x\left[Y \circ \theta_t \,|\, \mathcal{F}_t\right] = \mathbb{E}^{\xi_t}\left[Y\right], \tag{2.7}$$

for non-negative, $\mathcal{F}_\infty := \bigcup_{t\geq 0} \mathcal{F}_t$-measurable Y. By (2.4) and (2.5) one obtains that for all $x \in \mathbb{R}$ the processes $(X + x, \mathbf{P})$ and $(\xi, \mathbb{P}^x)$ are equivalent, i.e. these processes have the same finite dimensional distributions. Hence, one could interpret the process $(\xi, \mathbb{P}^x)$ as $(X, \mathbf{P})$ shifted by x. In particular $(\xi, \mathbb{P}) := (\xi, \mathbb{P}^0)$ is a Lévy process, too. For this reason we will usually work with the so-called universal Markov process (also called right-continuous realisation) $(\xi, (\mathbb{P}^x)_{x\in\mathbb{R}})$ corresponding to a Lévy process $(X, \mathbf{P})$ instead of the Lévy process itself. Sometimes we will also call $(\xi, (\mathbb{P}^x)_{x\in\mathbb{R}})$ a Lévy process.

It is also well-known that Lévy processes are strong Markov processes in the sense that (2.6) and (2.7) extend to $(\mathcal{F}_t)_{t\geq 0}$-stopping times T, i.e. it holds

$$\mathbb{1}_{\{T<\infty\}}\mathbb{E}^x\big[f(\xi_{T+s})\,|\,\mathcal{F}_T\big] = \mathbb{1}_{\{T<\infty\}}\mathbb{E}^{\xi_T}\big[f(\xi_s)\big], \tag{2.8}$$

for $f \in b\mathcal{B}(\mathbb{R})$ and

$$\mathbb{1}_{\{T<\infty\}}\mathbb{E}^x\left[Y\circ\theta_T\,|\,\mathcal{F}_T\right] = \mathbb{1}_{\{T<\infty\}}\mathbb{E}^{\xi_T}\left[Y\right], \tag{2.9}$$

for non-negative, $\mathcal{F}_\infty$-measurable Y.

Killed Lévy processes

Next, we model killing the process $(\xi, (\mathbb{P}^x)_{x\in\mathbb{R}})$ when it enters an open or closed set $B \subseteq \mathbb{R}$. For this let

$$T_B := \inf\left\{t \geq 0 : \xi_t \in B\right\},$$

which is known to be a $(\mathcal{F}_t)_{t\geq 0}$ stopping time. Further, we set

$$p_t^B(x, \mathrm{d}y) = \mathbb{P}^x(\xi_t \in \mathrm{d}y, t < T_B), \quad x, y \in \mathbb{R}\setminus B. \tag{2.10}$$

It it not hard to show that $(p_t^B)_{t\geq 0}$ defines a sub-Markov semigroup on the space $(\mathbb{R}\setminus B, \mathcal{B}(\mathbb{R}\setminus B))$ and one can extend it to a Markov semigroup on $((\mathbb{R}\setminus B)\cup\{\Delta\}, \mathcal{B}((\mathbb{R}\setminus B)\cup\{\Delta\}))$ via

$$p_t^B(x,\{\Delta\}) = \begin{cases} \mathbb{P}^x(t \geq T_B) & \text{if } x \in \mathbb{R}\setminus B \\ 1 & \text{if } x = \Delta \end{cases}.$$

To speak about open sets which are needed for the Borel σ-algebra $\mathcal{B}((\mathbb{R}\setminus B)\cup\{\Delta\})$, the topology on $(\mathbb{R}\setminus B)\cup\{\Delta\}$ is extended in the usual way. We see immediately that

$$p_t^B f(x) = \int\limits_{\mathbb{R}\setminus B} f(y)\,p_t(x,\mathrm{d}y) = \int\limits_{\mathbb{R}\setminus B} f(y)\,\mathbb{P}^x(\xi_t \in \mathrm{d}y, t < T_B) = \mathbb{E}^x\big[\mathbb{1}_{\{t<T_B\}}f(\xi_t)\big],$$

for $t \geq 0$, $x \in \mathbb{R}\setminus B$ and measurable $f : \mathbb{R}\setminus B \to \mathbb{R}$.
Markov theory implies that there are probability measures $(\mathbb{P}^{x,B})_{x\in\mathbb{R}\setminus B}$ on the space $(D, \mathcal{F})$ under which the canonical process ξ is a Markov process corresponding to the Markov semigroup $(p_t^B)_{t\geq 0}$, i.e.

$$\mathbb{E}^{x,B}\big[f(\xi_{t+s})\,|\,\mathcal{F}_s\big] = p_t^B f(\xi_s) = \mathbb{E}^{\xi_s}\big[f(\xi_t)\mathbb{1}_{\{t<T_B\}}\big],$$

for $t, s \geq 0, x \in \mathbb{R}\setminus B$ and $f \in b\mathcal{B}(\mathbb{R}\setminus B)$, and as a consequence

$$\begin{aligned}&\mathbb{P}^{x,B}(\xi_{t_1} \in \mathrm{d}y_1, \dots, \xi_{t_n} \in \mathrm{d}y_n, t_n < \zeta) \\ &\qquad = p^B_{t_n - t_{n-1}}(y_{n-1}, \mathrm{d}y_n)\ \dots\ p^B_{t_2-t_1}(y_1,\mathrm{d}y_2)\,p^B_{t_1}(x,\mathrm{d}y_1),\end{aligned}$$

for $t_n \geq \ldots \geq t_1 \geq 0$ and $x, y_1 \ldots, y_n \in \mathbb{R}\setminus B$. In particular it holds $\mathbb{P}^{x,B}(\xi_t \in B) = 0$ and

$$((\xi_t\mathbb{1}_{\{t<\zeta\}})_{t\geq 0}, \mathbb{P}^{x,B}) \overset{(d)}{=} ((\xi_t\mathbb{1}_{\{t<T_B\}})_{t\geq 0}, \mathbb{P}^x),$$

for all $x \in \mathbb{R}\setminus B$. Hence, we can interpret $(\xi, \mathbb{P}^{x,B})$ as the process $(\xi, \mathbb{P}^x)$ killed when it enters B.

Excessive functions and Doob-h-transform

We fix a Lévy process $(\xi, (\mathbb{P}^x)_{x\in\mathbb{R}})$, a Borel set B and the Markov semigroup $(p_t^B)_{t\geq 0}$ corresponding to the Lévy process killed when it enters B.

Definition 2.2.2. *A function $h : \mathbb{R} \setminus B \to [0, \infty)$ is called excessive for $(p_t^B)_{t\geq 0}$ if*

$$p_t^B h(x) \leq h(x), \tag{2.11}$$

and $\lim_{t\searrow 0} p_t^B h(x) = h(x)$ for all $x \in \mathbb{R} \setminus B$. An excessive function is called invariant if $p_t^B h(x) = h(x)$ for all $x \in \mathbb{R} \setminus B$.

Note that (2.11) is equivalent to

$$\mathbb{E}^x\big[\mathbb{1}_{\{t<T_B\}} h(\xi_t)\big] \leq h(x), \quad x \in \mathbb{R} \setminus B, \tag{2.12}$$

and invariance is equivalent to the inequality replaced by an equality. A nice feature is that a function is invariant (fulfils (2.12)) if and only if the process $(\mathbb{1}_{\{t<T_B\}} h(\xi_t))_{t\geq 0}$ is a (super-) martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$ under $\mathbb{P}^x$ for all $x \in \mathbb{R} \setminus B$. Indeed, suppose that h fulfils (2.12) and $t \geq s$, then thanks to the Markov property,

$$\mathbb{E}^x[\mathbb{1}_{\{t<T_B\}} h(\xi_t) \,|\, \mathcal{F}_s] = \mathbb{E}^{\xi_s}[\mathbb{1}_{\{t-s<T_B\}} h(\xi_{t-s})] \leq \mathbb{1}_{\{s<T_B\}} h(\xi_s).$$

If on the other hand $\mathbb{1}_{\{t<T_B\}} h(\xi_t)$ is a supermartingale, its expectation is monotone decreasing in t, in particular

$$\mathbb{E}^x[\mathbb{1}_{\{t<T_B\}} h(\xi_t)] \leq \mathbb{E}^x[\mathbb{1}_{\{0<T_B\}} h(\xi_0)] = h(x),$$

which is (2.12). The equivalence of invariance and $(\mathbb{1}_{\{t<T_B\}} h(\xi_t))_{t\geq 0}$ being a martingale can be shown analogously.

Let us now assume that $h : \mathbb{R} \setminus B \to [0, \infty)$ is an excessive function for $(p_t^B)_{t\geq 0}$.

Definition 2.2.3. *The sub-Markov semigroup $(p_t^{B,h})_{t\geq 0}$ defined by*

$$p_t^{B,h}(x, \mathrm{d}y) = \begin{cases} \frac{h(y)}{h(x)}\, p_t^B(x, \mathrm{d}y) & \text{if } h(x) > 0 \\ 0 & \text{if } h(x) = 0 \end{cases},$$

for $x, y \in \mathbb{R} \setminus B$ is called Doob-h-transform (or just h-transform) of $(p_t^B)_{t\geq 0}$.

That $(p_t^{B,h})_{t\geq 0}$ is a sub-Markov semigroup on $(\mathbb{R} \setminus B, \mathcal{B}(\mathbb{R} \setminus B))$ is not hard to show. Note that if h is strictly positive on $\mathbb{R} \setminus B$ and invariant, it holds $p_t^{B,h}(x, \mathbb{R} \setminus B) = 1$ for all $x \in \mathbb{R} \setminus B$. If h is not invariant, we follow the same procedure as for the killed sub-Markov semigroup and extend it to the space $((\mathbb{R} \setminus B) \cup \{\Delta\}, \mathcal{B}((\mathbb{R} \setminus B) \cup \{\Delta\}))$ via

$$p_t^{B,h}(x, \{\Delta\}) = \begin{cases} 1 - p_t^{B,h}(x, \mathbb{R} \setminus B) & \text{if } x \in \mathbb{R} \setminus B \\ 1 & \text{if } x = \Delta \end{cases}.$$

A natural question is if we can construct probability measures on the path space $(D, \mathcal{F})$ in the spirit we constructed the probability measures $(\mathbb{P}^{x,B})_{x\in\mathbb{R}\setminus B}$. The answer is given for example by Chung and Walsh [18], Chapter 11. The authors show that there exist probability measures $(\mathbb{P}_h^x)_{x\in\mathbb{R}\setminus B}$ on $(D, \mathcal{F})$ such that $\mathbb{P}_h^x(\xi_0 = x) = 1$ and that $(\xi, \mathbb{P}_h^x)$

is a strong Markov process with sub-Markov semigroup $(p_t^{B,h})_{t\geq 0}$ for all $x \in \mathbb{R} \setminus B$. In particular the finite dimensional distributions are

$$\begin{aligned}\mathbb{P}_h^x(\xi_{t_1} \in \mathrm{d}y_1, \ldots, \xi_{t_n} \in \mathrm{d}y_n, t_n < \zeta)\\ = p_{t_n - t_{n-1}}^h(y_{n-1}, \mathrm{d}y_n) \ldots p_{t_2-t_1}^h(y_1, \mathrm{d}y_2) p_{t_1}^h(x, \mathrm{d}y_1),\end{aligned} \tag{2.13}$$

for $0 \leq t_1 < \ldots < t_n$ and $x, y_1, \ldots, y_n \in \mathbb{R} \setminus B$.

Note that with $p_t^h(x, \mathbb{R} \setminus B) = 1$ for all $x \in \mathbb{R} \setminus B$ if h is strictly positive on $\mathbb{R} \setminus B$ and invariant we obtain $\mathbb{P}_h^x(\xi_t \in \mathbb{R} \setminus B) = 1$ for all $x \in \mathbb{R} \setminus B$ and $t \geq 0$ and as a consequence $\mathbb{P}^x(\zeta = \infty) = 1$. This means we transformed a Markov process which (possibly) hits Δ with positive probability to another Markov process which does not hit Δ almost surely. Including the step of killing the process when it enters B, we can even say that an $\mathbb{R}$-valued process which hits B with positive probability was transformed to an $\mathbb{R}$-valued process which does not hit B (by first killing and then h-transforming). This is one of the most crucial methods which we will use in this work.

For an excessive function h we call the process $(\xi, \mathbb{P}_h^x)$ also the h-transformed process of $(\xi, \mathbb{P}^{x,B})$ and sometimes we also call $\mathbb{P}_h^x$ the h-transform of the Lévy process killed on entering B. The for us most important properties of the h-transformed process are the characterisation of the finite-dimensional distributions via (2.13) and

$$\mathbb{P}_h^x(\Lambda, T < \zeta) = \frac{1}{h(x)} \mathbb{E}^x\big[\mathbb{1}_{\{T < T_B\}} h(\xi_T)\big], \quad \Lambda \in \mathcal{F}_T, \tag{2.14}$$

for all $(\mathcal{F}_t)_{t\geq 0}$-stopping times T. From Chung and Walsh [18] it is further known that the h-transformed process $(\xi, \mathbb{P}_h^x)$ is a strong Markov process in the sense that for all $(\mathcal{F}_t)_{t\geq 0}$-stopping times T, i.e. it holds

$$\mathbb{1}_{\{T<\zeta\}} \mathbb{E}_h^x\big[f(\xi_{T+s}) \,|\, \mathcal{F}_T\big] = \mathbb{1}_{\{T<\zeta\}} \mathbb{E}_h^{\xi_T}\big[f(\xi_s)\big], \tag{2.15}$$

for $f \in b\mathcal{B}(\mathbb{R})$ and

$$\mathbb{1}_{\{T<\zeta\}} \mathbb{E}_h^x\left[Y \circ \theta_T \,|\, \mathcal{F}_T\right] = \mathbb{1}_{\{T<\zeta\}} \mathbb{E}_h^{\xi_T}\left[Y\right], \tag{2.16}$$

for non-negative, $\mathcal{F}_\infty := \bigcup_{t\geq 0} \mathcal{F}_t$-measurable Y.

Next to invariant functions there is another special class of excessive functions. We already have seen easily that a function $h : \mathbb{R} \setminus B \to [0, \infty)$ is excessive for $(p_t^B)_{t\geq 0}$ if and only if

$$\mathbb{E}^x\big[\mathbb{1}_{\{t<T_B\}} h(\xi_t)\big] \leq h(x), \quad x \in \mathbb{R} \setminus B, t \geq 0. \tag{2.17}$$

With this characterisation we define the further subclass of excessive functions for killed Lévy processes.

Definition 2.2.4. *Let $(p_t^B)_{t\geq 0}$ the sub-Markov semigroup corresponding to a Lévy process killed on entering $B \in \mathcal{B}(\mathbb{R})$. An excessive function $h : \mathbb{R} \setminus B \to [0, \infty)$ is called harmonic if*

$$\mathbb{E}^x\big[\mathbb{1}_{\{T_{K^\mathrm{C}} < T_B\}} h(\xi_{T_{K^\mathrm{C}}})\big] = h(x), \quad x \in \mathbb{R} \setminus B, \tag{2.18}$$

for all sets K which are compact in $\mathbb{R} \setminus B$.

The crucial of a harmonic function is that the h-transformed process using a harmonic function leaves all subsets of $\mathbb{R} \setminus B$ which are compact in $\mathbb{R} \setminus B$. Indeed,

$$\mathbb{P}_h^x(T_{K^\complement} < \zeta) = \frac{1}{h(x)} \mathbb{E}^x\left[\mathbb{1}_{\{T_{K^\complement} < T_B\}} h(\xi_{T_{K^\complement}})\right] = \frac{h(x)}{h(x)} = 1, \tag{2.19}$$

where we used the harmonicity of h in the second equation.

Remark 2.2.5. *In many articles (including [27] and [28]) the authors use the notion of a harmonic function for a function which is invariant in the sense of Definition 2.2.2. In this work we will exclusively use the notions which we introduced in this section.*

2.3 Fluctuation theory for Lévy processes

We fix a realisation $(\xi, (\mathbb{P}^x)_{x\in\mathbb{R}})$ of a Lévy process and start by defining the resolvent measures of it,

$$U^q(x, \mathrm{d}y) := \mathbb{E}^x\Big[\int\limits_{[0,\infty)} e^{-qt} \mathbb{1}_{\{\xi_t \in \mathrm{d}y\}} \,\mathrm{d}t\Big], \quad x, y \in \mathbb{R}, q \geq 0.$$

The measure $U := U^0$ is called the potential of ξ and we see that $U(x, A)$ for a Borel set A represents the expected time the process started in $x \in \mathbb{R}$ spends in A.

Subordinators at first passage

Let us consider a 1-dimensional Lévy process $(\xi, \mathbb{P}) = (\xi, \mathbb{P}^0)$ and denote its Lévy triple by (γ, σ^2, Π). We call this Lévy process a subordinator if it is non-decreasing which forces

$$-(\gamma + \int\limits_{(0,1)} x\,\Pi(\mathrm{d}x)) \geq 0,\ \sigma^2 = 0,\ \Pi(-\infty, 0) = 0 \text{ and } \int\limits_{(0,\infty)} \min(1, x)\,\Pi(\mathrm{d}x) < \infty.$$

In this case it follows that $T_{[a,\infty)}$ is finite almost surely and a classic question is if there is an explicit expression for the distribution of $\xi_{T_{[a,\infty)}}$. The more general answer is the following: First we define $U(\mathrm{d}y) = U(0, \mathrm{d}y)$ and with this it holds:

$$\mathbb{P}^x(\xi_{T_{[a,\infty)}-} \in \mathrm{d}z, \xi_{T_{[a,\infty)}} \in \mathrm{d}y) = U(\mathrm{d}z - x)\,\Pi(\mathrm{d}y - z), \tag{2.20}$$

for $x \in \mathbb{R}$ and $x \leq z \leq a < y$ and $\mathbb{P}^x(\xi_{T_{[a,\infty)}-} < a = \xi_{T_{[a,\infty)}}) = 0$ for $a > x$. The remaining choices of the variables can be obtained by $\mathbb{P}^x(\xi_{T_{[a,\infty)}} = x) = 1$ if $x \geq a$. The value $\xi_{T_{[a,\infty)}} - a \geq 0$ is called overshoot over the level a and $a - \xi_{T_{[a,\infty)}-} \geq 0$ is called the undershoot.

By integration over z we get the following overshoot distribution:

$$\mathbb{P}^x(\xi_{T_{[a,\infty)}} \in \mathrm{d}y) = \int\limits_{z\in[x,a]} \Pi(\mathrm{d}y - z)U(\mathrm{d}z - x). \tag{2.21}$$

Duality

It should be clear that if $(\xi, \mathbb{P}^x)$ is a Lévy process then $(-\xi, \mathbb{P}^x)$ is so, too. We call $-\xi$ the dual Lévy process. Let us denote the semigroup corresponding to $(-\xi, \mathbb{P}^x)$ by $(\hat{p}_t)_{t\geq 0}$ which can be used to construct probability measures $(\hat{\mathbb{P}}^x)_{x\in\mathbb{R}}$ on $(D, \mathcal{F})$ such that $(-\xi, \mathbb{P}^x) \overset{(d)}{=} (\xi, \hat{\mathbb{P}}_x)$. The naming dual process comes from the fact that $(p_t)_{t\geq 0}$ and $(\hat{p}_t)_{t\geq 0}$ are dual with respect to the Lebesgue measure, i.e.

$$\int\limits_{\mathbb{R}} p_t f(x) g(x)\,\mathrm{d}x = \int\limits_{\mathbb{R}} f(x)\hat{p}_t g(x)\,\mathrm{d}x,$$

for all $t \geq 0$ and all non-negative, measurable functions $f, g : \mathbb{R} \to [0, \infty)$. One can even go further and denote the sub-Markov semigroups which are obtained by killing the process and its dual on hitting a given open or closed Borel set B by $(p_t^B)_{t\geq 0}$ and $(\hat{p}_t^B)_{t\geq 0}$. These semigroups are dual with respect to the Lebesgue measure as well, i.e.

$$\int\limits_{\mathbb{R}\setminus B} p_t^B f(x) g(x)\,\mathrm{d}x = \int\limits_{\mathbb{R}\setminus B} f(x)\hat{p}_t^B g(x)\,\mathrm{d}x,$$

for all $t \geq 0$ and all non-negative, measurable functions $f, g : \mathbb{R} \setminus B \to [0, \infty)$.

Local time at the maximum and ladder height process

It is classic that $(\sup_{s\leq t} \xi_s - \xi_t)_{t\geq 0}$ is a Markov process. Furthermore, it possesses a local time at 0. We call it $(L_t)_{t\geq 0}$ and refer the reader to Chapter 6 of [34] for more details. We just remark that it is a certain measure for the time the process $(\sup_{s\leq t} \xi_s - \xi_t)_{t\geq 0}$ spends close to 0. It is also called local time of $(\xi_t)_{t\geq 0}$ at the maximum.

Next we define the so called right-continuous inverse of $L = (L_t)_{t\geq 0}$ via

$$L_t^{-1} := \begin{cases} \inf\{s > 0 : L_s > t\} & \text{if } t < L_\infty \\ \infty & \text{otherwise} \end{cases},$$

for $t \geq 0$. We can interpret this process as the collected times which ξ spends at the maximum. Motivated by this we define $H = (H_t)_{t\geq 0}$ via

$$H_t := \begin{cases} \xi_{L_t^{-1}} & \text{if } t < L_\infty \\ \infty & \text{otherwise} \end{cases},$$

for $t \geq 0$, which we interpret as the process which collects the values when ξ is at its maximum. It is well-known that $(L_t^{-1}, H_t)_{t\geq 0}$ is a bivariate subordinator, that is a two-dimensional Lévy process whose components are non-decreasing. In particular both components are one-dimensional subordinators. We denote by μ_+ and γ_+ the Lévy measure and the drift of $(H, \mathbb{P}^0)$ and we define

$$U_+^q(\mathrm{d}x) = \mathbb{E}\Big[\int_{[0,\infty)} \mathbb{1}_{\{H_t\in \mathrm{d}x\}}\,\mathrm{d}t\Big],$$

for $q \geq 0$ and $x \geq 0$. We see that $U_+ := U_+^0$ is just the potential of $(H, \mathbb{P})$. The function $U_+(x) := U_+([0, x])$ is called the potential function of $(H, \mathbb{P})$. We denote by μ_-, γ_-,

$U_-^q, q \geq 0$ and U_- the analogous expressions for the process $(\xi, \hat{\mathbb{P}}^0)$. It is known that we can write μ_+ in terms of Π and U_- in the following way:

$$\mu_+(\mathrm{d}y) = \int\limits_{[0,\infty)} \Pi(z + \mathrm{d}y)\, U_-(\mathrm{d}z), \quad y \geq 0. \tag{2.22}$$

From the Wiener-Hopf factorisation (see e.g. [34], Chapter 6) it is known that the characteristic exponent of ξ can be uniquely written as

$$\Psi(\theta) = \Psi_+(\theta)\Psi_-(-\theta), \quad \theta \in \mathbb{R}, \tag{2.23}$$

where Ψ_+ and Ψ_- are characteristic exponents of two subordinators. Further, they can be identified as the characteristic exponent of the ladder height process and the dual ladder height process. For this reason we denote by $\Psi_+(\theta)$ and $\psi_+(\theta)$ (resp. $\Psi_-(\theta)$ and $\psi_-(\theta)$) for $\theta \in \mathbb{R}$ the characteristic exponent and the Laplace exponent of the ladder height process (the dual ladder height process respectively). If the Laplace exponents exist, one can translate (2.23) to the Laplace exponents of the corresponding Lévy processes. Moreover, the Laplace exponent of L^{-1} will be useful, too. We denote it by $\kappa(q) = -\log \mathbb{E}\big[e^{-qL_1^{-1}}\big]$ for $q \geq 0$ and by $\hat{\kappa}$ we denote its counterpart for the dual process.

Long-time behaviour

Here, we present results on the long-time behaviour of the Lévy process $(\xi, \mathbb{P})$. We say that a Lévy process $(\xi, \mathbb{P})$ of dimension 1 drifts to $+\infty$ $(-\infty)$ if $\lim_{t\to\infty} \xi_t = +\infty$ ($\lim_{t\to\infty} \xi_t = -\infty$) almost surely. We say that it oscillates if $\limsup_{t\to\infty} \xi_t = -\liminf_{t\to\infty} \xi_t = +\infty$. There is the following integral test which shows on the one hand that $(\xi, \mathbb{P})$ either drifts to $\pm\infty$ or oscillates and on the other hand we can say which of these possibilities holds.

(i) $(\xi, \mathbb{P})$ drifts to $+\infty$ if $\int_{[1,\infty)} \frac{1}{t}\mathbb{P}(\xi_t \leq 0)\,\mathrm{d}t < \infty$.

(ii) $(\xi, \mathbb{P})$ drifts to $-\infty$ if $\int_{[1,\infty)} \frac{1}{t}\mathbb{P}(\xi_t \geq 0)\,\mathrm{d}t < \infty$.

(iii) $(\xi, \mathbb{P})$ oscillates if it does not drift to $\pm\infty$.

Next, we turn to recurrence and transience. We call a Lévy process transient if

$$\mathbb{P}\Big(\int\limits_{[0,\infty)} \mathbb{1}_{\{|\xi_t|<a\}}\,\mathrm{d}t < \infty\Big) = 1 \quad \text{for all } a > 0 \tag{2.24}$$

and recurrent if

$$\mathbb{P}\Big(\int\limits_{[0,\infty)} \mathbb{1}_{\{|\xi_t|<a\}}\,\mathrm{d}t = \infty\Big) = 1 \quad \text{for all } a > 0. \tag{2.25}$$

It is classic that a Lévy process is either transient or recurrent. Further, the process ξ is transient if and only if $\lim_{t\to\infty} |\xi_t| = \infty$. In particular drifting to $\pm\infty$ implies transience but not the other way around because a transient Lévy process can also oscillate (we will later see an example). Moreover, a recurrent Lévy process oscillates.

General Lévy processes at first passage and stationary overshoots

Assume that ξ does not drift to $-\infty$, in particular we have that $T_{[a,\infty)}$ is finite almost surely. A nice application of Section 2.3 is that when ξ jumps over a level a for the first time it is at the maximum. Hence, the overshoot distribution of ξ over the level a is the same like the analogous for the ladder height process. Since the ladder height process is a subordinator we can apply (2.21) to deduce the overshoot distribution of ξ:

$$\begin{aligned}\mathbb{P}^x(\xi_{T_{[a,\infty)}} \in \mathrm{d}y) &= \mathbb{P}(\xi_{T_{[a-x,\infty)}} \in \mathrm{d}y - x)\\ &= \mathbb{P}(H_{T^H_{[a-x,\infty)}} \in \mathrm{d}y - x)\\ &= \int\limits_{z\in[x,a]} \mu_+(\mathrm{d}y - z)U_+(\mathrm{d}z - a),\end{aligned} \tag{2.26}$$

where $T^H_{[a,\infty)} = \inf\{t \geq 0 : H_t \in [a,\infty)\}$. For the common distribution of overshoot and undershoot this leading back to the ladder height process is not possible. But there is a general result of Doney and Kyprianou [23], called quintuple law which is the common distribution of five values concerning the overshoot and the overshoot time, including the overshoot itself and the undershoot. Since we are just interested in the overshoot-undershoot law we just state this distribution which can be extracted via integrating out the other three variables. The result is the following:

$$\begin{aligned}&\mathbb{P}^x(\xi_{T_{[a,\infty)}-} \in \mathrm{d}z, \xi_{T_{[a,\infty)}} \in \mathrm{d}y) \\ &\quad= \int\limits_{u\in[0,a-x]} \mathbb{1}_{\{a-z>v\}}\Pi(\mathrm{d}y - z)\, U_-(a - v - \mathrm{d}z)\, U_+(a - x - \mathrm{d}v),\end{aligned} \tag{2.27}$$

for $x < a$ and $z \leq a < y$. Furthermore, Bertoin and Savov [6] showed that if $\gamma_+ = 0$ it holds $\mathbb{P}^x(\xi_{T_{[a,\infty)}-} = a = \xi_{T_{[a,\infty)}}) = 0$. If $\gamma_+ > 0$, then U_+ has a continuous density u_+ which is continuous and strictly positive everywhere with $u_+(0) = 1/\gamma_+$ and it holds

$$\mathbb{P}^x(\xi_{T_{[a,\infty)}-} = a = \xi_{T_{[a,\infty)}}) = \gamma_+ u_+(a - x).$$

An interesting question is if this overshoot-undershoot distribution (2.27) converges in some sense for $x \to -\infty$, i.e. we let the initial position tend to $-\infty$ and ask ourself how the process then jumps over the level a. The answer is that this distribution converges weakly to a probability measure if the ladder height process has finite mean and in this case the limiting distribution is

$$\rho_a(\mathrm{d}z, \mathrm{d}y) = \frac{1}{\mathbb{E}[H_1]}\big(\gamma_+\delta_a(\mathrm{d}z)\delta_a(\mathrm{d}y) + \mathbb{1}_{\{z<y\}}U_-(a - z)\,\Pi(\mathrm{d}y - z)\,\mathrm{d}z\big). \tag{2.28}$$

One says that ξ has stationary overshoots and undershoots. This result in its full generality was obtained from Bertoin and Savov, [6]. But before some special cases were proven, for example the second marginal in the case of a subordinator, see Bertoin et al. [7]. Since we will use it later we give the second marginal of ρ_a which we call the stationary overshoot distribution which can be obtained by integrating over z:

$$\rho_a^+(\mathrm{d}y) = \frac{1}{\mathbb{E}[H_1]}\big(\gamma_+\delta_a(\mathrm{d}y) + \mathbb{1}_{\{y>a\}}\bar{\mu}_+(y - a)\mathrm{d}y\big), \tag{2.29}$$

where for a general measure ν on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ we denote by $\bar{\nu}$ its right-tail.

Killed potentials

At the beginning of this section the potential of a Lévy process was defined as the expected time which it stays in a given set. Here the so called killed potential will be defined as the expected time the process stays in a certain set before hitting a given open or closed set $B \subseteq \mathbb{R}$. More concrete:

$$U_B(x, \mathrm{d}y) := \mathbb{E}^x\Big[\int\limits_{[0,T_B)} \mathbb{1}_{\{\xi_t \in \mathrm{d}y\}}\,\mathrm{d}t\Big], \quad x, y \in \mathbb{R} \setminus B. \tag{2.30}$$

For the special $B = (-\infty, 0]$ there is the following formula from Bertoin [4], Theorem VI.20 if ξ is not a compound Poisson process:

$$U_{(-\infty,0]}(x, \mathrm{d}y) = k \int\limits_{[(x-y)^+,x]} U_+(\mathrm{d}y + z - x)\,U_-(\mathrm{d}z), \quad x, y > 0, \tag{2.31}$$

for some constant k which depends on the normalisation of the local time at the maximum. Later we will see that for the case of a stable process there is even an explicit formula for the case $B = [-1, 1]$.

2.4 Stable processes

Here, we introduce a special class of Lévy processes, called stable processes. It is based on Section 1.2.6 of [34], Section 7 of [4] and the review article [36]. As one can guess from the naming a stable process is (a realisation of) a Lévy process $(\xi, (\mathbb{P}^x)_{x\in\mathbb{R}})$ such that ξ_t has a stable distribution for all $t \geq 0$. A random variable Z is called stable if for all $n \in \mathbb{N}$ there exists $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$Z^{(1)} + \ldots + Z^{(n)} \overset{(d)}{=} a_n Z + b_n, \tag{2.32}$$

where $Z^{(1)}, \ldots, Z^{(n)}$ are independent copies of Z. It is known that $a_n = n^{\frac{1}{\alpha}}$ for some $\alpha \in (0, 2]$. Further, we call Z strictly stable if $b_n = 0$. From (2.32) we see that all stable random variables Z have an infinitely often divisible distribution and hence, there exists a Lévy process $(\xi_t)_{t\geq 0}$ such that

$$\xi_1 \overset{(d)}{=} Z.$$

In this case we call ξ a stable Lévy process and if $b_n = 0$ we call ξ a strictly stable process. It turns out that the difference between a stable process and a strictly stable process is just that in the stable case there could appear an additional drift which can not happen in the strictly stable case. Thus, we just consider the strictly stable case and henceforth we call it also stable.

Let us now fix a stable process $\xi = (\xi_t)_{t\geq 0}$ with $\alpha \in (0, 2]$. In the case $\alpha \in (0, 2]$ it is known that

$$\Psi(\theta) = \begin{cases} c|\theta|^\alpha(1 - \mathrm{i}\beta\tan(\frac{\pi\alpha}{2})\,\mathrm{sgn}(\theta)) & \text{if } \alpha \in (0,1) \cup (1,2) \\ c|\theta| + \mathrm{i}\gamma\theta & \text{if } \alpha = 1 \\ c\frac{\theta^2}{2} & \text{if } \alpha = 2 \end{cases}, \tag{2.33}$$

for some $\beta \in [-1,1]$, $c > 0$ and $\gamma \in \mathbb{R}$. Here $\operatorname{sgn}(x) = \mathbb{1}_{\{x>0\}} - \mathbb{1}_{\{x<0\}}$. We see that for $\alpha = 2$ the stable process is a linear Brownian motion with zero drift. In particular the drift and the Lévy measure in the Lévy-Khintchine formula are zero. For $\alpha = 1$ the process is the symmetric Cauchy process with an additional drift γ. Moreover, one can obtain by (2.33) for $\alpha \in (0,1) \cup (1,2)$ that $\sigma = 0$ and the Lévy measure has the following form:

$$\Pi(\mathrm{d}x) = c_+ x^{-(1+\alpha)} \mathbb{1}_{\{x>0\}} \mathrm{d}x + c_- |x|^{-(1+\alpha)} \mathbb{1}_{\{x<0\}} \mathrm{d}x, \tag{2.34}$$

where $c_+, c_- \geq 0$ with $c_+ + c_- > 0$ such that

$$\beta = \frac{c_+ - c_-}{c_+ + c_-}.$$

By (2.33) one can show that stable processes own the following scaling property:

$$(c\xi_{c^{-\alpha}t}, \mathbb{P}^x) \overset{(d)}{=} (\xi_t, \mathbb{P}^{cx}), \quad c > 0. \tag{2.35}$$

We obtain with this scaling property that $\rho := \mathbb{P}^0(\xi_t > 0)$ does not depend on $t \geq 0$. If $\alpha = 1$ and $\gamma = 0$ or $\alpha = 2$, it holds $\rho = 1/2$. If $\alpha \in (0,1) \cup (1,2)$, we can calculate ρ in terms of β:

$$\rho = \frac{1}{2} + \frac{1}{\pi\alpha} \arctan\big(\beta \tan(\frac{\pi\alpha}{2})\big).$$

We see that for $\alpha \in (0,1)$ it follows $\rho \in [0,1]$ and $\rho = 0$ ($\rho = 1$) appears if and only if $c_- = 0$ ($c_+ = 0$), i.e. the stable process does not make negative (positive) jumps. By $\rho = 0$ ($\rho = 1$) it follows that ξ is (the negative of) a subordinator. For $\alpha \in (1,2)$ it follows $\rho \in [1 - \frac{1}{\alpha}, \frac{1}{\alpha}]$ and again the boundary points belong to the cases $c_- = 0$ and $c_+ = 0$. For accordance with the literature we will henceforth use the particular choices of c_- and c_+ such that

$$\Pi(\mathrm{d}x) = \frac{\Gamma(\alpha+1)}{\pi} \Big(\sin(\pi\alpha\rho) x^{-(1+\alpha)} \mathbb{1}_{\{x>0\}} \mathrm{d}x + \sin(\pi\alpha\hat{\rho}) |x|^{-(1+\alpha)} \mathbb{1}_{\{x<0\}} \mathrm{d}x \Big), \tag{2.36}$$

where $\hat{\rho} = 1 - \rho$.

We should consider the results of Section 2.3 for the special case of a stable process. First it is known that $(H, \mathbb{P})$ is a stable subordinator with scaling index $\alpha\rho$ and $(H, \hat{\mathbb{P}})$ is a stable subordinator with scaling index $\alpha\hat{\rho}$. Furthermore, it holds

$$U_+(\mathrm{d}x) = \frac{1}{\Gamma(\alpha\rho)} x^{\alpha\rho - 1} \, \mathrm{d}x.$$

With the counterpart result for U_- one can obtain an explicit formula for the killed potential $U_{(-\infty,0]}$ via (2.31). There is also a result for the case of killing in an interval by Profeta and Simon [49] which was also discovered by Kyprianou et al. [38] for some special cases. To abbreviate let

$$\psi_{\alpha\rho}(x) = (x-1)^{\alpha\hat{\rho}-1}(x+1)^{\alpha\rho-1}, \quad |x| > 1,$$

and the analogous for ρ replaced by $\hat{\rho}$ and vice versa by $\psi_{\alpha\hat{\rho}}$. Moreover, we set $c_\alpha = 2^{1-\alpha}/(\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho}))$ and $z(x,y) = |1 - xy|/|x - y|$. The potential $U_{[-1,1]}(x, \mathrm{d}y)$ has the density

$$
\begin{aligned}
u_{[-1,1]}(x,y) &= c_\alpha\Big((y-x)^{\alpha-1}\int_1^{z(x,y)}\psi_{\alpha\rho}(v)\,\mathrm{d}v \\
&\quad -(\alpha-1)\int_1^y\psi_{\alpha\hat\rho}(v)\,\mathrm{d}v\int_1^x\psi_{\alpha\rho}(v)\,\mathrm{d}v\Big), \quad 1<x<y,\\
u_{[-1,1]}(x,y) &= c_\alpha\Big((x-y)^{\alpha-1}\int_1^{z(x,y)}\psi_{\alpha\hat\rho}(v)\,\mathrm{d}v \\
&\quad -(\alpha-1)\int_1^y\psi_{\alpha\hat\rho}(v)\,\mathrm{d}v\int_1^x\psi_{\alpha\rho}(v)\,\mathrm{d}v\Big), \quad 1<y<x,\\
u_{[-1,1]}(x,y) &= c_\alpha\frac{\sin(\pi\alpha\hat\rho)}{\sin(\pi\alpha\rho)}\Big((x-y)^{\alpha-1}\int_1^{z(x,y)}\psi_{\alpha\rho}(v)\,\mathrm{d}v \\
&\quad -(\alpha-1)\int_1^{|y|}\psi_{\alpha\rho}(v)\,\mathrm{d}v\int_1^x\psi_{\alpha\rho}(v)\,\mathrm{d}v\Big), \quad y<-1, 1<x,\\
u_{[-1,1]}(x,y) &= c_\alpha\frac{\sin(\pi\alpha\rho)}{\sin(\pi\alpha\hat\rho)}\Big((y-x)^{\alpha-1}\int_1^{z(x,y)}\psi_{\alpha\hat\rho}(v)\,\mathrm{d}v \\
&\quad -(\alpha-1)\int_1^y\psi_{\alpha\hat\rho}(v)\,\mathrm{d}v\int_1^{|x|}\psi_{\alpha\hat\rho}(v)\,\mathrm{d}v\Big), \quad x<-1, 1<y.
\end{aligned}
\tag{2.37}
$$

In the cases $x, y < -1$ we can apply duality to deduce $u_{[-1,1]}(x,y) = \hat{u}_{[-1,1]}(-x,-y)$ where $\hat{u}_{[-1,1]}$ has the same form as $u_{[-1,1]}$ with ρ and $\hat\rho$ switched.

Now we turn to transience and recurrence of stable processes. A (one-dimensional) stable process with index $\alpha < 1$ is transient, a stable process with $\alpha \geq 1$ is recurrent. For $\alpha > 1$ the process even hit points almost surely, i.e. it holds that $T_{\{x\}}$ is almost surely finite for all $x \in \mathbb{R}$. Furthermore, a stable process oscillates as long as $\rho \in (0,1)$ in the case $\alpha \neq 1$ and $\gamma = 0$ in the case $\alpha = 1$. This can be seen via the integral tests of Section 2.3.

2.5 Self-similar Markov processes and Lamperti-Kiu transform

One-dimensional stable processes do not just belong to the class of Lévy processes but also to the class of self-similar Markov processes. Lamperti [44] developed a pathwise one-to-one correspondence between positive self-similar Markov processes and Lévy processes via a time-change. In the last ten years this result was generalised to a one-to-one correspondence between real self-similar Markov processes and Markov additive processes (MAPs). Many works used the stable process as a self-similar Markov process and analysed it via the underlying MAP. We will do so as well and introduce the basic definitions.

Self-similar Markov processes and MAPs

A realisation of a Markov process $\xi = (\xi_t)_{t\geq 0}$ is called a self-similar Markov process (ssMp) if

$$((c\xi_{c^{-\alpha}t})_{t\geq 0}, \mathbb{P}^x) \overset{(d)}{=} ((\xi_t)_{t\geq 0}, \mathbb{P}^{cx}),$$

for all $c > 0$ and all $x \in \mathbb{R}$, where α is the index of self-similarlity. A Markov additive process (MAP) with respect to a filtration $(\mathcal{G}_t)_{t\geq 0}$ is a $\mathbb{R}\times E$-valued càdlàg process $(M, J) = (M_t, J_t)_{t\geq 0}$, where E is a finite space, J is a Markov chain in E and the following holds for all $i \in E$ and $s, t \geq 0$: given $\{J_t = i\}$, the pair $(M_{t+s} - M_t, J_{t+s})$ is independent of $\mathcal{G}_t$ and has the same distribution as $(M_s - M_0, J_s)$ given $\{J_0 = i\}$. Asmussen [3] provides the following alternative characterisation of MAPs in terms of Lévy processes, which helps us to get a clear image of the behavior of MAPs. The pair (M, J) is a MAP if and only if for all $i \in E$ there is an iid sequence $(M^{n,i})_{n\in\mathbb{N}}$ of Lévy processes and for all $i, j \in E$ there is an iid sequence of random variables $(U^n_{i,j})_{n\in\mathbb{N}}$, independent of J such that for the jump times $(\sigma_n)_{n\in\mathbb{N}}$ of J and the convention $\sigma_0 = 0$ it holds that

$$M_t = \mathbb{1}_{\{n\geq 1\}}\xi_{\sigma_n-} + U^n_{J_{\sigma_n-},J_{\sigma_n}} + M^{n,J_{\sigma_n}}_{t-\sigma_n}, \qquad t \in [\sigma_n, \sigma_{n+1}),\ n \geq 0.$$

In words, this characterisation shows that the dynamics of a MAP are governed by a set of Lévy processes with different characteristics and the choice of the Lévy process is governed by an underlying Markov chain.

The Lamperti-Kiu transform

A classic work of Lamperti, [44] provides a pathwise one-to-one connection between positive self-similar Markov processes and Lévy processes. This was generalised to the following result, proved by Chaumont et al. [17]. Given a self-similar Markov process ξ which starts from $x \in \mathbb{R}\backslash\{0\}$, there exists a MAP (M, J) on $\mathbb{R}\times\{\pm 1\}$ which starts from $(\log|x|, \mathrm{sgn}(x))$ such that

$$\xi_t = J_{\varphi_t} \exp\left(M_{\varphi_t}\right), \qquad 0 \leq t < T_0,$$

where $\varphi_t = \inf\{s > 0 : \int_0^s \exp(\alpha M_u)\,\mathrm{d}u > t\}$ and $T_0 = \inf\{t \geq 0 : \xi_t = 0\}$. The transformation offers the opportunity to obtain results for ssMps from the reservoir of proved and potentially provable results for MAPs as long as the time-change can be controlled. As an example, the 0-1 law for the longtime behaviour of a MAP readily implies a 0-1 law for the extinction of a ssMp. That is:

(i) $T_0 = +\infty$ a.s. if and only if the underlying MAP (M, J) oscillates or drifts to $+\infty$.

(ii) $T_0 < +\infty$ a.s. if and only if the underlying MAP (M, J) drifts to $-\infty$ or is killed.

To analyse self-similar Markov processes via the Lamperti-Kiu transform is a recent approach which was followed in the last decay for example by Caballero and Chaumont [14] and Dereich et al. [21] to start (positive) self-similar Markov processes from the origin.

Stable processes as self-similar Markov processes

In Section 2.4 we introduced stable processes as Lévy process which fulfil the scaling property (2.35). Since Lévy processes are Markov processes, stable processes are self-similar Markov processes. So a nearby approach to analyse stable process is to analyse the underlying MAPs. Kyprianou [35] and Kyprianou et al. [41] followed this approach and explored many useful results (For higher dimensions there are also results in Kyprianou et al. [40]). We do not state results of these articles here but we will give references at the points we will use them. Moreover, some killed stable processes (e.g. the stable process killed on hitting $(-\infty, 0]$) and transformations of killed Lévy processes (e.g. the stable process conditioned to stay positive) are self-similar Markov processes. Recent works on the analysis of the underlying MAP (Lévy process respectively because the self-similar Markov process in these cases is positive) are Caballero and Chaumont [13] or Caballero et al. [12].

2.6 Exemplary applications

In the last section of this chapter we review some examples mentioned in Chapter 1 in more detail, i.e. we present some known applications of Section 2.2 on killed Lévy processes. More concrete, we use $(p_t^B)_{t\geq 0}$ from Section 2.2 for some particular choices of the Borel set B, give one (or more) excessive functions and state the connection of the corresponding h-transformed process to the process conditioned to avoid B (resp. conditioned to hit B continuously from the outside).

Lévy processes conditioned to stay positive

We start with a Lévy process $(\xi, \mathbb{P}^x)$ which is not a compound Poisson process and does not drift to $-\infty$. From Chaumont and Doney [16] it is known that the dual potential function U_- is an invariant function for the process killed on the negative half-line, i.e. it is invariant for the sub-Markov semigroup

$$p_t^{(-\infty,0]}(x, \mathrm{d}y) = \mathbb{P}^x(\xi_t \in \mathrm{d}y, t < T_{(-\infty,0]}), \quad x, y > 0,$$

in the sense of Definition 2.2.2 in the case $B = (-\infty, 0]$. We denote the corresponding h-transformed probability measures by $(\mathbb{P}^x_\uparrow)_{x>0}$, i.e. it holds

$$\mathbb{P}^x_\uparrow(\Lambda, T < \infty) = \mathbb{E}^x\Big[\mathbb{1}_\Lambda \mathbb{1}_{\{T<T_{(-\infty,0]}\}} \frac{U_-(\xi_T)}{U_-(x)}\Big], \quad x > 0, \tag{2.38}$$

for $\Lambda \in \mathcal{F}_T$ and any $(\mathcal{F}_t)_{t\geq 0}$-stopping time T. The authors showed that the h-transformed process coincides with the process conditioned to stay positive in the usual sense

$$\mathbb{P}^x_\uparrow(\Lambda) = \lim_{q\searrow 0} \mathbb{P}^x(\Lambda, t < e_q \,|\, e_q < T_{(-\infty,0]}), \quad x > 0,$$

for $\Lambda \in \mathcal{F}_t$, $t \geq 0$. As an application they also showed that this process drifts to $+\infty$ in the sense that $\lim_{t\to\infty} \xi_t = +\infty$ almost surely under $\mathbb{P}^x_\uparrow$.

Lévy processes conditioned to hit 0 continuously from above

This part is based on Chaumont [15] and Silverstein [55] and again we consider the sub-Markov semigroup

$$p_t^{(-\infty,0]}(x, \mathrm{d}y) = \mathbb{P}^x(\xi_t \in \mathrm{d}y, t < T_{(-\infty,0]}), \quad x, y > 0,$$

where $(\xi, \mathbb{P}^x)$ is a Lévy process which does not drift to $-\infty$ such that the unkilled semigroup is absolutely continuous with respect to the Lebesgue measure and 0 is regular for $(-\infty, 0)$, in the sense that $\mathbb{P}^0(T_{(-\infty,0)} = 0) = 1$. It is known that in this case U_- has a density with respect to the Lebesgue measure which we denote by u_-. Silverstein [55] showed that this density is a harmonic function for $(p_t^{(-\infty,0]})_{t\geq 0}$ in the sense of Definition 2.2.4. Furthermore, Chaumont [15] used the harmonicity of u_- to show that under $\mathbb{P}^x_{u_-}$ the killing time ζ is finite almost surely and that the canonical process $(\xi_t)_{t\geq 0}$ has a left limit at the killing time $\xi_{\zeta-} = 0$, in particular $T_{(0,\eta)} < \zeta$ for all $\eta > 0$. Moreover, the author connected the corresponding h-transformed process to the following conditioned Lévy process:

$$\mathbb{P}^x_{u_-}(\Lambda, t < T_{(0,\eta)}) = \lim_{\varepsilon\searrow 0} \mathbb{P}^x(\Lambda, t < T_{(-\infty,\eta)} \,|\, \xi_{T_{(-\infty,0]}-} \leq \varepsilon), \quad \Lambda \in \mathcal{F}_t, \tag{2.39}$$

for all $\eta > 0$, $x > 0$ and $t \geq 0$ where $\underline{\xi}_t = \inf_{s \leq t} \xi_s$. This means that $(\xi, \mathbb{P}^x_{u_-})$ is the process conditioned to hit all upper neighbourhoods of 0 before the negative half-line. The process $(\xi, (\mathbb{P}^x_{u_-})_{x>0})$ is also called the Lévy process conditioned to hit 0 continuously.

Stable processes conditioned to avoid 0

For a stable process with scaling index $\alpha \in (0, 2)$ it is possible to show that the function $e : \mathbb{R} \setminus \{0\} \to (0, \infty)$ defined via

$$e(x) = \begin{cases} \sin(\pi\alpha\hat{\rho})x^{\alpha-1} & \text{if } x > 0 \\ \sin(\pi\alpha\rho)|x|^{\alpha-1} & \text{if } x < 0 \end{cases},$$

is an excessive function for the process killed on hitting 0, Kyprianou et. al [37]. Applications of the underlying h-transformed process have been found for instance in the study of entrance and exit at infinity of stochastic differential equations driven by stable processes, see Döring and Kyprianou [25].

For $\alpha > 1$ it is even possible to show that e is invariant and the corresponding h-transformed process $(\xi, (\mathbb{P}^x_\circ)_{x\in\mathbb{R}\setminus\{0\}})$ coincides with the process conditioned to avoid $\{0\}$ in a similar sense like (2.38), namely it holds

$$\mathbb{P}^x_\circ(\Lambda) = \lim_{s\to\infty} \mathbb{P}^x(\Lambda \,|\, t + s < T_{\{0\}}), \quad \Lambda \in \mathcal{F}_t, t \geq 0, \tag{2.40}$$

see Pantí [45] or Yano [58].

For all values of $\alpha \in (0, 2)$ there is a very helpful transformation between $(\xi, (\mathbb{P}^x_\circ)_{x\in\mathbb{R}\setminus\{0\}})$ and the dual process of the stable process. For symmetric stable process this was discovered by Bogdan and Żak [11], for general stable processes by Kyprianou [35] and for general self-similar Markov processes by Alili et al. [1]. We will call the statement the Riesz-Bogdan-Żak transform.

Theorem 2.6.1 (Riesz-Bogdan-Żak transform). *Let $(\xi, (\mathbb{P}^x)_{x\in\mathbb{R}})$ be a stable process with $\alpha \in (0, 2)$ and two-sided jumps and*

$$\eta_t = \inf\Big\{s > 0 : \int_0^s |\xi_u|^{-2\alpha}\mathbb{1}_{\{u<T_0\}}\,\mathrm{d}u > t\Big\}, \qquad t < \int_0^\infty |\xi_u|^{-2\alpha}\mathbb{1}_{\{u<T_0\}}\,\mathrm{d}u.$$

Then, for all $x \neq 0$, it holds that

$$\Big(\big(\frac{1}{\xi_{\eta_t}}\big)_{t\geq 0}, \hat{\mathbb{P}}^x\Big) \overset{(d)}{=} (\xi, \mathbb{P}^{\frac{1}{x}}_\circ).$$

The Riesz-Bogdan-Żak transformation tells that $(\xi, \mathbb{P}^x_\circ)$ has the same distribution as the spatial inverse of the dual process including a certain time-change.

3 Invariant measures for overshoots of Lévy processes

To warm up we take a closer look at overshoots and undershoots of Lévy processes since they will play a crucial role in Chapter 4. The pairs of over- and undershoots are considered as Markov process where the level of the overshoot models the time parameter. Here, the point of interest are stationary distributions for this Markov process. There are necessary and sufficient conditions on the Lévy process for the existence of a stationary distribution which in this case is known explicitly. We weaken the sufficient conditions and show that there is an invariant measure which is a generalisation of a stationary distribution. Moreover, this invariant measure is given explicitly.

3.1 Main results

Let $(\xi, \mathbb{P}^x)$ be a Lévy process which does not drift to $-\infty$, in particular it holds $T_{[a,\infty)} < \infty$ almost surely. With the strong Markov property it follows that $(Y_a)_{a\geq 0}$ with

$$Y_a := \xi_{T_{[a,\infty)}} - a,$$

defines a Markov process under $\mathbb{P}^x$ with state space $[0,\infty)$ with respect to the filtration $(\mathcal{F}_{T_{[a,\infty)}})_{a\geq 0}$. Indeed, if we define

$$p_a(x, \mathrm{d}y) := \mathbb{P}^x(Y_a \in \mathrm{d}y), \quad a, x, y \geq 0,$$

we can show the Chapman-Kolmogorov equation. Let $a, b \geq 0$, then it holds

$$\begin{aligned}
\int\limits_{[0,\infty)} p_a(z, \mathrm{d}y)\, p_b(x, \mathrm{d}z) &= \int\limits_{[0,\infty)} \mathbb{P}^z(\xi_{T_{[a,\infty)}} - a \in \mathrm{d}y)\, \mathbb{P}^x(\xi_{T_{[b,\infty)}} - b \in \mathrm{d}z) \\
&= \int\limits_{[b,\infty)} \mathbb{P}^{z-b}(\xi_{T_{[a,\infty)}} - a \in \mathrm{d}y)\, \mathbb{P}^x(\xi_{T_{[b,\infty)}} \in \mathrm{d}z) \\
&= \int\limits_{[b,\infty)} \mathbb{P}^z(\xi_{T_{[a+b,\infty)}} - (a+b) \in \mathrm{d}y)\, \mathbb{P}^x(\xi_{T_{[b,\infty)}} \in \mathrm{d}z) \\
&= \mathbb{P}^x(\xi_{T_{[a+b,\infty)}} - (a+b) \in \mathrm{d}y) = p_{a+b}(x, \mathrm{d}y),
\end{aligned}$$

where we used the strong Markov property of ξ in the last equality.

Our interest lies in stationary distributions for this Markov process, i.e. probability measures ρ_+ on $[0,\infty)$ such that

$$\rho_+(\mathrm{d}y) = \mathbb{P}^{\rho_+}(Y_a \in \mathrm{d}y) := \int\limits_{[0,\infty)} \mathbb{P}^z(Y_a \in \mathrm{d}y)\, \rho_+(\mathrm{d}z), \tag{3.1}$$

for all $a \geq 0$. In the case $\mathbb{E}[H_1] < +\infty$ it is known (see e.g. [6]) that

$$\rho_+(\mathrm{d}y) := \underset{z\to-\infty}{\text{w-lim}}\, \mathbb{P}^z(\xi_{T_{[0,\infty)}} \in \mathrm{d}y)$$

exists. Since ξ is not a compound Poisson process the map $z \mapsto \mathbb{P}^z(\xi_{T_{[a,\infty)}} \in A)$ is continuous for all open sets $A \in \mathcal{B}([a,\infty))$. So it follows with the strong Markov property for all $0 < a \leq y$:

$$\begin{aligned}
\mathbb{P}^{\rho_+}(\xi_{T_{[a,\infty)}} \in A) &= \lim_{z\to-\infty} \int_{[0,\infty)} \mathbb{P}^u(\xi_{T_{[a,\infty)}} \in A)\mathbb{P}^z(\xi_{T_{[0,\infty)}} \in \mathrm{d}u) \\
&= \lim_{z\to-\infty} \mathbb{E}^z\left[\mathbb{P}^{\xi_{T_{[0,\infty)}}}(\xi_{T_{[a,\infty)}} \in A)\right] \\
&= \lim_{z\to-\infty} \mathbb{P}^z(\xi_{T_{[a,\infty)}} \in A) \\
&= \lim_{z\to-\infty} \mathbb{P}^{z-a}(\xi_{T_{[0,\infty)}} \in A - a) \\
&= \rho_+(A-a),
\end{aligned} \tag{3.2}$$

which shows (3.1), i.e. ρ_+ is a stationary distribution for $(Y_a)_{a\geq 0}$. In this case one says that the Lévy process has stationary overshoots.

A recent application in this area are positive self-similar Markov processes (pssMp) started from the origin. A pssMp is a non-negative self-similar Markov process in the sense of Section 2.5, i.e. a Markov process which is absorbed by 0 and has the following scaling property for some $\alpha > 0$:

$$((cX_{c^{-\alpha}t})_{t\geq 0}, \mathbb{P}^x) \overset{(d)}{=} ((X_t)_{t\geq 0}, \mathbb{P}^{cx}), \quad x, c > 0.$$

The Lamperti transform (see e.g. [34], Chapter 13) tells that for any pssMp started from $x \in (0,\infty)$ there is a Lévy process started from $\log x$ such that

$$X_t = \exp(\xi_{\varphi_t}) \tag{3.3}$$

where $\varphi_t = \inf\left\{s \geq 0 : \int_0^s \exp(\alpha\xi_u)\,\mathrm{d}u > t\right\}$. A natural question is if (3.3) can be used to start a pssMp from 0, or in other words if $(X, \mathbb{P}^x)$ converges weakly for $x \searrow 0$. This question is answered by Caballero and Chaumont [14] and their result goes back exactly to our field of interest: A pssMp can be started from 0 if and only if the underlying Lévy process has stationary overshoots. For generalisations of this result see for example [21] where even real self-similar Markov process are started from the origin.

We come back to our setting. Very often one is not just interested in the overshoots but in the pair formed by the overshoots and undershoots, that is

$$(Y_a^-, Y_a^+) := (a - \xi_{T_{[a,\infty)}-}, \xi_{T_{[a,\infty)}} - a), \quad a \geq 0,$$

and consider the questions of stationary distributions for this pair. To modulate this we define a slightly different path space:

$$\tilde{D} := \{\omega : \{0-\} \cup [0,\infty) \to \mathbb{R} : \omega \text{ càdlàg on } [0,\infty)\}.$$

Further, $\mathbb{P}^{x^-,x^+}, x^-, x^+ \in \mathbb{R}$ are defined as the probability measure on $\tilde{D}$ equipped with the σ-algebra $\tilde{\mathcal{F}}$ generated by the Skorohod-topology such that $\mathbb{P}^{x^-,x^+}(\xi_{0-} = -x^-, \xi_0 = x^+) = 1$ and

$$\mathbb{P}^{x^-,x^+}(\xi_{t-} \in \mathrm{d}y^-, \xi_t \in \mathrm{d}y^+) = \mathbb{P}^{x^+}(\xi_{t-} \in \mathrm{d}y^-, \xi_t \in \mathrm{d}y^+),$$

for $t > 0$ and all $x^-x^+ \in \mathbb{R}$, where $\mathbb{P}^{x^+}$ is the usual probability measure on $(D, \mathcal{F})$ such that the canonical process ξ is a Lévy process started from x^+. Before we consider stationary distributions we should verify that $(Y_a^-, Y_a^+)_{a\geq 0}$ is a Markov process with state space $[0,\infty)^2$ under $\mathbb{P}^{x^-,x^+}$ for all $(x^-, x^+) \in \mathbb{R}^2$. We define

$$p_a((x^-, x^+), (\mathrm{d}y^-, \mathrm{d}y^+)) := \mathbb{P}^{x^-,x^+}(Y_a^- \in \mathrm{d}y^-, Y_a^+ \in \mathrm{d}y^+), \quad a, x^+, x^-, y^+, y^- \geq 0,$$

and check the Chapman-Kolmogorov equality: We have to show

$$\begin{aligned} &\int\limits_{[0,\infty)^2} p_a((z^-, z^+), (\mathrm{d}y^-, \mathrm{d}y^+))\, p_b((x^-, x^+), (\mathrm{d}z^-, \mathrm{d}z^+)) \\ &= p_{a+b}((x^-, x^+), (\mathrm{d}y^-, \mathrm{d}y^+)), \end{aligned} \tag{3.4}$$

for all $a, b \geq 0, x^-, x^+y^-, y^+ \geq 0$. For $x^+ \geq b$ this is simple since we then have

$$p_b((x^-, x^+), (\mathrm{d}z^-, \mathrm{d}z^+)) = \delta_{(x^-,x^+)}(b - \mathrm{d}z^-, \mathrm{d}z^+ + b).$$

So we restrict to $x^+ < b$. Note that in this case we can write

$$\mathbb{P}^{x^-,x^+}((Y_b^-, Y_b^+) \in (\mathrm{d}z^-, \mathrm{d}z^+)) = \mathbb{P}^{x^+}((Y_b^-, Y_b^+) \in (\mathrm{d}z^-, \mathrm{d}z^+)),$$

since $T_{[b,\infty)} > 0$ almost surely under $\mathbb{P}^{x^-,x^+}$. Hence, we have

$$\begin{aligned} &\int\limits_{[0,\infty)^2} p_a((z^-, z^+), (\mathrm{d}y^-, \mathrm{d}y^+))\, p_b((x^-, x^+), (\mathrm{d}z^-, \mathrm{d}z^+)) \\ &= \int\limits_{[0,\infty)^2} \mathbb{P}^{z^-,z^+}((Y_a^-, Y_a^+) \in (\mathrm{d}y^-, \mathrm{d}y^+))\, \mathbb{P}^{x^+}((Y_b^-, Y_b^+) \in (\mathrm{d}z^-, \mathrm{d}z^+)) \\ &= \int\limits_{[0,\infty)\times[a,\infty)} \mathbb{P}^{z^-,z^+}((Y_a^-, Y_a^+) \in (\mathrm{d}y^-, \mathrm{d}y^+))\, \mathbb{P}^{x^+}((Y_b^-, Y_b^+) \in (\mathrm{d}z^-, \mathrm{d}z^+)) \\ &\quad + \int\limits_{[0,\infty)\times[0,a)} \mathbb{P}^{z^-,z^+}((Y_a^-, Y_a^+) \in (\mathrm{d}y^-, \mathrm{d}y^+))\, \mathbb{P}^{x^+}((Y_b^-, Y_b^+) \in (\mathrm{d}z^-, \mathrm{d}z^+)). \end{aligned}$$

We care about both integrals separately and start with the first one. Since here $z^+ \geq a$ it holds $T_{[a,\infty)} = 0$ under $\mathbb{P}^{z^-,z^+}$. So we get

$$\begin{aligned} &\int\limits_{[0,\infty)\times[a,\infty)} \mathbb{P}^{z^-,z^+}((Y_a^-, Y_a^+) \in (\mathrm{d}y^-, \mathrm{d}y^+))\, \mathbb{P}^{x^+}((Y_b^-, Y_b^+) \in (\mathrm{d}z^-, \mathrm{d}z^+)) \\ &= \int\limits_{[0,\infty)\times[a,\infty)} \mathbb{P}^{z^-,z^+}((a - \xi_{0-}, \xi_0 - a) \in (\mathrm{d}y^-, \mathrm{d}y^+))\, \mathbb{P}^{x^+}((Y_b^-, Y_b^+) \in (\mathrm{d}z^-, \mathrm{d}z^+)) \\ &= \int\limits_{[0,\infty)\times[a,\infty)} \mathbb{1}_{\{(-z^-,z^+)\in(a-\mathrm{d}y^-,\mathrm{d}y^+-a)\}}\, \mathbb{P}^{x^+}((Y_b^-, Y_b^+) \in (\mathrm{d}z^-, \mathrm{d}z^+)) \\ &= \mathbb{P}^{x^+}(Y_b^- \in \mathrm{d}y^- - a, Y_b^+ \in \mathrm{d}y^+ - a, Y_b^+ \in [a,\infty)) \\ &= \mathbb{P}^{x^+}(b - \xi_{T_{[b,\infty)}-} \in \mathrm{d}y^- - a, \xi_{T_{[b,\infty)}} - b \in \mathrm{d}y^+ - a, \xi_{T_{[b,\infty)}} \in [a+b,\infty)) \end{aligned}$$

$$= \mathbb{P}^{x^+}(b - \xi_{T_{[b,\infty)}-} \in \mathrm{d}y^- - a, \xi_{T_{[b,\infty)}} - b \in \mathrm{d}y^+ - a, T_{[b,\infty)} = T_{[a+b,\infty)})$$
$$= \mathbb{P}^{x^+}(a + b - \xi_{T_{[a+b,\infty)}-} \in \mathrm{d}y^-, \xi_{T_{[a+b,\infty)}} - (a+b) \in \mathrm{d}y^+, T_{[b,\infty)} = T_{[a+b,\infty)})$$
$$= \mathbb{P}^{x^+}(Y^-_{a+b} \in \mathrm{d}y^-, Y^+_{a+b} \in \mathrm{d}y^+, T_{[b,\infty)} = T_{[a+b,\infty)}).$$

For the second integral we use the strong Markov property. Note that $T_{[a,\infty)} > 0$ a.s. under $\mathbb{P}^{z^-,z^+}$ and we can ignore again the left-limit of the initial value, i.e.

$$\begin{aligned}
&\int\limits_{[0,\infty)\times[0,a)} \mathbb{P}^{z^-,z^+}((Y^-_a, Y^+_a) \in (\mathrm{d}y^-, \mathrm{d}y^+))\, \mathbb{P}^{x^+}((Y^-_b, Y^+_b) \in (\mathrm{d}z^-, \mathrm{d}z^+)) \\
&= \int\limits_{[0,\infty)\times[0,a)} \mathbb{P}^{z^+}((Y^-_a, Y^+_a) \in (\mathrm{d}y^-, \mathrm{d}y^+))\, \mathbb{P}^{x^+}((Y^-_b, Y^+_b) \in (\mathrm{d}z^-, \mathrm{d}z^+)) \\
&= \mathbb{P}^{x^+}(Y^-_{a+b} \in \mathrm{d}y^-, Y^+_{a+b} \in \mathrm{d}y^+, T_{[b,\infty)} < T_{[a+b,\infty)}).
\end{aligned}$$

In the last step we used the spatial homogeneity for Lévy processes and the strong Markov property. Adding both terms leads to (3.4).

In [6] the authors showed that if $\mathbb{E}\,[H_1] < +\infty$ the distribution

$$\mathbb{P}^z(-\xi_{T_{[0,\infty)}-} \in \mathrm{d}y^-, \xi_{T_{[0,\infty)}} \in \mathrm{d}y^+) \tag{3.5}$$

converges weakly for $z \to -\infty$ to a probability measure $\rho(\mathrm{d}y^-, \mathrm{d}y^+)$ on $[0,\infty)^2$ (Note that ρ_+ is just the second marginal of ρ). Similar to (3.2) one can use the strong Markov property of Lévy processes to show that ρ is a stationary distribution for $(Y^-_a, Y^+_a)_{a\geq 0}$.

Now we consider the general case, i.e. $\mathbb{E}\,[H_1]$ can be infinite. In this setting it is possible that (3.5) does not converge for $z \to -\infty$. Our aim is to find an invariant measure $\nu(\mathrm{d}y^-, \mathrm{d}y^+)$ on $[0,\infty)^2$ for $(Y^-_a, Y^+_a)_{a\geq 0}$. That is a generalisation of a stationary distribution in the sense that it holds

$$\mathbb{P}^\nu(Y^-_a \in \mathrm{d}y^-, Y^+_a \in \mathrm{d}y^+) = \nu(\mathrm{d}y^-, \mathrm{d}y^+), \quad y^-, y^+ \geq 0,$$

as well, but ν does not have to be a probability measure but just σ-finite. It will turn out that we can extract such an invariant measure by the results on stationary distribution in the case $\mathbb{E}\,[H_1] < +\infty$ from [6]. But it is not possible to apply the strong Markov property like in (3.2) since there is no interpretation as a limit of a sequence of probability measures. Hence, the proof does not work like in the case $\mathbb{E}\,[H_1] < +\infty$. The crucial instead is that the stationary distribution ρ was calculated explicitely in [6] and has the following form:

$$\rho(\mathrm{d}y^-, \mathrm{d}y^+) = \frac{1}{\mathbb{E}\,[H_1]}\big[\gamma_+\delta_0(\mathrm{d}y^-)\delta_0(\mathrm{d}y^+) + \mathbb{1}_{\{-y^- < y^+\}} U_-(y^-)\,\Pi(y^- + \mathrm{d}y^+)\,\mathrm{d}y^-\big],$$

for $y^-, y^+ \geq 0$, where γ_+ is the drift of $(H, \mathbb{P})$, U_- is the potential function of $(H, \hat{\mathbb{P}})$ and Π is the Lévy measure of ξ. We see that the only problem is that this measure is a zero measure (or is not defined) if $\mathbb{E}\,[H_1] = +\infty$. So we just ignore the constant and define

$$\nu(\mathrm{d}y^-, \mathrm{d}y^+) := \gamma_+\delta_0(\mathrm{d}y^-)\delta_0(\mathrm{d}y^+) + \mathbb{1}_{\{-y^- < y^+\}} U_-(y^-)\,\Pi(y^- + \mathrm{d}y^+)\,\mathrm{d}y^-, \tag{3.6}$$

for $y^-, y^+ \geq 0$ to present the main results.

Theorem 3.1.1. *Assume that ξ is not a compound Poisson process and does not drift to $-\infty$. Then ν is an invariant measure for $(Y_a^-, Y_a^+)_{a\in[0,\infty)}$.*

Because overshoots are also considered without the corresponding undershoots we also give an invariant measure for the second marginal of $(Y_a^-, Y_a^+)_{a\in[0,\infty)}$. Denote $\nu_+(\mathrm{d}y) := \nu([0,\infty), \mathrm{d}y)$ and with the help of Theorem 7.8 of [34], which tells

$$\bar{\mu}_+(y) = \int\limits_{[0,\infty)} \bar{\Pi}(y+z)\, U_-(\mathrm{d}z), \quad y \geq 0,$$

it follows $\nu_+(\mathrm{d}y) := \gamma_+\delta_0(y) + \mathbb{1}_{(0,\infty)}(y)\bar{\mu}_+(y)\,\mathrm{d}y$. As a consequence of Theorem 3.1.1 an invariant measure for the Markov process just formed by the overshoots of ξ is presented.

Corollary 3.1.2. *Assume that ξ is not a compound Poisson process and does not drift to $-\infty$. Then ν_+ is an invariant measure for $(Y_a^+)_{a\in[0,\infty)}$.*

3.2 Proofs

Before we start with the proof we need a helping Lemma. We remind the reader of the definition of the killed potentials,

$$U_{(-\infty,0]}(x, \mathrm{d}y) = \mathbb{E}^x\Big[\int\limits_0^\infty \mathbb{1}_{\{\xi_t\in \mathrm{d}y\}}\,\mathrm{d}t\Big], \quad x, y \geq 0,$$

and

$$U_{(-\infty,0]}(\mu, \mathrm{d}y) = \int\limits_{(0,\infty)} U_{(-\infty,0]}(x, \mathrm{d}y)\,\mu(\mathrm{d}x), \quad y \geq 0,$$

for a measure μ on $([0,\infty), \mathcal{B}([0,\infty)))$. Further, we use the same notations for the dual process $-\xi$ by adding a hat on the top.

Lemma 3.2.1. *Denote $\lambda_a(\mathrm{d}y) = \mathbb{1}_{\{y\leq a\}}\nu_+(a-\mathrm{d}y)$. Then it holds*

$$\hat{U}_{(-\infty,0]}(\lambda_a, \mathrm{d}y) = U_-(y)\,\mathrm{d}y - \mathbb{1}_{[a,\infty)}(y)U_-(y-a)\,\mathrm{d}y, \quad y \geq 0,$$

for all $c > 0$.

Proof. From [4] Theorem VI.20 (the constant there we normalized to 1) it is known that

$$\hat{U}_{(-\infty,0]}(x, \mathrm{d}y) = \int\limits_{z\in[(x-y)^+,x]} U_-(\mathrm{d}y + z - x)\,U_+(\mathrm{d}z), \quad x, y \geq 0.$$

Now fix $y \geq 0$. Plugging in the measure λ_a it follows:

$$\begin{aligned}
\hat{U}_{(-\infty,0]}(\lambda_a, \mathrm{d}y) &= \int\limits_{x\in(-\infty,a)} \hat{U}_{(-\infty,0]}(x, \mathrm{d}y) \, \big(\gamma_+ \delta_{\{a\}}(\mathrm{d}x) + \bar{\mu}_+(a-x)\,\mathrm{d}x\big) \\
&= \gamma_+ \hat{U}_{(-\infty,0]}(a, \mathrm{d}y) + \int\limits_{x\in[0,c)} \hat{U}_{(-\infty,0]}(x, \mathrm{d}y) \bar{\mu}_+(a-x)\,\mathrm{d}x \\
&= \gamma_+ \int\limits_{z\in[(a-y)^+,a]} U_-(\mathrm{d}y + z - a)\, U_+(\mathrm{d}z) \\
&\quad + \int\limits_{x\in[0,a)} \bar{\mu}_+(a-x)\, U_{(-\infty,0]}(y, \mathrm{d}x)\,\mathrm{d}y \\
&= \gamma_+ \int\limits_{z\in[(a-y)^+,a]} U_-(\mathrm{d}y + z - a)\, U_+(\mathrm{d}z) \\
&\quad + \int\limits_{z\in[(y-a)^+,y]} \Big(\int\limits_{x\in[0,a)} \bar{\mu}_+(a-x)\, U_+(\mathrm{d}x + z - y)\Big)\, U_-(\mathrm{d}z)\,\mathrm{d}y.
\end{aligned} \tag{3.7}$$

In the third equality we used duality of ξ killed on entering $(-\infty, 0]$ and $\hat{\xi}$ killed on entering $(-\infty, 0]$, see Theorem II.5 of [4]. We treat both summands of (3.7) seperately and start with the first one. If $\gamma_+ = 0$, the term disappears (we will come back to this case at the end of the proof). If $\gamma_+ > 0$ it is known from Lemma 1 of [6] that U_+ has a density u_+ (w.r.t. the Lebesgue measure). We plug in, make some manipulations on the integral and use the duality between the ladder height H and its negative $-H$, i.e. in particular $\mathrm{d}z\, U_-(\mathrm{d}y - z) = U_-(y - \mathrm{d}z)\,\mathrm{d}y$:

$$\begin{aligned}
\gamma_+ \int\limits_{z\in[(a-y)^+,a]} U_-(\mathrm{d}y + z - a)\, U_+(\mathrm{d}z) &= \gamma_+ \int\limits_{z\in[(a-y)^+,a]} u_+(z)\, U_-(\mathrm{d}y + z - a)\,\mathrm{d}z \\
&= \gamma_+ \int\limits_{z\in[(y-a)^+,y]} u_+(a + z - y)\, U_-(\mathrm{d}z)\,\mathrm{d}y.
\end{aligned}$$

Now we use Theorem III.5 of [4] which says that

$$\mathbb{P}(H_{T^H_{[x,\infty)}} = x) = \gamma_+ u_+(x), \quad x \geq 0,$$

where $T^H_{[x,\infty)}$ should denote the overshoot time of H. So the first term in the last line of (3.7) equals

$$\begin{aligned}
&\gamma_+ \int\limits_{z\in[(y-a)^+,y]} u_+(a + z - y)\, U_-(\mathrm{d}z)\,\mathrm{d}y \\
&= \int\limits_{z\in[(y-a)^+,y]} \mathbb{P}(H_{T^H_{[a+z-y,\infty)}} = a + z - y)\, U_-(\mathrm{d}z)\,\mathrm{d}y.
\end{aligned}$$

Now we treat the second term and make at first some manipulations:

$$\int\limits_{z\in[(y-a)^+,y]} \Big(\int\limits_{x\in[0,a)} \bar{\mu}_+(a-x)\, U_+(\mathrm{d}x + z - y)\Big)\, U_-(\mathrm{d}z)\,\mathrm{d}y$$

$$= \int_{z\in[(y-a)^+,y]} \Big(\int_{x\in[(y-z)^+,a-y+z)} \bar{\mu}_+(a-x-y+z)\,U_+(\mathrm{d}x)\Big)\,U_-(\mathrm{d}z)\,\mathrm{d}y$$
$$= \int_{z\in[(y-a)^+,y]} \Big(\int_{x\in[0,a-y+z)} \bar{\mu}_+(a-x-y+z)\,U_+(\mathrm{d}x)\Big)\,U_-(\mathrm{d}z)\,\mathrm{d}y.$$

From Proposition III.2(i) of [4] we get

$$\int_{x\in[0,a-y+z)} \bar{\mu}_+(a-y+z-x)\,U_+(\mathrm{d}x) = \mathbb{P}(H_{T^H_{[a-y+z,\infty)}-} \in [0,a-y+z)).$$

So the second term in the last line of (3.7) equals

$$\int_{z\in[(y-a)^+,y]} \mathbb{P}(H_{T^H_{[a-y+z,\infty)}-} \in [0,a-y+z))\,U_-(\mathrm{d}z)\,\mathrm{d}y.$$

Now we add both terms and conclude with (3.7),

$$\begin{aligned}\hat{U}_{(-\infty,0]}(\lambda_a,\mathrm{d}y) = &\int_{z\in[(y-a)^+,y]} \mathbb{P}(H_{T^H_{[a+z-y,\infty)}} = a+z-y)\,U_-(\mathrm{d}z)\,\mathrm{d}y\\ &+\int_{z\in[(y-a)^+,y]} \mathbb{P}(H_{T^H_{[a-y+z,\infty)}-} \in [0,a-y+z))\,U_-(\mathrm{d}z)\,\mathrm{d}y.\end{aligned}$$

But now Proposition III.2(ii) of [4] gives us

$$\mathbb{P}(H_{T^H_{[a-y+z,\infty)}-} < H_{T^H_{[a+z-y,\infty)}} = a+z-y) = 0.$$

So the sum of the two probabilities in the integral is just 1. If $\gamma_+ = 0$, the first term vanishes as explained before. But in this case it holds $\mathbb{P}(H_{T^H_{[a+z-y,\infty)}} = a+z-y) = 0$ (Theorem III.4 of [4]). Because the sum of the two probabilities is always 1 we have also in the case $\gamma_+ = 0$ that $\mathbb{P}(H_{T^H_{[a-y+z,\infty)}-} \in [0,a-y+z)) = 1$. So we can conclude in all cases

$$\hat{U}_{(-\infty,0]}(\lambda_a,\mathrm{d}y) = \int_{z\in[(y-a)^+,y]} U_-(\mathrm{d}z)\,\mathrm{d}y = U_-(y)\,\mathrm{d}y - \mathbb{1}_{[a,\infty)}(y)U_-(y-a)\,\mathrm{d}y.$$

$\square$

Proof of Theorem 3.1.1. The strategy is to use the form of ν given in (3.6) and the explicitly known undershoot-overshoot distribution by the quintuple law for overshoots and undershoots of Lévy processes from Doney and Kyprianou [23]. We plug this in and calculate

$$\mathbb{P}^\nu(a-\xi_{T_{[a,\infty)}-} \in \mathrm{d}y^-, \xi_{T_{[a,\infty)}} - a \in \mathrm{d}y^+), \quad y^-, y^+, a \geq 0,$$

using fluctuation identities for Lévy processes. We have to show that this expression equals $\nu(\mathrm{d}y^-,\mathrm{d}y^+)$ for all $a \geq 0$ and $y^-, y^+ \geq 0$. For $a = 0$ this is obvious because in this case it holds $T_{[a,\infty)} = 0$ a.s. under $\mathbb{P}^\nu$. So we fix from now on $a > 0$. First we note

$$\mathbb{P}^\nu(a-\xi_{T_{[a,\infty)}-} \in \mathrm{d}y^-, \xi_{T_{[a,\infty)}} - a \in \mathrm{d}y^+)$$

$$= \int_{[0,\infty)^2} \mathbb{P}^{z^-,z^+}(a - \xi_{T_{[a,\infty)}-} \in \mathrm{d}y^-, \xi_{T_{[a,\infty)}} - a \in \mathrm{d}y^+)\, \nu(\mathrm{d}z^-, \mathrm{d}z^+)$$

$$= \int_{[0,\infty)\times[a,\infty)} \mathbb{P}^{z^-,z^+}(a - \xi_{T_{[a,\infty)}-} \in \mathrm{d}y^-, \xi_{T_{[a,\infty)}} - a \in \mathrm{d}y^+)\, \nu(\mathrm{d}z^-, \mathrm{d}z^+)$$

$$+ \int_{[0,\infty)\times[0,a)} \mathbb{P}^{z^-,z^+}(a - \xi_{T_{[a,\infty)}-} \in \mathrm{d}y^-, \xi_{T_{[a,\infty)}} - a \in \mathrm{d}y^+)\, \nu(\mathrm{d}z^-, \mathrm{d}z^+).$$

We start with the first summand and use that for $z^+ \geq a$ it holds $T_{[a,\infty)} = 0$ a.s. under $\mathbb{P}^{z^-,z^+}$. So we have

$$\begin{aligned}
&\int_{[0,\infty)\times[a,\infty)} \mathbb{P}^{z^-,z^+}(a - \xi_{T_{[a,\infty)}-} \in \mathrm{d}y^-, \xi_{T_{[a,\infty)}} - a \in \mathrm{d}y^+)\, \nu(\mathrm{d}z^-, \mathrm{d}z^+) \\
&= \int_{[0,\infty)\times[a,\infty)} \mathbb{P}^{z^-,z^+}(a - \xi_{0-} \in \mathrm{d}y^-, \xi_0 - a \in \mathrm{d}y^+)\, \nu(\mathrm{d}z^-, \mathrm{d}z^+) \\
&= \int_{[0,\infty)\times[a,\infty)} \mathbb{1}_{\{a+z^- \in \mathrm{d}y^-\}} \mathbb{1}_{\{z^+ - a \in \mathrm{d}y^+\}}\, \nu(\mathrm{d}z^-, \mathrm{d}z^+) \\
&= \int_{[0,\infty)\times[a,\infty)} \mathbb{1}_{\{z^- \in \mathrm{d}y^- - a\}} \mathbb{1}_{\{z^+ \in \mathrm{d}y^+ + a\}}\, \nu(\mathrm{d}z^-, \mathrm{d}z^+) \\
&= \mathbb{1}_{\{y^- \geq a\}} \nu(\mathrm{d}y^- - a, \mathrm{d}y^+ + a).
\end{aligned} \tag{3.8}$$

Now we treat the second integral an note this time that for $z^+ < a$ it holds $T_{[a,\infty)} > 0$ a.s. under $\mathbb{P}^{z^-,z^+}$. Using this we get

$$\begin{aligned}
&\int_{[0,\infty)\times[0,a)} \mathbb{P}^{z^-,z^+}(a - \xi_{T_{[a,\infty)}-} \in \mathrm{d}y^-, \xi_{T_{[a,\infty)}} - a \in \mathrm{d}y^+)\, \nu(\mathrm{d}z^-, \mathrm{d}z^+) \\
&= \int_{[0,\infty)\times[0,a)} \mathbb{P}^{z^+}(a - \xi_{T_{[a,\infty)}-} \in \mathrm{d}y^-, \xi_{T_{[a,\infty)}} - a \in \mathrm{d}y^+)\, \nu(\mathrm{d}z^-, \mathrm{d}z^+) \\
&= \int_{[0,a)} \mathbb{P}^{z^+}(a - \xi_{T_{[a,\infty)}-} \in \mathrm{d}y^-, \xi_{T_{[a,\infty)}} - a \in \mathrm{d}y^+)\, \nu_+(\mathrm{d}z^+),
\end{aligned}$$

where $\nu_+(\mathrm{d}z^+) := \nu([0,\infty), \mathrm{d}z^+)$ denotes the second marginal of ν. To go on we have to separate the cases that ξ creeps over the level a and that ξ jumps over a:

$$\begin{aligned}
&\int_{[0,a)} \mathbb{P}^{z^+}(a - \xi_{T_{[a,\infty)}-} \in \mathrm{d}y^-, \xi_{T_{[a,\infty)}} - a \in \mathrm{d}y^+)\, \nu_+(\mathrm{d}z^+) \\
&= \int_{[0,a)} \mathbb{P}^{z^+}(a - \xi_{T_{[a,\infty)}-} \in \mathrm{d}y^-, \xi_{T_{[a,\infty)}} - a \in \mathrm{d}y^+, \xi_{T_{[a,\infty)}-} = \xi_{T_{[a,\infty)}} = a)\, \nu_+(\mathrm{d}z^+) \\
&\quad + \int_{[0,a)} \mathbb{P}^{z^+}(a - \xi_{T_{[a,\infty)}-} \in \mathrm{d}y^-, \xi_{T_{[a,\infty)}} - a \in \mathrm{d}y^+, \xi_{T_{[a,\infty)}-} < \xi_{T_{[a,\infty)}})\, \nu_+(\mathrm{d}z^+).
\end{aligned}$$

Again we consider both terms separately. For the first one we need Lemma 1 of [6] which says that this term only appears if $\gamma_+ > 0$ and in this case U_+ has a density w.r.t. the Lebesgue measure which fulfils

$$\mathbb{P}^{z^+}(\xi_{T_{[a,\infty)}-} = \xi_{T_{[a,\infty)}} = a) = \gamma_+ u_+(a - z^+).$$

So we see (also in the case $\gamma_+ = 0$):

$$\begin{aligned}
&\int\limits_{[0,a)} \mathbb{P}^{z^+}(a - \xi_{T_{[a,\infty)}-} \in \mathrm{d}y^-, \xi_{T_{[a,\infty)}} - a \in \mathrm{d}y^+, \xi_{T_{[a,\infty)}-} = \xi_{T_{[a,\infty)}} = a)\, \nu_+(\mathrm{d}z^+) \\
&= \gamma_+ \delta_0(\mathrm{d}y^-)\delta_0(\mathrm{d}y^+) \int\limits_{[0,a)} u_+(a - z^+)\, \nu_+(\mathrm{d}z^+) \\
&= \gamma_+ \delta_0(\mathrm{d}y^-)\delta_0(\mathrm{d}y^+)\Big(\gamma_+ u_+(a) + \int\limits_{(0,a)} u_+(a - z^+)\bar{\mu}_+(z^+)\, \mathrm{d}z^+\Big) \\
&= \gamma_+ \delta_0(\mathrm{d}y^-)\delta_0(\mathrm{d}y^+)\Big(\mathbb{P}(H_{T^H{}_{[a,\infty)}} = a) + \int\limits_{(0,a)} \bar{\mu}_+(z^+)\, U_+(a - \mathrm{d}z^+)\Big) \\
&= \gamma_+ \delta_0(\mathrm{d}y^-)\delta_0(\mathrm{d}y^+)\big(\mathbb{P}(H_{T^H{}_{[a,\infty)}} = a) + \mathbb{P}(H_{T^H{}_{[a,\infty)}-} \in (0,a))\big) \\
&= \gamma_+ \delta_0(\mathrm{d}y^-)\delta_0(\mathrm{d}y^+).
\end{aligned} \tag{3.9}$$

The second last equation is Proposition III.2 of [4] and

$$\mathbb{P}(H_{T^H{}_{[a,\infty)}} = a) + \mathbb{P}(H_{T^H{}_{[a,\infty)}-} \in (0,a)) = 1,$$

because $\mathbb{P}(H_{T^H{}_{[a,\infty)}-} < a = H_{T^H{}_{[a,\infty)}}) = 0$. For the second term we use the mentioned quintuple law of [23] (in the second equation) to see

$$\begin{aligned}
&\mathbb{P}^{z^+}(a - \xi_{T_{[a,\infty)}-} \in \mathrm{d}y^-, \xi_{T_{[a,\infty)}} - a \in \mathrm{d}y^+, \xi_{T_{[a,\infty)}-} < \xi_{T_{[a,\infty)}}) \\
&= \mathbb{P}(a - z^+ - \xi_{T_{[a-z^+,\infty)}-} \in \mathrm{d}y^-, \xi_{T_{[a-z^+,\infty)}} - (a - z^+) \in \mathrm{d}y^+, \xi_{T_{[a-z^+,\infty)}-} < \xi_{T_{[a-z^+,\infty)}}) \\
&= \int\limits_{u \in [0,(a-z^+)\vee y^-]} \mathbb{1}_{\{a-z^+-y^- < y^+ + (a-z^+)\}}\, \Pi(\mathrm{d}y^+ + y^-)\, U_-(\mathrm{d}y^- - u)\, U_+(a - z^+ - \mathrm{d}u) \\
&= \int\limits_{u \in [0,(a-z^+)\vee y^-]} \mathbb{1}_{\{-y^- < y^+\}}\, \Pi(\mathrm{d}y^+ + y^-)\, U_-(\mathrm{d}y^- - u)\, U_+(a - z^+ - \mathrm{d}u) \\
&= \int\limits_{u \in [(a-z^+-y^-)_+,(a-z^+)]} \mathbb{1}_{\{-y^- < y^+\}}\, \Pi(\mathrm{d}y^+ + y^-)\, U_-(\mathrm{d}y^- + u - (a - z^+))\, U_+(\mathrm{d}u) \\
&= \mathbb{1}_{\{-y^- < y^+\}}\, \Pi(\mathrm{d}y^+ + y^-)\, \hat{U}_{(-\infty,0]}(a - z^+, \mathrm{d}y^-).
\end{aligned}$$

Now we plug-in an get:

$$\begin{aligned}
&\int\limits_{[0,a)} \mathbb{P}^{z^+}(a - \xi_{T_{[a,\infty)}-} \in \mathrm{d}y^-, \xi_{T_{[a,\infty)}} - a \in \mathrm{d}y^+, \xi_{T_{[a,\infty)}-} < \xi_{T_{[a,\infty)}})\, \nu_+(\mathrm{d}z^+) \\
&= \mathbb{1}_{\{-y^- < y^+\}}\, \Pi(\mathrm{d}y^+ + y^-) \int\limits_{[0,a)} \hat{U}_{(-\infty,0]}(a - z^+, \mathrm{d}y^-)\, \nu_+(\mathrm{d}z^+)
\end{aligned}$$

$$= \mathbb{1}_{\{-y^- < y^+\}} \, \Pi(\mathrm{d}y^+ + y^-) \int\limits_{[0,a)} \hat{U}_{(-\infty,0]}(z^+, \mathrm{d}y^-) \, \nu_+(a - \mathrm{d}z^+),$$

where $\hat{U}_{(-\infty,0]}$ is the potential of the dual process killed on the negative half-line. We used Theorem VI.20 of [4]. Now we apply Lemma 3.2.1 which tells us:

$$\begin{aligned}
&\int\limits_{[0,a)} \hat{U}_{(-\infty,0]}(z^+, \mathrm{d}y^-) \, \nu_+(a - \mathrm{d}z^+) \\
&= U_-(y^-) \, \mathrm{d}y^- - \mathbb{1}_{[a,\infty)}(y^-) U_-(y^- - a) \, \mathrm{d}y^-.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\int\limits_{[0,a)} \mathbb{P}^{z^+}(a - \xi_{T_{[a,\infty)}-} \in \mathrm{d}y^-, \xi_{T_{[a,\infty)}} - a \in \mathrm{d}y^+, \xi_{T_{[a,\infty)}-} < \xi_{T_{[a,\infty)}}) \, \nu_+(\mathrm{d}z^+) \\
&= \mathbb{1}_{\{-y^- < y^+\}} \, \Pi(\mathrm{d}y^+ + y^-) \big(U_-(y^-) \, \mathrm{d}y^- - \mathbb{1}_{\{y^- \geq a\}} U_-(y^- - a) \, \mathrm{d}y^- \big) \qquad (3.10) \\
&= \mathbb{1}_{\{-y^- < y^+\}} \, \Pi(\mathrm{d}y^+ + y^-) U_-(y^-) \, \mathrm{d}y^- - \mathbb{1}_{\{-y^- < y^+\}} \mathbb{1}_{\{y^- \geq a\}} \nu(\mathrm{d}y^- - a, \mathrm{d}y^+ + a) \\
&= \mathbb{1}_{\{-y^- < y^+\}} \, \Pi(\mathrm{d}y^+ + y^-) U_-(y^-) \, \mathrm{d}y^- - \mathbb{1}_{\{y^- \geq a\}} \nu(\mathrm{d}y^- - a, \mathrm{d}y^+ + a).
\end{aligned}$$

We conclude the proof by adding (3.8), (3.9) and (3.10):

$$\begin{aligned}
&\mathbb{P}^\nu(a - \xi_{T_{[a,\infty)}-} \in \mathrm{d}y^-, \xi_{T_{[a,\infty)}} - a \in \mathrm{d}y^+) \\
&= \mathbb{1}_{\{y^- \geq a\}} \nu(\mathrm{d}y^- - a, \mathrm{d}y^+ + a) + \gamma_+ \delta_0(\mathrm{d}y^-) \delta_0(\mathrm{d}y^+) \\
&\quad + \mathbb{1}_{\{-y^- < y^+\}} \, \Pi(\mathrm{d}y^+ + y^-) U_-(y^-) \, \mathrm{d}y^- - \mathbb{1}_{\{y^- \geq a\}} \nu(\mathrm{d}y^- - a, \mathrm{d}y^+ + a) \\
&= \gamma_+ \delta_0(\mathrm{d}y^-) \delta_0(\mathrm{d}y^+) + \mathbb{1}_{\{-y^- < y^+\}} \, \Pi(\mathrm{d}y^+ + y^-) U_-(y^-) \, \mathrm{d}y^- \\
&= \nu(a - \mathrm{d}y^-, \mathrm{d}y^+ - a).
\end{aligned}$$

□

Proof of Corollary 3.1.2. We just integrate ν with respect to y^- and note that according to [56] it holds

$$\mu_+(\mathrm{d}x) = \int\limits_{[0,\infty)} \Pi(z + \mathrm{d}x) \, U_-(\mathrm{d}z), \quad x \geq 0.$$

□

4 Lévy processes with finite variance conditioned to avoid an interval

In this chapter the object of interest is a Lévy process ξ which has finite variance. Our purpose is to find invariant functions for the Markov semigroup

$$p_t^{[a,b]}(x, \mathrm{d}y) = \mathbb{P}^x(\xi_t \in \mathrm{d}y, t < T_{[a,b]}), \quad t \geq 0, x, y \notin [a,b],$$

formed by the Lévy process killed on entering an interval $[a,b]$ with $a < b$. It will turn out that we can show invariance for even two functions and hence for all linear combination of them. In particular we will show that the corresponding h-transformed process for one particular choice of coefficients equals the process conditioned to avoid an interval in the spirit of (1.2). Moreover, we use the invariant functions to analyse the long-time behaviour of the h-transformed processes.

To state the conditions on the Lévy process for this chapter precisely we introduce the following assumption:

(A) $\quad$ ξ has zero mean and finite variance, and is not a compound Poisson process.

For certain auxiliary results, we will need to distinguish two cases:

$$(B) \qquad \Pi(b-a, \infty) > 0, \text{ i.e., upward jumps avoiding } [a,b] \text{ are possible}$$

and

$$(\hat{B}) \qquad \Pi(-\infty, a-b) > 0, \text{ i.e., downward jumps avoiding } [a,b] \text{ are possible.}$$

We will refer to these assumptions in the results of this chapter. Moreover, we will use that (A) implies that ξ is recurrent and in particular oscillating (Sato [53], Theorem 36.7).

4.1 Main results

Before stating the main results, some more notation is needed to define our invariant functions. We first define inductively the sequence of successive stopping times at which the process jumps crossing a or b:

$$\begin{aligned}\tau_0 &:= 0,\\ \tau_{k+1} &:= \inf\{t > \tau_k : \xi_{t-} > b, \xi_t \leq b\} \wedge \inf\{t > \tau_k : \xi_{t-} < a, \xi_t \geq a\}.\end{aligned}$$

Second, let $K^\dagger := \inf\{k \geq 1 : \tau_k = T_{[a,b]}\}$ be the index indicating the time at which the process hits the given interval, let

$$\nu_k^x(\mathrm{d}y) = \mathbb{P}^x(\xi_{\tau_k} \in \mathrm{d}y, \tau_k < \infty, k \leq K^\dagger), \quad x, y \in \mathbb{R} \setminus [a,b]$$

be the distribution of the position of ξ after its k-th jump across the interval, for $k \geq 0$.
It is important to note that each ν_k^x can be expressed explicitly in terms of the Lévy measures and potential measures of the ladder height processes. Indeed, ν_1^x is nothing but an overshoot distribution, for which a formula is given by (2.26), using that the overshoot of ξ has the same distribution as the overshoot of the corresponding ladder height subordinator H. Applying the strong Markov property successively yields explicit expressions for all other ν_k^x.

Theorem 4.1.1. *If Assumptions (A) and (B) hold, then the function*

$$h_+(x) := \begin{cases} \sum\limits_{k=0}^{\infty} \int\limits_{(b,\infty)} U_-(y-b)\, \nu_{2k}^x(\mathrm{d}y) & \text{if } x > b \\ \sum\limits_{k=0}^{\infty} \int\limits_{(b,\infty)} U_-(y-b)\, \nu_{2k+1}^x(\mathrm{d}y) & \text{if } x < a \end{cases}$$

is a positive invariant function for ξ killed on entering $[a,b]$, i.e.

$$\mathbb{E}^x\left[\mathbb{1}_{\{t<T_{[a,b]}\}} h_+(\xi_t)\right] = h_+(x), \quad t \geq 0, x \in \mathbb{R} \setminus [a,b].$$

If Assumption (B) is not satisfied, then h_+ is still invariant, but may not be positive. To be precise, when (B) fails, h_+ is positive on (b,∞) but zero on $(-\infty,a)$.
Similarly, under (A) and $(\hat{B})$, the function

$$h_-(x) := \begin{cases} \sum\limits_{k=0}^{\infty} \int\limits_{(-\infty,a)} U_+(a-y)\, \nu_{2k+1}^x(\mathrm{d}y) & \text{if } x > b \\ \sum\limits_{k=0}^{\infty} \int\limits_{(-\infty,a)} U_+(a-y)\, \nu_{2k}^x(\mathrm{d}y) & \text{if } x < a \end{cases}$$

is positive invariant as well. As above, when $(\hat{B})$ fails, h_- remains invariant, but is positive only on $(-\infty,a)$ and zero on (b,∞).
An important corollary of this discussion is the existence of positive invariant functions under the Assumption (A) only:

Corollary 4.1.2. *If Assumption (A) holds, then all linear combinations of h_+ and h_- with strictly positive coefficients are positive invariant functions.*

The invariant functions h_+ and h_- typically do not have a simple closed form (but Section 4.2 below for an example where they do). This would appear to reduce their applicability; however, we can use our definition to prove results on conditioning. We will show that the conditioning in the sense of (1.2) works and, as a consequence of general h-transform theory, that the conditioned process is strong Markov. Additionally, it turns out that the invariant functions are explicit enough to explain the limiting behaviour of trajectories under the conditioned law.

Remark 4.1.3. *Vysotsky [57] considered the analogous problem for a centred random walk $S = (S_n)_{n\in\mathbb{N}}$ with finite variance. He derived an invariant function V which is the discrete analogue of some linear combination of h_+ and h_-. Proving invariance in the discrete-time situation is less involved for the following reason. It is enough to show that $V(S)$ is a discrete-time martingale for which it is enough to derive the martingale property for one time-step. Since, in discrete-time, $1 \leq T_{[a,b]}$ for $x \notin [a,b]$ the computation is direct. The continuous-time situation of Lévy processes is much more delicate as $t \leq T_{[a,b]}$ does not hold almost surely for any $t > 0$.*

With the invariant functions h_+, h_- and their positive linear combinations it is now possible to h-transform the killed process as in Definition 2.2.3. We denote the corresponding probability measures on the path space introduced in section 2.2 with the positive invariant functions h_+ (resp. h_-) by $\mathbb{P}_+$ (resp. $\mathbb{P}_-$). We will show how to condition the Lévy process in order to obtain h-transforms with h_+ and h_-, and then derive the correct linear combination of h_+ and h_- corresponding to conditioning the Lévy process to avoid the interval in the sense of (1.2).

The next proposition gives a probabilistic representation of $\mathbb{P}_+^x$ by conditioning to avoid $[a,b]$ and staying above b at late times. The analogous conditioning under (A) and $(\hat{B})$ below the interval results in the h-transform $\mathbb{P}_-^x$.

Proposition 4.1.4. *Assume* (A) *and* (B). *Then*

$$\mathbb{P}_+^x(\Lambda) = \lim_{q \searrow 0} \mathbb{P}^x(\Lambda, t < e_q \,|\, e_q < T_{[a,b]}, \xi_{e_q} > b), \quad x \notin [a,b],$$

for $\Lambda \in \mathcal{F}_t$, $t \geq 0$.

To understand the Lévy process to avoid the interval without additional condition on the late values a natural guess is an h-transform using a linear combination of h_+ and h_-. Possible asymmetry of the Lévy process implies that different weights must be chosen for h_+ and h_-. It emerges that the right invariant function is

$$h := h_+ + Ch_-, \quad \text{where} \quad C = \lim_{q \searrow 0} \frac{\kappa(q)}{\hat{\kappa}(q)}. \tag{4.1}$$

Note that, if ξ oscillates and has finite variance, then $C \in (0,\infty)$ exists; see, for instance, Patie and Savov [46], Remark 2.21. From Corollary 4.1.2, it follows that h is a positive invariant function if we assume only (A). The h-transform of ξ killed in $[a,b]$ with h from (4.1) will be denoted by $\mathbb{P}_\updownarrow^x$. Our main result can now be formulated. Conditioning to avoid an interval is always possible for Lévy processes with second moments and the conditioned law corresponds to the h-transform with h from (4.1).

Theorem 4.1.5. *Assume* (A). *Then,*

$$\mathbb{P}_\updownarrow^x(\Lambda) = \lim_{q \searrow 0} \mathbb{P}^x(\Lambda, t < e_q \,|\, e_q < T_{[a,b]}), \quad x \notin [a,b],$$

for $\Lambda \in \mathcal{F}_t$, $t \geq 0$.

Typically the first property analysed for a conditioned process is the long-time behaviour. It is often the case that the conditioning turns a recurrent process into a transient process. Nonetheless, the limit behaviour under $\mathbb{P}_\pm^x$, and in particular $\mathbb{P}_\updownarrow^x$, is *a priori* unclear. Processes might be oscillating, diverge to $+\infty$ or $-\infty$, or might even diverge to both infinities with positive probability. The next proposition covers the case $\mathbb{P}_+^x$:

Proposition 4.1.6. *Assume* (A) *and* (B). *Then* $\mathbb{P}_+^x(\lim_{t \to \infty} \xi_t = +\infty) = 1$ *for all* $x \notin [a,b]$.

Analogously, assuming (A) and $(\hat{B})$ one can show that ξ drifts to $-\infty$ almost surely under $\mathbb{P}_-^x$. It remains to consider the behaviour of $(\xi, \mathbb{P}_\updownarrow^x)$. Our final theorem shows that Lévy processes with second moments conditioned to avoid an interval drift to either $+\infty$ or $-\infty$, each with (explicit) positive probabilities:

Theorem 4.1.7. *Assume (A). Then, $\mathbb{P}^x_\updownarrow$ is transient in the sense that*

$$\mathbb{E}^x_\updownarrow\Big[\int_{[0,\infty)} \mathbb{1}_{\{\xi_t\in K\}}\,\mathrm{d}t\Big] < \infty, \quad x \notin [a,b],$$

for all bounded $K \subseteq \mathbb{R}\setminus[a,b]$. More precisely,

$$\mathbb{P}^x_\updownarrow\big(\lim_{t\to\infty}\xi_t = +\infty\big) = \frac{h_+(x)}{h(x)} \qquad \textit{and} \qquad \mathbb{P}^x_\updownarrow\big(\lim_{t\to\infty}\xi_t = -\infty\big) = \frac{Ch_-(x)}{h(x)}, \quad x \notin [a,b],$$

so that, in particular, $\mathbb{P}^x_\updownarrow$-almost surely trajectories do not oscillate.

In chapter 5 we will consider the analogous problem for the case of a stable Lévy process. Since stable processes have infinite second moments, Theorem 4.1.7 does not apply, and it remains unclear if trajectories oscillate or diverge to $+\infty$ and $-\infty$ with positive probabilities. This is not merely a technical issue with our proof: for a stable process, the functions h_+ and h_-, as defined above, are actually infinite at every point of $\mathbb{R}\setminus[a,b]$; this can be shown directly using explicit formulas for the potential functions and overshoot distributions (see, e.g., Rogozin [52]).

4.2 An explicit example

When ξ is a Lévy process with no drift and two-sided exponential jumps, it is possible to compute the invariant functions h_+, h_- and h explicitly. Let

$$\xi_t = \sigma B_t + \sum_{i=1}^{N_t} Y_i, \quad t \geq 0, \tag{4.2}$$

where $\sigma \geq 0$, $(B_t)_{t\geq 0}$ is a standard Brownian motion, and $\sum_{i=1}^{N_t} Y_i$ is a compound Poisson process with rate $\lambda > 0$ and absolutely continuous jump distribution with density

$$f_Y(y) = \frac{1}{2}\eta\mathrm{e}^{-\eta y}\mathbb{1}_{\{y>0\}} + \frac{1}{2}\eta\mathrm{e}^{-\eta(-y)}\mathbb{1}_{\{y<0\}}.$$

For definiteness, let $\sigma = \sqrt{2}$ and $\lambda = 1$. The Laplace exponent ψ of ξ, given by $\mathbb{E}[\mathrm{e}^{-\theta\xi_t}] = \mathrm{e}^{-t\psi(\theta)}$, can be expressed, for $\theta \in (-\eta,\eta)$, by

$$\psi(\theta) = -\theta^2 - \frac{\theta^2}{(\eta+\theta)(\eta-\theta)} = \frac{\theta(\beta+\theta)}{\eta+\theta}\cdot\frac{(-\theta)(\beta-\theta)}{\eta-\theta}. \tag{4.3}$$

where $\beta = \sqrt{\eta^2+1} > \eta$. Note that ξ oscillates and has finite variance, so (A) holds, (B) and $(\hat{B})$ both hold as well. Let

$$\psi_+(\theta) = \psi_-(\theta) = \frac{\theta(\beta+\theta)}{\eta+\theta} = \theta + (\beta-\eta)\int_0^\infty (1-\mathrm{e}^{-\theta x})\eta\mathrm{e}^{-\eta x}\,\mathrm{d}x, \qquad \theta > -\eta,$$

which is the Laplace exponent of a subordinator with unit drift, jump rate $\beta-\eta$ and exponential jumps of parameter η. Since

$$\psi(\theta) = \psi_+(\theta)\psi_-(-\theta),$$

the uniqueness of the Wiener–Hopf factorisation (2.23) implies that ψ_+ and ψ_- are indeed the Laplace exponents of the ascending and descending ladder height subordinators, respectively.

Since

$$\int_{[0,\infty)} e^{-\theta x}\, U_-(\mathrm{d}x) = \int_{[0,\infty)} \mathrm{e}^{-\theta x}\, U_+(\mathrm{d}x) = \frac{1}{\psi_+(\theta)} = \frac{\eta+\theta}{\theta(\beta+\theta)} \tag{4.4}$$

by [34], equation (5.23) we can identify the potential measures

$$U_-(\mathrm{d}x) = U_+(\mathrm{d}x) = \Big(\frac{\eta}{\beta} + \frac{\beta-\eta}{\beta}\mathrm{e}^{-\beta x}\Big)\,\mathrm{d}x.$$

and the potential functions

$$U_-(x) = U_+(x) = \frac{\eta}{\beta}x + \frac{\beta-\eta}{\beta^2}(1-\mathrm{e}^{-\beta x}), \quad x \geq 0. \tag{4.5}$$

To find h_+ in closed form we first need to find the measures ν_k^x explicitly. This can in principle be done using the expressions we have just found for $U_\pm$ and the Lévy measures of the ladder height subordinators, but in fact the overshoot distributions have already been found in Kou and Wang [32], Corollary 3.1, where

$$\mathbb{P}^x(\xi_{T_{[a,\infty)}} \in \mathrm{d}y) = \frac{\eta(\beta-\eta)}{\beta}(1-\mathrm{e}^{-\beta(a-x)})\mathrm{e}^{-\eta(y-a)}, \quad x<a<y,$$

and

$$\mathbb{P}^x(\xi_{T_{(-\infty,b]}} \in \mathrm{d}y) = \frac{\eta(\beta-\eta)}{\beta}(1-\mathrm{e}^{-\beta(x-b)})\mathrm{e}^{-\eta(b-y)}, \quad x>b>y,$$

are proven. We now claim that

$$\nu_{2k+1}^x(\mathrm{d}y) = c^{2k}\nu_1^x(\mathrm{d}y), \quad x<a, y>b, \tag{4.6}$$

and

$$\nu_{2k+2}^x(\mathrm{d}y) = c^{2k}\nu_2^x(\mathrm{d}y), \quad x,y>b, \tag{4.7}$$

hold for all $k \geq 0$, where $c = \mathrm{e}^{-\eta(b-a)}(\beta-\eta)/(\beta+\eta)$. For proving this, note that

$$\begin{aligned}\int_{(b,\infty)} (1-\mathrm{e}^{-\beta(z-b)})\mathrm{e}^{-\eta(z-a)}\,\mathrm{d}z &= \int_{(-\infty,a)} (1-\mathrm{e}^{-\beta(a-z)})\mathrm{e}^{-\eta(b-z)}\,\mathrm{d}z\\ &= \mathrm{e}^{-\eta(b-a)}\frac{\beta}{\eta(\beta+\eta)}.\end{aligned}$$

For $k=0$ the claims are clearly correct. Next, note that for $x>b$:

$$\begin{aligned}\nu_2^x(\mathrm{d}y) &= \int_{(-\infty,a)} \mathbb{P}^z(\xi_{T_{[a,\infty)}} \in \mathrm{d}y)\,\mathbb{P}^x(\xi_{T_{(-\infty,b]}} \in \mathrm{d}z)\\ &= \Big(\frac{\eta(\beta-\eta)}{\beta}\Big)^2(1-\mathrm{e}^{-\beta(x-b)})\mathrm{e}^{-\eta(y-a)}\int_{(-\infty,a)} (1-\mathrm{e}^{-\beta(a-z)})\mathrm{e}^{-\eta(b-z)}\mathrm{d}z\end{aligned}$$

$$
\begin{aligned}
&= \Big(\frac{\eta(\beta-\eta)}{\beta}\Big)^2 (1-\mathrm{e}^{-\beta(x-b)})\mathrm{e}^{-\eta(y-a)}\mathrm{e}^{-\eta(b-a)}\frac{\beta}{\eta(\beta+\eta)}\\
&= c\frac{\eta(\beta-\eta)}{\beta}(1-\mathrm{e}^{-\beta(x-b)})\mathrm{e}^{-\eta(y-a)}.
\end{aligned}
$$

Now, let us assume the claims are correct for $k-1$, $k \geq 1$. Then, for $x<a, b<y$,

$$
\begin{aligned}
\nu^x_{2k+1}(\mathrm{d}y) &= \int\limits_{(b,\infty)} \nu^z_{2k}(\mathrm{d}y)\,\mathbb{P}^x(\xi_{T_{[a,\infty)}} \in \mathrm{d}z)\\
&= c^{2k-2} \int\limits_{(b,\infty)} \nu^z_{2}(\mathrm{d}y)\,\mathbb{P}^x(\xi_{T_{[a,\infty)}} \in \mathrm{d}z)\\
&= c^{2k-2} c\Big(\frac{\eta(\beta-\eta)}{\beta}\Big)^2 (1-\mathrm{e}^{-\beta(a-x)})\mathrm{e}^{-\eta(y-a)} \int\limits_{(b,\infty)} (1-\mathrm{e}^{-\beta(z-b)})\mathrm{e}^{-\eta(z-a)}\,\mathrm{d}z\\
&= c^{2k-1}\Big(\frac{\eta(\beta-\eta)}{\beta}\Big)^2 \mathrm{e}^{-\eta(b-a)}\frac{\beta}{\eta(\beta+\eta)}(1-\mathrm{e}^{-\beta(a-x)})\mathrm{e}^{-\eta(y-a)}\\
&= c^{2k-1}\Big(\frac{\beta-\eta}{\beta+\eta}\Big)\mathrm{e}^{-\eta(b-a)}\mathbb{P}^x(\xi_{T_{[a,\infty)}} \in \mathrm{d}y)\\
&= c^{2k}\nu^x_1(\mathrm{d}y),
\end{aligned}
$$

which is (4.6). Similarly we get, for $x, y > b$,

$$
\begin{aligned}
\nu^x_{2k+2}(\mathrm{d}y) &= \int\limits_{(-\infty,a)} \nu^z_{2k+1}(\mathrm{d}y)\,\mathbb{P}^x(\xi_{T_{(-\infty,b]}} \in \mathrm{d}z)\\
&= c^{2k} \int\limits_{(-\infty,a)} \nu^z_{1}(\mathrm{d}y)\,\mathbb{P}^x(\xi_{T_{(-\infty,b]}} \in \mathrm{d}z)\\
&= c^{2k} \int\limits_{(-\infty,a)} \nu^z_{1}(\mathrm{d}y)\,\nu^x_1(\mathrm{d}z)\\
&= c^{2k}\nu^x_2(\mathrm{d}y)
\end{aligned}
$$

which is (4.7).

Having formulas for U_- and all ν_k we can proceed to compute h_+. Combining (4.5), (4.6) and (4.7) standard integration shows, for $k \geq 1$,

$$
\begin{aligned}
\int\limits_{(b,\infty)} U_-(y-b)\,\nu^x_{2k+1}(\mathrm{d}y) &= c^{2k} \int\limits_{(b,\infty)} U_-(y-b)\,\nu^x_1(\mathrm{d}y)\\
&= c^{2k}\frac{2c}{\beta}(1-\mathrm{e}^{\beta(a-x)})\\
&= \frac{2c^{2k+1}}{\beta}(1-\mathrm{e}^{\beta(a-x)})
\end{aligned}
$$

for $x<a$ and

$$
\int\limits_{(b,\infty)} U_-(y-b)\,\nu^x_{2k+2}(\mathrm{d}y) = c^{2k} \int\limits_{(b,\infty)} U_-(y-b)\,\nu^x_2(\mathrm{d}y)
$$

$$
\begin{aligned}
&= c^{2k}\frac{2c^2}{\beta}(1-\mathrm{e}^{-\beta(x-b)})\\
&= \frac{2c^{2k+2}}{\beta}(1-\mathrm{e}^{-\beta(x-b)})
\end{aligned}
$$

for $x > b$. Hence, substituting in the definition of h_+ gives

$$
h_+(x) = \Big(\sum_{k=0}^{\infty} c^{2k+1}\Big)\frac{2}{\beta}(1-\mathrm{e}^{-\beta(a-x)}) = \frac{2c}{\beta(1-c^2)}(1-\mathrm{e}^{-\beta(a-x)})
$$

for $x < a$ and

$$
\begin{aligned}
h_+(x) &= \frac{\eta}{\beta}(x-b) + \frac{\beta-\eta}{\beta^2}(1-\mathrm{e}^{-\beta(x-b)}) + \Big(\sum_{k=0}^{\infty} c^{2k+2}\Big)\frac{2}{\beta}(1-\mathrm{e}^{-\beta(x-b)})\\
&= \frac{\eta}{\beta}(x-b) + \frac{\beta-\eta}{\beta^2}(1-\mathrm{e}^{-\beta(x-b)}) + \frac{2c^2}{\beta(1-c^2)}(1-\mathrm{e}^{-\beta(x-b)})\\
&= \frac{\eta}{\beta}(x-b) + \Big(\frac{\beta-\eta}{\beta^2} + \frac{2c^2}{\beta(1-c^2)}\Big)(1-\mathrm{e}^{-\beta(x-b)})
\end{aligned}
$$

for $x > b$. Analogously we obtain

$$
h_-(x) = \begin{cases} \frac{2c}{\beta(1-c^2)}(1-\mathrm{e}^{-\beta(x-b)}) & \text{if } x > b\\ \frac{\eta}{\beta}(a-x) + \Big(\frac{\beta-\eta}{\beta^2} + \frac{2c^2}{\beta(1-c^2)}\Big)(1-\mathrm{e}^{-\beta(a-x)}) & \text{if } x < a \end{cases},
$$

and, finally,

$$
h(x) = \begin{cases} \frac{\eta}{\beta}(x-b) + \Big(\frac{\beta-\eta}{\beta^2} + \frac{2(c+c^2)}{\beta(1-c^2)}\Big)(1-\mathrm{e}^{-\beta(x-b)}) & \text{if } x > b\\ \frac{\eta}{\beta}(a-x) + \Big(\frac{\beta-\eta}{\beta^2} + \frac{2(c+c^2)}{\beta(1-c^2)}\Big)(1-\mathrm{e}^{-\beta(a-x)}) & \text{if } x < a \end{cases},
$$

using that by symmetry $\kappa = \hat{\kappa}$ and consequently $C = \lim_{q\searrow 0}\kappa(q)/\hat{\kappa}(q) = 1$.

4.3 Proofs

Before going into the proofs let us discuss the form of the measures ν_k defined before. We assume in the theorems that ξ oscillates, hence, all appearing first hitting times are almost surely finite. Keeping in mind that on the event $\{K^\dagger > k\}$ the time τ_k is the time of the k^{th} jump across the interval. By the strong Markov property and $\nu_0^x(\mathrm{d}y) = \delta_x(\mathrm{d}y)$, we find the relations

$$
\begin{aligned}
\nu_{2k+1}^x(\mathrm{d}y) &= \int_{(b,\infty)} \mathbb{P}^z(\xi_{T_{(-\infty,b]}} \in \mathrm{d}y)\,\nu_{2k}^x(\mathrm{d}z) = \int_{(b,\infty)} \nu_1^z(\mathrm{d}y)\,\nu_{2k}^x(\mathrm{d}z),\\
\nu_{2k}^x(\mathrm{d}y) &= \int_{(-\infty,a)} \mathbb{P}^z(\xi_{T_{[a,\infty)}} \in \mathrm{d}y)\,\nu_{2k-1}^x(\mathrm{d}z) = \int_{(-\infty,a)} \nu_1^z(\mathrm{d}y)\,\nu_{2k-1}^x(\mathrm{d}z),
\end{aligned}
$$

for $x > b$, and

$$
\nu_{2k+1}^x(\mathrm{d}y) = \int_{(-\infty,a)} \mathbb{P}^z(\xi_{T_{[a,\infty)}} \in \mathrm{d}y)\,\nu_{2k}^x(\mathrm{d}z) = \int_{(-\infty,a)} \nu_1^z(\mathrm{d}y)\,\nu_{2k}^x(\mathrm{d}z),
$$

$$\nu_{2k}^x(\mathrm{d}y) = \int\limits_{(b,\infty)} \mathbb{P}^z(\xi_{T_{(-\infty,b]}} \in \mathrm{d}y)\, \nu_{2k-1}^x(\mathrm{d}z) = \int\limits_{(b,\infty)} \nu_1^z(\mathrm{d}y)\, \nu_{2k-1}^x(\mathrm{d}z),$$

for $x < a$. More generally, the strong Markov property also implies the relation

$$\int\limits_{(b,\infty)} \nu_l^z(\mathrm{d}y)\, \nu_{2k}^x(\mathrm{d}z) = \nu_{2k+l}^x(\mathrm{d}y) \qquad \text{and} \qquad \int\limits_{(-\infty,a)} \nu_l^z(\mathrm{d}y)\, \nu_{2k+1}^x(\mathrm{d}z) = \nu_{2k+l+1}^x(\mathrm{d}y) \tag{4.8}$$

for $x > b$ and $k, l \in \mathbb{N}$ and the analogous identities hold for $x < a$. It is important to note that (see e.g. Bertoin [4], Proposition III.2) analytic formulas exist for the overshoot distributions:

$$\mathbb{P}^x(\xi_{T_{[a,\infty)}} \in \mathrm{d}y) = \int\limits_{[x,a]} \mu_+(\mathrm{d}y - u)\, U_+(\mathrm{d}u - x), \quad x < a < y, \tag{4.9}$$

and, analogously,

$$\mathbb{P}^x(\xi_{T_{(-\infty,b]}} \in \mathrm{d}y) = \int\limits_{[b,x]} \mu_-(u - \mathrm{d}y)\, U_-(x - \mathrm{d}u), \quad x > b > y. \tag{4.10}$$

Hence, analytic expressions for the ν_k exist in the oscillating case even though these become more involved for big k due to the recursive definition. As an example, for $x > b$, we have

$$\begin{aligned} \nu_2^x(\mathrm{d}y) &= \int\limits_{(-\infty,a)} \mathbb{P}^z(\xi_{T_{[a,\infty)}} \in \mathrm{d}y)\, \mathbb{P}^x(\xi_{T_{(-\infty,b]}} \in \mathrm{d}z) \\ &= \int\limits_{(-\infty,a)} \Big[\int\limits_{[b,x]} \Big(\int\limits_{[x,a]} \mu_+(\mathrm{d}y - u)\, U_+(\mathrm{d}u - x) \Big) \mu_-(w - \mathrm{d}z) \Big] U_-(x - \mathrm{d}w). \end{aligned}$$

4.3.1 Finiteness of the invariant function

Since h_+ and h_- are defined by infinite series finiteness has to be proved. Along the way we deduce upper bounds that are needed in the sections below.

Note that Assumption (A) implies that $\mathbb{E}[H_1]$ and $\hat{\mathbb{E}}[H_1]$ are finite (see e.g. Doney [22], page 31, Corollary 4) and this will be crucial for the technical steps which are necessary to prove the following.

Proposition 4.3.1. *Assume (A), then there are constants $c_1, c_2, c_3 \geq 0$ such that*

$$h_+(x) \leq c_1 U_-(x-b)\mathbb{1}_{\{x>b\}} + c_2 U_+(a-x)\mathbb{1}_{\{x<a\}} + c_3, \quad x \notin [a,b],$$

in particular $h_+(x)$ is finite for all $x \in \mathbb{R} \setminus [a,b]$.

Before we start with the proof, we need a lemma which is intuitively clear, but needs a certain argumentation:

Lemma 4.3.2. *Let ξ be a Lévy process which is not the negative of a subordinator. Then, for all $y, z > 0$,*

$$\mathbb{P}(T_{(-\infty,-y]} > T_{[z,\infty)}) > 0.$$

Proof. Assume $\mathbb{P}(T_{(-\infty,-y]} \leq T_{[z,\infty)}) = 1$. Then it follows of course that $\mathbb{P}^x(T_{(-\infty,x-y]} \leq T_{[z,\infty)}) = 1$ for all $x < 0$. With the Markov property we get, for $s > 0$,

$$\begin{aligned}\mathbb{P}(T_{[z,\infty)} < s) &= \mathbb{E}\Big[\mathbb{P}^{\xi_{T_{(-\infty,-y]}}}(T_{[z,\infty)} < s)\Big] \\ &\leq \mathbb{P}^{-y}(T_{[z,\infty)} < s) \\ &= \mathbb{E}^{-y}\Big[\mathbb{P}^{\xi_{T_{(-\infty,-2y]}}}(T_{[z,\infty)} < s)\Big] \\ &\leq \mathbb{P}^{-2y}(T_{[z,\infty)} < s).\end{aligned}$$

Inductively we get $\mathbb{P}(T_{[z,\infty)} < s) \leq \mathbb{P}^{-ny}(T_{[z,\infty)} < s)$ for all $n \in \mathbb{N}$ and hence

$$\mathbb{P}(T_{[z,\infty)} < s) \leq \lim_{n\to\infty} \mathbb{P}^{-ny}(T_{[z,\infty)} < s) = 0.$$

With this we see

$$\mathbb{P}(T_{[z,\infty)} < +\infty) = \lim_{s\to\infty} \mathbb{P}(T_{[z,\infty)} < s) \leq \lim_{s\to\infty} \lim_{n\to\infty} \mathbb{P}^{-ny}(T_{[z,\infty)} < s) = 0,$$

but this cannot happen unless ξ is the negative of a subordinator. This concludes the proof. □

To prove Proposition 4.3.1 we will combine two statements. The discrete analogous statements were also used (with different arguments) by Vysotsky [57] to show finiteness of the invariant function in the discrete case.

Lemma 4.3.3. *Suppose that $\mathbb{E}[H_1] < \infty$, then*

$$\varphi_+ := \sup_{x<a} \mathbb{P}^x(\xi_{T_{[a,\infty)}} > b, T_{[a,\infty)} < \infty) < 1.$$

Proof. If ξ is the negative of a subordinator, it holds $\varphi_+ = 0$. So assume that ξ is not the negative of a subordinator, in particular we can apply Lemma 4.3.2.

We separate three regions of the range of x. First we consider very small x, i.e. we consider the limit of x tending to $-\infty$, then we consider the values of x which are close to a and last we treat the remaining values.

We begin with x close to $-\infty$. If ξ drifts to $-\infty$, then $\mathbb{P}^x(T_{[a,\infty]} < \infty) \to 0$ as $x \searrow -\infty$, and in particular $\mathbb{P}^x(\xi_{T_{[a,\infty)}} > b, T_{[a,\infty)} < \infty) \to 0$ also. Therefore there exist a $K < a$ and a $\varphi_1 < 1$ such that $\mathbb{P}^x(\xi_{T_{[a,\infty)}} > b, T_{[a,\infty)} < \infty) < \varphi_1$ when $x \leq K$.

If ξ oscillates or drifts to ∞, the bound for x close to $-\infty$ is more involved. Because $\mathbb{E}[H_1] < \infty$, ξ has stationary overshoots in the sense that the weak limit of $\mathbb{P}^x(\xi_{T_{[a,\infty)}} \in \mathrm{d}y)$ for $x \searrow -\infty$ exists. It can be expressed as

$$\operatorname*{w-lim}_{x\searrow-\infty} \mathbb{P}^x(\xi_{T_{[a,\infty)}} \in \mathrm{d}y) = \frac{1}{\mathbb{E}[H_1]}(\gamma_+\delta_a(\mathrm{d}y) + \bar{\mu}_+(y-a)\mathrm{d}y), \tag{4.11}$$

where γ_+ is the drift of $(H, \mathbb{P})$ and μ_+ its Lévy measure with the right-tail $\bar{\mu}_+$, see (2.29). Since weak convergence is equivalent to the pointwise convergence of the distribution function at continuity points, due to the explicit formula in (4.11) it holds that, for $b > a$,

$$\lim_{x\to-\infty} \mathbb{P}^x(\xi_{T_{[a,\infty)}} > b) = \frac{1}{\mathbb{E}[H_1]} \int_{(b,\infty)} \bar{\mu}_+(y-a)\mathrm{d}y$$

$$
\begin{aligned}
&= \frac{1}{\mathbb{E}[H_1]} \int\limits_{(b-a,\infty)} \bar{\mu}_+(y)\mathrm{d}y \\
&< \frac{1}{\mathbb{E}[H_1]} \int\limits_{(0,\infty)} \bar{\mu}_+(y)\mathrm{d}y \\
&\le 1
\end{aligned}
$$

Hence, also in this case there exist a $K < a$ and a $\varphi_1 < 1$ such that

$$\mathbb{P}^x(\xi_{T_{[a,\infty)}} > b) \le \varphi_1$$

for all $x \le K$. Now we have to treat the case $x \in (K, a)$. Therefore we separate two cases.
Case 1: The process ξ is regular upwards. First, we consider the limit for $x \to a$. Since ξ is regular upwards it holds

$$\lim_{x\to a} \mathbb{P}^x(\xi_{T_{[a,\infty)}} > b, T_{[a,\infty)} < \infty) < 1$$

and hence, there is some $\delta > 0$ such that

$$\varphi_2 := \sup_{x\in(a-\delta,a)} \mathbb{P}^x(\xi_{T_{[a,\infty)}} > b, T_{[a,\infty)} < \infty) < 1.$$

It remains to consider $x \in (K, a-\delta]$. First note that

$$
\begin{aligned}
&\mathbb{P}^x(\xi_{T_{[a,\infty)}} > b, T_{[a,\infty)} < \infty) \\
&= \mathbb{P}^x(\xi_{T_{[a,\infty)}} > b, T_{(-\infty,K]} < T_{[a,\infty)} < \infty) + \mathbb{P}^x(\xi_{T_{[a,\infty)}} > b, T_{(-\infty,K]} > T_{[a,\infty)}).
\end{aligned}
$$

For the first term we use the Markov property to get

$$
\begin{aligned}
\mathbb{P}^x(\xi_{T_{[a,\infty)}} > b, T_{(-\infty,K]} < T_{[a,\infty)} < \infty) &= \mathbb{E}^x\left[\mathbb{1}_{\{T_{(-\infty,K]}<T_{[a,\infty)}<\infty\}}\mathbb{P}^{\xi_{T_{(-\infty,K]}}}(\xi_{T_{[a,\infty)}} > b)\right] \\
&\le \varphi_1 \mathbb{P}^x(T_{(-\infty,K]} < T_{[a,\infty)} < \infty) \\
&\le \varphi_1 \mathbb{P}^x(T_{(-\infty,K]} < T_{[a,\infty)}).
\end{aligned}
$$

Together we have for all $x \in (K, a-\delta]$:

$$
\begin{aligned}
\mathbb{P}^x(\xi_{T_{[a,\infty)}} > b, T_{[a,\infty)} < \infty) &\le \sup_{x\in(K,a-\delta]} \Big(\mathbb{P}^x(\xi_{T_{[a,\infty)}} > b, T_{(-\infty,K]} < T_{[a,\infty)} < \infty) \\
&\qquad + \mathbb{P}^x(\xi_{T_{[a,\infty)}} > b, T_{(-\infty,K]} > T_{[a,\infty)})\Big) \\
&\le \sup_{x\in(K,a-\delta]} \big(\varphi_1 \mathbb{P}^x(T_{(-\infty,K]} < T_{[a,\infty)}) + \mathbb{P}^x(T_{(-\infty,K]} > T_{[a,\infty)})\big) \\
&=: \varphi_3.
\end{aligned}
$$

With Lemma 4.3.2 we get

$$\sup_{x\in(K,a-\delta)} \mathbb{P}^x(T_{(-\infty,K]} > T_{[a,\infty)}) = \mathbb{P}^{a-\delta}(T_{(-\infty,K]} > T_{[a,\infty)}) < 1$$

or, equivalently,

$$\inf_{x\in(K,a-\delta)} \mathbb{P}^x(T_{(-\infty,K]} < T_{[a,\infty)}) > 0.$$

Because of this it follows that

$$\varphi_3 < \sup_{x\in(K,a-\delta)} \left(\mathbb{P}^x(T_{(-\infty,K]} < T_{[a,\infty)}) + \mathbb{P}^x(T_{(-\infty,K]} > T_{[a,\infty)})\right) = 1.$$

Case 2: The process ξ is not regular upwards. In this case it holds

$$\sup_{x\in(K,a)} \mathbb{P}^x(T_{[a,\infty)} < T_{(-\infty,K]}) < 1.$$

or equivalently

$$\inf_{x\in(K,a)} \mathbb{P}^x(T_{(-\infty,K]} < T_{[a,\infty)}) > 0. \tag{4.12}$$

We split up again

$$\begin{aligned}
&\mathbb{P}^x(\xi_{T_{[a,\infty)}} > b, T_{[a,\infty)} < \infty)\\
&= \mathbb{P}^x(\xi_{T_{[a,\infty)}} > b, T_{(-\infty,K]} < T_{[a,\infty)} < \infty) + \mathbb{P}^x(\xi_{T_{[a,\infty)}} > b, T_{(-\infty,K]} > T_{[a,\infty)}).
\end{aligned}$$

For the first term we use the Markov property to get

$$\begin{aligned}
\mathbb{P}^x(\xi_{T_{[a,\infty)}} > b, T_{(-\infty,K]} < T_{[a,\infty)} < \infty) &= \mathbb{E}^x\left[\mathbb{1}_{\{T_{(-\infty,K]}<T_{[a,\infty)}<\infty\}}\mathbb{P}^{\xi_{T_{(-\infty,K]}}}(\xi_{T_{[a,\infty)}} > b)\right]\\
&\leq \varphi_1\mathbb{P}^x(T_{(-\infty,K]} < T_{[a,\infty)} < \infty)\\
&\leq \varphi_1\mathbb{P}^x(T_{(-\infty,K]} < T_{[a,\infty)}).
\end{aligned}$$

Together we have for all $x \in (K,a)$:

$$\begin{aligned}
\mathbb{P}^x(\xi_{T_{[a,\infty)}} > b, T_{[a,\infty)} < \infty) &\leq \sup_{x\in(K,a)} \Big(\mathbb{P}^x(\xi_{T_{[a,\infty)}} > b, T_{(-\infty,K]} < T_{[a,\infty)} < \infty)\\
&\qquad\qquad + \mathbb{P}^x(\xi_{T_{[a,\infty)}} > b, T_{(-\infty,K]} > T_{[a,\infty)})\Big)\\
&\leq \sup_{x\in(K,a)} \left(\varphi_1\mathbb{P}^x(T_{(-\infty,K]} < T_{[a,\infty)}) + \mathbb{P}^x(T_{(-\infty,K]} > T_{[a,\infty)})\right)\\
&=: \varphi_3.
\end{aligned}$$

From (4.12) follows that

$$\varphi_3 < \sup_{x\in(K,a)} \left(\mathbb{P}^x(T_{(-\infty,K]} < T_{[a,\infty)}) + \mathbb{P}^x(T_{(-\infty,K]} > T_{[a,\infty)})\right) = 1.$$

For the general case (both, regular upwards and not) set $\varphi_+ := \max(\varphi_1, \varphi_2, \varphi_3) < 1$. $\square$

Analogously to the lemma before it holds

$$\varphi_- := \sup_{x>b} \mathbb{P}^x(\xi_{T_{(-\infty,b]}} < a, T_{(-\infty,b]} < \infty) < 1,$$

provided that $\hat{\mathbb{E}}[H_1] < \infty$. The second Lemma which we need to prove Proposition 4.3.1 is the following:

Lemma 4.3.4. *Assume ξ oscillates and $\hat{\mathbb{E}}[H_1] < \infty$. For all $\lambda \in (0,1)$ there exists a constant $C_+(\lambda) > 0$ such that*

$$\mathbb{E}^x\left[U_+(a - \xi_{T_{(-\infty,b]}})\mathbb{1}_{\{\xi_{T_{(-\infty,b]}}<a\}}\right] \leq \lambda U_-(x-b) + C_+(\lambda)$$

for all $x > b$.

Proof. We start to show that

$$\int_{(K,\infty)} U_+(y)\,\mu_-(\mathrm{d}y) < +\infty$$

for all $K > 0$. For that we estimate $U_+(y)$ for $y > K$ with Proposition III.1 of Bertoin [4] which says that there are constants $c_1, c_2 \geq 0$ such that

$$U_+(x) \leq c_1 \left(\Phi\left(\frac{1}{x}\right)\right)^{-1} \quad \text{and} \quad \Phi(x) \geq c_2 x \left(I\left(\frac{1}{x}\right) + \gamma_+\right)$$

for all $x > 0$, where $\Phi(\lambda) = \mathbb{E}\big[\int_{[0,\infty)} e^{-\lambda H_t}\,\mathrm{d}t\big]$ and $I(x) = \int_{(0,x]} \bar{\mu}_+(y)\,\mathrm{d}y$. We combine these two statements as follows:

$$U_+(x) \leq c_1\Big(\Phi\big(\frac{1}{x}\big)\Big)^{-1} \leq c_1\Big(c_2\frac{1}{x}(I(x)+\gamma_+)\Big)^{-1} = \frac{c_1}{c_2}\frac{x}{I(x)+\gamma_+} \leq \frac{c_1}{c_2}\frac{x}{I(K)} = c_K x$$

for all $x > K$, where $c_K = \frac{c_1}{c_2 I(K)}$. Hence, by assumption,

$$\int_{(K,\infty)} U_+(y)\,\mu_-(\mathrm{d}y) \leq c_K \int_{(K,\infty)} y\,\mu_-(\mathrm{d}y) \leq c_K \hat{\mathbb{E}}\left[H_1\right] < +\infty$$

for all $K > 0$. The second inequality can be seen from $\hat{\mathbb{E}}\left[H_1\right] = \int_{(0,\infty)} y\,\mu_-(\mathrm{d}y) + \gamma_-$ because H is a subordinator. Now, for fixed $\lambda \in (0,1)$, choose $K = K(\lambda) > 0$ such that

$$\int_{(K,\infty)} U_+(y)\,\mu_-(\mathrm{d}y) < \lambda. \tag{4.13}$$

To prove the claim let us first split as

$$\begin{aligned}
&\mathbb{E}^x\Big[U_+(a-\xi_{T_{(-\infty,b]}})\mathbb{1}_{\{\xi_{T_{(-\infty,b]}}<a\}}\Big] \\
&= \mathbb{E}^x\Big[U_+(a-\xi_{T_{(-\infty,b]}})\mathbb{1}_{\{\xi_{T_{(-\infty,b]}}\in[a-K,a)\}}\Big] + \mathbb{E}^x\Big[U_+(a-\xi_{T_{(-\infty,b]}})\mathbb{1}_{\{\xi_{T_{(-\infty,b]}}\in(-\infty,a-K)\}}\Big]
\end{aligned}$$

and estimate the first summand, using monotonicity of U_+, as

$$\mathbb{E}^x\Big[U_+(a-\xi_{T_{(-\infty,b]}})\mathbb{1}_{\{\xi_{T_{(-\infty,b]}}\in[a-K,a)\}}\Big] \leq U_+(K).$$

Applying the overshoot formula (4.10) the second summand can be treated in the following way:

$$\begin{aligned}
&\mathbb{E}^x\Big[U_+(a-\xi_{T_{(-\infty,b]}})\mathbb{1}_{\{\xi_{T_{(-\infty,b]}}\in(-\infty,a-K)\}}\Big] \\
&= \int_{(-\infty,a-K)} U_+(a-y)\,\mathbb{P}^x(\xi_{T_{(-\infty,b]}}\in\mathrm{d}y) \\
&= \int_{[b,x]} \Big(\int_{(-\infty,a-K)} U_+(a-y)\,\mu_-(w-\mathrm{d}y)\Big)\,U_-(x-\mathrm{d}w)
\end{aligned}$$

$$
\begin{aligned}
&= \int\limits_{[b,x]} \Big(\int\limits_{(K+w-a,\infty)} U_+(y-w+a)\,\mu_-(\mathrm{d}y)\Big)\, U_-(x-\mathrm{d}w) \\
&\le \int\limits_{[b,x]} \Big(\int\limits_{(K,\infty)} U_+(y)\,\mu_-(\mathrm{d}y)\Big)\, U_-(x-\mathrm{d}w) \\
&\le \lambda U_-(x-b).
\end{aligned}
$$

Defining $C_+(\lambda) := U_+(K)$ we proved

$$
\mathbb{E}^x\big[U_+(a-\xi_{T_{(-\infty,b]}})\mathbb{1}_{\{\xi_{T_{(-\infty,b]}}<a\}}\big] \le \lambda U_-(x-b) + C_+(\lambda)
$$

for all $x > b$. □

Analogously to the lemma above one can show in the case that ξ oscillates and $\mathbb{E}\,[H_1] < \infty$ that for all $\lambda \in (0,1)$ there exists a constant $C_-(\lambda) > 0$ such that

$$
\mathbb{E}^x\big[U_-(\xi_{T_{[a,\infty)}} - b)\mathbb{1}_{\{\xi_{T_{[a,\infty)}}>b\}}\big] \le \lambda U_+(a-x) + C_-(\lambda), \quad x < a.
$$

Now we are ready to combine Lemmas 4.3.3 and 4.3.4 to show finiteness of $h_+(x)$. The idea how to combine them was also used by Vysotsky [57].

Proof of Proposition 4.3.1. Let $\lambda \in (0,1)$ be arbitrary. In the first step we use the finiteness of $\mathbb{E}\,[H_1]$ and $\hat{\mathbb{E}}\,[H_1]$ combined with Lemmas 4.3.3 and 4.3.4 to find an upper bound for

$$
\int\limits_{(b,\infty)} U_-(y-b)\,\nu^x_{2k}(\mathrm{d}y), \quad x > b.
$$

Set $\varphi = \max(\varphi_+, \varphi_-)$ and note by Lemma 4.3.3 that for $x > b$ and $k \ge 1$:

$$
\begin{aligned}
\nu^x_{2k-1}(-\infty,a) &= \int\limits_{(b,\infty)} \mathbb{P}^y(\xi_{T_{(-\infty,b]}} < a)\,\nu^x_{2k-2}(\mathrm{d}y) \\
&\le \varphi \nu^x_{2k-2}(b,\infty) \\
&= \varphi\Big(\mathbb{1}_{\{k=1\}} + \mathbb{1}_{\{k\ge2\}} \int\limits_{(-\infty,a)} \mathbb{P}^y(\xi_{T_{[a,\infty)}} > b)\,\nu^x_{2k-3}(\mathrm{d}y)\Big) \\
&\le \varphi\Big(\mathbb{1}_{\{k=1\}} + \varphi \mathbb{1}_{\{k\ge2\}}\nu^x_{2k-3}(-\infty,a)\Big).
\end{aligned}
$$

Inductively we get

$$
\nu^x_{2k-1}(-\infty,a) \le \varphi^{2k-1}
$$

for $x > b$ and $k \ge 1$. Analogously for $k \ge 1$ we can show

$$
\nu^x_{2k}(b,\infty) \le \varphi^{2k}
$$

for $x > b$ and

$$
\nu^x_{2k-1}(b,\infty) \le \varphi^{2k-1} \quad \text{and} \quad \nu^x_{2k}(-\infty,a) \le \varphi^{2k-1}
$$

for $x < a$. Now set $C(\lambda) = \max(C_-(\lambda), C_+(\lambda))$ and use Lemma 4.3.4 for $k \ge 1$ to find

$$
\int\limits_{(b,\infty)} U_-(y-b)\,\nu^x_{2k}(\mathrm{d}y) = \int\limits_{(-\infty,a)} \Big(\int\limits_{(b,\infty)} U_-(y-b)\,\mathbb{P}^v(\xi_{T_{[a,\infty)}} \in \mathrm{d}y)\Big)\,\nu^x_{2k-1}(\mathrm{d}v)
$$

$$\leq \int_{(-\infty,a)} \lambda U_+(a-v)\nu^x_{2k-1}(\mathrm{d}v) + C(\lambda)\,\nu^x_{2k-1}(-\infty,a)$$
$$\leq \lambda \int_{(-\infty,a)} U_+(a-v)\,\nu^x_{2k-1}(\mathrm{d}v) + C(\lambda)\varphi^{2k-1}.$$

We estimate the first term in the same way by

$$\lambda^2 \int_{(b,\infty)} U_-(b-y)\nu^x_{2k-2}(\mathrm{d}y) + C(\lambda)\lambda\varphi^{2k-2}$$

and hence,

$$\int_{(b,\infty)} U_-(y-b)\,\nu^x_{2k}(\mathrm{d}y) \leq \lambda^2 \int_{(b,\infty)} U_-(b-y)\,\nu^x_{2k-2}(\mathrm{d}y) + C(\lambda)(\varphi^{2k-1} + \lambda\varphi^{2k-2}).$$

Going on with this procedure until ν^x_0 we see

$$\int_{(b,\infty)} U_-(y-b)\,\nu^x_{2k}(\mathrm{d}y) \leq U_-(x-b)\lambda^{2k} + C(\lambda)\sum_{i=0}^{2k-1}\varphi^i\lambda^{2k-1-i}$$
$$= U_-(x-b)\lambda^{2k} + C(\lambda)\lambda^{2k-1}\sum_{i=0}^{2k-1}\Big(\frac{\varphi}{\lambda}\Big)^i.$$

Now note

$$\lambda^{2k-1}\sum_{i=0}^{2k-1}\Big(\frac{\varphi}{\lambda}\Big)^i = \lambda^{2k-1}\frac{\big(\frac{\varphi}{\lambda}\big)^{2k}-1}{\frac{\varphi}{\lambda}-1} = \frac{\varphi^{2k}-\lambda^{2k}}{\varphi-\lambda}$$

and hence

$$\int_{(b,\infty)} U_-(y-b)\,\nu^x_{2k}(\mathrm{d}y) \leq U_-(x-b)\lambda^{2k} + \frac{C(\lambda)}{\varphi-\lambda}(\varphi^{2k}-\lambda^{2k})$$

for $k \geq 1$ (for $k = 0$ we get obviously $U_-(x-b)$ as upper bound). In the same way we get for $x < a$:

$$\int_{(b,\infty)} U_-(y-b)\,\nu^x_{2k+1}(\mathrm{d}y) \leq U_+(a-x)\lambda^{2k+1} + \frac{C(\lambda)}{\varphi-\lambda}(\varphi^{2k+1}-\lambda^{2k+1})$$

for $k \geq 0$ (here we get an upper bound dependent on U_+ because the number of steps is odd). All together we get

$$h_+(x)$$
$$\leq \mathbb{1}_{(b,\infty)}(x)U_-(x-b)\sum_{k=0}^{\infty}\lambda^{2k} + \mathbb{1}_{(-\infty,a)}(x)U_+(a-x)\sum_{k=0}^{\infty}\lambda^{2k+1} + \frac{C(\lambda)}{\varphi-\lambda}\sum_{k=0}^{\infty}(\varphi^k-\lambda^k)$$
$$= \frac{1}{1-\lambda^2}U_-(x-b)\mathbb{1}_{(b,\infty)}(x) + \frac{\lambda}{1-\lambda^2}U_+(a-x)\mathbb{1}_{(-\infty,a)}(x) + \frac{C(\lambda)}{\varphi-\lambda}\Big(\frac{1}{1-\varphi} - \frac{1}{1-\lambda}\Big)$$

which finishes the proof of Proposition 4.3.1. □

4.3.2 Invariance of h_+ and h_-

In this section we give the proof of Theorem 4.1.1. Define, for $q \geq 0$ and $x \notin [a,b]$, the auxiliary functions

$$h_+^q(x) := \begin{cases} \sum\limits_{k=0}^{\infty} \int\limits_{(b,\infty)} U_-^q(y-b)\, \nu_{2k}^x(\mathrm{d}y) & \text{if } x > b \\ \sum\limits_{k=0}^{\infty} \int\limits_{(b,\infty)} U_-^q(y-b)\, \nu_{2k+1}^x(\mathrm{d}y) & \text{if } x < a \end{cases}$$

$$= \begin{cases} \sum\limits_{k=0}^{\infty} \mathbb{E}^x\Big[U_-^q(\xi_{\tau_{2k}} - b)\mathbb{1}_{\{K^\dagger \geq 2k, \tau_{2k} < \infty\}}\Big] & \text{if } x > b \\ \sum\limits_{k=0}^{\infty} \mathbb{E}^x\Big[U_-^q(\xi_{\tau_{2k+1}} - b)\mathbb{1}_{\{K^\dagger \geq 2k+1, \tau_{2k+1} < \infty\}}\Big] & \text{if } x < a \end{cases},$$

where $U_-^q(\mathrm{d}x) := \hat{\mathbb{E}}\big[\int_{[0,\infty)} e^{-qt}\mathbb{1}_{\{H_t \in \mathrm{d}x, L_t^{-1} < \infty\}}\,\mathrm{d}t\big]$ is the q-potential of the dual ladder height process. It follows immediately that $h_+^q(x) \leq h_+(x)$ for all $x \notin [a,b]$ and by monotone convergence that h_+^q converges pointwise to h_+ for $q \searrow 0$.

Proposition 4.3.5. *Assume (A) and let e_q be independent exponentially distributed random variables with parameter $q > 0$. Then, for $x \notin [a,b]$,*

$$\frac{1}{\hat{\kappa}(q)}\mathbb{P}^x\big(e_q < T_{[a,b]}, \xi_{e_q} > b\big) \leq h_+^q(x), \quad q > 0, \tag{4.14}$$

and

$$\lim_{q \searrow 0} \frac{1}{\hat{\kappa}(q)}\mathbb{P}^x\big(e_q < T_{[a,b]}, \xi_{e_q} > b\big) = h_+(x). \tag{4.15}$$

To prove this crucial proposition we need a small lemma which is basically just the strong Markov property:

Lemma 4.3.6. *Let be $s \geq 0$ and $k \geq 0$. Then it holds*

$$\int\limits_{(b,\infty)} \mathbb{P}^y(s < T_{(-\infty,b]})\, \nu_{2k}^x(\mathrm{d}y) = \mathbb{P}^x\left(s < \tau_{2k+1} - \tau_{2k}, K^\dagger \geq 2k+1\right)$$

and

$$\int\limits_{(-\infty,a)} \mathbb{P}^y(s < T_{[a,\infty)})\, \nu_{2k+1}^x(\mathrm{d}y) = \mathbb{P}^x\left(s < \tau_{2k+2} - \tau_{2k+1}, K^\dagger \geq 2k+2\right)$$

for $x > b$ and

$$\int\limits_{(-\infty,a)} \mathbb{P}^y(s < T_{[a,\infty)})\, \nu_{2k}^x(\mathrm{d}y) = \mathbb{P}^x\left(s < \tau_{2k+1} - \tau_{2k}, K^\dagger \geq 2k+1\right)$$

and

$$\int\limits_{(b,\infty)} \mathbb{P}^y(s < T_{(-\infty,b]})\, \nu_{2k+1}^x(\mathrm{d}y) = \mathbb{P}^x\left(s < \tau_{2k+2} - \tau_{2k+1}, K^\dagger \geq 2k+2\right)$$

for $x < a$.

Proof. We focus on the case $x > b$ and prove the first equality. We use the strong Markov property (2.9) in the version including the shift-operator. Here, we set $T = \tau_{2k}$ and $Y = \mathbb{1}_{\{s<T_{(-\infty,b]}\}}$. It is clear that Y is non-negative and that Y is $\mathcal{F}_\infty$-measurable can be seen as follows:

$$\{s < T_{(-\infty,b]}\} = \{T_{(-\infty,b]} \leq s\}^{\mathrm{C}} \in \mathcal{F}_s \subseteq \mathcal{F}_\infty.$$

With (2.9) we obtain for our choice of Y:

$$\mathbb{P}^{\xi_{\tau_{2k}}}\big(s < T_{(-\infty,b]}\big) = \mathbb{E}^x\big[\mathbb{1}_{\{s<T_{(-\infty,b]}\}} \circ \theta_{\tau_{2k}} \,|\, \mathcal{F}_{\tau_{2k}}\big].$$

Using this we get

$$\begin{aligned}
&\int\limits_{(b,\infty)} \mathbb{P}^y(s < T_{(-\infty,b]})\, \nu_{2k}^x(\mathrm{d}y) \\
&= \mathbb{E}^x\Big[\mathbb{1}_{\{\xi_{\tau_{2k}}>b, K^\dagger \geq 2k\}} \mathbb{P}^{\xi_{\tau_{2k}}}(s < T_{(-\infty,b]})\Big] \\
&= \mathbb{E}^x\Big[\mathbb{1}_{\{\xi_{\tau_{2k}}>b, K^\dagger \geq 2k\}} \mathbb{E}^x\big[\mathbb{1}_{\{s<T_{(-\infty,b]}\}} \circ \theta_{\tau_{2k}} \,|\, \mathcal{F}_{\tau_{2k}}\big]\Big] \\
&= \mathbb{E}^x\Big[\mathbb{1}_{\{\tau_{2k}<T_{[a,b]}\}} \mathbb{P}^x(s + \tau_{2k} < \tau_{2k+1} \,|\, \mathcal{F}_{\tau_{2k}})\Big] \\
&= \mathbb{E}^x\big[\mathbb{P}^x(\tau_{2k} < T_{[a,b]}, s < \tau_{2k+1} - \tau_{2k} \,|\, \mathcal{F}_{\tau_{2k}})\big] \\
&= \mathbb{P}^x(\tau_{2k} < T_{[a,b]}, s < \tau_{2k+1} - \tau_{2k}) \\
&= \mathbb{P}^x(K^\dagger \geq 2k+1, s < \tau_{2k+1} - \tau_{2k}).
\end{aligned}$$

We used that $\{\xi_{\tau_{2k}} > b\} \in \mathcal{F}_{\tau_{2k}}$ and $\{\tau_{2k} < T_{[a,b]}\} \in \mathcal{F}_{\tau_{2k}} \cap \mathcal{F}_{T_{[a,b]}} \subseteq \mathcal{F}_{\tau_{2k}}$ which can be seen by Theorem 1.3.6 of [18]. The remaining claims follow analogously. □

Now we continue the proof of Proposition 4.3.5 for which we use the identity

$$\hat{\kappa}(q) U_-^q(x) = \mathbb{P}^x(e_q < T_{(-\infty,0]}), \quad x > 0, q > 0, \tag{4.16}$$

proved by Kyprianou [34], Section 13.2.1 for a general Lévy process.

Proof of Proposition 4.3.5. We only consider the case $x > b$ and start to prove the bounds

$$1 \leq \frac{\hat{\kappa}(q) h_+^q(x)}{\mathbb{P}^x(e_q < T_{[a,b]}, \xi_{e_q} > b)} \leq \frac{1}{\mathbb{P}^x(e_q \geq T_{[a,b]})}. \tag{4.17}$$

To derive the lower bound we define $\tilde{\tau}_k = \min(\tau_k, T_{[a,b]})$. It follows, in particular, that $\tilde{\tau}_k = \tau_k$ on $K^\dagger \geq k$ and $\tilde{\tau}_{k+1} - \tilde{\tau}_k = 0$ on $K^\dagger \leq k$. For the next chain of equalities we use (4.16), Lemma 4.3.6 and the lack of memory property of e_q:

$$\begin{aligned}
\hat{\kappa}(q) \int\limits_{(b,\infty)} U_-^q(y-b)\, \nu_{2k}^x(\mathrm{d}y) &= \int\limits_{(b,\infty)} \mathbb{P}^y(e_q < T_{(-\infty,b]})\, \nu_{2k}^x(\mathrm{d}y) \\
&= \mathbb{P}^x(\tau_{2k+1} - \tau_{2k} > e_q, K^\dagger \geq 2k+1) \\
&= \mathbb{P}^x(\tilde{\tau}_{2k+1} - \tilde{\tau}_{2k} > e_q) \\
&= \mathbb{P}^x(\tilde{\tau}_{2k+1} > e_q | e_q \geq \tilde{\tau}_{2k}) \\
&= \frac{\mathbb{P}^x(e_q \in [\tilde{\tau}_{2k}, \tilde{\tau}_{2k+1}))}{\mathbb{P}^x(e_q \geq \tilde{\tau}_{2k})}.
\end{aligned}$$

Furthermore, it holds that

$$\mathbb{P}^x(e_q \geq \tilde{\tau}_{2k}) \geq \mathbb{P}^x(e_q \geq T_{[a,b]})$$

because $\tilde{\tau}_{2k} \leq T_{[a,b]}$. So we obtain

$$\mathbb{P}^x(e_q \in [\tilde{\tau}_{2k}, \tilde{\tau}_{2k+1})) \leq \hat{\kappa}(q) \int_{(b,\infty)} U_-^q(y-b)\,\nu_{2k}^x(\mathrm{d}y) \leq \frac{\mathbb{P}^x(e_q \in [\tilde{\tau}_{2k}, \tilde{\tau}_{2k+1}))}{\mathbb{P}^x(e_q \geq T_{[a,b]})}. \tag{4.18}$$

Before proving the bounds of (4.17) we note that

$$\begin{aligned}
\mathbb{P}^x(e_q < T_{[a,b]}, \xi_{e_q} > b) &= \mathbb{P}^x(e_q < \lim_{k\to\infty} \tilde{\tau}_k, \xi_{e_q} > b) \\
&= \mathbb{P}^x\Big(\bigcup_{k=0}^{\infty}\{e_q \in [\tilde{\tau}_k, \tilde{\tau}_{k+1}), \xi_{e_q} > b\}\Big) \\
&= \mathbb{P}^x\Big(\bigcup_{k=0}^{\infty}\{e_q \in [\tilde{\tau}_{2k}, \tilde{\tau}_{2k+1})\}\Big) \\
&= \sum_{k=0}^{\infty} \mathbb{P}^x(e_q \in [\tilde{\tau}_{2k}, \tilde{\tau}_{2k+1})).
\end{aligned} \tag{4.19}$$

The first equality follows from the definition of $\tilde{\tau}_k$ and the facts that $T_{[a,b]} < \infty$ almost surely (because ξ is recurrent under Assumption (A)) and that τ_k diverges to $+\infty$ almost surely. The third one is due to the fact that for $x > b$ the process remains above b only in the intervals $[\tilde{\tau}_{2k}, \tilde{\tau}_{2k+1})$. With (4.19), summing (4.18) over k yields

$$\begin{aligned}
\hat{\kappa}(q)h_+^q(x) &= \sum_{k=0}^{\infty} \hat{\kappa}(q) \int_{(b,\infty)} U_-^q(y-b)\,\nu_{2k}^x(\mathrm{d}y) \\
&\in \Big[\mathbb{P}^x(e_q < T_{[a,b]}, \xi_{e_q} > b), \frac{\mathbb{P}^x(e_q < T_{[a,b]}, \xi_{e_q} > b)}{\mathbb{P}^x(e_q \geq T_{[a,b]})}\Big]
\end{aligned}$$

which is (4.17). Since ξ is recurrent $\mathbb{P}^x(e_q \geq T_{[a,b]})$ converges to 1 for $q \searrow 0$, hence, (4.17) implies the claim. □

The key for the proof of Theorem 4.1.1 are the relations in Proposition 4.3.5. We use them in a similar way Chaumont and Doney [16] proved invariance of a certain function for the Lévy process killed on the negative half-line.

Proof of Theorem 4.1.1. First note that (B) guarantees that $h_+(x)$ is strictly positive for all $x \in \mathbb{R} \setminus [a,b]$, which is not the case for $x < a$ when (B) fails. From now on Assumption (B) won't be used anymore. For $x \in \mathbb{R} \setminus [a,b]$ and $t \geq 0$ we have to show

$$\mathbb{E}^x\big[\mathbb{1}_{\{t<T_{[a,b]}\}} h_+(\xi_t)\big] = h_+(x).$$

First we show that the left-hand side is smaller or equal to the right-hand side. This can be done applying Proposition 4.3.5 in the first step and Fatou's Lemma in the second one:

$$\mathbb{E}^x\big[\mathbb{1}_{\{t<T_{[a,b]}\}} h_+(\xi_t)\big] = \mathbb{E}^x\Big[\mathbb{1}_{\{t<T_{[a,b]}\}} \lim_{q\searrow 0} \frac{1}{\hat{\kappa}(q)} \mathbb{P}^{\xi_t}(e_q < T_{[a,b]}, \xi_{e_q} > b)\Big]$$

$$
\begin{aligned}
&\leq \lim_{q\searrow 0} \frac{1}{\hat\kappa(q)} \mathbb{E}^x\Big[\mathbb{1}_{\{t<T_{[a,b]}\}}\mathbb{P}^{\xi_t}(e_q < T_{[a,b]}, \xi_{e_q} > b)\Big] \qquad (4.20)\\
&= \lim_{q\searrow 0} \frac{q}{\hat\kappa(q)} \int\limits_{(0,\infty)} e^{-qs}\mathbb{E}^x\Big[\mathbb{1}_{\{t<T_{[a,b]}\}}\mathbb{P}^{\xi_t}(s < T_{[a,b]}, \xi_s > b)\Big]\,\mathrm{d}s\\
&= \lim_{q\searrow 0} \frac{q}{\hat\kappa(q)} \int\limits_{(0,\infty)} e^{-qs}\mathbb{P}^x(s+t < T_{[a,b]}, \xi_{s+t} > b)\,\mathrm{d}s\\
&= \lim_{q\searrow 0} \frac{q}{\hat\kappa(q)} e^{qt}\int\limits_{(t,\infty)} e^{-qs}\mathbb{P}^x(s < T_{[a,b]}, \xi_s > b)\,\mathrm{d}s\\
&= \lim_{q\searrow 0} \frac{q}{\hat\kappa(q)} e^{qt}\int\limits_{(0,\infty)} e^{-qs}\mathbb{P}^x(s < T_{[a,b]}, \xi_s > b)\,\mathrm{d}s\\
&\quad - \lim_{q\searrow 0} \frac{q}{\hat\kappa(q)} e^{qt}\int\limits_{(0,t]} e^{-qs}\mathbb{P}^x(s < T_{[a,b]}, \xi_s > b)\,\mathrm{d}s\\
&= \lim_{q\searrow 0} \frac{1}{\hat\kappa(q)} e^{qt}\mathbb{P}^x(e_q < T_{[a,b]}, \xi_{e_q} > b)\\
&\quad - \lim_{q\searrow 0} \frac{q}{\hat\kappa(q)} e^{qt}\int\limits_{(0,t]} e^{-qs}\mathbb{P}^x(s < T_{[a,b]}, \xi_s > b)\,\mathrm{d}s\\
&= h_+(x) - \lim_{q\searrow 0} \frac{q}{\hat\kappa(q)} e^{qt}\int\limits_{(0,t]} e^{-qs}\mathbb{P}^x(s < T_{[a,b]}, \xi_s > b)\,\mathrm{d}s\\
&= h_+(x).
\end{aligned}
$$

The last equality follows because, according to Kyprianou [34], Section 13.2.1, it holds that $\lim_{q\searrow 0}\frac{q}{\hat\kappa(q)} = 0$ if ξ oscillates. To show the equality it remains to show that we can replace the inequality in (4.20) by an equality. To apply the dominated convergence theorem, we use Proposition 4.3.5 which says also that

$$
\frac{1}{\hat\kappa(q)}\mathbb{P}^{\xi_t}(e_q < T_{[a,b]}, \xi_{e_q} > b) \leq h_+^q(\xi_t) \leq h_+(\xi_t)
$$

for all $q > 0$. Furthermore, we have just seen that

$$
\mathbb{E}^x\big[\mathbb{1}_{\{t<T_{[a,b]}\}}h_+(\xi_t)\big] \leq h_+(x) < \infty.
$$

So we can apply dominated convergence to switch the limit and the integral. □

4.3.3 Conditioning and h-transforms

The aim of this section is to prove Proposition 4.1.4 and Theorem 4.1.5.

Proof of Proposition 4.1.4. Integrating out e_q, using Proposition 4.3.5 and the Markov property, gives

$$
\begin{aligned}
&\lim_{q\searrow 0}\mathbb{P}^x(\Lambda, t < e_q \,|\, e_q < T_{[a,b]}, \xi_{e_q} > b)\\
=&\lim_{q\searrow 0}\frac{1}{\mathbb{P}^x(e_q < T_{[a,b]}, \xi_{e_q} > b)}\int\limits_{(t,\infty)} qe^{-qs}\mathbb{P}^x\left(\Lambda, s < T_{[a,b]}, \xi_s > b\right)\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{h_+(x)}\lim_{q\searrow 0}\frac{e^{-qt}}{\hat\kappa(q)}\int_{(0,\infty)} qe^{-qs}\mathbb{P}^x\left(\Lambda, s+t<T_{[a,b]},\xi_{s+t}>b\right)\,\mathrm{d}s\\
=&\frac{1}{h_+(x)}\lim_{q\searrow 0}\frac{1}{\hat\kappa(q)}\int_{(0,\infty)} qe^{-qs}\mathbb{E}^x\left[\mathbb{1}_\Lambda\mathbb{1}_{\{t<T_{[a,b]}\}}\mathbb{P}^{\xi_t}(s<T_{[a,b]},\xi_s>b)\right]\mathrm{d}s\\
=&\frac{1}{h_+(x)}\lim_{q\searrow 0}\frac{1}{\hat\kappa(q)}\mathbb{E}^x\left[\mathbb{1}_\Lambda\mathbb{1}_{\{t<T_{[a,b]}\}}\mathbb{P}^{\xi_t}(e_q<T_{[a,b]},\xi_{e_q}>b)\right].
\end{aligned}
$$

From Proposition 4.3.5 we also know $\frac{1}{\hat\kappa(q)}\mathbb{P}^{\xi_t}(e_q<T_{[a,b]},\xi_{e_q}>b)\le h_+(\xi_t)$ for all $q>0$ and $\mathbb{1}_\Lambda\mathbb{1}_{\{t<T_{[a,b]}\}}h_+(\xi_t)$ is integrable since h_+ is invariant. So we can use dominated convergence to conclude

$$
\begin{aligned}
&\lim_{q\searrow 0}\mathbb{P}^x(\Lambda,t<e_q|e_q<T_{[a,b]},\xi_{e_q}>b)\\
=&\frac{1}{h_+(x)}\mathbb{E}^x\left[\mathbb{1}_\Lambda\mathbb{1}_{\{t<T_{[a,b]}\}}\lim_{q\searrow 0}\frac{1}{\hat\kappa(q)}\mathbb{P}^{\xi_t}(e_q<T_{[a,b]},\xi_{e_q}>b)\right]\\
=&\mathbb{E}^x\left[\mathbb{1}_\Lambda\mathbb{1}_{\{t<T_{[a,b]}\}}\frac{h_+(\xi_t)}{h_+(x)}\right],
\end{aligned}
$$

where we used again Proposition 4.3.5 in the final equality. Hence, conditioning is possible and coincides with the h-transform with h_+ which confirms Proposition 4.1.4. $\square$

For the proof of Theorem 4.1.5 we will use a corollary of Proposition 4.3.5.

Corollary 4.3.7. *Assume (A) and let e_q be an independent exponentially distributed random variable with parameter $q>0$. Then, for $x\notin[a,b]$, we have*

$$\mathbb{P}^x(e_q<T_{[a,b]})\le\hat\kappa(q)h_+^q(x)+\kappa(q)h_-^q(x),\quad q>0,\tag{4.21}$$

and

$$\lim_{q\searrow 0}\frac{1}{\hat\kappa(q)}\mathbb{P}^x(e_q<T_{[a,b]})=h_+(x)+Ch_-(x),\tag{4.22}$$

where $C=\lim_{q\searrow 0}\frac{\kappa(q)}{\hat\kappa(q)}$.

Proof. Let be $x\notin[a,b]$. With Proposition 4.3.5 and its counterpart for h_- we have

$$\mathbb{P}^x(e_q<T_{[a,b]},\xi_{e_q}>b)\le\hat\kappa(q)h_+^q(x)\quad\text{ and }\quad\mathbb{P}^x(e_q<T_{[a,b]},\xi_{e_q}<a)\le\kappa(q)h_-^q(x)$$

from which the first claim follows. Furthermore, we have again with Proposition 4.3.5:

$$\lim_{q\searrow 0}\frac{1}{\hat\kappa(q)}\mathbb{P}^x(e_q<T_{[a,b]},\xi_{e_q}>b)=h_+(x)$$

and

$$\lim_{q\searrow 0}\frac{1}{\kappa(q)}\mathbb{P}^x(e_q<T_{[a,b]},\xi_{e_q}<a)=h_-(x).$$

With this we get

$$\lim_{q\searrow 0}\frac{1}{\hat\kappa(q)}\mathbb{P}^x(e_q<T_{[a,b]})$$

$$= \lim_{q\searrow 0} \frac{1}{\hat\kappa(q)} \mathbb{P}^x(e_q < T_{[a,b]}, \xi_{e_q} > b) + \lim_{q\searrow 0} \frac{\kappa(q)}{\hat\kappa(q)} \frac{1}{\kappa(q)} \mathbb{P}^x(e_q < T_{[a,b]}, \xi_{e_q} < a)$$
$$= h_+(x) + Ch_-(x)$$

and the proof is complete. □

Proof of Theorem 4.1.5. We follow a similar strategy as in the proof of Proposition 4.1.4. First note that since $\lim_{q\searrow 0} \kappa(q)/\hat\kappa(q)$ exists, the ratio is bounded for $q \in (0,1)$ by some $\beta > 0$. Hence, with Corollary 4.3.7 we get

$$\frac{1}{\hat\kappa(q)} \mathbb{P}^y(e_q < T_{[a,b]}) \le h_+^q(y) + \frac{\kappa(q)}{\hat\kappa(q)} h_-^q(y) \le h_+(y) + \beta h_-(y)$$

for all $y \notin [a,b]$. So we use dominated convergence and the second part of Corollary 4.3.7 to get

$$\begin{aligned}
&\lim_{q\searrow 0} \mathbb{P}^x(\Lambda, t < e_q | e_q < T_{[a,b]}) \\
&= \lim_{q\searrow 0} \frac{1}{\mathbb{P}^x(e_q < T_{[a,b]})} \int_{(t,\infty)} q e^{-qs} \mathbb{P}^x\left(\Lambda, s < T_{[a,b]}\right) \mathrm{d}s \\
&= \frac{1}{h_+(x) + Ch_-(x)} \lim_{q\searrow 0} \frac{e^{-qt}}{\hat\kappa(q)} \int_{(0,\infty)} q e^{-qs} \mathbb{P}^x\left(\Lambda, s + t < T_{[a,b]}\right) \mathrm{d}s \\
&= \frac{1}{h_+(x) + Ch_-(x)} \lim_{q\searrow 0} \frac{e^{-qt}}{\hat\kappa(q)} \int_{(0,\infty)} q e^{-qs} \mathbb{E}^x\left[\mathbb{1}_\Lambda \mathbb{1}_{\{t < T_{[a,b]}\}} \mathbb{P}^{\xi_t}(s < T_{[a,b]})\right] \mathrm{d}s \\
&= \frac{1}{h_+(x) + Ch_-(x)} \lim_{q\searrow 0} \frac{1}{\hat\kappa(q)} \mathbb{E}^x\left[\mathbb{1}_\Lambda \mathbb{1}_{\{t < T_{[a,b]}\}} \mathbb{P}^{\xi_t}(e_q < T_{[a,b]})\right] \\
&= \frac{1}{h_+(x) + Ch_-(x)} \mathbb{E}^x\left[\mathbb{1}_\Lambda \mathbb{1}_{\{t < T_{[a,b]}\}} \lim_{q\searrow 0} \frac{1}{\hat\kappa(q)} \mathbb{P}^{\xi_t}(e_q < T_{[a,b]})\right] \\
&= \frac{1}{h_+(x) + Ch_-(x)} \mathbb{E}^x\left[\mathbb{1}_\Lambda \mathbb{1}_{\{t < T_{[a,b]}\}} \left(h_+(\xi_t) + Ch_-(\xi_t)\right)\right].
\end{aligned}$$

□

4.3.4 Long-time behaviour

Finally, we analyse the transience behaviour of the conditioned processes constructed in the previous section.

Proof of Proposition 4.1.6. **Step 1:** We show that ξ under $\mathbb{P}_+^x$ is almost surely bounded from below. First note that, for $x < a$,

$$\begin{aligned}
\mathbb{E}^x\left[\mathbb{1}_{\{T_{[a,\infty)} < T_{[a,b]}\}} h_+(\xi_{T_{[a,\infty)}})\right] &= \int_{(b,\infty)} h_+(y)\, \nu_1^x(\mathrm{d}y) \\
&= \sum_{k=0}^{\infty} \int_{(b,\infty)} \int_{(b,\infty)} U_-(z-b)\, \nu_{2k}^y(\mathrm{d}z)\, \nu_1^x(\mathrm{d}y)
\end{aligned}$$

$$= \sum_{k=0}^{\infty} \int_{(b,\infty)} U_-(z-b)\, \nu_{2k+1}^x(\mathrm{d}z)$$
$$= h_+(x).$$

For the first equality we used $\nu_1^x(\mathrm{d}y) = \mathbb{P}^x(\xi_{T_{[a,\infty)}} \in \mathrm{d}y, T_{[a,\infty)} < T_{[a,b]})$ for $x < a$, in the second we plugged-in the definition of $h_+(y)$ for $y > b$ and used Fubini's theorem, in the third we used (4.8) and for the final equality we used the definition of $h_+(x)$ for $x < a$. Since $\xi_{T_{(-\infty,c]}} < a$ for $c < a$ it follows, for all $x \in \mathbb{R} \setminus [a,b]$, that

$$\begin{aligned}
&\mathbb{P}_+^x(T_{(-\infty,c]} < \infty \text{ for all } c < a)\\
&= \lim_{c\to-\infty} \mathbb{P}_+^x(T_{(-\infty,c]} < \infty)\\
&= \frac{1}{h_+(x)} \lim_{c\to-\infty} \mathbb{E}^x\Big[\mathbb{1}_{\{T_{(-\infty,c]}<T_{[a,b]}\}} h_+(\xi_{T_{(-\infty,c]}})\Big]\\
&= \frac{1}{h_+(x)} \lim_{c\to-\infty} \mathbb{E}^x\Big[\mathbb{1}_{\{T_{(-\infty,c]}<T_{[a,b]}\}} \mathbb{E}^{\xi_{T_{(-\infty,c]}}}\big[\mathbb{1}_{\{T_{[a,\infty)}<T_{[a,b]}\}} h_+(\xi_{T_{[a,\infty)}})\big]\Big]\\
&= \frac{1}{h_+(x)} \lim_{c\to-\infty} \mathbb{E}^x\Big[\mathbb{1}_{\{T_{(-\infty,c]}<T_{[a,b]}\}} \mathbb{E}^x\big[(\mathbb{1}_{\{T_{[a,\infty)}<T_{[a,b]}\}} h_+(\xi_{T_{[a,\infty)}})) \circ \theta_{T_{(-\infty,c]}} \,|\, \mathcal{F}_{T_{(-\infty,c]}}\big]\Big],
\end{aligned}$$

where we used (2.14) in the second equality and again the strong Markov property (2.16) with $Y = \mathbb{1}_{\{T_{[a,\infty)}<T_{[a,b]}\}} h_+(\xi_{T_{[a,\infty)}})$ in the final equality. According to Theorem 1.3.6 of Chung and Walsh [18] it holds that

$$\{T_{(-\infty,c]} < T_{[a,b]}\} \in \mathcal{F}_{T_{(-\infty,c]}} \cap \mathcal{F}_{T_{[a,b]}} \subseteq \mathcal{F}_{T_{(-\infty,c]}}.$$

So we continue for all $x \in \mathbb{R} \setminus [a,b]$ with

$$\begin{aligned}
&\mathbb{E}^x\Big[\mathbb{1}_{\{T_{(-\infty,c]}<T_{[a,b]}\}} \mathbb{E}^x\big[(\mathbb{1}_{\{T_{[a,\infty)}<T_{[a,b]}\}} h_+(\xi_{T_{[a,\infty)}})) \circ \theta_{T_{(-\infty,c]}} \,|\, \mathcal{F}_{T_{(-\infty,c]}}\big]\Big]\\
&= \mathbb{E}^x\Big[\mathbb{E}^x\big[\mathbb{1}_{\{T_{(-\infty,c]}<T_{[a,b]}\}} (\mathbb{1}_{\{T_{[a,\infty)}<T_{[a,b]}\}} h_+(\xi_{T_{[a,\infty)}})) \circ \theta_{T_{(-\infty,c]}} \,|\, \mathcal{F}_{T_{(-\infty,c]}}\big]\Big]\\
&= \mathbb{E}^x\Big[\mathbb{1}_{\{T_{(-\infty,c]}<T_{[a,b]}\}} \big((\mathbb{1}_{\{T_{[a,\infty)}<T_{[a,b]}\}} h_+(\xi_{T_{[a,\infty)}})) \circ \theta_{T_{(-\infty,c]}}\big)\Big].
\end{aligned}$$

Now consider just $x < a$ and observe

$$\begin{aligned}
&\mathbb{E}^x\Big[\mathbb{1}_{\{T_{(-\infty,c]}<T_{[a,b]}\}} \big((\mathbb{1}_{\{T_{[a,\infty)}<T_{[a,b]}\}} h_+(\xi_{T_{[a,\infty)}})) \circ \theta_{T_{(-\infty,c]}}\big)\Big]\\
&= \sum_{k=0}^{\infty} \mathbb{E}^x\Big[\mathbb{1}_{\{T_{(-\infty,c]}\in[\tilde\tau_{2k},\tilde\tau_{2k+1})\}} \big((\mathbb{1}_{\{T_{[a,\infty)}<T_{[a,b]}\}} h_+(\xi_{T_{[a,\infty)}})) \circ \theta_{T_{(-\infty,c]}}\big)\Big]\\
&= \sum_{k=0}^{\infty} \mathbb{E}^x\Big[\mathbb{1}_{\{T_{(-\infty,c]}\in[\tilde\tau_{2k},\tilde\tau_{2k+1})\}} \mathbb{1}_{\{\tilde\tau_{2k+1}<T_{[a,b]}\}} h_+(\xi_{\tilde\tau_{2k+1}})\Big]\\
&= \sum_{k=0}^{\infty} \mathbb{E}^x\Big[\mathbb{1}_{\{T_{(-\infty,c]}\in[\tau_{2k},\tau_{2k+1})\}} \mathbb{1}_{\{\tau_{2k+1}<T_{[a,b]}\}} h_+(\xi_{\tau_{2k+1}})\Big],
\end{aligned}$$

where $\tilde\tau_k = \min(\tau_k, T_{[a,b]})$ as in the proof of Proposition 4.3.5. Combining the above computations gives

$$\mathbb{P}_+^x(T_{(-\infty,c]} < \infty \text{ for all } c < a) \tag{4.23}$$

$$= \frac{1}{h_+(x)} \lim_{c\to-\infty} \sum_{k=0}^{\infty} \mathbb{E}^x \Big[\mathbb{1}_{\{T_{(-\infty,c]}\in[\tau_{2k},\tau_{2k+1})\}} \mathbb{1}_{\{\tau_{2k+1}<T_{[a,b]}\}} h_+(\xi_{\tau_{2k+1}}) \Big]$$

for $x < a$. Our aim is to switch the limit and the sum. In order to justify the dominated convergence theorem it is enough to verify

$$\sum_{k=0}^{\infty} \mathbb{E}^x \Big[\mathbb{1}_{\{\tau_{2k+1}<T_{[a,b]}\}} h_+(\xi_{\tau_{2k+1}}) \Big] < \infty.$$

With Proposition 4.3.1 we have

$$\begin{aligned}
&\mathbb{E}^x \Big[\mathbb{1}_{\{\tau_{2k+1}<T_{[a,b]}\}} h_+(\xi_{\tau_{2k+1}}) \Big] \\
&\le c_1 \mathbb{E}^x \Big[\mathbb{1}_{\{\tau_{2k+1}<T_{[a,b]}\}} U_-(\xi_{\tau_{2k+1}} - b) \Big] + c_3 \mathbb{P}^x(\tau_{2k+1} < T_{[a,b]}) \\
&\le c_1 \mathbb{E}^x \Big[\mathbb{1}_{\{K^\dagger \ge 2k+1\}} U_-(\xi_{\tau_{2k+1}} - b) \Big] + c_3 \nu^x_{2k+1}((b,\infty)) \\
&\le c_1 \mathbb{E}^x \Big[\mathbb{1}_{\{K^\dagger \ge 2k+1\}} U_-(\xi_{\tau_{2k+1}} - b) \Big] + c_3 \varphi^{2k}
\end{aligned}$$

where c_1, c_3 and φ are the constants from Proposition 4.3.1 and its proof. It follows that

$$\begin{aligned}
&\sum_{k=0}^{\infty} \mathbb{E}^x \big[\mathbb{1}_{\{\tau_{2k+1}<T_{[a,b]}\}} h_+(\xi_{\tau_{2k+1}}) \big] \\
&\le c_1 \sum_{k=0}^{\infty} \mathbb{E}^x \big[\mathbb{1}_{\{K^\dagger \ge 2k+1\}} U_-(\xi_{\tau_{2k+1}} - b) \big] + c_3 \sum_{k=0}^{\infty} \varphi^{2k} \\
&= c_1 h_+(x) + \frac{c_3}{1-\varphi^2} < \infty.
\end{aligned}$$

So we can switch the limit and the integral in (4.23). With the same upper bound for every summand for itself we can even move the limit inside the expectation. Hence,

$$\begin{aligned}
&\mathbb{P}^x_+(T_{(-\infty,c]} < \infty \text{ for all } c < a) \\
&= \frac{1}{h_+(x)} \sum_{k=0}^{\infty} \mathbb{E}^x \Big[\lim_{c\to-\infty} \mathbb{1}_{\{T_{(-\infty,c]}\in[\tau_{2k},\tau_{2k+1})\}} \mathbb{1}_{\{\tau_{2k+1}<T_{[a,b]}\}} h_+(\xi_{\tau_{2k+1}}) \Big].
\end{aligned}$$

Since ξ oscillates (which implies $\tau_k < \infty$ $\mathbb{P}^x$-almost surely) we obtain that $\mathbb{1}_{\{T_{(-\infty,c]}\in[\tau_{2k},\tau_{2k+1})\}}$ converges to 0 almost surely under $\mathbb{P}^x$ for $c \to -\infty$. Hence,

$$\mathbb{P}^x_+(T_{(-\infty,c]} < \infty \text{ for all } c < a) = 0$$

for $x < a$. For $x > b$ it is proved analogously that

$$\begin{aligned}
&\mathbb{P}^x_+(T_{(-\infty,c]} < \infty \text{ for all } c < a) \\
&= \frac{1}{h_+(x)} \lim_{c\to-\infty} \sum_{k=0}^{\infty} \mathbb{E}^x \big[\mathbb{1}_{\{T_{(-\infty,c]}\in[\tau_{2k+1},\tau_{2k+2})\}} \mathbb{1}_{\{\tau_{2k+2}<T_{[a,b]}\}} h_+(\xi_{\tau_{2k+2}})) \big]
\end{aligned}$$

and, with the above argumentation, we also find that $\mathbb{P}^x_+(T_{(-\infty,c]} < \infty \text{ for all } c < a) = 0$ for $x > b$. This finishes the arguments for Step 1.

Step 2: In the second step we show that ξ is transient under $\mathbb{P}_+^x$, i.e. only spends finite time in sets of the form $[d,a)\cup(b,c]$ for $d<a$ and $c>b$. Actually, we even show that the expected occupation is finite:

$$
\begin{aligned}
&\mathbb{E}_+^x\Big[\int_{[0,\infty)} \mathbb{1}_{\{\xi_t\in[d,a)\cup(b,c]\}}\mathrm{d}t\Big]\\
&=\int_{[0,\infty)} \mathbb{P}_+^x(\xi_t\in[d,a)\cup(b,c])\,\mathrm{d}t\\
&=\int_{[0,\infty)} \mathbb{E}^x\Big[\mathbb{1}_{\{\xi_t\in[d,a)\cup(b,c]\}}\mathbb{1}_{\{t<T_{[a,b]}\}}\frac{h_+(\xi_t)}{h_+(x)}\Big]\,\mathrm{d}t\\
&\le\frac{1}{h_+(x)}\sup_{y\in[d,a)\cup(b,c]}h_+(y)\int_{[0,\infty)}\mathbb{E}^x\Big[\mathbb{1}_{\{\xi_t\in[d,a)\cup(b,c]\}}\mathbb{1}_{\{t<T_{[a,b]}\}}\Big]\,\mathrm{d}t.
\end{aligned}
\tag{4.24}
$$

Recalling Proposition 4.3.1, $\sup_{y\in[d,a)\cup(b,c]}h_+(y)$ is finite and it remains to show finiteness of

$$
\int_{[0,\infty)}\mathbb{E}^x\Big[\mathbb{1}_{\{\xi_t\in[d,a)\cup(b,c]\}}\mathbb{1}_{\{t<T_{[a,b]}\}}\Big]\,\mathrm{d}t
$$

which is just the potential of $[d,a)\cup(b,c]$ of the process killed on entering $[a,b]$. To abbreviate we denote the potential of $(\xi,\mathbb{P}^x)$ killed on entering a Borel set B by $U^B(x,\mathrm{d}y)$. It follows

$$
U^{[a,b]}(x,[d,a)\cup(b,c])=\sum_{k=0}^{\infty}\Big(U^{(-\infty,b]}(\nu_{2k}^x,(b,c])+U^{[a,\infty)}(\nu_{2k+1}^x,[d,a))\Big).
$$

To compute the righthand side we apply (2.31) (resp. Proposition VI.20 of Bertoin [4]) for $y>b$:

$$
\begin{aligned}
U^{(-\infty,b]}(y,(b,c])&=U^{(-\infty,0]}(y-b,(0,c-b])\\
&=\int_{(0,c-b]}\int_{[(y-b-u)^+,y-b]}U_+(\mathrm{d}u+v-(y-b))\,U_-(\mathrm{d}v)\\
&=\int_{[0,y-b]}\Big(\int_{(0,c-b]}\mathbb{1}_{\{u\ge y-b-v\}}U_+(\mathrm{d}u-(y-b-v))\Big)\,U_-(\mathrm{d}v)\\
&=\int_{[0,y-b]}U_+(c+v-y)\,U_-(\mathrm{d}v)\\
&\le U_+(c-b)U_-(y-b).
\end{aligned}
$$

It holds analogously that $U^{[a,\infty)}(y,[d,a))\le U_-(a-d)U_+(a-y)$ for $y>a$. So we have

$$
\begin{aligned}
U_{[a,b]}(x,[d,a)\cup(b,c])\le U_+(c-b)&\sum_{k=0}^{\infty}\int_{(b,\infty)}U_-(y-b)\,\nu_{2k}^x(\mathrm{d}y)\\
&+U_-(a-d)\sum_{k=0}^{\infty}\int_{(-\infty,a)}U_+(a-y)\,\nu_{2k+1}^x(\mathrm{d}y)
\end{aligned}
$$

$$= U_+(c-b)h_+(x) + U_-(a-d))h_-(x) < \infty.$$

It follows in particular that the time the process $(\xi, \mathbb{P}^x_+)$ spends in sets of the form $[d,a) \cup (b,c]$ is finite almost surely. Together with the first result that the process is bounded below almost surely and that the process is conservative it follows that $\lim_{t\to\infty} \xi_t = +\infty$ almost surely under $\mathbb{P}^x_+$. □

Proof of Theorem 4.1.7. The proof strategy is similar to the one above. Transience of the conditioned process is verified again by computing the occupation measure using the representation of the conditioned process as h-transform. The computation is in analogy to (4.24), using that $h = h_+ + Ch_-$ is bounded by Proposition 4.3.1.

Next, recall from the counterpart of Proposition 4.1.6 for $\mathbb{P}^x_-$ that under $(\hat{B})$,

$$\mathbb{P}^x_-(T_{(-\infty,c]} < \infty) = 1, \quad c < a$$

for all $x \in \mathbb{R} \setminus [a,b]$. Since (2.14) implies

$$\mathbb{P}^x_-(T_{(-\infty,c]} < \infty) = \frac{1}{h_-(x)} \mathbb{E}^x \left[\mathbb{1}_{\{T_{(-\infty,c]} < T_{[a,b]}\}} h_-(\xi_{T_{(-\infty,c]}}) \right]$$

we deduce

$$\mathbb{E}^x \left[\mathbb{1}_{\{T_{(-\infty,c]} < T_{[a,b]}\}} h_-(\xi_{T_{(-\infty,c]}}) \right] = h_-(x), \quad c < a \tag{4.25}$$

for all $x \in \mathbb{R} \setminus [a,b]$ under $(\hat{B})$. If $(\hat{B})$ fails, we know

$$h_-(x) = \begin{cases} 0 & \text{if } x > b \\ U_+(a-x) & \text{if } x < a \end{cases}.$$

Let us check if (4.25) holds in this case, too. If $x > b$, the left-hand side of (4.25) is 0 (because there are no jumps bigger than $b-a$), as well as the right-hand side. For $x < a$ the measure $\mathbb{P}^x_-$ corresponds to the process conditioned to stay below a which is known to drift to $-\infty$ (see Chaumont and Doney [16]). In particular it holds

$$\mathbb{P}^x_-(T_{(-\infty,c]} < \infty) = 1, \quad c < a$$

from which we can deduce (4.25) in the same way as before. So (4.25) holds for all $x \in \mathbb{R} \setminus [a,b]$ just under (A).

Again using (2.14) yields

$$\begin{aligned}
&\mathbb{P}^x_\updownarrow(T_{(-\infty,c]} < \infty) \\
&= \frac{1}{h(x)} \Big(\mathbb{E}^x \big[\mathbb{1}_{\{T_{(-\infty,c]} < T_{[a,b]}\}} h_+(\xi_{T_{(-\infty,c]}}) \big] + \mathbb{E}^x \big[\mathbb{1}_{\{T_{(-\infty,c]} < T_{[a,b]}\}} Ch_-(\xi_{T_{(-\infty,c]}}) \big] \Big) \\
&= \frac{1}{h(x)} \mathbb{E}^x \big[\mathbb{1}_{\{T_{(-\infty,c]} < T_{[a,b]}\}} h_+(\xi_{T_{(-\infty,c]}}) \big] + \frac{Ch_-(x)}{h(x)}.
\end{aligned}$$

In the proof of Proposition 4.1.6 we have already seen that $\mathbb{E}^x \big[\mathbb{1}_{\{T_{(-\infty,c]} < T_{[a,b]}\}} h_+(\xi_{T_{(-\infty,c]}}) \big]$ vanishes for $c \to -\infty$, hence,

$$\mathbb{P}^x_\updownarrow(\xi \text{ is unbounded below}) = \mathbb{P}^x_\updownarrow(T_{(-\infty,c]} < \infty \text{ for all } c < a) = \frac{Ch_-(x)}{h(x)}.$$

So we get

$$\mathbb{P}^x_\updownarrow(\xi \text{ is bounded below}) = 1 - \frac{Ch_-(x)}{h(x)} = \frac{h_+(x)}{h(x)}$$

and, because of transience,

$$\frac{h_+(x)}{h(x)} = \mathbb{P}^x_\updownarrow(\xi \text{ is bounded below}) = \mathbb{P}^x_\updownarrow(\lim_{t\to\infty} \xi_t = \infty).$$

Analogously one derives $\mathbb{P}^x_\updownarrow(\lim_{t\to\infty} \xi_t = \infty) = \frac{Ch_-(x)}{h(x)}$ and the proof is complete. $\square$

4.4 Extension to transient Lévy processes

When conditioning a process to avoid an interval, the most interesting case is when the process is recurrent; if it is transient, it may avoid the interval with positive probability, and things become simpler. On the other hand, the conditionings in Proposition 4.1.4, to avoid the interval while finishing above (or below) it, may still be non-trivial. In this section, we drop Assumption (A), and require only that ξ is not a compound Poisson process and does not oscillate. In particular, we do not assume that ξ has finite second moments; only for the study of h_- do we need further conditions.

Without loss of generality, we assume from now on that ξ drifts to $+\infty$, and indicate which of our results still hold and which need modification. Under this assumption, the function h defined by (4.1) simplifies to h_+. This can be seen from the fact that $\kappa(0) = 0 < \hat{\kappa}(0)$, which implies $C = \lim_{q\searrow 0} \frac{\kappa(q)}{\hat{\kappa}(q)} = 0$.

4.4.1 Study of $h = h_+$

For the study of h (which is now equal to h_+) we need to distinguish two cases based on whether or not condition (B) is satisfied.

Condition (B) holds

Since the Lévy process is transient, the event $\{T_{[a,b]} = \infty\}$ has positive probability for every starting point. The conditioning simplifies dramatically and our results are still valid, as we now demonstrate. Let $\ell(x) := \mathbb{P}^x(T_{[a,b]} = \infty)$ for $x \notin [a,b]$. This is easily seen to be invariant using the strong Markov property:

$$\begin{aligned}
\mathbb{E}^x\big[\mathbb{1}_{\{t<T_{[a,b]}\}}\ell(\xi_t)\big] &= \mathbb{E}^x\big[\mathbb{1}_{\{t<T_{[a,b]}\}}\mathbb{P}^{\xi_t}(T_{[a,b]} = \infty)\big] \\
&= \lim_{s\to\infty} \mathbb{E}^x\big[\mathbb{1}_{\{t<T_{[a,b]}\}}\mathbb{P}^{\xi_t}(T_{[a,b]} > s)\big] \\
&= \lim_{s\to\infty} \mathbb{P}^x(T_{[a,b]} > t+s) \\
&= \mathbb{P}^x(T_{[a,b]} = \infty).
\end{aligned} \tag{4.26}$$

Transience ensures that ℓ is a positive invariant function. We next show that ℓ is indeed a multiple of $h = h_+$. To do so we will use the identity $\hat{\kappa}(q)U^q_-(x) = \mathbb{P}^x(e_q < T_{(-\infty,0]})$, where e_q is an independent exponentially distributed random variable with parameter $q > 0$ (see Kyprianou [34], Section 13.2.1 for a general Lévy process). Since ξ drifts to $+\infty$, we have $\hat{\kappa}(0) > 0$, and hence

$$\hat{\kappa}(0)U_-(x) = \mathbb{P}^x(T_{(-\infty,0]} = \infty), \quad x > 0.$$

The idea is to separate the two-sided entrance problem in infinitely many one-sided entrance problems and use the strong Markov property to combine them. For $x > b$, using the strong Markov property, we find

$$\begin{aligned}
&\mathbb{P}^x(T_{[a,b]} = \infty)\\
&= \mathbb{P}^x(T_{(-\infty,b]} = \infty) + \mathbb{P}^x(T_{[a,b]} = \infty, T_{(-\infty,b]} < \infty)\\
&= \mathbb{P}^x(T_{(-\infty,b]} = \infty) + \mathbb{E}^x\Big[\mathbb{1}_{\{T_{(-\infty,b]}<\infty, \xi_{T_{(-\infty,b]}}<a\}}\mathbb{P}^{\xi_{T_{(-\infty,b]}}}(T_{[a,b]} = \infty)\Big]\\
&= \hat{\kappa}(0)U_-(x-b) + \mathbb{E}^x\Big[\mathbb{1}_{\{T_{(-\infty,b]}<\infty, \xi_{T_{(-\infty,b]}}<a\}}\mathbb{E}^{\xi_{T_{(-\infty,b]}}}\big[\mathbb{1}_{\{\xi_{T_{[a,\infty)}}>b\}}\mathbb{P}^{\xi_{T_{[a,\infty)}}}(T_{[a,b]} = \infty)\big]\Big]\\
&= \hat{\kappa}(0)U_-(x-b) + \int\limits_{(b,\infty)} \mathbb{P}^y(T_{[a,b]} = \infty)\,\nu_2^x(\mathrm{d}y).
\end{aligned}$$

Now we split up $\mathbb{P}^y(T_{[a,b]} = \infty)$ in the same manner, i.e.,

$$\mathbb{P}^y(T_{[a,b]} = \infty) = \hat{\kappa}(0)U_-(y-b) + \int\limits_{(b,\infty)} \mathbb{P}^z(T_{[a,b]} = \infty)\,\nu_2^x(\mathrm{d}z).$$

Using $\int\limits_{(b,\infty)} \nu_2^z(\mathrm{d}y)\,\nu_2^x(\mathrm{d}z) = \nu_4^x(\mathrm{d}y)$ from (4.8) yields

$$\begin{aligned}
&\mathbb{P}^x(T_{[a,b]} = \infty)\\
&= \hat{\kappa}(0)\Big(U_-(x-b) + \int\limits_{(b,\infty)} U_-(y-b)\,\nu_2^x(\mathrm{d}y)\Big) + \int\limits_{(b,\infty)} \mathbb{P}^y(T_{[a,b]} = \infty)\,\nu_4^x(\mathrm{d}y).
\end{aligned}$$

By induction the following series representation is obtained:

$$\mathbb{P}^x(T_{[a,b]} = \infty) = \hat{\kappa}(0)\sum_{k=0}^{\infty}\int\limits_{(b,\infty)} U_-(y-b)\,\nu_{2k}^x(\mathrm{d}y).$$

For $x < a$ a similar computation can be carried out, and we obtain

$$\begin{aligned}
\ell(x) = \mathbb{P}^x(T_{[a,b]} = \infty) &= \begin{cases} \hat{\kappa}(0)\sum\limits_{k=0}^{\infty}\int\limits_{(b,\infty)} U_-(y-b)\,\nu_{2k}^x(\mathrm{d}y) & \text{if } x > b\\ \hat{\kappa}(0)\sum\limits_{k=0}^{\infty}\int\limits_{(b,\infty)} U_-(y-b)\,\nu_{2k+1}^x(\mathrm{d}y) & \text{if } x < a\end{cases}\\
&= \hat{\kappa}(0)h_+(x) = \hat{\kappa}(0)h(x).
\end{aligned}$$

Theorem 4.1.1: This is a consequence of the discussion above.

Theorem 4.1.5: Since we condition here on a positive probability event, the h-transform and the conditioning are related in a standard way, using the strong Markov property and integrating out e_q:

$$\begin{aligned}
\mathbb{E}^x\Big[\mathbb{1}_\Lambda\mathbb{1}_{\{t<T_{[a,b]}\}}\frac{\ell(\xi_t)}{\ell(x)}\Big] &= \frac{1}{\mathbb{P}^x(T_{[a,b]} = \infty)}\mathbb{E}^x\Big[\mathbb{1}_\Lambda\mathbb{1}_{\{t<T_{[a,b]}\}}\mathbb{P}^{\xi_t}(T_{[a,b]} = \infty)\Big]\\
&= \lim_{q\searrow 0}\frac{1}{\mathbb{P}^x(e_q < T_{[a,b]})}\mathbb{E}^x\Big[\mathbb{1}_\Lambda\mathbb{1}_{\{t<T_{[a,b]}\}}\mathbb{P}^{\xi_t}(e_q < T_{[a,b]})\Big]
\end{aligned}$$

$$\begin{aligned}
&= \lim_{q \searrow 0} \frac{\mathbb{P}^x(\Lambda, t + e_q < T_{[a,b]})}{\mathbb{P}^x(e_q < T_{[a,b]})} \\
&= \lim_{q \searrow 0} \frac{e^{qt}\mathbb{P}^x(\Lambda, t < e_q < T_{[a,b]})}{\mathbb{P}^x(e_q < T_{[a,b]})} \\
&= \lim_{q \searrow 0} \mathbb{P}^x(\Lambda, t < e_q \,|\, e_q < T_{[a,b]}),
\end{aligned}$$

for $\Lambda \in \mathcal{F}_t$, $t \geq 0$.

Proposition 4.1.4: The conditioning of Proposition 4.1.4 is equivalent to the conditioning of Theorem 4.1.5, since the additional condition to stay above the interval at late time vanishes in the limit due to the transience towards $+\infty$. Since $h = h_+$ the result of Proposition 4.1.4 follows.

Proposition 4.1.6 and Theorem 4.1.7: Since the conditioned measure is a restriction of the original one, the long-time behaviour of the conditioned process is identical to that of the original process. Hence, the statements of Proposition 4.1.6 and Theorem 4.1.7 hold.

Condition (B) fails

The definition of h_+ in this case simplifies to

$$h_+(x) = \begin{cases} U_-(x-b) & \text{if } x > b \\ 0 & \text{if } x < a \end{cases}.$$

This function is plainly not positive everywhere. It is nonetheless invariant for the process killed on entering $[a, b]$. The conditionings in Theorem 4.1.5 and Proposition 4.1.4 can still be carried out but, as we now prove, the results are somewhat different.

Let $h_\uparrow : (b, \infty) \to [0, \infty)$ be given by $h_\uparrow(x) = U_-(x-b)$, the restriction of h_+ to (b, ∞). As shown by Chaumont and Doney [16], this function is invariant for the process ξ killed on entering $(-\infty, b]$, and the h-transform of this process using $h_\uparrow$ is the process ξ conditioned to avoid $(-\infty, b]$. We will write $(\mathbb{P}^x_\uparrow)_{x\in(b,\infty)}$ for the probabilities associated with this Markov process.

Consider now the conditioning of Proposition 4.1.4. When $x > b$ the process cannot cross below the set $[a, b]$ and return above it without hitting the set. Therefore, we have that

$$\lim_{q \searrow 0} \mathbb{P}^x(\Lambda, t < e_q \mid e_q < T_{[a,b]}, \xi_{e_q} > b) = \lim_{q \searrow 0} \mathbb{P}^x(\Lambda, t < e_q \mid e_q < T_{(-\infty,b]}) = \mathbb{P}^x_\uparrow(\Lambda),$$

the last equality being due to Chaumont and Doney [16]. For $x < a$, $\mathbb{P}^x(e_q < T_{[a,b]}, \xi_{e_q} > b) = 0$ for every $q > 0$, so the conditioning does not have any sense. In total, the conditioning of Proposition 4.1.4 reduces to conditioning ξ to avoid $(-\infty, b)$.

We turn next to the conditioning in Theorem 4.1.5. Let us define $h_\downarrow : (-\infty, a) \to [0, \infty)$ by $h_\downarrow(x) = U_+(a-x)$, which is a positive invariant function for the process killed on entering $[a, \infty)$ resulting in the process conditioned to avoid $[a, \infty)$ when h-transformed with $h_\downarrow$. As before, we write $(\mathbb{P}^x_\downarrow)_{x\in(-\infty,a)}$ for the probabilities associated with the conditioned process, which is killed at its lifetime ζ. By the same reasoning in the case where (B) holds, $\lim_{q\searrow 0} \mathbb{P}^x(T_{[a,b]} > e_q) = \hat{\kappa}(0)h_+(x) = \hat{\kappa}(0)h_\uparrow(x)$ when $x > b$; and, when $x < a$, using the asymptotics of $T_{[a,\infty)}$ which we have already seen, we obtain $\mathbb{P}^x(T_{[a,b]} > e_q) = \mathbb{P}^x(T_{[a,\infty)} > e_q) \sim \kappa(q)U_+(a-x)$ as $q \searrow 0$, since ξ cannot jump over $[a, b]$ from below. If

$x > b$, and $\Lambda \in \mathcal{F}_t$, the same technique as in the proof of Theorem 4.1.5 gives rise to the calculation

$$\begin{aligned}
&\lim_{q\searrow 0} \mathbb{P}^x(\Lambda, t < e_q \mid e_q < T_{[a,b]}) \\
&= \frac{1}{\hat{\kappa}(0)h_\uparrow(x)} \mathbb{E}^x \left[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{[a,b]}\}} \lim_{q\searrow 0} \mathbb{P}^{\xi_t}(e_q < T_{[a,b]}) \right] \\
&= \frac{1}{\hat{\kappa}(0)h_\uparrow(x)} \mathbb{E}^x \left[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{[a,b]}\}} \lim_{q\searrow 0} \left(\mathbb{1}_{\{t<T_{(-\infty,b]}\}} \hat{\kappa}(0) h_\uparrow(\xi_t) + \mathbb{1}_{\{t>T_{(-\infty,b]}\}} \kappa(q) U_+(a-\xi_t) \right) \right] \\
&= \frac{1}{h_\uparrow(x)} \mathbb{E}^x [h_+(\xi_t) \mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{(-\infty,b]}\}}] = \mathbb{P}^x_\uparrow(\Lambda).
\end{aligned}$$

Similarly, if $x < a$, we obtain $\lim_{q\searrow 0} \mathbb{P}^x(\Lambda, t < e_q \mid e_q < T_{[a,b]}) = \mathbb{P}^x_\downarrow(\Lambda, t < \zeta)$.

This shows that the conditioning from Theorem 4.1.5 leads not to a single Doob-h-transform of a killed Lévy process, but rather to a Markov process which behaves entirely differently depending on whether it is started above or below the interval. The long-time behaviour can be deduced from Chaumont and Doney [16]: the conditioned process approaches $+\infty$ when started above b, and is killed when started below a.

4.4.2 Study of h_-

This section is kept informal; the claims can be proved by an adaptation of arguments developed in Section 4.3.

In order to study h_- we need to assume that $\mathbb{E}[H_1] < \infty$ and $\hat{\mathbb{E}}[H_1] < \infty$. Note that here the descending ladder height subordinator has finite lifetime ζ, so we understand $\hat{\mathbb{E}}[H_1] = \hat{\mathbb{E}}[H_1 \mathbb{1}_{1<\zeta}]$. The function h_- is merely excessive, in the sense that

$$\mathbb{E}^x[\mathbb{1}_{\{t<T_{[a,b]}\}} h_-(\xi_t)] \leq h_-(x), \quad x \in \mathbb{R} \setminus [a,b].$$

We may still define the excessive transform

$$\mathbb{P}^x_-(\Lambda, t < \zeta) = \mathbb{E}^x \left[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{[a,b]}\}} \frac{h_-(\xi_t)}{h_-(x)} \right], \quad x \in \mathbb{R} \setminus [a,b],$$

but the transformed process is now a killed Markov process, with lifetime ζ.

The dual version of the conditioning of Proposition 4.1.4 is then given by

$$\mathbb{P}^x_-(\Lambda, t < \zeta) = \lim_{q\searrow 0} \mathbb{P}^x(\Lambda, t < e_q \mid e_q < T_{[a,b]}, \xi_{e_q} < a), \quad x \in \mathbb{R} \setminus [a,b], \tag{4.27}$$

and gives rise to a killed strong Markov process. This is a generalization of the subordinator conditioned to stay below a level as studied in Kyprianou et al. [39].

5 Stable processes conditioned to avoid an interval

Here, we consider the same problem as in Chapter 4 but for a stable Lévy process. This means we focus on two main problems: On the one hand we would like to find an invariant function for ξ killed on entering an interval and on the other hand we would like to connect the h-transformed process to the process conditioned to avoid the interval in the spirit of (1.1). The techniques are different from the ones in Chapter 4 since stable processes do not have finite variance which was one of the main assumptions. Instead we use the deep factorisation of the stable process (Kyprianou [35] and Kyprianou et al. [39]) which is the analysis of the stable process using the underlying MAP via the Lamperti-Kiu transform.

5.1 Main results

We fix an α-stable Lévy process $(\xi, (\mathbb{P}^x)_{x\in\mathbb{R}})$ with scaling index $\alpha \in (0,2)$ which is not (the negative of) a subordinator (in the case $\alpha \in (1,2)$) and has both sided jumps. This forces $\rho := \mathbb{P}(\xi_t \geq 0)$ to be in $(0,1)$ when $\alpha < 1$ and to be in $(1-1/\alpha, 1/\alpha)$ when $\alpha > 1$. In the case $\alpha = 1$ we assume the drift a to be 0, and hence $\rho = 1/2$. Before stating the main results of this chapter, note that we can easily reduce our analysis for the interval $[a,b]$ to the interval $[-1,1]$. Indeed, suppose that h is a invariant function for $(p_t^{[-1,1]})_{t\geq 0}$. Then the function

$$h_{[a,b]}(x) := h\Big(\frac{2}{b-a}x - \frac{b+a}{b-a}\Big), \quad x \in \mathbb{R}\backslash[a,b],$$

is invariant for $(p_t^{[a,b]})_{t\geq 0}$. With the help of stationary independent increments and the scaling property for stable processes we have

$$\begin{aligned}
\mathbb{E}^{\xi}\Big[\mathbb{1}_{\{t<T_{[a,b]}\}} h\Big(\frac{2}{b-a}\xi_t - \frac{b+a}{b-a}\Big)\Big] &= \mathbb{E}^{\xi}\Big[\mathbb{1}_{\{t<T_{[a,b]}\}} h\Big(\frac{2}{b-a}\big(\xi_t - \frac{b+a}{2}\big)\Big)\Big] \\
&= \mathbb{E}^{x-\frac{b+a}{2}}\Big[\mathbb{1}_{\{t<T_{[-\frac{b-a}{2},\frac{b-a}{2}]}\}} h\Big(\frac{2}{b-a}\xi_t\Big)\Big] \\
&= \mathbb{E}^{\frac{2}{b-a}(x-\frac{b+a}{2})}\Big[\mathbb{1}_{\{(\frac{b-a}{2})^{-\alpha}t<T_{[-1,1]}\}} h\Big(\xi_{(\frac{b-a}{2})^{-\alpha}t}\Big)\Big] \\
&= h\Big(\frac{2}{b-a}x - \frac{b+a}{b-a}\Big).
\end{aligned}$$

As a consequence we will focus for the rest of the chapter on the case $[a,b] = [-1,1]$. As a prelude to our first theorem, let us introduce the function

$$\psi_{\alpha\rho}(z) = (z-1)^{\alpha\hat{\rho}-1}(z+1)^{\alpha\rho-1}, \quad z > 1,$$

and the analogous expression $\psi_{\alpha\hat{\rho}}$, for which ρ is replaced with $\hat{\rho} := 1-\rho$. The first result identifies positive invariant functions for the stable processes killed on entering $[-1,1]$.

Theorem 5.1.1. *Let ξ be a two-side stable process with $\alpha \in (0,2)$, then*

$$h(x) := \begin{cases} \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat{\rho})} \int\limits_1^x \psi_{\alpha\rho}(z)\,\mathrm{d}z & \text{if } x > 1 \\ \frac{\Gamma(1-\alpha\hat{\rho})}{\Gamma(\alpha\rho)} \int\limits_1^{|x|} \psi_{\alpha\hat{\rho}}(z)\,\mathrm{d}z & \text{if } x < -1 \end{cases},$$

is a invariant function for $(p_t^{[-1,1]})_{t\geq 0}$.

In the case $\alpha = 1$ we can write the above invariant function in explicit detail:

$$h(x) = \int_1^{|x|} (z^2-1)^{-\frac{1}{2}}\,\mathrm{d}z = \log\left(|x| + (x^2-1)^{\frac{1}{2}}\right), \qquad |x| > 1. \tag{5.1}$$

The next result addresses our main motivation, namely to give a precise meaning to conditioning the stable processes to avoid an interval and to characterise the resulting process. Combining the invariant functions with classical results of Blumenthal et al. [10] and Port [48] we can prove that the usual conditioning procedure works and at the same time identify the conditioned processes as h-transforms of the killed processes with the invariant functions from Theorem 5.1.1.

Theorem 5.1.2. *Let ξ be a two-side stable process with $\alpha \in (0,2)$, then*

$$\lim_{s\to\infty} \mathbb{P}^x(\Lambda \,|\, t+s < T_{[-1,1]}) = \mathbb{P}^x_\updownarrow(\Lambda) := \mathbb{E}^x\left[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{[-1,1]}\}} \frac{h(\xi_t)}{h(x)}\right]$$

for $t \geq 0$, $x \notin [-1,1]$ and $\Lambda \in \mathcal{F}_t$.

The reader will note that conditioning to avoid an interval is much simpler when $\alpha < 1$ thanks to transience of the stable process. Indeed, the conditioning is a conditioning on a positive probability event. Moreover, the probability is exactly proportional to the invariant function h from Theorem 5.1.1.

As part of our characterisation of the conditioned process we show that it is transient. This is in analogy to Lévy processes conditioned to be positive which almost surely drift to infinity, see Chaumont and Doney [16].

Theorem 5.1.3. *Let ξ be stable process with both positive and negative jumps with $\alpha \in (0,2)$, then the conditioned process is transient in the sense that*

$$\mathbb{P}^x_\updownarrow\Big(\int_0^\infty \mathbb{1}_{\{\xi_t\in K\}}\,\mathrm{d}t < +\infty\Big) = 1, \qquad |x| > 1,$$

for all compact subsets K of $\mathbb{R}\backslash[-1,1]$.

Another question which we answered in Chapter 4 for Lévy processes with finite variance was the question of drifting and oscillating. In the case of a stable processes with self-similarity index $\alpha < 1$ we can answer this question. The conditioned process oscillates, in the sense that

$$\limsup_{t\to\infty} \xi_t = -\liminf_{t\to\infty} \xi_t = \infty$$

almost surely under $\mathbb{P}^x_{\updownarrow}$. This can be seen in the proofs because when $\alpha < 1$ the conditioning procedure is just a conditioning on a positive probability event, i.e. we restrict the probability measure to the paths which do not hit the interval. Since a stable process oscillates also the paths on which we restricted oscillate and hence, the conditioned process oscillates.

5.2 Sketch of proofs

The methods to prove our two main theorems differ for the three parameter regimes $\alpha \in (0,1)$, $\alpha = 1$ and $\alpha \in (1,2)$, where different background machinery will be employed.

Techniques for $\alpha \in (0,1)$: Transience for $\alpha < 1$ implies that $\{T_{[-1,1]} = +\infty\}$ has strictly positive probability, irrespective of the point of issue. We apply a recent formula from Kyprianou et al. [41], which gives the the law of the point of closest reach of 0 of the stable process. From the law of the point of closest reach we can precisely compute h which, up to a multiplicative constant, is equal to $\mathbb{P}^x(T_{[-1,1]} = +\infty)$, $x \in \mathbb{R}\backslash[-1,1]$. Since here we condition on a positive probability event the limit theorem is simple.

Techniques for $\alpha = 1$: We employ the fact that stable processes are self-similar Markov processes and self-similar Markov processes can be represented as time-space transformation of Markov additive processes (MAPs) via the Lamperti-Kiu representation, see Section 2.5.

In the appendix of Dereich et al. [21] the authors analysed questions on fluctuation theory for MAPs. In particular, analogously to Lévy processes, it was proved that the potential function U^- of the ascending ladder height process (see Section 5.3.1 below) of the so-called dual process to the MAP is invariant for the MAP killed on entering the negative half-line when it oscillates or drifts to $+\infty$. This is the generalisation of the Lévy process conditioned to stay positive, see Section 2.6.

Since stable processes are ssMps, the Lamperti-Kiu transform is applicable to the stable process ξ. The underlying MAP was characterised in Kyprianou [35] and Kyprianou et al. [41]. For $\alpha = 1$ it holds that $T_0 = +\infty$ almost surely (which is a consequence of the fact that all points are polar for the Cauchy process), hence, the underlying MAP oscillates or drifts to $+\infty$ (in fact the former is the case). In [41] the authors were able to calculate explicit densities for the aforementioned invariant function of the MAP underlying the Cauchy process. Since the Lamperti-Kiu transform tells us that killing the MAP on the negative half-line is equivalent to killing the stable process in $[-1,1]$, an appropriate spatial transform of this MAP turns out to be the the key to prove Theorem 5.1.1 in the Cauchy setting.

Techniques for $\alpha \in (1,2)$: For $\alpha > 1$ the stable process visits points almost surely, irrespective of its point of issue. We consider the stable process killed on hitting 0. From Section 2.6 we know that

$$e(x) := \begin{cases} \sin(\pi\alpha\hat{\rho})\,|x|^{\alpha-1} & \text{if } x > 0 \\ \sin(\pi\alpha\rho)\,|x|^{\alpha-1} & \text{if } x < 0 \end{cases},$$

is invariant for the stable process killed on hitting 0. Moreover, its corresponding Doob-h-transform with $e(x)$, $x \neq 0$, corresponds to the stable process conditioned to avoid 0 in a sense that also conforms to the general notion highlighted in (1.1). We will show that $h^\circ(x) := e(x)^{-1}h(x), |x| > 1$, with h defined in Theorem 5.1.1, is invariant for the stable process conditioned to avoid the origin and killed on entering $[-1,1]$. In that case, by the compounding effect of Doob-h-transforms, it must be the case that $h(x) = e(x)h^\circ(x)$ is invariant for ξ killed on entering $[-1,1]$. The tool we use to show invariance of $h^\circ(x)$ for the stable process conditioned to avoid 0 and killed on entering $[-1,1]$ is the Riesz–Bogdan–Żak transform (Theorem 2.6.1). Let therefore

$$\mathbb{P}^x_\circ(\Lambda) := \mathbb{E}^x\Big[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{\{0\}}\}}\frac{e(X_t)}{e(x)}\Big], \quad \Lambda \in \mathcal{F}_t, t \geq 0,$$

be the law of the stable process conditioned to avoid 0. We will use the Riesz-Bogdan-Żak-transform to show that $\mathbb{P}^x_\circ(T_{[-1,1]} = +\infty) > 0$ and to calculate this probability explicitly using a very recent formula for the distribution of the point of furthest reach of a stable process prior to hitting the origin.

5.3 Proofs

5.3.1 Invariance of h

The proofs for the cases $\alpha \in (0,1)$, $\alpha = 1$ and $\alpha \in (1,2)$ are completely different in nature but all rely on recent new explicit formulas for stable processes obtained through the Lamperti-Kiu representation.

The case $\alpha \in (0,1)$

Since ξ is transient, $\mathbb{P}^x(T_{[-1,1]} = \infty) > 0$ for all $x \notin [-1,1]$ and the function

$$g(x) = \mathbb{P}^x(T_{[-1,1]} = \infty), \quad x \in \mathbb{R}\backslash[-1,1],$$

is strictly positive. The following standard argument shows that g is a invariant function for the process killed on entering $[-1,1]$:

$$\begin{aligned}
\mathbb{E}^x\big[\mathbb{1}_{\{t<T_{[-1,1]}\}}g(\xi_t)\big] &= \mathbb{E}^x\big[\mathbb{1}_{\{t<T_{[-1,1]}\}}\mathbb{P}^{\xi_t}(T_{[-1,1]} = \infty)\big] \\
&= \lim_{s\to\infty}\mathbb{E}^x\big[\mathbb{1}_{\{t<T_{[-1,1]}\}}\mathbb{P}^{\xi_t}(T_{[-1,1]} > s)\big] \\
&= \lim_{s\to\infty}\mathbb{P}^x(T_{[-1,1]} > s+t) \\
&= \mathbb{P}^\xi(T_{[-1,1]} = \infty) \\
&= g(x),
\end{aligned}$$

where we used dominated convergence in the second equation and the Markov property in the third. It remains to show that $h(x)$ equals $\mathbb{P}^x(T_{[-1,1]} = \infty)$ up to some multiplicative constant. Proposition 1.1 of Kyprianou et al. [41] gives us the distribution of the point of closest reach to 0 of the stable process from which the probability of missing $[-1,1]$ can be computed readily. Define $\underline{m}$ as the time such that $|\xi_t| \geq |\xi_{\underline{m}}|$ for all $t \geq 0$ and let $x > 1$, then using the aforesaid result in [41],

$$\begin{aligned}
\mathbb{P}^x(T_{[-1,1]} = \infty) &= \mathbb{P}^x(\xi_{\underline{m}} > 1) + \mathbb{P}^x(\xi_{\underline{m}} < -1) \\
&= \frac{\Gamma(1-\alpha\rho)}{\Gamma(1-\alpha)\Gamma(\alpha\hat{\rho})}\bigg(\int_1^x \frac{x+z}{(2z)^\alpha}(x-z)^{\alpha\hat{\rho}-1}(x+z)^{\alpha\rho-1}\,\mathrm{d}z
\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_{-x}^{-1} \frac{x+z}{(-2z)^\alpha}(x+z)^{\alpha\hat\rho-1}(x-z)^{\alpha\rho-1}\,\mathrm{d}z\Big) \\
&= \frac{2\Gamma(1-\alpha\rho)}{2^\alpha\Gamma(1-\alpha)\Gamma(\alpha\hat\rho)} x\int_1^x \frac{1}{z^2}\Big(\frac{\xi}{z}-1\Big)^{\alpha\hat\rho-1}\Big(\frac{\xi}{z}+1\Big)^{\alpha\rho-1}\,\mathrm{d}z \\
&= \frac{2^{1-\alpha}\Gamma(1-\alpha\rho)}{\Gamma(1-\alpha)\Gamma(\alpha\hat\rho)} \int_1^x (u-1)^{\alpha\hat\rho-1}(u+1)^{\alpha\rho-1}\,\mathrm{d}u \\
&= \frac{2^{1-\alpha}\Gamma(1-\alpha\rho)}{\Gamma(1-\alpha)\Gamma(\alpha\hat\rho)} \int_1^x \psi_{\alpha\rho}(u)\,\mathrm{d}u \\
&= \frac{2^{1-\alpha}}{\Gamma(1-\alpha)} h(x),
\end{aligned}
$$

where in the third equality we have substituted $u = x/z$. If $x < -1$, we apply duality to deduce

$$\mathbb{P}^x(T_{[-1,1]} = \infty) = \hat{\mathbb{P}}^{-x}(T_{[-1,1]} = \infty),$$

from which an analogous calculation for $x < -1$ yields the claim.

The case $\alpha = 1$

Let $(\mathbf{P}^{x,i})_{x\in\mathbb{R},i\in\{\pm1\}}$ be the family of probability measures on the space of $\mathbb{R}\times\{\pm1\}$-valued càdlàg paths under which the canonical process (M,J) has the distribution of the Markov additive process (MAP) which underlies the stable process (seen as an ssMp) via the Lamperti-Kiu transform. More precisely, this means that, under $\mathbf{P}^{\log|x|,\mathrm{sgn}(x)}$, the transformation $(J_{\varphi_t}\exp(M_{\varphi_t}))_{t\geq 0}$ is a ssMp with law $\mathbb{P}^x$, i.e. the stable process.

We will need to introduce some terminology from Dereich et al. [21] in order to talk about the ladder height processes of (M,J). To this end, let $Y_t = M_t - \underline{M}_t$, $t \geq 0$, where $\underline{M}_t = \inf_{s\leq t} M_s$. Following ideas that are well known from the theory of Lévy processes, it is straightforward to show that, as a pair, the process (Y,J) is a strong Markov process. It was shown in the Appendix of Dereich et al. [21] that there exists a local time of (Y,J) in the set $\{0\}\times\{-1,1\}$, say $L := (L_t)_{t\geq 0}$. Moreover, the process $(L^{-1},H^-,J^-) := (L_t^{-1},H_t^-,J_t^-)_{t\geq 0}$ is a (possibly killed) Markov additive bivariate subordinator (meaning that it is a possibly killed MAP for which the components $(L_t^{-1})_{t\geq 0}$ and $(H_t^-)_{t\geq 0}$ are increasing), where

$$H_t^- := M_{L_t^{-1}} \text{ and } J_t^- := J_{L_t^{-1}}, \qquad \text{if } L_t^{-1} < \infty,$$

and $H_t^- := \Delta$ and $J_t^- := \Delta$ otherwise, for some cemetery state Δ.

The process (H^-,J^-) is called the descending ladder MAP. Next define

$$U_{i,j}^-(x) = \mathbf{E}^{0,i}\Big[\int_0^\infty \mathbb{1}_{\{H_t^-\leq x, J_t^-=j\}}\,\mathrm{d}t\Big]$$

which we call potential function of (H^-,J^-) and

$$U_i^-(x) := U_{i,-1}^-(x) + U_{i,1}^-(x) = \mathbf{E}^{0,i}\Big[\int_0^\infty \mathbb{1}_{\{H_t^-\leq x\}}\,\mathrm{d}t\Big].$$

In the case $\alpha = 1$ the stable process ξ does not hit points, i.e. $T_0 = +\infty$ almost surely. It follows from the Lamperti-Kiu transfrom that the underlying MAP oscillates or drifts to $+\infty$ and in this case the function

$$(x,i) \mapsto U_i^-(x)$$

is invariant for the MAP killed on entering the negative half-line, i.e.

$$\mathbf{E}^{x,i}\left[\mathbb{1}_{\{t<\tau_{(-\infty,0]}\}}U^-_{J_t}(M_t)\right]=U^-_i(x)$$

for all $x>0$, where $\tau_{(-\infty,0]}:=\inf\{t\geq 0: M_t\leq 0\}$ (see Theorem 29 of Dereich et al. [21]). Using this we see

$$\begin{aligned}
&\mathbb{E}^x\left[\mathbb{1}_{\{t<T_{[-1,1]}\}}U^-_{\operatorname{sgn}(\xi_t)}(\log|\xi_t|)\right]\\
&=\mathbf{E}^{\log|x|,\operatorname{sgn}(x)}\left[\mathbb{1}_{\{\varphi_t<\tau_{(-\infty,0]}\}}U^-_{\operatorname{sgn}(J_{\varphi_t}\exp(M_{\varphi_t}))}(\log|J_{\varphi_t}\exp(M_{\varphi_t})|)\right]\\
&=\mathbf{E}^{\log|x|,\operatorname{sgn}(x)}\left[\mathbb{1}_{\{\varphi_t<\tau_{(-\infty,0]}\}}U^-_{\operatorname{sgn}(J_{\varphi_t})}(M_{\varphi_t})\right].
\end{aligned}$$

Denote the natural enlargement of the filtration induced by (M,J) by $(\mathcal{G}_t)_{t\geq 0}$. An important fact that we will need is that for all $t\geq 0$ the time-change φ_t is a stopping-time with respect to $(\mathcal{G}_v)_{v\geq 0}$ which is almost surely finite with respect to $\mathbf{P}^{x,i}$, for all $x\in\mathbb{R}$ and $i\in\{-1,1\}$. To prove that we first need the following lemma.

Lemma 5.3.1. *With the notation above and $t\geq 0$ the following holds with $a\wedge b=\min\{a,b\}$:*

(i) $\{\varphi_t<\tau_{(-\infty,0]}\}=\{\varphi_t<\tau_{(-\infty,0]}\}\cap\{\varphi_t\wedge t=\varphi_t\}$.

(ii) $\{\varphi_t\wedge t<\tau_{(-\infty,0]}\}=\{\varphi_t<\tau_{(-\infty,0]}\}\cap\{\varphi_t\wedge t=\varphi_t\}$.

Proof. For $t=0$ the claims are trivial, so we focus on $t>0$. Before we start to prove the two claims note that $t<\tau_{(-\infty,0]}$ implies

$$\int_0^t\exp(\alpha M_u)\,\mathrm{d}u>\int_0^t\exp(0)\,\mathrm{d}u=t.$$

Hence, by the definition of φ_t as an inverse function we find

$$\{t<\tau_{(-\infty,0]}\}\subseteq\{\varphi_t\leq t\}. \tag{5.2}$$

(i) Of course it is sufficient to show $\{\varphi_t<\tau_{(-\infty,0]}\}\subseteq\{\varphi_t\leq t\}$ because $\{\varphi_t\leq t\}=\{\varphi_t\wedge t=\varphi_t\}$. Using (5.2) this follows from

$$\begin{aligned}
\{\varphi_t<\tau_{(-\infty,0]}\}&=\{\varphi_t<\tau_{(-\infty,0]}\leq t\}\cup\{\varphi_t<\tau_{(-\infty,0]},t<\tau_{(-\infty,0]}\}\\
&\subseteq\{\varphi_t\leq t\}\cup\{t<\tau_{(-\infty,0]}\}\\
&\subseteq\{\varphi_t\leq t\}.
\end{aligned}$$

(ii) The right-hand side is a subset of the left-hand side since

$$\{\varphi_t<\tau_{(-\infty,0]}\}\subseteq\{\varphi_t\wedge t<\tau_{(-\infty,0]}\}.$$

To show the other direction we decompose the left-hand side as follows

$$\{\varphi_t\wedge t<\tau_{(-\infty,0]}\}=\{\varphi_t<\tau_{(-\infty,0]}\}\cup\{t<\tau_{(-\infty,0]}\}$$

and show that both parts of this union are subsets of $\{\varphi_t<\tau_{(-\infty,0]}\}\cap\{\varphi_t\wedge t=\varphi_t\}$. The first part is just (i). With (5.2) we see furthermore that

$$\begin{aligned}
\{t<\tau_{(-\infty,0]}\}&=\{t<\tau_{(-\infty,0]}\}\cap\{\varphi_t\leq t\}\\
&\subseteq\{\varphi_t<\tau_{(-\infty,0]}\}\cap\{\varphi_t\leq t\}
\end{aligned}$$

which shows that the second part is a subset of $\{\varphi_t<\tau_{(-\infty,0]}\}\cap\{\varphi_t\wedge t=\varphi_t\}$, too. $\square$

Taking account of Lemma 5.3.1, we can work with the strong Markov property to show the following.

Lemma 5.3.2. *For all $t \geq 0$ the time-change φ_t is a stopping-time with respect to $(\mathcal{G}_v)_{v\geq 0}$.*

Proof. Reminding the definition of φ_t and using that all paths are càdlàg, it holds that

$$\begin{aligned}\{\varphi_t \leq v\} &= \Big\{\exists s \leq v : \int_0^s \exp(M_u)\,\mathrm{d}u > t\Big\} \\ &= \Big\{\exists s \in [0,v]\cap\mathbb{Q} : \int_0^s \exp(M_u)\,\mathrm{d}u > t\Big\} \\ &= \bigcup_{s\in[0,v]\cap\mathbb{Q}} \Big\{ \int_0^s \exp(M_u)\,\mathrm{d}u > t,\ J_s \in \{-1,1\} \Big\}.\end{aligned} \tag{5.3}$$

Moreover, the integral can be approximated by $\int_0^s \exp(M_u)\,\mathrm{d}u = \lim_{n\to\infty} \frac{s}{2^n}\sum_{k=0}^{2^n-1}\exp(M_{\frac{sk}{2^n}})$. With this we can obtain that

$$\omega \mapsto \Big(\int_0^s \exp(M_u(\omega))\,\mathrm{d}u, J_s\Big)$$

is $\mathcal{G}_s$-measurable as a pointwise limit of $\mathcal{G}_s$-measurable functions. Combining with (5.3) implies $\{\varphi_t \leq v\} \in \mathcal{G}_v$. □

Now we go on with the proof of Theorem 5.1.1. Applying Lemma 5.3.1 it follows that

$$\begin{aligned}&\mathbf{E}^{\log|x|,\mathrm{sgn}(x)}\Big[\mathbb{1}_{\{\varphi_t<\tau_{(-\infty,0]}\}}U^-_{J_{\varphi_t}}(M_{\varphi_t})\Big] \\ &= \mathbf{E}^{\log|x|,\mathrm{sgn}(x)}\Big[\mathbb{1}_{\{\varphi_t<\tau_{(-\infty,0]}\}\cap\{\varphi_t\wedge t=\varphi_t\}}U^-_{J_{\varphi_t}}(M_{\varphi_t})\Big] \\ &= \mathbf{E}^{\log|x|,\mathrm{sgn}(x)}\Big[\mathbb{1}_{\{\varphi_t<\tau_{(-\infty,0]}\}\cap\{\varphi_t\wedge t=\varphi_t\}}U^-_{J_{\varphi_t\wedge t}}(M_{\varphi_t\wedge t})\Big] \\ &= \mathbf{E}^{\log|x|,\mathrm{sgn}(x)}\Big[\mathbb{1}_{\{\varphi_t\wedge t<\tau_{(-\infty,0]}\}}U^-_{J_{\varphi_t\wedge t}}(M_{\varphi_t\wedge t})\Big].\end{aligned}$$

Thanks to the Markov property, we have that invariance of $(x,i)\mapsto U^-_i(x)$ for the MAP killed on entering $(-\infty,0]$ is equivalent to $(\mathbb{1}_{\{t<\tau_{(-\infty,0]}\}}U^-_{J_t}(M_t))_{t\geq 0}$ being a $\mathbf{P}^{\log|x|,\mathrm{sgn}(x)}$-martingale with respect to $(\mathcal{G}_t)_{t\geq 0}$. Applying the optional stopping theorem to the bounded stopping times $\varphi_t\wedge t$ we deduce that the last expression equals $U^-_{\mathrm{sgn}(x)}(\log|x|)$. Hence, $U^-_{\mathrm{sgn}(x)}(\log|x|)$, $|x|>1$, is a invariant function for the stable process killed on entering $[-1,1]$.

To finish the proof of Theorem 5.1.1 we have to show that

$$h(x) = U^-_{\mathrm{sgn}(x)}(\log|x|), \qquad x\in\mathbb{R}\backslash[-1,1],$$

up to a multiplicative constant, where h is the explicit function from Theorem 5.1.1. Kyprianou et al. [41], Corollary 1.6, found explicit densities for $U^-_i(\mathrm{d}x)$. Using these results we have, for $x>1$, that

$$\begin{aligned}U^-_{\mathrm{sgn}(x)}(\log|x|) &= \int_0^{\log x}(1-\mathrm{e}^{-z})^{-\frac{1}{2}}(1+\mathrm{e}^{-z})^{\frac{1}{2}} + (1-\mathrm{e}^{-z})^{\frac{1}{2}}(1+\mathrm{e}^{-z})^{-\frac{1}{2}}\,\mathrm{d}z \\ &= \int_0^{\log x}(1-\mathrm{e}^{-z})^{-\frac{1}{2}}(1+\mathrm{e}^{-z})^{-\frac{1}{2}}((1+\mathrm{e}^{-z})+(1-\mathrm{e}^{-z}))\,\mathrm{d}z\end{aligned}$$

$$
\begin{aligned}
&= 2\int_0^{\log x} (1-\mathrm{e}^{-2z})^{-\frac{1}{2}}\,\mathrm{d}z \\
&= 2\int_1^{x} \frac{1}{z}(1-z^{-2})^{-\frac{1}{2}}\,\mathrm{d}z \\
&= 2\int_1^{x} (u^2-1)^{-\frac{1}{2}}\,\mathrm{d}u \\
&= 2h(x).
\end{aligned}
$$

By symmetry the claim for $x < -1$ follows analogously.

The case $\alpha \in (1,2)$

The idea of the argument is as follows. The multiplication of invariant functions corresponding to the concatenation of h-transforms gives a new invariant function. In our setting, recall from Section 5.2 that

$$
e(x) := \begin{cases} \sin(\pi\alpha\hat{\rho})\,|x|^{\alpha-1} & \text{if } x > 0 \\ \sin(\pi\alpha\rho)\,|x|^{\alpha-1} & \text{if } x < 0 \end{cases},
$$

is invariant for the process killed on hitting the origin and the h-transformed process delivers the Markov probabilities $\mathbb{P}^x_\circ$, $x \neq 0$. We will show that $h^\circ(x) := e(x)^{-1}h(x)$, $x \in \mathbb{R}\backslash[-1,1]$, with h defined in Theorem 5.1.1, is the probability of avoiding $[-1,1]$ for $\mathbb{P}^x_\circ$ ($\mathbb{P}^x_\circ$ is transient) and as a consequence is invariant for $\mathbb{P}^x_\circ$ killed in $[-1,1]$:

$$
\mathbb{P}^x \xrightarrow{\text{killing at 0, } h\text{-transform with } |x|^{\alpha-1}} \mathbb{P}^x_\circ \xrightarrow{\text{killing in [-1,1], } h\text{-transform with } |x|^{1-\alpha}h(x)} \mathbb{P}^x_\updownarrow .
$$

From this idea it turns out that $h(x) = e(x)h^\circ(x)$ is invariant for $\mathbb{P}^x$ killed on entering $[-1,1]$. Later we will prove that conditioning $\mathbb{P}^x$ to avoid $[-1,1]$ is nothing but conditioning $\mathbb{P}^x$ to avoid zero and then to condition $\mathbb{P}^x_\circ$ to avoid $[-1,1]$.

Remark 5.3.3. *Our argument resonates with the work of Hirano [29] who looked at the conditioning of a Lévy process to stay positive. Under the assumption of a positive Cramér number, Hirano conditioned a Lévy process that drifts to $-\infty$ to stay positive by first Esscher transforming, which is equivalent to conditioning to 'drift towards $+\infty$', and then conditioning the Esscher transform to stay positive.*

In the following lemma the Riesz–Bogdan–Żak transform is used to identify the invariant function for $\mathbb{P}^x_\circ$ killed in $[-1,1]$.

Lemma 5.3.4. *Define $h^\circ(x) = e(x)^{-1}h(x)$, then*

(i) $\mathbb{P}^x_\circ(T_{[-1,1]} = +\infty) = \frac{\pi(\alpha-1)}{\Gamma(1-\alpha\hat{\rho})\Gamma(1-\alpha\rho)}h^\circ(x)$ *for all* $x \notin [-1,1]$,

(ii) the function h° is invariant for $(p_t^{\circ,[-1,1]})_{t\geq 0}$, where

$$
p_t^{\circ,[-1,1]}f(x) := \mathbb{E}^x_\circ\big[\mathbb{1}_{\{t<T_{[-1,1]}\}}f(\xi_t)\big].
$$

Proof. (i) Let $\underline{m}$ be the $[0,\infty)$-valued time such that $|\xi_{\underline{m}}| \leq |\xi_t|$ for all $t \geq 0$ (point of closest reach) and $\overline{m}$ the time such that $|\xi_{\overline{m}}| \geq |\xi_t|$ for all $t \leq T_0$ (point of furthest reach).

Then we see for $x \notin [-1,1]$ using the Riesz–Bogdan–Żak transform in the second equation:

$$\begin{aligned}\mathbb{P}^x_\circ(T_{[-1,1]} = +\infty) &= \mathbb{P}^x_\circ(|\xi_{\underline{m}}| > 1)\\ &= \hat{\mathbb{P}}^{\frac{1}{x}}\Big(|\frac{1}{\xi_{\overline{m}}}| > 1\Big)\\ &= \hat{\mathbb{P}}^{\frac{1}{x}}(|\xi_{\overline{m}}| < 1)\\ &= \hat{\mathbb{P}}^{\frac{1}{x}}\Big(\xi_{\overline{m}} \in \big[\frac{1}{|x|}, 1)\Big) + \hat{\mathbb{P}}^{\frac{1}{x}}\Big(\xi_{\overline{m}} \in \big(-1, -\frac{1}{|x|}\big]\Big).\end{aligned} \tag{5.4}$$

Now we use Proposition 1.2 of Kyprianou et al. [41] which gives an explicit expression for the distribution of $\xi_{\overline{m}}$ under $\hat{\mathbb{P}}^{\frac{1}{x}}$. We consider only the case $x > 1$, the case $x < -1$ is similar with ρ replaced by $\hat{\rho}$ and x replaced by $|x|$. We start to compute the first summand in (5.4) as

$$\begin{aligned}&\hat{\mathbb{P}}^{\frac{1}{x}}\Big(\xi_{\overline{m}} \in \big[\frac{1}{\xi}, 1)\Big)\\ &= \frac{\alpha-1}{2}\int_{\frac{1}{x}}^{1} z^{-\alpha}\Big[\big(z+\frac{1}{x}\big)^{\alpha\rho}\big(z-\frac{1}{x}\big)^{\alpha\hat{\rho}-1} - (\alpha-1)\frac{1}{x^{\alpha-1}}\int_1^{zx}\psi_{\alpha\rho}(v)\,\mathrm{d}v\Big]\,\mathrm{d}z\\ &= \frac{\alpha-1}{2}\int_1^x \frac{1}{x}\Big(\frac{x}{z}\Big)^{\alpha}\Big[\big(\frac{z}{x}+\frac{1}{x}\big)^{\alpha\rho}\big(\frac{z}{x}-\frac{1}{x}\big)^{\alpha\hat{\rho}-1} - (\alpha-1)x^{1-\alpha}\int_1^{z}\psi_{\alpha\rho}(v)\,\mathrm{d}v\Big]\,\mathrm{d}z\\ &= \frac{\alpha-1}{2}\int_1^x z^{-\alpha}\Big[(z+1)^{\alpha\rho}(z-1)^{\alpha\hat{\rho}-1} - (\alpha-1)\int_1^{z}\psi_{\alpha\rho}(v)\,\mathrm{d}v\Big]\,\mathrm{d}z\\ &= \frac{\alpha-1}{2}\int_1^x u^{-\alpha}\Big[(u+1)\psi_{\alpha\rho}(u) - (\alpha-1)\int_1^{u}\psi_{\alpha\rho}(v)\,\mathrm{d}v\Big]\,\mathrm{d}u,\end{aligned}$$

where for the third equality we have have made the change variable $z = u/x$. We can similarly compute the second summand in (5.4) as

$$\begin{aligned}&\hat{\mathbb{P}}^{\frac{1}{x}}\Big(\xi_{\overline{m}} \in \big(-1, -\frac{1}{x}\big]\Big)\\ &= \frac{\alpha-1}{2}\int_{-1}^{-\frac{1}{x}} (-z)^{-\alpha}\Big[\big(-z-\frac{1}{x}\big)^{\alpha\hat{\rho}}\big(-z+\frac{1}{x}\big)^{\alpha\rho-1} - (\alpha-1)\frac{1}{x^{\alpha-1}}\int_1^{-zx}\psi_{\alpha\rho}(v)\,\mathrm{d}v\Big]\,\mathrm{d}z\\ &= \frac{\alpha-1}{2}\int_1^x \frac{1}{x}\Big(\frac{x}{z}\Big)^{\alpha}\Big[\big(\frac{z}{x}-\frac{1}{x}\big)^{\alpha\hat{\rho}}\big(\frac{z}{x}+\frac{1}{x}\big)^{\alpha\rho-1} - (\alpha-1)x^{1-\alpha}\int_1^{z}\psi_{\alpha\rho}(v)\,\mathrm{d}v\Big]\,\mathrm{d}z\\ &= \frac{\alpha-1}{2}\int_1^x z^{-\alpha}\Big[(z-1)^{\alpha\hat{\rho}}(z+1)^{\alpha\rho-1} - (\alpha-1)\int_1^{z}\psi_{\alpha\rho}(v)\,\mathrm{d}v\Big]\,\mathrm{d}z\\ &= \frac{\alpha-1}{2}\int_1^x u^{-\alpha}\Big[(u-1)\psi_{\alpha\rho}(u) - (\alpha-1)\int_1^{u}\psi_{\alpha\rho}(v)\,\mathrm{d}v\Big]\,\mathrm{d}u,\end{aligned}$$

and, adding the two terms together, it follows that

$$\mathbb{P}^x_\circ(T_{[-1,1]} = +\infty) = (\alpha-1)\int_1^x \Big[z^{1-\alpha}\psi_{\alpha\rho}(z) - (\alpha-1)z^{-\alpha}\int_1^z \psi_{\alpha\rho}(v)\,\mathrm{d}v\Big]\,\mathrm{d}z.$$

Integration by parts yields

$$\begin{aligned}\int_1^x \Big[z^{-\alpha}\int_1^z \psi_{\alpha\rho}(v)\,\mathrm{d}v\Big]\,\mathrm{d}z &= \Big[\frac{1}{1-\alpha}z^{1-\alpha}\int_1^z \psi_{\alpha\rho}(v)\,\mathrm{d}v\Big]_{z=1}^{x} - \int_1^x \frac{1}{1-\alpha}z^{1-\alpha}\psi_{\alpha\rho}(z)\,\mathrm{d}z\\ &= -\frac{1}{\alpha-1}x^{1-\alpha}\int_1^x \psi_{\alpha\rho}(v)\,\mathrm{d}v + \int_1^x \frac{1}{\alpha-1}z^{1-\alpha}\psi_{\alpha\rho}(z)\,\mathrm{d}z,\end{aligned}$$

hence,

$$\mathbb{P}_\circ^x(T_{[-1,1]} = +\infty) = (\alpha-1)x^{1-\alpha}\int_1^x \psi_{\alpha\rho}(v)\,\mathrm{d}v$$

for $x > 1$. On the other hand we have for $x > 1$

$$\begin{aligned} h^\circ(x) = e(x)^{-1}h(x) &= \frac{1}{\sin(\pi\alpha\hat{\rho})}\frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat{\rho})}x^{1-\alpha}\int_1^x \psi_{\alpha\rho}(v)\,\mathrm{d}v \\ &= \frac{1}{\pi}\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})\frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat{\rho})}x^{1-\alpha}\int_1^x \psi_{\alpha\rho}(v)\,\mathrm{d}v \\ &= \frac{1}{\pi}\Gamma(1-\alpha\hat{\rho})\Gamma(1-\alpha\rho)x^{1-\alpha}\int_1^x \psi_{\alpha\rho}(v)\,\mathrm{d}v. \end{aligned}$$

Hence we have

$$\mathbb{P}_\circ^x(T_{[-1,1]} = +\infty) = \frac{\pi(\alpha-1)}{\Gamma(1-\alpha\hat{\rho})\Gamma(1-\alpha\rho)}e(x)^{-1}h(x).$$

For $x < -1$ we can show analogously

$$\mathbb{P}_\circ^x(T_{[-1,1]} = +\infty) = (\alpha-1)x^{1-\alpha}\int_1^{|x|} \psi_{\alpha\hat{\rho}}(v)\,\mathrm{d}v,$$

and,

$$h^\circ(x) = e(x)^{-1}h(x) = \frac{1}{\pi}\Gamma(1-\alpha\hat{\rho})\Gamma(1-\alpha\rho)x^{1-\alpha}\int_1^{|x|} \psi_{\alpha\hat{\rho}}(v)\,\mathrm{d}v,$$

which yields to the claim for $x < -1$.

(ii) Using (i) the argument is classical and essentially the same as the one given in Section 5.3.1. For the sake of brevity we leave the details to the reader. $\square$

With Lemma 5.3.4 in hand, it is now straight forward to verify the invariance of h using the definition of $\mathbb{P}_\circ^x$ as a change of measure. Indeed, we note that

$$\begin{aligned} \mathbb{E}^x\big[\mathbb{1}_{\{t<T_{[-1,1]}\}}h(\xi_t)\big] &= \mathbb{E}^x\big[\mathbb{1}_{\{t<T_{[-1,1]}\}}h^\circ(\xi_t)e(X_t)\big] \\ &= e(x)\mathbb{E}^{\circ,x}\big[\mathbb{1}_{\{t<T_{[-1,1]}\}}h^\circ(\xi_t)\big] \\ &= e(x)h^\circ(x) \\ &= h(x) \end{aligned}$$

for all $x \in \mathbb{R}\setminus[-1,1]$. Hence, h is invariant for the killed process.

5.3.2 Conditioning and h-transforms

To identify the conditioned processes as h-transforms we follow a classical argument based on the Markov property. The argument needs two ingredients. First is the existence of an asymptotic tail distribution of the kind

$$\lim_{s\to\infty} f(s)\,\mathbb{P}^x(s < T_{[-1,1]}) = h(x). \tag{5.5}$$

Second is the invariance h.

For $\alpha < 1$ the argument is straightforward with $f = 1$ and the explicit formula for $h(x) = \mathbb{P}^x(T_{[-1,1]} = \infty)$ already derived in Section 5.3.1. Much more interesting are the cases $\alpha = 1$ and $\alpha > 1$. In both cases, the asymptotic tail distributions are given in classical fluctuation theory results due to Blumenthal et al. [10] and Port [48]. Unfortunately, in both cases it is unclear if the limit is invariant (the limiting expression only implies h is excessive). To justify invariance we use the results from the previous section: for $\alpha = 1$ the asymptotic tail distribution of Blumenthal et al. is precisely given by (5.1) which we proved to be invariant using the Lamperti-Kiu representation. For $\alpha > 1$ the limit has a particular form that we can identify (using a recent result of Profeta and Simon [49] already given in Section 2.4) as our invariant function h from the previous section.

To summarize the approach, our results only give the invariance whereas the known fluctuation theory only gives the needed tail asymptotics. Combining both the limiting procedure from Theorem 5.1.2 can be performed and the limit is identified as h-transform with h from Theorem 5.1.1.

The case $\alpha \in (0, 1)$

We start with the simplest case $\alpha \in (0, 1)$ where we condition on a positive probability event. As seen in the proof of Theorem 5.1.1 the probability of this event is the invariant function up to a multiplicative constant. That is to say, with h defined as in Theorem 5.1.1, we have

$$h(x) = \frac{\Gamma(1-\alpha)}{2^{1-\alpha}} \mathbb{P}^x(T_{[-1,1]} = \infty), \quad |x| > 1.$$

It follows immediately that h is bounded. Hence, for $\Lambda \in \mathcal{F}_t$, we get with the Markov property and dominated convergence

$$\begin{aligned}
\lim_{s\to\infty} \mathbb{P}^x(\Lambda \,|\, t+s < T_{[-1,1]}) &= \lim_{s\to\infty} \frac{\mathbb{E}^x\left[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{[-1,1]}\}} \mathbb{P}^{\xi_t}(T_{[-1,1]} > s)\right]}{\mathbb{P}^x(T_{[-1,1]} > t+s)} \\
&= \frac{\mathbb{E}^x\left[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{[-1,1]}\}} \mathbb{P}^{\xi_t}(T_{[-1,1]} = \infty)\right]}{\mathbb{P}^x(T_{[-1,1]} = \infty)} \\
&= \mathbb{E}^x\left[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{[-1,1]}\}} \frac{h(\xi_t)}{h(x)}\right] \\
&= \mathbb{P}^x_\updownarrow(\Lambda).
\end{aligned}$$

The case $\alpha = 1$

According to Blumenthal et al. [10], Corollary 3, the tail asymptotics of first hitting times are

$$\lim_{s\to\infty} \mathbb{P}^x(s < T_{[-1,1]}) \log(s) = h(x) \tag{5.6}$$

with h from (5.1). In the previous section we proved that h is invariant.

Using Fatou's Lemma in the second equality and the strong Markov property in the third equality we get, for $\Lambda \in \mathcal{F}_t$,

$$\begin{aligned}
\mathbb{P}^x_\updownarrow(\Lambda) &= \frac{1}{h(x)} \mathbb{E}^x\left[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{[-1,1]}\}} \lim_{s\to\infty} \log(s) \mathbb{P}^{\xi_t}(s < T_{[-1,1]})\right] \\
&\leq \liminf_{s\to\infty} \frac{\log(s)}{h(x)} \mathbb{E}^x\left[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{[-1,1]}\}} \mathbb{P}^{\xi_t}(s < T_{[-1,1]})\right]
\end{aligned}$$

$$
\begin{aligned}
&= \liminf_{s\to\infty} \frac{\log(s)}{h(x)} \mathbb{P}^{\xi}(\Lambda, t+s < T_{[-1,1]}) \\
&= \liminf_{s\to\infty} \frac{\log(s)}{\log(t+s)\mathbb{P}^x(t+s < T_{[-1,1]})} \mathbb{P}^{\xi}(\Lambda, t+s < T_{[-1,1]}) \\
&= \liminf_{s\to\infty} \mathbb{P}^x(\Lambda \,|\, t+s < T_{[-1,1]}).
\end{aligned}
$$

Since $\Lambda^{\mathrm{C}} \in \mathcal{F}_t$ and we can apply the same calculation for Λ^{C} we also get

$$
\mathbb{P}^x_{\updownarrow}(\Lambda^{\mathrm{C}}) \leq \liminf_{s\to\infty} \mathbb{P}^x(\Lambda^{\mathrm{C}} \,|\, t+s < T_{[-1,1]}) = 1 - \limsup_{s\to\infty} \mathbb{P}^x(\Lambda \,|\, t+s < T_{[-1,1]}).
$$

At this point in the argument, it is important that we have already proved invariance of h, i.e. that $\mathbb{P}^x_{\updownarrow}$ is a probability measure. Hence, we can write $\mathbb{P}^x_{\updownarrow}(\Lambda^{\mathrm{C}}) = 1 - \mathbb{P}^x_{\updownarrow}(\Lambda)$. This leads us to

$$
\mathbb{P}^x_{\updownarrow}(\Lambda) \geq \limsup_{s\to\infty} \mathbb{P}^x(\Lambda \,|\, t+s < T_{[-1,1]})
$$

and combining both inequalities the claim follows.

The case $\alpha \in (1,2)$

According to Port [48] the tail asymptotics of first hitting times are

$$
\lim_{s\to\infty} s^{1-\frac{1}{\alpha}} \mathbb{P}^x(s < T_{[-1,1]}) = c_{\alpha\rho} \lim_{y\to\infty} u_{[-1,1]}(x,y), \tag{5.7}
$$

for some constant $c_{\alpha\rho} > 0$ and $u_{[-1,1]}(x,y)$ is the density of the potential of the stable process killed on entering $[-1,1]$. Profeta and Simon [49] derived the already in Section 2.4 mentioned explicit formulas for $u_{[-1,1]}(x,y)$, namely

$$
\begin{aligned}
u_{[-1,1]}(x,y) = c_\alpha (y-x)^{\alpha-1} \Big(&\int_1^{z(x,y)} \psi_{\alpha\rho}(v)\,\mathrm{d}v \\
&- (\alpha-1)(y-x)^{1-\alpha} \int_1^y \psi_{\alpha\hat{\rho}}(v)\,\mathrm{d}v \int_1^x \psi_{\alpha\rho}(v)\,\mathrm{d}v \Big),
\end{aligned} \tag{5.8}
$$

$1 < x < y$ and

$$
\begin{aligned}
u_{[-1,1]}(x,y) = c_\alpha \frac{\sin(\pi\alpha\rho)}{\sin(\pi\alpha\hat{\rho})} (y-x)^{\alpha-1} \Big(&\int_1^{z(x,y)} \psi_{\alpha\hat{\rho}}(v)\,\mathrm{d}v \\
&- (\alpha-1)(y-x)^{1-\alpha} \int_1^y \psi_{\alpha\hat{\rho}}(v)\,\mathrm{d}v \int_1^{|x|} \psi_{\alpha\hat{\rho}}(v)\,\mathrm{d}v \Big),
\end{aligned} \tag{5.9}
$$

$x < -1, 1 < y$. In the formulas we used the abbreviations $z(x,y) = |xy-1|/|y-x|$ and $c_\alpha = 2^{1-\alpha}/(\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho}))$.

In order to combine Port's asymptotic formula (5.7) with our Theorem 5.1.1 we need to send y to infinity in Profeta's and Simon's formula (5.8).

Lemma 5.3.5. *If h is defined as in Theorem 5.1.1, then*

$$
\lim_{y\to\infty} u_{[-1,1]}(x,y) = K_{\alpha\rho} h(x), \qquad x \notin [-1,1],
$$

where $K_{\alpha\rho} = \frac{2c_\alpha(1-\alpha\hat{\rho})\Gamma(\alpha\hat{\rho})}{\Gamma(1-\alpha\rho)} \int\limits_1^\infty \psi_{\alpha\hat{\rho}}(v) \frac{1}{v+1}\,\mathrm{d}v.$

Proof. We start with $x > 1$. First note that

$$\lim_{y\to\infty} \int_1^{z(x,y)} \psi_{\alpha\rho}(v)\,\mathrm{d}v = \int_1^{\xi} \psi_{\alpha\rho}(v)\,\mathrm{d}v.$$

On the other hand we see with l'Hopital's rule

$$\lim_{y\to\infty} (\alpha-1)(y-x)^{1-\alpha} \int_1^{y} \psi_{\alpha\hat{\rho}}(v)\,\mathrm{d}v = \lim_{y\to\infty} (y-x)^{2-\alpha}\psi_{\alpha\hat{\rho}}(y) = 1.$$

Hence, $u_{[-1,1]}(x,y)$ is a product of two functions, one tending to $+\infty$, the other tending to 0. Applying l'Hopital's rule again to the whole term we get

$$\begin{aligned}
&\lim_{y\to\infty} c_\alpha^{-1} u_{[-1,1]}(x,y)\\
&= \lim_{y\to\infty} \frac{1}{(1-\alpha)(y-x)^{-\alpha}}\Big[\frac{1-x^2}{(y-x)^2}\psi_{\alpha\rho}(z(x,y))\\
&\quad - (\alpha-1)\int_1^x \psi_{\alpha\rho}(v)\,\mathrm{d}v\Big((1-\alpha)(y-x)^{-\alpha}\int_1^y \psi_{\alpha\hat{\rho}}(v)\,\mathrm{d}v + \psi_{\alpha\hat{\rho}}(y)(y-x)^{1-\alpha}\Big)\Big]\\
&= \lim_{y\to\infty} \frac{(1-x^2)\psi_{\alpha\rho}(z(x,y))}{(1-\alpha)(y-x)^{2-\alpha}} + \int_1^x \psi_{\alpha\rho}(v)\,\mathrm{d}v\Big((1-\alpha)\int_1^y \psi_{\alpha\hat{\rho}}(v)\,\mathrm{d}v + \psi_{\alpha\hat{\rho}}(y)(y-x)\Big).
\end{aligned}$$

The first summand converges to 0 since $z(x,y)$ converges to x and $\alpha < 2$. So it remains to show that

$$\psi_{\alpha\hat{\rho}}(y)(y-x) - (\alpha-1)\int_1^y \psi_{\alpha\hat{\rho}}(v)\,\mathrm{d}v$$

converges to some positive constant which is independent of x. It is clear that both terms tend to $+\infty$ with order $\alpha-1$. We rewrite the term in the following way:

$$\begin{aligned}
&\psi_{\alpha\hat{\rho}}(y)(y-x) - (\alpha-1)\int_1^y \psi_{\alpha\hat{\rho}}(v)\,\mathrm{d}v\\
&= \psi_{\alpha\hat{\rho}}(y)(y-1) - \psi_{\alpha\hat{\rho}}(y)(x-1) - (\alpha-1)\int_1^y \psi_{\alpha\hat{\rho}}(v)\,\mathrm{d}v.
\end{aligned}$$

Of course $\psi_{\alpha\hat{\rho}}(y)(x-1)$ converges to 0. So it remains to show that

$$\psi_{\alpha\hat{\rho}}(y)(y-1) - (\alpha-1)\int_1^y \psi_{\alpha\hat{\rho}}(v)\,\mathrm{d}v$$

converges. Note that we can write

$$\psi'_{\alpha\hat{\rho}}(y) = \psi_{\alpha\hat{\rho}}(y)\big[(\alpha\rho-1)(v-1)^{-1} + (\alpha\hat{\rho}-1)(v+1)^{-1}\big],$$

and with this we see that

$$\begin{aligned}
&\psi_{\alpha\hat{\rho}}(y)(y-1) - (\alpha-1)\int_1^y \psi_{\alpha\hat{\rho}}(v)\,\mathrm{d}v\\
&= \int_1^y \big(\psi'_{\alpha\hat{\rho}}(v)(v-1) + \psi_{\alpha\hat{\rho}}(v)\big)\,\mathrm{d}v - (\alpha-1)\int_1^y \psi_{\alpha\hat{\rho}}(v)\,\mathrm{d}v\\
&= \int_1^y \psi_{\alpha\hat{\rho}}(v)\Big((\alpha\rho-1) + (\alpha\hat{\rho}-1)\frac{v-1}{v+1} + 1 - (\alpha-1)\Big)\,\mathrm{d}v
\end{aligned}$$

$$= 2(1-\alpha\hat{\rho})\int_1^y \psi_{\alpha\hat{\rho}}(v)\frac{1}{v+1}\,\mathrm{d}v.$$

Since $\psi_{\alpha\hat{\rho}}(v)/(v+1)$ behaves like $(v-1)^{\alpha\rho-1}$ for $v \searrow 0$ and like $v^{\alpha-3}$ for $v \to +\infty$, it follows that $\int_1^\infty \psi_{\alpha\hat{\rho}}(v)/(v+1)\,\mathrm{d}v \in (0,\infty)$ because $\alpha\rho \in (0,1)$ and $\alpha - 3 < -1$. Now it follows that

$$\begin{aligned}\lim_{y\to\infty} u_{[-1,1]}(x,y) &= c_\alpha(1-\alpha\hat{\rho})\int_1^\infty \psi_{\alpha\hat{\rho}}(v)\frac{1}{v+1}\,\mathrm{d}v\int_1^x \psi_{\alpha\rho}(v)\,\mathrm{d}v\\ &= \frac{2c_\alpha(1-\alpha\hat{\rho})\Gamma(\alpha\hat{\rho})}{\Gamma(1-\alpha\rho)}\int_1^\infty \psi_{\alpha\hat{\rho}}(v)\frac{1}{v+1}\,\mathrm{d}v\,h(x)\\ &= K_{\alpha\rho}h(x).\end{aligned}$$

For $x < -1$ we can use (5.9) to do a similar calculation which yields to

$$\begin{aligned}\lim_{y\to\infty} u_{[-1,1]}(x,y) &= c_\alpha\frac{\sin(\pi\alpha\rho)}{\sin(\pi\alpha\hat{\rho})}(1-\alpha\hat{\rho})\int_1^\infty \psi_{\alpha\hat{\rho}}(v)\frac{1}{v+1}\,\mathrm{d}v\int_1^{|x|} \psi_{\alpha\hat{\rho}}(v)\,\mathrm{d}v\\ &= c_\alpha\frac{\sin(\pi\alpha\rho)\Gamma(\alpha\rho)}{\sin(\pi\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})}(1-\alpha\hat{\rho})\int_1^\infty \psi_{\alpha\hat{\rho}}(v)\frac{1}{v+1}\,\mathrm{d}v\,h(x).\end{aligned}$$

Since $\frac{\sin(\pi\alpha\rho)\Gamma(\alpha\rho)}{\sin(\pi\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} = \frac{\Gamma(\alpha\hat{\rho})}{\Gamma(1-\alpha\rho)}$ it follows

$$\lim_{y\to\infty} u_{[-1,1]}(x,y) = K_{\alpha\rho}h(x)$$

also if $x < -1$. □

The proof of Theorem 5.1.2 in this regime of α can now be copied from Section 5.3.2 replacing $\log(s)$ by $s^{1-\frac{1}{\alpha}}$ in (5.6). The desired invariance of $\lim_{s\to\infty} s^{1-\frac{1}{\alpha}}\mathbb{P}^x(s < T_{[-1,1]})$, as a function of x, comes from Lemma 5.3.5 combined with Theorem 5.1.1.

5.3.3 Transience

Since h is invariant and hence, the h-transformed process is conservative it is sufficient to show that the potential of the h-transformed process

$$\mathbb{E}^x_\updownarrow\Big[\int_0^\infty \mathbb{1}_{\{\xi_t\in[-d,d]\}}\,\mathrm{d}t\Big]$$

is finite for all $d > 0$. To prove this note that

$$\begin{aligned}\mathbb{E}^x_\updownarrow\Big[\int_0^\infty \mathbb{1}_{\{\xi_t\in[-d,d]\}}\,\mathrm{d}t\Big] &= \int_0^\infty \mathbb{E}^x\Big[\mathbb{1}_{\{\xi_t\in[-d,d]\}}\mathbb{1}_{\{t<T_{[-1,1]}\}}\frac{h(\xi_t)}{h(x)}\Big]\,\mathrm{d}t\\ &\le \sup_{y\in[-d,d]\setminus[-1,1]}\frac{h(y)}{h(x)}\,\mathbb{E}^x\Big[\int_0^{T_{[-1,1]}} \mathbb{1}_{\{\xi_t\in[-d,d]\}}\,\mathrm{d}t\Big]\\ &= \sup_{y\in[-d,d]\setminus[-1,1]}\frac{h(y)}{h(x)}\,U_{[-1,1]}\big(x,[-d,d]\big),\end{aligned}$$

where $U_{[-1,1]}\big(x,\mathrm{d}y\big)$ is the potential of the stable process killed on entering $[-1,1]$. The explicit form of h implies

$$\sup_{y\in[-d,d]\setminus[-1,1]} h(y) = \max(h(-d),h(d)) < \infty.$$

Hence, it is sufficient to show finiteness of the potentials $U_{[-1,1]}(x,[-d,d])$.

The case $\alpha \in (0,1)$

The unkilled process is transient, i.e. $\lim_{t\to\infty} |\xi_t| = +\infty$ a.s. under $\mathbb{P}^x$. According to Theorem I.19 of Bertoin [4] this implies that the unkilled potential is finite, hence,

$$U_{[-1,1]}(x,[-d,d]) = \mathbb{E}^x\Big[\int_0^{T_{[-1,1]}} \mathbb{1}_{\{\xi_t\in[-d,d]\}}\,\mathrm{d}t\Big] \leq \mathbb{E}^x\Big[\int_0^{\infty} \mathbb{1}_{\{\xi_t\in[-d,d]\}}\,\mathrm{d}t\Big] < +\infty.$$

The case $\alpha = 1$

We use the explicit formula for the killed potential density $u_{[-1,1]}(x,y)$ from Profeta and Simon [49], see Section 5.3.2:

$$u_{[-1,1]}(x,y) = c\int_1^{z(x,y)} \psi_{\frac{1}{2}}(v)\,\mathrm{d}v = c\log\Big(z(x,y) + (z(x,y)^2-1)^{\frac{1}{2}}\Big),$$

where $z(x,y) = \left|\frac{xy-1}{x-y}\right|$ and c is some positive constant. Let $x > 1$ and assume without loss of generality that $x < d$ (if $x \geq d$ the last summand of the following term vanishes). We have

$$U_{[-1,1]}(x,[-d,d]) = U_{[-1,1]}(x,[-d,-1)) + U_{[-1,1]}(x,(1,x]) + U_{[-1,1]}(x,(x,d]).$$

In the following we estimate separately the three summands. The first summand is

$$U_{[-1,1]}(x,[-d,-1)) = c\int_{-d}^{-1} \log\big(z(x,y) + (z(x,y)^2-1)^{\frac{1}{2}}\big)\mathrm{d}y.$$

Since $x > 1$, the function $y \mapsto z(x,y)$ is bounded on $[-d,-1)$ and, hence,

$$y \mapsto \log\big(z(x,y) + (z(x,y)^2-1)^{\frac{1}{2}}\big)$$

is bounded on $[-d,-1)$, too. For the second summand we compute with the explicit formula:

$$\begin{aligned}
U_{[-1,1]}(x,(1,x]) &= c\int_1^x \log\big(z(x,y) + (z(x,y)^2-1)^{\frac{1}{2}}\big)\,\mathrm{d}y\\
&\leq c\int_1^x \log(2z(x,y))\,\mathrm{d}y\\
&= c\int_1^x \log\Big(\frac{2(xy-1)}{x-y}\Big)\,\mathrm{d}y\\
&\leq c(x-1)\log\big(2(x^2-1)\big) + c\int_1^x \log\Big(\frac{1}{x-y}\Big)\,\mathrm{d}y < \infty.
\end{aligned}$$

It remains to show finiteness of the third summand:

$$\begin{aligned}
U_{[-1,1]}(x,(x,d]) &= c\int_x^d \log\big(z(x,y) + (z(x,y)^2-1)^{\frac{1}{2}}\big)\,\mathrm{d}y\\
&\leq c\int_x^d \log(2z(x,y))\,\mathrm{d}y\\
&= c\int_x^d \log\Big(\frac{2(xy-1)}{y-x}\Big)\,\mathrm{d}y
\end{aligned}$$

$$\leq c(d-x)\log(2(xd-1)) + c\int_x^d \log\Big(\frac{1}{y-x}\Big)\,\mathrm{d}y < \infty,$$

as above. These estimates show that $U_{[-1,1]}(x,[-d,d])$ is finite if $x>1$. If $x<-1$, we get from the symmetry of ξ that $U_{[-1,1]}(x,[-d,d]) = U_{[-1,1]}(-x,[-d,d])$ and the claim follows from the arguments given for $x>1$.

The case $\alpha \in (1,2)$

As for $\alpha = 1$ we use the formula of Profeta and Simon [49] stated in Section 2.4. For $x<-1$ it holds that $u_{[-1,1]}(x,y) = \hat{u}_{[-1,1]}(-x,-y)$ where $\hat{u}_{[-1,1]}$ is the analogue expression for the dual process.
We start with the case $x>1$. From the explicit formula we see that $y \mapsto u_{[-1,1]}(x,y)$ is continuous on $(-\infty,-1)\cup(1,x)\cup(x,\infty)$. Since we have $\lim_{y\to\pm1} z(x,y) = 1$, it holds $\lim_{y\to\pm1} u_{[-1,1]}(x,y) = 0$. Hence, to show finiteness of $U_{[-1,1]}(x,[-d,d])$ it is sufficient to show that the limits of $u_{[-1,1]}(x,y)$ for $y \searrow x$ and $y \nearrow x$ exist. For the existence of the limit for $y \searrow x$ it suffices to show that

$$\lim_{y\searrow x}(y-x)^{\alpha-1}\int_1^{z(x,y)} \psi_{\alpha\rho}(v)\,\mathrm{d}v$$

exists. The first factor converges obviously to 0 and the second to $+\infty$. Applying l'Hopital's rule gives

$$\begin{aligned}
&\lim_{y\searrow x}(y-x)^{\alpha-1}\int_1^{z(x,y)} \psi_{\alpha\rho}(v)\,\mathrm{d}v\\
&= \lim_{y\searrow x}\frac{\frac{1-x^2}{(y-x)^2}\psi_{\alpha\rho}(z(x,y))}{(1-\alpha)(y-x)^{-\alpha}}\\
&= \lim_{y\searrow x}\frac{x^2-1}{\alpha-1}(y-x)^{\alpha-2}\psi_{\alpha\rho}(z(x,y))\\
&= \frac{x^2-1}{\alpha-1}\lim_{y\searrow x}(y-x)^{\alpha-2}(z(x,y)-1)^{\alpha\hat{\rho}-1}(z(x,y)+1)^{\alpha\rho-1}\\
&= \frac{x^2-1}{\alpha-1}\lim_{y\searrow x}(xy-1-(y-x))^{\alpha\hat{\rho}-1}(xy-1+(y-x))^{\alpha\rho-1}\\
&= \frac{(x^2-1)^{\alpha-1}}{\alpha-1},
\end{aligned}$$

hence, $\lim\limits_{y\searrow x} u_{[-1,1]}(x,y)$ exists. To show existence of the limit for $y\nearrow x$ it is sufficient to show that

$$\lim_{y\nearrow x}(x-y)^{\alpha-1}\int_1^{z(x,y)} \psi_{\alpha\hat{\rho}}(v)\,\mathrm{d}v$$

exists. But this follows from an analogous calculation. For $x<-1$ we use $u_{[-1,1]}(x,y) = \hat{u}_{[-1,1]}(-x,-y)$ and apply the result for $x>1$.

6 Stable processes conditioned to hit an interval continuously

Here, we tackle the problem of Section 1.2 when ξ is a stable process and B is the interval $[-1,1]$. In Chapter 5 we already found a positive invariant function for the stable processes killed on entering $[-1,1]$, i.e. a function $h : \mathbb{R} \setminus [-1,1] \to (0,\infty)$ such that

$$\mathbb{E}^x\big[\mathbb{1}_{\{t<T_{[-1,1]}\}} h(\xi_t)\big] = h(x), \quad x \notin [-1,1], t \geq 0. \tag{6.1}$$

The invariant function was used to condition the stable processes to avoid the interval and to relate the conditioned processes to their h-transformed path measure:

$$\mathbb{E}^x\Big[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{[-1,1]}\}} \frac{h(\xi_t)}{h(x)}\Big] = \lim_{s\to\infty} \mathbb{P}^x(\Lambda \,|\, s+t < T_{[-1,1]}), \quad x \notin [-1,1], t \geq 0,$$

for $\Lambda \in \mathcal{F}_t$. As for other processes conditioned to avoid sets the conditioned stable processes are transient. As a counterpart, the present chapter studies the question if stable processes can also be conditioned to hit the interval continuously in finite time.

We follow the strategy introduced in Section 1.2. First, we find harmonic functions for the stable process killed on entering an interval, i.e. excessive functions $v : \mathbb{R}\setminus[-1,1] \to (0,\infty)$ which fulfil

$$\mathbb{E}^x\Big[\mathbb{1}_{\{T_{K^C}<T_{[-1,1]}\}} v(\xi_{T_{K^C}})\Big] = v(x), \quad x \notin [-1,1], \tag{6.2}$$

for all compact $K \subseteq \mathbb{R} \setminus [-1,1]$. From these harmonic functions we define h-transformed measures which we then identify as the limiting measures of suitable conditionings that force the process to be absorbed at the boundary of the interval. The different possible cases of absorption at the top or the bottom of the interval will be reflected in the existence of different harmonic functions and their linear combinations.

6.1 Main results

As mentioned the main results of this chapter are two-fold. We first identify new harmonic functions and then connect the underlying h-transformed processes to some conditioned processes.

6.1.1 Harmonic functions

In this first section we identify two (minimal) harmonic functions. Let us define two functions $v_1, v_2 : \mathbb{R} \setminus [-1,1] \to (0,\infty)$ by

$$v_1(x) := \begin{cases} \sin(\pi\alpha\hat{\rho})\Big[(x+1)\psi_{\alpha\rho}(x) - (\alpha-1)_+ \int\limits_1^x \psi_{\alpha\rho}(u)\,\mathrm{d}u\Big] & \text{if } x>1 \\ \sin(\pi\alpha\rho)\Big[(|x|-1)\psi_{\alpha\hat{\rho}}(|x|) - (\alpha-1)_+ \int\limits_1^{|x|} \psi_{\alpha\hat{\rho}}(u)\,\mathrm{d}u\Big] & \text{if } x<-1 \end{cases},$$

and

$$v_{-1}(x) := \begin{cases} \sin(\pi\alpha\hat{\rho})\Big[(x-1)\psi_{\alpha\rho}(x) - (\alpha-1)_+ \int\limits_1^x \psi_{\alpha\rho}(u)\,\mathrm{d}u\Big] & \text{if } x>1 \\ \sin(\pi\alpha\rho)\Big[(|x|+1)\psi_{\alpha\hat{\rho}}(|x|) - (\alpha-1)_+ \int\limits_1^{|x|} \psi_{\alpha\hat{\rho}}(u)\,\mathrm{d}u\Big] & \text{if } x<-1 \end{cases}.$$

The appearing auxiliary functions

$$\psi_{\alpha\rho}(x) = (x-1)^{\alpha\hat{\rho}-1}(x+1)^{\alpha\rho-1}, \quad x>1,$$

already played a crucial rule to condition the stable processes to avoid an interval in Chapter 5. For the function $\psi_{\alpha\hat{\rho}}$ the positivity parameter ρ is replaced by $\hat{\rho}$, and vice versa.

Here is the main result of this section:

Theorem 6.1.1. *Let ξ be a stable process with index $\alpha \in (0,2)$ which has jumps in both directions. Then v_1 and v_{-1} are harmonic functions for ξ killed on first hitting the interval $[-1,1]$.*

A harmonic function is in particular excessive, hence, a new measure can be defined as an h-transform with the harmonic function. In what follows we will denote the h-transforms with v_1, v_{-1} and $v := v_1 + v_{-1}$ by $\mathbb{P}^x_{v_1}$, $\mathbb{P}^x_{v_{-1}}$ and $\mathbb{P}^x_v$.

6.1.2 Stable processes absorbed from above (or below)

The purpose of this section is to analyse the h-transformed process $(\xi, \mathbb{P}^x_{v_1})$. Since all results for $(\xi, \mathbb{P}^x_{v_{-1}})$ are analogous (replacing ρ and $\hat{\rho}$) without loss of generality we only discuss $(\xi, \mathbb{P}^x_{v_1})$. Two questions will be our main concern:

- Is the process killed in finite time and, if so, what is the limiting behavior at the killing time?
- How to characterize $\mathbb{P}^x_{v_1}$ through a limiting conditioning of $\mathbb{P}^x$?

The first question can be answered for all α simultaneously using properties of the explicit form of v_1:

Proposition 6.1.2. *Let ξ be an α-stable process with $\alpha \in (0,2)$ and both sided jumps, then*

$$\mathbb{P}^x_{v_1}(\zeta < \infty, \xi_{\zeta-} = 1) = 1, \quad x \notin [-1,1].$$

To answer the second question we need to distinguish the recurrent and the transient cases:

The case $\alpha < 1$: The probability that ξ never hits the interval $[-1,1]$ is positive because the stable process is transient. To condition ξ to be absorbed by $[-1,1]$ from above without hitting the interval we first condition on $\{T_{[-1,1]} = \infty\}$ and then on some event which describes the absorption from above. The most plausible event is $T_{(1,1+\varepsilon)}$ being finite for small $\varepsilon > 0$. Another possibility refers to the so-called point of closest reach. Let therefore $\underline{m}$ be the time such that $|\xi_{\underline{m}}| \leq |\xi_t|$ for all $t \geq 0$. Then $\xi_{\underline{m}}$ is called the point of closest reach of 0. The polarity of points for $\alpha < 1$ implies $\xi_{\underline{m}} \neq 0$ almost surely under $\mathbb{P}^x$ for all starting points $x \neq 0$. With these definitions one could also think of conditioning on the event $\{\xi_{\underline{m}} \in (1, 1+\varepsilon)\}$ which is contained in $\{T_{[-1,1]} = \infty, T_{(1,1+\varepsilon)} < \infty\}$ and, indeed, this is the right choice.

The case $\alpha \geq 1$: The first hitting time $T_{[-1,1]}$ is finite almost surely, hence, a different conditioning is needed. Since $T_{(-1-\varepsilon,1+\varepsilon)}$ is finite as well the good conditioning is to condition $\xi_{T_{(-1-\varepsilon,1+\varepsilon)}}$ to be in $(1, 1+\varepsilon)$ and then let ε tend to 0.

The techniques we use for the conditioning center around the recent results on the so-called deep factorisation of stable processes, see e.g. Kyprianou [35] and Kyprianou et al. [41] and hitting distributions of stable processes, see Kyprianou et al. [38]. In particular, results on the distribution of the point of closest reach in the case $\alpha < 1$ and the distribution of the first hitting time of the interval $(-1,1)$ in the case $\alpha \geq 1$ are the keys to prove our results.

We come to the first characterisation of the h-transform $\mathbb{P}^x_{v_1}$ as the process conditioned to be absorbed by $[-1,1]$ from above in a meaningful way.

Theorem 6.1.3. *Let ξ be an α-stable process with $\alpha \in (0,1)$ and both sided jumps. Then it holds, for all $x \notin [-1,1]$ and $\Lambda \in \mathcal{F}_t$, that*

$$\mathbb{P}^x_{v_1}(\Lambda, t < \zeta) = \lim_{\delta \searrow 0} \lim_{\varepsilon \searrow 0} \mathbb{P}^x(\Lambda, t < T_{(-(1+\delta),1+\delta)} \,|\, \xi_{\underline{m}} \in (1, 1+\varepsilon)).$$

In fact, we prove a slightly more general statement which has precisely the form of a self-similar Markov process conditioned to be absorbed at the origin in Kyprianou et al. [37] and a Lévy process conditioned to be absorbed at the origin from above in Chaumont [15]:

$$\mathbb{P}^x_{v_1}(\Lambda, t < T_{(-(1+\delta),1+\delta)}) = \lim_{\varepsilon \searrow 0} \mathbb{P}^x(\Lambda, t < T_{(-(1+\delta),1+\delta)} \,|\, \xi_{\underline{m}} \in (1, 1+\varepsilon)) \tag{6.3}$$

for all $\delta > 0$.

In the case $\alpha \geq 1$ the h-transform belongs to a different conditioned process.

Theorem 6.1.4. *Let ξ be an α-stable process with $\alpha \in [1,2)$ and both sided jumps. Then it holds, for all $x \notin [-1,1]$ and $\Lambda \in \mathcal{F}_t$, that*

$$\mathbb{P}^x_{v_1}(\Lambda, t < \zeta) = \lim_{\delta \searrow 0} \lim_{\varepsilon \searrow 0} \mathbb{P}^x(\Lambda, t < T_{(-(1+\delta),1+\delta)} \,|\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (1, 1+\varepsilon)).$$

With this result we can interpret the h-transformed process as the original process conditioned to approach the interval $[-1,1]$ continuously from above.

For $\alpha > 1$ we can even find a second characterisation of $\mathbb{P}^x_{v_1}$ as conditioned process. We need to introduce the stable process conditioned to avoid 0 (see e.g. Pantí [45] or Yano

[58] for general Lévy processes or Section 2.6). As a reminder define $e : \mathbb{R} \setminus \{0\} \to (0, \infty)$ via

$$e(x) = \begin{cases} \sin(\pi\alpha\hat{\rho})x^{\alpha-1} & \text{if } x > 0 \\ \sin(\pi\alpha\rho)|x|^{\alpha-1} & \text{if } x < 0 \end{cases},$$

which is known to be a positive invariant function for the process killed on hitting 0 when $\alpha > 1$. Denote the underlying h-transform by $\mathbb{P}^x_\circ$, i.e.

$$\mathbb{P}^x_\circ(\Lambda) = \mathbb{E}^x\Big[\mathbb{1}_{\{t<T_{\{0\}}\}}\frac{e(\xi_t)}{e(x)}\Big], \quad x \neq 0, \Lambda \in \mathcal{F}_t,$$

which can be shown to correspond to conditioning the stable process to avoid the origin. We can use $\mathbb{P}^x_\circ$ to give a conditioning analogously to the case $\alpha < 1$ also in the case $\alpha > 1$. But here the conditioning does not refer to the original process but to the process conditioned to avoid 0.

Theorem 6.1.5. *Let ξ be an α-stable process with $\alpha \in (1, 2)$ and both sided jumps. Then it holds, for all $x \notin [-1, 1]$ and $\Lambda \in \mathcal{F}_t$, that*

$$\mathbb{P}^x_{v_1}(\Lambda, t < \zeta) = \lim_{\delta\searrow 0}\lim_{\varepsilon\searrow 0}\mathbb{P}^x_\circ(\Lambda, t < T_{(-(1+\delta),1+\delta)} \,|\, \xi_{\underline{m}} \in (1, 1 + \varepsilon)).$$

It is quite remarkable to compare Theorem 6.1.5 and Theorem 6.1.3. Since conditioning to avoid a point has no effect for $\alpha < 1$ both theorems coincide. First condition to avoid the origin (trivial for $\alpha < 1$) then condition to approach 1 from above yields $\mathbb{P}^x_{v_1}$. The case $\alpha = 1$ differs from $\alpha \neq 1$ in this respect because 0 is polar and the conditioning to approach the interval from above is not well-defined because $\xi_{\underline{m}} = 0$ almost surely.

6.1.3 Stable processes absorbed without restrictions

In this section we want to analyse the h-transforms $(\xi, \mathbb{P}^x_v)$ with $v = v_1 + v_{-1}$. The two main aspects are the same as in Section 6.1.2. First we want to analyse the behaviour of the paths of $(\xi, \mathbb{P}^x_v)$ at the killing time if it is finite. Second we give characterisations of the h-transformed process as the original process conditioned on similar events as in Section 6.1.2.

In the case $\alpha < 1$ this works as one would expect, namely the h-transform using v corresponds to the process conditioned on $\{|\xi_{\underline{m}}| \in (1, 1 + \varepsilon)\}$ for ε tending to 0. For $\alpha \geq 1$ we won't find a representation $(\xi, \mathbb{P}^x_v)$ as a conditioned process. Nonetheless we can show that the process conditioned to be absorbed by $[-1, 1]$ without any restrictions on the side of the interval of which it is absorbed, equals $(\xi, \mathbb{P}^x_{v_1})$ or $(\xi, \mathbb{P}^x_{v_{-1}})$ depending on some relation on ρ. This means that the process conditioned to be absorbed without any restrictions coincides with one of the processes conditioned to be absorbed from one side.

Here is the result on the behaviour at the killing time:

Proposition 6.1.6. *Let ξ be an α-stable process with $\alpha \in (0, 2)$ and both sided jumps, then*

$$\mathbb{P}^x_v(\zeta < \infty, |\xi_{\zeta-}| = 1) = 1, \quad x \notin [-1, 1].$$

As before we want to connect the h-transformed process to some conditioned process. Again we have to separate the cases $\alpha < 1$ and $\alpha \geq 1$ and for $\alpha > 1$ we give an alternative conditioned process. The event we condition on is bigger than in Section 6.1.2 in all cases. We start with the asymptotic in the case $\alpha < 1$ and the characterisation of $(\xi, \mathbb{P}^x_v)$ as conditioned process as one would expect with the knowledge of Theorem 6.1.3.

Theorem 6.1.7. *Let ξ be an α-stable process with $\alpha \in (0,1)$ and both sided jumps. Then it holds, for all $x \notin [-1,1]$ and $\Lambda \in \mathcal{F}_t$, that*

$$\mathbb{P}_v^x(\Lambda, t < \zeta) = \lim_{\delta \searrow 0} \lim_{\varepsilon \searrow 0} \mathbb{P}^x(\Lambda, t < T_{(-(1+\delta),1+\delta)} \,|\, |\underline{\xi_m}| \in (1, 1+\varepsilon)).$$

As we already mentioned, in the case $\alpha \geq 1$ the process conditioned to be absorbed by the interval without restriction on the side of absorption is the same as the process conditioned to be absorbed from one side, the side depending on ρ.

Theorem 6.1.8. *Let ξ be an α-stable process with $\alpha \in [1,2)$ and both sided jumps. Then it holds, for all $x \notin [-1,1]$ and $\Lambda \in \mathcal{F}_t$, that*

$$\begin{aligned}&\lim_{\delta \searrow 0} \lim_{\varepsilon \searrow 0} \mathbb{P}^x(\Lambda, t < T_{(-(1+\delta),1+\delta)} \,|\, \xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \notin [-1,1]) \\ &= \begin{cases} \mathbb{P}_{v_1}^x(\Lambda, t < \zeta) & \text{if } \rho \leq \frac{1}{2} \\ \mathbb{P}_{v_{-1}}^x(\Lambda, t < \zeta) & \text{if } \rho > \frac{1}{2} \end{cases}.\end{aligned}$$

We conclude with the alternative characterisation for the h-transform for $\alpha > 1$. Again the conditioning refers to the stable process conditioned to avoid 0 and the event we condition on is the same as in the case $\alpha < 1$.

Theorem 6.1.9. *Let ξ be an α-stable process with $\alpha \in (1,2)$ and both sided jumps. Then it holds, for all $x \notin [-1,1]$ and $\Lambda \in \mathcal{F}_t$, that*

$$\mathbb{P}_v^x(\Lambda, t < \zeta) = \lim_{\delta \searrow 0} \lim_{\varepsilon \searrow 0} \mathbb{P}_\circ^x(\Lambda, t < T_{(-(1+\delta),1+\delta)} \,|\, |\underline{\xi_m}| \in (1, 1+\varepsilon)).$$

6.2 Proofs

6.2.1 Harmonic functions

In this section we prove Theorem 6.1.1. First we give an idea how to extract the right harmonic functions. The potential measure of ξ killed when it enters $[-1,1]$ is defined as

$$U_{[-1,1]}(x, \mathrm{d}y) := \mathbb{E}^x\left[\int_0^{T_{[-1,1]}} \mathbb{1}_{\{\xi_t \in \mathrm{d}y\}} \,\mathrm{d}t\right], \quad x, y \notin [-1,1].$$

It is known that the potential measure has a density with respect to the Lebesgue measure (also known as Green's function), i.e.

$$U_{[-1,1]}(x, \mathrm{d}y) = u_{[-1,1]}(x,y)\,\mathrm{d}y,$$

where $u_{[-1,1]} : (\mathbb{R} \setminus [-1,1])^2 \to [0,\infty)$ is explicitely known from Profeta and Simon [49]. Moreover, Kunita and Watanabe [33] showed that $x \mapsto u_{[-1,1]}(x,y)$ is harmonic for all $y \notin [-1,1]$ and, heuristically speaking, the corresponding h-transform should be the process conditioned to be absorbed by y. Since our aim is to condition the process to be absorbed from 1 we will consider the limit when y tends to 1. But from the formulas of [49] we see immediately that $u_{[-1,1]}(x,y)$ converges to 0 for y tending to 1. So there are two difficulties. The first one is that we need to renormalise $u_{[-1,1]}(x,y)$ such that it converges pointwise for $y \searrow 1$ to some function in x and second we need to argue why in this case the limit of the (scaled) harmonic function is harmonic again.

To abbreviate we denote

$$c_{\alpha\rho} := 2^{\alpha\rho}\frac{\pi\alpha\rho\Gamma(\alpha\rho)}{\Gamma(1-\alpha\hat{\rho})} \quad \text{and} \quad c_{\alpha\hat{\rho}} := 2^{\alpha\hat{\rho}}\frac{\pi\alpha\hat{\rho}\Gamma(\alpha\hat{\rho})}{\Gamma(1-\alpha\rho)}.$$

The first auxiliary result establishes a pointwise connection between v_1 and the potential density $u_{[-1,1]}$ which will be very important for the proof of harmonicity of v_1. From Profeta and Simon [49] we know that $y \mapsto u_{[-1,1]}(x,y)$ has a pole in x (for $\alpha < 1$) but is also integrable at x. Hence, defining $u_{[-1,1]}(x,x) := 0$ does not change anything for the potential of the process killed on entering $[-1,1]$.

Lemma 6.2.1. *Whenever $x > y > 1$ or $x < -1, y > 1$, it holds that*

$$\begin{aligned} v_1(x) &= 2^{\alpha\hat{\rho}-1}c_{\alpha\rho}\frac{u_{[-1,1]}(x,y)}{g(y)} \\ &\quad - \Big(\sin(\pi\alpha\hat{\rho})\mathbb{1}_{\{x>1\}} + \sin(\pi\alpha\rho)\mathbb{1}_{\{x<-1\}}\Big) \\ &\quad \times \frac{(1-\alpha\hat{\rho})|x-y|^{\alpha-1}}{g(y)} \int_1^{z(x,y)} (u-1)^{\alpha\rho}(u+1)^{\alpha\hat{\rho}-2}\,\mathrm{d}u \\ &\quad + (\alpha-1)_+\Big(\sin(\pi\alpha\hat{\rho})\mathbb{1}_{\{x>1\}}\int_1^x \psi_{\alpha\rho}(u)\,\mathrm{d}u + \sin(\pi\alpha\rho)\mathbb{1}_{\{x<-1\}}\int_1^{|x|}\psi_{\alpha\hat{\rho}}(u)\,\mathrm{d}u\Big) \\ &\quad \times \Big(\frac{\alpha\rho}{g(y)}\int_1^y \psi_{\alpha\hat{\rho}}(u)\,\mathrm{d}u - 1\Big), \end{aligned}$$

where $g(y) = (y-1)^{\alpha\rho}(y+1)^{\alpha\hat{\rho}-1} = (y-1)\psi_{\alpha\hat{\rho}}(y)$.

Proof. We use the explicit expression for $u_{[-1,1]}(x,y)$ from Profeta and Simon [49], where the expression

$$z(x,y) = \frac{|xy-1|}{|x-y|}, \quad x,y \notin [-1,1], x \neq y,$$

appears frequently. Before we start we note that

$$z(x,y) - 1 = \begin{cases} \frac{(x+1)(y-1)}{x-y} & \text{if } x > y > 1 \\ \frac{(|x|-1)(y-1)}{y-x} & \text{if } x < -1, y > 1 \end{cases}$$

and

$$z(x,y) + 1 = \begin{cases} \frac{(x-1)(y+1)}{x-y} & \text{if } x > y > 1 \\ \frac{(|x|+1)(y+1)}{y-x} & \text{if } x < -1, y > 1 \end{cases}.$$

Furthermore, with integration by parts we get

$$\int_1^{z(x,y)} \psi_{\alpha\rho}(u)\,\mathrm{d}u$$

$$
\begin{aligned}
&= \int_1^{z(x,y)} (u-1)^{\alpha\hat{\rho}-1}(u+1)^{\alpha\rho-1}\,\mathrm{d}u \\
&= \frac{1}{\alpha\hat{\rho}}\Big[(u-1)^{\alpha\hat{\rho}}(u+1)^{\alpha\rho-1}\Big]_1^{z(x,y)} - \frac{\alpha\rho-1}{\alpha\hat{\rho}} \int_1^{z(x,y)} (u-1)^{\alpha\hat{\rho}}(u+1)^{\alpha\rho-2}\,\mathrm{d}u \\
&= \frac{1}{\alpha\hat{\rho}}\Big((z(x,y)-1)^{\alpha\hat{\rho}}(z(x,y)+1)^{\alpha\rho-1}\Big) + \frac{1-\alpha\rho}{\alpha\hat{\rho}} \int_1^{z(x,y)} (u-1)^{\alpha\hat{\rho}}(u+1)^{\alpha\rho-2}\,\mathrm{d}u
\end{aligned}
$$

and analogously

$$
\begin{aligned}
&\int_1^{z(x,y)} \psi_{\alpha\hat{\rho}}(u)\,\mathrm{d}u \\
&= \frac{1}{\alpha\rho}\Big((z(x,y)-1)^{\alpha\rho}(z(x,y)+1)^{\alpha\hat{\rho}-1}\Big) + \frac{1-\alpha\hat{\rho}}{\alpha\rho} \int_1^{z(x,y)} (u-1)^{\alpha\rho}(u+1)^{\alpha\hat{\rho}-2}\,\mathrm{d}u.
\end{aligned}
$$

We use the explicit form for $u_{[-1,1]}(x,y)$ given in [49] and plug in to see, for $x>y>1$,

$$
\begin{aligned}
&\frac{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}{2^{1-\alpha}} u_{[-1,1]}(x,y) \\
&= (x-y)^{\alpha-1} \int_1^{z(x,y)} \psi_{\alpha\hat{\rho}}(u)\,\mathrm{d}u - (\alpha-1)_+ \int_1^{y} \psi_{\alpha\hat{\rho}}(u)\,\mathrm{d}u \int_1^{x} \psi_{\alpha\rho}(u)\,\mathrm{d}u \\
&= \frac{(x-y)^{\alpha-1}}{\alpha\rho}(z(x,y)-1)^{\alpha\rho}(z(x,y)+1)^{\alpha\hat{\rho}-1} \\
&\quad + \frac{(1-\alpha\hat{\rho})(x-y)^{\alpha-1}}{\alpha\rho} \int_1^{z(x,y)} (u-1)^{\alpha\rho}(u+1)^{\alpha\hat{\rho}-2}\,\mathrm{d}u \\
&\quad - (\alpha-1)_+ \int_1^{y} \psi_{\alpha\hat{\rho}}(u)\,\mathrm{d}u \int_1^{x} \psi_{\alpha\rho}(u)\,\mathrm{d}u \\
&= \frac{1}{\alpha\rho}((x+1)(y-1))^{\alpha\rho}((x-1)(y+1))^{\alpha\hat{\rho}-1} \\
&\quad + \frac{(1-\alpha\hat{\rho})(x-y)^{\alpha-1}}{\alpha\rho} \int_1^{z(x,y)} (u-1)^{\alpha\rho}(u+1)^{\alpha\hat{\rho}-2}\,\mathrm{d}u \\
&\quad - (\alpha-1)_+ \int_1^{y} \psi_{\alpha\hat{\rho}}(u)\,\mathrm{d}u \int_1^{x} \psi_{\alpha\rho}(u)\,\mathrm{d}u \\
&= \frac{1}{\alpha\rho}(y-1)^{\alpha\rho}(y+1)^{\alpha\hat{\rho}-1}\Big(\frac{1}{\sin(\pi\alpha\hat{\rho})}v_1(x) + (\alpha-1)_+ \int_1^{x} \psi_{\alpha\rho}(u)\,\mathrm{d}u\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(1-\alpha\hat\rho)(x-y)^{\alpha-1}}{\alpha\rho}\int_1^{z(x,y)}(u-1)^{\alpha\rho}(u+1)^{\alpha\hat\rho-2}\,\mathrm{d}u\\
&-(\alpha-1)_+\int_1^y\psi_{\alpha\hat\rho}(u)\,\mathrm{d}u\int_1^x\psi_{\alpha\rho}(u)\,\mathrm{d}u.
\end{aligned}
$$

Solving the equation with respect to v_1 and using $\sin(\pi\alpha\hat\rho)=\frac{\pi}{\Gamma(\alpha\hat\rho)\Gamma(1-\alpha\hat\rho)}$ yields the claim for $x>y>1$. For $x<-1, y>1$ we get similarly:

$$
\begin{aligned}
&\frac{\sin(\pi\alpha\hat\rho)}{\sin(\pi\alpha\rho)}\frac{\Gamma(\alpha\rho)\Gamma(\alpha\hat\rho)}{2^{1-\alpha}}u_{[-1,1]}(x,y)\\
&=(y-x)^{\alpha-1}\int_1^{z(x,y)}\psi_{\alpha\hat\rho}(u)\,\mathrm{d}u-(\alpha-1)_+\int_1^y\psi_{\alpha\hat\rho}(u)\,\mathrm{d}u\int_1^{|x|}\psi_{\alpha\hat\rho}(u)\,\mathrm{d}u\\
&=\frac{(y-x)^{\alpha-1}}{\alpha\rho}(z(x,y)-1)^{\alpha\rho}(z(x,y)+1)^{\alpha\hat\rho-1}\\
&\quad+\frac{(1-\alpha\hat\rho)(y-x)^{\alpha-1}}{\alpha\rho}\int_1^{z(x,y)}(u-1)^{\alpha\rho}(u+1)^{\alpha\hat\rho-2}\,\mathrm{d}u\\
&\quad-(\alpha-1)_+\int_1^y\psi_{\alpha\hat\rho}(u)\,\mathrm{d}u\int_1^{|x|}\psi_{\alpha\hat\rho}(u)\,\mathrm{d}u\\
&=\frac{1}{\alpha\rho}((|x|-1)(y-1))^{\alpha\rho}((|x|+1)(y+1))^{\alpha\hat\rho-1}\\
&\quad+\frac{(1-\alpha\hat\rho)(y-x)^{\alpha-1}}{\alpha\rho}\int_1^{z(x,y)}(u-1)^{\alpha\rho}(u+1)^{\alpha\hat\rho-2}\,\mathrm{d}u\\
&\quad-(\alpha-1)_+\int_1^y\psi_{\alpha\hat\rho}(u)\,\mathrm{d}u\int_1^{|x|}\psi_{\alpha\hat\rho}(u)\,\mathrm{d}u\\
&=\frac{1}{\alpha\rho}(y-1)^{\alpha\rho}(y+1)^{\alpha\hat\rho-1}\Big(\frac{1}{\sin(\pi\alpha\rho)}v_1(x)+(\alpha-1)_+\int_1^{|x|}\psi_{\alpha\hat\rho}(u)\,\mathrm{d}u\Big)\\
&\quad+\frac{(1-\alpha\hat\rho)(x-y)^{\alpha-1}}{\alpha\rho}\int_1^{z(x,y)}(u-1)^{\alpha\rho}(u+1)^{\alpha\hat\rho-2}\,\mathrm{d}u\\
&\quad-(\alpha-1)_+\int_1^y\psi_{\alpha\hat\rho}(u)\,\mathrm{d}u\int_1^{|x|}\psi_{\alpha\hat\rho}(u)\,\mathrm{d}u.
\end{aligned}
$$

Again, solving with respect to $v_1(x)$ leads to the claim. □

Corollary 6.2.2. *It holds that*

$$v_1(x)=c_{\alpha\rho}\lim_{y\searrow1}\frac{u_{[-1,1]}(x,y)}{(y-1)^{\alpha\rho}},\quad x\in\mathbb{R}\setminus[-1,1].$$

Proof. We consider the expression from Lemma 6.2.1 and let y tend to 1 from above. It is sufficient to show that

$$\begin{aligned}
&-\Big(\sin(\pi\alpha\hat\rho)\mathbb{1}_{\{x>1\}}+\sin(\pi\alpha\rho)\mathbb{1}_{\{x<-1\}}\Big)\\
&\quad\times\frac{(1-\alpha\hat\rho)|x-y|^{\alpha-1}}{g(y)}\int_1^{z(x,y)}(u-1)^{\alpha\rho}(u+1)^{\alpha\hat\rho-2}\,\mathrm{d}u\\
&+(\alpha-1)_+\Big(\sin(\pi\alpha\hat\rho)\mathbb{1}_{\{x>1\}}\int_1^x\psi_{\alpha\rho}(u)\,\mathrm{d}u+\sin(\pi\alpha\rho)\mathbb{1}_{\{x<-1\}}\int_1^{|x|}\psi_{\alpha\hat\rho}(u)\,\mathrm{d}u\Big)\\
&\quad\times\Big(\frac{\alpha\rho}{g(y)}\int_1^y\psi_{\alpha\hat\rho}(u)\,\mathrm{d}u-1\Big)
\end{aligned}$$

converges to 0 for $y\searrow 1$. For that it is of course sufficient to show that

$$\frac{1}{g(y)}\int_1^{z(x,y)}(u-1)^{\alpha\rho}(u+1)^{\alpha\hat\rho-2}\,\mathrm{d}u\quad\text{and}\quad\frac{\alpha\rho}{g(y)}\int_1^y\psi_{\alpha\hat\rho}(u)\,\mathrm{d}u-1$$

converge to 0 for $y\searrow 1$. Both claims can be seen readily with l'Hopital's rule. □

Now we prove harmonicity of v_1.

Proof of Theorem 6.1.1. To show excessiveness we define the measure

$$\eta(\mathrm{d}x):=v_1(x)\,\mathrm{d}x\qquad\text{on }\mathbb{R}\setminus[-1,1].$$

We will show that η is an excessive measure for the dual process killed on entering the intervall, i.e. η is σ-finite and it holds that

$$\int_{\mathbb{R}\setminus[-1,1]}\hat{\mathbb{P}}^x(\xi_t\in A,t<T_{[-1,1]})\,\eta(\mathrm{d}x)\le\eta(A),$$

for all $A\in\mathcal{B}(\mathbb{R}\setminus[-1,1])$ and $t\ge 0$. From Theorem XII.71 of Dellacherie and Meyer [19] it is known that if an excessive measure has a density with respect to the duality measure (which is the Lebesgue measure also for killed Lévy processes, see Bertoin [4], Theorem II.5), then this density is an excessive function for the dual process killed on hitting $[-1,1]$. Hence, by showing that η is an excessive measure for the dual process killed on hitting $[-1,1]$, it follows that v_1 is an excessive function for the original process killed on entering the interval.

To show that η is excessive for the dual process, first note that η is σ-finite because v_1 is continuous on $\mathbb{R}\setminus[-1,1]$. Next, for the dual process, we note that

$$\hat U_{[-1,1]}(y,\mathrm{d}x)=u_{[-1,1]}(x,y)\,\mathrm{d}x,\quad x,y\in\mathbb{R}\setminus[-1,1],$$

where $\hat U_{[-1,1]}$ is the potential of the dual process killed on entering $[-1,1]$ (see Theorem XII.72 of Dellacherie and Meyer [19] for a general Markov process). Let $A\in\mathcal{B}(\mathbb{R}\setminus[-1,1])$

be compact, use Corollary 6.2.2 in the first equation and Fatou's Lemma in the second one:

$$\begin{aligned}
&\frac{1}{c_{\alpha\rho}} \int\limits_{\mathbb{R}\setminus[-1,1]} \hat{\mathbb{P}}^x(\xi_t \in A, t < T_{[-1,1]})\, \eta(\mathrm{d}x) \\
&= \int\limits_{\mathbb{R}\setminus[-1,1]} \hat{\mathbb{P}}^x(\xi_t \in A, t < T_{[-1,1]}) \lim_{y \searrow 1} \frac{u_{[-1,1]}(x,y)}{(y-1)^{\alpha\rho}}\, \mathrm{d}x \\
&\leq \liminf_{y \searrow 1} \frac{1}{(y-1)^{\alpha\rho}} \int\limits_{\mathbb{R}\setminus[-1,1]} \hat{\mathbb{P}}^x(\xi_t \in A, t < T_{[-1,1]}) u_{[-1,1]}(x,y)\, \mathrm{d}x \\
&\leq \liminf_{y \searrow 1} \frac{1}{(y-1)^{\alpha\rho}} \int\limits_{\mathbb{R}\setminus[-1,1]} \hat{\mathbb{P}}^x(\xi_t \in A, t < T_{[-1,1]})\, \hat{U}_{[-1,1]}(y, \mathrm{d}x) \\
&= \liminf_{y \searrow 1} \frac{1}{(y-1)^{\alpha\rho}} \int\limits_0^\infty \Big(\int\limits_{\mathbb{R}\setminus[-1,1]} \hat{\mathbb{P}}^x(\xi_t \in A, t < T_{[-1,1]})\, \hat{\mathbb{P}}^y(\xi_s \in \mathrm{d}x, s < T_{[-1,1]}) \Big)\, \mathrm{d}s \\
&= \liminf_{y \searrow 1} \frac{1}{(y-1)^{\alpha\rho}} \int\limits_0^\infty \hat{\mathbb{P}}^y(\xi_{t+s} \in A, t+s < T_{[-1,1]})\, \mathrm{d}s \\
&= \liminf_{y \searrow 1} \frac{1}{(y-1)^{\alpha\rho}} \int\limits_t^\infty \hat{\mathbb{P}}^y(\xi_s \in A, s < T_{[-1,1]})\, \mathrm{d}s \\
&\leq \liminf_{y \searrow 1} \frac{1}{(y-1)^{\alpha\rho}} \int\limits_0^\infty \hat{\mathbb{P}}^y(\xi_s \in A, s < T_{[-1,1]})\, \mathrm{d}s \\
&\leq \liminf_{y \searrow 1} \frac{1}{(y-1)^{\alpha\rho}} \int\limits_A \hat{u}_{[-1,1]}(y,x)\, \mathrm{d}x \\
&\leq \liminf_{y \searrow 1} \int\limits_A \frac{u_{[-1,1]}(x,y)}{(y-1)^{\alpha\rho}}\, \mathrm{d}x.
\end{aligned}$$

From Corollary 6.2.2 we know that $(u_{[-1,1]}(x,y))/((y-1)^{\alpha\rho})$ converges for $y \searrow 1$ for all $x \in \mathbb{R}\setminus[-1,1]$, in particular the function $y \mapsto (u_{[-1,1]}(x,y))/((y-1)^{\alpha\rho})$ is bounded on $(1,\varepsilon)$ with $\varepsilon < \inf A \cap (1,\infty)$ for all $x \in A$. But since A is compact $(u_{[-1,1]}(x,y))/((y-1)^{\alpha\rho})$ is uniformly bounded for $x \in A$. Hence, we can apply dominated convergence to deduce:

$$\begin{aligned}
\frac{1}{c_{\alpha\rho}} \int\limits_{\mathbb{R}\setminus[-1,1]} \hat{\mathbb{P}}^x(\xi_t \in A)\, \eta(\mathrm{d}x) &\leq \int\limits_A \lim_{y \searrow 1} \frac{u_{[-1,1]}(x,y)}{(y-1)^{\alpha\rho}}\, \mathrm{d}x \\
&= \frac{1}{c_{\alpha\rho}} \int_A v_1(x)\, \mathrm{d}x \\
&= \frac{1}{c_{\alpha\rho}} \eta(A).
\end{aligned}$$

Hence, we proved that η is an excessive measure and as mentioned above it follows with Theorem XII.71 of [19] that v_1 is an excessive function.

Now we show the characterising property of harmonicity, i.e.

$$\mathbb{E}^x\big[\mathbb{1}_{\{T_{K^\mathrm{C}}<T_{[-1,1]}\}}v_1(\xi_{T_{K^\mathrm{C}}})\big] = v_1(x), \quad x \in \mathbb{R}\setminus[-1,1],$$

for all $K \subseteq \mathbb{R}\setminus[-1,1]$ which are compact in $\mathbb{R}\setminus[-1,1]$. If $x \in K^\mathrm{C} = (\mathbb{R}\setminus[-1,1])\setminus K$, the claim is clear. So we assume $x \in K$. The idea is to use the connection between v_1 and $u_{[-1,1]}$ from Lemma 6.2.1 and Proposition 6.2 (ii) of Kunita and Watanabe [33]. The second tells us that $x \mapsto u_{[-1,1]}(x,y)$ is harmonic on $(\mathbb{R}\setminus[-1,1])\setminus\{y\}$ for all $y \in \mathbb{R}\setminus[-1,1]$, i.e.

$$\mathbb{E}^x\big[\mathbb{1}_{\{T_{K^\mathrm{C}}<T_{[-1,1]}\}}u_{[-1,1]}(\xi_{T_{K^\mathrm{C}}},y)\big] = u_{[-1,1]}(x,y), \quad x,y \in \mathbb{R}\setminus[-1,1], x\neq y,$$

for all $K \subseteq \mathbb{R}\setminus[-1,1]$ which are compact in $\mathbb{R}\setminus[-1,1]\setminus\{y\}$.
Let us fix $x \notin [-1,1]$ and since y tends to 1 we can assume $x \neq y$ and $y \notin K$. We use monotone convergence twice and plug in the result of Lemma 6.2.1:

$$\begin{aligned}
&\mathbb{E}^x\big[\mathbb{1}_{\{T_{K^\mathrm{C}}<T_{[-1,1]}\}}v_1(\xi_{T_{K^\mathrm{C}}})\big]\\
&= \lim_{\varepsilon\searrow 0}\mathbb{E}^x\big[\mathbb{1}_{\{\xi_{T_{K^\mathrm{C}}}>1+\varepsilon \text{ or } \xi_{T_{K^\mathrm{C}}}<-1\}}v_1(\xi_{T_{K^\mathrm{C}}})\big]\\
&= \lim_{\varepsilon\searrow 0}\lim_{y\searrow 1}\mathbb{E}^x\big[\mathbb{1}_{\{\xi_{T_{K^\mathrm{C}}}>y+\varepsilon \text{ or } \xi_{T_{K^\mathrm{C}}}<-1\}}v_1(\xi_{T_{K^\mathrm{C}}})\big]\\
&= \lim_{\varepsilon\searrow 0}\lim_{y\searrow 1}\frac{2^{\alpha\hat\rho-1}c_{\alpha\rho}}{g(y)}\mathbb{E}^x\big[\mathbb{1}_{\{\xi_{T_{K^\mathrm{C}}}>y+\varepsilon \text{ or } \xi_{T_{K^\mathrm{C}}}<-1\}}u_{[-1,1]}(\xi_{T_{K^\mathrm{C}}},y)\big]\\
&\quad - \lim_{\varepsilon\searrow 0}\lim_{y\searrow 1}\mathbb{E}^x\Big[\big(\sin(\pi\alpha\hat\rho)\mathbb{1}_{\{\xi_{T_{K^\mathrm{C}}}>y+\varepsilon\}}+\sin(\pi\alpha\rho)\mathbb{1}_{\{\xi_{T_{K^\mathrm{C}}}<-1\}}\big) \qquad (6.4)\\
&\qquad \times \frac{(1-\alpha\hat\rho)|\xi_{T_{K^\mathrm{C}}}-y|^{\alpha-1}}{g(y)}\int_1^{z(\xi_{T_{K^\mathrm{C}}},y)}(u-1)^{\alpha\rho}(u+1)^{\alpha\hat\rho-2}\,\mathrm{d}u\Big]\\
&\quad + \lim_{\varepsilon\searrow 0}\lim_{y\searrow 1}(\alpha-1)_+\Big(\frac{\alpha\rho}{g(y)}\int_1^y\psi_{\alpha\hat\rho}(u)\,\mathrm{d}u-1\Big)\\
&\qquad \times \mathbb{E}^x\Big[\sin(\pi\alpha\hat\rho)\mathbb{1}_{\{\xi_{T_{K^\mathrm{C}}}>y+\varepsilon\}}\int_1^{\xi_{T_{K^\mathrm{C}}}}\psi_{\alpha\rho}(u)\,\mathrm{d}u+\sin(\pi\alpha\rho)\mathbb{1}_{\{\xi_{T_{K^\mathrm{C}}}<-1\}}\int_1^{|\xi_{T_{K^\mathrm{C}}}|}\psi_{\alpha\hat\rho}(u)\,\mathrm{d}u\Big]
\end{aligned}$$

We care about these three summands separately. We start with the last one which just appears if $\alpha > 1$. From the proof of Corollary 6.2.2 we already know that

$$\frac{\alpha\rho}{g(y)}\int_1^y\psi_{\alpha\hat\rho}(u)\,\mathrm{d}u-1$$

converges to 0 for $y \searrow 0$. Furthermore, we get with monotone convergence:

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\lim_{y\searrow 1}\mathbb{E}^x\Big[\sin(\pi\alpha\hat\rho)\mathbb{1}_{\{\xi_{T_{K^\mathrm{C}}}>y+\varepsilon\}}\int_1^{\xi_{T_{K^\mathrm{C}}}}\psi_{\alpha\rho}(u)\,\mathrm{d}u+\sin(\pi\alpha\rho)\mathbb{1}_{\{\xi_{T_{K^\mathrm{C}}}<-1\}}\int_1^{|\xi_{T_{K^\mathrm{C}}}|}\psi_{\alpha\hat\rho}(u)\,\mathrm{d}u\Big]\\
&= \mathbb{E}^x\Big[\sin(\pi\alpha\hat\rho)\mathbb{1}_{\{\xi_{T_{K^\mathrm{C}}}>1\}}\int_1^{\xi_{T_{K^\mathrm{C}}}}\psi_{\alpha\rho}(u)\,\mathrm{d}u+\sin(\pi\alpha\rho)\mathbb{1}_{\{\xi_{T_{K^\mathrm{C}}}<-1\}}\int_1^{|\xi_{T_{K^\mathrm{C}}}|}\psi_{\alpha\hat\rho}(u)\,\mathrm{d}u\Big]
\end{aligned}$$

$$= \frac{\pi}{\Gamma(1-\alpha\rho)\Gamma(1-\alpha\hat{\rho})}\mathbb{E}^x\Big[\mathbb{1}_{\{T_{K^\complement}<T_{[-1,1]}\}}h(\xi_{T_{K^\complement}})\Big],$$

where h is the invariant function which appears in Chapter 5. But since the h-transformed process with this invariant function is transient with infinite lifetime (see Theorem 1.3 in that article) it leaves all compact sets almost surely. Hence, we have

$$\begin{aligned}\mathbb{E}^x\Big[\mathbb{1}_{\{T_{K^\complement}<T_{[-1,1]}\}}\frac{h(\xi_{T_{K^\complement}})}{h(x)}\Big] &= \mathbb{P}^x_h(T_{K^\complement}<\zeta)\\ &= \mathbb{P}^x_h(T_{K^\complement}<\infty)\\ &= 1,\end{aligned}$$

thus, $\mathbb{E}^x\big[\mathbb{1}_{\{T_{K^\complement}<T_{[-1,1]}\}}h(\xi_{T_{K^\complement}})\big] = h(x)<\infty$. It follows that the third term of (6.4) is 0. So it remains to consider the first and the second summand of (6.4). With Proposition 6.2 (ii) of Kunita and Watanabe [33] and Corollary 6.2.2 we see for the first term:

$$\begin{aligned}&\lim_{\varepsilon\searrow 0}\lim_{y\searrow 1}\frac{2^{\alpha\hat{\rho}-1}c_{\alpha\rho}}{g(y)}\mathbb{E}^x\big[\mathbb{1}_{\{T_{K^\complement}<T_{[-1,1]}\}}\mathbb{1}_{\{\xi_{T_{K^\complement}}>y+\varepsilon \text{ or } \xi_{T_{K^\complement}}<-1\}}u_{[-1,1]}(\xi_{T_{K^\complement}},y)\big]\\
&=\lim_{y\searrow 1}\frac{2^{\alpha\hat{\rho}-1}c_{\alpha\rho}}{g(y)}\mathbb{E}^x\big[\mathbb{1}_{\{T_{K^\complement}<T_{[-1,1]}\}}u_{[-1,1]}(\xi_{T_{K^\complement}},y)\big]\\
&\quad-\lim_{\varepsilon\searrow 0}\lim_{y\searrow 1}\frac{2^{\alpha\hat{\rho}-1}c_{\alpha\rho}}{g(y)}\mathbb{E}^x\big[\mathbb{1}_{\{T_{K^\complement}<T_{[-1,1]}\}}\mathbb{1}_{\{\xi_{T_{K^\complement}}\in(1,y+\varepsilon)\}}u_{[-1,1]}(\xi_{T_{K^\complement}},y)\big]\\
&=\lim_{y\searrow 1}\frac{2^{\alpha\hat{\rho}-1}c_{\alpha\rho}}{g(y)}u_{[-1,1]}(x,y)\\
&\quad-\lim_{\varepsilon\searrow 0}\lim_{y\searrow 1}\frac{2^{\alpha\hat{\rho}-1}c_{\alpha\rho}}{g(y)}\mathbb{E}^x\big[\mathbb{1}_{\{T_{K^\complement}<T_{[-1,1]}\}}\mathbb{1}_{\{\xi_{T_{K^\complement}}\in(1,y+\varepsilon)\}}u_{[-1,1]}(\xi_{T_{K^\complement}},y)\big]\\
&=v_1(x)-\lim_{\varepsilon\searrow 0}\lim_{y\searrow 1}\frac{2^{\alpha\hat{\rho}-1}c_{\alpha\rho}}{g(y)}\mathbb{E}^x\big[\mathbb{1}_{\{T_{K^\complement}<T_{[-1,1]}\}}\mathbb{1}_{\{\xi_{T_{K^\complement}}\in(1,y+\varepsilon)\}}u_{[-1,1]}(\xi_{T_{K^\complement}},y)\big].\end{aligned}$$

Hence, to prove harmonicity of v_1 it suffices to show

$$\lim_{y\searrow 1}\mathbb{E}^x\Big[\mathbb{1}_{\{T_{K^\complement}<T_{[-1,1]},|\xi_{T_{K^\complement}}-y|>\varepsilon\}}\frac{|\xi_{T_{K^\complement}}-y|^{\alpha-1}}{g(y)}\int_1^{z(\xi_{T_{K^\complement}},y)}(u-1)^{\alpha\rho}(u+1)^{\alpha\hat{\rho}-2}\,\mathrm{d}u\Big]=0 \tag{6.5}$$

for all $\varepsilon>0$ and

$$\lim_{\varepsilon\searrow 0}\lim_{y\searrow 1}\frac{1}{g(y)}\mathbb{E}^x\big[\mathbb{1}_{\{T_{K^\complement}<T_{[-1,1]}\}}\mathbb{1}_{\{\xi_{T_{K^\complement}}\in(1,y+\varepsilon)\}}u_{[-1,1]}(\xi_{T_{K^\complement}},y)\big]=0. \tag{6.6}$$

We start with (6.5). First we note that

$$\begin{aligned}
\int_1^{z(\xi_{T_{K^\complement}},y)} (u-1)^{\alpha\rho}(u+1)^{\alpha\hat\rho-2}\,\mathrm{d}u &\le (z(\xi_{T_{K^\complement}},y)-1)^{\alpha\rho}\int_1^{z(\xi_{T_{K^\complement}},y)}(u+1)^{\alpha\hat\rho-2}\,\mathrm{d}u\\
&\le C_1(z(\xi_{T_{K^\complement}},y)-1)^{\alpha\rho}\\
&=\begin{cases} C_1\frac{(\xi_{T_{K^\complement}}+1)^{\alpha\rho}(y-1)^{\alpha\rho}}{|\xi_{T_{K^\complement}}-y|^{\alpha\rho}} & \text{if } \xi_{T_{K^\complement}}>y+\varepsilon\\ C_1\frac{(|\xi_{T_{K^\complement}}|-1)^{\alpha\rho}(y-1)^{\alpha\rho}}{|\xi_{T_{K^\complement}}-y|^{\alpha\rho}} & \text{if } \xi_{T_{K^\complement}}<-1\end{cases} \qquad (6.7)\\
&\le C_1\frac{(|\xi_{T_{K^\complement}}|+1)^{\alpha\rho}(y-1)^{\alpha\rho}}{|\xi_{T_{K^\complement}}-y|^{\alpha\rho}}
\end{aligned}$$

where $C_1=\int_1^\infty (u+1)^{\alpha\hat\rho-2}\,\mathrm{d}u<\infty$. With that we get on $\{|\xi_{T_{K^\complement}}-y|>\varepsilon\}$ (without loss of generality we assume $y<2$):

$$\begin{aligned}
&\frac{|\xi_{T_{K^\complement}}-y|^{\alpha-1}}{g(y)}\int_1^{z(\xi_{T_{K^\complement}},y)}(u-1)^{\alpha\rho}(u+1)^{\alpha\hat\rho-2}\,\mathrm{d}u\\
&\le C_1\frac{|\xi_{T_{K^\complement}}-y|^{\alpha-1}}{g(y)}\frac{(|\xi_{T_{K^\complement}}|+1)^{\alpha\rho}(y-1)^{\alpha\rho}}{|\xi_{T_{K^\complement}}-y|^{\alpha\rho}}\\
&=C_1|\xi_{T_{K^\complement}}-y|^{\alpha\hat\rho-1}(|\xi_{T_{K^\complement}}|+1)^{\alpha\rho}(y+1)^{1-\alpha\hat\rho}\\
&\le C_1|\xi_{T_{K^\complement}}-y|^{\alpha\hat\rho-1}(|\xi_{T_{K^\complement}}-y|^{\alpha\rho}+(y+1)^{\alpha\rho})(y+1)^{1-\alpha\hat\rho}\\
&=C_1(y+1)^{1-\alpha\hat\rho}(|\xi_{T_{K^\complement}}-y|^{\alpha-1}+|\xi_{T_{K^\complement}}-y|^{\alpha\hat\rho-1}(y+1)^{\alpha\rho})\\
&\le C_1 3^{1-\alpha\hat\rho}(\varepsilon^{\alpha-1}+3^{\alpha\rho}\varepsilon^{\alpha\hat\rho-1})\\
&\le C_1 3^{1+\alpha\rho-\alpha\hat\rho}(\varepsilon^{\alpha-1}+\varepsilon^{\alpha\hat\rho-1})=:C_\varepsilon
\end{aligned}$$

Hence, we can use dominated convergence to switch the y-limit and the expectation in (6.5). The following calculation on $\{|\xi_{T_{K^\complement}}-y|>\varepsilon\}$ shows that the integrand converges pointwise to 0 which shows (6.5):

$$\begin{aligned}
&\frac{|\xi_{T_{K^\complement}}-y|^{\alpha-1}}{g(y)}\int_1^{z(\xi_{T_{K^\complement}},y)}(u-1)^{\alpha\rho}(u+1)^{\alpha\hat\rho-2}\,\mathrm{d}u\\
&\le 2^{\alpha\hat\rho-2}\frac{|\xi_{T_{K^\complement}}-y|^{\alpha-1}}{g(y)}\int_1^{z(\xi_{T_{K^\complement}},y)}(u-1)^{\alpha\rho}\,\mathrm{d}u\\
&=\frac{2^{\alpha\hat\rho-2}}{\alpha\rho+1}\frac{|\xi_{T_{K^\complement}}-y|^{\alpha-1}}{g(y)}(z(\xi_{T_{K^\complement}},y)-1)^{\alpha\rho+1}\\
&\le\frac{2^{\alpha\hat\rho-2}}{\alpha\rho+1}\frac{|\xi_{T_{K^\complement}}-y|^{\alpha-1}}{(y-1)^{\alpha\rho}(y+1)^{\alpha\hat\rho-1}}\frac{(|\xi_{T_{K^\complement}}|+1)^{\alpha\rho+1}(y-1)^{\alpha\rho+1}}{|\xi_{T_{K^\complement}}-y|^{\alpha\rho+1}}\\
&=\frac{2^{\alpha\hat\rho-2}}{\alpha\rho+1}(|\xi_{T_{K^\complement}}-y|^{\alpha\hat\rho-2}(|\xi_{T_{K^\complement}}|+1)^{\alpha\rho+1}(y-1)(y+1)^{1-\alpha\hat\rho}
\end{aligned}$$

$$\xrightarrow{y\searrow 1} 0$$

where we used the same estimate for $z(\xi_{T_{K^\mathrm{C}}}, y) - 1$ as in (6.7). This shows (6.5).

Now we show (6.6). We define $a = \min(\inf(K \cap (1,\infty), -\sup(K \cap (-\infty, -1))$. Sinc y tends to 1 and ε to 0 we can assume $a > y + \varepsilon$. It follows that $\xi_{T_{K^\mathrm{C}}} \in (1, y+\varepsilon)$ is just possible if $T_{K^\mathrm{C}} = T_{(-a,a)}$. So we have

$$\begin{aligned}
&\mathbb{E}^x\big[\mathbb{1}_{\{T_{K^\mathrm{C}} < T_{[-1,1]}\}}\mathbb{1}_{\{\xi_{T_{K^\mathrm{C}}} \in (1,y+\varepsilon)\}} u_{[-1,1]}(\xi_{T_{K^\mathrm{C}}}, y)\big] \\
&\le \mathbb{E}^x\big[\mathbb{1}_{\{\xi_{T_{(-a,a)}} \in (1,y+\varepsilon), T_{(-a,a)} < \infty\}} u_{[-1,1]}(\xi_{T_{(-a,a)}}, y)\big].
\end{aligned}$$

Further, $\xi_{T_{(-a,a)}} = y$ happens with zero probability and with this follows

$$\begin{aligned}
&\mathbb{E}^x\big[\mathbb{1}_{\{\xi_{T_{(-a,a)}} \in (1,y+\varepsilon), T_{(-a,a)} < \infty\}} u_{[-1,1]}(\xi_{T_{(-a,a)}}, y)\big] \\
&= \mathbb{E}^x\big[\mathbb{1}_{\{\xi_{T_{(-a,a)}} \in (1,y), T_{(-a,a)} < \infty\}} u_{[-1,1]}(\xi_{T_{(-a,a)}}, y)\big] \\
&\quad + \mathbb{E}^x\big[\mathbb{1}_{\{\xi_{T_{(-a,a)}} \in (y,\varepsilon), T_{(-a,a)} < \infty\}} u_{[-1,1]}(\xi_{T_{(-a,a)}}, y)\big].
\end{aligned}$$

With the formulas for $u_{[-1,1]}$ of Profeta and Simon [49] we get for $\xi_{T_{(-a,a)}} \in (1,y)$:

$$\begin{aligned}
u_{[-1,1]}(\xi_{T_{(-a,a)}}, y) &\le \frac{2^{1-\alpha}}{\Gamma(\alpha\rho)\Gamma(\alpha\hat\rho)}(y - \xi_{T_{(-a,a)}})^{\alpha-1} \int_1^{z(\xi_{T_{(-a,a)}}, y)} (u-1)^{\alpha\hat\rho-1}(u+1)^{\alpha\rho-1}\,\mathrm{d}u \\
&\le \frac{2^{-\alpha\hat\rho}}{\alpha\hat\rho\Gamma(\alpha\rho)\Gamma(\alpha\hat\rho)}(y - \xi_{T_{(-a,a)}})^{\alpha-1}(z(\xi_{T_{(-a,a)}}, y) - 1)^{\alpha\hat\rho} \\
&\le \frac{2^{-\alpha\hat\rho}}{\alpha\hat\rho\Gamma(\alpha\rho)\Gamma(\alpha\hat\rho)}(y - \xi_{T_{(-a,a)}})^{\alpha-1}\Big(\frac{(\xi_{T_{(-a,a)}} - 1)(y+1)}{y - \xi_{T_{(-a,a)}}}\Big)^{\alpha\hat\rho} \\
&= \frac{1}{\alpha\hat\rho\Gamma(\alpha\rho)\Gamma(\alpha\hat\rho)}(y - \xi_{T_{(-a,a)}})^{\alpha\rho-1}(\xi_{T_{(-a,a)}} - 1)^{\alpha\hat\rho} \\
&\le \frac{1}{\alpha\hat\rho\Gamma(\alpha\rho)\Gamma(\alpha\hat\rho)}(y - \xi_{T_{(-a,a)}})^{\alpha\rho-1}(y - 1)^{\alpha\hat\rho}.
\end{aligned}$$

It follows for $x > a$ with Theorem 1.1 of Kyprianou et al. [38] and the scaling property:

$$\begin{aligned}
&\mathbb{E}^x\left[\mathbb{1}_{\left\{\xi_{T_{(-a,a)}} \in (1,y), T_{(-a,a)} < \infty\right\}} u_{[-1,1]}(\xi_{T_{(-a,a)}}, y)\right] \\
&\le \frac{1}{\Gamma(\alpha\rho)\Gamma(\alpha\hat\rho)}(y-1)^{\alpha\hat\rho}\mathbb{E}^x\left[\mathbb{1}_{\left\{\xi_{T_{(-a,a)}} \in (1,y), T_{(-a,a)} < \infty\right\}}(y - \xi_{T_{(-a,a)}})^{\alpha\rho-1}\right] \\
&\le \frac{\sin(\pi\alpha\hat\rho)}{\pi\Gamma(\alpha\rho)\Gamma(\alpha\hat\rho)}(x+a)^{\alpha\rho}(x-a)^{\alpha\hat\rho}(y-1)^{\alpha\hat\rho}\int_{(1,y)} \frac{(y-u)^{\alpha\rho-1}}{(a+u)^{\alpha\rho}(a-u)^{\alpha\hat\rho}(x-u)}\,\mathrm{d}u \\
&\le \frac{a\sin(\pi\alpha\hat\rho)}{\pi\Gamma(\alpha\rho)\Gamma(\alpha\hat\rho)}\frac{(x+a)^{\alpha\rho}(x-a)^{\alpha\hat\rho}}{(a+1)^{\alpha\rho}(a-y)^{\alpha\hat\rho}(x-y)}(y-1)^{\alpha\hat\rho}\int_{(1,y)}(y-u)^{\alpha\rho-1}\mathrm{d}u \\
&= \frac{a\sin(\pi\alpha\hat\rho)}{\pi\alpha\rho\Gamma(\alpha\rho)\Gamma(\alpha\hat\rho)}\frac{(x+a)^{\alpha\rho}(x-a)^{\alpha\hat\rho}}{(a+1)^{\alpha\rho}(a-y)^{\alpha\hat\rho}(x-y)}(y-1)^{\alpha\hat\rho}(y-1)^{\alpha\rho}.
\end{aligned}$$

With this estimate we see immediately

$$\lim_{y\searrow 1}\frac{1}{(y-1)^{\alpha\rho}}\mathbb{E}^x\big[\mathbb{1}_{\{\xi_{T_{(-a,a)}}\in(1,y)\}}u_{[-1,1]}(\xi_{T_{(-a,a)}},y)\big]=0$$

for $x>1$. For $x<-1$ we use Theorem 1.1 of Kyprianou et al. [38] in a similar way to deduce the analogous claim. Similarly we get for $\xi_{T_{(-a,a)}}\in(y,y+\varepsilon)$ (without loss of generality $y+\varepsilon<2$):

$$\begin{aligned}
u_{[-1,1]}(\xi_{T_{(-a,a)}},y)&\leq\frac{2^{1-\alpha}}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}(\xi_{T_{(-a,a)}}-y)^{\alpha-1}\int\limits_1^{z(\xi_{T_{K^{\mathrm{C}}}},y)}(u-1)^{\alpha\rho-1}(u+1)^{\alpha\hat{\rho}-1}\,\mathrm{d}u\\
&\leq\frac{2^{-\alpha\rho}}{\alpha\rho\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}(\xi_{T_{(-a,a)}}-y)^{\alpha-1}(z(\xi_{T_{(-a,a)}},y)-1)^{\alpha\rho}\\
&=\frac{2^{-\alpha\rho}}{\alpha\rho\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}(\xi_{T_{(-a,a)}}-y)^{\alpha\hat{\rho}-1}(\xi_{T_{(-a,a)}}+1)^{\alpha\rho}(y-1)^{\alpha\rho}\\
&\leq\frac{2^{-\alpha\rho}3^{\alpha\rho}}{\alpha\rho\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}(y-1)^{\alpha\rho}(\xi_{T_{(-a,a)}}-y)^{\alpha\hat{\rho}-1}.
\end{aligned}$$

Define $C_2:=\frac{2^{-\alpha\rho}3^{\alpha\rho}}{\alpha\rho\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}$ and we get again with Theorem 1.1 of [38] for $x>1$

$$\begin{aligned}
&\mathbb{E}^x\big[\mathbb{1}_{\{\xi_{T_{(-a,a)}}\in(y,y+\varepsilon),T_{(-a,a)}<\infty\}}u_{[-1,1]}(\xi_{T_{(-a,a)}},y)\big]\\
&\leq C_2(y-1)^{\alpha\rho}\mathbb{E}^x\big[\mathbb{1}_{\{\xi_{T_{(-a,a)}}\in(y,y+\varepsilon),T_{(-a,a)}<\infty\}}(\xi_{T_{(-a,a)}}-y)^{\alpha\hat{\rho}-1}\big]\\
&\leq\frac{C_2a\sin(\pi\alpha\hat{\rho})}{\pi}(x+a)^{\alpha\rho}(x-a)^{\alpha\hat{\rho}}(y-1)^{\alpha\rho}\int\limits_{(y,y+\varepsilon)}\frac{(u-y)^{\alpha\hat{\rho}-1}}{(a+u)^{\alpha\rho}(a-u)^{\alpha\hat{\rho}}(x-u)}\,\mathrm{d}u\\
&\leq\frac{C_2a\sin(\pi\alpha\hat{\rho})}{\pi}\frac{(x+a)^{\alpha\rho}(x-a)^{\alpha\hat{\rho}}(y-1)^{\alpha\rho}}{(a+y)^{\alpha\rho}(a-(y+\varepsilon))^{\alpha\hat{\rho}}(x-(y+\varepsilon))}\int\limits_{(y,y+\varepsilon)}(u-y)^{\alpha\hat{\rho}-1}\mathrm{d}u\\
&=\frac{C_2a\sin(\pi\alpha\hat{\rho})}{\pi}\frac{(x+a)^{\alpha\rho}(x-a)^{\alpha\hat{\rho}}}{(a+y)^{\alpha\rho}(a-(y+\varepsilon))^{\alpha\hat{\rho}}(x-(y+\varepsilon))}\frac{(y-1)^{\alpha\rho}\varepsilon^{\alpha\hat{\rho}}}{\alpha\hat{\rho}}.
\end{aligned}$$

So we have:

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\lim_{y\searrow 1}\frac{1}{(y-1)^{\alpha\rho}}\mathbb{E}^x\big[\mathbb{1}_{\{T_{K^{\mathrm{C}}}<T_{[-1,1]}\}}\mathbb{1}_{\{\xi_{T_{K^{\mathrm{C}}}}\in(1,y+\varepsilon)\}}u_{[-1,1]}(\xi_{T_{K^{\mathrm{C}}}},y)\big]\\
&\leq\frac{C_2\sin(\pi\alpha\hat{\rho})}{\pi\alpha\hat{\rho}}\lim_{\varepsilon\searrow 0}\lim_{y\searrow 1}\Big[\frac{(x+a)^{\alpha\rho}(x-a)^{\alpha\hat{\rho}}}{(a+y)^{\alpha\rho}(a-(y+\varepsilon))^{\alpha\hat{\rho}}(x-(y+\varepsilon))}\varepsilon^{\alpha\hat{\rho}}\Big]\\
&=0.
\end{aligned}$$

The claim for $x<-1$ follows again similarly. This shows (6.6) and hence, we have harmonicity of v_1. □

Remark 6.2.3. *If $\alpha\leq 1$, another (maybe more elegant) way of proving harmonicity of v_1 is to prove that the renewal densities of the MAP which corresponds to the stable process via the Lamperti-Kiu transform (for explicit expressions see Corollary 1.6 of [41]) are harmonic functions for the MAP killed on entering the negative half-line. This claim should be true since Silverstein [55] proved the analogous claim for a Lévy process which does not drift to $-\infty$. One can show that v_1 and v_{-1} are just these renewal densities (the argument replaced by the logarithm). Via the Lamperti-Kiu transform one could obtain harmonicity of v_1 and v_{-1} for the stable process killed in $[-1,1]$.*

6.2.2 Behaviour at the killing time

Before we start with the proofs we should discuss more elementary properties of v_1 and v_{-1}. First, it can be seen immediately that v_1 has a pole in 1 and v_{-1} has a pole in -1 and hence $v := v_1 + v_{-1}$ has poles in 1 and -1. Further, v_1 is bounded on $(-\infty,-1) \cup (K,\infty)$ for all $K > 1$. For $\alpha \leq 1$ this is obvious and for $\alpha > 1$ this can be seen via showing that v_1 converges for $x \to \pm\infty$ (a similar convergence was shown in [27] in the proof of Lemma 3.3). Similarly v_{-1} is bounded on $(-\infty,-K) \cup (1,\infty)$ for all $K > 1$. It follows obviously that v is bounded on $(-\infty,-K_1) \cup (K_2,\infty)$ for all $K_1, K_2 > 1$.

For the first results we need to define the potential of the h-transformed process via

$$U_{v_1}(x, \mathrm{d}y) = \mathbb{E}^x_{v_1}\Big[\int_0^\zeta \mathbb{1}_{\{\xi_t \in \mathrm{d}y\}}\,\mathrm{d}t\Big], \quad x, y \notin [-1,1],$$

which is the expected time the process $(\xi, \mathbb{P}^x_{v_1})$ stays in $\mathrm{d}y$ until it is killed. With a Fubini flip we obtain

$$U_{v_1}(x, \mathrm{d}y) = \frac{v_1(y)}{v_1(x)}\, U_{[-1,1]}(x, \mathrm{d}y) = \frac{v_1(y)}{v_1(x)} u_{[-1,1]}(x,y)\,\mathrm{d}y.$$

The following result shows on the one hand that the h-transformed process is almost surely bounded and second that the expected time the process stays in a set of the form $[-b,-1) \cup (1,b]$ is finite.

Lemma 6.2.4. *Let ξ be an α-stable process with $\alpha \in (0,2)$ and both sided jumps. Then it holds for $x \notin [-1,1]$:*

(i) $\mathbb{P}^x_{v_1}(T_{(-\infty,-d]\cup[d,\infty)} < \zeta\, \forall d > 1) = 0.$

(ii) $U_{v_1}(x, [-b,-1) \cup (1,b]) < \infty$ *for all* $b > 1$.

Proof. (i) We already noticed that v_1 is bounded on $(-\infty,-K) \cup (K,\infty)$ for all $K > 1$. So we obtain, applying dominated convergence in the last equality,

$$\begin{aligned}
\mathbb{P}^x_{v_1}(T_{(-\infty,-d]\cup[d,\infty)} < \zeta\, \forall d > 1) &= \lim_{d\to\infty} \mathbb{P}^x_{v_1}(T_{(-\infty,-d]\cup[d,\infty)} < \zeta) \\
&= \lim_{d\to\infty} \mathbb{E}^x\Big[\mathbb{1}_{\{T_{(-\infty,-d]\cup[d,\infty)} < T_{[-1,1]}\}} \frac{v_1(\xi_{T_{(-\infty,-d]\cup[d,\infty)}})}{v_1(x)}\Big] \\
&= \mathbb{E}^x\Big[\lim_{d\to\infty} \mathbb{1}_{\{T_{(-\infty,-d]\cup[d,\infty)} < T_{[-1,1]}\}} \frac{v_1(\xi_{T_{(-\infty,-d]\cup[d,\infty)}})}{v_1(x)}\Big].
\end{aligned}$$

In the case $\alpha < 1$ we use that $v_1(y)$ converges to 0 for $y \to \pm\infty$. If $\alpha \geq 1$, we see that $\mathbb{1}_{\{T_{(-\infty,-d]\cup[d,\infty)} < T_{[-1,1]}\}}$ converges to 0 almost surely since $(\xi, \mathbb{P}^x)$ is recurrent. This shows (i).

(ii) It holds

$$U_{v_1}(x, [-b,-1) \cup (1,b]) = \frac{1}{v_1(x)} \int_{[-b,-1)\cup(1,b]} v_1(y) u_{[-1,1]}(x,y)\,\mathrm{d}y.$$

Since v_1 is bounded and $u_{[-1,1]}(x,\cdot)$ is integrable on all compact intervals, the only points where this integral could be infinite, are the boundary points 1 and -1. From the explicit formulas of [49] we see that $u_{[-1,1]}(x,y)$ converges to 0 for $y \to \pm 1$. Further, $v_1(y)$ behaves as $(y-1)^{\alpha\hat{\rho}-1}$ for $y \searrow 1$ and as $(|y|-1)^{\alpha\rho}$ for $y \nearrow -1$. Since $\alpha\rho, \alpha\hat{\rho} \in (0,1)$ these arguments shows $U_{v_1}(x,[-b,-1)\cup(1,b]) < \infty$. $\square$

Combining the two statements of Lemma 6.2.4 we can show Proposition 6.1.2.

Proof of Proposition 6.1.2. We show that $\mathbb{P}^x_{v_1}(\zeta < \infty) = 1$ and $\mathbb{P}^x_{v_1}(\xi_{\zeta-} = 1) = 1$ and start with the first equality. From Lemma 6.2.4 (ii) we know

$$\mathbb{P}^x_{v_1}\Big(\int_0^\zeta \mathbb{1}_{\{\xi_t \in [-b,-1)\cup(1,b]\}}\,\mathrm{d}t < \infty\Big) = 1$$

for all $b > 1$. By the continuity of probability measures we see

$$\begin{aligned}
&\mathbb{P}^x_{v_1}\Big(\int_0^\zeta \mathbb{1}_{\{\xi_t \in [-b,-1)\cup(1,b]\}}\,\mathrm{d}t < \infty \,\forall b > 1\Big)\\
&= \lim_{b\to\infty} \mathbb{P}^x_{v_1}\Big(\int_0^\zeta \mathbb{1}_{\{\xi_t \in [-b,-1)\cup(1,b]\}}\,\mathrm{d}t < \infty\Big) = 1.
\end{aligned}$$

On the other hand Lemma 6.2.4 (i) yields

$$\mathbb{P}^x_{v_1}(\exists d > 1 : T_{(-\infty,-d]\cup[d,\infty)} \geq \zeta) = 1.$$

Since the intersection of two events with probability 1 has again probability 1 it follows:

$$\begin{aligned}
\mathbb{P}^x_{v_1}(\zeta < \infty) &= \mathbb{P}^x_{v_1}\Big(\int_0^\zeta \mathbb{1}_{\{\xi_t \in \mathbb{R}\setminus[-1,1]\}}\,\mathrm{d}t < \infty\Big)\\
&\geq \mathbb{P}^x_{v_1}\Big(\int_0^\zeta \mathbb{1}_{\{\xi_t \in [-b,-1)\cup(1,b]\}}\,\mathrm{d}t < \infty \,\forall b > 1\,,\, \exists d > 1 : T_{(-\infty,-d]\cup[d,\infty)} \geq \zeta\Big)\\
&= 1.
\end{aligned}$$

To prove $\mathbb{P}^x_{v_1}(\xi_{\zeta-} = 1) = 1$ we use a procedure which is inspired by Chaumont [15]. Using that v_1 is harmonic (Theorem 6.1.1) we see for $x \notin [-1,1]$ and

$$M_{a,b} = (-\infty,-b)\cup(-a,-1)\cup(1,1+\varepsilon)\cup(b,\infty)$$

with $1 < a < b$ and $\varepsilon > 0$ (obviously the complement of $M_{a,b}$ is compact in $\mathbb{R}\setminus[-1,1]$):

$$\mathbb{E}^x\big[\mathbb{1}_{\{T_{M_{a,b}} < T_{[-1,1]}\}} v_1(\xi_{T_{M_{a,b}}})\big] = v_1(x).$$

It follows that

$$\mathbb{P}^x_{v_1}(T_{M_{a,b}} < \zeta) = \frac{1}{v_1(x)}\mathbb{E}^x\big[\mathbb{1}_{\{T_{M_{a,b}} < T_{[-1,1]}\}} v_1(\xi_{T_{M_{a,b}}})\big] = 1.$$

From Lemma 6.2.4 we know on the one hand

$$\mathbb{P}^x_{v_1}(T_{(-\infty,-b)\cup(b,\infty)} < \zeta \,\forall\, b > 1) = 0. \tag{6.8}$$

On the other hand we see, applying dominated convergence using that v_1 is bounded on $(-\infty,-1)$,

$$\begin{aligned}\mathbb{P}^x_{v_1}(T_{(-a,-1)} < \zeta\,\forall\, a>1) &= \lim_{a\searrow 1}\mathbb{P}^x_{v_1}(T_{(-a,-1)}<\zeta)\\ &= \lim_{a\searrow 1}\frac{1}{v_1(x)}\mathbb{E}^x\big[\mathbb{1}_{\{T_{(-a,-1)}<T_{[-1,1]}\}}v_1(\xi_{T_{(-a,-1)}})\big]\\ &= \frac{1}{v_1(x)}\mathbb{E}^x\big[\lim_{a\searrow 1}\mathbb{1}_{\{T_{(-a,-1)}<T_{[-1,1]}\}}v_1(\xi_{T_{(-a,-1)}})\big]\\ &= 0.\end{aligned} \tag{6.9}$$

In the last step we used that $v_1(y)$ converges to 0 for $y\nearrow -1$. Note that this argument does not work if $(-a,-1)$ is replaced by $(1,a)$ because v_1 has a pole in 1. Now we plug in (6.8) and (6.9) to obtain for all $\varepsilon>0$:

$$\begin{aligned}&\mathbb{P}^x_{v_1}(T_{(1,1+\varepsilon)}<\zeta)\\ &=\mathbb{P}^x_{v_1}(\{T_{(1,1+\varepsilon)}<\zeta\}\cup\{T_{(-a,-1)}<\zeta\,\forall\, a>1\}\cup\{T_{(-\infty,-b)\cup(b,\infty)}<\zeta\,\forall\, b>1\})\\ &=\lim_{b\to\infty}\lim_{a\searrow 1}\mathbb{P}^x_{v_1}(\big\{T_{(1,1+\varepsilon)}<\zeta\big\}\cup\big\{T_{(-a,-1)}<\zeta\big\}\cup\big\{T_{(-\infty,-b)\cup(b,\infty)}<\zeta\big\})\\ &=\lim_{b\to\infty}\lim_{a\searrow 1}\mathbb{P}^x_{v_1}(T_{M_{a,b}}<\zeta)\\ &=1.\end{aligned}$$

With this in hand we show the final claim that $\xi_{\zeta-}=1$ almost surely under $\mathbb{P}^x_{v_1}$. By $(1,1+\delta)^{\mathrm{C}}$ we mean as usual $\mathbb{R}\setminus[-1,1]\setminus(1,1+\delta)$.

$$\begin{aligned}\mathbb{P}^x_{v_1}(\xi_{\zeta-}=1) &= \mathbb{P}^x_{v_1}(\forall\,\delta>0\,\exists\,\varepsilon\in(0,\delta]\,:\,\xi_t\in(1,1+\delta)\,\forall\, t\in[T_{(1,1+\varepsilon)},\zeta))\\ &=\lim_{\delta\searrow 0}\lim_{\varepsilon\searrow 0}\mathbb{P}^x_{v_1}(\xi_t\in(1,1+\delta)\,\forall\, t\in[T_{(1,1+\varepsilon)},\zeta))\\ &=\lim_{\delta\searrow 0}\lim_{\varepsilon\searrow 0}\mathbb{E}^x_{v_1}\big[\mathbb{P}^{\xi_{T_{(1,1+\varepsilon)}}}_{v_1}(\xi_t\in(1,1+\delta)\,\forall\, t\in[0,\zeta))\big]\\ &=\lim_{\delta\searrow 0}\lim_{\varepsilon\searrow 0}\mathbb{E}^x_{v_1}\big[\mathbb{P}^{\xi_{T_{(1,1+\varepsilon)}}}_{v_1}(T_{(1,1+\delta)^{\mathrm{C}}}\geq\zeta)\big]\\ &=1-\lim_{\delta\searrow 0}\lim_{\varepsilon\searrow 0}\mathbb{E}^x_{v_1}\big[\mathbb{P}^{\xi_{T_{(1,1+\varepsilon)}}}_{v_1}(T_{(1,1+\delta)^{\mathrm{C}}}<\zeta)\big]\\ &=1-\lim_{\delta\searrow 0}\mathbb{E}^x_{v_1}\big[\lim_{\varepsilon\searrow 0}\mathbb{P}^{\xi_{T_{(1,1+\varepsilon)}}}_{v_1}(T_{(1,1+\delta)^{\mathrm{C}}}<\zeta)\big]\\ &=1-\lim_{\delta\searrow 0}\mathbb{E}^x_{v_1}\big[\lim_{\varepsilon\searrow 0}\mathbb{P}^{1+\varepsilon}_{v_1}(T_{(1,1+\delta)^{\mathrm{C}}}<\zeta)\big].\end{aligned} \tag{6.10}$$

In the second equality we used that $T_{(1,1+\varepsilon)}<\zeta$ almost surely and in the third equality we used the strong Markov property of $(\xi,\mathbb{P}^x_{v_1})$. Let us consider the ε-limit inside the expectation. Using the definition of $\mathbb{P}^x_{v_1}$ we see:

$$\mathbb{P}^{1+\varepsilon}_{v_1}(T_{(1,1+\delta)}<\zeta)=\frac{1}{v_1(1+\varepsilon)}\mathbb{E}^x\big[\mathbb{1}_{\{T_{(1,1+\delta)^{\mathrm{C}}}<T_{[-1,1]}\}}v_1(\xi_{T_{(1,1+\delta)^{\mathrm{C}}}})\big].$$

Since for fixed $\delta>0$ the function v_1 is bounded on $(-\infty,-1)\cup(1+\delta,\infty)$ and $\lim_{\varepsilon\searrow 0}v_1(1+\varepsilon)=\infty$ it follows that

$$\lim_{\varepsilon\searrow 0}\mathbb{P}^{1+\varepsilon}_{v_1}(T_{(1,1+\delta)}<\zeta)=0$$

and with (6.10) we conclude $\mathbb{P}^x_{v_1}(\xi_{\zeta-}=1)=1$.

□

Proposition 6.1.6 can be proved similarly to Proposition 6.1.2 using the following lemma:

Lemma 6.2.5. *Let ξ be an α-stable process with $\alpha \in (0,2)$ and both sided jumps. Then it holds:*

(i) $\mathbb{P}_v^x(T_{(-\infty,-d]\cup[d,\infty)} < \zeta\,\forall d > 1) = 0.$

(ii) $U_v(x, [-b,-1) \cup (1,b]) < \infty$ *for all* $b > 1$.

Proof. The proof is analogous to the one of Lemma 6.2.4. □

The proof of Proposition 6.1.6 consists of combining these two statements as in the proof of Proposition 6.1.2.

6.2.3 Conditioning and h-transform

To connect the h-transform with the conditioned process we need some connection between the harmonic function and the asymptotic probability of the event we condition on. We have to separate the cases $\alpha < 1$ and $\alpha \geq 1$.

The case $\alpha < 1$

Proposition 6.2.6. *Let ξ be an α-stable process with $\alpha \in (0,1)$ and both sided jumps. Then it holds:*

$$\frac{\pi\Gamma(1-\alpha\rho)\Gamma(1-\alpha\hat{\rho})}{2^\alpha\Gamma(1-\alpha)} v_1(x) = \lim_{\varepsilon\searrow 0}\frac{1}{\varepsilon}\mathbb{P}^x(\xi_{\underline{m}} \in (1,1+\varepsilon)), \quad x \in \mathbb{R}\setminus[-1,1], \tag{6.11}$$

and

$$\frac{\pi\Gamma(1-\alpha\rho)\Gamma(1-\alpha\hat{\rho})}{2^\alpha\Gamma(1-\alpha)} v(x) = \lim_{\varepsilon\searrow 0}\frac{1}{\varepsilon}\mathbb{P}^x(|\xi_{\underline{m}}| \in (1,1+\varepsilon)), \quad x \in \mathbb{R}\setminus[-1,1]. \tag{6.12}$$

Proof. The proof is based on Proposition 1.1 of [41] where we find an explicit expression for the distribution of $\xi_{\underline{m}}$. For $x > 1$ this gives

$$\begin{aligned}
\mathbb{P}^x(\xi_{\underline{m}} \in (1,1+\varepsilon)) &= \frac{2^{-\alpha}\Gamma(1-\alpha\rho)}{\Gamma(1-\alpha)\Gamma(\alpha\hat{\rho})}\int_1^{x\wedge(1+\varepsilon)} z^{-\alpha}(x-z)^{\alpha\hat{\rho}-1}(x+z)^{\alpha\rho}\,\mathrm{d}z \\
&= \frac{2^{-\alpha}\Gamma(1-\alpha\rho)}{\Gamma(1-\alpha)\Gamma(\alpha\hat{\rho})}\int_1^{x\wedge(1+\varepsilon)} z^{-1}\big(\frac{x}{z}-1\big)^{\alpha\hat{\rho}-1}\big(\frac{x}{z}+1\big)^{\alpha\rho}\,\mathrm{d}z \\
&= \frac{2^{-\alpha}\Gamma(1-\alpha\rho)}{\Gamma(1-\alpha)\Gamma(\alpha\hat{\rho})}\int_{\frac{x}{x\wedge(1+\varepsilon)}}^{x} \frac{x}{z^2}\frac{z}{x}(z-1)^{\alpha\hat{\rho}-1}(z+1)^{\alpha\rho}\,\mathrm{d}z \\
&= \frac{2^{-\alpha}\Gamma(1-\alpha\rho)}{\Gamma(1-\alpha)\Gamma(\alpha\hat{\rho})}\int_{1\vee\frac{x}{1+\varepsilon}}^{x} (1+\frac{1}{z})\psi_{\alpha\rho}(z)\,\mathrm{d}z.
\end{aligned}$$

Applying l'Hopital's rule to the first calculation we obtain:

$$\begin{aligned}
\frac{\Gamma(1-\alpha)\Gamma(\alpha\hat{\rho})}{2^{-\alpha}\Gamma(1-\alpha\rho)} \lim_{\varepsilon\searrow 0} \frac{1}{\varepsilon}\mathbb{P}^x(\xi_{\underline{m}} \in (1,1+\varepsilon)) &= \lim_{\varepsilon\searrow 0} \frac{1}{\varepsilon} \int\limits_{\frac{x}{1+\varepsilon}}^{x} \big(1+\frac{1}{z}\big)\psi_{\alpha\rho}(z)\,\mathrm{d}z \\
&= \lim_{\varepsilon\searrow 0} \frac{x}{(1+\varepsilon)^2}\big(1+\frac{1+\varepsilon}{x}\big)\psi_{\alpha\rho}\big(\frac{x}{1+\varepsilon}\big) \\
&= (x+1)\psi_{\alpha\rho}(x) \\
&= \frac{1}{\sin(\pi\alpha\hat{\rho})} v_1(x).
\end{aligned}$$

Since $\sin(\pi\alpha\hat{\rho}) = \pi/(\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho}))$ this shows (6.11) for $x>1$. For $x<-1$ we first use duality to deduce:

$$\begin{aligned}
\mathbb{P}^x(\xi_{\underline{m}} \in (1,1+\varepsilon)) &= \hat{\mathbb{P}}^{-x}(\xi_{\underline{m}} \in (-1-\varepsilon,-1)) \\
&= \frac{2^{-\alpha}\Gamma(1-\alpha\hat{\rho})}{\Gamma(1-\alpha)\Gamma(\alpha\rho)} \int\limits_{1\vee\frac{-x}{1+\varepsilon}}^{-x} \big(1-\frac{1}{z}\big)\psi_{\alpha\hat{\rho}}(z)\,\mathrm{d}z,
\end{aligned}$$

where the second equality is verified using a similar calculation as above. Hence, it follows, for $x<-1$, that

$$\begin{aligned}
\frac{\Gamma(1-\alpha)\Gamma(\alpha\rho)}{2^{-\alpha}\Gamma(1-\alpha\hat{\rho})} \lim_{\varepsilon\searrow 0} \frac{1}{\varepsilon}\mathbb{P}^x(\xi_{\underline{m}} \in (1,1+\varepsilon)) &= \lim_{\varepsilon\searrow 0} \frac{1}{\varepsilon} \int\limits_{\frac{-x}{1+\varepsilon}}^{-x} (1-\frac{1}{z})\psi_{\alpha\hat{\rho}}(z)\,\mathrm{d}z \\
&= \lim_{\varepsilon\searrow 0} \frac{-x}{(1+\varepsilon)^2}\big(1-\frac{1+\varepsilon}{-x}\big)\psi_{\alpha\hat{\rho}}\big(\frac{-x}{1+\varepsilon}\big) \\
&= (-x-1)\psi_{\alpha\hat{\rho}}(-x) \\
&= \frac{1}{\sin(\pi\alpha\rho)} v_1(x).
\end{aligned}$$

Again we use $\sin(\pi\alpha\rho) = \frac{\pi}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)}$ to obtain (6.11) for $x<-1$.

Similarly, (6.12) can be deduced as follows. Analogously to the proof of the first equation we can show

$$\frac{\pi\Gamma(1-\alpha\rho)\Gamma(1-\alpha\hat{\rho})}{2^{\alpha}\Gamma(1-\alpha)} v_{-1}(x) = \lim_{\varepsilon\searrow 0} \frac{1}{\varepsilon}\mathbb{P}^x(\xi_{\underline{m}} \in (-(1+\varepsilon)),-1)), \quad x \in \mathbb{R}\setminus[-1,1].$$

Since we defined $v(x) = v_1(x) + v_{-1}(x)$ this shows (6.12). □

Now we are ready to prove the connection between the h-transform and the conditioned process.

Proof of Theorems 6.1.3 and 6.1.7. We start with $x>1$. First note for $\delta > \varepsilon > 0$:

$$\begin{aligned}
&\mathbb{P}^x(\Lambda, t < T_{(-(1+\delta),1+\delta)}, \xi_{\underline{m}} \in (1,1+\varepsilon)) \\
&= \mathbb{P}^x(\Lambda, t < T_{(-(1+\delta),1+\delta)}, t < \underline{m}, \xi_{\underline{m}} \in (1,1+\varepsilon)).
\end{aligned}$$

With the tower property of the conditional expectation and the Markov property in the version including the shift-operator (see (2.7)) it holds:

$$\begin{aligned}
&\mathbb{P}^x(\Lambda, t < T_{(-(1+\delta),1+\delta)}, t < \underline{m}, \xi_{\underline{m}} \in (1, 1+\varepsilon)) \\
&= \mathbb{E}^x\Big[\mathbb{1}_\Lambda \mathbb{E}^x\big[\mathbb{1}_{\{t<T_{(-(1+\delta),1+\delta)}\}} \mathbb{1}_{\{\xi_{\underline{m}}\in(1,1+\varepsilon)\}} \,|\, \mathcal{F}_t\big]\Big] \\
&= \mathbb{E}^x\Big[\mathbb{1}_\Lambda \mathbb{E}^x\big[\mathbb{1}_{\{t<T_{(-(1+\delta),1+\delta)}\}} (\mathbb{1}_{\{\xi_{\underline{m}}\in(1,1+\varepsilon)\}} \circ \theta_t) \,|\, \mathcal{F}_t\big]\Big] \\
&= \mathbb{E}^x\Big[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{(-(1+\delta),1+\delta)}\}} \mathbb{E}^x\big[\mathbb{1}_{\{\xi_{\underline{m}}\in(1,1+\varepsilon)\}} \circ \theta_t \,|\, \mathcal{F}_t\big]\Big] \\
&= \mathbb{E}^x\Big[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{(-(1+\delta),1+\delta)}\}} \mathbb{P}^{\xi_t}(\xi_{\underline{m}} \in (1, 1+\varepsilon))\Big].
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\mathbb{P}^x(\Lambda, t < T_{(-(1+\delta),1+\delta)}, \xi_{\underline{m}} \in (1, 1+\varepsilon)) \\
&= \mathbb{E}^x\Big[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{(-(1+\delta),1+\delta)}\}} \mathbb{P}^{\xi_t}(\xi_{\underline{m}} \in (1, 1+\varepsilon))\Big], \quad |x| > \delta > \varepsilon.
\end{aligned}$$

With the help of this application of the Markov property we obtain

$$\begin{aligned}
&\mathbb{P}^x(\Lambda, t < T_{(-(1+\delta),1+\delta)} \,|\, \xi_{\underline{m}} \in (1, 1+\varepsilon)) \\
&= \frac{\mathbb{P}^x(\Lambda, t < T_{(-(1+\delta),1+\delta)}, \xi_{\underline{m}} \in (1, 1+\varepsilon))}{\mathbb{P}^x(\xi_{\underline{m}} \in (1, 1+\varepsilon))} \\
&= \mathbb{E}^x\Big[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{(-(1+\delta),1+\delta)}\}} \frac{\mathbb{P}^{\xi_t}(\xi_{\underline{m}} \in (1, 1+\varepsilon))}{\mathbb{P}^x(\xi_{\underline{m}} \in (1, 1+\varepsilon))}\Big].
\end{aligned}$$

Now we would like to replace the ratio inside the expectation by $v_1(\xi_t)/v_1(x)$ with Proposition 6.2.6 when ε tends to 0. For that we need to argue why we can move the ε-limit inside the integral. Without loss of generality we assume $|x| > 1 + \delta > 1 + \varepsilon$. Note that for $y > 1 + \delta$ we have again with Proposition 1.1 of [41]:

$$\begin{aligned}
\mathbb{P}^y(|\xi_{\underline{m}}| \in (1, 1+\varepsilon)) &= 2\frac{2^{-\alpha}\Gamma(1-\alpha\rho)}{\Gamma(1-\alpha)\Gamma(\alpha\hat{\rho})} \int_{\frac{y}{1+\varepsilon}}^{y} \psi_{\alpha\rho}(z)\,\mathrm{d}z \\
&\le \frac{2^{1-\alpha}\Gamma(1-\alpha\rho)}{\Gamma(1-\alpha)\Gamma(\alpha\hat{\rho})}\Big(y - \frac{y}{1+\varepsilon}\Big)\psi_{\alpha\rho}\Big(\frac{y}{1+\varepsilon}\Big) \\
&= \frac{2^{1-\alpha}\Gamma(1-\alpha\rho)}{\Gamma(1-\alpha)\Gamma(\alpha\hat{\rho})} \frac{y\varepsilon}{1+\varepsilon}\psi_{\alpha\rho}\Big(\frac{y}{1+\varepsilon}\Big) \\
&= \frac{2^{1-\alpha}\Gamma(1-\alpha\rho)}{\Gamma(1-\alpha)\Gamma(\alpha\hat{\rho})} \frac{\varepsilon}{2\sin(\pi\alpha\hat{\rho})} v\Big(\frac{y}{1+\varepsilon}\Big).
\end{aligned}$$

Now let ε be so small that $\frac{1+\delta}{1+\varepsilon} > 1 + \frac{\delta}{2}$ and define

$$C_\delta := \sup_{|u|>1+\frac{\delta}{2}} v(u),$$

which is finite because of the properties of v. So we can estimate on the event $\{t < T_{[-1,1+\delta]}, \xi_t \ge 1 + \delta\}$:

$$\frac{1}{\varepsilon}\mathbb{P}^{\xi_t}(\xi_{\underline{m}} \in (1, 1+\varepsilon)) \le \frac{2^{-\alpha}\Gamma(1-\alpha\rho)}{\Gamma(1-\alpha)\Gamma(\alpha\hat{\rho})\sin(\pi\alpha\hat{\rho})} v\Big(\frac{\xi_t}{1+\varepsilon}\Big)$$

$$\leq \frac{2^{-\alpha}\Gamma(1-\alpha\rho)}{\Gamma(1-\alpha)\Gamma(\alpha\hat{\rho})\sin(\pi\alpha\hat{\rho})}C_\delta.$$

On $\{t < T_{(-(1+\delta),1+\delta)}, \xi_t \leq -(1+\delta)\}$ an analogous argumentation shows

$$\frac{1}{\varepsilon}\mathbb{P}^{\xi_t}(\xi_{\underline{m}} \in (1,1+\varepsilon)) \leq \frac{2^{-\alpha}\Gamma(1-\alpha\hat{\rho})}{\Gamma(1-\alpha)\Gamma(\alpha\rho)\sin(\pi\alpha\rho)}C_\delta.$$

So we can use dominated convergence as follows:

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\mathbb{P}^x(\Lambda, t < T_{(-(1+\delta),1+\delta)} \mid \xi_{\underline{m}} \in (1,1+\varepsilon)) \\
&= \lim_{\varepsilon\searrow 0}\frac{\varepsilon}{\mathbb{P}^x(\xi_{\underline{m}} \in (1,1+\varepsilon))}\lim_{\varepsilon\searrow 0}\mathbb{E}^x\Big[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{(-(1+\delta),1+\delta)}\}}\frac{\mathbb{P}^{\xi_t}(\xi_{\underline{m}} \in (1,1+\varepsilon))}{\varepsilon}\Big] \\
&= \lim_{\varepsilon\searrow 0}\frac{\varepsilon}{\mathbb{P}^x(\xi_{\underline{m}} \in (1,1+\varepsilon))}\mathbb{E}^x\Big[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{(-(1+\delta),1+\delta)}\}}\lim_{\varepsilon\searrow 0}\frac{\mathbb{P}^{\xi_t}(\xi_{\underline{m}} \in (1,1+\varepsilon))}{\varepsilon}\Big] \\
&= \mathbb{E}^x\Big[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{(-(1+\delta),1+\delta)}\}}\frac{v_1(\xi_t)}{v_1(x)}\Big]. \\
&= \mathbb{P}^x_{v_1}(\Lambda, t < T_{(-(1+\delta),1+\delta)}).
\end{aligned}$$

In the last step we used Proposition 6.2.6. This proves Theorem 6.1.3.

The proof of Theorem 6.1.7 is similar. Applying the Markov property in the shift-operator-version we get

$$\begin{aligned}
&\mathbb{P}^x(\Lambda, t < T_{(-(1+\delta),1+\delta)} \mid |\xi_{\underline{m}}| \in (1,1+\varepsilon)) \\
&= \frac{\mathbb{P}^x(\Lambda, t < T_{(-(1+\delta),1+\delta)}, |\xi_{\underline{m}}| \in (1,1+\varepsilon))}{\mathbb{P}^x(|\xi_{\underline{m}}| \in (1,1+\varepsilon))} \\
&= \mathbb{E}^x\Big[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{(-(1+\delta),1+\delta)}\}}\frac{\mathbb{P}^{\xi_t}(|\xi_{\underline{m}}| \in (1,1+\varepsilon))}{\mathbb{P}^x(|\xi_{\underline{m}}| \in (1,1+\varepsilon))}\Big].
\end{aligned}$$

In the proof of Theorem 6.1.3 we already found an integrable dominating function for $\mathbb{P}^{\xi_t}(|\xi_{\underline{m}}| \in (1,1+\varepsilon))/\varepsilon$. So we can use dominated convergence as follows:

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\mathbb{P}^x(\Lambda, t < T_{(-(1+\delta),1+\delta)} \mid |\xi_{\underline{m}}| \in (1,1+\varepsilon)) \\
&= \lim_{\varepsilon\searrow 0}\frac{\varepsilon}{\mathbb{P}^x(|\xi_{\underline{m}}| \in (1,1+\varepsilon))}\lim_{\varepsilon\searrow 0}\mathbb{E}^x\Big[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{(-(1+\delta),1+\delta)}\}}\frac{\mathbb{P}^{\xi_t}(|\xi_{\underline{m}}| \in (1,1+\varepsilon))}{\varepsilon}\Big] \\
&= \lim_{\varepsilon\searrow 0}\frac{\varepsilon}{\mathbb{P}^x(|\xi_{\underline{m}}| \in (1,1+\varepsilon))}\mathbb{E}^x\Big[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{(-(1+\delta),1+\delta)}\}}\lim_{\varepsilon\searrow 0}\frac{\mathbb{P}^{\xi_t}(|\xi_{\underline{m}}| \in (1,1+\varepsilon))}{\varepsilon}\Big] \\
&= \mathbb{E}^x\Big[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{(-(1+\delta),1+\delta)}\}}\frac{v(\xi_t)}{v(x)}\Big]. \\
&= \mathbb{P}^x_{v}(\Lambda, t < T_{(-(1+\delta),1+\delta)}),
\end{aligned}$$

where we used Proposition 6.2.6 in the last equality. □

The case $\alpha \geq 1$

The strategy for $\alpha \geq 1$ is as in the case $\alpha < 1$. First we need a relation between v_1 and the asymptotic probability we want to condition on. This event looks a bit different from the one in the case $\alpha < 1$.

Proposition 6.2.7. *Let ξ be an α-stable process with $\alpha \in [1,2)$ and both sided jumps, then*

$$\frac{1-\alpha\hat{\rho}}{2^{\alpha\rho}\pi} v_1(x) = \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon^{1-\alpha\hat{\rho}}} \mathbb{P}^x(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (1, 1+\varepsilon)), \quad x \in \mathbb{R} \setminus [-1,1].$$

Proof. Using the scaling property and Theorem 1.1 of [38] we get for $x > 1+\varepsilon$:

$$\begin{aligned}
&\frac{\pi}{\sin(\pi\alpha\hat{\rho})} \mathbb{P}^x(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (1, 1+\varepsilon)) \\
&= \frac{\pi}{\sin(\pi\alpha\hat{\rho})} \mathbb{P}^{\frac{x}{1+\varepsilon}}\Big(\xi_{T_{(-1,1)}} \in \big(\frac{1}{1+\varepsilon}, 1\big)\Big) \\
&= \big(\frac{x}{1+\varepsilon}+1\big)^{\alpha\rho}\big(\frac{x}{1+\varepsilon}-1\big)^{\alpha\hat{\rho}} \int_{\frac{1}{1+\varepsilon}}^{1} (1+y)^{-\alpha\rho}(1-y)^{-\alpha\hat{\rho}}\big(\frac{x}{1+\varepsilon}-y\big)^{-1} \,\mathrm{d}y \\
&\quad -(\alpha-1)\int_{1}^{\frac{x}{1+\varepsilon}} \psi_{\alpha\rho}(u)\,\mathrm{d}u \int_{\frac{1}{1+\varepsilon}}^{1} (1+y)^{-\alpha\rho}(1-y)^{-\alpha\hat{\rho}} \,\mathrm{d}y.
\end{aligned} \tag{6.13}$$

With l'Hopital's rule and the integration rule of Leibnitz we see:

$$\begin{aligned}
&\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon^{1-\alpha\hat{\rho}}} \int_{\frac{1}{1+\varepsilon}}^{1} (1+y)^{-\alpha\rho}(1-y)^{-\alpha\hat{\rho}}\big(\frac{x}{1+\varepsilon}-y\big)^{-1} \,\mathrm{d}y \\
&= \lim_{\varepsilon \searrow 0} \frac{\varepsilon^{\alpha\hat{\rho}}}{1-\alpha\hat{\rho}} \frac{1}{(1+\varepsilon)^2}\big(1+\frac{1}{1+\varepsilon}\big)^{-\alpha\rho}\big(1-\frac{1}{1+\varepsilon}\big)^{-\alpha\hat{\rho}}\big(\frac{x-1}{1+\varepsilon}\big)^{-1} \\
&\qquad + \lim_{\varepsilon \searrow 0} \frac{\varepsilon^{\alpha\hat{\rho}}}{1-\alpha\hat{\rho}} \int_{\frac{1}{1+\varepsilon}}^{1} (1+y)^{-\alpha\rho}(1-y)^{-\alpha\hat{\rho}} \frac{x}{(x-y(1+\varepsilon))^2} \,\mathrm{d}y \\
&= \frac{2^{-\alpha\rho}}{1-\alpha\hat{\rho}}(x-1)^{-1}
\end{aligned} \tag{6.14}$$

and further,

$$\begin{aligned}
&\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon^{1-\alpha\hat{\rho}}} \int_{\frac{1}{1+\varepsilon}}^{1} (1+y)^{-\alpha\rho}(1-y)^{-\alpha\hat{\rho}} \,\mathrm{d}y \\
&= \lim_{\varepsilon \searrow 0} \frac{\varepsilon^{\alpha\hat{\rho}}}{1-\alpha\hat{\rho}} \frac{1}{(1+\varepsilon)^2}\big(1+\frac{1}{1+\varepsilon}\big)^{-\alpha\rho}\big(1-\frac{1}{1+\varepsilon}\big)^{-\alpha\hat{\rho}} \\
&= \frac{2^{-\alpha\rho}}{1-\alpha\hat{\rho}}.
\end{aligned} \tag{6.15}$$

Now we plug in (6.14) and (6.15) in (6.13) and get

$$\begin{aligned}
&\lim_{\varepsilon \searrow 0} \frac{\pi}{\varepsilon^{1-\alpha\hat{\rho}}} \mathbb{P}^x(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (1, 1+\varepsilon)) \\
&= \frac{2^{-\alpha\rho}}{1-\alpha\hat{\rho}} \sin(\pi\alpha\hat{\rho})\Big[(x+1)^{\alpha\rho}(x-1)^{\alpha\hat{\rho}}(x-1)^{-1} - (\alpha-1)\int_1^x \psi_{\alpha\rho}(u)\,\mathrm{d}u\Big]
\end{aligned}$$

$$= \frac{2^{-\alpha\rho}}{1-\alpha\hat\rho} v_1(x).$$

For $x < -1$ we note that

$$\mathbb{P}^x(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (1, 1+\varepsilon)) = \hat{\mathbb{P}}^{|x|}(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (-(1+\varepsilon), -1)),$$

use again Theorem 1.1 of [38] and do a similar calculation as above to deduce

$$\begin{aligned}
&\frac{\pi}{\sin(\pi\alpha\rho)} \mathbb{P}^x(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (1, 1+\varepsilon)) \\
&= \Big(\frac{|x|}{1+\varepsilon}+1\Big)^{\alpha\hat\rho}\Big(\frac{|x|}{1+\varepsilon}-1\Big)^{\alpha\rho} \int_{-1}^{-\frac{1}{1+\varepsilon}} (1+y)^{-\alpha\hat\rho}(1-y)^{-\alpha\rho}\Big(\frac{|x|}{1+\varepsilon}-y\Big)^{-1} \,\mathrm{d}y \\
&\quad - (\alpha-1) \int_1^{\frac{|x|}{1+\varepsilon}} \psi_{\alpha\hat\rho}(u)\,\mathrm{d}u \int_{-1}^{-\frac{1}{1+\varepsilon}} (1+y)^{-\alpha\hat\rho}(1-y)^{-\alpha\rho}\,\mathrm{d}y.
\end{aligned}$$

A substitution on the integrals and the same limiting arguments as in the case $x > 1$ show

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0} \frac{\pi}{\varepsilon^{1-\alpha\hat\rho}} \mathbb{P}^x(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (1, 1+\varepsilon)) \\
&= \frac{2^{-\alpha\rho}}{1-\alpha\hat\rho} \sin(\pi\alpha\rho)\Big[(|x|+1)^{\alpha\hat\rho}(|x|-1)^{\alpha\rho}(|x|+1)^{-1} - (\alpha-1)\int_1^{|x|} \psi_{\alpha\hat\rho}(u)\,\mathrm{d}u\Big] \\
&= \frac{2^{-\alpha\rho}}{1-\alpha\hat\rho} v_1(x).
\end{aligned}$$

□

Proof of Theorems 6.1.4 and 6.1.8. First we note by a similar application of the Markov property as in the proof of Theorem 6.1.3:

$$\begin{aligned}
&\mathbb{P}^x(\Lambda, t < T_{(-(1+\delta),1+\delta)}, \xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (1, 1+\varepsilon)) \\
&= \mathbb{E}^x\Big[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{(-(1+\delta),1+\delta)}\}} \mathbb{P}^{\xi_t}(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (1, 1+\varepsilon))\Big]
\end{aligned}$$

and hence,

$$\begin{aligned}
&\mathbb{P}^x(\Lambda, t < T_{(-(1+\delta),1+\delta)} \,|\, \xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (1, 1+\varepsilon)) \\
&= \mathbb{E}^x\Big[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{(-(1+\delta),1+\delta)}\}} \frac{\mathbb{P}^{\xi_t}(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (1, 1+\varepsilon))}{\mathbb{P}^x(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (1, 1+\varepsilon))}\Big].
\end{aligned}$$

Again we want to move the ε-limit inside the integral and use Proposition 6.2.7. First we use (6.13):

$$\begin{aligned}
&\frac{\pi}{\sin(\pi\alpha\hat\rho)} \mathbb{P}^y(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (1, 1+\varepsilon)) \\
&= \Big(\frac{y}{1+\varepsilon}+1\Big)^{\alpha\rho}\Big(\frac{y}{1+\varepsilon}-1\Big)^{\alpha\hat\rho} \int_{\frac{1}{1+\varepsilon}}^{1} (1+u)^{-\alpha\rho}(1-u)^{-\alpha\hat\rho}\Big(\frac{y}{1+\varepsilon}-u\Big)^{-1} \,\mathrm{d}u
\end{aligned}$$

$$- (\alpha-1)\int_1^{\frac{y}{1+\varepsilon}} \psi_{\alpha\rho}(w)\,\mathrm{d}w \int_{\frac{1}{1+\varepsilon}}^{1} (1+u)^{-\alpha\rho}(1-u)^{-\alpha\hat{\rho}}\,\mathrm{d}u$$

$$\leq \Big[\big(\frac{y}{1+\varepsilon}+1\big)^{\alpha\rho}\big(\frac{y}{1+\varepsilon}-1\big)^{\alpha\hat{\rho}-1} - (\alpha-1)\int_1^{\frac{y}{1+\varepsilon}} \psi_{\alpha\rho}(w)\,\mathrm{d}w\Big] \int_{\frac{1}{1+\varepsilon}}^{1} (1+u)^{-\alpha\rho}(1-u)^{-\alpha\hat{\rho}}\,\mathrm{d}u$$

$$= \frac{1}{\sin(\pi\alpha\hat{\rho})} v_1\big(\frac{y}{1+\varepsilon}\big) \int_{\frac{1}{1+\varepsilon}}^{1} (1+u)^{-\alpha\rho}(1-u)^{-\alpha\hat{\rho}}\,\mathrm{d}u.$$

Further,

$$\begin{aligned}
\varepsilon^{\alpha\hat{\rho}-1}\int_{\frac{1}{1+\varepsilon}}^{1} (1+u)^{-\alpha\rho}(1-u)^{-\alpha\hat{\rho}}\,\mathrm{d}u &\leq \varepsilon^{\alpha\hat{\rho}-1}\int_{\frac{1}{1+\varepsilon}}^{1} (1-u)^{-\alpha\hat{\rho}}\,\mathrm{d}u \\
&= \frac{\varepsilon^{\alpha\hat{\rho}-1}}{1-\alpha\hat{\rho}}\Big(\frac{\varepsilon}{1+\varepsilon}\Big)^{1-\alpha\hat{\rho}} \\
&\leq \frac{1}{1-\alpha\hat{\rho}}.
\end{aligned}$$

Let be ε so small that $\frac{1+\delta}{1+\varepsilon} > 1+\frac{\delta}{2}$ and define

$$C_\delta = \sup_{|u|\geq 1+\frac{\delta}{2}} v_1(u).$$

Then it follows

$$\frac{\pi}{\varepsilon^{1-\alpha\hat{\rho}}}\mathbb{P}^y(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (1,1+\varepsilon)) \leq \frac{1}{1-\alpha\hat{\rho}} v_1\Big(\frac{y}{1+\varepsilon}\Big) \leq \frac{C_\delta}{1-\alpha\hat{\rho}}.$$

Similarly, we get for $y < -(1+\varepsilon)$:

$$\begin{aligned}
\frac{\pi}{\varepsilon^{1-\alpha\hat{\rho}}}\mathbb{P}^y(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (1,1+\varepsilon)) &\leq \frac{1}{1-\alpha\hat{\rho}} v_1\Big(\frac{y}{1+\varepsilon}\Big) \\
&\leq \frac{C_\delta}{1-\alpha\hat{\rho}}.
\end{aligned}$$

So we can apply dominated convergence to deduce

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0} \mathbb{P}^x(\Lambda, t < T_{(-(1+\delta),1+\delta)}\,|\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (1,1+\varepsilon)) \\
&= \lim_{\varepsilon\searrow 0} \frac{\varepsilon^{1-\alpha\hat{\rho}}}{\mathbb{P}^x(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (1,1+\varepsilon))} \\
&\quad\times \lim_{\varepsilon\searrow 0} \mathbb{E}^x\Big[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{(-(1+\delta),1+\delta)}\}} \frac{\mathbb{P}^{\xi_t}(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (1,1+\varepsilon))}{\varepsilon^{1-\alpha\hat{\rho}}}\Big] \\
&= \lim_{\varepsilon\searrow 0} \frac{\varepsilon^{1-\alpha\hat{\rho}}}{\mathbb{P}^x(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (1,1+\varepsilon))}
\end{aligned}$$

$$
\begin{aligned}
&\times \mathbb{E}^x\Big[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{(-(1+\delta),1+\delta)}\}} \lim_{\varepsilon\searrow 0} \frac{\mathbb{P}^{\xi_t}(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (1,1+\varepsilon))}{\varepsilon^{1-\alpha\hat\rho}}\Big]\\
&= \mathbb{E}^x\Big[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{(-(1+\delta),1+\delta)}\}} \frac{v_1(\xi_t)}{v_1(x)}\Big]\\
&= \mathbb{P}^x_{v_1}(\Lambda, t< T_{(-(1+\delta),1+\delta)}),
\end{aligned}
$$

where we used Proposition 6.2.7 in the second last equality. This finishes the proof of Theorem 6.1.4.

To prove Theorem 6.1.8 we first not that one can show analogously to the proof of Proposition 6.2.7:

$$
\lim_{\varepsilon\searrow 0} \varepsilon^{\alpha\rho-1}\mathbb{P}^x(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (-(1+\varepsilon),-1)) = \frac{1-\alpha\rho}{2^{\alpha\hat\rho}\pi} v_{-1}(x), \quad x\notin[-1,1].
$$

We assume without loss of generality $\rho\le\hat\rho$ (i.e. $\rho\le 1/2$) and in particular it holds that

$$
\begin{aligned}
&\lim_{\varepsilon\searrow 0} \varepsilon^{\alpha\hat\rho-1}\mathbb{P}^x(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (-(1+\varepsilon),-1))\\
&= \lim_{\varepsilon\searrow 0} \varepsilon^{\alpha(\hat\rho-\rho)}\varepsilon^{\alpha\rho-1}\mathbb{P}^x(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (-(1+\varepsilon),-1))\\
&= 0.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\lim_{\varepsilon\searrow 0} \frac{\mathbb{P}^{\xi_t}(|\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}}| \in (1,1+\varepsilon))}{\mathbb{P}^x(|\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}}| \in (1,1+\varepsilon))}\\
&= \lim_{\varepsilon\searrow 0} \frac{\mathbb{P}^{\xi_t}(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (1,1+\varepsilon)) + \mathbb{P}^{\xi_t}(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (-(1+\varepsilon),-1))}{\mathbb{P}^x(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (1,1+\varepsilon)) + \mathbb{P}^x(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (-(1+\varepsilon),-1))}\\
&= \lim_{\varepsilon\searrow 0} \frac{\varepsilon^{\alpha\hat\rho-1}\mathbb{P}^{\xi_t}(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (1,1+\varepsilon)) + \varepsilon^{\alpha\hat\rho-1}\mathbb{P}^{\xi_t}(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (-(1+\varepsilon),-1))}{\varepsilon^{\alpha\hat\rho-1}\mathbb{P}^x(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (1,1+\varepsilon)) + \varepsilon^{\alpha\hat\rho-1}\mathbb{P}^x(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (-(1+\varepsilon),-1))}\\
&= \lim_{\varepsilon\searrow 0} \frac{\varepsilon^{\alpha\hat\rho-1}\mathbb{P}^{\xi_t}(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (1,1+\varepsilon))}{\varepsilon^{\alpha\hat\rho-1}\mathbb{P}^x(\xi_{T_{(-(1+\varepsilon),1+\varepsilon)}} \in (1,1+\varepsilon))}\\
&= \frac{v_1(\xi_t)}{v_1(x)}.
\end{aligned}
$$

For $\rho > 1/2$, following the same argument, the first summands vanish instead of the second. To finish the proof of Theorem 6.1.8 the dominated convergence argument can be transferred from the proof of Theorem 6.1.4. □

The alternative characterisation for $\alpha > 1$

As before we start with the needed asymptotic probability of the event we want to condition on which is, as already mentioned, the same as in the case $\alpha < 1$ but under the law of the process conditioned to avoid 0.

Proposition 6.2.8. *Let ξ be an α-stable process with $\alpha\in(1,2)$ and both sided jumps, then*

$$
\frac{\alpha-1}{2} v_1(x) = \lim_{\varepsilon\searrow 0} \frac{e(x)}{\varepsilon}\mathbb{P}^x_\circ(\xi_{\underline{m}} \in (1,1+\varepsilon)), \quad x\notin[-1,1], \tag{6.16}
$$

and

$$\frac{\alpha-1}{2}v(x)=\lim_{\varepsilon\searrow 0}\frac{e(x)}{\varepsilon}\mathbb{P}^x_\circ(|\xi_{\underline{m}}|\in(1,1+\varepsilon)),\quad x\notin[-1,1]. \tag{6.17}$$

Proof. We use the so-called point of furthest reach before hitting 0. Let $\overline{m}$ be the time such that $|\xi_t|\leq|\xi_{\overline{m}}|$ for all $t\leq T_0$. The Riesz-Bogdan-Żak (Theorem 2.6.1) tells us that the process conditioned to avoid 0 is the spatial inverse of the original (i.e. not h-transformed) dual process including a certain time-change. Since the time change does not play any role for the value $\xi_{\overline{m}}$ we can extract the distribution of the point of closest reach of the process conditioned to avoid 0 from the distribution of the point of furthest reach of the original dual process, i.e.

$$\mathbb{P}^x_\circ(\xi_{\underline{m}}\in(1,1+\varepsilon))=\hat{\mathbb{P}}^{\frac{1}{x}}\Big(\xi_{\overline{m}}\in\Big(\frac{1}{1+\varepsilon},1\Big)\Big).$$

Combining this with Proposition 1.2 of Kyprianou et al. [41] where one can find an explicit expression for the distribution of the point of furthest reach before hitting 0, we get for $x>1$:

$$\begin{aligned}
&\frac{2}{\alpha-1}\mathbb{P}^x_\circ(\xi_{\underline{m}}\in(1,1+\varepsilon))\\
=&\int\limits_{\frac{1}{x}\vee\frac{1}{1+\varepsilon}}^{1} u^{-\alpha}\Big[\big(u+\frac{1}{x}\big)^{\alpha\rho}\big(u-\frac{1}{x}\big)^{\alpha\hat{\rho}-1}-(\alpha-1)x^{1-\alpha}\int\limits_1^{ux}\psi_{\alpha\rho}(w)\,\mathrm{d}w\Big]\,\mathrm{d}u\\
=&\int\limits_{1\vee\frac{x}{1+\varepsilon}}^{x}\frac{1}{x}\Big(\frac{x}{u}\Big)^\alpha\Big[\big(\frac{u}{x}+\frac{1}{x}\big)^{\alpha\rho}\big(\frac{u}{x}-\frac{1}{x}\big)^{\alpha\hat{\rho}-1}-(\alpha-1)x^{1-\alpha}\int\limits_1^{u}\psi_{\alpha\rho}(w)\,\mathrm{d}w\Big]\,\mathrm{d}u\\
=&\int\limits_{1\vee\frac{x}{1+\varepsilon}}^{x}u^{-\alpha}\Big[(u+1)^{\alpha\rho}(u-1)^{\alpha\hat{\rho}-1}-(\alpha-1)\int\limits_1^{u}\psi_{\alpha\rho}(w)\,\mathrm{d}w\Big]\,\mathrm{d}u\\
=&\int\limits_{1\vee\frac{x}{1+\varepsilon}}^{x}u^{-\alpha}\Big[(u+1)\psi_{\alpha\rho}(u)-(\alpha-1)\int\limits_1^{u}\psi_{\alpha\rho}(w)\,\mathrm{d}w\Big]\,\mathrm{d}u\\
=&\frac{1}{\sin(\pi\alpha\hat{\rho})}\int\limits_{1\vee\frac{x}{1+\varepsilon}}^{x}u^{-\alpha}v_1(u)\,\mathrm{d}u.
\end{aligned}$$

With l'Hopital's rule we get

$$\begin{aligned}
\lim_{\varepsilon\searrow 0}\frac{2}{\alpha-1}\frac{1}{\varepsilon}\mathbb{P}^x_\circ(\xi_{\underline{m}}\in(1,1+\varepsilon))&=\frac{1}{\sin(\pi\alpha\hat{\rho})}\lim_{\varepsilon\searrow 0}\Big[\frac{x}{(1+\varepsilon)^2}\Big(\frac{x}{1+\varepsilon}\Big)^{-\alpha}v_1\Big(\frac{x}{1+\varepsilon}\Big)\Big]\\
&=\frac{1}{\sin(\pi\alpha\hat{\rho})}x^{1-\alpha}v_1(x)\\
&=\frac{v_1(x)}{e(x)}.
\end{aligned}$$

This shows (6.16) for $x>1$. For $x<-1$ the equality (6.16) follows similarly.

To show the second claim we use

$$\frac{\alpha-1}{2}v_{-1}(x)=\lim_{\varepsilon\searrow 0}\frac{e(x)}{\varepsilon}\mathbb{P}^x_\circ(\xi_{\underline{m}}\in(-(1+\varepsilon),-1)),\quad x\notin[-1,1],$$

which follows from a computation similar to (6.16). Using that $v=v_1+v_{-1}$ the second claim follows. $\square$

Proof of Theorems 6.1.5 and 6.1.9. Since the process conditioned to avoid 0 is a strong Markov process (this follows by general theory on h-transforms, see e.g. Chung and Walsh [18]) we can use arguments analogous to the case $\alpha<1$ to obtain, for all $x\notin[-1,1]$,

$$\begin{aligned}&\mathbb{P}^x_\circ(\Lambda,t<T_{(-(1+\delta),1+\delta)}\,|\,\xi_{\underline{m}}\in(1,1+\varepsilon))\\&=\mathbb{E}^x_\circ\left[\mathbb{1}_\Lambda\mathbb{1}_{\{t<T_{(-(1+\delta),1+\delta)}\}}\frac{\mathbb{P}^{\xi_t}_\circ(\xi_{\underline{m}}\in(1,1+\varepsilon))}{\mathbb{P}^x_\circ(\xi_{\underline{m}}\in(1,1+\varepsilon))}\right].\end{aligned}$$

In the proof of Proposition 6.2.8 we have already seen that

$$\mathbb{P}^y_\circ(\xi_{\underline{m}}\in(1,1+\varepsilon))=\frac{\alpha-1}{2\sin(\pi\alpha\hat{\rho})}\int_{1\vee\frac{y}{1+\varepsilon}}^{y}u^{-\alpha}v_1(u)\,\mathrm{d}u$$

for $y>1+\varepsilon$. Analogously we can show

$$\mathbb{P}^y_\circ(\xi_{\underline{m}}\in(-(1+\varepsilon),-1))=\frac{\alpha-1}{2\sin(\pi\alpha\hat{\rho})}\int_{1\vee\frac{y}{1+\varepsilon}}^{y}u^{-\alpha}v_{-1}(u)\,\mathrm{d}u$$

for $y>1+\varepsilon$ and hence, we have

$$\mathbb{P}^y_\circ(|\xi_{\underline{m}}|\in(1,1+\varepsilon))=\frac{\alpha-1}{2\sin(\pi\alpha\hat{\rho})}\int_{1\vee\frac{y}{1+\varepsilon}}^{y}u^{-\alpha}v(u)\,\mathrm{d}u$$

for $y>1+\varepsilon$. Now we fix $\delta>0$ and assume that ε is so small that $\frac{1+\delta}{1+\varepsilon}\geq 1+\frac{\delta}{2}$. We define again $C_\delta:=\sup_{|u|\geq 1+\frac{\delta}{2}}v(u)$ which is finite. Note that, for $y>1+\delta$, we have:

$$\begin{aligned}\mathbb{P}^y_\circ(|\xi_{\underline{m}}|\in(1,1+\varepsilon))&=\frac{\alpha-1}{2\sin(\pi\alpha\hat{\rho})}\int_{\frac{y}{1+\varepsilon}}^{y}u^{-\alpha}v(u)\,\mathrm{d}u\\&\leq\frac{\alpha-1}{2\sin(\pi\alpha\hat{\rho})}\frac{y\varepsilon}{1+\varepsilon}\Big(\frac{y}{1+\varepsilon}\Big)^{-\alpha}\sup_{u\in[\frac{y}{1+\varepsilon},\infty)}v(u)\\&\leq\frac{C_\delta(\alpha-1)}{2\sin(\pi\alpha\hat{\rho})}\frac{\varepsilon}{(1+\varepsilon)^{1-\alpha}}y^{1-\alpha}\\&\leq\frac{C_\delta(\alpha-1)}{2\sin(\pi\alpha\hat{\rho})}\varepsilon(1+\delta)^{\alpha-1}y^{1-\alpha}.\end{aligned}$$

So we can estimate on $\{t<T_{[-(1+\delta),1+\delta]},\xi_t>1\}$:

$$\frac{\mathbb{P}^{\xi_t}_\circ(\xi_{\underline{m}}\in(1,1+\varepsilon))}{\varepsilon}\leq\frac{C_\delta(\alpha-1)}{2\sin(\pi\alpha\hat{\rho})}(1+\delta)^{\alpha-1}\xi_t^{1-\alpha}.$$

On $\{t < T_{[-(1+\delta),1+\delta]}, \xi_t < -1\}$ an analogous argument shows

$$\frac{\mathbb{P}^{\xi_t}(\xi_{\underline{m}} \in (1,1+\varepsilon))}{\varepsilon} \le \frac{C_\delta(\alpha-1)}{2\sin(\pi\alpha\rho)}(1+\delta)^{\alpha-1}|\xi_t|^{1-\alpha}.$$

Further, it holds that

$$\begin{aligned}\frac{1}{\sin(\pi\alpha\hat{\rho})}\mathbb{E}^x_\circ\big[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{[-(1+\delta),1+\delta]},\xi_t>1\}}\xi_t^{1-\alpha}\big] &= \frac{1}{e(x)}\mathbb{E}^x\big[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{[-(1+\delta),1+\delta]},\xi_t>1\}}\xi_t^{\alpha-1}\xi_t^{1-\alpha}\big]\\ &\le \frac{1}{e(x)}\end{aligned}$$

and, analogously,

$$\frac{1}{\sin(\pi\alpha\rho)}\mathbb{E}^x_\circ\big[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{[-(1+\delta),1+\delta]},\xi_t<-1\}}|\xi_t|^{1-\alpha}\big] \le \frac{1}{e(x)}.$$

So we can use dominated convergence and the Markov property as follows:

$$\begin{aligned}&\lim_{\varepsilon\searrow 0}\mathbb{P}^x_\circ(\Lambda, t<T_{[-(1+\delta),1+\delta]}\,|\,\xi_{\underline{m}}\in(1,1+\varepsilon))\\ &=\lim_{\varepsilon\searrow 0}\frac{\varepsilon}{\mathbb{P}^x_\circ(\xi_{\underline{m}}\in(1,1+\varepsilon))}\\ &\qquad\times\lim_{\varepsilon\searrow 0}\mathbb{E}^x_\circ\Big[\mathbb{1}_\Lambda\mathbb{1}_{\{t<T_{[-(1+\delta),1+\delta]}\}}\frac{\mathbb{P}^{\xi_t}_\circ(\xi_{\underline{m}}\in(1,1+\varepsilon))}{\varepsilon}\Big]\\ &=\lim_{\varepsilon\searrow 0}\frac{\varepsilon}{\mathbb{P}^x_\circ(\xi_{\underline{m}}\in(1,1+\varepsilon))}\\ &\qquad\times\mathbb{E}^x_\circ\Big[\mathbb{1}_\Lambda\mathbb{1}_{\{t<T_{[-(1+\delta),1+\delta]}\}}\lim_{\varepsilon\searrow 0}\frac{\mathbb{P}^{\xi_t}_\circ(\xi_{\underline{m}}\in(1,1+\varepsilon))}{\varepsilon}\Big]\\ &=\mathbb{E}^x_\circ\Big[\mathbb{1}_\Lambda\mathbb{1}_{\{t<T_{[-(1+\delta),1+\delta]}\}}\frac{e(x)v_1(\xi_t)}{e(\xi_t)v_1(x)}\Big]\\ &=\mathbb{E}^x\Big[\mathbb{1}_\Lambda\mathbb{1}_{\{t<T_{[-(1+\delta),1+\delta]}\}}\frac{v_1(\xi_t)}{v_1(x)}\Big].\end{aligned}$$

In the second last second step we used Proposition 6.2.8. Theorem 6.1.9 can be proven similarly using the same dominating function. $\square$

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