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Duality for pathwise superhedging in continuous time

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Abstract We provide a model-free pricing–hedging duality in continuous time. For a frictionless market consisting of d risky assets with continuous price trajectories, we show that the purely analytic problem of finding the minimal superhedging price of a path-dependent European option has the same value as the purely probabilistic problem of finding the supremum of the expectations of the option over all martingale measures. The superhedging problem is formulated with simple trading strategies, the claim is the limit inferior of continuous functions, which allows upper and lower semi-continuous claims, and superhedging is required in the pathwise sense on a σ-compact sample space of price trajectories. If the sample space is stable under stopping, the probabilistic problem reduces to finding the supremum over all martingale measures with compact support. As an application of the general results, we deduce dualities for Vovk's outer measure and semi-static superhedging with finitely many securities.

Keywords Pathwise superhedging · Pricing–hedging duality · Vovk's outer measure · Semi-static hedging · Martingale measures · σ-compactness

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1 Introduction

Given the space $C([0,T],\mathbb{R}^d)$ of all continuous price trajectories, the superhedging problem of a contingent claim $X\colon C([0,T],\mathbb{R}^d)\to\mathbb{R}$ consists of finding the infimum over all $\lambda\in\mathbb{R}$ such that there exists a trading strategy H which satisfies

$$\lambda+(H\cdot S)_T(\omega)\geq X(\omega),\qquad \omega\in C([0,T],\mathbb{R}^d), \tag{1.1}$$

where $(H\cdot S)_T(\omega)$ denotes the capital gain by trading according to the strategy H in the underlying assets $S_t(\omega):=\omega(t)$.

In the classical framework of mathematical finance, one commonly postulates a model for the price evolution by fixing a probability measure P such that S is a semimartingale and defines $(H\cdot S)_T$ as the stochastic integral $\int_0^T H_t\,\mathrm{d}S_t$. Then a consequence of the fundamental theorem of asset pricing states that the infimum over all λ such that there are admissible predictable integrands H fulfilling inequality (1.1) is equal to the supremum of $E_Q[X]$ over all absolutely continuous local martingale measures Q; see Delbaen and Schachermayer [19, Sect. 9.5]. Here, the superhedging (i.e., inequality (1.1)) is assumed to hold P-almost surely and the set of absolutely continuous local martingale measures is non-empty, which is guaranteed by the exclusion of some form of arbitrage; see [19, Corollary 9.1.2] for the precise formulation.

More recently, alternative possibilities to specify the superhedging requirement without referring to a fixed model have been proposed. For instance, if an investor takes into account a class $\mathcal{P}$ of probabilistic models, then superhedging is naturally required to hold $\mathcal{P}$-quasi surely, i.e., P-almost surely for all considered models $P\in\mathcal{P}$. The pioneering works of Lyons [33] and Avellaneda et al. [4] on Knightian uncertainty in mathematical finance consider models with uncertain volatility in continuous time. The study of the pricing–hedging duality in this setting has given rise to a rich literature starting with the capacity-theoretic approach of Denis and Martini [20]. Further, Peng [39, Theorem 2.4] obtains the duality using stochastic control techniques, whereas Soner et al. [47, 46, 45] rely on supermartingale decomposition results under individual models and eventually build on aggregation results to derive the duality under model uncertainty. This approach has been extended by Neufeld and Nutz [37] to cover measurable claims using the theory of analytic sets; see also Biagini et al. [15] for a robust fundamental theorem of asset pricing under a model ambiguity version of the no-arbitrage-of-the-first-kind condition $\mathrm{NA}_1(\mathcal{P})$, and Nutz [38] for the case of jump diffusions.

In the present work, we focus on the pathwise/model-free approach and assume that the superhedging requirement (1.1) has to hold pointwise for all price trajectories in a given set $\Omega\subseteq C([0,T],\mathbb{R}^d)$. In this pathwise setting, finding the minimal superhedging price turns out to be a purely analytic problem whose formulation is independent of any probabilistic assumptions. This is in contrast to the above-mentioned approaches working with a fixed model, under Knightian uncertainty or in a quasi-sure setting. Notice that the pathwise approach corresponds to the quasi-sure approach when $\mathcal{P}$ contains all Dirac measures, which in continuous time is excluded; see e.g. [15, Corollary 3.5].

In the now classical paper [30], Hobson first addressed the problem of pathwise superhedging for the lookback option. His analysis was based on some sharp pathwise martingale inequalities and has motivated Beiglböck et al. [11] to introduce the martingale optimal transport problem in discrete time. Here, the investor takes static positions in some liquidly traded vanilla options and dynamic positions in the stocks. The rationale is that information on the price of options translates into the knowledge of some marginals of the martingale measures; see also [1, 7, 18, 17, 16, 12] for further developments in this direction. In continuous time, the duality for the martingale optimal transport has been obtained by Galichon et al. [26] and Possamaï et al. [41] in the quasi-sure setting. The pathwise formulation was studied by Dolinsky and Soner [21] using a discretisation of the sample space. These results have been extended by Hou and Obłój [31], who in particular allow incorporation of investor's beliefs (of possible price paths) by relying on the notion of "prediction set" due to Mykland [36].

Following this consideration in our analysis, we also assume that the investor does not deem every continuous path plausible, but focuses instead on a prediction set $\Omega \subseteq C([0,T],\mathbb{R}^d)$ that is required to be σ-compact (i.e., at most a countable union of compact sets), and define the pathwise superhedging problem on the sample space Ω. Moreover, restricting the set of possible price paths has the financially desirable effect of reducing the superhedging price. See also Aksamit et al. [3] and Acciaio and Larsson [2] for other treatments of belief and information in robust superhedging, and Dolinsky and Soner [22] and Guo et al. [29] for extensions of the pathwise formulation to the Skorokhod space.

In the continuous-time setting, already the definition of a pathwise "stochastic integral" is a non-trivial issue. We circumvent this problem by working with simple strategies and consider as "stochastic" integral the pointwise limit inferior of pathwise integrals against simple strategies, an approach that was proposed by Perkowski and Prömel [40] to define an outer measure allowing to study stochastic integration under model ambiguity. This outer measure is very similar in spirit to that of Vovk [48] and can be seen as the value of a pathwise superhedging problem; cf. Sect. 2.1 for details and Beiglböck et al. [10] and Vovk [49] for existing duality results in this setting.

Formally, the superhedging price of a contingent claim $X\colon \Omega \to [-\infty,+\infty]$ is defined as the infimum over all $\lambda \in \mathbb{R}$ such that there exists a sequence (H^n) of simple strategies which satisfies

$$\lambda + \liminf_{n\to\infty}(H^n \cdot S)_T(\omega) \geq X(\omega) \qquad \text{for all } \omega \in \Omega$$

and the admissibility condition $\lambda + (H^n \cdot S)_t(\omega) \geq 0$ for all $n \in \mathbb{N}$, $\omega \in \Omega$ and $t \in [0,T]$. If X is the limit inferior of a sequence of continuous functions, then under the assumptions that Ω is σ-compact and contains all its stopped paths, we show that the superhedging price coincides with the supremum of $E_Q[X]$ over all martingale measures Q. Furthermore, this duality is generalised to the case when X is unbounded from above and when Ω does not contain all its stopped paths. In addition to providing a way around the technical difficulty posed by the definition of pathwise stochastic integrals, the superhedging in terms of limit inferior turns out to be necessary to guarantee the duality on a sufficiently large space; see Remark 2.6 for a counterexample.

Our main contributions to the pathwise pricing–hedging duality in continuous time and with finitely many risky assets are as follows. While in the current literature (see e.g. [31, 21, 29]) pathwise duality results hold for uniformly continuous options, the proposed method allows much less regular claims (including for example European options, spread options, continuously and discretely monitored Asian options, lookback options, certain types of barrier options, and options on realised variance). In particular, this implies a duality for Vovk's outer measure on closed sets. A related duality result was given by Vovk [49], however, under an additional closedness assumption on the set of attainable outcomes. Moreover, our pricing–hedging duality holds for every prediction set Ω which is σ-compact. Let us remark that the assumption of σ-compactness is an essential ingredient of the presented method to get the pricing–hedging duality. We show in Sect. 3.1 that typical price trajectories for various popular financial models such as local, stochastic or even rough volatility models belong to the σ-compact space of Hölder-continuous functions. In the related work [31], the pricing–hedging duality holds for an approximate version of the superhedging price which requires the superhedging on an enlarged prediction set $\Omega^\varepsilon := \{\omega \in C([0,T],\mathbb{R}^d) : \inf_{\omega'\in\Omega} \|\omega - \omega'\|_\infty \le \varepsilon\} \supseteq \Omega$ for any given $\varepsilon > 0$.

The article is organised as follows. In Sect. 2, we present the main results (Theorems 2.1 and 2.7) and some direct applications. Section 3 contains a detailed discussion of feasible choices for the underlying sample space. The proofs of the main results are carried out in Sect. 4. A criterion for the sample path regularity of stochastic processes and the construction of a counterexample are given in the Appendix.

2 Main results

Let $C([0,T],\mathbb{R}^d)$ be the space of continuous functions $\omega\colon [0,T] \to \mathbb{R}^d$, where $T > 0$ is a finite time horizon and $d \in \mathbb{N}$. **Throughout the entire paper**, $\Omega \subseteq C([0,T],\mathbb{R}^d)$ is a non-empty metric space, that is, $\Omega \neq \emptyset$ and there is a fixed metric d on Ω. We consider on Ω the topology $\mathcal{T}$ which is induced by d and the Borel σ-algebra which is generated by the open sets with respect to d. The metric space Ω is called *σ-compact* if there exists a countable sequence of compact (with respect to d) sets $K_n \subseteq \Omega$ such that $\Omega = \bigcup_n K_n$. A map $X\colon \Omega \to \mathbb{R}$ is said to be continuous if X is continuous with respect to d and the Euclidean distance on $\mathbb{R}$.

The canonical process $S\colon [0,T]\times\Omega \to \mathbb{R}^d$ given by $S_t(\omega) := \omega(t)$ generates the raw filtration $\mathcal{F}_t^0 := \sigma(S_s, s \le t\wedge T)$, $t \ge 0$, i.e., $\mathcal{F}_t^0$ is the smallest σ-algebra that makes all S_s with $s \le t$ measurable. Furthermore, let $(\mathcal{F}_t)$ be the right-continuous version of the raw filtration $(\mathcal{F}_t^0)$, defined by $\mathcal{F}_t := \bigcap_{s>t} \mathcal{F}_s^0$ for all $t \in [0,T]$. Denote by $\mathcal{M}(\Omega)$ the set of all Borel probability measures Q on Ω such that the canonical process S is a Q-martingale, and by

$$\mathcal{M}_c(\Omega) := \{Q \in \mathcal{M}(\Omega) : Q[K] = 1 \text{ for some compact } K \subseteq \Omega\}$$

the subset of all martingale measures with compact support. Define

$$C_{\delta\sigma} := \left\{ X\colon \Omega \to [-\infty,+\infty] : \begin{array}{l} X = \liminf_n X_n \text{ for a sequence } (X_n) \text{ such that} \\ X_n\colon \Omega \to \mathbb{R} \text{ is bounded and continuous} \end{array} \right\}.$$

Note that $C_{\delta\sigma}$ contains all upper and lower semicontinuous functions from Ω to $\mathbb{R}$.

A process $H \colon [0,T] \times \Omega \to \mathbb{R}^d$ is called *simple predictable* if it is of the form

$$H_t(\omega) = \sum_{n=1}^{N} h_n(\omega) \mathbf{1}_{(\tau_n(\omega), \tau_{n+1}(\omega)]}(t), \qquad (t,\omega) \in [0,T] \times \Omega,$$

where $N \in \mathbb{N}$, $0 \leq \tau_1 \leq \cdots \leq \tau_{N+1} \leq T$ are stopping times with respect to the filtration $(\mathcal{F}_t)$, and $h_n \colon \Omega \to \mathbb{R}^d$ are bounded $\mathcal{F}_{\tau_n}$-measurable functions. The set of all simple predictable processes is denoted by $\mathcal{H}^f := \mathcal{H}^f(\Omega)$. For a simple predictable $H \in \mathcal{H}^f$, the pathwise stochastic integral

$$(H \cdot S)_t(\omega) := \sum_{n=1}^{N} h_n(\omega) \big(S_{\tau_{n+1}(\omega) \wedge t}(\omega) - S_{\tau_n(\omega) \wedge t}(\omega) \big)$$

is well defined for all $t \in [0,T]$ and all $\omega \in \Omega$. Similarly, the pathwise stochastic integral $H \cdot S$ is also well defined for every $H \colon [0,T] \times \Omega \to \mathbb{R}^d$ in the set $\mathcal{H} := \mathcal{H}(\Omega)$ of processes of the form

$$H_t(\omega) = \sum_{n=1}^{\infty} h_n(\omega) \mathbf{1}_{(\tau_n(\omega), \tau_{n+1}(\omega)]}(t),$$

where $0 \leq \tau_1 \leq \tau_2 \leq \cdots$ are stopping times such that for each $\omega \in \Omega$ there exists an $N(\omega) \in \mathbb{N}$ with $\tau_k(\omega) = T$ for all $k \geq N(\omega)$, and $h_n \colon \Omega \to \mathbb{R}$ are bounded $\mathcal{F}_{\tau_n}$-measurable functions.

We introduce the following two assumptions, which we shall use frequently.

(A1) Ω is σ-compact, the metric on Ω induces a topology finer than (or equal to) that induced by the maximum norm $\|\omega\|_\infty := \max_{t \in [0,T]} |\omega(t)|$, and for each Borel probability Q on Ω and every bounded $\mathcal{F}_t^0$-measurable function h, there exists a sequence of $\mathcal{F}_t^0$-measurable continuous functions (h_n) which converges Q-almost surely to h.

(A2) For every $\omega \in \Omega$ and each $t \in [0,T]$, the stopped path $\omega^t(\cdot) := \omega(\cdot \wedge t)$ is in Ω and the function $[0,T] \times \Omega \ni (t,\omega) \mapsto \omega^t$ is continuous.

If Ω is a σ-compact metric space for the metric and the topology induced by the maximum norm, then (A1) is always satisfied; see Remark 4.1. Now we are ready to state the main results of this paper. The proofs are given in Sect. 4.

Theorem 2.1 *Suppose that* (A1) *and* (A2) *hold and let* $Z \colon \Omega \to [0,+\infty)$ *be a continuous function such that* $Z(\omega^s) \leq Z(\omega^t)$ *for all* $\omega \in \Omega$ *and* $0 \leq s \leq t \leq T$. *Then for every* $X \in C_{\delta\sigma}$ *which satisfies* $X(\omega) \geq -Z(\omega)$ *for all* $\omega \in \Omega$, *one has*

$$\inf \left\{ \lambda \in \mathbb{R} : \begin{array}{l} \textit{there is a sequence } (H^n) \textit{ in } \mathcal{H}^f \textit{ such that} \\ \lambda + (H^n \cdot S)_t(\omega) \geq -Z(\omega^t) \textit{ for all } (t,\omega) \in [0,T] \times \Omega \\ \textit{and } \lambda + \liminf_n (H^n \cdot S)_T(\omega) \geq X(\omega) \textit{ for all } \omega \in \Omega \end{array} \right\}$$

$$= \sup_{Q \in \mathcal{M}_c(\Omega)} E_Q[X]. \tag{2.1}$$

Moreover, the equality (2.1) *also holds if* $\mathcal{H}^f$ *is replaced by* $\mathcal{H}$, *or* $\mathcal{M}_c(\Omega)$ *is replaced by* $\mathcal{M}_Z(\Omega) := \{Q \in \mathcal{M}(\Omega) : E_Q[Z] < +\infty\}$.

Remark 2.2 (i) By continuity of Z, one has $\mathcal{M}_c(\Omega) \subseteq \mathcal{M}_Z(\Omega)$. In particular, if $X(\omega) \geq -Z(\omega)$ for all $\omega \in \Omega$, the expectation $E_Q[X]$ is well defined under every $Q \in \mathcal{M}_Z(\Omega)$.

(ii) Note that $Z(\omega) := \max_{t\in[0,T]} |\omega(t)|^p$ for $p \geq 0$ satisfies $Z(\omega^s) \leq Z(\omega^t)$ for every $\omega \in \Omega$ and $0 \leq s \leq t \leq T$.

(iii) If $Z \geq \|\cdot\|_\infty$, then $E_Q[\max_{t\in[0,T]} |S_t|] < +\infty$ for every Borel probability measure Q which integrates Z. Hence the set of all local martingale measures which integrate Z coincides with $\mathcal{M}_Z(\Omega)$.

In particular, for $Z = 0$ the previous theorem reads as follows.

Corollary 2.3 *Suppose that* (A1) *and* (A2) *hold. Then for every* $X \in C_{\delta\sigma}$ *with* $X \geq 0$, *one has*

$$\inf\left\{\lambda \in \mathbb{R} : \begin{array}{l} \text{there is a sequence } (H^n) \text{ in } \mathcal{H} \text{ such that} \\ \lambda + (H^n \cdot S)_t(\omega) \geq 0 \text{ for all } (t,\omega) \in [0,T]\times\Omega \\ \text{and } \lambda + \liminf_n (H^n \cdot S)_T(\omega) \geq X(\omega) \text{ for all } \omega \in \Omega \end{array}\right\}$$
$$= \sup_{Q\in\mathcal{M}(\Omega)} E_Q[X].$$

The arguments in the proof of Theorem 2.1 in combination with a regularity result for martingale measures on $C([0,T],\mathbb{R}^d)$ (see Lemma 4.7 below) yield the following pricing–hedging duality on the entire space $C([0,T],\mathbb{R})$.

Corollary 2.4 *Let* $\Omega = C([0,T],\mathbb{R}^d)$. *Then*

$$\inf\left\{\lambda \in \mathbb{R} : \begin{array}{l} \text{for every compact } K \subseteq \Omega, \text{ there are } H \in \mathcal{H}^f \text{ and } c \geq 0 \\ \text{such that } \lambda + (H \cdot S)_T(\omega) \geq -c \text{ for all } \omega \in \Omega \text{ and} \\ \lambda + (H \cdot S)_T(\omega) \geq X(\omega) \text{ for all } \omega \in K \end{array}\right\}$$
$$= \sup_{Q\in\mathcal{M}(\Omega)} E_Q[X]$$

for every bounded upper semicontinuous function $X : \Omega \to \mathbb{R}$.

Remark 2.5 Suppose $\Omega \subseteq C([0,T],\mathbb{R})$ satisfies the assumptions (A1) and (A2). Let $(\pi_n)_{n\in\mathbb{N}}$ be a refining sequence of partitions of $[0,T]$ with mesh converging to zero. The pathwise quadratic variation of a path $\omega \in \Omega$ is defined by

$$\langle\omega\rangle_t := \liminf_n \langle\omega\rangle_t^n, \qquad \text{where } \langle\omega\rangle_t^n := \sum_{[u,v]\in\pi_n} \big(\omega(u\wedge t) - \omega(v\wedge t)\big)^2, \tag{2.2}$$

for $t \in [0,T]$. Then for every continuous function $\xi : \Omega \times \mathbb{R} \to \mathbb{R}$ which is bounded from below, one has

$$X(\omega) := \liminf_n \xi(\omega, \langle\omega\rangle_T^n) \in C_{\delta\sigma}.$$

Hence the pathwise pricing–hedging duality in Theorem 2.1 holds for this claim. This shows that the class $C_{\delta\sigma}$ includes in particular the financial derivatives in the scope of [10], i.e., options on realised variance, among many others.

Remark 2.6 While the pathwise pricing–hedging duality results in [21, 31] hold for sufficiently regular claims when trading is limited to simple predictable processes (i.e., without the "lim inf" as in our definition), the following example shows the necessity of "lim inf" for claims in $C_{\delta\sigma}$. Let Ω be the set of all Hölder-continuous functions starting at zero with values in $[0, 1]$ and equipped with the distance induced by the maximum norm. There exist a refining deterministic sequence $(\pi_n)_{n\in\mathbb{N}}$ of partitions with mesh size going to zero and a function $\tilde{\omega} \in \Omega$ such that

- $0 \le \tilde{\omega}(t) \le 1$ for all $t \in [0, T]$,
- $\langle\tilde{\omega}\rangle_t := \lim_n \langle\tilde{\omega}\rangle_t^n$ exists for all $t \in [0, T)$ and $\lim_{t\to T}\langle\tilde{\omega}\rangle_t = +\infty$,

where $\langle\tilde{\omega}\rangle_t^n$ is defined as in (2.2). For the existence of such a function $\tilde{\omega}$, we refer to Lemma A.3. We fix now the above sequence $(\pi_n)_{n\in\mathbb{N}}$ and denote by $\langle\omega\rangle_t$ the corresponding quadratic variation along $(\pi_n)_{n\in\mathbb{N}}$ defined as in (2.2) for all $\omega \in \Omega$. Furthermore, let us consider the option $X(\cdot) := \langle\cdot\rangle_T \in C_{\delta\sigma}$.

Firstly, we get by Itô's formula and Fatou's lemma that

$$\sup_{Q\in\mathcal{M}(\Omega)} E_Q[X] \le 1.$$

Secondly, we observe that

$$\begin{aligned}&\inf\{\lambda \ge 0 : \text{there is } H \in \mathcal{H}^f \text{ such that } \lambda + (H\cdot S)_T(\omega) \ge X(\omega) \text{ for all } \omega \in \Omega\}\\ &= +\infty. \end{aligned} \tag{2.3}$$

Indeed, assume that there exist (even more generally) a predictable process H of bounded variation and a constant $\lambda_0 > 0$ such that

$$\lambda_0 + (H\cdot S)_T(\omega) \ge X(\omega) \qquad \text{for all } \omega \in \Omega, \tag{2.4}$$

where $(H\cdot S)_T(\omega)$ denotes the classical Riemann–Stieltjes integral defined using the integration by parts formula. For $\tilde{\omega}$ we get

$$(H\cdot S)_T(\tilde{\omega}) \le \|\tilde{\omega}\|_\infty \|H(\tilde{\omega})\|_{1\text{-var};[0,T]} \le \|H(\tilde{\omega})\|_{1\text{-var};[0,T]} < +\infty,$$

where $\|H(\tilde{\omega})\|_{1\text{-var};[0,T]}$ denotes the bounded variation seminorm of H. Because $X(\tilde{\omega}) = +\infty$, this implies that (2.4) cannot hold for any $\lambda_0 \in \mathbb{R}$ and therefore establishes (2.3).

Hence, there exists a duality gap if the superhedging is restricted to trading strategies of bounded variation as in [21, 31]. However, the pricing–hedging duality using the limit inferior of simple predictable processes holds true since Ω and X satisfy all assumptions of Theorem 2.1; see Sect. 3 below.

If Ω does not contain all its stopped paths, then the following version of Theorem 2.1 holds true.

Theorem 2.7 *Let $Z\colon \Omega \to [1,+\infty)$ be a function with compact sublevel sets $\{Z \le c\}$ for all $c \in \mathbb{R}$ and such that $Z(\omega) \ge \|\omega\|_\infty$ for all $\omega \in \Omega$. If* (A1) *holds true and $\mathcal{M}_Z(\Omega) \neq \emptyset$, then*

$$\inf\left\{\lambda \in \mathbb{R} : \begin{array}{l} \textit{there are } c \ge 0 \textit{ and a sequence } (H^n) \textit{ in } \mathcal{H}^f \textit{ such that} \\ (H^n \cdot S)_T(\omega) \ge -cZ(\omega) \textit{ for all } \omega \in \Omega \textit{ and} \\ \lambda + \liminf_n (H^n \cdot S)_T(\omega) \ge X(\omega) \textit{ for all } \omega \in \Omega \end{array}\right\}$$
$$= \sup_{Q \in \mathcal{M}_Z(\Omega)} E_Q[X]$$

for every $X \in C_{\delta\sigma}$ which is bounded from below.

2.1 Relation to Vovk's outer measure

In recent years (see e.g. [48, 49] and the references therein), Vovk introduced on different path spaces an outer measure, defined as the minimal superhedging price, which allows quantifying the path behaviour of "typical price paths" in frictionless financial markets without any reference measure.

In order to recall Vovk's outer measure on a set $\Omega \subseteq C([0,T],\mathbb{R}^d)$ endowed with the maximum norm, we write $\mathcal{H}_\lambda$ for the set of λ-admissible strategies, i.e., the set of all $H \in \mathcal{H}$ such that $(H \cdot S)_t(\omega) \ge -\lambda$ for all $(t,\omega) \in [0,T] \times \Omega$. Furthermore, we define the set of processes

$$\mathcal{V}_\lambda := \left\{\mathrm{H} := (H^k)_{k \in \mathbb{N}} : H^k \in \mathcal{H}_{\lambda_k},\ \lambda_k > 0,\ \sum_{k=1}^{\infty} \lambda_k = \lambda\right\}$$

for an initial capital $\lambda \in (0,+\infty)$. Note that for every $\mathrm{H} = (H^k) \in \mathcal{V}_\lambda$, all $\omega \in \Omega$ and all $t \in [0,T]$, the corresponding capital process

$$(\mathrm{H} \cdot S)_t(\omega) := \sum_{k=1}^{\infty} (H^k \cdot S)_t(\omega) = \sum_{k=1}^{\infty} \big(\lambda_k + (H^k \cdot S)_t(\omega)\big) - \lambda$$

is well defined and takes values in $[-\lambda,+\infty]$. Then Vovk's outer measure on Ω is given by

$$\overline{Q}_\Omega(A) := \inf\left\{\lambda > 0 : \begin{array}{l} \text{there is } \mathrm{H} \in \mathcal{V}_\lambda \text{ such that} \\ \lambda + (\mathrm{H} \cdot S)_T(\omega) \ge \mathbf{1}_A(\omega) \text{ for all } \omega \in \Omega \end{array}\right\}.$$

A slight modification of $\overline{Q}_\Omega$ was introduced in Perkowski and Prömel [40], namely

$$\overline{P}_\Omega(A) := \inf\left\{\lambda > 0 : \begin{array}{l} \text{there is } (H^n) \text{ in } \mathcal{H}_\lambda \text{ such that} \\ \lambda + \liminf_{n\to\infty} (H^n \cdot S)_T(\omega) \ge \mathbf{1}_A(\omega) \text{ for all } \omega \in \Omega \end{array}\right\}$$

for $A \subseteq \Omega$. The latter definition seems to be more in the spirit of superhedging prices in semimartingale models as discussed in [40, Sects. 2.1 and 2.2]. Notice that even if it would be convenient to just minimise over simple strategies rather than over the limit (inferior) along sequences of simple strategies in both definitions of outer measures, the latter is essential to obtain the desired countable subadditivity of both outer measures.

Remark 2.8 In case that $\Omega = C([0,T],\mathbb{R}^d)$ with the maximum norm, one would expect that the outer measures $\overline{Q}_\Omega$ and $\overline{P}_\Omega$ coincide. However, currently it is only known that

$$\sup_{Q\in\mathcal{M}(\Omega)} Q[A] \le \overline{P}_\Omega(A) \le \overline{Q}_\Omega(A), \tag{2.5}$$

where $A \subseteq C([0,T],\mathbb{R}^d)$ is a Borel-measurable set; see [48, Lemma 6.2] and [40, Lemma 2.9]. In the special case of $\Omega = C([0,+\infty),\mathbb{R})$ and for time-superinvariant sets $A \subseteq C([0,+\infty),\mathbb{R})$, the inequalities in (2.5) turn out to be true equalities. See Vovk [48, Sects. 2 and 3] and Beiglböck et al. [10, Sect. 2] for the precise definitions and statements in this context.

By restricting the outer measure $\overline{P}_\Omega$ to a σ-compact space Ω satisfying assumptions (A1) and (A2), we get the following duality result for the slightly modified version of Vovk's outer measure as a direct application of Theorem 2.1.

Proposition 2.9 *Under the assumptions on Ω of Theorem 2.1, one has*

$$\overline{P}_\Omega(A) = \sup_{Q\in\mathcal{M}(\Omega)} Q[A]$$

for all closed subsets $A \subseteq \Omega$.

Proof For every closed subset $A \subseteq \Omega$, it follows from Corollary 2.3 that

$$\begin{aligned}\overline{P}_\Omega(A) &= \inf\left\{\lambda>0 : \begin{array}{l}\text{there is a sequence } (H^n) \text{ in } \mathcal{H} \text{ such that}\\ \lambda + (H^n\cdot S)_t(\omega) \ge 0 \text{ for all } (t,\omega)\in[0,T]\times\Omega \text{ and}\\ \lambda + \liminf_n (H^n\cdot S)_T(\omega) \ge \mathbf{1}_A(\omega) \ \text{ for all } \omega\in\Omega\end{array}\right\}\\ &= \sup_{Q\in\mathcal{M}(\Omega)} E_Q[\mathbf{1}_A] = \sup_{Q\in\mathcal{M}(\Omega)} Q[A],\end{aligned}$$

because $\mathbf{1}_A$ is upper semicontinuous. □

Remark 2.10 Recently, Vovk [49] obtained a similar duality for open sets by adjusting the definition of the outer measure $\overline{P}_\Omega$. More precisely, his new definition of outer measure allows superhedging with all processes in the "liminf-closure" of capital processes generated by sequences of λ-admissible simple strategies; see [49, Sect. 2 and Theorem 2] for more details.

2.2 Semi-static superhedging

Let us fix a continuous function $Z\colon \Omega \to [1,+\infty)$ such that $Z(\omega^s) \le Z(\omega^t)$ for all $\omega \in \Omega$ and $0 \le s \le t \le T$, and consider a finite number of securities with (discounted) continuous payoffs $G_1,\dots,G_K$ such that $|G_i| \le cZ$ for $i = 1,\dots,K$ and some $c \geq 0$. We assume that these securities can be bought and sold at prices $g_k \in \mathbb{R}$ and satisfy the no-arbitrage condition

$$(g_1, \dots, g_K) \in \operatorname{ri}\{(E_Q[G_1], \dots, E_Q[G_K]) : Q \in \mathcal{M}_c(\Omega)\} \tag{2.6}$$

where ri denotes the relative interior. Then the following semi-static hedging duality holds.

Proposition 2.11 *Suppose the assumptions* (A1) *and* (A2) *are satisfied and the securities with payoffs $G_1, \dots, G_K$ satisfy the static no-arbitrage condition* (2.6). *Then for every upper semicontinuous function $X : \Omega \to \mathbb{R}$ which satisfies $|X| \leq cZ$ for some $c \geq 0$, one has*

$$\begin{aligned}
&\inf \left\{ \lambda \in \mathbb{R} : \begin{array}{l} \text{there are } c \geq 0, \alpha \in \mathbb{R}^K \text{ and a sequence } (H^n) \text{ in } \mathcal{H}^f \text{ such that} \\ \lambda + (H^n \cdot S)_t(\omega) \geq -cZ(\omega^t) \text{ for all } (t,\omega) \in [0,T] \times \Omega \text{ and} \\ \lambda + \sum_{k=1}^K \alpha_k (G_k(\omega) - g_k) + \liminf_n (H^n \cdot S)_T(\omega) \geq X(\omega) \\ \text{for all } \omega \in \Omega \end{array} \right\} \\
&= \sup_{Q \in \mathcal{M}_c^G(\Omega)} E_Q[X],
\end{aligned} \tag{2.7}$$

where $\mathcal{M}_c^G(\Omega) := \{Q \in \mathcal{M}_c(\Omega) : E_Q[G_k] = g_k \text{ for } k = 1, \dots, K\}$.

Proof For every $Y : \Omega \to \mathbb{R}$ which satisfies $|Y| \leq cZ$ for some $c \geq 0$, we define

$$\phi(Y) := \inf \left\{ \lambda \in \mathbb{R} : \begin{array}{l} \text{there are } c \geq 0 \text{ and a sequence } (H^n) \text{ in } \mathcal{H}^f \text{ such that} \\ \lambda + (H^n \cdot S)_t(\omega) \geq -cZ(\omega^t) \text{ for } (t,\omega) \in [0,T] \times \Omega \\ \text{and } \lambda + \liminf_n (H^n \cdot S)_T(\omega) \geq Y(\omega) \text{ for all } \omega \in \Omega \end{array} \right\},$$

and we remark that by interchanging two infima, the left-hand side of (2.7) can be expressed as $\inf_{\alpha \in \mathbb{R}^K} \phi(X - \sum_{k=1}^K \alpha_k (G_k - g_k))$. Further, Theorem 2.1 yields

$$\phi\Big(X - \sum_{k=1}^K \alpha_k (G_k - g_k)\Big) = \sup_{Q \in \mathcal{M}_c(\Omega)} E_Q\Big[X - \sum_{k=1}^K \alpha_k (G_k - g_k)\Big]$$

for every $\alpha \in \mathbb{R}^K$. Now define the function

$$J : \mathcal{M}_c(\Omega) \times \mathbb{R}^K \to \mathbb{R}, \qquad J(Q,\alpha) := E_Q[X] - \sum_{k=1}^K \alpha_k E_Q[G_k - g_k].$$

It is immediate that $J(Q, \cdot)$ is convex for every $Q \in \mathcal{M}_c(\Omega)$ and that $J(\cdot, \alpha)$ is concave for each $\alpha \in \mathbb{R}^K$ since $\mathcal{M}_c(\Omega)$ is convex. Therefore, it follows exactly as in step (a) of the proof of [5, Theorem 2.1] that the assumption (2.6) of 0 being in the relative interior of

$$\{(E_Q[G_1 - g_1], \dots, E_Q[G_K - g_K]) : Q \in \mathcal{M}_c(\Omega)\}$$

can be used to show that all the requirements of the minimax theorem [44, Theorem 4.1] are satisfied. Hence, one gets

$$\begin{aligned}\inf_{\alpha\in\mathbb{R}^K}\phi\Big(X-\sum_{k=1}^K\alpha_k(G_k-g_k)\Big)&=\inf_{\alpha\in\mathbb{R}^K}\sup_{Q\in\mathcal{M}_c(\Omega)}J(Q,\alpha)\\&=\sup_{Q\in\mathcal{M}_c(\Omega)}\inf_{\alpha\in\mathbb{R}^K}J(Q,\alpha)=\sup_{Q\in\mathcal{M}_c^G(\Omega)}E_Q[X],\end{aligned}$$

where the first equality follows from Theorem 2.1 and the last by

$$\inf_{\alpha\in\mathbb{R}^K}J(Q,\alpha)=\begin{cases}E_Q[X], & \text{if } Q\in\mathcal{M}_c^G(\Omega),\\ -\infty, & \text{if } Q\in\mathcal{M}_c(\Omega)\setminus\mathcal{M}_c^G(\Omega).\end{cases}$$

The proof is complete. □

3 Discussion of σ-compact spaces

By definition, the σ-compactness of the metric space $\Omega\subseteq C([0,T],\mathbb{R}^d)$ with metric d requires to find a covering of Ω by compact sets K^m, $m\in\mathbb{N}$. It is an easy consequence of the Arzelà–Ascoli theorem (see e.g. [25, Theorem 1.4]) that these K^m have to be bounded, closed and equicontinuous.

In the next lemma, we provide an easy-to-check criterion for a set Ω of continuous functions to be σ-compact. This leads to many interesting examples of such $\Omega\subseteq C([0,T],\mathbb{R}^d)$ appearing in the context of (classical) financial modelling; see Sect. 3.1.

Lemma 3.1 *For $n\in\mathbb{N}$, let $c_n\colon[0,T]^2\to[0,+\infty)$ be a continuous function with $c_n(t,t)=0$ for $t\in[0,T]$ and define the norm*

$$\|\omega\|_{c_n,\alpha}:=|\omega(0)|+\sup_{s,t\in[0,T]}\frac{|\omega(t)-\omega(s)|}{c_n(s,t)^\alpha},\qquad \omega\in C([0,T],\mathbb{R}^d),$$

with $\alpha\in(0,1]$ and the convention $\frac{0}{0}:=0$. Then the spaces

$$\Omega_n:=\{\omega\in C([0,T],\mathbb{R}^d):\|\omega\|_{c_n,1}<+\infty\},\qquad n\in\mathbb{N},$$

are σ-compact with respect to the norm $\|\cdot\|_{c_n,\alpha}$ for $\alpha\in(0,1)$ and in particular with respect to the maximum norm $\|\cdot\|_\infty$. Moreover, the set $\Omega:=\bigcup_{n\in\mathbb{N}}\Omega_n$ is σ-compact with respect to the maximum norm $\|\cdot\|_\infty$.

Proof For $m,n\in\mathbb{N}$, we observe that

$$\Omega_n=\bigcup_{m\in\mathbb{N}}K_n^m\qquad\text{with } K_n^m:=\{\omega\in C([0,T],\mathbb{R}^d):\|\omega\|_{c_n,1}\le m\}.$$

In order to show the σ-compactness of Ω_n with respect to $\|\cdot\|_\infty$, we show that each K_n^m is compact. Due to the Arzelà–Ascoli theorem, it is sufficient to show that each K_n^m is bounded, equicontinuous and closed. As to boundedness, for every $\omega \in K_n^m$, we have

$$\|\omega\|_\infty \le |\omega(0)| + \sup_{t\in[0,T]} |\omega(t) - \omega(0)| \le |\omega(0)| + m \sup_{t\in[0,T]} c_n(0,t).$$

Next, because c_n is continuous on a compact set and $c_n(t,t) = 0$ for $t \in [0,T]$, there exists for every $\varepsilon > 0$ a $\delta > 0$ such that $|c_n(s,t)| < \varepsilon/m$ for $|t-s| \le \delta$. Hence for every $\omega \in K_n^m$ and $s,t \in [0,T]$ with $|t-s| \le \delta$, we get $|\omega(t) - \omega(s)| \le \varepsilon$, which yields equicontinuity. Finally, for closedness, we show that if $(\omega_k) \subseteq K_n^m$ converges uniformly to ω, then $\omega \in K_n^m$. Indeed, this can be seen by

$$|\omega(0)| + \frac{|\omega(t) - \omega(s)|}{c_n(s,t)} = \lim_{k\to\infty} \left(|\omega_k(0)| + \frac{|\omega_k(t) - \omega_k(s)|}{c_n(s,t)} \right) \le m.$$

The σ-compactness of Ω_n with respect to $\|\cdot\|_{c_n,\alpha}$ for $\alpha \in (0,1)$ follows by the fact that uniform convergence in each K_n^m implies convergence with respect to $\|\cdot\|_{c_n,\alpha}$, which is a consequence of the interpolation inequality, for $s,t \in [0,T]$,

$$\frac{|\omega(t) - \omega(s)|}{c_n(s,t)^\alpha} = \left(\frac{|\omega(t) - \omega(s)|}{c_n(s,t)} \right)^\alpha |\omega(t) - \omega(s)|^{1-\alpha} \le 2\|\omega\|_{c_n,1}^\alpha \|\omega\|_\infty^{1-\alpha}.$$

Finally, Ω is σ-compact (with respect to $\|\cdot\|_\infty$) as a countable union of σ-compact sets. □

From the previous lemma, it is easy to deduce that many well-known function spaces $\Omega \subseteq C([0,T], \mathbb{R}^d)$ are σ-compact spaces. To state the next corollary, we recall that a function $c\colon [0,T]^2 \to [0,+\infty)$ is called a *control function* if c is continuous, superadditive, i.e., $c(s,t) + c(t,u) \le c(s,u)$ for $0 \le s \le t \le u \le T$, and $c(t,t) = 0$ for every $t \in [0,T]$.

Corollary 3.2 (i) *The space $C^\alpha([0,T], \mathbb{R}^d)$ of α-Hölder-continuous functions, i.e.,*

$$C^\alpha([0,T], \mathbb{R}^d) := \left\{ \omega \in C([0,T], \mathbb{R}^d) : \sup_{s,t\in[0,T]} \frac{|\omega(t) - \omega(s)|}{|t-s|^\alpha} < +\infty \right\}$$

for $\alpha \in (0,1]$, is σ-compact with respect to $\|\cdot\|_\infty$ and for $\beta \in (0,\alpha)$ with respect to the Hölder norm $\|\cdot\|_\beta$ defined by

$$\|\omega\|_\beta := |\omega(0)| + \sup_{s,t\in[0,T]} \frac{|\omega(t) - \omega(s)|}{|t-s|^\beta} \qquad \text{for } \omega \in C^\alpha([0,T], \mathbb{R}^d).$$

(ii) *The space $C^{\text{Hölder}}([0,T], \mathbb{R}^d) := \bigcup_{\alpha\in(0,1]} C^\alpha([0,T], \mathbb{R}^d)$ of all Hölder-continuous functions is σ-compact with respect to the maximum norm $\|\cdot\|_\infty$.*

(iii) *The fractional Sobolev space $W^{\delta,p}([0,T], \mathbb{R}^d)$ with $\delta - 1/p > 0$, given by*

$$W^{\delta,p}([0,T], \mathbb{R}^d) := \left\{ \omega \in C([0,T], \mathbb{R}^d) : \int_{[0,T]^2} \frac{|\omega(t) - \omega(s)|^p}{|t-s|^{\delta p+1}} \, \mathrm{d}s \, \mathrm{d}t < +\infty \right\}$$

for $\delta \in (0,1)$ and $p \in [1,+\infty)$, is σ-compact with respect to $\|\cdot\|_\infty$.

(iv) *The space* $C^{p\text{-var},c}([0,T],\mathbb{R}^d)$, *which is a subspace of continuous functions with finite* p*-variation, given by*

$$C^{p\text{-var},c}([0,T],\mathbb{R}^d) := \left\{\omega \in C([0,T],\mathbb{R}^d) : \sup_{s,t\in[0,T]} \frac{|\omega(t)-\omega(s)|}{c(s,t)^{1/p}} < +\infty\right\}$$

for $p \in [1,+\infty)$ *and a control function* c, *is* σ*-compact with respect to* $\|\cdot\|_\infty$ *and for* $p' \in (p,+\infty)$ *with respect to the* p'*-variation norm* $\|\cdot\|_{p'\text{-var}}$ *defined by*

$$\|\omega\|_{p'\text{-var}} := |\omega(0)| + \sup_{0\le t_0\le\cdots\le t_n\le T,\, n\in\mathbb{N}} \left(\sum_{i=0}^{n-1} |\omega(t_{i+1})-\omega(t_i)|^{p'}\right)^{1/p'}.$$

Proof (i) and (ii) follow directly by Lemma 3.1 and the fact that

$$C^\alpha([0,T],\mathbb{R}^d) \subseteq C^{\frac{1}{n}}([0,T],\mathbb{R}^d) \qquad \text{for } \alpha \in [n^{-1},(n-1)^{-1}],\, n\in\mathbb{N}.$$

(iii) Classical Sobolev embedding results, see e.g. [25, Corollary A.2], imply that

$$W^{\delta,p}([0,T],\mathbb{R}^d) \subseteq C^{\delta-1/p}([0,T],\mathbb{R}^d) \quad \text{and} \quad \|\omega\|_{\delta-1/p} \le C(\delta,p)\|\omega\|_{W^{\delta,p}}$$

for $\omega \in W^{\delta,p}([0,T],\mathbb{R}^d)$ with $\delta - 1/p > 0$ and for a constant $C(\delta,p) > 0$ depending only on δ and p. Here $\|\cdot\|_{W^{\delta,p}}$ denotes the fractional Sobolev seminorm; see (A.1) below. Hence, to obtain the stated σ-compactness from Lemma 3.1, it remains to show that if a sequence $(\omega_k) \subseteq W^{\delta,p}([0,T],\mathbb{R}^d)$ with $\|\omega\|_{W^{\delta,p}} \le K$ for some constant $K > 0$ converges uniformly to a function ω, then $\|\omega\|_{W^{\delta,p}} \le K$. However, this is a simple consequence of Fatou's lemma.

(iv) The σ-compactness with respect to $\|\cdot\|_\infty$ and $\|\cdot\|_{c,\alpha}$ for $\alpha \in (0,1)$ follows again by Lemma 3.1. The σ-compactness with respect to $\|\cdot\|_{p'\text{-var}}$ can be deduced from the inequality

$$\|\omega\|_{p'\text{-var}} \le \|\omega\|_{c,\frac{1}{p}}\, c(0,T)^{1/p}$$

for $\omega \in C^{p\text{-var},c}([0,T],\mathbb{R}^d)$ and $p' \in (p,+\infty)$. □

Remark 3.3 (i) The function spaces in Corollary 3.2 satisfy also the first part of assumption (A2): for every $\omega \in \Omega$ and $t \in [0,T]$, the stopped path $\omega^t(\cdot) := \omega(\cdot \wedge t)$ is in Ω. For the Hölder-type spaces, this is fairly easy to verify, and for the Sobolev space, we refer to [28, Lemma 1.5.1.8]. Hence all these function spaces equipped with the maximum norm satisfy the assumptions (A1) and (A2); see also Remark 4.1.

(ii) From the perspective of (completely) model-free financial mathematics, it might be desirable to consider the space $C^{p\text{-var}}([0,T],\mathbb{R}^d)$ of all continuous functions possessing finite p-variation for $p > 2$ since this space includes the support of all martingale measures. Unfortunately, the elementary covering used in the proof of Lemma 3.1 cannot work as the unit ball in $C^{p\text{-var}}([0,T],\mathbb{R}^d)$ is not compact; see e.g. [34, Example 3.4].

3.1 Examples from mathematical finance

As mentioned in the introduction, the prediction set Ω can be interpreted to contain all the price paths that an investor believes could possibly appear in a financial market. Hence it is natural to choose Ω in a way that it includes those price paths coming from financial models which have been proved to provide fairly reasonable underlying price processes.

Example 3.4 A natural assumption coming from semimartingale models is to consider a prediction set Ω_{QV} of continuous paths possessing pathwise quadratic variation in the sense of Föllmer [24]. We refer e.g. to the work [43] (and the references therein) for such frameworks. To be more precise, fix a refining sequence of partitions $(\pi_n)_{n\in\mathbb{N}}$ with mesh size going to zero and consider the prediction set

$$\Omega_{\mathrm{QV}} := \{\omega \in C^\alpha([0,T],\mathbb{R}) : \omega(0) = 0 \text{ and } \|\omega\|_{QV} < C\}$$

for $\alpha \in (0,1)$ and some constant $C > 0$, where

$$\|\omega\|_{\mathrm{QV}} := \sup_{n\in\mathbb{N}} \Bigg(\sum_{[s,t]\in\pi_n} |\omega(t) - \omega(s)|^2 \Bigg)^{1/2}.$$

Note that Ω_{QV} is σ-compact with respect to the norm $\|\cdot\|_\infty$. Indeed, we have $\Omega_{\mathrm{QV}} = \bigcup_{n\in\mathbb{N}} \Omega_n$ with

$$\Omega_n := \{\omega \in C^\alpha([0,T],\mathbb{R}) : \|\omega\|_\alpha \le n \text{ and } \|\omega\|_{\mathrm{QV}} \le C - 1/n\},$$

where Ω_n is a compact set for each $n \in \mathbb{N}$. In order to see the compactness of Ω_n, we observe that the condition $\|\omega\|_\alpha \le n$ ensures that the set Ω_n is equicontinuous and uniformly bounded, and furthermore, every sequence $(\omega_m) \subseteq \Omega_n$ possesses a subsequence which converges in the maximum norm to a function $\omega \in C^\alpha([0,T],\mathbb{R})$ with $\|\omega\|_\alpha \le n$; cf. Lemma 3.1. The required bound $\|\omega\|_{\mathrm{QV}} \le C - 1/n$ follows by the same estimates as used for the proof of [25, Proposition 5.28].

Let us consider for instance a simple lookback option

$$X(\omega) := \max_{t\in[0,T]} |\omega(t)|$$

on the market Ω_{QV}. Using a pathwise version of the Burkholder–Davis–Gundy inequality (see [13, Theorem 2.1]), we get

$$X(\omega) \le \liminf_n \max_{t\in\pi^n} |S_t(\omega)| \le 6\sqrt{C} + \liminf_n (H^n \cdot S)_T(\omega)$$

for all $\omega \in \Omega_{\mathrm{QV}}$ and some sequence (H^n) of simple predictable processes. From this, we can conclude that the superhedging price is less than or equal to $6\sqrt{C}$, using the definitions from Theorem 2.1. Note that the superhedging price on the entire space $C([0,T],\mathbb{R})$ has to be $+\infty$ if we aim to have the duality between the superhedging price and the supremum of $E_Q[X]$ over all martingale measures Q.

Example 3.5 Instead of using a financial model based on semimartingales, there is a rich literature on financial modelling using fractional Brownian motion because of its favourable time-series properties; see e.g. [42] and the references therein.

This motivates the choice of prediction set $\Omega := \{\omega \in C^H([0,T],\mathbb{R}) : \omega(0) = 0\}$ as it contains the sample paths of fractional Brownian motion with Hurst index $H \in (0,1)$. If $H > 1/2$, for every upper semicontinuous claim $X\colon \Omega \to [0,+\infty]$, we can apply our pathwise pricing–hedging duality (Theorem 2.1) to see that the superhedging price is given by

$$\begin{aligned}\phi(X) &:= \inf\left\{\lambda \in \mathbb{R} : \begin{array}{l}\text{there is a sequence } (H^n) \text{ in } \mathcal{H}^f \text{ such that}\\ \lambda + (H^n \cdot S)_t(\omega) \geq 0 \text{ for all } (t,\omega) \in [0,T] \times \Omega\\ \text{and } \lambda + \liminf_n (H^n \cdot S)_T(\omega) \geq X(\omega) \;\text{ for all } \omega \in \Omega\end{array}\right\}\\ &= X(0),\end{aligned}$$

where 0 stands for the constant path equal to 0, since the Dirac measure at 0 is the only martingale measure in $\mathcal{M}_c(\Omega)$. Notice that the pathwise superhedging price considering the entire space $C([0,T],\mathbb{R})$ is $\sup_{\omega \in C([0,T],\mathbb{R})} |X(\omega)|$ for many options X.

Note that it is actually a delicate question under which conditions non-semimartingale models are almost surely arbitrage-free. However, even for prediction sets like Ω, on which one expects arbitrage in a probabilistic sense, the pathwise superreplication price turns out to be finite.

Remark 3.6 Prediction sets can naturally be modelled by means of the pathwise quadratic variation (2.2). For instance, the typical price paths of the Black–Scholes model are given by the prediction set

$$\Omega = \left\{\omega \in C([0,T],\mathbb{R}) : \omega(0) = s_0 \text{ and } \langle\omega\rangle_\cdot = \int_0^\cdot \sigma^2 \omega(t)^2 \,\mathrm{d}t\right\}, \qquad s_0 \in \mathbb{R}.$$

However, prediction sets of this form are not σ-compact in general and the duality results of this paper do not apply. As shown in Bartl et al. [8], a pathwise pricing–hedging duality on such prediction sets can still be obtained, but it requires a modified superhedging price which allows investing directly in the quadratic variation. This new superhedging price of a contingent claim X is defined as the infimum over all $\lambda \in \mathbb{R}$ for which there exist sequences (H^n) and (G^n) of simple predictable strategies satisfying

$$\lambda + \liminf_{n\to\infty}\left((H^n \cdot S)_T(\omega) + \left(G^n \cdot \int S \,\mathrm{d}S\right)_T(\omega)\right) \geq X(\omega) \qquad \text{for all } \omega \in \Omega$$

and the admissibility condition $\lambda + (H^n \cdot S)_t(\omega) + (G^n \cdot \int S\,\mathrm{d}S)_t(\omega) \geq 0$ for all $n \in \mathbb{N}$, $\omega \in \Omega$ and $t \in [0,T]$. The key idea is to extend the market model, consider a two-dimensional price process $(S, \int S\,dS)$ on the product space $C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R})$ and adapt the duality results (and their proofs) of the present paper accordingly. For a detailed discussion on prediction sets depending on pathwise quadratic variation, we refer to [8].

In the following, we present several examples coming from the modelling of financial markets which satisfy the assumptions (A1) and (A2) and are concentrated on a σ-compact metric spaces $\Omega \subseteq C([0,T],\mathbb{R}^d)$. For simplicity, we consider one-dimensional processes and denote by W a one-dimensional Brownian motion on a probability space $(\tilde{\Omega},\mathcal{F},P)$. However, all arguments extend straightforwardly to multidimensional settings.

Example 3.7 A classical example from mathematical finance is the famous *Black–Scholes model*, which is given by

$$\mathrm{d}S_t = \sigma S_t \,\mathrm{d}W_t + \mu S_t \,\mathrm{d}t, \qquad t \in [0,T],$$

for $\mu \in \mathbb{R}$ and $\sigma > 0$. In this case, the price process S is a so-called geometric Brownian motion, which possesses the same sample path regularity as a Brownian motion. Hence, one has almost surely $S \in C^\alpha([0,T],\mathbb{R})$ and $S \in W^{\alpha-\frac{1}{q},q}([0,T],\mathbb{R})$ for every $\alpha \in (0,1/2)$ and $q > 2$; cf. Corollary A.2.

Example 3.8 Other examples are *local volatility models*

$$\mathrm{d}S_t = \sigma(t,S_t)\,\mathrm{d}W_t, \qquad S_0 = s_0, t \in [0,T],$$

for a volatility function $\sigma : [0,T]\times\mathbb{R}\to\mathbb{R}$. For these classes of models, one again has $S \in \Omega := C^\alpha([0,T],\mathbb{R})$ a.s. for every $\alpha < 1/2$ if $s_0 \in \mathbb{R}$ and σ is Lipschitz-continuous and satisfies the linear growth condition $|\sigma(t,x)|^2 \leq K(1+|x|^2)$ for $(t,x) \in [0,T]\times\mathbb{R}$ and a positive constant $K > 0$. Indeed, the Hölder regularity of S can be deduced from Corollary A.2 combined with the estimate

$$\begin{aligned} E_P\bigg[\int_0^T |\sigma(s,S_s)|^q \,\mathrm{d}s\bigg] &\leq \tilde{C} E_P\bigg[\int_0^T (1+|S_s|)^q \,\mathrm{d}s\bigg] \\ &\leq \tilde{C}'\bigg(1+\int_0^T E_P[|S_s|^q]\,\mathrm{d}s\bigg) \leq C, \end{aligned}$$

for constants $\tilde{C},\tilde{C}' > 0$ and $C = C(q,K,T,S_0) > 0$ and for every $q \geq 2$, where the last inequality follows by the L^q-estimate in [35, Theorem 4.1].

Example 3.9 A frequently applied generalisation of the Black–Scholes model is given by *stochastic volatility models*

$$\mathrm{d}S_t = \sigma_t S_t \,\mathrm{d}W_t + \mu_t S_t \,\mathrm{d}t, \qquad S_0 = s_0, t \in [0,T], \tag{3.1}$$

for $s_0 \in \mathbb{R}$ and predictable real-valued processes μ and σ. This type of linear stochastic differential equation can be explicitly solved by

$$S_t := s_0 \exp\bigg(\int_0^t \Big(\mu_s - \frac{\sigma_s^2}{2}\Big)\,\mathrm{d}s + \int_0^t \sigma_s \,\mathrm{d}W_s\bigg), \qquad t \in [0,T].$$

Based on Corollary A.2, one can easily deduce the sample path regularity of the price process S. Indeed, for $q \in (2, +\infty)$, $\alpha \in (0, 1/2 - 1/(2q))$ and $\delta := \alpha - 1/q$, if $E_P[\int_0^T |\mu_s|^q \,\mathrm{d}s] < +\infty$ and $E_P[\int_0^T |\sigma_s|^{2q} \,\mathrm{d}s] < +\infty$, then

$$S \in C^\alpha([0,T],\mathbb{R}) \quad \text{and} \quad S \in W^{\delta,q}([0,T],\mathbb{R}), \qquad \text{a.s.} \tag{3.2}$$

For example, the Heston model is a stochastic volatility model in which the volatility process σ satisfies such a bound.

In the context of *stochastic volatility modelling with Knightian uncertainty,* one usually replaces the fixed volatility process σ by a class of volatility processes. For example, the seminal works [4] and [33] on volatility uncertainty require the volatility processes σ to be such that $\sigma_t \in [\sigma_{\min}, \sigma_{\max}]$ for all $t \in [0,T]$ and some constants $\sigma_{\min}, \sigma_{\max} > 0$ with $\sigma_{\min} < \sigma_{\max}$. Therefore, due to the bounds on the volatility, all possible price paths considered in [4] and [33] belong to the function spaces as stated in (3.2).

Example 3.10 (Rough volatility models) Recently, investigating time series of volatility using high-frequency data, Gatheral et al. [27] showed that the log-volatility behaves essentially like a fractional Brownian motion with Hurst exponent H close to 0.1. This new insight has led to various fractional extensions of classical volatility models (see e.g. [27, 9, 14, 23]) which nicely lead to price paths belonging to the σ-compact metric space of Hölder-continuous functions. Indeed, if the stochastic volatility σ fulfils for some $M > 0$ and $q > r \geq 1$ the bound

$$E_P[|\sigma_t - \sigma_s|^q] \leq |t-s|^{\frac{q}{r}} \qquad \text{for } s,t \in [0,T] \text{ and } \sigma_0 \in \mathbb{R}, \tag{3.3}$$

then we observe that

$$E_P\bigg[\int_0^T |\sigma_s|^q \,\mathrm{d}s\bigg] \leq C\big(|\sigma_0|^q + E_P[\|\sigma\|_\beta^q]\big) < +\infty,$$

for some constant $C = C(q, M, T) > 0$ and $\beta \in (0, 1/r - 1/q)$. Note that condition (3.3) is exactly the condition usually required by the Kolmogorov continuity criterion (cf. Theorem A.1), which is frequently used to verify the Hölder regularity of a stochastic process. In particular, every rough volatility model satisfying (3.3) with associated price process given by (3.1) generates price paths possessing Hölder regularity as provided in (3.2). For example, a simple fractional Brownian motion with Hurst index H fulfils the bound (3.3) with $q \in [2, +\infty)$ and $r = H$, and the rough Heston model as introduced by El Euch and Rosenbaum [23, (1.3)] fulfils the bound (3.3) with $q \in [2, +\infty)$ and $1/r = \alpha - 1/2$ for $\alpha \in (1/2, 1)$, where α denotes the parameter specified in the rough Heston model [23, (1.3)].

Example 3.11 The most general case of *volatility uncertainty* is usually provided by simultaneously considering all processes of the type

$$S_t = \int_0^t \sqrt{\sigma_s}\,\mathrm{d}W_s, \qquad t \in [0,T],$$

for strictly positive and predictable processes σ; see [37, 41]. While they can deal with all σ such that $\int_0^T \sigma_s \,\mathrm{d}s < +\infty$ a.s., we have seen in Example 3.9 that we can deal with all volatility processes σ such that $E_P[\int_0^T \sigma_s^q \,\mathrm{d}s] < +\infty$ for $q \in (1, +\infty)$.

Another subclass of price processes S leading to σ-compact sets of price paths is given by all processes S with corresponding volatility process σ such that $\sigma \leq f$ for some deterministic integrable function $f\colon [0,T] \to (0,+\infty)$. Indeed, defining the quadratic variation of S by $\langle S\rangle_t = \int_0^t \sigma_s \,\mathrm{d}s$ for $t \in [0,T]$ and using the Dambis–Dubins–Schwarz theorem, one has $S_t = B_{\langle S\rangle_t}$ for a suitable Brownian motion B. Based on this observation, it is easy to derive that

$$S \in C^{p\text{-var},c}([0,T],\mathbb{R}) \quad \text{a.s.,} \qquad \text{with } c(s,t) := \int_s^t f(r)\,\mathrm{d}r, s,t \in [0,T],$$

and $p > 2$. Recall that $C^{p\text{-var},c}([0,T],\mathbb{R})$ is σ-compact by Corollary 3.2.

4 Proofs of the main results

Denote by $C_b := C_b((\Omega,\mathrm{d}),\mathbb{R})$ the set of all bounded continuous (with respect to d) functions $X\colon \Omega \to \mathbb{R}$.

Remark 4.1 If Ω is a σ-compact space endowed with the maximum norm, then (A1) is always satisfied.

Proof Fix $t \in [0,T]$, a bounded $\mathcal{F}_t^0$-measurable function h and a Borel probability Q. Define $\pi : \Omega \to C([0,t],\mathbb{R}^d)$, $\pi(\omega)(s) := \omega(s)$, and set $\Omega_t := \pi(\Omega)$ endowed with the maximum norm $\|\omega\|_\infty := \max_{s\in[0,t]} |\omega(s)|$. By σ-compactness, there exist compact sets K_n, $n \in \mathbb{N}$, such that $\Omega = \bigcup_n K_n$. Further, since $\Omega_t = \bigcup_n \pi(K_n)$ and $\pi(K_n)$ is compact by continuity of π, it follows that Ω_t is σ-compact and therefore separable. Standard arguments show that $\mathcal{F}_t^0 = \{\pi^{-1}(B) : B \in \mathcal{B}(\Omega_t)\}$, where $\mathcal{B}(\Omega_t)$ denotes the Borel sets of Ω_t. Hence, $h = \tilde{h} \circ \pi$ for some Borel function $\tilde{h}\colon \Omega_t \to \mathbb{R}$. Again by σ-compactness of Ω_t, the probability measure $\tilde{Q} := Q \circ \pi^{-1}$ is tight and thus regular, i.e., Borel sets can be approximated from inside in $\tilde{Q}$-measure by compact subsets. In particular, there exists a sequence of continuous functions $\tilde{h}_n\colon \Omega_t \to \mathbb{R}$ such that $\tilde{h}_n \to \tilde{h}$ $\tilde{Q}$-almost surely, which in turn implies $h_n := \tilde{h}_n \circ \pi \to \tilde{h} \circ \pi = h$ Q-almost surely. □

The following lemma is an immediate consequence of standard results about discrete-time local martingales (see [32, Theorems 1 and 2]), which we recall for later reference.

Lemma 4.2 *If $Q \in \mathcal{M}(\Omega)$ and $H \in \mathcal{H}^f$ is such that $E_Q[(H \cdot S)_T^-] < +\infty$, then $(H \cdot S)_T$ is Q-integrable and $E_Q[(H \cdot S)_T] = 0$.*

Next we need to establish some auxiliary results.

Lemma 4.3 *Let $d = 1$, $0 \leq s < t \leq T$, $m > 0$ and define*

$$\tau := \inf\{r \geq s : S_r > m \text{ or } S_r \leq -m\} \wedge T.$$

Then the function $\omega \mapsto S_{\tau(\omega)\wedge t}(\omega)$ is lower semicontinuous with respect to the maximum norm.

Proof Define $\tau_+ := \inf\{r \geq s : S_r > m\} \wedge T$ and $\tau_- := \inf\{r \geq s : S_r \leq -m\} \wedge T$, so that $\tau = \tau_+ \wedge \tau_-$. Moreover, fix ω and a sequence (ω_n) such that $\|\omega_n - \omega\|_\infty \to 0$. We claim that

$$\limsup_n \tau_+(\omega_n) \leq \tau_+(\omega) \qquad \text{and} \qquad \liminf_n \tau_-(\omega_n) \geq \tau_-(\omega).$$

Indeed, assume without loss of generality that $r := \tau_+(\omega) < T$. Then by definition, for every $\varepsilon > 0$, there is $\delta \in (0, \varepsilon)$ such that $\omega(r + \delta) > m$. Therefore $\omega_n(r + \delta) > m$ for eventually all n, showing that $\tau_+(\omega_n) \leq r + \varepsilon$ for eventually all n. As ε was arbitrary, the first part of the claim follows. Next, we may assume without loss of generality that $r := \tau_-(\omega) > s$. Then necessarily $\omega(u) > -m$ for $u \in [s, r)$. By continuity of ω and since $\|\omega_n - \omega\|_\infty \to 0$, for every $\varepsilon > 0$, we have $\omega_n(u) > -m$ for all $u \in [s, r - \varepsilon]$ and therefore $\tau_-(\omega_n) \geq r - \varepsilon$ for eventually all n. As ε was arbitrary, the second part of the claim follows. In the sequel, we prove the lower semicontinuity of S_t^τ.

(a) If $S_t^\tau(\omega) > m$, then $\tau(\omega) = \tau_+(\omega) = s$ and $\omega(s) > m$. In particular, $\omega_n(s) > m$ and $\tau_+(\omega_n) = s$ for eventually all n, hence

$$\lim_n S_t^\tau(\omega_n) = \lim_n \omega_n(s) = \omega(s) = S_t^\tau(\omega).$$

(b) If $S_t^\tau(\omega) = m$, then either $\tau_+(\omega) < t$ or $\tau_+(\omega) \geq t$. In the first case, it follows that $\tau_+(\omega) < \tau_-(\omega)$ so that $\tau_+(\omega_n) < \tau_-(\omega_n)$ and $\tau_+(\omega_n) < t$ for all but finitely many n by the first part of the proof, and therefore

$$\liminf_n S_t^\tau(\omega_n) = \liminf_n \omega_n\big(\tau_+(\omega_n)\big) = m = S_t^\tau(\omega).$$

On the other hand, if $\tau_+(\omega) \geq t$, then $\omega(t) = m$ and $\omega(r) > -m$ for $r \in [s, t]$. This implies that $\tau_-(\omega_n) \geq t$ for eventually all n and therefore

$$\liminf_n S_t^\tau(\omega_n) = \liminf_n \omega_n\big(t \wedge \tau_+(\omega_n)\big) = m = S_t^\tau(\omega).$$

(c) If $S_t^\tau(\omega) \in (-m, m)$, then either $\tau(\omega) > t$ or $\tau(\omega) = t$ (in which case necessarily $t = T$). In the latter case, it follows that $\omega(r) > -m$ for $r \in [s, T]$, hence $\tau_-(\omega_n) = T$ for eventually all n and thus

$$\liminf_n S_t^\tau(\omega_n) = \liminf_n \omega_n\big(t \wedge \tau_+(\omega_n)\big) \geq \omega(t) = S_t^\tau(\omega).$$

If $\tau(\omega) > t$, then again $\tau_-(\omega_n) > t$ for eventually all n so that the same argument shows that $\liminf_n S_t^\tau(\omega_n) \geq S_t^\tau(\omega)$.

(d) If $S_t^\tau(\omega) = -m$, then $\omega(s) \geq -m$. Assume that $\liminf_n S_t^\tau(\omega_n) < -m$. Then there is a subsequence still denoted by (ω_n) such that $\tau(\omega_n) = \tau_-(\omega_n) = s$. However, this contradicts $\liminf_n S_t^\tau(\omega_n) = \lim_n \omega_n(s) = \omega(s) \geq -m$.

(e) If $S_t^\tau(\omega) < -m$, then $\tau_-(\omega) = s$ and $\omega(s) < -m$. This implies $\omega_n(s) < -m$ and therefore $\tau_-(\omega_n) = s$ for eventually all n, so that

$$\lim_n S_t^\tau(\omega_n) = \lim_n \omega_n(s) = \omega(s) = S_t^\tau(\omega). \qquad \square$$

Proposition 4.4 *Assume that* (A1) *holds true. Then for any Borel probability measure Q on Ω which is not a local martingale measure, there exist $X \in C_b$ and $H \in \mathcal{H}^f$ such that $X \leq (H \cdot S)_T$ and $E_Q[X] > 0$.*

Proof Notice that S is a local martingale if and only if each component is a local martingale, which means we may assume without loss of generality that $d = 1$.

We prove that if $E_Q[X] \leq 0$ for all $X \in G$ with

$$G := \{X \in C_b : X \leq (H \cdot S)_T \text{ for some } H \in \mathcal{H}^f\},$$

then Q is a local martingale measure, i.e., for every $m \in \mathbb{N}$, the stopped process

$$S_t^\tau := S_{t \wedge \tau}, \qquad \text{where } \tau := \inf\{t \geq 0 : |S_t| \geq m\} \wedge T,$$

is a martingale. Fix $m \in \mathbb{N}$, $0 \leq s < t \leq T$ and define the stopping times

$$\sigma := \inf\{r \geq s : |S_r| \geq m\} \wedge T,$$

$$\sigma_\varepsilon := \inf\{r \geq s : S_r > m - \varepsilon \text{ or } S_r \leq \varepsilon - m\} \wedge T$$

for $0 < \varepsilon \leq 1$. First note that by continuity of S and right-continuity of $(\mathcal{F}_t)$, one has that σ_ε, σ and τ are in fact stopping times. By Lemma 4.3, the function $\omega \mapsto S_{t \wedge \sigma_\varepsilon(\omega)}(\omega)$ is lower semicontinuous with respect to $\|\cdot\|_\infty$ for every ε. In particular, for every continuous $\mathcal{F}_s^0$-measurable function $h\colon \Omega \to [0, 1]$, it holds that

$$(H \cdot S)_T \text{ is lower semicontinuous,} \quad \text{where } H := h\mathbf{1}_{]\!]s, \sigma_\varepsilon \wedge t]\!]} \in \mathcal{H}^f.$$

Since additionally $|S_t^{\sigma_\varepsilon} - S_s| \leq 2m$, there exists a sequence of continuous functions $X_n\colon \Omega \to [-2m, 2m]$ with $X_n \leq (H \cdot S)_T$ which increases pointwise to $(H \cdot S)_T$. Since $X_n \in G$ for all n, it follows that

$$E_Q[h(S_t^{\sigma_\varepsilon} - S_s)] = E_Q[(H \cdot S)_T] = \sup_n E_Q[X_n] \leq 0.$$

By assumption (A1), for every bounded and $\mathcal{F}_s^0$-measurable function h, there exists a sequence of continuous $\mathcal{F}_s^0$-measurable functions $h_n\colon \Omega \to [0, 1]$ which converges Q-almost surely to h, in particular

$$E_Q[h(S_t^{\sigma_\varepsilon} - S_s)] = \lim_n E_Q[h_n(S_t^{\sigma_\varepsilon} - S_s)] \leq 0.$$

The fact that σ_ε increases to σ as ε tends to 0 (and therefore $S_t^{\sigma_\varepsilon} \to S_t^\sigma$ by continuity of S) shows that

$$E_Q[h(S_t^\sigma - S_s)] = \lim_{\varepsilon\to 0} E_Q[h(S_t^{\sigma_\varepsilon} - S_s)] \le 0.$$

Furthermore, notice that $\sigma = \tau$ on $\{\tau \ge s\}$ so that $\mathbf{1}_{\{\tau\ge s\}}(S_t^\sigma - S_s) = S_t^\tau - S_s^\tau$. Since τ is the hitting time of a closed set, it is also a stopping time with respect to the raw filtration $(\mathcal{F}_t^0)$, so that $h\mathbf{1}_{\{\tau\ge s\}}\colon \Omega \to [0,1]$ is $\mathcal{F}_s^0$-measurable. This shows that

$$E_Q[h(S_t^\tau - S_s^\tau)] = E_Q[(h\mathbf{1}_{\{\tau\ge s\}})(S_t^\sigma - S_s)] \le 0,$$

which implies $E_Q[S_t^\tau|\mathcal{F}_s^0] \le S_s^\tau$, i.e., S^τ is a supermartingale with respect to the raw filtration $(\mathcal{F}_t^0)$. Finally, using that S^τ is bounded and $\mathcal{F}_s \subseteq \mathcal{F}_{s+\varepsilon}^0$ yields

$$\begin{aligned} E_Q[S_t^\tau - S_s^\tau|\mathcal{F}_s] &= \lim_{\varepsilon\to 0} E_Q[S_t^\tau - S_{s+\varepsilon}^\tau|\mathcal{F}_s] \\ &= \lim_{\varepsilon\to 0} E_Q\big[E_Q[S_t^\tau - S_{s+\varepsilon}^\tau|\mathcal{F}_{s+\varepsilon}^0]\big|\mathcal{F}_s\big] \le 0 \end{aligned}$$

which shows that S^τ is a supermartingale.

By similar arguments, one can also show that S^τ is a submartingale (and thus a martingale). Indeed, replace h by a continuous $\mathcal{F}_s^0$-measurable function $\tilde{h}\colon \Omega \to [-1,0]$ and the stopping times σ_ε by $\tilde{\sigma}_\varepsilon := \inf\{r \ge s : S_r \ge m - \varepsilon \text{ or } S_r < \varepsilon - m\} \wedge T$ for $\varepsilon > 0$. The same arguments as in Lemma 4.3 show that $\omega \mapsto S_{t\wedge\tilde{\sigma}_\varepsilon(\omega)}(\omega)$ is upper semicontinuous, which implies that $(H \cdot S)_T$ is lower semicontinuous for $H := \tilde{h}\mathbf{1}_{]\!]s,\tilde{\sigma}_\varepsilon\wedge t]\!]} \in \mathcal{H}^f$. The rest follows in the same way as before. □

Lemma 4.5 *Assume that* (A1) *and* (A2) *hold true. Then there exists an increasing sequence of non-empty compacts* (K_n) *such that* $\Omega = \bigcup_n K_n$ *and* $\omega^t \in K_n$ *for every* $(t,\omega) \in [0,T] \times K_n$.

Proof By assumption, $\Omega = \bigcup_n K_n'$ for some non-empty compacts (K_n'), where we assume without loss of generality that $K_n' \subseteq K_{n+1}'$ for every n. Define the function $\rho\colon [0,T]\times\Omega \to \Omega$, $(t,\omega) \mapsto \omega^t$ which, again by assumption, is continuous. Therefore $K_n := \{\omega^t : t \in [0,T],\ \omega \in K_n'\} = \rho([0,T], K_n')$ has the desired properties. □

Lemma 4.6 *Assume that* (A1) *and* (A2) *hold true and fix a sequence of compacts* (K_j) *as in Lemma 4.5. Further fix a continuous function* $Z\colon \Omega \to \mathbb{R}$, $H \in \mathcal{H}^f$ *and* $n \in \mathbb{N}$. *If* $(H\cdot S)_T(\omega) \ge -Z(\omega)$ *for all* $\omega \in K_j$, *then* $(H \cdot S)_t(\omega) \ge -Z(\omega^t)$ *for all* $(t,\omega) \in [0,T]\times K_j$.

Proof Fix $H = \sum_{n=1}^N h_n \mathbf{1}_{]\!]\tau_n,\tau_{n+1}]\!]} \in \mathcal{H}^f$, $\omega \in K_j$ and $t \in [0,T)$ (for $t = T$ the statement holds by assumption). We may assume that $\tau_{N+1} = T$ by adding an additional stopping time and setting $h_N \equiv 0$. Further, fix $\varepsilon > 0$ with $t + \varepsilon \le T$ and $m \in \mathbb{N}$ such that $\tau_m(\omega^{t+\varepsilon}) \le t \le \tau_{m+1}(\omega^{t+\varepsilon})$. Then

$$\begin{aligned}
&(H \cdot S)_t(\omega^{t+\varepsilon}) - (H \cdot S)_T(\omega^{t+\varepsilon}) \\
&= h_m(\omega^{t+\varepsilon})\big(S_t(\omega^{t+\varepsilon}) - S_{\tau_{m+1}(\omega^{t+\varepsilon})}(\omega^{t+\varepsilon})\big) \\
&\quad - \sum_{n=m+1}^{N} h_m(\omega^{t+\varepsilon})\big(S_{\tau_{n+1}(\omega^{t+\varepsilon})}(\omega^{t+\varepsilon}) - S_{\tau_n(\omega^{t+\varepsilon})}(\omega^{t+\varepsilon})\big)
\end{aligned}$$

as well as

$$|S_t(\omega^{t+\varepsilon}) - S_{\tau_{m+1}(\omega^{t+\varepsilon})}(\omega^{t+\varepsilon})| \le \delta(\varepsilon)$$

and

$$|S_{\tau_{n+1}(\omega^{t+\varepsilon})}(\omega^{t+\varepsilon}) - S_{\tau_n(\omega^{t+\varepsilon})}(\omega^{t+\varepsilon})| \le \delta(\varepsilon)$$

for all $n \ge m+1$, where $\delta(\varepsilon) := \max_{r,s\in[t,t+\varepsilon]} |\omega(r) - \omega(s)|$. Let C be a constant such that $|h_n| \le C$. Then, since $\lim_{\varepsilon\downarrow 0} \delta(\varepsilon) = 0$, we have

$$|(H \cdot S)_t(\omega^{t+\varepsilon}) - (H \cdot S)_T(\omega^{t+\varepsilon})| \le NC\delta(\varepsilon) \longrightarrow 0$$

as $\varepsilon \downarrow 0$. Since $\mathcal{F}_t \subseteq \mathcal{F}^0_{t+\varepsilon}$, it follows that $(H \cdot S)_t(\omega) = (H \cdot S)_t(\omega^{t+\varepsilon})$ for all $\varepsilon > 0$, so that

$$(H \cdot S)_t(\omega) = \lim_{\varepsilon\downarrow 0}(H \cdot S)_T(\omega^{t+\varepsilon}) \ge \liminf_{\varepsilon\downarrow 0} -Z(\omega^{t+\varepsilon}) = -Z(\omega^t),$$

since $\omega^{t+\varepsilon} \in K_j$ for all $\varepsilon > 0$ and $\varepsilon \mapsto Z(\omega^{t+\varepsilon})$ is continuous by assumption. □

We have now all ingredients at hand to prove the main results of the present paper.

Proof of Theorem 2.1 Fix a continuous function $Z : \Omega \to [0, +\infty)$ and a sequence (K_n) of compact sets as in Lemma 4.5.

Step (a). Fix $n \in \mathbb{N}$ and define

$$\phi_n(X) := \inf\left\{\lambda \in \mathbb{R} : \begin{array}{l} \text{there are } H \in \mathcal{H}^f \text{ and } c \in \mathbb{R} \text{ such that} \\ (H \cdot S)_T \ge c \text{ on } \Omega \text{ and } \lambda + (H \cdot S)_T \ge X \text{ on } K_n \end{array}\right\}$$

for $X : \Omega \to \mathbb{R}$. By Lemma 4.2, it follows that

$$\phi_n(X) \ge \sup_{Q\in\mathcal{M}(K_n)} E_Q[X] \tag{4.1}$$

for every Borel measurable X which is bounded from below on K_n. Let $\bar\omega \in K_n$ be the constant path $t \mapsto \bar\omega(t) := \omega(0)$ for some $\omega \in K_n$. Since the Dirac measure $\delta_{\bar\omega}$ assigning probability 1 to $\bar\omega$ belongs to $\mathcal{M}(K_n)$, it follows that ϕ_n is real-valued on C_b and $\phi_n(m) = m$ for every $m \in \mathbb{R}$.

Further, it is straightforward to check that ϕ_n is convex and increasing in the sense that $\phi_n(X) \le \phi_n(Y)$ whenever $X \le Y$. Moreover, ϕ_n is continuous from above on C_b, i.e., $\phi_n(X_k) \downarrow \phi_n(0)$ for every sequence (X_k) in C_b such that $X_k \downarrow 0$. To see this, fix such a sequence (X_k) and let $\varepsilon > 0$ be arbitrary. By Dini's lemma, one has $X_k \le \varepsilon$ on K_n for all k large enough so that $\phi_n(X_k) \le \varepsilon$ for all such k, which shows that $\phi_n(X_k) \downarrow 0$. It follows from [6, Theorem 2.2] that

$$\phi_n(X) = \max_{Q \in ca^+(\Omega)} \big(E_Q[X] - \phi_n^*(Q)\big) \tag{4.2}$$

for all $X \in C_b$, where $\phi_n^*(Q) := \sup_{X \in C_b}(E_Q[X] - \phi_n(X))$ and $ca^+(\Omega)$ denotes the set of non-negative countably additive Borel measures on Ω. We claim that

$$\phi_n^*(Q) = \begin{cases} 0, & \text{if } Q \in \mathcal{M}(K_n), \\ +\infty, & \text{else}, \end{cases} \tag{4.3}$$

for all $Q \in ca^+(\Omega)$. First notice that (4.1) implies $\phi_n^*(Q) \le 0$ whenever $Q \in \mathcal{M}(K_n)$. Since in addition $\phi_n(0) = 0$, it follows that $\phi_n^*(Q) = 0$. On the other hand, if $Q \notin \mathcal{M}(K_n)$, then $\phi_n^*(Q) = +\infty$. Indeed, if Q is not a probability, then $\phi_n(m) = m$ implies that $\phi_n^*(Q) \ge \sup_{m \in \mathbb{R}}(mQ(\Omega) - m) = +\infty$. Similarly, since K_n^c is open, there exists a sequence (X_k) of bounded continuous functions such that $X_k \uparrow +\infty \mathbf{1}_{K_n^c}$ with the convention $0 \cdot (+\infty) := 0$. By definition, $\phi_n(X_k) \le 0$ for all k, from which it follows that

$$\phi_n^*(Q) \ge \sup_k E_Q[X_k] = +\infty E_Q[\mathbf{1}_{K_n^c}].$$

It remains to show that if Q is a probability with $Q[K_n] = 1$ but not a martingale measure, then $\phi_n^*(Q) = +\infty$. Note that compactness of K_n implies boundedness of K_n with respect to $\|\cdot\|_\infty$, and therefore Q is also not a local martingale measure. Thus Proposition 4.4 yields the existence of $X \in C_b$ and $H \in \mathcal{H}^f$ such that $X \le (H \cdot S)_T$ and $E_Q[X] > 0$. Since $\phi_n(mX) \le 0$ for all $m > 0$, it follows that $\phi_n^*(Q) \ge \sup_{m>0}(E_Q[mX] - \phi_n(mX)) = +\infty$.

Now fix some upper semicontinuous X which is bounded from above (i.e., we have $X = X \wedge m$ for some $m > 0$) and satisfies $X \ge -Z$. We claim that

$$\phi_n(X) = \max_{Q \in \mathcal{M}(K_n)} E_Q[X]. \tag{4.4}$$

To see this, let (X_k) be a sequence in C_b such that $X_k \downarrow X$. By (4.2) and (4.3), there exist $Q_k \in \mathcal{M}(K_n)$ such that $\phi_n(X_k) = E_{Q_k}[X_k]$. Since $\mathcal{M}(K_n)$ is (sequentially) compact in the weak topology induced by the continuous bounded functions, we may assume, possibly after passing to a subsequence, that $Q_k \to Q$ for some $Q \in \mathcal{M}(K_n)$. For every $\varepsilon > 0$, there exists k' such that $E_Q[X_{k'}] \le E_Q[X] + \varepsilon$. Choose $k \ge k'$ such that $E_{Q_k}[X_{k'}] \le E_Q[X_{k'}] + \varepsilon$. Then

$$E_{Q_k}[X_k] \le E_{Q_k}[X_{k'}] \le E_Q[X_{k'}] + \varepsilon \le E_Q[X] + 2\varepsilon$$

so that

$$\begin{aligned}\phi_n(X) \le \lim_k \phi_n(X_k) = \lim_k E_{Q_k}[X_k] \le E_Q[X] + 2\varepsilon \\ \le \sup_{R \in \mathcal{M}(K_n)} E_R[X] + 2\varepsilon \le \phi_n(X) + 2\varepsilon,\end{aligned}$$

where the last inequality follows from (4.1). This shows (4.4).

Step (b). For $X\colon \Omega \to (-\infty, +\infty]$, define

$$\phi(X) := \inf\left\{ \lambda \in \mathbb{R} : \begin{array}{c} \text{there is } (H^n) \text{ in } \mathcal{H}^f \text{ such that} \\ \lambda + (H^n \cdot S)_t(\omega) \geq -Z(\omega^t) \text{ for } (t,\omega) \in [0,T] \times \Omega \\ \text{and } \lambda + \liminf_n (H^n \cdot S)_T \geq X \text{ on } \Omega \end{array} \right\}.$$

Let $X \in C_{\delta\sigma}$ be such that $X \geq -Z$, and let (Y_n) be a sequence of upper semicontinuous functions which increases pointwise to X. Define $X_n := (Y_n \wedge n) \vee (-Z)$ which is still upper semicontinuous and increases to X. We claim that $\sup_n \phi_n(X_n) = \phi(X)$. First observe that for every $Q \in \mathcal{M}_c(\Omega)$, Fatou's lemma and Lemma 4.2 imply

$$\lambda = \lambda + \liminf_n E_Q[(H^n \cdot S)_T] \geq E_Q\big[\lambda + \liminf_n (H^n \cdot S)_T\big] \geq E_Q[X]$$

for every $\lambda \in \mathbb{R}$ and every sequence (H^n) in $\mathcal{H}^f$ such that $\lambda + \liminf_n (H^n \cdot S)_T \geq X$ and $\lambda + (H^n \cdot S)_T \geq -mZ$ for all n and some $m \geq 0$. Hence, one gets

$$\phi(X) \geq \sup_{Q \in \mathcal{M}_c(\Omega)} E_Q[X] \geq \sup_n \sup_{Q \in \mathcal{M}(K_n)} E_Q[X_n] = \sup_n \phi_n(X_n), \tag{4.5}$$

where the last equality follows from (4.4).

On the other hand, let $m > \sup_n \phi_n(X_n)$ so that by definition, for each n, there exists $H^n \in \mathcal{H}^f$ such that $m + (H^n \cdot S)_T \geq X_n \geq -Z$ on K_n. Thus it follows from Lemma 4.6 that

$$m + (H^n \cdot S)_t(\omega) \geq -Z(\omega^t) \qquad \text{for all } (t,\omega) \in [0,T] \times K_n. \tag{4.6}$$

Fix $\varepsilon > 0$. Define the stopping times

$$\sigma_n(\omega) := \inf\{t \in [0,T] : m + \varepsilon + (H^n \cdot S)_t(\omega) + Z(\omega^t) = 0\} \wedge T$$

and notice that

$$(H^n \cdot S)_{\sigma_n} = (\tilde{H}^n \cdot S)_T \tag{4.7}$$

for

$$\tilde{H}^n := \sum_{i=1}^N h_i^n \mathbf{1}_{\{\sigma_n \geq \tau_i\}} \mathbf{1}_{]\!]\tau_i \wedge \sigma_n, \tau_{i+1} \wedge \sigma_n]\!]} \in \mathcal{H}^f,$$

where $H^n = \sum_{i=1}^N h_i^n \mathbf{1}_{]\!]\tau_i, \tau_{i+1}]\!]}$. Fix $\omega \in \Omega$. Then $\omega \in K_j$ for some $j \in \mathbb{N}$ and therefore it follows by (4.6) that $\sigma_n(\omega) = T$ whenever $n \geq j$. Hence, we have

$$m + \varepsilon + (\tilde{H}^n \cdot S)_T(\omega) = m + \varepsilon + (H^n \cdot S)_T(\omega) \geq X_n(\omega) \qquad \text{for } n \geq j.$$

As ω was arbitrary, it follows that $\liminf_n (m + \varepsilon + (\tilde{H}^n \cdot S)_T) \geq X$. Moreover, it follows from (4.7) that

$$m + \varepsilon + (\tilde{H}^n \cdot S)_t(\omega) \geq -Z(\omega^{t \wedge \sigma_n(\omega)}) \geq -Z(\omega^t) \qquad \text{for all } (t,\omega) \in [0,T] \times \Omega,$$

which shows that $\phi(X) \le m + \varepsilon$. Finally, since $m > \sup_n \phi_n(X_n)$ and $\varepsilon > 0$ was arbitrary, we conclude that $\phi(X) \le \sup_n \phi_n(X_n)$, which shows that all inequalities in (4.5) are equalities. In particular, $\phi(X) = \sup_{Q \in \mathcal{M}_c(\Omega)} E_Q[X]$ which shows (2.1).

Step (c). We finally show that $\mathcal{M}_c(\Omega)$ can be replaced by the set $\mathcal{M}_Z(\Omega)$, and $\mathcal{H}^f$ by $\mathcal{H}$. To that end, fix $X\colon \Omega \to (-\infty, +\infty]$ satisfying $X \ge -Z$ for some $\lambda \in \mathbb{R}$, $Q \in \mathcal{M}_Z(\Omega)$, and (H^n) in $\mathcal{H}$ such that $\lambda + (H^n \cdot S)_t(\omega) \ge -Z(\omega^t)$ for all $(t,\omega) \in [0,T] \times \Omega$ and $\lambda + \liminf_n (H^n \cdot S)_T \ge X$. Define

$$H^{n,K} := \sum_{k=1}^{K} h_k^n \mathbf{1}_{]\!]\tau_k^n, \tau_{k+1}^n]\!]} \in \mathcal{H}^f \qquad \text{and} \qquad H^n = \sum_{k=1}^{\infty} h_k^n \mathbf{1}_{]\!]\tau_k^n, \tau_{k+1}^n]\!]}.$$

Then we obtain

$$\lambda + (H^{n,K} \cdot S)_T(\omega) = \lambda + (H^n \cdot S)_{\tau_{K+1}^n(\omega)}(\omega) \ge -Z(\omega^{\tau_{K+1}^n(\omega)}) \ge -Z(\omega),$$

where the last inequality holds by assumption. Hence, by Lemma 4.2 and Fatou's lemma, it follows that

$$\begin{aligned}\lambda &= \lambda + \liminf_n \liminf_K E_Q[(H^{n,K} \cdot S)_T] \ge \liminf_n E_Q\big[\lambda + \liminf_K (H^{n,K} \cdot S)_T\big] \\ &= \liminf_n E_Q[\lambda + (H^n \cdot S)_T] \ge E_Q\big[\lambda + \liminf_n (H^n \cdot S)_T\big] \ge E_Q[X].\end{aligned}$$

This shows that

$$\begin{aligned}&\inf\left\{\lambda \in \mathbb{R} : \begin{array}{c}\text{there is a sequence } (H^n) \text{ in } \mathcal{H}^f \text{ such that} \\ \lambda + (H^n \cdot S)_t(\omega) \ge -Z(\omega^t) \text{ for all } (t,\omega) \in [0,T] \times \Omega \\ \text{and } \lambda + \liminf_n (H^n \cdot S)_T(\omega) \ge X(\omega) \text{ for all } \omega \in \Omega\end{array}\right\} \\ &\ge \inf\left\{\lambda \in \mathbb{R} : \begin{array}{c}\text{there is a sequence } (H^n) \text{ in } \mathcal{H} \text{ such that} \\ \lambda + (H^n \cdot S)_t(\omega) \ge -Z(\omega^t) \text{ for all } (t,\omega) \in [0,T] \times \Omega \\ \text{and } \lambda + \liminf_n (H^n \cdot S)_T(\omega) \ge X(\omega) \text{ for all } \omega \in \Omega\end{array}\right\} \\ &\ge \sup_{Q \in \mathcal{M}_Z(\Omega)} E_Q[X] \ge \sup_{Q \in \mathcal{M}_c(\Omega)} E_Q[X],\end{aligned}$$

where the first and last terms coincide by the previous steps (a) and (b). □

The proof of Corollary 2.4 is a consequence of the following lemma.

Lemma 4.7 *Let* $\Omega = C([0,T],\mathbb{R}^d)$, $Q \in \mathcal{M}(\Omega)$ *and* $X\colon \Omega \to \mathbb{R}$ *be bounded and Borel. For every* $\varepsilon > 0$, *there exist* $K \subseteq \Omega$ *compact and* $\tilde{Q} \in \mathcal{M}(K)$ *such that* $|E_Q[X] - E_{\tilde{Q}}[X]| \le \varepsilon$. *In particular,*

$$\sup_{Q \in \mathcal{M}(\Omega)} E_Q[X] = \sup_{Q \in \mathcal{M}_c(\Omega)} E_Q[X].$$

Proof If $X = 0$, there is nothing to prove. Otherwise, since Ω is a Polish space, there exists $K \subseteq \Omega$ compact such that $Q(K^c) \le \varepsilon / \|X\|_\infty$. By an Arzelà–Ascoli type theorem [25, Theorem 1.4], there exist $a \in \mathbb{R}$ and a continuous increasing function $f\colon [0,+\infty) \to [0,+\infty)$ such that

$$K \subseteq \tilde{K} := \big\{\omega \in \Omega : \|\omega\|_\infty \le a \text{ and } |\omega(t) - \omega(s)| \le f(|t-s|) \text{ for } s,t \in [0,T]\big\}$$

and $\tilde{K}$ is compact. Now define the stopping time

$$\tau := \inf\big\{t \ge 0 : |S_t| > a \text{ or } |S_t - S_s| > f(|t-s|) \text{ for some } s \in \mathbb{Q} \cap [0,t]\big\} \wedge T$$

so that $\tilde{K} = \{\tau = T\}$. Then for $\tilde{Q} := Q \circ (S^\tau)^{-1} \in \mathcal{M}(\tilde{K})$, we have

$$|E_{\tilde{Q}}[X] - E_Q[X]| \le |E_Q[X(S^\tau)1_{K^c}]| + E_Q[X1_{K^c}] \le 2\varepsilon.$$

In particular, $\sup_{Q \in \mathcal{M}(\Omega)} E_Q[X] = \sup_{Q \in \mathcal{M}_c(\Omega)} E_Q[X]$. □

Proof of Corollary 2.4 Denote by $\mathcal{K}$ the set of all compact subsets $K \subseteq \Omega$. For $K \in \mathcal{K}$, define $\tilde{K} := \{\omega^t : t \in [0,T] \text{ and } \omega \in K\}$ which is compact due to (the proof of) Lemma 4.5. For $K \in \mathcal{K}$ and every bounded upper semicontinuous function $X\colon \Omega \to \mathbb{R}$, define

$$\phi_K(X) := \inf\left\{\lambda \in \mathbb{R} : \begin{array}{l} \text{there are } H \in \mathcal{H}^f \text{ and } c \ge 0 \text{ such that} \\ \lambda + (H\cdot S)_T(\omega) \ge -c \text{ for all } \omega \in \Omega \text{ and} \\ \lambda + (H\cdot S)_T(\omega) \ge X(\omega) \text{ for all } \omega \in K \end{array}\right\}.$$

Then we have

$$\begin{aligned} \sup_{K\in\mathcal{K}} \phi_K(X) = \sup_{K\in\mathcal{K}} \phi_{\tilde{K}}(X) &= \sup_{K\in\mathcal{K}} \max_{Q\in\mathcal{M}(\tilde{K})} E_Q[X] \\ &= \sup_{K\in\mathcal{K}} \max_{Q\in\mathcal{M}(K)} E_Q[X] = \sup_{Q\in\mathcal{M}_c(\Omega)} E_Q[X]. \end{aligned}$$

The first and third equalities follow from $K \subseteq \tilde{K}$, the second one follows from $\phi_{\tilde{K}}(X) = \max_{Q\in\mathcal{M}(\tilde{K})} E_Q[X]$ as in (4.4) for every $K \in \mathcal{K}$ and the last equality follows by the definition of $\mathcal{M}_c(\Omega)$. Now use Lemma 4.7 to conclude. □

Proof of Theorem 2.7 **Step (a).** For $n \in \mathbb{N}$ and every function $X\colon \Omega \to \mathbb{R}$, define

$$\phi_n(X) := \inf\left\{\lambda \in \mathbb{R} : \begin{array}{l} \text{there are } H \in \mathcal{H}^f \text{ and } c > 0 \text{ such that} \\ (H\cdot S)_T \ge -c \text{ and } \lambda + (H\cdot S)_T \ge X - Z/n \end{array}\right\}.$$

It follows from Lemma 4.2 that $\phi_n(X) \ge \sup_{Q\in\mathcal{M}_Z(\Omega)}(E_Q[X] - E_Q[Z]/n)$ for every Borel function X which is bounded from below. Moreover, if (X_k) is a sequence in C_b decreasing pointwise to 0, then $\phi(X_n) \downarrow \phi(0)$. Indeed, fix an arbitrary $\varepsilon > 0$ and $H \in \mathcal{H}^f$ with $(H\cdot S)_T \ge -c$ for some $c \ge 0$ such that

$$\varepsilon + \phi_n(0) + (H\cdot S)_T + Z/n \ge 0.$$

Now define $\tilde{c} := \|X_1\|_\infty - \varepsilon - \phi_n(0) + c$ so that $\tilde{c} + \varepsilon + \phi_n(0) + (H \cdot S)_T \geq X_1$. Since $\{Z \leq \tilde{c}n\}$ is compact, it follows from Dini's lemma that $X_k \mathbf{1}_{\{Z \leq \tilde{c}n\}} \leq \varepsilon$ for k large enough. Hence we get

$$\begin{aligned} X_k \leq X_k \mathbf{1}_{\{Z \leq \tilde{c}n\}} + X_1 \mathbf{1}_{\{Z > \tilde{c}n\}} &\leq \varepsilon + \big(\varepsilon + \phi_n(0) + (H \cdot S)_T + Z/n\big)\mathbf{1}_{\{Z > \tilde{c}n\}} \\ &\leq 2\varepsilon + \phi_n(0) + (H \cdot S)_T + Z/n \end{aligned}$$

so that $\phi_n(X_k) \leq \phi_n(0) + 2\varepsilon$ for k large enough, which shows that $\phi_n(X_k) \downarrow \phi_n(0)$. Now a computation similar to the one in the proof of Theorem 2.1 shows that

$$\phi_n(X) = \max_{Q \in \mathcal{M}_Z(\Omega)} \big(E_Q[X] - E_Q[Z]/n\big) \tag{4.8}$$

for every bounded upper semicontinuous function $X\colon \Omega \to \mathbb{R}$. Indeed, first notice that since $Z \geq \|\cdot\|_\infty$ by assumption, the set $\mathcal{M}_Z(\Omega)$ coincides with the set of all local martingale measures which integrate Z. Therefore, the same arguments as in the proof of Theorem 2.1 show that

$$\phi_n^*(Q) := \sup_{X \in C_b} \big(E_Q[X] - \phi_n(X)\big) = \begin{cases} E_Q[Z]/n, & \text{if } Q \in \mathcal{M}_Z(\Omega), \\ +\infty, & \text{else,} \end{cases}$$

and thus that (4.8) is true, at least whenever $X \in C_b$. For the extension to upper semicontinuous functions, notice that $\phi(X) = \max_{Q \in \Lambda_{2c}}(E_Q[X] - E_Q[Z]/n)$ for every $X \in C_b$ satisfying $|X| \leq c$, where $\Lambda_{2c} := \{\phi_n^* \leq 2c\}$. Using the fact that Z has compact sublevel sets and Proposition 4.4, it follows that Λ_c is (sequentially) compact. The rest follows analogously to the proof of Theorem 2.1.

Step (b). For $X \in C_{\delta\sigma}$, define

$$\phi(X) := \inf \left\{ \lambda \in \mathbb{R} : \begin{array}{c} \text{there are } (H^n) \text{ in } \mathcal{H}^f \text{ and } c \geq 0 \\ \text{such that } (H^n \cdot S)_T \geq -cZ \text{ for all } n \\ \text{and } \lambda + \liminf_n (H^n \cdot S)_T \geq X \end{array} \right\}.$$

Fix $X \in C_{\delta\sigma}$ bounded from below and X_n upper semicontinuous bounded from below such that $X = \sup_n X_n$. Then it follows from Fatou's lemma and Lemma 4.2 that

$$\begin{aligned} \phi(X) \geq \sup_{Q \in \mathcal{M}_Z(\Omega)} E_Q[X] &= \sup_{Q \in \mathcal{M}_Z(\Omega)} \Big(\sup_n E_Q[X_n] - E_Q[Z]/n\Big) \\ &= \sup_n \sup_{Q \in \mathcal{M}_Z(\Omega)} (E_Q[X_n] - E_Q[Z]/n) = \sup_n \phi_n(X_n). \end{aligned}$$

On the other hand, if $m > \sup_n \phi_n(X_n)$, then for every n, there exists $H^n \in \mathcal{H}^f$ such that $m + (H^n \cdot S)_T \geq X_n - Z/n$. Hence $(H^n \cdot S)_T \geq -cZ$ for $c := \|X_1 \wedge 0\|_\infty + m + 1$ and $m + \liminf_n (H^n \cdot S)_T \geq \liminf_n (X_n - Z/n) = X$, which completes the proof. □

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Appendix

A.1 Kolmogorov continuity criterion

In this section, we briefly recall a version of the so-called Kolmogorov continuity criterion, which provides a sufficient condition for Hölder and Sobolev regularity of stochastic processes. The presented version is a slight reformulation of [25, Theorem A.10].

Let $(\tilde{\Omega}, \mathcal{F}, P)$ be a probability space, $X\colon [0,T]\times\tilde{\Omega}\to\mathbb{R}^d$ a stochastic process, $T\in(0,+\infty)$, $(\mathbb{R}^d, |\cdot|)$ the Euclidean space and W a d-dimensional Brownian motion.

Theorem A.1 *Let $q > r \geq 1$ and suppose there exists a constant $M > 0$ such that*

$$E_P[|X_t - X_s|^q] \leq M|t-s|^{\frac{q}{r}} \qquad \textit{for all } s,t\in[0,T].$$

Then for any $\alpha\in[0, 1/r - 1/q)$ and with $\delta := \alpha + 1/q$, there exists a constant $C = C(r,q,\alpha,T)$ such that

$$E_P\big[\|X\|_\alpha^q\big] \leq CM \qquad \textit{and} \qquad E_P\big[\|X\|_{W^{\delta,q}}^q\big] \leq CM,$$

where we recall the seminorms

$$\|X\|_\alpha := \sup_{s,t\in[0,T]} \frac{|X_t - X_s|}{|t-s|^\alpha}, \qquad \|X\|_{W^{\delta,q}} := \left(\int_{[0,T]^2} \frac{|X_t - X_s|^q}{|t-s|^{\delta q+1}}\,\mathrm{d}s\,\mathrm{d}t\right)^{\frac{1}{q}}. \quad (\text{A.1})$$

Applying Theorem A.1 to Itô processes yields the following regularity criterion.

Corollary A.2 *Let X be a d-dimensional Itô process of the form*

$$X_t = x_0 + \int_0^t a_s\,\mathrm{d}W_s, \qquad t\in[0,T],$$

for a predictable process $a\colon [0,T]\times\tilde{\Omega}\to\mathbb{R}^{d\times d}$ and $x_0\in\mathbb{R}^d$. Suppose $q\in(2,+\infty)$, $\alpha\in(0, 1/2 - 1/(2q))$ and $\delta = \alpha - 1/q$. If $E_P[\int_0^T |a_s|^q\,\mathrm{d}s] < +\infty$, then

$$X\in C^\alpha([0,T],\mathbb{R}^d) \quad \textit{and} \quad X\in W^{\delta,q}([0,T],\mathbb{R}^d), \qquad P\textit{-a.s.}$$

Proof Using the Burkholder–Davis–Gundy and Jensen's inequalities gives

$$E_P[|X_t - X_s|^q] \le E_P\bigg[\bigg(\int_s^t |a_r|^2\,\mathrm{d}r\bigg)^{q/2}\bigg] \le E_P\bigg[\int_0^T |a_r|^q\,\mathrm{d}r\bigg]|t-s|^{q(\frac{1}{2}-\frac{1}{q})}.$$

Therefore Theorem A.1 implies the assertion. □

A.2 Construction of counterexample

The example (see Remark 2.6) showing that bounded variation strategies, and in particular simple trading strategies, are not rich enough to obtain the pathwise pricing–hedging duality was based on a Hölder-continuous function with exploding quadratic variation. The existence of such a function is ensured by the following lemma.

Lemma A.3 *There exist a function $\tilde{\omega} \in C^{1/4}([0,T],\mathbb{R})$ for some $T>0$ and a refining sequence of partitions $(\tilde{\pi}_n)_{n\in\mathbb{N}}$ of the interval $[0,T]$ such that*

$$0 \le \tilde{\omega}(t) \le 1, \qquad t \in [0,T],$$

$$\langle\tilde{\omega}\rangle_t := \lim_n \langle\tilde{\omega}\rangle_t^n, \qquad \textit{where } \langle\tilde{\omega}\rangle_t^n := \sum_{[u,v]\in\tilde{\pi}_n} \big(\tilde{\omega}(u\wedge t) - \tilde{\omega}(v\wedge t)\big)^2,$$

exists for every $t\in[0,T)$, and $\langle\tilde{\omega}\rangle_t \to \infty$ as $t\to T$.

Proof For $\omega \in C([0,T],\mathbb{R})$, let (π_n) be the refining sequence of partition consisting of the dyadic points $\mathbb{D}_n := \{k2^{-n} : k\in\mathbb{N}_0\}$ with $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$, and consider the first hitting time of 0 given by $\tau(\omega) := \inf\{t>0 : \omega(t)=0\}$.

Recalling the properties of a Brownian motion W, we know that the event of a Brownian motion W starting at 0, $\tau\ge 1$ and $W_t\in(0,1)$ for $t\in(0,\tau)$ has a strictly positive probability. This fact ensures the existence of a constant $T_0>1$ and a (nowhere constant) function $f\in C^\beta([0,T_0],\mathbb{R})$ for every $\beta\in(0,1/2)$ such that

(i) $f(0)=f(T_0)=0$;
(ii) $0\le f(t)\le 1$ for all $t\in[0,T_0]$;
(iii) the pathwise quadratic variation given by $\langle f\rangle_t := \lim_n \langle f\rangle_t^n$ exists along (π_n) for every $t\in[0,T_0]$ (as limit in uniform convergence), and $\langle f\rangle_{T_0}>0$.

Without loss of generality, we may assume $T_0=1$ since it is always possible to modify f to ensure this without losing the other properties.

Setting $T := \sum_{n\in\mathbb{N}_0} n^{-2} < \infty$ and iteratively $t_n := t_{n-1}+n^{-2}$ for $n\in\mathbb{N}$ with $t_0=0$, we define

$$\tilde{\omega}(t) := n^{-1/2} f\big(n^2(t-t_{n-1})\big) \qquad \text{for } t\in[t_{n-1},t_n),$$

with $\tilde{\omega}(T):=0$.

Let us first show that $\tilde{\omega}\in C^{1/4}([0,T],\mathbb{R})$. For $s,t\in[0,T)$, there exist $n,m\in\mathbb{N}$ such that $s\in[t_{n-1},t_n]$ and $t\in[t_{m-1},t_m]$. Therefore, we get

$$
\begin{aligned}
|\tilde{\omega}(t)-\tilde{\omega}(s)| &\leq \sum_{k=n}^{m} |\tilde{\omega}(t_k \wedge t)-\tilde{\omega}(t_{k-1} \vee s)| \\
&\leq |\tilde{\omega}(t_m \wedge t)-\tilde{\omega}(t_{m-1} \vee s)| + |\tilde{\omega}(t_n \wedge t)-\tilde{\omega}(t_{n-1} \vee s)| \\
&\leq L_f m^{-1/2} \Big| m^2 \big((t_m \wedge t)-(t_{m-1} \vee s)\big)\Big|^{1/4} \\
&\quad + L_f n^{-1/2} \Big| n^2 \big((t_n \wedge t)-\omega(t_{n-1} \vee s)\big)\Big|^{1/4} \\
&\leq 2L_f |t-s|^{1/4},
\end{aligned}
$$

where $L_f > 0$ denotes the 1/4-Hölder norm of f. If $0 \leq s < t = T$, choose $n, m \in \mathbb{N}$ such that $s \in [t_{n-1}, t_n]$ and $m^{-1/2} \leq |t-s|^{1/4}$. This time, we get

$$
\begin{aligned}
|\tilde{\omega}(t)-\tilde{\omega}(s)| &\leq |\tilde{\omega}(T)-\tilde{\omega}(t_{m-1} \vee s)| + |\tilde{\omega}(t_n \wedge t)-\tilde{\omega}(t_{n-1} \vee s)| \\
&\leq m^{-1/2} + L_f n^{-1/2} \Big| n^2 \big((t_n \wedge t)-\omega(t_{n-1} \vee s)\big)\Big|^{1/4} \\
&\leq (1+L_f)|t-s|^{1/4}.
\end{aligned}
$$

Based on these two estimates, we see that $\tilde{\omega} \in C^{1/4}([0,T],\mathbb{R})$.

To obtain the desired properties of the quadratic variation, we define the partition $\tilde{\pi}_m$ for $m \in \mathbb{N}$ as follows. For $n \leq m$, $\tilde{\pi}_m$ restricted to $[t_{n-1}, t_n]$ consists of the point

$$
\tau_k^m := \inf\{t \geq \tau_{k-1}^m : \tilde{\omega}(t) = f(k2^{-m})\} \qquad \text{and} \qquad \tau_0^m := t_{n-1};
$$

for $n \geq m$, choose $\tilde{\pi}_m$ restricted to $[t_{n-1}, t_n]$ to be empty, and T is included in $\tilde{\pi}_m$. Note that $(\tilde{\pi}_m)$ is a refining sequence of partitions. Furthermore, by the construction of $(\tilde{\pi}_m)$, the pathwise quadratic variation of $\tilde{\omega}$ exists along $(\tilde{\pi}_m)$ for all $t \in [0,T)$, and for t_n, we observe that

$$
\langle \tilde{\omega} \rangle_{t_n} = \sum_{k=1}^{n} \frac{\langle f \rangle_{T_0}}{n}
$$

which goes to infinity as $t_n \to T$ or in other words as $n \to \infty$. □

References

1. Acciaio, B., Beiglböck, M., Penkner, F., Schachermayer, W.: A model-free version of the fundamental theorem of asset pricing and the super-replication theorem. Math. Finance **26**, 233–251 (2016)
2. Acciaio, B., Larsson, M.: Semi-static completeness and robust pricing by informed investors. Ann. Appl. Probab. **27**, 2270–2304 (2017)
3. Aksamit, A., Hou, Z., Obłój, J.: Robust framework for quantifying the value of information in pricing and hedging. Preprint (2018). Available online at https://arxiv.org/abs/1605.02539
4. Avellaneda, M., Levy, A., Parás, A.: Pricing and hedging derivative securities in markets with uncertain volatilities. Appl. Math. Finance **2**, 73–88 (1995)
5. Bartl, D.: Exponential utility maximization under model uncertainty for unbounded endowments. Ann. Appl. Probab. **29**, 577–612 (2019)
6. Bartl, D., Cheridito, P., Kupper, M.: Robust expected utility maximization with medial limits. J. Math. Anal. Appl. **471**, 752–775 (2019)

7. Bartl, D., Cheridito, P., Kupper, M., Tangpi, L.: Duality for increasing convex functionals with countably many marginal constraints. Banach J. Math. Anal. **11**, 72–89 (2017)
8. Bartl, D., Kupper, M., Neufeld, A.: Pathwise superhedging on prediction sets. Preprint (2018). Available online at https://arxiv.org/abs/1711.02764
9. Bayer, C., Friz, P., Gatheral, J.: Pricing under rough volatility. Quant. Finance **16**, 887–904 (2016)
10. Beiglböck, M., Cox, A.M.G., Huesmann, M., Perkowski, N., Prömel, D.J.: Pathwise superreplication via Vovk's outer measure. Finance Stoch. **21**, 1141–1166 (2017)
11. Beiglböck, M., Henry-Labordère, P., Penkner, F.: Model-independent bounds for option prices—a mass transport approach. Finance Stoch. **17**, 477–501 (2013)
12. Beiglböck, M., Nutz, M., Touzi, N.: Complete duality for martingale transport on the line. Ann. Probab. **45**, 3038–3074 (2017)
13. Beiglböck, M., Siorpaes, P.: Pathwise versions of the Burkholder–Davis–Gundy inequality. Bernoulli **21**, 360–373 (2015)
14. Bennedsen, M., Lunde, A., Pakkanen, M.S.: Decoupling the short- and long-term behavior of stochastic volatility. Preprint (2017). Available online at https://arxiv.org/abs/1610.00332
15. Biagini, S., Bouchard, B., Kardaras, C., Nutz, M.: Robust fundamental theorem for continuous processes. Math. Finance **27**, 963–987 (2017)
16. Burzoni, M., Frittelli, M., Hou, Z., Maggis, M., Obłój, J.: Pointwise arbitrage pricing theory in discrete time. Math. Oper. Res. (2019). Forthcoming, available online at https://doi.org/10.1287/moor.2018.0956
17. Burzoni, M., Frittelli, M., Maggis, M.: Model-free superhedging duality. Ann. Appl. Probab. **27**, 1452–1477 (2017)
18. Cheridito, P., Kupper, M., Tangpi, L.: Duality formulas for robust pricing and hedging in discrete time. SIAM J. Financ. Math. **8**, 738–765 (2017)
19. Delbaen, F., Schachermayer, W.: The Mathematics of Arbitrage. Springer, Berlin (2006)
20. Denis, L., Martini, C.: A theoretical framework for the pricing of contingent claims in the presence of model uncertainty. Ann. Appl. Probab. **16**, 827–852 (2006)
21. Dolinsky, Y., Soner, H.M.: Martingale optimal transport and robust hedging in continuous time. Probab. Theory Relat. Fields **160**, 391–427 (2014)
22. Dolinsky, Y., Soner, H.M.: Martingale optimal transport in the Skorokhod space. Stoch. Process. Appl. **125**, 3893–3931 (2015)
23. El Euch, O., Rosenbaum, M.: The characteristic function of rough Heston models. Math. Finance **29**, 3–38 (2019)
24. Föllmer, H.: Calcul d'Itô sans probabilités. In: Azéma, J., Yor, M. (eds.) Séminaire de Probabilités XV. Lecture Notes in Math., vol. 850, pp. 143–150. Springer, Berlin (1981)
25. Friz, P., Victoir, N.: Multidimensional Stochastic Processes as Rough Paths. Theory and Applications. Cambridge University Press, Cambridge (2010)
26. Galichon, A., Henry-Labordère, P., Touzi, N.: A stochastic control approach to no-arbitrage bounds given marginals, with an application to lookback options. Ann. Appl. Probab. **24**, 312–336 (2014)
27. Gatheral, J., Jaisson, T., Rosenbaum, M.: Volatility is rough. Quant. Finance **18**, 933–949 (2018)
28. Grisvard, P.: Elliptic Problems in Nonsmooth Domains. Monographs and Studies in Mathematics, vol. 24. Pitman (Advanced Publishing Program), Boston (1985)
29. Guo, G., Tan, X., Touzi, N.: Tightness and duality of martingale transport on the Skorokhod space. Stoch. Process. Appl. **127**, 927–956 (2017)
30. Hobson, D.: Robust hedging of the lookback option. Finance Stoch. **2**, 329–347 (1998)
31. Hou, Z., Obłój, J.: Robust pricing–hedging dualities in continuous time. Finance Stoch. **22**, 511–567 (2018)
32. Jacod, J., Shiryaev, A.N.: Local martingales and the fundamental asset pricing theorems in the discrete-time case. Finance Stoch. **2**, 259–273 (1998)
33. Lyons, T.J.: Uncertain volatility and the risk-free synthesis of derivatives. Appl. Math. Finance **2**, 117–133 (1995)
34. Maligranda, L.: Weakly compact operators and interpolation. Acta Appl. Math. **27**, 79–89 (1992)
35. Mao, X.: Stochastic Differential Equations and Applications, 2nd edn. Horwood Publishing Limited, Chichester (2008)
36. Mykland, P.A.: Financial options and statistical prediction intervals. Ann. Stat. **31**, 1413–1438 (2003)
37. Neufeld, A., Nutz, M.: Superreplication under volatility uncertainty for measurable claims. Electron. J. Probab. **18**(48), 1–14 (2013)
38. Nutz, M.: Robust superhedging with jumps and diffusion. Stoch. Process. Appl. **125**, 4543–4555 (2015)

39. Peng, S.: Nonlinear expectation and stochastic calculus under uncertainty. Preprint (2010). Available online at https://arxiv.org/abs/1002.4546
40. Perkowski, N., Prömel, D.J.: Pathwise stochastic integrals for model free finance. Bernoulli **22**, 2486–2520 (2016)
41. Possamaï, D., Royer, G., Touzi, N.: On the robust superhedging of measurable claims. Electron. Commun. Probab. **18**(95), 1–13 (2013)
42. Rostek, S., Schöbel, R.: A note on the use of fractional Brownian motion for financial modeling. Econ. Model. **30**, 30–35 (2013)
43. Schied, A., Voloshchenko, I.: Pathwise no-arbitrage in a class of delta hedging strategies. Probab. Uncertain. Quant. Risk **1**(3), 1–25 (2016)
44. Sion, M.: On general minimax theorems. Pac. J. Math. **8**(1), 171–176 (1958)
45. Soner, H.M., Touzi, N., Zhang, J.: Quasi-sure stochastic analysis through aggregation. Electron. J. Probab. **16**(67), 1844–1879 (2011)
46. Soner, H.M., Touzi, N., Zhang, J.: Wellposedness of second order backward SDEs. Probab. Theory Relat. Fields **153**, 149–190 (2012)
47. Soner, H.M., Touzi, N., Zhang, J.: Dual formulation of second order target problems. Ann. Appl. Probab. **23**, 308–347 (2013)
48. Vovk, V.: Continuous-time trading and the emergence of probability. Finance Stoch. **16**, 561–609 (2012)
49. Vovk, V.: Another example of duality between game-theoretic and measure-theoretic probability. Preprint (2016). Available online at https://arxiv.org/abs/1608.02706