Change points and uniform confidence for spot volatility

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Abstract

In this dissertation we consider issues of high-frequency statistics, whereas our data is generated by discretization of noisy and pure Itô semimartingales. In the first part of this thesis, we present results and tools from stochastic calculus, high-frequency statistics and extreme value theory, being essential for all subsequent parts. Based on noisy Itô semimartingale observations, in the second part of this work limit theorems are proved, which are necessary to tackle change-point questions in the volatility adequately. Furthermore, the consistency of a change-point test is proved as well as consistency of the associated change-point estimator.

In the third part of the work weak limit theorems for extreme value statistics are proved, which are appropriate for constructing uniform confidence bands for the volatility process. The respective extreme value statistics are based on pure Itô semimartingale observations without microstructure noise.

The final part of the work contains weak limit theorems, which are appropriate to construct uniform confidence bands for observations based on data with microstructure noise.

Zusammenfassung

In dieser Dissertation befassen wir uns mit Fragestellungen der Hochfrequenzstatistik, wobei unsere Datenbasis die Diskretisierung von verrauschten wie auch reinen Itô Semimartingalen ist. Im ersten Teil dieser Dissertation stellen wir Resultate und Hilfsmittel aus der stochastischen Analysis, Hochfrequenzstatistik und Extremwerttheorie vor, die wesentlich für alle nachfolgenden Teile sind. Im zweiten Teil dieser Arbeit werden Grenzwertsätze bewiesen, die notwendig sind, um Fragen von Strukturbrüchen in der Volatilität adäquat behandeln zu können. Darüber hinaus wird sowohl die Konsistenz eines Strukturbruchtests als auch die Konsistenz von Schätzern nachgewiesen, die den jeweiligen Zeitpunkt des Strukturbruchs schätzen.

Im dritten Teil der Arbeit werden schwache Grenzwertsätze für Extremwertstatistiken bewiesen, die geeignet sind, gleichmäßige Konfidenzbänder für den Volatilitätsprozess zu konstruieren. Die jeweiligen Extremwertstatistiken basieren auf reinen Itô Semimartingalen, die keinem Rauschen unterliegen.

Der letzte Teil der Arbeit enthält schwache Grenzwertsätze, die geeignet sind, gleichmäßige Konfidenzbänder für Beobachtungen zu konstruieren, die auf verrauschten Daten basieren.

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Contents

1. Introduction

1.1. Motivation

At least since the celebrated work by Black and Scholes [15] and the independent work of Merton [47], in which the authors have presented the famous Black-Scholes-Merton formula for option pricing, stochastic calculus and stochastic methods are an integral part of financial mathematics. The famous partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{BSM}$$

provides a link between the option value V, the underlying asset price S, the risk free interest rate r and the volatility σ^2. In (BSM) the asset price S is driven by the geometric Brownian motion stochastic differential equation

$$dS = \mu S dt + \sigma S dW \tag{GBM}$$

with μ being the drift rate. The theoretical foundation of the differential dW is given in the groundbreaking papers Itô [32] and Itô [31], in which the author constructs stochastic integrals

$$\int_0^t \sigma_s \, dW_s$$

for adaptive, locally bounded integrands $(\sigma_t)_{t\geq 0}$ and presents stochastic differential equations, respectively. With the further extension and generalization of stochastic integration by the 'Strasbourg school of probability', c.f. Emery and Yor [21], the Brownian motion W has been replaced by a so called *semimartingale* X. As a result, general semimartingales and the modern theory of stochastic integration have found their way to financial mathematics. For the latter relation, the *Fundamental theorem of asset pricing* due to Delbaen and Schachermayer [20] constitutes the striking bridge. This result, being fundamental and very deep, states, roughly speaking, that the asset price S can be modeled by a semimartingale if and only if the financial market fulfills the so called NFLVR property. More precisely, the following holds.

Fundamental Theorem of Asset Pricing. *Let S be a locally bounded semimartingale. There exists an equivalent martingale measure $\mathbb{Q}$ for S if and only if S satisfies the No-Free Lunch with Vanishing Risk (NFLVR) condition.*

We refer to Theorem 9.1.1 in Delbaen and Schachermayer [20] and to Theorem 9.7.2. for the conclusion NFLVR $\Rightarrow$ S = semimartingale. Due to the results described above, semimartingales possess an outstanding relevance in applications. This central significance of these stochastic processes motivates their statistical and probabilistic inference. We will pursue statistical inference in the high-frequency regime, that is we have data

$$\left(X_{i\Delta_n}\right)_{i=0\ldots,\Delta_n^{-1}},$$

generated by a stochastic process X with equidistant observation points $i\Delta_n$, $i = 0,\ldots,\Delta_n^{-1}$. This work will provide further insight on volatility inference and estimation of a given asset price X. More precisely, we will develop *pathwise* techniques and results for functionals of the second characteristic of the semimartingale X. The spot volatility, interpreted as a measure of dispersion in applications, is basically a functional of the quadratic variation of the semimartingale X. While in the classical partial differential equation (BSM) the volatility is neither random nor time dependent, empirical evidence suggests to work with much more general volatility models. In a first step, we will replace the geometric Brownian motion S in (GBM) by a general, possibly discontinuous Itô semimartingale $(X_t)_{t\in[0,1]}$, i.e. a semimartingale with characteristics being continuous with respect to the Lebesgue measure. Secondly, considering high-frequency data, the existence of microstructure noise is empirically evident, c.f. Chapter 8 in Schmidt-Hieber [57], such that even more general processes $(Y_t)_{t\in[0,1]}$, where Y_t is a noisy version of X_t with exogenous additive noise, are considered. Statistical inference on certain functionals of the volatility process $(\sigma_t^2)_{t\in[0,1]}$ has attracted a huge interest in recent years. A considerable part of these research results have been gathered and comprehensively presented in the remarkable work Jacod and Protter [35]. Those results essentially comprise consistency and associated central limit theorems concerning functionals of the spot volatility process $(\sigma_t^2)_{t\in[0,1]}$. In this thesis we will go a step further extending the classical results based on linear statistics to non-linear versions. We will pursue inference via extreme value limits of certain functionals, tackling change-point questions based on the process $(Y_t)_{t\in[0,1]}$ and uniform confidence for spot volatility extending and improving results in recent literature based on $(X_t)_{t\in[0,1]}$ as well as for $(Y_t)_{t\in[0,1]}$.

1.2. Outline of the thesis

This thesis is organized as follows.

In Chapter 2 we present the basic objects and collect the necessary tools from stochastic calculus, high-frequency statistics, Skorohod embedding and extreme value limit theorems, being necessary proving the main results of the thesis.

Chapter 3 contains the first part of the thesis, where we will tackle change-point inference for noisy high-frequency data. Inference on structural breaks for discrete-time stochastic processes, particularly in time series analysis, is a very active research field within mathematical statistics. Parametric and nonparametric approaches are usually based on limit theorems relying on extreme value theory, cf. Csörgő and Horváth [19] for an overview. Whereas the latter is usually concerned with i.i.d. data, important contributions beyond that case are presented in Wu and Zhao [61], proving limit theorems for nonparametric change-point analysis under weak dependence. These results serve as an important ingredient not only for this chapter but also for the whole work. Except the classical works like Müller [48] and Müller and Stadtmüller [49] so far inference on structural breaks for continuous-time stochastic processes has attracted less attention. Let us mention the very recent work by Bücher et al. [17], which also deals with questions of detecting structural breaks of certain continuous-time stochastic processes. Our target of inference is the volatility process. Understanding the structure and dynamics of stochastic volatility processes is a highly important issue in finance and econometrics. Due to the outstanding role of volatility for quantifying financial risk, there is a vast literature on these topics.

In Chapter 4 we will start with inference on uniform spot volatility estimation. Whereas the results sketched in Chapter 2 allow for *pointwise* inference, there is a lack of appropriate tools for uniform inference. More precisely, taking into account the limit theorems in Jacod and Todorov [38], it is easy to conclude a feasible central limit theorem, which enables the construction of an interval $C_{1-\alpha}(t)$ such that for a given $t \in [0,1]$

$$\mathbb{P}\left[\sigma_t^2 \in C_{1-\alpha}(t)\right] \longrightarrow 1-\alpha\,.$$

Whereas a *confidence band* $C_{1-\alpha}(t)$ fulfills

$$\mathbb{P}\left[\sigma_t^2 \in C_{1-\alpha}(t)\,\forall t \in [0,1]\right] \longrightarrow 1-\alpha\,,$$

i.e. confidence intervals are sets with only local coverage and confidence bands are sets with simultaneous coverage.
The key tool to construct asymptotic confidence sets are weak limit theorems based on different types of statistics. Concerning the construction of confidence intervals the limit theorems are usually provided for linear statistics, measuring the point-wise deviation between the estimator and the unknown parameter. In contrast, the construction of confidence bands relies on non-linear statistics measuring the global deviation between the estimator and the unknown parameter. Beyond the construction of confidence bands, uniform weak limit theorems allow for tackling testing problems, which are not accessible with point-wise versions. For example, they allow to validate a certain parametric model for the spot volatility process $\left(\sigma_t^2\right)_{t\in[0,1]}$ such as goodness of fit tests or change-point tests.

The final Chapter 5 provides a further extension of the previous Chapter 4. Though the theory in Chapter 4 covers a large class of stochastic volatility models and a quite general data generating price process $(X_t)_{t\in[0,1]}$ including general, infinite jump activity, it still leaves a gap concerning microstructure noise effects, as the estimators (4.1) and (4.2) are both not noise robust. That is, the limit theorems 4.12 and 4.15

do not hold, if we pass from $(X_t)_{t\in[0,1]}$ to $(Y_t)_{t\in[0,1]}$. Therefore we will generalize these estimators using methods presented in Chapters 2 and 3.
To the best of our knowledge, the very relevant but mathematically involved topic of uniform spot volatility estimation in a noisy Itô semimartingale framework has not been rigorously discussed in the existing literature. The only work, which we are aware of, considering uniform estimation aspects of spot volatility estimation, is Kanaya and Kristensen [40]. Concerning the noisy framework in Subsection 3.4 the authors only discuss possible future research directions, without providing any rigorous arguments or results. We intend to close this gap in Chapter 5.

2. Some basics and tools on high frequency statistics

In this chapter we will summarize and present the tools and results, which are used in the main part of the thesis. Starting with fundamentals of stochastic calculus, we will pass to important limit theorems and conclude the chapter with very recent methods on volatility estimation. We will not present any proofs but will refer to the corresponding literature. Unless otherwise stated, all stochastic objects are defined on a complete filtered probability space $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t\in[0,1]}, \mathbb{P})$.

2.1. Basics in stochastic calculus

Throughout the thesis we will pursue inference on certain characteristics of a stochastic process $(X_t)_{t\in[0,1]}$. Due to the fundamental theorem of asset pricing, which has already been mentioned in the introduction, this process, modeling the price of an asset, is necessarily a so called *semimartingale*.

Definition 2.1 (Semimartingale). *A semimartingale is a stochastic process X of the form*

$$X = X_0 + M + A \tag{2.1}$$

where X_0 is finite valued and $\mathcal{F}_0$-measurable, where M is a local martingale and A is a finite variation process.

Remark 2.2. Semimartingales are undoubtedly the most important class of stochastic processes in stochastic calculus, probability theory and mathematical finance. On the one hand they are the basis of modern stochastic integration due to the celebrated Bichteler-Dellacherie Theorem, c.f. Protter [52], and on the other hand they are the key building blocks in asset pricing theory.

Remark 2.3. It is a deep result in probability theory, that every semimartingale X exhibits a representation in terms of random measures and truncation function κ:

$$X = X_0 + X^c + \kappa \star (\mu - \nu) + (x - \kappa(x)) \star \mu + B\,. \tag{2.2}$$

The triplet $\big(B, C, \nu\big)$ is called the *characteristic triplet* of the semimartingale X, with

- $C = [X^c, X^c]$ being the quadratic variation of the continuous local martingale part X^c,

- B being the predictable process of locally finite variation, c.f. Definition 2.6 in Jacod and Shiryaev [37] and
- ν the compensator of the jump measure μ, c.f. Theorem 1.8 in Jacod and Shiryaev [37].

In contrast to the defining decomposition (2.1), the refined one in (2.2) is unique and is more appropriate for further inference.

Whereas there exists a vast literature on probabilistic results for general semimartingales, statistical inference, especially high-frequency statistics, with respect to these processes has not been developed yet. Though, we refer to Aït-Sahalia and Jacod [3] for discussions, including the general case. Throughout the whole work, the price process $(X_t)_{t\in[0,1]}$ will be a so called Itô semimartingale and we will stick to the univariate case.

Definition 2.4 (Itô semimartingale)**.** *A semimartingale X is an Itô semimartingale, if its characteristics $\big(B, C, \nu\big)$ are absolutely continuous with respect to the Lebesgue measure.*

In addition to the general representation (2.2) of an Itô semimartingale, there is the so called Grigelionis representation, which will be used in subsequent parts of this work:

$$X_t = X_0 + \int_0^t a_s\, ds + \int_0^t \sigma_s\, dW_s + J_t\,, \tag{2.3}$$

with

$$J_t = \int_0^t \int_{\mathbb{R}} \kappa(\delta(s,x))(\mu-\nu)(ds,dx) + \int_0^t \int_{\mathbb{R}} \overline{\kappa}(\delta(s,x))\mu(ds,dx) \tag{2.4}$$

and $\overline{\kappa} = x - \kappa(x)$. For a detailed discussion and presentation of this decomposition, we refer to Subsection 1.4.3 in Aït-Sahalia and Jacod [1]. We will conclude this section with two fundamental inequalities in martingale theory, which will be key tools calculating upper bounds for certain functionals of increments of local martingales.

Theorem 2.5 (Doob's L^p inequality)**.** *Let $\big(X_t\big)_{t\in[0,1]}$ be a right-continuous martingale (or nonnegative submartingale) and $J = [u,v] \subset [0,1]$. If $p \in (1,\infty)$, $p^{-1} + q^{-1} = 1$ and $X_t \in L^p$, then*

$$\left\| \sup_{t\in J} |X_t| \right\|_p \le \sup_{t\in J} \left\| X_t \right\|_p \,. \tag{2.5}$$

For a proof, we refer to Theorem 5.1.3 in Borodin [16].
A deeper and an extremely useful inequality is given by the Burkholder-Davis-Gundy inequality. We use the notation

$$X_\infty = \lim_{t\longrightarrow\infty} X_t$$

and

$$X_t^* = \sup_{s \le t} |X_s|$$

for any stochastic process X_t. The inequality essentially states, that the norms

$$X \mapsto \mathbb{E}\big[(X_\infty^*)^p\big]^{1/p} \text{ and}$$
$$X \mapsto \|[X,X]_\infty\|_p$$

are equivalent on the space of continuous local martingales. For our purposes, the following simplified version is sufficient.

Theorem 2.6 (Burkholder-Davis-Gundy). *For any $p \in (0, +\infty)$ there exist two positive constants c_p, C_p such that, for all continuous local martingales M with $M_0 = 0$, the following inequality holds for all $t > 0$:*

$$c_p \mathbb{E}\left[[M,M]_t^{p/2}\right] \le \mathbb{E}\left[\left(M_t^*\right)^p\right] \le C_p \mathbb{E}\left[[M,M]_t^{p/2}\right] . \tag{2.6}$$

For a proof of this version, we refer to Theorem 4.1 and Corollary 4.2 in Revuz and Yor [55].

Remark 2.7. Note that the equivalence of norms, stated in Theorem 2.6, does not extend to general semimartingales, c.f. Revuz and Yor [55], Exercise 1.13 in Chapter IV. Therefore, bounding moments of a general semimartingale, we have to consider the local martingale part and the finite variation part separately. There are versions of Theorem 2.6 for general, discontinuous local martingales, c.f. Protter [52]. The continuous version is sufficient for applications in this work.

2.2. Stable convergence and central limit theorems

The fundamental framework, on which all results in this work are based on, is high-frequency data. More precisely, given any data generating stochastic process $(X_t)_{t\in[0,1]}$, indexed with the unit interval, we record data

$$(X_{i\Delta_n})_{i=0,\dots\Delta_n^{-1}} \tag{2.7}$$

with $\Delta_n^{-1} = n \in \mathbb{N}$. The infill asymptotics regime implies $\Delta_n \longrightarrow 0$. In (2.7) we have the most simple observation scheme, i.e. equidistant and deterministic. For more general discretization schemes we refer to Jacod and Protter [35]. In the general framework, using a semimartingale X as a data generating process, proving central limit theorems for functionals of (2.7) provides so called non-feasible central limit theorems. More precisely, given a statistic Φ, there is a weak limit like

$$\Phi\left((X_{i\Delta_n})_{i=0,\dots\Delta_n^{-1}}\right) \xrightarrow{d} UV\,, \tag{2.8}$$

with a random variable $V > 0$ and $U \sim N(0,1)$ such that U and V are independent. Limits of this type are usually called *mixed normal*, in symbols $MN(0, V^2)$. In general, the distribution of the random variable V is not known, such that using (2.8) for further statistical inference, e.g. testing and confidence, is not useful. The natural idea, to construct an estimator V_n of V, based on (2.7), doesn't allow to conclude

$$\Phi\left((X_{i\Delta_n})_{i=0,\ldots\Delta_n^{-1}}\right)/V_n \xrightarrow{d} N(0,1)\,. \tag{2.9}$$

However, the concept of *stable convergence* is exactly the right mode of convergence, ensuring (2.9).

Definition 2.8 (Stable convergence)**.** *Let Y_n be a sequence of random variables defined on $\big(\Omega, \mathbb{F}, \mathbb{P}\big)$ taking values in a Polish space $\big(E, \mathcal{E}\big)$. We say that Y_n converges stably with limit Y, written $Y_n \xrightarrow{st} Y$, where Y is defined on an extension $\big(\Omega', \mathbb{F}', \mathbb{P}'\big)$, if and only if for any bounded, continuous function g and any bounded $\mathbb{F}$-measurable random variable Z it holds that*

$$\mathbb{E}\big[g(Y_n)Z\big] \longrightarrow \mathbb{E}'\big[g(Y)Z\big] \tag{2.10}$$

as $n \longrightarrow +\infty$.

Remark 2.9. The extension of the original probability space is necessary, since $\mathbb{F}$-measurability of Y would imply $Y_n \xrightarrow{\mathbb{P}} Y$, c.f. Lemma 2.3 in Podolskij and Vetter [51].

A suitable choice of $\big(E, \mathcal{E}\big)$ shows that Y_n is allowed to be a sequence of stochastic processes. Furthermore, from (2.10) it is obvious, that this concept is an extension of usual weak convergence. For further properties and results on stable convergence, we refer the reader to Jacod and Protter [35] and Podolskij and Vetter [51]. Let us only emphasize, that proving a stable version of (2.8),

$$\Phi\left((X_{i\Delta_n})_{i=0,\ldots\Delta_n^{-1}}\right) \xrightarrow{st} UV\,,$$

and having a consistent estimator of V,

$$V_n \xrightarrow{\mathbb{P}} V\,,$$

ensures

$$\Phi\left((X_{i\Delta_n})_{i=0,\ldots\Delta_n^{-1}}\right)/V_n \xrightarrow{d} N(0,1)\,.$$

Proving functional stable convergence is, in general, an involved task. There is a general result due to Jean Jacod, which is the basis of possibly every functional central limit theorem in high-frequency statistics. For the sake of completeness, we want to cite this result in its simplified form. Therefore, we consider functionals of the form

$$Y_t^n = \sum_{i=1}^{[t/\Delta_n]} \chi_{i,n}\,,$$

with $\chi_{i,n}$ being a triangular array of $\mathcal{F}_{i\Delta_n}$-measurable and square integrable random variables. Finally, we use the notation $\Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n}$ for any stochastic process X.

Theorem 2.10 (Jacod's Theorem)**.** *Assume there exist a continuous square integrable local martingale M, absolutely continuous processes F, G and a continuous process B with finite variation such that the following conditions are satisfied:*

$$\sup_{t\in[0,1]} \left| \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}[\chi_{i,n} \mid \mathcal{F}_{(i-1)\Delta_n}] - B_t \right| \xrightarrow{\mathbb{P}} 0\,,$$

$$\sum_{i=1}^{[t/\Delta_n]} (\mathbb{E}[\chi_{i,n}^2 \mid \mathcal{F}_{(i-1)\Delta_n}] - \mathbb{E}^2[\chi_{i,n} \mid \mathcal{F}_{(i-1)\Delta_n}]) \xrightarrow{\mathbb{P}} F_t = \int_0^t (v_s^2 + w_s^2)\, ds\,,$$

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}[\chi_{i,n} \Delta_i^n M \mid \mathcal{F}_{(i-1)\Delta_n}] \xrightarrow{\mathbb{P}} G_t = \int_0^t v_s\, ds\,,$$

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}[\chi_{i,n}^2 \mathbb{1}_{\{|\chi_{i,n}|>\varepsilon\}} \mid \mathcal{F}_{(i-1)\Delta_n}] \xrightarrow{\mathbb{P}} 0 \quad \forall\varepsilon > 0\,,$$

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}[\chi_{i,n} \Delta_i^n N \mid \mathcal{F}_{(i-1)\Delta_n}] \xrightarrow{\mathbb{P}} 0\,,$$

where $(v_s)_{s\in[0,1]}$ and $(w_s)_{s\in[0,1]}$ are predictable processes and the last convergence holds for all bounded $\mathcal{F}_t$-martingales with $N_0 = 0$ and $[M,N] \equiv 0$. Then we obtain the stable convergence of processes:

$$Y_t^n \xrightarrow{st} Y_t = B_t + \int_0^t v_s\, dM_s + \int_0^t w_s\, dW_s'\,,$$

where W' is a Brownian motion defined on an extension of the original probability space and independent of $\mathbb{F}$.

For a proof of this result we refer to Theorem 2.1 and 3.2 in Jacod [34] or Chapter IX in Jacod and Shiryaev [37].

Remark 2.11. (1) More general versions of Theorem 2.10 can be found in Jacod [34] or Chapter IX in Jacod and Shiryaev [37]. Those generalizations contain limit results beyond the square integrability of $\chi_{i,n}$ and the absolute continuity of the processes F and G, as well as multidimensional extensions.

(2) We want to emphasize, that there is no similar result for sequences of 'simple' random variables, i.e. proving stable convergence in this setting, is usually pursued by directly checking the convergence (2.10).

(3) Theorem 2.10 can be considered as a stable functional extension of the classical pointwise results for martingale sequences presented in Hall and Heyde [28].

(4) For a wide range of applications the random variables $\chi_{i,n}$ are functionals of an Itô semimartingale. In this case the local martingale M in Theorem 2.10 can be chosen to be another Brownian motion $\overline{W}$.

(5) The extension of the original probability space is necessary in order to ensure the existence of an independent Brownian motion. Furthermore, the extension in Theorem 2.10 is a so called *very good extension*. Such extensions ensure that the martingales M, N remain martingales on the larger probability space. Furthermore, a semimartingale on the extension remains a semimartingale with the same characteristic triplet and the same Grigelionis representation.

(6) The very good extension in Theorem 2.10 is of Wiener type, i.e. the second factor of the larger probability space is the canonical Wiener space and the process W' is the canonical Wiener process. This construction ensures the independence of the new Brownian motion W'.

(7) Further statistical inference using the limit in Theorem 2.10 is only possible to a limited extent, since the distribution of the limit process Y is usually not accessible. Fortunately, for a wide range of applications, it holds that $B \equiv 0$ and $v \equiv 0$, i.e. after a proper rescaling and an application of stable convergence properties a feasible central limit theorem follows.

2.3. Volatility estimation

This section is devoted to introduce and summarize methods and results on volatility estimation. In the first subsection we will repeat some tools for high-frequency data based on direct Itô semimartingale observations. The second subsection provides techniques and results for high-frequency data with observation noise.

2.3.1. Direct observation without microstructure noise

Based on data (2.7) we are interested in estimation of, possibly infinite dimensional, functionals of the spot volatility process $\big(\sigma_t^2\big)_{t\in[0,1]}$.

Remark 2.12. (1) The reason to restrict to functionals of the square of the volatility process $(\sigma_t)_{t\in[0,1]}$ is due to the fact that the quadratic variation process $[X, X]$ can be estimated and the latter includes the squared process σ_t^2.

(2) It is a tempting question whether it is possible to estimate and pursue statistical inference on functionals of the other characteristics of the Itô semimartingale X. Whereas there exists a rich literature on the compensator ν, functionals of the drift process $(a_t)_{t\in[0,1]}$ are, in general, not identifiable. We refer the reader to Aït-Sahalia and Jacod [3] for an extensive discussion of these questions.

Starting with a continuous Itô semimartingale X, i.e. $\mu \equiv 0$ in (2.3), a useful class of estimators applied for statistical inference on volatility are functionals of the form

$$V\big(f,X\big)_t^n = \Delta_n \sum_{i=1}^{[t/\Delta_n]} f\left(\frac{\Delta_i^n X}{\sqrt{\Delta_n}}\right), \tag{2.11}$$

with some smooth function $f : \mathbb{R} \longrightarrow \mathbb{R}$. The rescaling factor $\Delta_n^{-1/2}$ is due to the self similarity of the Brownian motion. There exists a very rich literature on limit theorems for $V\big(f,X\big)_t^n$, whereas the most prominent and important subclasses are so called *power variations* given by (2.11) with $f(x) = |x|^p$ and $p > 0$.

Example 2.13 (Consistency)**.** For a continuous function f with polynomial growth, càglàd process $(a_t)_{t\in[0,1]}$ and an adapted process $(\sigma_t)_{t\in[0,1]}$ being càdlàg, it holds that

$$V(f,X)_t^n \overset{\text{u.c.p.}}{\longrightarrow} \int_0^t \rho_{\sigma_s}(f)ds\,, \tag{2.12}$$

with $\overset{\text{u.c.p.}}{\longrightarrow}$ denoting uniform convergence in probability, the operator ρ defined by

$$\rho_{\sigma_s}(f) = \rho_x(f)\big|_{x=\sigma_s}$$

and

$$\rho_x(f) = \mathbb{E}\left[f\big(xU\big)\right],$$

with $U \sim N(0,1)$ and for every $x \in \mathbb{R}$.

In (2.12) the drift process $(a_t)_{t\in[0,1]}$ does not appear in the limit, which is mainly due to the Burkholder-Davis-Gundy inequality (2.6) and standard upper bounds for Lebesgue integrals.
In order to construct confidence intervals with respect to functionals of the volatility process we need associated central limit theorems.

Example 2.14 (Stable limit for power variations)**.** The undoubtedly most important class of estimators for applications is obtained, if $f(x) = |x|^p$, for some $p > 0$. If $(\sigma_t)_{t\in[0,1]}$ itself is an Itô semimartingale, then the following stable limit holds:

$$\Delta_n^{-1/2}\left(V(f,X)_t^n - m_p\int_0^t |\sigma_s|^p\,ds\right) \overset{st}{\longrightarrow} \sqrt{m_{2p}-m_p^2}\int_0^t |\sigma_s|^p\,dW_s'\,, \tag{2.13}$$

with $m_p \equiv \mathbb{E}\left[|U|^p\right]$, $U \sim N(0,1)$ and a Brownian motion W' independent of $\mathbb{F}$ defined on an extension of the original probability space. The properties of stable convergence imply

$$\frac{\Delta_n^{-1/2}\Big(V(f,X)_t^n - m_p\int_0^t |\sigma_s|^p\,ds\Big)}{\sqrt{\frac{m_{2p}-m_p^2}{m_{2p}}V(f^2,X)_t^n}} \overset{st}{\longrightarrow} N(0,1)\,. \tag{2.14}$$

Remark 2.15. Though, we will not pursue inference on *integrated* volatility, but *spot* volatility, the class of statistics (2.11) will provide a good starting point for the construction of our main statistics, as they will be a modification of (2.11).

We refer the reader to Chapter 5 in Jacod and Protter [35] for more central limit theorems of this type.

When the observed process X exhibits jumps, the situation becomes quite different. First of all, let us define a non-rescaled version, $\overline{V}(f,X)_t^n$, of $V(f,X)_t^n$ given by

$$\overline{V}(f,X)_t^n = \sum_{i=1}^{[t/\Delta_n]} f\left(\Delta_i^n X\right) . \tag{2.15}$$

Let ΔX_s be the size of the jump of X at time s, i.e.

$$\Delta X_s = X_s - X_{s-} .$$

Starting with the most important class of test functions f, given by $f(x) = |x|^p$ for $p \geq 2$, we have the following stochastic convergence for any $t \geq 0$

$$\overline{V}(f,X)_t^n \xrightarrow{\mathbb{P}} \begin{cases} \sum_{s\leq t} |\Delta X_s|^p & \text{if } p > 2\,, \\ [X,X]_t & \text{if } p = 2\,. \end{cases} \tag{2.16}$$

For a sketch of the proof of the stochastic convergence (2.16) we refer to Podolskij and Vetter [51].

Remark 2.16. (1) Note that the quadratic variation process $[X,X]_t$ can be decomposed via

$$[X,X]_t = [X^c,X^c]_t + \sum_{s\leq t} |\Delta X_s|^2 \ ,$$

i.e. the objects of interest, (integrated) functionals of spot volatility, are not directly accessible via those estimators if $p = 2$, as the jump part pops up in the limit.

(2) Similar limit theorems and associated central limit theorems are available for more general test functions f. Those limit theorems crucially depend on the behaviour of the test function f around 0. We refer to Jacod and Protter [35] for extensive discussions and numerous results.

Focusing on inference for spot volatility, one has to eliminate the jumps in the limit process. There are basically two approaches to overcome this difficulty, so called *truncated power variations* and *multipower variations*. Since we will only use the first one in subsequent parts of this work, we refer to Barndorff-Nielsen and Shephard [7] for the multipower variation approach. For a comparison of both estimators we refer the reader to Jacod and Reiss [36]. We stick to the most important case in applications, namely $f(x) = x^2$. For a sequence v_n with $v_n \propto 1/n^{\varpi}$ for some $\varpi \in (0, 1/2)$ we define

$$V(f, v_n, X)_t^n = \sum_{i=1}^{[t/\Delta_n]} (\Delta_i^n X)^2 \mathbb{1}_{\{|\Delta_i^n X| \leq v_n\}}. \tag{2.17}$$

We refer to Mancini [44] for further discussions. Once truncation provides to eliminate the jumps asymptotically, a similar central limit theorem as (2.8) is fulfilled by $V(f, v_n, X)_t^n$ given some conditions on the jump activity. Concerning the latter, we refer to Theorem 6.9 in Aït-Sahalia and Jacod [4].

2.3.2. Microstructure noise and spectral statistics

Though the data generating Itô semimartingale X in (2.3) is a very flexible and general model, it is empirically evident, that it is not sufficient to reflect every aspect of high-frequency data. This is mainly due to the *microstructure noise* effects, which contaminate the data (2.7). Therefore, instead of observing a discretization of the price process $(X_t)_{t\in[0,1]}$, we observe a different process $(Y_t)_{t\in[0,1]}$, which is not a semimartingale and is given by a superposition

$$Y_t = X_t + \varepsilon_t\,, \tag{2.18}$$

with a white noise process $(\varepsilon_t)_{t\in[0,1]}$, modeling the microstructure noise. In order to incorporate microstructure noise, we have to extend the original probability space. We set $\mathcal{G}_t = \mathcal{F}_t^{(0)} \otimes \sigma(\varepsilon_s : s \leq t)$. The data generating process $(Y_t)_{t\in[0,1]}$ is defined on the filtered probability space $\big(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,1]}, \mathbb{P}\big)$, which is a very good extension. For the details of the construction we refer to Chapter 16 in Jacod and Protter [35]. Therefore, our statistical tools presented in Subsection 2.3.1 have to be modified and extended in order to address high-frequency data

$$(Y_{i\Delta_n})_{i=0,\ldots,\Delta_n^{-1}}\ . \tag{2.19}$$

The reason why this approach of modeling is more appropriate is due to phenomena like rounding errors or bid-ask spreads. We refer the reader to Chapter 8 in Schmidt-Hieber [57] and Chapter 7 Aït-Sahalia and Jacod [1] for extensive discussions of these issues and on possible structures of the noise process. Since for all t

$$\Delta_n V(f, Y)_t^n \xrightarrow{\mathbb{P}} 2t\mathbb{E}\left[\varepsilon_t^2\right]\,,$$

if $\mu \equiv 0$ and $f(x) = x^2$, the estimators (2.11) provide consistent estimation of characteristics of the noise term, but not of X. Several statistical methods have been developed in order to tackle this problem of asymptotic dominance of noise. Beside the pre-averaging method there are multiscale estimators and realized kernels methods. We refer to Vetter [60] for a discussion of these approaches and corresponding references. All these three methods have in common that they attain the optimal rate of convergence, without being efficient, i.e. they do not attain the smallest possible asymptotic variance coinciding with a lower bound. In Reiß [54] the so called spectral approach is proposed, which outperforms the three methods mentioned above, since it not only attains the optimal rate of convergence but is also efficient. Since we will use this method in this work, we want to introduce its key objects and recent results. Therefore we pick a sequence h_n with

$$h_n \propto n^{-1/2} \log(n) \tag{2.20}$$

and $h_n^{-1} \in \mathbb{N}$.
The observation interval $[0,1]$ is split into h_n^{-1} bins of length h_n, such that each bin is given by

$$[(k-1)h_n, kh_n], \quad k = 1, \ldots, h_n^{-1}.$$

Furthermore, we consider the $L^2([0,1])$ orthonormal systems, given by

$$\Phi_{jk}(t) = \Phi_j(t-(k-1)h_n)$$
$$\varphi_{jk}(t) = \varphi_j(t-(k-1)h_n)$$

with

$$\Phi_j(t) = \sqrt{\frac{2}{h_n}} \sin\left(j\pi h_n^{-1} t\right) \mathbb{1}_{[0,h_n]}(t), \quad j \geq 0, 0 \leq t \leq 1,$$
$$\varphi_j(t) = 2n\sqrt{\frac{2}{h_n}} \sin\left(\frac{j\pi}{2nh_n}\right) \cos\left(j\pi h_n^{-1} t\right) \mathbb{1}_{[0,h_n]}(t).$$

We define, for any stochastic process $(L_t)_{t\in[0,1]}$, the *spectral statistics*

$$S_{jk}(L) := \sum_{i=1}^{n} \Delta_i^n L \, \Phi_{jk}\left(\frac{i}{n}\right).$$

The squared volatility $\sigma^2_{(k-1)h_n}$ can be estimated locally by a parametric estimator through oracle versions of bias corrected linear combinations of the squared spectral statistics,

$$\hat{\sigma}^2_{(k-1)h_n} = \sum_{j=1}^{\lfloor nh_n \rfloor - 1} w_{jk} \left(S_{jk}^2(Y) - [\varphi_{jk}, \varphi_{jk}]_n \frac{\eta^2}{n} \right), \tag{2.21}$$

with variance minimizing oracle weights w_{jk}, given by

$$w_{jk} = \frac{\left(\sigma^2_{(k-1)h_n} + \frac{\eta^2}{n}[\varphi_{jk}, \varphi_{jk}]_n\right)^{-2}}{\sum_{m=1}^{\lfloor nh_n \rfloor - 1} \left(\sigma^2_{(k-1)h_n} + \frac{\eta^2}{n}[\varphi_{mk}, \varphi_{mk}]_n\right)^{-2}}. \tag{2.22}$$

The empirical scalar products $[f,g]_n$, for any functions f and g, are given by

$$[f,g]_n = \frac{1}{n}\sum_{j=1}^{n} f\left(\frac{i-\frac{1}{2}}{n}\right) g\left(\frac{i-\frac{1}{2}}{n}\right).$$

We want to conclude this subsection with recent limit theorems presented in Altmeyer and Bibinger [5], extending the theory in Reiß [54] to general, stochastic volatility. We set

$$\widehat{\mathbf{IV}}_{n,t} = \sum_{k=1}^{\lfloor th_n^{-1}\rfloor} \sum_{j=1}^{\lfloor nh_n\rfloor -1} w_{jk}\left(S_{jk}^2(Y) - [\varphi_{jk},\varphi_{jk}]_n \frac{\hat{\eta}^2}{n}\right) \tag{2.23}$$

with $\hat{\eta}^2$ being a $\sqrt{n}$- rate estimator of $\mathbb{E}\left[\varepsilon_t^2\right]$ proposed in Zhang et al. [62] and given by

$$\hat{\eta}^2 = \frac{1}{2n}\sum_{i=1}^{n}(Y_{i\Delta_n} - Y_{(i-1)\Delta_n})^2 .$$

The following assumptions are imposed for the asymptotic theory in Altmeyer and Bibinger [5].

Assumption 2.17. *The volatility process σ is assumed to fulfill at least one of the following properties.*

(σ-1) There exists a random variable L with at least four finite moments, i.e. with $\mathbb{E}[L^4] < \infty$ such that $t \mapsto \sigma_t$ is almost surely α-Hölder continuous on $[0,1]$ for $\alpha > \frac{1}{2}$ and Hölder constant L, i.e. $|\sigma_t - \sigma_s| \le L\,|t-s|^\alpha$, $0 \le t,s \le 1$, almost surely.

(σ-2) The process σ is itself an Itô semimartingale, i.e. there exist a random variable σ_0 and adapted càdlàg processes $\widetilde{b} = (\widetilde{b}_t)_{t\in[0,1]}, \widetilde{\sigma} = (\widetilde{\sigma}_t)_{t\in[0,1]}$ and $\widetilde{\eta} = (\widetilde{\eta}_t)_{t\in[0,1]}$ such that

$$\sigma_t = \sigma_0 + \int_0^t \widetilde{b}_s\,ds + \int_0^t \widetilde{\sigma}_s\,dW_s + \int_0^t \widetilde{\eta}_s\,dW'_s .$$

W' is an $(\mathcal{F}_t)_{t\in[0,1]}$-Brownian motion, which is independent of W. Furthermore, suppose there exists a constant $\kappa > 0$ such that $|\sigma_t| > \kappa$ uniformly for $0 \le t \le 1$. For the drift process, assume there exists a random variable L' with $\mathbb{E}[(L')^2] < \infty$ such that $t \mapsto b_t$ is almost surely ν-Hölder continuous on $[0,1]$ for $\nu > 0$ and Hölder constant L', i.e. $|b_t - b_s| \le L'\,|t-s|^\nu$, almost surely.

We need some assumptions on the noise process.

Assumption 2.18. *The microstructure noise process ε is assumed to fulfill the following properties.*

(ε-1) *The stochastic process* $(\varepsilon_t)_{t\in[0,1]}$ *is a centered white noise process with variance* η^2*.*

(ε-2) *It has finite eighth moment.*

Theorem 2.19 (Functional stable limit for spectral estimators)**.** *Given the Assumptions 2.17 and 2.18 the following functional stable weak convergence holds,*

$$n^{1/4}\left(\widehat{\boldsymbol{IV}}_{n,t} - \int_0^t \sigma_s^2\, ds\right) \xrightarrow{st} \int_0^t \sqrt{8\eta\,|\sigma_s^3|}\, dB_s \tag{2.24}$$

as $n \longrightarrow \infty$ *on the Skorohod space* $\mathcal{D}\,[0,1]$ *with another Brownian Motion* B*, defined on an extension of the original space and* $\big(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,1]}, \mathbb{P}\big)$ *and being independent of* $\mathcal{G}$*.*

For a proof we refer to Altmeyer and Bibinger [5].

Remark 2.20. (1) Not only the convergence rate in (2.24) is optimal, but also the asymptotic variance coincides with the best possible lower bound, i.e. the spectral statistics provide optimal integrated spot volatility estimation.

(2) The convergence (2.24) will serve as a key tool handling the change point detection problem in the parametric case, i.e. if there is a constant σ, such that $(\sigma_t)_{t\in[0,1]}$ is deterministic and it holds that $\sigma_t \equiv \sigma$.

(3) After a proper rescaling of the left hand side of (2.24) and using the properties of stable convergence, one can immediately conclude a feasible central limit theorem.

(4) Similar limit theorems for higher dimensions d are also available in Altmeyer and Bibinger [5], including the sophisticated local method of moments estimator.

2.4. Strong invariance principles

Almost every result, which is presented in this work, is based on *uniform* limit theorems. Key tools proving these kind of limits are so called *strong invariance principles*, which are usually based on Skorohod embedding techniques. This subsection is meant to review those results, which are used to prove our main results. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent, identically distributed random variables on a probability space $\big(\Omega, \mathcal{A}, \mathbb{P}\big)$. Moreover, let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of independent $N(0,1)$ distributed random variables on a suitable, possibly different, extension $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\mathbb{P}})$. We set

$$S_n = \sum_{k=1}^n X_k \quad \text{and} \quad T_n = \sum_{k=1}^n Y_k\,.$$

Let $H(x) > 0$, $x > 0$ be a monotone increasing, continuous function having the following properties

(i) $\frac{H(x)}{x^{3+\delta}}$ is monotone increasing for some $\delta > 0$ and all $x > x_0$,

(ii) $\frac{\log(H(x))}{x}$ is monotone decreasing for $x > x_0$.

Theorem 2.21. *Let $H(x)$ satisfy (i) and (ii). Define a sequence K_n by the equation $H(K_n) = n$. If $H(|X_1|) \in L^1$, then there exists a construction of $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ and a constant $C > 0$ such that*

$$\mathbb{P}\left[\limsup_{n \longrightarrow +\infty} \frac{|S_n - T_n|}{K_n} \leq C\right] = 1\,.$$

For a proof based on dyadic construction we refer to Theorem 3 in Komlós et al. [42].

Remark 2.22. (1) We want to emphasize that according to Komlós et al. [42] Theorem 2.21 also holds for not necessarily identically distributed $X_1, X_2, \ldots$, which will be important for some results presented in the next chapter.

(2) An application of Theorem 2.21 with $H(x) = x^r$ with $r > 3$ yields

$$|S_n - T_n| = \mathcal{O}_{a.s.}(n^{1/r})\,,$$

which is not only an improvement of the classical results due to Strassen [58], but is also shown to be *optimal*.

Calculating precise upper bounds for the probability of the event

$$\sup_{k \leq n} |S_k - T_k| > x\,,$$

for $x > K_n$, will be important in the sequel and is the content of the following result.

Theorem 2.23. *Let $H(x)$ satisfy the conditions of Theorem 2.21 and $H(|X_1|) \in L^1$. Then for any x, such that $K_n < x < C_1\sqrt{n\log(n)}$ (more generally $x > K_n$, $x^2/\log(H(x)) < C_1 n$) there exist two finite sequences $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ such that*

$$\mathbb{P}\left[\sup_{k \leq n} |S_k - T_k| > x\right] \leq C_2 \frac{n}{H(ax)}\,,$$

where C_1, C_2 and a are positive constants depending only on the distribution function of X_1.

For a proof we refer to Theorem 4 in Komlós et al. [42].

Remark 2.24. It is often more convenient to reformulate the results presented above using a standard Brownian Motion $(\mathbb{B}(t))_{t \in [0,\infty)}$ as it allows to use stochastic calculus tools. Therefore, we set

$$S(t) = S_{[t]} \quad t \in \mathbb{R}_+\,.$$

Then, (2) in Remark 2.22 translates to

$$S(T) - \mathbb{B}(T) = o_{a.s.}(T^{1/r}) .$$

as $T \longrightarrow \infty$. Controlling the discretization error $\sup_{k \le t \le k+1} |\mathbb{B}(t) - \mathbb{B}(k)|$ is usually pursued using Lévy's modulus of continuity, c.f. Revuz and Yor [55].

Remark 2.25. The quality of the strong approximation results, which have been presented so far, crucially depends on the existence of moments. If the existence of a moment generating function can be ensured, then a better approximation can be proven. More precisely, if

$$\mathbb{E}\left[\exp(tX_1)\right] < \infty$$

for $|t| < t_0$, $t_0 > 0$, then it holds that

$$S(T) - \mathbb{B}(T) = \mathcal{O}_{a.s.}(\log(T))$$

as $T \longrightarrow \infty$. This result, combined with the fact that the χ^2-squared distribution exhibits a moment generating function, will ensure the quite general results on uniform confidence for spot volatility in subsequent parts of this work. Let us insist that $\mathcal{O}_{a.s.}(\log(T))$ in the above approximation can not be replaced by $o_{a.s.}(\log(T))$ unless X_1 itself is normally distributed.

2.5. Some limit theorems for extreme value statistics

This last section of this preparatory chapter is meant to gather recent and classical limit theorems for extreme value statistics.

Lemma 2.26. *Let $(Z_i)_{i\in\mathbb{N}}$ be a family of independent and $N(0,1)$ distributed random variables. We set $Y_i = |Z_i - Z_{i-1}|$ and $\gamma_n = [4\log(n) - 2\log(\log(n))]^{1/2}$. Then the following weak convergence holds,*

$$\sqrt{\log(n)}\left(\max_{1\le i\le n-1} Y_i - \gamma_n\right) \xrightarrow{d} V\,, \tag{2.25}$$

with

$$\mathbb{P}\left[V \le x\right] = \exp(-\pi^{-1/2}\exp(-x))\,, \tag{2.26}$$

For a proof we refer to Lemma 1 in Wu and Zhao [61].
Proving uniform limit theorems, we need a more general result for sequences of Gaussian processes which has been proven in the classical work Bickel and Rosenblatt [14]. Therefore, let $Y^T(\cdot)$ be a sequence of separable Gaussian processes with mean $\mu^T(\cdot)$ such that $Y^T(\cdot) - \mu^T(\cdot)$ is stationary. Let $r(\cdot)$ be the covariance function of Y^T,

$$M^T = \sup\left\{Y^T(t) \,:\, 0 \le t \le T\right\}\,,$$
$$m^T = \inf\left\{Y^T(t) \,:\, 0 \le t \le T\right\}$$

and $b^T(t) = \mu^T(t)(2\log(T))^{1/2}$.

Theorem 2.27 (Uniform limit for stationary and Gaussian sequences)**.** *Suppose that*

(1) $b^T(t)$ *is uniformly bounded in* t *and* T *on* $[0,T]$ *as* $T \longrightarrow \infty$.

(2) $b^T(t) \longrightarrow b(t)$ *uniformly on* $[0,T]$ *as* $T \longrightarrow \infty$.

(3) $T^{-1}\boldsymbol{\lambda}\big(\{t : b(t) \le x, 0 \le t \le T\}\big) \longrightarrow \eta(x)$ *the cumulative distribution function of a probability measure as* $T \longrightarrow \infty$, *with* $\boldsymbol{\lambda}$ *being the Lebesgue measure.*

(4) $b(\cdot)$ *is uniformly continuous on* $\mathbb{R}$.

(5) $r(t) = 1 - C\,|t|^\alpha + o(|t|^\alpha)$, $0 < \alpha \le 2$ *with a constant* C *as* $T \longrightarrow \infty$.

(6) $\int_0^\infty r^2(t)\,dt < \infty$.

Let

$$d_t = (2\log(t))^{1/2} + \frac{1}{(2\log(t))^{1/2}} \times \left[\left(\frac{1}{\alpha} - \frac{1}{2}\right)\log\log(t) + \log(2\pi)^{-1/2}(C^{1/\alpha}H_\alpha 2^{(2-\alpha)/2\alpha})\right],$$

where

$$H_\alpha = \lim_{T\longrightarrow\infty}\frac{1}{T}\int_0^\infty \exp(s)\mathbb{P}[\sup_{0\le t\le T} Y_t > s]\,ds \tag{2.27}$$

and Y *being a Gaussian process with*

$$\mathbb{E}[Y_t] = -|t|^\alpha$$

and

$$\mathbb{C}ov(Y_{t_1}, Y_{t_2}) = |t_1|^\alpha + |t_2|^\alpha - |t_1 - t_2|^\alpha.$$

Then

$$U_T = (2\log(T))^{1/2}(M^T - d_T) \quad \textit{and} \quad V_T = (2\log(T))^{1/2}(m^T - d_T)$$

are asymptotically independent with,

$$\mathbb{P}[U_T < z] \longrightarrow \exp(-\lambda_1\exp(-z)),$$
$$\mathbb{P}[V_T < z] \longrightarrow \exp(-\lambda_2\exp(-z));$$

where

$$\lambda_1 = \int \exp(z)\,d\eta(z), \lambda_2 = \int \exp(-z)\,d\eta(z).$$

For a proof of Theorem 2.27 we refer to Theorem A1 in Bickel and Rosenblatt [14]. An immediate consequence of the theorem above is the following corollary for the absolute value, which we will use in subsequent parts of this work.

Corollary 2.28. *If* $\overline{M}^T = \sup\left\{|Y^T(t)| : 0 \leq t \leq T\right\}$ *then under the conditions of Theorem 2.27,*

$$\mathbb{P}[(2\log(T))^{1/2}(\overline{M}^T - d_T) < x] \longrightarrow \exp(-(\lambda_1 + \lambda_2)\exp(-x))\,,$$

as $T \longrightarrow \infty$.

Remark 2.29. The bridge constituting the connection between our general semimartingale framework and the limit theorems presented in this section is the strong invariance principles presented in Section 2.4. The underlying idea is to provide several approximation steps and to exploit the independence of Brownian increments, such that the i.i.d. results presented in this section become applicable. This procedure has already been pursued in Bibinger et al. [10] proving limit theorems based on high-frequency data without observation noise.

3. Change-point inference on spot volatility

This chapter is organized as follows: First, we will give a very short treatment of the parametric case, using limit theorems presented in the previous chapter and will proceed with the general Itô semimartingale case. We formulate the testing problem and the corresponding test statistic. We begin with the test for a continuous semimartingale $(X_t)_{t\in[0,1]}$ which is then extended to the general case utilizing truncation techniques. We present the asymptotic theory including the limit theorem under the null hypothesis, consistency of the test and consistent estimation of the change point under the alternative hypothesis. We conduct a Monte Carlo simulation study. The main insight is that the new test considerably increases the power compared to (optimally) skip sampling the noisy data to lower frequencies and applying the not noise robust method by Bibinger et al. [10] directly. The last section gathers the proofs. This chapter, except the first Section 3.1, has been published in Bibinger and Madensoy [11].

3.1. The parametric case

In this short section we will present a concise discussion of the parametric case, using the limit theorem presented in Theorem 2.19. The simplified parametric model implies that the volatility process $(\sigma_t)_{t\in[0,1]}$ is neither random nor time varying, that is

$$\sigma_t \equiv \sigma$$

for some constant σ. Furthermore, we drop the drift and jump component of the Itô semimartingale $(X_t)_{t\in[0,1]}$, i.e.

$$a_t \equiv 0\,,$$

and

$$\mu \equiv 0\,,$$

using the notation introduced in (2.3). This implies that the data generating process $(Y_t)_{t\in[0,1]}$,

$$Y_t = X_t + \varepsilon_t\,,$$

is given by

$$X_t = \int_0^t \sigma \, dW_s$$

and with $(\varepsilon_t)_{t\in[0,1]}$ fulfilling Assumption 2.18. Based on high-frequency data (2.19) we define the statistic $Q_{m,n}$, given by

$$Q_{m,n} = \sqrt{h_n} \sum_{k=1}^{m} \left(\hat{\sigma}^2_{(k-1)h_n} - h_n \sum_{k=1}^{h_n^{-1}} \hat{\sigma}^2_{(k-1)h_n} \right) , \quad m \in \left\{1, \ldots, h_n^{-1}\right\} .$$

With $m = \lfloor t h_n^{-1} \rfloor$, it holds that

$$\begin{aligned} Q_{m,n} &= \sqrt{h_n} \sum_{k=1}^{\lfloor t h_n^{-1} \rfloor} \hat{\sigma}^2_{(k-1)h_n} - \sqrt{h_n} h_n \lfloor t h_n^{-1} \rfloor \sum_{k=1}^{h_n^{-1}} \hat{\sigma}^2_{(k-1)h_n} \\ &= \frac{1}{\sqrt{h_n}} \left(h_n \sum_{k=1}^{\lfloor t h_n^{-1} \rfloor} \hat{\sigma}^2_{(k-1)h_n} - \int_0^t \sigma^2 \, ds \right) \\ &\qquad - \frac{t}{\sqrt{h_n}} \left(h_n \sum_{k=1}^{h_n^{-1}} \hat{\sigma}^2_{(k-1)h_n} - \int_0^1 \sigma^2 \, ds \right) + \mathcal{O}_{\mathbb{P}}(\sqrt{h_n}) \\ &\propto \frac{n^{1/4}}{\sqrt{\log(n)}} \left(\sum_{k=1}^{\lfloor t h_n^{-1} \rfloor} h_n \hat{\sigma}^2_{(k-1)h_n} - \int_0^t \sigma^2 \, ds \right) \\ &\qquad - \frac{t n^{1/4}}{\sqrt{\log(n)}} \left(\sum_{k=1}^{h_n^{-1}} h_n \hat{\sigma}^2_{(k-1)h_n} - \int_0^1 \sigma^2 \, ds \right) + \mathcal{O}_{\mathbb{P}}(\sqrt{h_n}) . \end{aligned}$$

We will apply Theorem 2.19. Therefore, taking into account, that according to the calculations presented in Altmeyer and Bibinger [5], the factor $1/\sqrt{\log(n)}$ provides the right rescaling with respect to the variance. This yields

$$\begin{aligned} &\frac{n^{1/4}}{\sqrt{\log(n)}} \left(\sum_{k=1}^{\lfloor t h_n^{-1} \rfloor} h_n \hat{\sigma}^2_{(k-1)h_n} - \int_0^t \sigma^2 \, ds \right) \\ &\qquad - \frac{t n^{1/4}}{\sqrt{\log(n)}} \left(\sum_{k=1}^{h_n^{-1}} h_n \hat{\sigma}^2_{(k-1)h_n} - \int_0^1 \sigma^2 \, ds \right) + \mathcal{O}_{\mathbb{P}}(\sqrt{h_n}) \xrightarrow{st} \gamma B_t^\circ , \end{aligned}$$

with a Brownian bridge $(B_t^\circ)_{t\in[0,1]}$ independent of $\mathcal{G}$. It is

$$\gamma^2 = 8\eta \left|\sigma\right|^3 ,$$

which is, in general, unknown, due to η and σ being unknown parameters. We have to construct an estimator $\hat{\gamma}_n^2$ of γ^2. It is well-known that η^2 can be estimated in this model with $\sqrt{n}$-rate by either a rescaled realized volatility or from the negative first-lag autocovariances of the noisy increments. Under Assumption 2.18 Zhang et al. [62] provide a rate-optimal consistent estimator for η^2:

$$\hat{\eta}^2 = \frac{1}{2n}\sum_{i=1}^{n}(Y_{i\Delta_n} - Y_{(i-1)\Delta_n})^2 = \eta^2 + \mathcal{O}_{\mathbb{P}}\big(n^{-1/2}\big)\,. \tag{3.1}$$

Furthermore, since

$$\widehat{\mathbf{IV}}_{n,1} \xrightarrow{\mathbb{P}} \sigma^2\,,$$

we set

$$\widehat{\gamma}_n^2 = 8\sqrt{\hat{\eta}^2}(\widehat{\mathbf{IV}}_{n,1})^{3/2}\,.$$

Therefore, after proper rescaling, we can conclude that the limit process is a standard Brownian Bridge. Testing for jumps in the volatility parameter, we can use the test statistic T_n given by

$$T_n = \sup_{m=1,\ldots,h_n^{-1}} \left|(\widehat{\gamma}_n^2)^{-1/2} Q_{m,n}\right|\,.$$

As $n \longrightarrow +\infty$ this converges to a Kolmogorov Smirnov distributed random variable. Concerning the quantiles of the latter, we refer to Marsaglia et al. [46].

3.2. The general nonparametric case

This section tackles the change-point detection question for general volatility processes $(\sigma_t)_{t\in[0,1]}$. We will develop a test for volatility jumps. We aim to test for some càdlàg squared volatility process $(\sigma_t^2)_{t\in[0,1]}$ hypotheses of the form

$$\begin{aligned} &H_0 : \text{there is no jump in } \sigma_t^2 \quad \text{vs.} \\ &H_1 : \text{there is at least one } \theta\in(0,1) \text{ such that } \big|\sigma_\theta^2 - \lim_{s\to\theta, s<\theta}\sigma_s^2\big| > 0\,. \end{aligned} \tag{3.2}$$

It is standard in the theory of statistics of high-frequency data to address such questions *path-wise*. This means that H_0 and H_1 are formulated for one particular path of the squared volatility $(\sigma_t^2(\omega))_{t\in[0,1]}$ and we strive to make a decision based on discrete observations of the given path of $(Y_t(\omega))_{t\in[0,1]}$. The semimartingale $(X_t)_{t\in[0,1]}$ is defined on a filtered probability space $(\Omega^{(0)}, \mathcal{F}^{(0)}, (\mathcal{F}^{(0)})_{t\in[0,1]}, \mathbb{P}^{(0)})$.

3.2.1. Assumptions and test statistics

We need further assumptions on the coefficient processes of (X_t).

Assumption 3.1 (The processes a and σ)**.** *The processes a and σ are locally bounded. σ is almost surely strictly positive, that is,* $\inf_{t\in[0,1]}\sigma_t^2 \geq K^- > 0$.

Our notation for jump processes follows Jacod and Protter [35].

Assumption 3.2. *Suppose* $\sup_{\omega,x}|\delta(s,x)|/\gamma(x)$ *is locally bounded for some deterministic non-negative function γ which satisfies for some $r\in[0,2]$:*

$$\int_{\mathbb{R}}(1\wedge\gamma^r(x))\lambda(dx) < \infty\,. \tag{3.3}$$

Remark 3.3. The smaller r the more restrictive Assumption 3.2. The case $r=0$ is tantamount to jumps of finite activity.

On the null hypothesis we allow for very general and rough continuous stochastic volatility processes.

Hypothesis (H_0-$\mathfrak{a}$). *Under the null hypothesis, the modulus of continuity*

$$w_\delta(\sigma)_t = \sup_{s,r\leq t}\{|\sigma_s-\sigma_r| : |s-r|<\delta\}$$

is locally bounded in the sense that there exists $\mathfrak{a}>0$ and a sequence of stopping times $T_n\to\infty$, such that $w_\delta(\sigma)_{(T_n\wedge 1)}\leq L_n\delta^{\mathfrak{a}}$, for some $\mathfrak{a}>0$ and some (almost surely finite) random variables L_n.

The regularity exponent $\mathfrak{a}\in(0,1]$ is selected for the testing problem. The test can be repeated for different values also. The regularity exponent coincides with a usual Hölder exponent when L_n is a fix constant. Integrating a sequence L_n enables us to include stochastic volatility processes in our theory. Since stochastic processes as Brownian motion are not in some fix Hölder class, it is crucial to work with (slightly) more general smoothness classes determined by the exponent $\mathfrak{a}>0$ and by L_n. Observe that if

$$\mathbb{E}\left[\left|\sigma_t^2-\sigma_s^2\right|^{\mathfrak{b}}\right]\leq C\,|t-s|^{\gamma+\mathfrak{a}\mathfrak{b}}\ \text{, for some } \mathfrak{b},\ C>0 \text{ and } \gamma>1\,,$$

then the Kolmogorov Čentsov Theorem implies that

$$\lim_{n\to+\infty}\mathbb{P}\Big(\sup_{\substack{s,t\in[0,1]\\|t-s|\leq\delta}}\left|\sigma_t^2-\sigma_s^2\right|\leq L_n\delta^{\mathfrak{a}}\Big)=1$$

if $L_n\to+\infty$ arbitrarily slowly. In particular, we can impose that $L_n=\mathcal{O}(\log(n))$ for our derivation of upper bounds in the sections below. The null hypothesis is the same as in Assumption 3.1 of Bibinger et al. [10]. Our test distinguishes the null hypothesis from alternative hypotheses of the following type.

Alternative (H_1-$\mathfrak{a}$). *Under the alternative hypothesis, there exists at least one* $\theta \in (0,1)$*, such that*

$$\left|\Delta\sigma_\theta^2\right| = \left|\sigma_\theta^2 - \lim_{s\to\theta, s<\theta} \sigma_s^2\right| = \delta > 0\,.$$

We suppose that $\sigma_t^2 = \sigma_t^{2,(c)} + \sigma_t^{2,(j)}$*, where* $(\sigma_t^{2,(c)})_{t\in[0,1]}$ *satisfies (*H_0-$\mathfrak{a}$*). The jump component* $(\sigma_t^{2,(j)})_{t\in[0,1]}$ *is a pure-jump semimartingale which satisfies Assumption 3.2 with* $r \le 2$*.*

In particular, the alternative hypothesis does not restrict to only one jump. We establish a consistent test when *at least* one non-negligible jump is present. Multiple jumps and quite general jump components are possible. Consistency of our test only requires that in a small vicinity of θ, $(\sigma_t^{2,(c)})$ and $(\sigma_t^{2,(j)} - \Delta\sigma_\theta^2 \mathbb{1}_{[\theta,1]}(t))$ are sufficiently regular such that the jump $\Delta\sigma_\theta^2$ is detected. Bibinger et al. [10] impose in their Theorem 4.3 the condition that all volatility jumps are positive. This condition is replaced here by the semimartingale assumption on $(\sigma_t^{2,(j)})_{t\in[0,1]}$. Both ensure that $\Delta\sigma_\theta^2$ can not be compensated by opposite jumps in an asymptotically small vicinity. In order to incorporate microstructure noise, we have to extend the original probability space. We set $\mathcal{G}_t = \mathcal{F}_t^{(0)} \otimes \sigma\left(\varepsilon_s : s \le t\right)$. As we have already explained in the previous chapter, the data generating process $(Y_t)_{t\in[0,1]}$ is defined on the filtered probability space $\left(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,1]}, \mathbb{P}\right)$. For the noise process we impose further assumptions being more restrictive than Assumption 2.18.

Assumption 3.4 (The noise process)**.** *The stochastic process* $(\varepsilon_t)_{t\in[0,1]}$ *is defined on* $\left(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,1]}, \mathbb{P}\right)$ *and fulfills the following conditions.*

(1) $(\varepsilon_t)_{t\in[0,1]}$ *is a centered white noise process,* $\mathbb{E}[\varepsilon_t] \equiv 0$*, and with*

$$\mathbb{E}\left[\varepsilon_t^2\right] = \eta^2\,.$$

(2) The following moment condition holds.

$$\mathbb{E}\left[|\varepsilon_t|^m\right] < \infty, \text{ for all } m \in \mathbb{N}. \tag{3.4}$$

Remark 3.5. The moment condition (3.4) is standard in related literature, see for instance Assumption (WN) of Aït-Sahalia and Jacod [1], p. 221 or Assumption 16.1.1 of Jacod and Protter [35], but in a certain sense purely technical. Let us stress that in our setting, we do impose as less assumptions as possible on the volatility process $(\sigma_t)_{t\in[0,1]}$. More precisely the regularity under (H_0-$\mathfrak{a}$), for arbitrarily small $\mathfrak{a} \in (0,1]$, requires the existence of all moments in (3.4). More precisely, the smaller $\mathfrak{a}$ the larger m has to be chosen. Nevertheless, we point out that the moment condition is not that restrictive for standard models of volatility. In the usual case, for instance, where $(\sigma_t)_{t\in[0,1]}$ itself is assumed to be an Itô semimartingale, when $\mathfrak{a} \approx 1/2$, only the existence of moments up to order $m = 8$ has to be imposed.

Remark 3.6. While Assumption 3.4 is in line with standard conditions on the additive noise component in the literature, possible generalizations with respect to the structure of the noise process $(\varepsilon_t)_{t\in[0,1]}$ in three directions are of interest: *serial dependence, heterogeneity and endogeneity.* Such generalizations are also motivated by stylized facts in econometrics, see Hansen and Lunde [29] for a detailed discussion. For instance, Chapter 16 in Jacod and Protter [35] includes conditional i.i.d. noise, endogenous as it may depend (in a certain way) on (X_t), in the theory of pre-average estimators. This allows to model phenomena as noise by price discreteness (rounding). Bibinger and Winkelmann [13] provide some first extensions of spectral spot volatility estimation to serially correlated and heterogeneous noise. Though the possible extensions appear to be relevant for applications, we work in the framework formulated in Assumption 3.4, mainly due to the lack of groundwork sufficient for the present work. Since we exploit some ingredients from previous works on spectral volatility estimation, particularly the form of the efficient asymptotic variance based on Altmeyer and Bibinger [5], a generalization of our results requires non-trivial generalizations of these ingredients first. Furthermore, more general noise processes ask for extensive work on the estimation of the local long-run variance replacing (3.1). This topic, however, is beyond the scope of this work. Let us remark that it is as well not obvious how to apply strong embedding principles in these cases to generalize our proofs. Since Wu and Zhao [61] provide strong approximation results for weakly dependent time series, we nevertheless conjecture that certain generalizations in the three directions are possible.

In this subsection we construct the test first for the model $(X_t)_{t\in[0,1]}$ without jumps, that is, the process (J_t) in (2.3) fulfills

$$J_t \equiv 0.$$

The construction of the test is based on a combination of the techniques by Altmeyer and Bibinger [5] introduced in Subsection 2.3.2 and Bibinger et al. [10]. In order to do so, we split the observation interval $[0,1]$ by some "big blocks" with length $\alpha_n h_n$:

$$[i\alpha_n h_n, (i+1)\,\alpha_n h_n]\,, \quad i = 0,\ldots,\lfloor(\alpha_n h_n)^{-1}\rfloor - 1,$$

where $(\alpha_n)_{n\in\mathbb{N}}$ is some $\mathbb{N}$-valued sequence fulfilling as $n\to+\infty$:

$$\sqrt{\alpha_n}\,(\alpha_n h_n)^{\mathfrak{a}}\,\sqrt{\log(n)} \longrightarrow 0 \text{ and } h_n^{-\varpi}/\alpha_n \longrightarrow 0 \tag{3.5}$$

for some $\varpi > 0$ and the regularity exponent $\mathfrak{a}\in(0,1]$ under the null hypothesis (H_0-$\mathfrak{a}$). Using spectral estimators and averaging within each big block $[i\alpha_n h_n, (i+1)\,\alpha_n h_n]$ provides a *consistent* estimator for $\sigma^2_{i\alpha_n h_n}$:

$$\overline{RV}_{n,i} = \frac{1}{\alpha_n}\sum_{\ell=1}^{\alpha_n}\hat{\sigma}^2_{h_n(i\alpha_n+(\ell-1))}, \quad i = 0,\ldots,\lfloor(\alpha_n h_n)^{-1}\rfloor - 1\,. \tag{3.6}$$

A feasible adaptive estimation is obtained by a two-stage method where $\hat{\eta}^2$ from (3.1) and

$$\frac{1}{\alpha_n}\sum_{l=k-\alpha_n\vee 0}^{(k-1)\vee(\alpha_n-1)}\sum_{j=1}^{J}\frac{1}{J}\left(S_{jl}^2(Y)-[\varphi_{jl},\varphi_{jl}]_n\frac{\hat{\eta}^2}{n}\right)=\sigma^2_{(k-1)h_n}+\mathcal{O}_{\mathbb{P}}(1) \tag{3.7}$$

are inserted in the oracle weights to derive feasible estimated weights $\hat{w}_{jk}$. The result (3.7) has been established and used in previous works on spectral volatility estimation, see Bibinger and Winkelmann [13]. The pilot volatility estimator (3.7) is an average of squared bias corrected spectral statistics over J Fourier frequencies and α_n bins. For some fix $J \in \mathbb{N}$ and an optimal choice of $\alpha_n \propto n^{\mathfrak{a}/(2\mathfrak{a}+1)}/\log(n)$, it renders a rate-optimal estimator for which the $\mathcal{O}_{\mathbb{P}}(1)$-term in (3.7) is $\mathcal{O}_{\mathbb{P}}(n^{-\mathfrak{a}/(4\mathfrak{a}+2)})$. A sub-optimal choice of α_n will not affect our results, however. Other weights than (2.22) do not yield an asymptotically efficient estimator with minimal asymptotic variance. With estimated versions of the optimal weights (2.22), Altmeyer and Bibinger [5] show that a Riemann sum over the estimates (2.21) yields a quasi-efficient estimator for the integrated squared volatility. Hence, we use the statistics (2.21) with exactly these weights and the orthogonal sine basis (Φ_{jk}) motivated by the efficiency results of Reiß [53]. Finally, with adaptive versions of the local volatility estimators (3.6)

$$\overline{RV}^{ad}_{n,i}=\frac{1}{\alpha_n}\sum_{\ell=1}^{\alpha_n}\hat{\sigma}^{2,ad}_{h_n(i\alpha_n+(\ell-1))},\quad i=0,\ldots,\lfloor(\alpha_n h_n)^{-1}\rfloor-1\,, \tag{3.8}$$

$$\hat{\sigma}^{2,ad}_{(k-1)h_n}=\sum_{j=1}^{\lfloor nh_n\rfloor-1}\hat{w}_{jk}\left(S_{jk}^2(Y)-[\varphi_{jk},\varphi_{jk}]_n\frac{\hat{\eta}^2}{n}\right),$$

our test statistic is given by

$$\overline{V}_n=\max_{i=0,\ldots,\lfloor(\alpha_n h_n)^{-1}\rfloor-2}\left|\frac{\overline{RV}^{ad}_{n,i}-\overline{RV}^{ad}_{n,i+1}}{\sqrt{8\hat{\eta}}\,\left|\overline{RV}^{ad}_{n,i+1}\right|^{3/4}}\right|, \tag{3.9}$$

where $\hat{\eta}=\sqrt{\hat{\eta}^2}$, with $\hat{\eta}^2$ from (3.1). We write the absolute value in the denominator, since due to the bias correction in (2.21) the statistics $(\overline{RV}_{n,i})$ and $(\overline{RV}^{ad}_{n,i})$, $i = 0,\ldots,\lfloor(\alpha_n h_n)^{-1}\rfloor-1$ are not guaranteed to be positive.

Remark 3.7. (1) The construction of the test statistic (3.9) is based on the idea to compare the values of the spot volatility process $\left(\sigma_t^2\right)_{t\in[0,1]}$ on contiguous intervals $[i\alpha_n h_n,(i+1)\,\alpha_n h_n]$ and $[(i+1)\,\alpha_n h_n,(i+2)\,\alpha_n h_n]$ and to reject the null hypothesis of no jumps, if the test statistic $\overline{V}_n$ fulfills $\overline{V}_n \geq c_n$ for some accurate sequence c_n.

(2) The statistic (3.9) significantly differs from the statistic V_n given in Equation (13) of Bibinger et al. [10] beyond replacing spot volatility estimates by noise-robust spot volatility estimates. Though both statistics are quotients, the underlying

structure of them is different. Whereas in Bibinger et al. [10] the simple structure of the (asymptotic) variance of spot volatility estimates allows to use statistics based on their quotients, (3.9) is based on differences rescaled with their estimated variances. The statistics which are used to wipe out the influence of the noise process imply that volatility does not simply "cancel out" in our case as in Proposition A.3 of Bibinger et al. [10]. The construction of (3.9) is particularly appropriate from an implementation point of view, since it scales to obtain an asymptotic distribution-free test and makes it possible to avoid pre-estimation of higher order moments.

In order to increase the performance of the statistic, we also include a statistic $\overline{V}_n^{ov}$ based on *overlapping* big blocks:

$$\overline{V}_n^{ov} = \max_{i=\alpha_n,\dots,h_n^{-1}-\alpha_n} \left| \frac{\overline{RV}_{n,i}^{ov} - \overline{RV}_{n,i+\alpha_n}^{ov}}{\sqrt{8\hat{\eta}}\,\left|\overline{RV}_{n,i+\alpha_n}^{ov}\right|^{3/4}} \right| \tag{3.10}$$

with $\overline{RV}_{n,i}^{ov}$ given by

$$\overline{RV}_{n,i}^{ov} = \frac{1}{\alpha_n} \sum_{\ell=i-\alpha_n+1}^{i} \hat{\sigma}_{(\ell-1)h_n}^{2,ad}, \quad i = \alpha_n, \dots, h_n^{-1}.$$

3.2.2. The discontinuous case

In this subsection we generalize the method to be robust in the presence of jumps in (2.3). When $(\sigma_t^2)_{t\in[0,1]}$ is our target of inference, the jumps are a nuisance quantity. In order to eliminate jumps of $(X_t)_{t\in[0,1]}$ in the approach, we consider truncated spot volatility estimates

$$\overline{RV}_{n,i}^{tr} = \frac{1}{\alpha_n} \sum_{\ell=i-\alpha_n+1}^{i} \hat{\sigma}_{(\ell-1)h_n}^{2,ad} \mathbb{1}_{\{|\hat{\sigma}_{(\ell-1)h_n}^{2,ad}| \le h_n^{\tau-1}\}}, \quad i = \alpha_n, \dots, h_n^{-1}, \tag{3.11}$$

with a truncation exponent $\tau \in (0,1)$. Truncated volatility estimators have been introduced first for integrated volatility estimation by Mancini [44] and Jacod [33]. We define the test statistics with the truncated spot volatility estimates (3.11)

$$\overline{V}_n^{\tau} = \max_{i=1,\dots,\lfloor(\alpha_n h_n)^{-1}\rfloor-1} \left| \frac{\overline{RV}_{n,i\alpha_n}^{tr} - \overline{RV}_{n,(i+1)\alpha_n}^{tr}}{\sqrt{8\hat{\eta}}\,\left|\overline{RV}_{n,(i+1)\alpha_n}^{tr}\right|^{3/4}} \right|, \tag{3.12a}$$

$$\overline{V}_n^{ov,\tau} = \max_{i=\alpha_n,\dots,h_n^{-1}-\alpha_n} \left| \frac{\overline{RV}_{n,i}^{tr} - \overline{RV}_{n,i+\alpha_n}^{tr}}{\sqrt{8\hat{\eta}}\,\left|\overline{RV}_{n,i+\alpha_n}^{tr}\right|^{3/4}} \right|. \tag{3.12b}$$

3.2.3. Limit theorem I: The continuous case

The hypothesis test formulated above is based on asymptotic results for the statistics $\overline{V}_n$ and $\overline{V}_n^{ov}$.

Theorem 3.8. *Set* $m_n = \lfloor(\alpha_n h_n)^{-1}\rfloor$, $\gamma_{m_n} = [4\log(m_n) - 2\log(\log(m_n))]^{1/2}$ *and assume that* $J_t \equiv 0$. *If Assumptions 3.1 and 3.4 hold and* α_n *satisfies condition* (3.5), *then we have under (*H_0-$\mathfrak{a}$*) that*

$$\sqrt{\log(m_n)}\left(\sqrt{\alpha_n}\overline{V}_n - \gamma_{m_n}\right) \xrightarrow{\text{d}} V\,, \tag{3.13}$$

where V *follows an extreme value distribution with distribution function*

$$\mathbb{P}(V \le x) = \exp(-\pi^{-1/2}\exp(-x)).$$

Theorem 3.8 is a key tool tackling the testing problem which is based on non-overlapping big blocks. The following result covers the case of overlapping big blocks.

Corollary 3.9. *Given the assumptions of Theorem 3.8, the following weak convergence holds under (*H_0-$\mathfrak{a}$*):*

$$\sqrt{\log(m_n)}\sqrt{\alpha_n}\overline{V}_n^{ov} - 2\log(m_n) - \frac{1}{2}\log(\log(m_n)) - \log(3) \xrightarrow{\text{d}} V, \tag{3.14}$$

with V *as in Theorem 3.8.*

3.2.4. Limit theorem II: The general case

We extend this result to the setup with jumps in $(X_t)_{t\in[0,1]}$ when using truncated functionals.

Proposition 3.10. *Let* m_n *and* γ_{m_n} *be the sequences defined in Theorem 3.8. Suppose* $\alpha_n = \kappa h_n^{-\beta}$ *for a constant* κ *and with* $0 < \beta < 1$, *Assumption 3.1, Assumption 3.4 and Assumption 3.2 with*

$$r < \min\Big(2 - \frac{\beta}{\tau}, 2\tau^{-1}(1-\beta), \tau^{-1}, \frac{3}{4}\big(1 + \tau - \frac{\beta}{2}\big)\Big)\,. \tag{3.15}$$

Then we have under (H_0-$\mathfrak{a}$*) that*

$$\sqrt{\log(m_n)}\left(\sqrt{\alpha_n}\,\overline{V}_n^{\tau} - \gamma_{m_n}\right) \xrightarrow{\text{d}} V\,, \tag{3.16a}$$

$$\sqrt{\log(m_n)}\sqrt{\alpha_n}\,\overline{V}_n^{ov,\tau} - 2\log(m_n) - \frac{1}{2}\log(\log(m_n)) - \log(3) \xrightarrow{\text{d}} V\,, \tag{3.16b}$$

with V *as in Theorem 3.8.*

It is natural that we derive the same limit results as above, since the truncation aims to eliminate the nuisance jumps. Proposition 3.10 gives rather minimal conditions, in particular (3.15), under that we can guarantee that the truncation works in this sense.

Remark 3.11. Condition (3.15) ensures that different error terms in the proof of Proposition 3.10 are asymptotically negligible. Though we state it in terms of upper bounds on the jump activity r, it rather puts restrictions on the interplay between r, τ and β. Given $\mathfrak{a}$ from (H_0-$\mathfrak{a}$), we choose β close to $2\mathfrak{a}/(2\mathfrak{a}+1)$ to attain the highest possible power of the test. This results in $0 < \beta < 2/3$, where the case $\beta \approx 1/2$ for $\mathfrak{a} = 1/2$ appears the most relevant one including a test for jumps in a semimartingale volatility process. Rewritten in terms of bounds on τ, (3.15) gives:

$$\max\left(\frac{\beta}{2-r}, \frac{4}{3}r + \frac{\beta}{2} - 1\right) < \tau < \min\left(r^{-1}, 2r^{-1}(1-\beta)\right).$$

For finite activity, $r = 0$, we only have mild lower bounds on the choice of τ. Usually a choice of τ close to 1 is advocated in previous works on truncated volatility estimation. For $\beta \approx 1/2$ this requires $r < 1$. The different error terms under noise for the maximum obtained here actually suggest that $\tau = 3/4$ is an even better choice when we require only $r < 4/3$. Overall the conditions on the jumps are not much more restrictive than required for central limit theorems of linear volatility estimators, see Chapter 13 of Jacod and Protter [35]. Compared to Proposition 3.5 of Bibinger et al. [10], we relax the conditions on (J_t) by a more sophisticated strategy of our proof. In particular, we do not have to restrict to a Lévy-type process with independent increments, since we work with Doob's submartingale maximal inequality instead of Kolmogorov's maximal inequality. With this strategy it is also possible to generalize the result in Proposition 3.5 of Bibinger et al. [10].

3.2.5. Construction and consistency of the hypothesis test

Based on the limit results presented in Subsections 3.2.3 and 3.2.4 we can summarize the following rejection rules. Thereto, let c_α be the $(1-\alpha)$-quantile of the Gumbel-type limit law $\mathbb{P}_V$ of V in the limit theorems. Since the latter is absolutely continuous with respect to the Lebesgue measure, there is a unique solution given by

$$c_\alpha = -\log\left(-\log\left(1-\alpha\right)\right) - \frac{1}{2}\log\left(\pi\right).$$

(R) Based on Theorem 3.8 and the notations used there we

$$\text{reject } \ H_0\text{-}\mathfrak{a} \text{ if } \ \overline{V}_n \geq \alpha_n^{-1/2}\left((\log(m_n))^{-1/2} c_\alpha + \gamma_{m_n}\right). \tag{3.17}$$

(R^{ov}) Based on Corollary 3.9 and the notations used there we

$$\text{reject } \ H_0\text{-}\mathfrak{a} \text{ if } \ \overline{V}_n^{ov} \geq \frac{\left(c_\alpha + 2\log\left(m_n\right) + \frac{1}{2}\log\left(\log\left(m_n\right)\right) + \log\left(3\right)\right)}{\left(\log(m_n)\alpha_n\right)^{1/2}}. \tag{3.18}$$

(R^{τ}) Based on Proposition 3.10 and the notations used there we

$$\text{reject } \ H_0\text{-}\mathfrak{a} \text{ if } \ \overline{V}_n^{\tau} \geq \alpha_n^{-1/2}\left((\log(m_n))^{-1/2} c_\alpha + \gamma_{m_n}\right). \tag{3.19}$$

($R^{ov,\tau}$) Based on Proposition 3.10 and the notations used there we

$$\text{reject } \ H_0\text{-}\mathfrak{a} \text{ if } \ \overline{V}_n^{ov,\tau} \geq \frac{\left(c_\alpha + 2\log\left(m_n\right) + \frac{1}{2}\log\left(\log\left(m_n\right)\right) + \log\left(3\right)\right)}{\left(\log(m_n)\alpha_n\right)^{1/2}}. \tag{3.20}$$

Theorem 3.12. *Suppose Assumption 3.1, Assumption 3.4, and Assumption 3.2 with* (3.15) *in the case with jumps. The decision rules* (3.17)*,* (3.18)*,* (3.19) *and* (3.20) *provide consistent tests to distinguish the null hypothesis* (H_0-$\mathfrak{a}$) *from the alternative hypothesis* (H_1-$\mathfrak{a}$) *for the testing problem* (3.2)*.*

Consistency of the test means that under the alternative hypothesis, if for some $\theta \in (0,1)$ we have that $\left|\sigma_\theta^2 - \sigma_{\theta-}^2\right| = \delta > 0$ for some fix $\delta > 0$, the power of the test, for instance by (3.17), tends to one as $n \to \infty$:

$$\mathbb{P}_{H_1}\big(\overline{V}_n \geq \alpha_n^{-1/2}\big(\,(\log(m_n))^{-1/2}\,c_\alpha + \gamma_{m_n}\big)\big) \overset{n\to\infty}{\longrightarrow} 1\,.$$

Theorem 3.8 ensures that (3.17) facilitates an asymptotic level-α-test that correctly controls the type 1 error, that is

$$\mathbb{P}_{H_0}\big(\overline{V}_n \geq \alpha_n^{-1/2}\big(\,(\log(m_n))^{-1/2}\,c_\alpha + \gamma_{m_n}\big)\big) \overset{n\to\infty}{\longrightarrow} \alpha\,.$$

Thereby, even for small $\mathfrak{a} > 0$ the test can distinguish continuous volatility paths from paths with jumps.

Remark 3.13. The rate $\sqrt{\log(m_n)\alpha_n}$ in (3.13), (3.14), (3.16a) and (3.16b) determines how fast the power of the test increases in the sample size n. The convergence rate, for α_n close to the upper bound in (3.5) is close to $n^{\mathfrak{a}/(4\mathfrak{a}+2)}$. The latter coincides with the optimal convergence rate for spot volatility estimation under noise, see Munk and Schmidt-Hieber [50]. In light of the lower bound for the testing problem without noise established in Bibinger et al. [10] and the relation of the models with and without noise studied in Gloter and Jacod [26], we conjecture that the above test yields an asymptotic minimax-optimal decision rule. A formal generalization of the proof for the detection boundary from Theorem 4.1 of Bibinger et al. [10] to our setting however appears not to be feasible, since it heavily exploits simple χ^2-approximations of squared increments.

3.2.6. Consistent estimation of the change point

In this subsection we present an estimator for the change point θ, which is of importance, once we have decided to reject (H_0-$\mathfrak{a}$). Therefore, we suppose (H_1-$\mathfrak{a}$) and that there exists one $\theta \in (0,1)$ with $|\Delta\sigma_\theta^2| > 0$. The aim is to estimate θ, in general referred to as the change point or break date in change-point statistics, which here gives the time of the volatility jump. We suggest the estimator $\hat{\theta}_n$, given by

$$\hat{\theta}_n = h_n \operatorname{argmax}_{i=\alpha_n,\ldots,h_n^{-1}-\alpha_n} \overline{V}^{\diamond}_{n,i}\,, \tag{3.21}$$

where

$$\overline{V}^{\diamond}_{n,i} = \alpha_n^{-1/2}\left|\sum_{\ell=i-\alpha_n+1}^{i} \hat{\sigma}^{2,ad}_{(\ell-1)h_n} - \sum_{\ell=i+1}^{i+\alpha_n} \hat{\sigma}^{2,ad}_{(\ell-1)h_n}\right|\,.$$

It is sufficient to use these modified non-rescaled versions of the statistics in (3.10). We prove the following consistency result for our estimator.

Proposition 3.14. *Given the assumptions of Theorem 3.8, that is, Assumptions 3.1 and 3.4, $J_t \equiv 0$ and α_n satisfies* (3.5)*, and assume that (H_1-$\mathfrak{a}$) applies with one jump time $\theta \in (0,1)$. For $\Delta\sigma_\theta^2 = \delta \neq 0$ it holds that*

$$\left|\hat{\theta}_n - \theta\right| = \mathcal{O}_{\mathbb{P}}\left(h_n|\delta^{-1}|\sqrt{\alpha_n \log(n)}\right) .$$

In particular, $\hat{\theta}_n \xrightarrow{\mathbb{P}} \theta$.

Remark 3.15. Put another way, we can detect jump times associated with sequences of jump sizes $\delta_n \to 0$ as $n \to +\infty$ as long as $h_n^{-1}(\alpha_n \log(n))^{-1/2} = \scriptstyle{\mathcal{O}}(\delta_n)$ in the sense of weak consistency. Choosing α_n as small as possible, such that (3.5) is satisfied, yields the best possible rate, while for the testing problem in Theorem 3.8 we select α_n as large as possible. In the optimal case, a jump with fix size $\delta \neq 0$ can be detected with a convergence rate close to h_n^{-1}. This provides important information how precisely volatility jump times can be located under noisy observations. With jumps in (X_t), we conjecture that an analogous results holds true under the conditions of Proposition 3.10. A sequential application of our methods allows for testing and the estimation of multiple change points. The extension of the estimation from the one change to the multiple change-point alternative is accomplished similarly to Algorithm 4.9 from paragraph 4.2.2. in Bibinger et al. [10].

3.2.7. Simulations

In this subsection we investigate the finite-sample performance of the new method in a simulation study. We also analyze the efficiency gains of our noise-robust approach based on the spectral volatility estimation methodology in comparison to simply skip sampling the data and applying the non noise-robust method from Bibinger et al. [10]. Skip sampling the data, which means we only consider every 60th datapoint, reduces the dilution by the noise and is a standard way to deal with high-frequency data in practice. We consider $n = 30,000$ observations of (2.18), a typical sample size of high-frequency returns over one trading day. The noise is centered and normally distributed with a realistic magnitude, $\eta = 0.005$, see, for instance, Bibinger et al. [9]. We implement the same volatility model as in Section 5 of Bibinger et al. [10], where

$$\sigma_t = \left(\int_0^t c \cdot \rho \, dW_s + \int_0^t \sqrt{1-\rho^2} \cdot c \, dW_s^{\perp}\right) \cdot v_t \tag{3.22}$$

is a semimartingale volatility process fluctuating around the seasonality function

$$v_t = 1 - 0.2 \sin\left(\tfrac{3}{4}\pi \, t\right), \, t \in [0,1] \, , \tag{3.23}$$

where $c = 0.1$ and $\rho = 0.5$, with $W^{\perp}$ a standard Brownian motion independent of W. We set $X_0 = 4$ and the drift $a = 0.1$. We perform the simulations in R using an Euler-Maruyama discretization scheme.

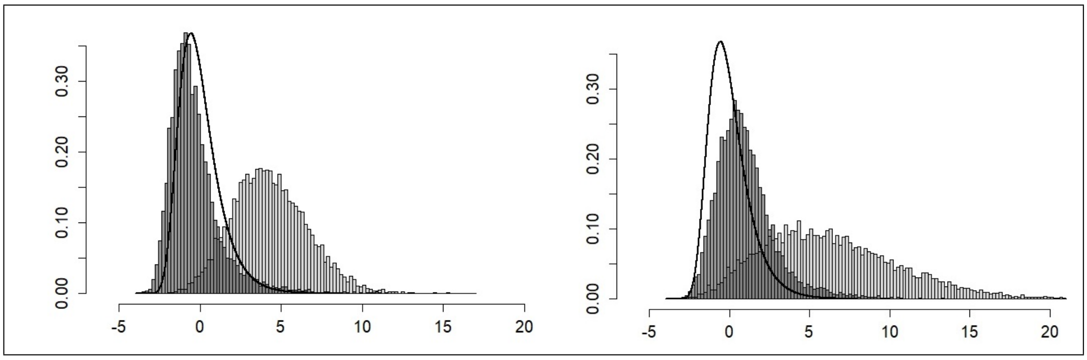

Figure 3.1.: Left: Histogram of statistics left-hand side in (3.16b) for $h_n^{-1} = 120$ and $\alpha_n = 15$, $n = 30,000$, under null hypothesis and alternative hypothesis and limit law density marked by the line. Right: Histogram of corresponding not noise-robust statistics from Bibinger et al. [10] applied after (most efficiently) skip sampling to a subset with $n_{skip} = 500$ observations, $k_n = 125$, and limit law density marked by the line.

Performance of the test, comparison to skip sampling, bootstrap adjustment and sensitivity analysis

Concerning the jumps of (X_t) and (σ_t) under the alternative hypothesis, we implement two different model configurations. In order to grant a good comparison to Bibinger et al. [10] in the evaluation of the efficiency gains by our method instead of a skip-sample approach, we will adopt the setup from Section 5 of Bibinger et al. [10]. There, under the alternative hypothesis, the volatility admits one jump of size 0.2 at time $t = 2/3$. The jump size equals the range of the expected continuous movement. Under the alternative hypothesis, (X_t) admits a jump at the same time $t = 2/3$. Under the null hypothesis and the alternative hypothesis, (X_t) also jumps at some uniformly drawn time. All price jumps are normally distributed with expected size 0.5 and variance 0.1. More general jumps are considered below.

We consider the test statistic (3.12b) with overlapping blocks and truncation. The simulations below confirm that it outperforms the non-overlapping version (3.12a). We set $h_n^{-1} = 120$ and $\alpha_n = 15$. Robustness with respect to different choices of h_n and α_n is discussed below. For the truncation, we set $\tau = 3/4$ according to Remark 3.11. In all cases, we compute the adaptive feasible statistics and do not make use of the generated volatility paths to derive the weights (2.22). We rather rely on the two-stage method and insert (3.7) with $J = 20$ and (3.1) in the statistics. The spectral estimates from (2.21) are computed as sums up to the spectral cut-off $J_n = 50$, smaller than $\lfloor nh_n \rfloor - 1 = 254$, as the fast decay of the weights (2.22) in j, compare also (3.40), renders higher frequencies completely negligible. The investigated test statistics will be identically feasible in data applications.

Figure 3.1 visualizes the empirical distribution from 10,000 Monte Carlo iterations under the null hypothesis and the alternative hypothesis. The left plot shows our

statistics while the right plot gives the results for the statistics from Bibinger et al. [10] applied to a skip sample of 500 observations. The skip-sampling frequency has been chosen to maximize the performance of these statistics. While they are reasonably robust to minor modifications, too large samples lead to an explosion of the statistics also under the null hypothesis and much smaller samples result in poor power. The length of the smoothing window k_n for the statistics given in Equation (24) of Bibinger et al. [10] is set $k_n = 125$, adopted from the simulations in Bibinger et al. [10]. In the optimal case, null and alternative hypothesis are reasonably well distinguished by the skip-sampling method – but the two plots confirm that our approach improves the finite-sample power considerably. For the spectral approach, 88% of the outcomes under H_1 exceed the 90%-decile of the empirical distribution under H_0. For the optimized skip-sample approach this number reduces to 75%. The approximation of the limit law appears somewhat imprecise. The relevant high quantiles, however, fit their empirical counterparts quite well.
Nevertheless, we propose a bootstrap procedure to fit the distribution of $\overline{V}_n^{ov,\tau}$ under H_0 with improved finite-sample accuracy. We start with an estimator for the spot volatility $\overline{RV}^{tr}_{n,i}, \quad i = \alpha_n, \ldots, h_n^{-1}$, from (3.11), using the same h_n^{-1} and α_n as for the test. We also define and compute $\overline{RV}^{tr}_{n,i}, \quad i = 1, \ldots, \alpha_n - 1$, averaging over the available number of blocks, smaller than α_n, back in time. In order to smooth the random fluctuations of the spot volatility pre-estimates, we apply a filter to the estimates of length 30 with equal weights and denote $\tilde{\sigma}^2_{n,i}, \quad i = 1, \ldots, h_n^{-1}$, the resulting estimated volatility path. At the boundaries we interpolate linearly to $\overline{RV}^{tr}_{n,1}$ and $\overline{RV}^{tr}_{n,h_n^{-1}}$, respectively. Repeating each entry $nh_n = 250$ times, we obtain a (bin-wise constant) estimator $\tilde{\sigma}^2_{n,i}, \quad i = 1, \ldots, n$. For two sequences of i.i.d. standard normals $\{Z_i\}_{1\le i\le n}, \{E_i\}_{1\le i\le n}$, and $X_0^* = Y_0^* = Y_0$, denote with

$$X_i^* = X_{i-1}^* + \sqrt{\frac{\tilde{\sigma}^2_{n,i}}{n}} \cdot Z_i \;, Y_i^* = X_i^* + \hat{\eta} \cdot E_i \;, 1 \le i \le n \,,$$

a pseudo path Y^* generated with the estimated volatility path and estimated noise variance and the (Z_i, E_i). We can iterate the procedure as a Monte Carlo simulation and produce $N = 10,000$ different pseudo paths Y^* using independent generalizations of random variables (Z_i, E_i). With

$$\hat{\sigma}^{2*}_{(k-1)h_n} = \sum_{j=1}^{\lfloor nh_n \rfloor - 1} \hat{w}_{jk} \left(S^2_{jk}(Y^*) - [\varphi_{jk}, \varphi_{jk}]_n \frac{\hat{\eta}^2}{n} \right) ,$$

$$\overline{RV}^*_{n,i} = \frac{1}{\alpha_n} \sum_{\ell = i-\alpha_n+1}^{i} \hat{\sigma}^{2*}_{(\ell-1)h_n} \mathbb{1}_{\{|\hat{\sigma}^{2*}_{(\ell-1)h_n}| \le h_n^{\tau-1}\}}, \quad i = \alpha_n, \ldots, h_n^{-1} \,,$$

we derive the pseudo test statistic

$$\hat{V}_n^\dagger = \max_{i=\alpha_n,\ldots,h_n^{-1}-1} \left| \frac{\overline{RV}^*_{n,i} - \overline{RV}^*_{n,i+1}}{\sqrt{8\hat{\eta}}\, \left|\overline{RV}^*_{n,i+1}\right|^{3/4}} \right| .$$

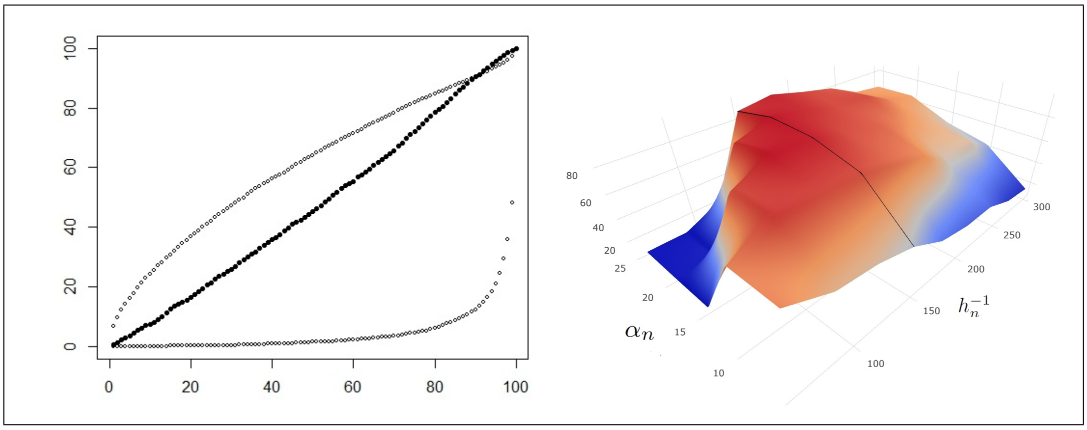

Figure 3.2.: Left: Empirical size and power of the new test for $h_n^{-1} = 120$ and $\alpha_n = 15$, $n = 30{,}000$, by comparing empirical percentiles to ones of limit law under H_0 and H_1 (light points). Empirical percentiles compared to bootstrapped percentiles under H_0 (dark points). Right: Percentage of exceedances under H_1 of the 90% empirical quantile under H_0 for $h_n^{-1} = 60, 90, 120, 150, 180, 210, 240, 270, 300$ and $\alpha_n = 5, 10, 15, 20, 25$. The line gives the marginal curve for $h_n^{-1} = 120$.

In fact, the truncation with the indicator function is obsolete, since we do not have jumps in the pseudo samples. For a test, we can use the approximative (conditional) quantiles

$$\hat{q}_\alpha(\hat{V}_n^\dagger|\mathcal{F}) = \inf\{x \geq 0 \, : \, \mathbb{P}(\hat{V}_n^\dagger \leq x|\mathcal{F}) \geq \alpha\}$$

and compute $\hat{q}_\alpha(\hat{V}_n^\dagger|\mathcal{F})$ based on Monte Carlo approximation. We reject H_0 when

$$\overline{V}_n^{ov,\tau} > \hat{q}_{1-\alpha}(\hat{V}_n^\dagger|\mathcal{F})\,.$$

In the left plot of Figure 3.2 the black dots compare the empirical percentiles of the left-hand side in (3.16b), the standardized versions of $\overline{V}_n^{ov,\tau}$, under H_0 to the ones of the bootstrap, i.e. $\hat{q}_\alpha(\hat{V}_n^\dagger|\mathcal{F})$. The finite-sample accuracy of the bootstrap for the distribution under H_0 is significantly better than the limit law (light points). Since the high percentiles of bootstrap and limit law are quite close, the power of both tests is comparable. For a level $\alpha = 10\%$ test, we obtain approx. 88% power using the limit law and 89% power using the bootstrap. For a level $\alpha = 5\%$ test, we obtain approx. 79% and 75%, respectively.
Finally, we consider different parameter configurations (h_n^{-1}, α_n). Since we can exploit the bootstrap to ensure a good fit under H_0, we concentrate on the ability of $\overline{V}_n^{ov,\tau}$ to distinguish hypothesis and alternative. To quantify the ability to separate H_0 and H_1, we visualize the relative number of exceedances under H_1 of the 90% empirical quantile under H_0. We plot the percentage numbers in the right plot of Figure 3.2 over a grid of different values for (h_n^{-1}, α_n). Additionally, we draw the marginal curve for $h_n^{-1} = 120$

at points $\alpha_n = 5, 10, 15, 20, 25$ (black line). Choosing a different (reasonable) quantile under H_0 does not change the shape of the surface with respect to the values of (h_n^{-1}, α_n). Figure 3.2 confirms that the test is reasonably robust with respect to different values of the tuning parameters. For h_n^{-1} sufficiently large, setting α_n between 15 and 20 yields the highest power. For h_n^{-1}, values between 60 and 180 grant a good performance. Hence, we choose $h_n^{-1} = 120$ and $\alpha_n = 15$ as suitable configuration for the simulation study.

Comparison of tests with overlapping and non-overlapping statistics

We illustrate the improvement in the power of the test based on (3.12b) compared to the non-overlapping version (3.12a). Here, we use a prominent general model for jumps of (X_t) often considered in related literature, including Jacod and Todorov [38], with a predictable compensator $\nu(ds, dz) = (\mathbb{1}_{\{z\in[-1,-0.2]\cup[0.2,1]\}})/1.6\,dt\,dz$. Since jumps of very small absolute sizes are not generated, the truncation works well and we do not see a manipulation of the empirical distribution of the test statistics due to errors in the truncation step. We investigate the power of the tests for different volatility-jump sizes under the alternative, $\Delta\sigma_\theta^2 = (10 + 5\cdot i)/100, i = 1, \ldots, 7$. The volatility-jump time θ is randomly generated in each run according to a uniform distribution on $(\alpha_n h_n, 1 - \alpha_n h_n)$. Note that not excluding the boundary intervals $[0, \alpha_n h_n] \cup [1 - \alpha_n h_n, 1]$ would slightly reduce the power in all configurations, since the test is not able to detect jumps in these boundary blocks. In order to include common price and volatility jumps, we add an additional price jump at θ with uniformly distributed size as according to ν above. We keep to the parameters $h_n^{-1} = 120$, $\alpha_n = 15$ and $\tau = 3/4$ and compute the adaptive statistics as in the previous paragraph in $10,000$ iterations.

Figure 3.3 confirms that the test using (3.12b) with overlapping statistics has a significantly higher power than the test based on (3.12a) and non-overlapping statistics. The largest difference for $\Delta\sigma_\theta^2 = 0.2$ is 17.8% at 10% testing level and for $\Delta\sigma_\theta^2 = 0.25$, 14.8% at 5% testing level. Thus, for volatility jumps with moderate absolute size in the range considered in Figure 3.3, the overlapping statistics attain relevant efficiency gains. The location of the volatility jump – when the boundaries are excluded – does not affect the power of the tests. Figure 3.3 illustrates increasing power of both tests as $\Delta\sigma_\theta^2$ gets larger. It also reveals that volatility jumps with $\Delta\sigma_\theta^2 \leq 0.15$ are difficult to detect in our setting where this corresponds only to approximately 20 times the average absolute increment $|\Delta_i^n Y|$. Due to the required smoothing over blocks we cannot expect to detect such small volatility jumps with good power. We can further report a better accuracy of the theoretical limit law under H_0 from Proposition 3.10 for the empirical distribution of the statistics with overlapping compared to non-overlapping blocks. The average amount of realizations of simulated statistics (3.12a) exceeding the theoretical 90%-percentile is 9.99% and exceeding the 95%-percentile 6.41%. For the statistics (3.12b) these values are 21.00% and 11.11%, respectively.

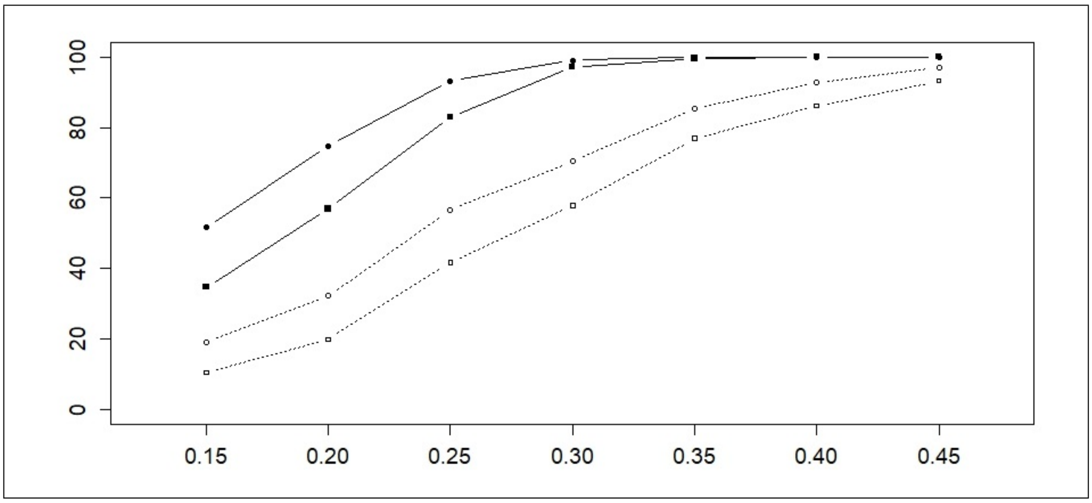

Figure 3.3.: Empirical power of the test based on (3.12b) with overlapping statistics (dark, solid) and (3.12a) with non-overlapping statistics (light, dashed) for the level 10% (points) and 5% (squares) under the alternative hypothesis as function of the volatility-jump size $\Delta\sigma_\theta^2$. The plot gives empirical percentiles exceeding the bootstrapped percentiles under the null hypothesis for $h_n^{-1} = 120$, $\alpha_n = 15$ and $n = 30,000$.

3.2.8. Proofs

Since the proofs of the results stated in Subsections 3.2.3 and 3.2.4 are quite long, we want to sketch the key ideas of the proof shortly.
Starting with the *continuous case*, for the results given in Theorem 3.8 and Corollary 3.9, the main ingredients are described as follows. In the *first step* we carry out the crucial approximation where we show that the error, replacing the true log-price increments of $(X_t)_{t\in[0,1]}$ by Brownian increments multiplied with a locally constant approximated volatility, is negligible. More precisely, we show that the spectral statistics $S_{jk}(Y)$ are adequately approximated through $\sigma_{\lfloor\alpha_n^{-1}(k-1)\rfloor\alpha_n h_n} S_{jk}(W) + S_{jk}(\varepsilon)$ with the volatility approximated constant over the big blocks. The analogues of $\overline{RV}_{n,i}$ after the approximation are denoted $\overline{Z}_{n,i}$, given in (3.24).
In the *second step*, we conduct a time shift with respect to the volatility in $\overline{Z}_{n,i+1}$ to approximate the volatility by the same constant in the differences $\overline{Z}_{n,i} - \overline{Z}_{n,i+1}$.
The *third step* is to replace the estimated asymptotic standard deviation in the denominator in (3.9) by its stochastic limit. The latter step is essentially completed by a Taylor expansion. Finally, we establish in a *fourth step* that the difference between the statistics using (3.6) with oracle weights and the statistics using (3.8) with adaptive weights is sufficiently small to extend the results to the feasible statistics.
The approximation steps combine Fourier analysis for the spectral estimation with methods from stochastic calculus. Disentangling the approximation errors of maxi-

mum statistics requires a deeper study than for linear statistics. After an appropriate decomposition of the terms, we frequently use Burkholder, Jensen, Rosenthal and Minkowski inequalities to derive upper bounds.
The final step is to apply strong invariance principles from Section 2.4 and to apply results from Sakhanenko [56] to conclude with Lemma 1 and Lemma 2, respectively, in Wu and Zhao [61]. Concerning the non-overlapping statistics we need Lemma 1, whereas the overlapping case needs the more involved limit result presented in Lemma 2 of Wu and Zhao [61].
In order to prove Proposition 3.10, we show that under the stated conditions the jump robust statistics provide the same limit as in the continuous case. That is, the jumps do not affect the limit at all. We decompose the additional error term by truncation in several terms of different structure which we prove to be asymptotically negligible under the mild conditions (3.15) on the jump activity and its interplay with the truncation and smoothing parameters. We use Doob's maximal submartingale inequality to bound one crucial remainder without imposing a more restrictive Lévy structural assumption as has been used in Bibinger et al. [10].

Proof of Theorem 3.8

For notational convenience we replace

$$\max_{i=0,\dots,\lfloor(\alpha_n h_n)^{-1}\rfloor-2} \quad \text{by} \quad \max_i ,$$

and

$$\min_{i=0,\dots,\lfloor(\alpha_n h_n)^{-1}\rfloor-2} \quad \text{by} \quad \min_i ,$$

respectively.
We also introduce the following notation, adapting the elements of the spectral statistics on each big block. Set

$$\Phi_{ij\ell}\left(t\right) = \Phi_j\left(t - \left(h_n(i\alpha_n + (\ell-1)\right)\right)\right)$$

and

$$\varphi_{ij\ell}\left(t\right) = \varphi_j\left(t - \left(h_n(i\alpha_n + (\ell-1)\right)\right)\right) \, .$$

Furthermore, we define the big block-wise spectral statistics

$$S_{ij\ell}\left(L\right) = \sum_{\nu=1}^{n} \Delta_i^n \Phi_{ij\ell}\left(\frac{\nu}{n}\right)$$

and the associated variance minimizing oracle weights

$$w_{ij\ell} = \frac{\left(\sigma^2_{h_n(i\alpha_n+(\ell-1))} + \frac{\eta^2}{n}\left[\varphi_{ij\ell},\varphi_{ij\ell}\right]_n\right)^{-2}}{\sum_{j=1}^{\lfloor nh_n\rfloor-1}\left(\sigma^2_{h_n(i\alpha_n+(\ell-1))} + \frac{\eta^2}{n}\left[\varphi_{ij\ell},\varphi_{ij\ell}\right]_n\right)^{-2}} \, .$$

We further introduce the bias correction terms

$$\mu_{ij\ell} = [\varphi_{ij\ell}, \varphi_{ij\ell}]_n \, \frac{\eta^2}{n} \, .$$

We can strengthen the assumptions presented in Assumption 3.1 and (H_0-$\mathfrak{a}$) as follows. We replace local boundedness of $(\sigma_t)_{t\in[0,1]}$, $(a_t)_{t\in[0,1]}$, and the modulus of continuity $(w_\delta(\sigma)_t)_{t\in[0,1]}$ under (H_0-$\mathfrak{a}$) by global boundedness. We refer to Section 4.4.1 of Jacod and Protter [35] for a proof and the construction through localization. We set

$$\overline{U}_n = \max_{i=0,\dots,\lfloor(\alpha_n h_n)^{-1}\rfloor-2} \left| \frac{\overline{Z}_{n,i} - \overline{Z}_{n,i+1}}{\left|\overline{Z}_{n,i+1}\right|^{3/4}} \right| ,$$

with

$$\overline{Z}_{n,i} := \frac{1}{\alpha_n} \sum_{\ell=1}^{\alpha_n} \sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{ij\ell} \Big(\left(\sigma_{i\alpha_n h_n} S_{ij\ell}(W) + S_{ij\ell}(\varepsilon)\right)^2 - \mu_{ij\ell} \Big) \, . \tag{3.24}$$

We fix some constants $K^+, K^- > 0$, such that almost surely

$$K^- < \inf_{t\in[0,1]} \sigma_t^2 \quad \text{and} \quad K^+ > \sup_{t\in[0,1]} \sigma_t^2 \, .$$

Finally, we will use an universal constant C which may change from line to line. We will write C_p to indicate that the constant depends on an external parameter p. The constant will never depend on n. The *first step* outlined in the sketch of the key ideas is accomplished in the next proposition.

Proposition 3.16. *Given the assumptions of Theorem 3.8, it holds under (H_0-$\mathfrak{a}$) that*

$$\sqrt{\alpha_n \log\left(h_n^{-1}\right)} \max_i \left| \left| \frac{\overline{RV}_{n,i} - \overline{RV}_{n,i+1}}{\left|\overline{RV}_{n,i+1}\right|^{3/4}} \right| - \left| \frac{\overline{Z}_{n,i} - \overline{Z}_{n,i+1}}{\left|\overline{Z}_{n,i+1}\right|^{3/4}} \right| \right| \xrightarrow{\mathbb{P}} 0 \, .$$

Proof of Proposition 3.16.

Since $h_n \propto n^{-1/2}\log(n)$ we can proceed as follows. The reverse triangle inequality and the decomposition

$$\begin{aligned} &\frac{\overline{RV}_{n,i} - \overline{RV}_{n,i+1}}{\left|\overline{RV}_{n,i+1}\right|^{3/4}} - \frac{\overline{Z}_{n,i} - \overline{Z}_{n,i+1}}{\left|\overline{Z}_{n,i+1}\right|^{3/4}} \\ &= \frac{\overline{RV}_{n,i}}{\left|\overline{RV}_{n,i+1}\right|^{3/4}} - \frac{\overline{RV}_{n,i}}{\left|\overline{Z}_{n,i+1}\right|^{3/4}} + \frac{\overline{RV}_{n,i} - \overline{Z}_{n,i}}{\left|\overline{Z}_{n,i+1}\right|^{3/4}} - \frac{\overline{RV}_{n,i+1}}{\left|\overline{RV}_{n,i+1}\right|^{3/4}} + \frac{\overline{Z}_{n,i+1} - \overline{RV}_{n,i+1}}{\left|\overline{Z}_{n,i+1}\right|^{3/4}} \end{aligned}$$

yield the following decomposition:

$$\begin{aligned} &\max_i \left| \overline{RV}_{n,i} \left(\frac{1}{\left|\overline{RV}_{n,i+1}\right|^{3/4}} - \frac{1}{\left|\overline{Z}_{n,i+1}\right|^{3/4}} \right) \right| + \max_i \left| \frac{\overline{RV}_{n,i} - \overline{Z}_{n,i}}{\left|\overline{Z}_{n,i+1}\right|^{3/4}} \right| \\ &\quad + \max_i \left| \overline{RV}_{n,i+1} \left(\frac{1}{\left|\overline{Z}_{n,i+1}\right|^{3/4}} - \frac{1}{\left|\overline{RV}_{n,i+1}\right|^{3/4}} \right) \right| + \max_i \left| \frac{\overline{Z}_{n,i+1} - \overline{RV}_{n,i+1}}{\left|\overline{Z}_{n,i+1}\right|^{3/4}} \right| \\ &=: (\mathbf{I}) + (\mathbf{II}) + (\mathbf{III}) + (\mathbf{IV}) \, . \end{aligned} \tag{3.25}$$

Starting with (**II**) in (3.25) we proceed as follows.
For all $\delta > 0$ and $\kappa^- > 0$, such that $\kappa^- \in (0, K^-)$, the following holds:

$$
\begin{aligned}
&\mathbb{P}\left[\max_i \left|\frac{\sqrt{\alpha_n \log(n)}\left(\overline{RV}_{n,i} - \overline{Z}_{n,i}\right)}{\left|\overline{Z}_{n,i+1}\right|^{3/4}}\right| > \delta\right] \\
&= \mathbb{P}\left[\max_i \left|\frac{\sqrt{\alpha_n \log(n)}\left(\overline{RV}_{n,i} - \overline{Z}_{n,i}\right)}{\left|\overline{Z}_{n,i+1}\right|^{3/4}}\right| > \delta,\ \min_i \left|\overline{Z}_{n,i+1}\right| \geq K^- - \kappa^-\right] \\
&+ \mathbb{P}\left[\max_i \left|\frac{\sqrt{\alpha_n \log(n)}\left(\overline{RV}_{n,i} - \overline{Z}_{n,i}\right)}{\left|\overline{Z}_{n,i+1}\right|^{3/4}}\right| > \delta,\ \min_i \left|\overline{Z}_{n,i+1}\right| < K^- - \kappa^-\right] \\
&\leq \mathbb{P}\left[\max_i \sqrt{\alpha_n \log(n)} \left|\overline{RV}_{n,i} - \overline{Z}_{n,i}\right| > \delta\left(K^- - \kappa^-\right)^{3/4}\right] \\
&\qquad\qquad + \mathbb{P}\left[\min_i \left|\overline{Z}_{n,i+1}\right| < K^- - \kappa^-\right] \\
&=: A_n + B_n. \qquad (3.26)
\end{aligned}
$$

In (3.26) we dropped the dependence on the constants δ and K^- for notational convenience. We start with the term A_n. We split the term into various summands in the following way:

$$
\begin{aligned}
\overline{RV}_{n,i} - \overline{Z}_{n,i} =& \frac{1}{\alpha_n} \sum_{\ell=1}^{\alpha_n} \sum_{j=1}^{\lfloor nh_n \rfloor - 1} w_{ij\ell}\left(S_{ij\ell}^2(X) - \sigma_{i\alpha_n h_n}^2 S_{ij\ell}^2(W)\right) \\
&+ \frac{2}{\alpha_n} \sum_{\ell=1}^{\alpha_n} \sum_{j=1}^{\lfloor nh_n \rfloor - 1} w_{ij\ell} S_{ij\ell}(\varepsilon)\left(S_{ij\ell}(X) - \sigma_{i\alpha_n h_n} S_{ij\ell}(W)\right).
\end{aligned}
$$

That yields

$$
\begin{aligned}
A_n \leq \mathbb{P}\Bigg[&\max_i \Bigg|\frac{\sqrt{\log(n)}}{\sqrt{\alpha_n}} \sum_{\ell=1}^{\alpha_n} \sum_{j=1}^{\lfloor nh_n \rfloor - 1} w_{ij\ell} \\
&\qquad \times \left(S_{ij\ell}^2(X) - \sigma_{i\alpha_n h_n}^2 S_{ij\ell}^2(W)\right)\Bigg| > \frac{\delta\left(K^- - \kappa^-\right)^{3/4}}{2}\Bigg] \\
+\mathbb{P}\Bigg[&\max_i \Bigg|\frac{\sqrt{\log(n)}}{\sqrt{\alpha_n}} \sum_{\ell=1}^{\alpha_n} \sum_{j=1}^{\lfloor nh_n \rfloor - 1} w_{ij\ell} S_{ij\ell}(\varepsilon) \\
&\qquad \times \left(S_{ij\ell}(X) - \sigma_{i\alpha_n h_n} S_{ij\ell}(W)\right)\Bigg| > \frac{\delta\left(K^- - \kappa^-\right)^{3/4}}{4}\Bigg] \\
=:&\ A_n^1 + A_n^2. \qquad (3.27)
\end{aligned}
$$

In order to handle $A^1_{n,}$, we rewrite the spectral statistics $S_{ij\ell}(L)$ for any stochastic process $(L_t)_{t\in[0,1]}$, using step functions $\xi^{(n)}_{ij\ell}$ given by

$$\xi^{(n)}_{ij\ell}(t) := \sum_{\nu=1}^{n} \Phi_{ij\ell}\left(\frac{\nu}{n}\right) \mathbb{1}_{\left(\frac{\nu-1}{n},\frac{\nu}{n}\right]}(t)$$

which yield

$$S_{ij\ell}(L) = \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n} \xi^{(n)}_{ij\ell}(s)\, dL_s$$

for any semimartingale $L = (L_t)_{t\in[0,1]}$.
By virtue of the Itô process structure of (X_t), we obtain that

$$\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n} \xi^{(n)}_{ij\ell}(s)\, dX_s = \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n} \xi^{(n)}_{ij\ell}(s)\, a_s\, ds + \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n} \xi^{(n)}_{ij\ell}(s)\, \sigma_s\, dW_s\,.$$

Itô's formula yields

$$\begin{aligned} S^2_{ij\ell}(X) =& 2\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n} \left(\widetilde{X}_\tau - \widetilde{X}_{h_n(i\alpha_n+(\ell-1))-n^{-1}}\right) \xi^{(n)}_{ij\ell}(\tau)\, \sigma_\tau\, dW_\tau \\ &+ 2\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n} \left(\widetilde{X}_\tau - \widetilde{X}_{h_n(i\alpha_n+(\ell-1))-n^{-1}}\right) \xi^{(n)}_{ij\ell}(\tau)\, a_\tau\, d\tau \\ &+ \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n} \left(\xi^{(n)}_{ij\ell}(\tau)\right)^2 \sigma^2_\tau\, d\tau \end{aligned}$$

with

$$\widetilde{X}_t := X_0 + \int_0^t \xi^{(n)}_{ij\ell}(s)\, a_s\, ds + \int_0^t \xi^{(n)}_{ij\ell}(s)\, \sigma_s\, dW_s.$$

Similarly,

$$S^2_{ij\ell}(W) = 2\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n} \widetilde{W}_\tau\, \xi^{(n)}_{ij\ell}(\tau)\, dW_\tau + \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n} \left(\xi^{(n)}_{ij\ell}(\tau)\right)^2 d\tau$$

with

$$\widetilde{W}_t := \int_0^t \xi^{(n)}_{ij\ell}(s)\, dW_s\,.$$

For notational brevity, we suppress the dependence of $\widetilde{X}_t$ and $\widetilde{W}_t$, respectively on (i, j, ℓ, n). We bound A^1_n via

$$A^1_n \le A^{1,1}_n + A^{1,2}_n + A^{1,3}_n\,,$$

where

$$A_n^{1,1} = \mathbb{P}\Bigg[\max_i \Bigg|\frac{\sqrt{\log(n)}}{\sqrt{\alpha_n}}\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell} \times \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n}\Big(\widetilde{X}_\tau - \widetilde{X}_{h_n(i\alpha_n+(\ell-1))-n^{-1}}\Big)\xi_{ij\ell}^{(n)}(\tau)\,a_\tau\,d\tau\Bigg| > \frac{\delta\left(K^{-}-\kappa^{-}\right)^{3/4}}{12}\Bigg],$$

$$A_n^{1,2} = \mathbb{P}\Bigg[\max_i \Bigg|\frac{\sqrt{\log(n)}}{\sqrt{\alpha_n}}\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n}\Big(\xi_{ij\ell}^{(n)}(\tau)\Big)^2 \times \left(\sigma_\tau^2-\sigma_{i\alpha_n h_n}^2\right)\,d\tau\Bigg| > \frac{\delta\left(K^{-}-\kappa^{-}\right)^{3/4}}{6}\Bigg],$$

and

$$A_n^{1,3} = \mathbb{P}\Bigg[\max_i \Bigg|\frac{\sqrt{\log(n)}}{\sqrt{\alpha_n}}\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n}\xi_{ij\ell}^{(n)}(\tau) \times \Big(\Big(\widetilde{X}_\tau - \widetilde{X}_{h_n(i\alpha_n+(\ell-1))-n^{-1}}\Big)\sigma_\tau - \sigma_{i\alpha_n h_n}^2\widetilde{W}_\tau\Big)\,dW_\tau\Bigg| > \frac{\delta\left(K^{-}-\kappa^{-}\right)^{3/4}}{12}\Bigg].$$

Starting with $A_n^{1,1}$ we employ Markov's inequality, applied to the function $z \mapsto |z|^r$, $r > 0$ and $r \in \mathbb{N}$:

$$\mathbb{P}\Bigg[\Bigg|\frac{\sqrt{\log(n)}}{\sqrt{\alpha_n}}\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n}\Big(\widetilde{X}_\tau - \widetilde{X}_{h_n(i\alpha_n+(\ell-1))-n^{-1}}\Big) \times \xi_{ij\ell}^{(n)}(\tau)\,a_\tau\,d\tau\Bigg| > \frac{\delta(K^{-}-\kappa^{-})^{3/4}}{12}\Bigg]$$

$$\le C_r\,(\log(n))^{r/2}\,\alpha_n^{-r/2} \times \mathbb{E}\Bigg[\Bigg|\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n}\Big(\widetilde{X}_\tau - \widetilde{X}_{h_n(i\alpha_n+(\ell-1))-n^{-1}}\Big)\xi_{ij\ell}^{(n)}(\tau)\,a_\tau\,d\tau\Bigg|^r\Bigg].$$

The identity

$$\mathbb{E}\Bigg[\Bigg|\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n}\Big(\widetilde{X}_\tau - \widetilde{X}_{h_n(i\alpha_n+(\ell-1))-n^{-1}}\Big)\xi_{ij\ell}^{(n)}(\tau)\,a_\tau\,d\tau\Bigg|^r\Bigg]$$

$$= \alpha_n^r\,\mathbb{E}\Bigg[\Bigg|\sum_{\ell=1}^{\alpha_n}\frac{1}{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell} \times \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n}\Big(\widetilde{X}_\tau - \widetilde{X}_{h_n(i\alpha_n+(\ell-1))-n^{-1}}\Big)\xi_{ij\ell}^{(n)}(\tau)\,a_\tau\,d\tau\Bigg|^r\Bigg]$$

implies, together with Jensen's inequality, that

$$\begin{aligned}
&\alpha_n^{r/2}\mathbb{E}\Bigg[\Bigg|\sum_{\ell=1}^{\alpha_n}\frac{1}{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell} \\
&\qquad\qquad\times\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n}\Big(\widetilde{X}_\tau-\widetilde{X}_{h_n(i\alpha_n+(\ell-1))-n^{-1}}\Big)\,\xi_{ij\ell}^{(n)}(\tau)\,a_\tau\,d\tau\Bigg|^r\Bigg] \\
&\le \alpha_n^{r/2-1}\sum_{\ell=1}^{\alpha_n}\mathbb{E}\Bigg[\Bigg|\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell} \\
&\qquad\qquad\times\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n}\Big(\widetilde{X}_\tau-\widetilde{X}_{h_n(i\alpha_n+(\ell-1))-n^{-1}}\Big)\,\xi_{ij\ell}^{(n)}(\tau)\,a_\tau\,d\tau\Bigg|^r\Bigg] \\
&\le \alpha_n^{r/2-1}\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell} \\
&\qquad\qquad\times\mathbb{E}\Bigg[\Bigg|\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n}\Big(\widetilde{X}_\tau-\widetilde{X}_{h_n(i\alpha_n+(\ell-1))-n^{-1}}\Big)\,\xi_{ij\ell}^{(n)}(\tau)\,a_\tau\,d\tau\Bigg|^r\Bigg].
\end{aligned}$$

Concerning the second inequality we have taken into account that $\sum_j w_{ij\ell}=1$, in order to apply Jensen's inequality a second time.
We employ the generalized Minkowski inequality for double measure integrals, which implies

$$\begin{aligned}
&\mathbb{E}\Bigg[\Bigg|\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n}\Big(\widetilde{X}_\tau-\widetilde{X}_{h_n(i\alpha_n+(\ell-1))-n^{-1}}\Big)\,\xi_{ij\ell}^{(n)}(\tau)\,a_\tau\,d\tau\Bigg|^r\Bigg] \\
&\le\Bigg(\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n}\mathbb{E}\Bigg[\Bigg|\Big(\widetilde{X}_\tau-\widetilde{X}_{h_n(i\alpha_n+(\ell-1))-n^{-1}}\Big)\,\xi_{ij\ell}^{(n)}(\tau)\,a_\tau\Bigg|^r\Bigg]^{1/r}d\tau\Bigg)^r.
\end{aligned}\tag{3.28}$$

In order to bound the expectation in (3.28), we apply Burkolder's inequality to the local martingale part. The general case can be handled via the elementary inequality $|a+b|^p\le 2^p\left(|a|^p+|b|^p\right)$ and the standard bound for Lebesgue integrals

$$\int_\Omega f(s)\,d\mu(s)\le\mu(\Omega)\sup_s|f(s)|\;,\tag{3.29}$$

applied to the finite variation part. Taking into account that the quadratic variation process, $\big([\widetilde{X},\widetilde{X}]_t\big)_{t\in[0,1]}$, is given by

$$[\widetilde{X},\widetilde{X}]_t=\int_0^t(\xi_{ij\ell}^{(n)}(s))^2\sigma_s^2\,ds$$

yields

$$\begin{aligned}\mathbb{E}\left[\left|\widetilde{X}_\tau - \widetilde{X}_{h_n(i\alpha_n+(\ell-1))-n^{-1}}\right|^r\right] \leq & C_r\mathbb{E}\left[\left(\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau} (\xi_{ij\ell}^{(n)}(s))^2\sigma_s^2\, ds\right)^{r/2}\right] \\ \leq & C_r\mathbb{E}\left[\left(\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n} (\xi_{ij\ell}^{(n)}(s))^2\sigma_s^2\, ds\right)^{r/2}\right] \\ \leq & C_r h_n^{r/2} h_n^{-r/2} = \mathcal{O}(1)\,. \end{aligned} \tag{3.30}$$

(3.30) is a consequence of (3.29),

$$\xi_{ij\ell}^{(n)}(x) = \mathcal{O}\left(\frac{1}{\sqrt{h_n}}\right)\,, \tag{3.31}$$

and the global boundedness of $\left(\sigma_t^2\right)_{t\in[0,1]}$. Consequently, the above yields

$$\begin{aligned}&\frac{(\log(n))^{r/2}}{\alpha_n^{1-r/2}}\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell} \\ &\times \mathbb{E}\left[\left|\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n}\left(\widetilde{X}_\tau - \widetilde{X}_{h_n(i\alpha_n+(\ell-1))-n^{-1}}\right)\xi_{ij\ell}^{(n)}(\tau)\, a_\tau\, d\tau\right|^r\right] \\ &\leq (\log(n))^{r/2}\,\alpha_n^{r/2-1}\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell} h_n^r h_n^{-r/2} = \mathcal{O}\left(\log(n)^{r/2}\,\alpha_n^{r/2} h_n^{r/2}\right)\,.\end{aligned}$$

Taking into account that

$$\alpha_n h_n = \mathcal{O}\left(n^{-\frac{1}{4\mathfrak{a}+2}}\,(\log(n))^{1-\frac{2\mathfrak{a}}{2\mathfrak{a}+1}}\right)\,,$$

we can conclude, if $r > 2$, that

$$A_n^{1,1} = \mathcal{O}\left((\alpha_n h_n)^{-1}\log(n)^{r/2}\,\alpha_n^{r/2} h_n^{r/2}\right) = \mathcal{o}\,(1)\ ,\ \text{as } n\to\infty\,.$$

For the term $A_n^{1,2}$, we start with

$$\begin{aligned}&\frac{\sqrt{\log(n)}}{\sqrt{\alpha_n}}\left|\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n}(\xi_{ij\ell}^{(n)}(\tau))^2\left(\sigma_\tau^2-\sigma_{i\alpha_n h_n}^2\right)\, d\tau\right| \\ &\leq \frac{\sqrt{\log(n)}}{\sqrt{\alpha_n}}\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n}(\xi_{ij\ell}^{(n)}(\tau))^2\left|\sigma_\tau^2-\sigma_{i\alpha_n h_n}^2\right|\, d\tau\,.\end{aligned} \tag{3.32}$$

In (3.32) the triangle inequality and Jensen's inequality are applied. Combining the regularity under (H_0-$\mathfrak{a}$) and (3.5) gives

$$\begin{aligned}
&\frac{\sqrt{\log(n)}}{\sqrt{\alpha_n}} \sum_{\ell=1}^{\alpha_n} \sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{ij\ell} \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n} (\xi_{ij\ell}^{(n)}(\tau))^2 \left|\sigma_\tau^2 - \sigma_{i\alpha_n h_n}^2\right| d\tau \\
&= \mathcal{O}_{\mathbb{P}}\left(\sqrt{\log(n)}\sqrt{\alpha_n}\,(\alpha_n h_n)^{\mathfrak{a}}\right), \text{ uniformly in } i \\
&= \scriptstyle{\mathcal{O}}\displaystyle(1)\,, \text{ as } n \to \infty.
\end{aligned}$$

Concerning $A_n^{1,3}$ we use further decompositions rewriting

$$\left(\widetilde{X}_\tau - \widetilde{X}_{h_n(i\alpha_n+(\ell-1))-n^{-1}}\right)\sigma_\tau - \widetilde{W}_\tau \sigma_{i\alpha_n h_n}^2 \tag{3.33}$$

in the following way:

$$\begin{aligned}
&\left(\widetilde{X}_\tau - \widetilde{X}_{h_n(i\alpha_n+(\ell-1))-n^{-1}}\right)\sigma_\tau - \widetilde{W}_\tau \sigma_{i\alpha_n h_n}^2 \\
&= \sigma_\tau \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau} \xi_{ij\ell}^{(n)}(s)\, a_s\, ds \\
&+ \sigma_\tau \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau} \xi_{ij\ell}^{(n)}(s)\, \sigma_s\, dW_s \\
&\qquad - \sigma_{i\alpha_n h_n} \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau} \xi_{ij\ell}^{(n)}(s)\, \sigma_s\, dW_s \\
&+ \sigma_{i\alpha_n h_n} \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau} \xi_{ij\ell}^{(n)}(s)\, \sigma_s\, dW_s \\
&\qquad - \sigma_{i\alpha_n h_n} \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau} \xi_{ij\ell}^{(n)}(s)\, \sigma_{i\alpha_n h_n}\, dW_s \\
&= \sigma_\tau \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau} \xi_{ij\ell}^{(n)}(s)\, a_s\, ds \qquad (3.34) \\
&+ \left(\sigma_\tau - \sigma_{i\alpha_n h_n}\right) \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau} \xi_{ij\ell}^{(n)}(s)\, \sigma_s\, dW_s \qquad (3.35) \\
&+ \sigma_{i\alpha_n h_n} \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau} \xi_{ij\ell}^{(n)}(s) \left(\sigma_s - \sigma_{i\alpha_n h_n}\right) dW_s\,. \qquad (3.36)
\end{aligned}$$

Using this decomposition, we can bound $A_n^{1,3}$ via

$$A_n^{1,3} \leq A_n^{1,3,1} + A_n^{1,3,2} + A_n^{1,3,3}\,.$$

We start with the probability involving the summand (3.34). We have to bound the probability

$$\mathbb{P}\Bigg[\Bigg|\frac{\sqrt{\log(n)}}{\sqrt{\alpha_n}}\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1}\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n} w_{ij\ell}\xi_{ij\ell}^{(n)}(\tau)\,\sigma_\tau \times\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau}\xi_{ij\ell}^{(n)}(s)\,a_s\,ds\,dW_\tau\Bigg| > \frac{\delta\left(K^- - \kappa^-\right)^{3/4}}{36}\Bigg]$$
$$\leq C_r\left(\frac{\sqrt{\log(n)}}{\sqrt{\alpha_n}}\right)^r\mathbb{E}\Bigg[\Bigg|\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n}\xi_{ij\ell}^{(n)}(\tau) \times\sigma_\tau\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau}\xi_{ij\ell}^{(n)}(s)\,a_s\,ds\,dW_\tau\Bigg|^r\Bigg],$$

where we have applied Markov's inequality with some exponent $r > 0$ and $r \in \mathbb{N}$. Set

$$c_n^{i,1}(\tau) = \sum_{\ell=1}^{\alpha_n}\sigma_\tau\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\xi_{ij\ell}^{(n)}(\tau)\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau}\xi_{ij\ell}^{(n)}(s)\,a_s\,ds \times\mathbb{1}_{(h_n(i\alpha_n+(\ell-1))-n^{-1},i\alpha_n h_n+\ell h_n]}(\tau)\,.$$

In order to apply Itô isometry, we set $r = 2m$, with some $m > 0$ and $m \in \mathbb{N}$. We derive that

$$\mathbb{E}\Bigg[\Bigg|\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n}\xi_{ij\ell}^{(n)}(\tau)\,\sigma_\tau\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau}\xi_{ij\ell}^{(n)}(s)\,a_s ds dW_\tau\Bigg|^r\Bigg]$$
$$= \mathbb{E}\Bigg[\Bigg(\int_{i\alpha_n h_n}^{(i+1)\alpha_n h_n}\left(c_n^{i,1}(\tau)\right)^2 d\tau\Bigg)^m\Bigg] \leq \Bigg(\int_{i\alpha_n h_n}^{(i+1)\alpha_n h_n}\mathbb{E}\big[(c_n^{i,1}(\tau))^{2m}\big]^{1/m}\,d\tau\Bigg)^m,$$

where we have again used the Minkowski inequality for double measure integrals. Since $(i\alpha_n h_n + (\ell_1 - 1)\,h_n, i\alpha_n h_n + \ell_1 h_n]$ and $(\alpha_n h_n + (\ell_2 - 1)\,h_n, i\alpha_n h_n + \ell_2 h_n]$ are disjoint, if $\ell_1 \neq \ell_2$ and τ is fixed, we get

$$\mathbb{E}\big[(c_n^{i,1}(\tau))^{2m}\big] = \sum_{\ell=1}^{\alpha_n}\mathbb{E}\Bigg[\sigma_\tau^{2m}\Bigg(\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\xi_{ij\ell}^{(n)}(\tau)\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau}\xi_{ij\ell}^{(n)}(s)\,a_s\,ds\Bigg)^{2m}\Bigg] \times\mathbb{1}_{(h_n(i\alpha_n+(\ell-1))-n^{-1},i\alpha_n h_n+\ell h_n]}(\tau)\,.$$

We proceed with Jensen's inequality, which yields

$$\sum_{\ell=1}^{\alpha_n}\mathbb{E}\Bigg[\sigma_\tau^{2m}\Bigg(\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\xi_{ij\ell}^{(n)}(\tau)\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau}\xi_{ij\ell}^{(n)}(s)\,a_s\,ds\Bigg)^{2m}$$
$$\times\,\mathbb{1}_{(h_n(i\alpha_n+(\ell-1))-n^{-1},i\alpha_n h_n+\ell h_n]}(\tau)\Bigg]$$
$$\leq\sum_{\ell=1}^{\alpha_n}\mathbb{E}\Bigg[\sigma_\tau^{2m}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\Bigg(\xi_{ij\ell}^{(n)}(\tau)\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau}\xi_{ij\ell}^{(n)}(s)\,a_s\,ds\Bigg)^{2m}$$
$$\times\,\mathbb{1}_{(h_n(i\alpha_n+(\ell-1))-n^{-1},i\alpha_n h_n+\ell h_n]}(\tau)\Bigg]\,.$$

Using (3.31), global boundedness of the volatility and (3.30) we can conclude that

$$\mathbb{E}\Big[\left(c_n^{i,1}(\tau)\right)^{2m}\Big]=\mathcal{O}(1)\,.$$

Consequently, we can conclude as follows using (3.30):

$$\left(\int_{i\alpha_n h_n}^{(i+1)\alpha_n h_n}\mathbb{E}\left[\left(c_n^{i,1}(\tau)\right)^{2m}\right]^{1/m}d\tau\right)^m=\mathcal{O}\left((\alpha_n h_n)^m\right)\,. \tag{3.37}$$

That yields the following bound for $A_n^{1,3,1}$

$$A_n^{1,3,1}=\mathcal{O}\left((\log(n))^m\,h_n^m\,(\alpha_n h_n)^{-1}\right)=\mathcal{o}(1)\,,\text{ as } n\to\infty\,,$$

for m sufficiently large.
We proceed with the probability $A_n^{1,3,2}$ involving the term (3.35). We first get the standard bound by the Markov inequality with some exponent $r>0$ and $r\in\mathbb{N}$:

$$\mathbb{P}\Bigg[\frac{\sqrt{\log(n)}}{\sqrt{\alpha_n}}\Bigg|\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1}w_{ij\ell}\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n}\xi_{ij\ell}^{(n)}(\tau)\left(\sigma_\tau-\sigma_{i\alpha_n h_n}\right)$$
$$\times\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau}\xi_{ij\ell}^{(n)}(s)\,\sigma_s\,dW_s dW_\tau\Bigg|>\frac{\delta\left(K^{-}-\kappa^{-}\right)^{3/4}}{36}\Bigg]$$
$$\leq C_r\left(\frac{\sqrt{\log(n)}}{\sqrt{\alpha_n}}\right)^r\mathbb{E}\Bigg[\Bigg|\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1}w_{ij\ell}\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n}\xi_{ij\ell}^{(n)}(\tau)$$
$$\times\left(\sigma_\tau-\sigma_{i\alpha_n h_n}\right)\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau}\xi_{ij\ell}^{(n)}(s)\,\sigma_s\,dW_s dW_\tau\Bigg|^r\Bigg]\,.$$

We define

$$c_n^{i,2}(\tau)=\sum_{\ell=1}^{\alpha_n}\left(\sigma_\tau-\sigma_{i\alpha_n h_n}\right)\sum_{j=1}^{\lfloor nh_n\rfloor-1}w_{ij\ell}\xi_{ij\ell}^{(n)}(\tau)\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau}\xi_{ij\ell}^{(n)}(s)\,\sigma_s\,dW_s$$
$$\times\mathbb{1}_{(h_n(i\alpha_n+(\ell-1))-n^{-1},i\alpha_n h_n+\ell h_n]}(\tau)\,.$$

In order to apply Itô isometry, we set $r = 2m$, with $m > 0$ and $m \in \mathbb{N}$. We obtain that

$$\mathbb{E}\Big[\Big|\int_{i\alpha_n h_n}^{(i+1)\alpha_n h_n} c_n^{i,2}(\tau)\, dW_\tau\Big|^{2m}\Big] = \mathbb{E}\Big[\Big(\int_{i\alpha_n h_n}^{(i+1)\alpha_n h_n} \big(c_n^{i,2}(\tau)\big)^2\, d\tau\Big)^m\Big] \leq \Big(\int_{i\alpha_n h_n}^{(i+1)\alpha_n h_n} \mathbb{E}\Big[\big(c_n^{i,2}(\tau)\big)^{2m}\Big]^{1/m} d\tau\Big)^m .$$

We have via Jensen's inequality

$$\begin{aligned}
&\mathbb{E}\Big[\big(c_n^{i,2}(\tau)\big)^{2m}\Big] \\
&= \mathbb{E}\Big[\sum_{\ell=1}^{\alpha_n} (\sigma_\tau - \sigma_{i\alpha_n h_n})^{2m} \Big(\sum_{j=1}^{\lfloor nh_n\rfloor -1} w_{ij\ell}\xi_{ij\ell}^{(n)}(s) \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau} \xi_{ij\ell}^{(n)}(s)\,\sigma_s\, dW_s\Big)^{2m} \\
&\qquad \times \mathbb{1}_{(h_n(i\alpha_n+(\ell-1))-n^{-1}, i\alpha_n h_n+\ell h_n]}(\tau)\Big] \\
&\leq \mathbb{E}\Big[\sum_{\ell=1}^{\alpha_n} (\alpha_n h_n)^{2m\mathfrak{a}} \sum_{j=1}^{\lfloor nh_n\rfloor -1} w_{ij\ell}\big(\xi_{ij\ell}^{(n)}(\tau)\big)^{2m} \Big(\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau} \xi_{ij\ell}^{(n)}(s)^2\,\sigma_s^2\, ds\Big)^{m} \\
&\qquad \times \mathbb{1}_{(h_n(i\alpha_n+(\ell-1))-n^{-1}, i\alpha_n h_n+\ell h_n]}(\tau)\Big] \\
&= \mathcal{O}\Big((\alpha_n h_n)^{2m\mathfrak{a}}\, h_n^{-m}\Big) ,
\end{aligned}$$

by the regularity under (H_0-$\mathfrak{a}$). Overall we can deduce for $A_n^{1,3,2}$ that

$$A_n^{1,3,2} = \mathcal{O}\Big((\alpha_n h_n)^{-1}(\alpha_n h_n)^{2m\mathfrak{a}}(\log(n))^m\Big) = \mathcal{o}(1)\ , \text{ as } n\to\infty ,$$

if $m \in \mathbb{N}$ sufficiently large. Proceeding with $A_n^{1,3,3}$, we have with $r > 0$ and $r \in \mathbb{N}$:

$$\begin{aligned}
&\mathbb{P}\Big[\Big|\frac{\sqrt{\log(n)}}{\sqrt{\alpha_n}} \sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor -1} w_{ij\ell} \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n} \xi_{ij\ell}^{(n)}(\tau)\,\sigma_{i\alpha_n h_n} \\
&\qquad \times \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n} \xi_{ij\ell}^{(n)}(s)\,(\sigma_s - \sigma_{i\alpha_n h_n})\, dW_s dW_\tau\Big| > \frac{\delta\,(K^- - \kappa^-)^{3/4}}{36}\Big] \qquad (3.38) \\
&\leq C_r\Big(\frac{\sqrt{\log(n)}}{\sqrt{\alpha_n}}\Big)^r \mathbb{E}\Big[\Big|\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor -1} w_{ij\ell} \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n} \xi_{ij\ell}^{(n)}(\tau)\,\sigma_{i\alpha_n h_n} \\
&\qquad \times \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n} \xi_{ij\ell}^{(n)}(s)\,(\sigma_s - \sigma_{i\alpha_n h_n})\, dW_s dW_\tau\Big|^r\Big].
\end{aligned}$$

Analogously, we set

$$c_n^{i,3}(\tau) := \sum_{\ell=1}^{\alpha_n} \sigma_{i\alpha_n h_n} \sum_{j=1}^{\lfloor nh_n\rfloor -1} w_{ij\ell}\xi_{ij\ell}^{(n)}(\tau) \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau} \xi_{ij\ell}^{(n)}(s)\,(\sigma_s - \sigma_{i\alpha_n h_n})\, dW_s \times \mathbb{1}_{(h_n(i\alpha_n+(\ell-1))-n^{-1}, i\alpha_n h_n+\ell h_n]}(\tau) .$$

With $r = 2m$, $r > 0$ and $r \in \mathbb{N}$ we apply Itô isometry and Minkowski inequality.

$$\mathbb{E}\Big[\Big(\int_{i\alpha_n h_n}^{(i+1)\alpha_n h_n} c_n^{i,3}(\tau)\, dW_\tau\Big)^{2m}\Big] = \mathbb{E}\Big[\Big(\int_{i\alpha_n h_n}^{(i+1)\alpha_n h_n} \left(c_n^{i,3}(\tau)\right)^2\, d\tau\Big)^{m}\Big]$$
$$\leq \left(\int_{i\alpha_n h_n}^{(i+1)\alpha_n h_n} \mathbb{E}\left[\left(c_n^{i,3}(\tau)\right)^{2m}\right]^{1/m} d\tau\right)^m .$$

Since $(i\alpha_n h_n + (\ell_1 - 1)\, h_n, i\alpha_n h_n + \ell_1 h_n]$ and $(i\alpha_n h_n + (\ell_2 - 1)\, h_n, i\alpha_n h_n + \ell_2 h_n]$ are disjoint if $\ell_1 \neq \ell_2$ and τ is fixed, we get

$$\mathbb{E}\left[\left(c_n^{i,3}(\tau)\right)^{2m}\right]$$
$$= \mathbb{E}\Big[\sum_{\ell=1}^{\alpha_n} \sigma_{i\alpha_n h_n}^{2m}\Big(\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\xi_{ij\ell}^{(n)}(\tau)\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau} \xi_{ij\ell}^{(n)}(s)\,(\sigma_s - \sigma_{i\alpha_n h_n})\, dW_s\Big)^{2m}$$
$$\times \mathbb{1}_{(h_n(i\alpha_n+(\ell-1))-n^{-1}, i\alpha_n h_n+\ell h_n]}(\tau)\Big]$$
$$\leq \mathbb{E}\Big[\sum_{\ell=1}^{\alpha_n} \sigma_{i\alpha_n h_n}^{2m} \sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}(\xi_{ij\ell}^{(n)}(\tau))^{2m}$$
$$\times \Big(\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau} \xi_{ij\ell}^{(n)}(s)\,(\sigma_s - \sigma_{i\alpha_n h_n})\, dW_s\Big)^{2m}$$
$$\times \mathbb{1}_{(h_n(i\alpha_n+(\ell-1))-n^{-1}, i\alpha_n h_n+\ell h_n]}(\tau)\Big] ,$$

where we applied Jensen's inequality. Proceeding with Burkholder's inequality and (3.29), we get

$$\mathbb{E}\Big[\sum_{\ell=1}^{\alpha_n} \sigma_{i\alpha_n h_n}^{2m} \sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}(\xi_{ij\ell}^{(n)}(\tau))^{2m}\Big(\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau} \xi_{ij\ell}^{(n)}(s)\,(\sigma_s - \sigma_{i\alpha_n h_n})\, dW_s\Big)^{2m}$$
$$\times \mathbb{1}_{(h_n(i\alpha_n+(\ell-1))-n^{-1}, i\alpha_n h_n+\ell h_n]}(\tau)\Big]$$
$$\leq K_m \mathbb{E}\Big[\sum_{\ell=1}^{\alpha_n} \sigma_{i\alpha_n h_n}^{2m} \sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}(\xi_{ij\ell}^{(n)}(\tau))^{2m}$$
$$\times \Big(\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau} \left(\xi_{ij\ell}^{(n)}(s)\right)^2 (\sigma_s - \sigma_{i\alpha_n h_n})^2\, ds\Big)^{m}$$
$$\times \mathbb{1}_{(h_n(i\alpha_n+(\ell-1))-n^{-1}, i\alpha_n h_n+\ell h_n]}(\tau)\Big]$$
$$= \mathcal{O}\left((\alpha_n h_n)^{2m\mathfrak{a}}\, h_n^{-m}\right) ,$$

which gives the following bound concerning $A_n^{1,3,3}$:

$$A_n^{1,3,3} = \mathcal{O}\left((\alpha_n h_n)^{-1}(\log(n))^m (\alpha_n h_n)^{2m\mathfrak{a}}\right) = \mathcal{o}(1) \ , \ \text{as } n \to \infty ,$$

if m is sufficiently large. We have completed the third term $A_n^{1,3,3}$ and so $A_n^{1,3}$. Overall the term A_n^1 has shown to be negligible. We proceed with A_n^2 from (3.27). Therefore, we take into account that

$$\begin{aligned} &S_{ij\ell}(X) - \sigma_{i\alpha_n h_n} S_{ij\ell}(W) \\ &= \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n} \xi_{ij\ell}^{(n)}(s)\,dX_s - \sigma_{i\alpha_n h_n} \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n} \xi_{ij\ell}^{(n)}(s)\,dW_s \\ &= \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n} \xi_{ij\ell}^{(n)}(s)\,(\sigma_s - \sigma_{i\alpha_n h_n})\,dW_s + \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n} \xi_{ij\ell}^{(n)}(s) a_s\,ds\,. \end{aligned} \tag{3.39}$$

Using this identity, we bound A_n^2 by

$$A_n^2 \le A_n^{2,1} + A_n^{2,2}\,,$$

where the probability $A_n^{2,1}$ is based on the local martingale part in (3.39) and $A_n^{2,2}$ is based on the finite variation part. The elementary inequality $|a+b|^p \le 2^p\left(|a|^p + |b|^p\right)$ allows to split the discussion of A_n^2. Starting with $A_n^{2,1}$ we proceed as follows using Markov's inequality with an exponent $r > 0$ and $r \in \mathbb{N}$.

$$\begin{aligned} &\mathbb{P}\left[\left|\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell} S_{ij\ell}(\varepsilon) \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n} \xi_{ij\ell}^{(n)}(s)\,(\sigma_s - \sigma_{i\alpha_n h_n})\,dW_s\right| > \frac{\delta\left(K^- - \kappa^-\right)^{3/4}}{8}\right] \\ &\le C_r \mathbb{E}\left[\left|\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell} S_{ij\ell}(\varepsilon) \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n} \xi_{ij\ell}^{(n)}(s)\,(\sigma_s - \sigma_{i\alpha_n h_n})\,dW_s\right|^r\right] \end{aligned}$$

We define

$$\begin{aligned} c_n^{i,4}(\tau) = \frac{1}{h_n}\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell} S_{ij\ell}(\varepsilon) \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau} \xi_{ij\ell}^{(n)}(s)\,(\sigma_s - \sigma_{i\alpha_n h_n})\,dW_s \\ \times \mathbb{1}_{(h_n(i\alpha_n+(\ell-1))-n^{-1},\, i\alpha_n h_n+\ell h_n]}(\tau)\,, \end{aligned}$$

such that with $r = 2m$, $m > 0$ and $m \in \mathbb{N}$

$$\begin{aligned} &= \mathbb{E}\left[\left|\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell} S_{ij\ell}(\varepsilon) \int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n} \xi_{ij\ell}^{(n)}(s)\,(\sigma_s - \sigma_{i\alpha_n h_n})\,dW_s\right|^r\right] \\ &= \mathbb{E}\left[\left|\int_{i\alpha_n h_n}^{(i+1)\alpha_n h_n} c_n^{i,4}(\tau)\,d\tau\right|^{2m}\right] \le \left(\int_{i\alpha_n h_n}^{(i+1)\alpha_n h_n} \mathbb{E}\left[\left(c_n^{i,4}(\tau)\right)^{2m}\right]^{1/2m} d\tau\right)^{2m}. \end{aligned}$$

In order to bound this expectation, we split the j-sum using the elementary inequality $|a+b|^p \le 2^p\left(|a|^p + |b|^p\right)$ and that the weights fulfill the following growth behaviour:

$$w_{jk} \propto \begin{cases} 1, & \text{for } j \le \sqrt{n} h_n \\ j^{-4} n^2 h_n^4, & \text{for } j > \sqrt{n} h_n\,. \end{cases} \tag{3.40}$$

That yields

$$\begin{aligned}
&\mathbb{E}\left[\left(c_n^{i,4}(\tau)\right)^{2m}\right] \\
&= \mathbb{E}\Bigg[\frac{1}{h_n^{2m}}\sum_{\ell=1}^{\alpha_n}\Bigg(\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}S_{ij\ell}(\varepsilon)\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n}\xi_{ij\ell}^{(n)}(s)\left(\sigma_s-\sigma_{i\alpha_n h_n}\right)dW_s\Bigg)^{2m}\Bigg] \\
&\qquad\qquad\qquad\qquad \times \mathbb{1}_{(h_n(i\alpha_n+(\ell-1))-n^{-1},i\alpha_n h_n+\ell h_n]}(\tau) \\
&\le \frac{C_m}{h_n^{2m}}\sum_{\ell=1}^{\alpha_n}\mathbb{E}\Bigg[\Bigg(\sum_{j=1}^{\sqrt{n}h_n} S_{ij\ell}(\varepsilon)\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau}\xi_{ij\ell}^{(n)}(s)\left(\sigma_s-\sigma_{i\alpha_n h_n}\right)dW_s\Bigg)^{2m}\Bigg] \\
&\qquad\qquad\qquad\qquad \times \mathbb{1}_{(h_n(i\alpha_n+(\ell-1))-n^{-1},i\alpha_n h_n+\ell h_n]}(\tau) \\
&+ \frac{C_m}{h_n^{2m}}\sum_{\ell=1}^{\alpha_n}\mathbb{E}\Bigg[\Bigg(\sum_{j=\sqrt{n}h_n+1}^{\lfloor nh_n\rfloor-1} j^{-4}nh_nS_{ij\ell}(\varepsilon)\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{\tau}\xi_{ij\ell}^{(n)}(s)\left(\sigma_s-\sigma_{i\alpha_n h_n}\right)dW_s\Bigg)^{2m}\Bigg] \\
&\qquad\qquad\qquad\qquad \times \mathbb{1}_{(h_n(i\alpha_n+(\ell-1))-n^{-1},i\alpha_n h_n+\ell h_n]}(\tau)\,.
\end{aligned}$$

Since

$$j^{-4}n^2h_n^4 = \mathcal{O}(1) \quad \text{for} \quad \sqrt{n}h_n \le j \le nh_n\,,$$

it is sufficient to consider the first summand only.
The calculations pursued in Lemma 2 in Bibinger and Winkelmann [13] imply the following, using the fact, that $(\varepsilon_t)_{t\in[0,1]}$ is independent of $\mathcal{F}^{(0)}$.

$$\begin{aligned}
&\mathbb{E}\Bigg[\frac{1}{h_n^{2m}}\sum_{\ell=1}^{\alpha_n}\Bigg(\sum_{j=1}^{\sqrt{n}h_n} w_{ij\ell}S_{ij\ell}(\varepsilon)\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n}\xi_{ij\ell}^{(n)}(s)\left(\sigma_s-\sigma_{i\alpha_n h_n}\right)dW_s\Bigg)^{2m}\Bigg] \\
&\le C_m\mathbb{E}\Bigg[\Bigg(\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n}\left(\xi_{ij\ell}^{(n)}(s)\right)^2\left(\sigma_s-\sigma_{i\alpha_n h_n}\right)^2 ds\Bigg)^{m}\Bigg] \\
&= \mathcal{O}\left(\left(\alpha_n h_n\right)^{2m\mathfrak{a}}h_n^{-2m}\right)\,,
\end{aligned}$$

such that

$$\mathbb{E}\left[\left(c_n^{i,4}(\tau)\right)^2\right] = \mathcal{O}\left(\left(\alpha_n h_n\right)^{2m\mathfrak{a}}h_n^{-2m}\right)\,.$$

We can conclude that

$$\mathbb{E}\left[\left|\int_{i\alpha_n h_n}^{(i+1)\alpha_n h_n} c_n^{i,4}(\tau)\,d\tau\right|^{2m}\right] = \mathcal{O}\left(\left(\alpha_n h_n\right)^{2m\mathfrak{a}}\alpha_n^{2m}\right)\,.$$

Overall we get

$$A_n^{2,1} = \mathcal{O}\left(\left(\alpha_n h_n\right)^{2m\mathfrak{a}}\left(\log(n)\right)^m\alpha_n^m\left(\alpha_n h_n\right)^{-1}\right) = \mathcal{o}(1)\,,\ \text{as } n\to\infty\,,$$

if $m \in \mathbb{N}$ is sufficiently large.
The term $A_n^{2,2,}$ can be handled easier, using (3.29) instead of Burkholder's inequality.
Overall it is shown that $A_n^2 = \mathcal{o}(1)$.
We can proceed with B_n from (3.26). Note that

$$\mathbb{P}\left[\min_i \left|\overline{Z}_{n,i+1}\right| < K^- - \kappa^-\right] \leq \mathbb{P}\left[\min_i \overline{Z}_{n,i+1} < K^- - \kappa^-\right]$$
$$\leq \sum_{i=0}^{\lfloor(\alpha_n h_n)^{-1}\rfloor-2} \mathbb{P}\left[\overline{Z}_{n,i+1} < K^- - \kappa^-\right].$$

It is sufficient to bound the probability

$$\begin{aligned}&\mathbb{P}\left[\overline{Z}_{n,i+1} < K^- - \kappa^-\right]\\ =&\mathbb{P}\left[\frac{1}{\alpha_n}\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\left(\left(\sigma_{i\alpha_n h_n} S_{ij\ell}(W) + S_{ij\ell}(\varepsilon)\right)^2 - \mu_{ij\ell}\right) < K^- - \kappa^-\right]\\ =&\mathbb{P}\left[\frac{1}{\alpha_n}\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\left(\left(\sigma_{i\alpha_n h_n} S_{ij\ell}(W) + S_{ij\ell}(\varepsilon)\right)^2 - \mu_{ij\ell}\right) - \sigma^2_{i\alpha_n h_n} < K^- - \sigma^2_{i\alpha_n h_n} - \kappa^-\right].\end{aligned}$$

Note that $K^- - \sigma^2_{i\alpha_n h_n} < 0$ and $\kappa^- > 0$, such that we can proceed with Markov's inequality with an exponent $r > 0$ and the elementary inequality $|a+b+c|^r \leq 3^r\left(|a|^r + |b|^r + |c|^r\right)$:

$$\begin{aligned}&\mathbb{P}\left[\frac{1}{\alpha_n}\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\left(\left(\sigma_{i\alpha_n h_n} S_{ij\ell}(W) + S_{ij\ell}(\varepsilon)\right)^2 - \mu_{ij\ell}\right) - \sigma^2_{i\alpha_n h_n} < K^- - \sigma^2_{i\alpha_n h_n} - \kappa^-\right]\\ &\leq \mathbb{P}\left[\frac{1}{\alpha_n}\left|\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\left(\left(\sigma_{i\alpha_n h_n} S_{ij\ell}(W) + S_{ij\ell}(\varepsilon)\right)^2 - \mu_{ij\ell}\right) - \sigma^2_{i\alpha_n h_n}\right| > \kappa^-\right]\\ &\leq C_r\mathbb{E}\left[\left|\frac{1}{\alpha_n}\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\sigma^2_{i\alpha_n h_n}\left(S^2_{ij\ell}(W) - 1\right)\right|^r\right]\\ &+ C_r\mathbb{E}\left[\left|\frac{1}{\alpha_n}\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\sigma_{i\alpha_n h_n} S_{ij\ell}(W)\, S_{ij\ell}(\varepsilon)\right|^r\right]\\ &+ C_r\mathbb{E}\left[\left|\frac{1}{\alpha_n}\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\left(S^2_{ij\ell}(\varepsilon) - \mu_{ij\ell}\right)^2\right|^r\right]\\ &= C_r\mathbb{E}\left[\frac{1}{\alpha_n^{r/2}}\left|\frac{1}{\sqrt{\alpha_n}}\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\left(S^2_{ij\ell}(W) - 1\right)\right|^r\right]\\ &+ C_r\mathbb{E}\left[\frac{1}{\alpha_n^{r/2}}\left|\frac{1}{\sqrt{\alpha_n}}\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell} S_{ij\ell}(W)\, S_{ij\ell}(\varepsilon)\right|^r\right]\end{aligned}$$

$$+ C_r \mathbb{E}\left[\frac{1}{\alpha_n^{r/2}} \left| \frac{1}{\sqrt{\alpha_n}} \sum_{\ell=1}^{\alpha_n} \sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{ij\ell} \left(S_{ij\ell}^2(\varepsilon) - \mu_{ij\ell} \right)^2 \right|^r \right]$$
$$= \mathcal{O}\left(\alpha_n^{-r/2} \right),$$

by the classical central limit theorem. This implies

$$\mathbb{P}\left[\min_i \left| \overline{Z}_{n,i+1} \right| < K^{-} - \kappa^{-} \right] = \mathcal{O}\left(\alpha_n^{-r/2} (\alpha_n h_n)^{-1} \right) = \mathcal{o}(1), \text{ as } n \to \infty,$$

if $r > 0$ sufficiently large. Thus, we have completed the term B_n, and so the term (**II**). We proceed with (**I**) from (3.25). It holds that

$$\overline{RV}_{n,i} \left(\frac{1}{\left| \overline{RV}_{n,i+1} \right|^{3/4}} - \frac{1}{\left| \overline{Z}_{n,i+1} \right|^{3/4}} \right) = \overline{RV}_{n,i} \left(\frac{\left| \overline{Z}_{n,i+1} \right|^{3/4} - \left| \overline{RV}_{n,i+1} \right|^{3/4}}{\left| \overline{RV}_{n,i+1} \right|^{3/4} \left| \overline{Z}_{n,i+1} \right|^{3/4}} \right),$$

such that for every $\delta > 0$ we have

$$\mathbb{P}\left[\max_i \sqrt{\alpha_n \log(n)} \left| \overline{RV}_{n,i} \left(\frac{1}{\left| \overline{RV}_{n,i+1} \right|^{3/4}} - \frac{1}{\left| \overline{Z}_{n,i+1} \right|^{3/4}} \right) \right| > \delta \right]$$
$$= \mathbb{P}\left[\max_i \sqrt{\alpha_n \log(n)} \left| \overline{RV}_{n,i} \left(\frac{\left| \overline{Z}_{n,i+1} \right|^{3/4} - \left| \overline{RV}_{n,i+1} \right|^{3/4}}{\left| \overline{RV}_{n,i+1} \right|^{3/4} \left| \overline{Z}_{n,i+1} \right|^{3/4}} \right) \right| > \delta, \right.$$
$$\left. \min_i \left| \overline{RV}_{n,i+1} \right| \left| \overline{Z}_{n,i+1} \right| \geq \frac{(K^{-} - \kappa^{-})^2}{2} \right]$$
$$+ \mathbb{P}\left[\max_i \sqrt{\alpha_n \log(n)} \left| \overline{RV}_{n,i} \left(\frac{\left| \overline{Z}_{n,i+1} \right|^{3/4} - \left| \overline{RV}_{n,i+1} \right|^{3/4}}{\left| \overline{RV}_{n,i+1} \right|^{3/4} \left| \overline{Z}_{n,i+1} \right|^{3/4}} \right) \right| > \delta, \right.$$
$$\left. \min_i \left| \overline{RV}_{n,i+1} \right| \left| \overline{Z}_{n,i+1} \right| < \frac{(K^{-} - \kappa^{-})^2}{2} \right]$$
$$\leq \mathbb{P}\left[\max_i \sqrt{\alpha_n \log(n)} \left| \overline{RV}_{n,i} \left(\left| \overline{Z}_{n,i+1} \right|^{3/4} - \left| \overline{RV}_{n,i+1} \right|^{3/4} \right) \right| > 4\delta \left(K^{-} - \kappa^{-} \right)^{3/2} \right] \tag{3.41}$$
$$+ \mathbb{P}\left[\min_i \left| \overline{RV}_{n,i+1} \right| \left| \overline{Z}_{n,i+1} \right| < \frac{(K^{-} - \kappa^{-})^2}{4} \right]. \tag{3.42}$$

We start with the second probability (3.42):

$$\mathbb{P}\left[\min_i \left| \overline{RV}_{n,i+1} \right| \left| \overline{Z}_{n,i+1} \right| < \frac{(K^{-} - \kappa^{-})^2}{4} \right]$$
$$\leq \mathbb{P}\left[\min_i \left| \overline{RV}_{n,i+1} \right| < \frac{K^{-} - \kappa^{-}}{2} \right] + \mathbb{P}\left[\min_i \left| \overline{Z}_{n,i+1} \right| < \frac{K^{-} - \kappa^{-}}{2} \right].$$

The second probability has already been considered, since

$$\mathbb{P}\Big[\min_i \left|\overline{Z}_{n,i+1}\right| < \frac{K^- - \kappa^-}{2}\Big] = \mathbb{P}\Big[\min_i \overline{Z}_{n,i+1} < \frac{K^- - \kappa^-}{2}\Big] = \mathcal{O}\left(B_n\right) .$$

Concerning the first one, it holds that

$$\begin{aligned}
&\mathbb{P}\Big[\min_i \left|\overline{RV}_{n,i+1}\right| < \frac{K^- - \kappa^-}{2}\Big] \leq \mathbb{P}\Big[\min_i \overline{RV}_{n,i+1} < \frac{K^- - \kappa^-}{2}\Big] \\
&= \mathbb{P}\Big[\min_i \overline{RV}_{n,i+1} < \frac{K^- - \kappa^-}{2}, \max_i \left|\overline{RV}_{n,i+1} - \overline{Z}_{n,i+1}\right| \leq \frac{K^- - \kappa^-}{2}\Big] \\
&+ \mathbb{P}\Big[\min_i \overline{RV}_{n,i+1} < \frac{K^- - \kappa^-}{2}, \max_i \left|\overline{RV}_{n,i+1} - \overline{Z}_{n,i+1}\right| > \frac{K^- - \kappa^-}{2}\Big] \\
&\leq \mathbb{P}\left[\min_i \overline{Z}_{n,i+1} < K^- - \kappa^-\right] + \mathbb{P}\Big[\max_i \left|\overline{RV}_{n,i+1} - \overline{Z}_{n,i+1}\right| > \frac{K^- - \kappa^-}{2}\Big] .
\end{aligned}$$

Since

$$\mathbb{P}\Big[\max_i \left|\overline{RV}_{n,i+1} - \overline{Z}_{n,i+1}\right| > \frac{K^- - \kappa^-}{2}\Big] = \mathcal{O}\left(A_n\right) ,$$

we can proceed with (3.41). For every $\delta > 0$ and $\kappa^+ \in (K^+, \infty)$, it holds that

$$\begin{aligned}
&\mathbb{P}\left[\max_i \sqrt{\alpha_n \log(n)} \left|\overline{RV}_{n,i} \left(\left|\overline{Z}_{n,i+1}\right|^{3/4} - \left|\overline{RV}_{n,i+1}\right|^{3/4}\right)\right| > 4\delta\left(K^- - \kappa^-\right)^{3/2}\right] \\
&= \mathbb{P}\left[\max_i \sqrt{\alpha_n \log(n)} \left|\overline{RV}_{n,i} \left(\left|\overline{Z}_{n,i+1}\right|^{3/4} - \left|\overline{RV}_{n,i+1}\right|^{3/4}\right)\right| > 4\delta\left(K^- - \kappa^-\right)^{3/2}, \right. \qquad (3.43) \\
&\qquad\qquad \left. \max_i \overline{Z}_{n,i} \leq K^+ + \kappa^+\right] \\
&+ \mathbb{P}\left[\max_i \sqrt{\alpha_n \log(n)} \left|\overline{RV}_{n,i} \left(\left|\overline{Z}_{n,i+1}\right|^{3/4} - \left|\overline{RV}_{n,i+1}\right|^{3/4}\right)\right| > 4\delta\left(K^- - \kappa^-\right)^{3/2}, \right. \qquad (3.44) \\
&\qquad\qquad \left. \max_i \overline{Z}_{n,i} > K^+ + \kappa^+\right] .
\end{aligned}$$

We start with (3.43).

$$\begin{aligned}
&\mathbb{P}\left[\max_i \sqrt{\alpha_n \log(n)} \left|\overline{RV}_{n,i} \left(\left|\overline{Z}_{n,i+1}\right|^{3/4} - \left|\overline{RV}_{n,i+1}\right|^{3/4}\right)\right| > 4\delta\left(K^- - \kappa^-\right)^{3/2}, \right. \\
&\qquad\qquad \left. \max_i \overline{Z}_{n,i} \leq K^+ + \kappa^+\right] \\
&\leq \mathbb{P}\Big[\max_i \left|\overline{RV}_{n,i}\right| > 2\left(K^+ + \kappa^+\right)\Big]
\end{aligned}$$

$$+\,\mathbb{P}\left[\max_i \sqrt{\alpha_n \log(n)}\left|\left(\left|\overline{Z}_{n,i+1}\right|^{3/4} - \left|\overline{RV}_{n,i+1}\right|^{3/4}\right)\right| > \frac{2\delta\left(K^- - \kappa^-\right)^{3/2}}{K^+ + \kappa^+}\right]$$

$$\le \mathbb{P}\left[\max_i \left|\overline{RV}_{n,i} - \overline{Z}_{n,i}\right| + \left|\overline{Z}_{n,i}\right| > 2\left(K^+ + \kappa^+\right)\right]$$

$$+\,\mathbb{P}\left[\max_i \sqrt{\alpha_n \log(n)}\left|\left(\left|\overline{Z}_{n,i+1}\right|^{3/4} - \left|\overline{RV}_{n,i+1}\right|)^{3/4}\right)\right| > \frac{2\delta\left(K^- - \kappa^-\right)^{3/2}}{K^+ + \kappa^+}\right]$$

$$\le \mathbb{P}\left[\max_i \left|\overline{RV}_{n,i} - \overline{Z}_{n,i}\right| > K^+ + \kappa^+\right] + \mathbb{P}\left[\max_i \left|\overline{Z}_{n,i}\right| > K^+ + \kappa^+\right] \tag{3.45}$$

$$+\,\mathbb{P}\left[\max_i \sqrt{\alpha_n \log(n)}\left|\left(\left|\overline{Z}_{n,i+1}\right|^{3/4} - \left|\overline{RV}_{n,i+1}\right|^{3/4}\right)\right| > \frac{2\delta\left(K^- - \kappa^-\right)^{3/2}}{K^+ + \kappa^+}\right] \tag{3.46}$$

Note that

$$\mathbb{P}\left[\max_i \left|\overline{Z}_{n,i}\right| > K^+ + \kappa^+\right] \le \sum_{i=0}^{\lfloor(\alpha_n h_n)^{-1}\rfloor - 2} \mathbb{P}\left[\left|\overline{Z}_{n,i}\right| > K^+ + \kappa^+\right]$$

holds. We proceed with the triangle inequality and using that $K^+ - \sigma^2_{i\alpha_n h_n} > 0$ uniformly in i,

$$\begin{aligned}
\mathbb{P}\left[\max_i \left|\overline{Z}_{n,i}\right| > K^+ + \kappa^+\right] \le\ & \mathbb{P}\left[\left|\frac{1}{\alpha_n}\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor - 1} w_{ij\ell}\sigma^2_{i\alpha_n h_n}\left(S_{ij\ell}(W) - 1\right)\right| > \kappa^+\right] \\
&+ \mathbb{P}\left[\left|\frac{1}{\alpha_n}\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor - 1} w_{ij\ell}\sigma_{i\alpha_n h_n} S_{ij\ell}(W)\, S_{ij\ell}(\varepsilon)\right| > \kappa^+\right] \\
&+ \mathbb{P}\left[\left|\frac{1}{\alpha_n}\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor - 1} w_{ij\ell}\left(S_{ij\ell}(\varepsilon) - \mu_{ij\ell}\right)\right| > \kappa^+\right].
\end{aligned}$$

Applying the Markov inequality, bounding the volatility from above, and concluding with a classical central limit theorem argument, yields the bound

$$\mathbb{P}\left[\left|\overline{Z}_{n,i}\right| > K^+ + \kappa^+\right] = \mathcal{O}\left(\alpha_n^{-r/2}\right),$$

such that

$$\mathbb{P}\left[\max_i \left|\overline{Z}_{n,i}\right| > K^+ + \kappa^+\right] = \mathcal{O}\left(\alpha_n^{-r/2}\left(\alpha_n h_n\right)^{-1}\right) = \mathcal{o}(1)\,,\ \text{as } n \to \infty\,,$$

holds if the exponent $r > 0$ is sufficiently large. This completes (3.45), since the first probability therein is included in A_n.
We proceed with (3.46). The discussion of this term can be traced back to A_n with a

Taylor expansion. More precisely, we set $\psi(x) = x^{3/4}$ and expand around the point $\left|\overline{Z}_{n,i+1}\right|$,

$$\begin{aligned}\psi\left(\left|\overline{RV}_{n,i+1}\right|\right) - \psi\left(\left|\overline{Z}_{n,i+1}\right|\right) &= \psi'\left(\left|\overline{Z}_{n,i+1}\right|\right)\left(\left|\overline{RV}_{n,i+1}\right| - \left|\overline{Z}_{n,i+1}\right|\right) \\ &\quad + \left(\left|\overline{RV}_{n,i+1}\right| - \left|\overline{Z}_{n,i+1}\right|\right)\mathcal{R}\left(\left|\overline{RV}_{n,i+1}\right| - \left|\overline{Z}_{n,i+1}\right|\right) .\end{aligned}$$

Since $\psi'\left(\left|\overline{Z}_{n,i+1}\right|\right) = \mathcal{O}_{\mathbb{P}}(1)$, and since the remainder $\mathcal{R}$ is negligible,

$$\mathcal{R}\left(\left|\overline{RV}_{n,i+1}\right| - \left|\overline{Z}_{n,i+1}\right|\right) = o_{\mathbb{P}}(1) ,$$

by the reverse triangle inequality and the estimates for A_n. Therefore, only

$$\left|\overline{RV}_{n,i+1}\right| - \left|\overline{Z}_{n,i+1}\right|$$

is crucial. But, using the reverse triangle inequality again this has already been worked out in A_n, too. So we have completed (3.46) and so (3.43). We proceed with (3.44). It holds that

$$\begin{aligned}&\mathbb{P}\left[\max_i \sqrt{\alpha_n \log(n)}\left|\overline{RV}_{n,i}\left(\left|\overline{Z}_{n,i+1}\right|^{3/4} - \left|\overline{RV}_{n,i+1}\right|^{3/4}\right)\right| > 4\delta\left(K^{-}-\kappa^{-}\right)^{3/2},\right. \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\left.\max_i \overline{Z}_{n,i} > K^{+} + \kappa^{+}\right] \\ &\leq \mathbb{P}\left[\max_i \overline{Z}_{n,i} > K^{+} + \kappa^{+}\right] \leq \mathbb{P}\left[\max_i \left|\overline{Z}_{n,i}\right| > K^{+} + \kappa^{+}\right] .\end{aligned}$$

Thus, this probability has already been considered within (3.45). Therefore, we also have completed (3.44), such that we are done with (**I**). The terms (**III**) and (**IV**) in (3.25) are only shifted in i. So we have finished the proof of Proposition 3.16. □

For the *second step* described in the outline of the proof we approximate the volatility locally constant over two consecutive blocks by shifting the index of $\sigma_{(i+1)\alpha_n h_n}$ in $\overline{Z}_{n,i+1}$ as follows: $i+1 \mapsto i$. We set

$$\widetilde{Z}_{n,i} := \frac{1}{\alpha_n}\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\left(\left(\sigma_{(i-1)\alpha_n h_n} S_{ij\ell}(W) + S_{ij\ell}(\varepsilon)\right)^2 - \mu_{ij\ell}\right) .$$

Proposition 3.17. *Given the assumptions of Theorem 3.8, it holds under (H_0-$\mathfrak{a}$) that*

$$\sqrt{\alpha_n \log\left(h_n^{-1}\right)}\max_i \left|\left|\frac{\overline{Z}_{n,i} - \overline{Z}_{n,i+1}}{\left|\overline{Z}_{n,i+1}\right|^{3/4}}\right| - \left|\frac{\overline{Z}_{n,i} - \widetilde{Z}_{n,i+1}}{\left|\widetilde{Z}_{n,i+1}\right|^{3/4}}\right|\right| \xrightarrow{\mathbb{P}} 0 ,$$

as $n \longrightarrow +\infty$.

Proof of Proposition 3.17.
The decomposition

$$\frac{\overline{Z}_{n,i}-\overline{Z}_{n,i+1}}{\left|\overline{Z}_{n,i+1}\right|^{3/4}}-\frac{\overline{Z}_{n,i}-\widetilde{Z}_{n,i+1}}{\left|\widetilde{Z}_{n,i+1}\right|^{3/4}}$$
$$=\frac{\overline{Z}_{n,i}}{\left|\overline{Z}_{n,i+1}\right|^{3/4}}-\frac{\overline{Z}_{n,i}}{\left|\widetilde{Z}_{n,i+1}\right|^{3/4}}+\frac{\overline{Z}_{n,i+1}}{\left|\widetilde{Z}_{n,i+1}\right|^{3/4}}-\frac{\overline{Z}_{n,i+1}}{\left|\overline{Z}_{n,i+1}\right|^{3/4}}+\frac{\widetilde{Z}_{n,i+1}-\overline{Z}_{n,i+1}}{\left|\overline{Z}_{n,i+1}\right|^{3/4}}$$

yields, via the triangle inequality, the three terms

$$\max_i\left|\overline{Z}_{n,i}\left(\frac{1}{\left|\overline{Z}_{n,i+1}\right|^{3/4}}-\frac{1}{\left|\widetilde{Z}_{n,i+1}\right|^{3/4}}\right)\right|+\max_i\left|\overline{Z}_{n,i+1}\left(\frac{1}{\left|\widetilde{Z}_{n,i+1}\right|^{3/4}}-\frac{1}{\left|\overline{Z}_{n,i+1}\right|^{3/4}}\right)\right|$$
$$+\max_i\left|\frac{\widetilde{Z}_{n,i+1}-\overline{Z}_{n,i+1}}{\left|\overline{Z}_{n,i+1}\right|^{3/4}}\right|$$
$$=: (\mathbf{I})+(\mathbf{II})+(\mathbf{III})\,.$$

We start with (**III**). For any $\delta>0$ it holds that

$$\mathbb{P}\left[\max_i\left|\frac{\sqrt{n\log(n)}\left(\widetilde{Z}_{n,i+1}-\overline{Z}_{n,i+1}\right)}{\left|\widetilde{Z}_{n,i+1}\right|^{3/4}}\right|>\delta\right]$$
$$=\mathbb{P}\left[\max_i\left|\frac{\sqrt{n\log(n)}\left(\widetilde{Z}_{n,i+1}-\overline{Z}_{n,i+1}\right)}{\left|\widetilde{Z}_{n,i+1}\right|^{3/4}}\right|>\delta,\min_i\left|\widetilde{Z}_{n,i+1}\right|\geq K^{-}-\kappa^{-}\right]$$
$$+\mathbb{P}\left[\max_i\left|\frac{\sqrt{n\log(n)}\left(\widetilde{Z}_{n,i+1}-\overline{Z}_{n,i+1}\right)}{\left|\widetilde{Z}_{n,i+1}\right|^{3/4}}\right|>\delta,\min_i\left|\widetilde{Z}_{n,i+1}\right|< K^{-}-\kappa^{-}\right]$$
$$\leq\mathbb{P}\left[\max_i\sqrt{n\log(n)}\left|\widetilde{Z}_{n,i+1}-\overline{Z}_{n,i+1}\right|>\delta\left(K^{-}-\kappa^{-}\right)^{3/4}\right]\tag{3.47}$$
$$+\mathbb{P}\left[\min_i\left|\widetilde{Z}_{n,i+1}\right|<K^{-}-\kappa^{-}\right]\,.\tag{3.48}$$

The probability (3.48) has already been done, since it only differs by a shift in i with respect to the volatility from the term in Proposition 3.16. We continue with (3.47).

It holds that

$$\widetilde{Z}_{n,i} - \overline{Z}_{n,i} = \left(\sigma^2_{(i-1)\alpha_n h_n} - \sigma^2_{i\alpha_n h_n}\right) \frac{1}{\alpha_n} \sum_{\ell=1}^{\alpha_n} \sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{ij\ell} S^2_{ij\ell}(W)$$
$$+ \left(\sigma_{(i-1)\alpha_n h_n} - \sigma_{i\alpha_n h_n}\right) \frac{1}{\alpha_n} \sum_{\ell=1}^{\alpha_n} \sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{ij\ell} S_{ij\ell}(W)\, S_{ij\ell}(\varepsilon)\,.$$

That yields

$$\mathbb{P}\left[\max_i \sqrt{\alpha_n \log(n)} \left|\widetilde{Z}_{n,i+1} - \overline{Z}_{n,i+1}\right| > \delta\big(K^- - \kappa^-\big)^{3/4}\right]$$
$$\leq \mathbb{P}\left[\max_i \sqrt{\alpha_n \log(n)} \left(\sigma^2_{i\alpha_n h_n} - \sigma^2_{(i+1)\alpha_n h_n}\right) \times \frac{1}{\alpha_n}\left|\sum_{\ell=1}^{\alpha_n} \sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{(i+1)j\ell} S^2_{(i+1)j\ell}(W)\right| > \frac{\big(K^- - \kappa^-\big)^{3/4}}{2}\right]$$
$$+ \mathbb{P}\left[\max_i \sqrt{\alpha_n \log(n)} \left(\sigma_{i\alpha_n h_n} - \sigma_{(i+1)\alpha_n h_n}\right) \times \frac{1}{\alpha_n}\left|\sum_{\ell=1}^{\alpha_n} \sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{(i+1)j\ell} S_{(i+1)j\ell}(W)\, S_{(i+1)j\ell}(\varepsilon)\right| > \frac{\big(K^- - \kappa^-\big)^{3/4}}{4}\right]$$
$$\leq \mathbb{P}\left[\max_i \sqrt{\alpha_n \log(n)} \left|\sigma^2_{i\alpha_n h_n} - \sigma^2_{(i+1)\alpha_n h_n}\right| > \frac{\big(K^- - \kappa^-\big)^{3/4}}{4}\right]$$
$$+ \mathbb{P}\left[\max_i \frac{1}{\alpha_n} \sum_{\ell=1}^{\alpha_n} \sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{(i+1)j\ell} S^2_{(i+1)j\ell}(W) > 2\right].$$

Concerning the first term it holds that

$$\max_i \sqrt{\alpha_n \log(n)} \left|\sigma^2_{i\alpha_n h_n} - \sigma^2_{(i+1)\alpha_n h_n}\right| = \mathcal{O}_{\mathbb{P}}\left(\sqrt{\alpha_n \log(n)}\,(\alpha_n h_n)^{\mathfrak{a}}\right), \text{ uniformly in } i$$
$$= \scriptstyle{\mathcal{O}}_{\mathbb{P}}(1)\,, \text{ as } n \to \infty\,, \text{ by (3.5).}$$

It remains to show that

$$\mathbb{P}\left[\max_i \frac{1}{\alpha_n} \sum_{\ell=1}^{\alpha_n} \sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{(i+1)j\ell} S^2_{(i+1)j\ell}(W) > 2\right] = \scriptstyle{\mathcal{O}}(1)\,.$$

We conclude with a classical central limit theorem argument, using Markov's inequality with $r > 0$.

$$\begin{aligned}
&\mathbb{P}\left[\max_i \frac{1}{\alpha_n}\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{(i+1)j\ell}S^2_{(i+1)j\ell}(W) > 2\right] \\
&= \mathbb{P}\left[\max_i \frac{1}{\alpha_n}\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{(i+1)j\ell}\big(S^2_{(i+1)j\ell}(W)-1\big) > 1\right] \\
&\leq (\alpha_n h_n)^{-1}\,\mathbb{E}\left[\frac{1}{\alpha_n^{r/2}}\left|\frac{1}{\sqrt{\alpha_n}}\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{(i+1)j\ell}\big(S^2_{(i+1)j\ell}(W)-1\big)\right|^r\right] \\
&= \mathcal{O}\left((\alpha_n h_n)^{-1}\,\alpha_n^{-r/2}\right) = \mathcal{o}(1)\,,\ \text{as } n\to\infty\,,
\end{aligned}$$

with $r > 0$ sufficiently large. We have completed (3.47) and so (**III**).
We proceed with (**I**). For any $\delta > 0$ it holds that

$$\begin{aligned}
&\mathbb{P}\left[\max_i\left|\overline{Z}_{n,i}\left(\frac{1}{\left|\overline{Z}_{n,i+1}\right|^{3/4}}-\frac{1}{\left|\widetilde{Z}_{n,i+1}\right|^{3/4}}\right)\right| > \delta\right] \\
&= \mathbb{P}\left[\max_i\left|\overline{Z}_{n,i}\left(\frac{\left|\widetilde{Z}_{n,i+1}\right|^{3/4}-\left|\overline{Z}_{n,i+1}\right|^{3/4}}{\left|\overline{Z}_{n,i+1}\right|^{3/4}\left|\widetilde{Z}_{n,i+1}\right|^{3/4}}\right)\right| > \delta\right] \\
&= \mathbb{P}\left[\max_i\left|\overline{Z}_{n,i}\left(\frac{\left|\widetilde{Z}_{n,i+1}\right|^{3/4}-\left|\overline{Z}_{n,i+1}\right|^{3/4}}{\left|\overline{Z}_{n,i+1}\right|^{3/4}\left|\widetilde{Z}_{n,i+1}\right|^{3/4}}\right)\right| > \delta, \min_i\left|\overline{Z}_{n,i+1}\right|\left|\widetilde{Z}_{n,i+1}\right| \geq \left(K^{-}-\kappa^{-}\right)^2\right] \\
&+ \mathbb{P}\left[\max_i\left|\overline{Z}_{n,i}\left(\frac{\left|\widetilde{Z}_{n,i+1}\right|^{3/4}-\left|\overline{Z}_{n,i+1}\right|^{3/4}}{\left|\overline{Z}_{n,i+1}\right|^{3/4}\left|\widetilde{Z}_{n,i+1}\right|^{3/4}}\right)\right| > \delta, \min_i\left|\overline{Z}_{n,i+1}\right|\left|\widetilde{Z}_{n,i+1}\right| < \left(K^{-}-\kappa^{-}\right)^2\right] \\
&\leq \mathbb{P}\left[\max_i\left|\overline{Z}_{n,i}\left(\left|\widetilde{Z}_{n,i+1}\right|^{3/4}-\left|\overline{Z}_{n,i+1}\right|^{3/4}\right)\right| > \delta\left(K^{-}-\kappa^{-}\right)^{3/2}\right] &(3.49)\\
&+ \mathbb{P}\left[\min_i\left|\overline{Z}_{n,i+1}\right|\left|\widetilde{Z}_{n,i+1}\right| < \left(K^{-}-\kappa^{-}\right)^2\right]. &(3.50)
\end{aligned}$$

We start with (3.50).

$$\begin{aligned}
&\mathbb{P}\left[\min_i\left|\overline{Z}_{n,i+1}\right|\left|\widetilde{Z}_{n,i+1}\right| < \left(K^{-}-\kappa^{-}\right)^2\right] \\
&\leq \mathbb{P}\left[\min_i\left|\overline{Z}_{n,i+1}\right| < K^{-}-\kappa^{-}\right] + \mathbb{P}\left[\min_i\left|\widetilde{Z}_{n,i+1}\right| < K^{-}-\kappa^{-}\right].
\end{aligned}$$

Only the second probability has to be considered. But since the involved statistic only differs by a shift in the volatility, we can bound the latter from below and argue with the central limit theorem. So we have completed (3.50) and continue with (3.49).
We handle (3.49) via a Taylor expansion. Expanding the function $\psi(x) = x^{3/4}$ around the point $|\widetilde{Z}_{n,i+1}|$ yields the desired result using the procedure for (**III**). We will omit the details for (**II**), since it only differs by a shift in i. So Proposition 3.17 is proven. □

We do a further approximation step, replacing the denominator in Proposition 3.17 by its limit. This is the *third step* outlined in the proof sketch above. Here we use the estimator $\hat{\eta}^2$ from (3.1).

Proposition 3.18. *Given the assumptions of Theorem 3.8, it holds under (H_0-$\mathfrak{a}$) that*

$$\sqrt{\alpha_n \log\left(h_n^{-1}\right)} \max_i \left| \left| \frac{\overline{Z}_{n,i} - \widetilde{Z}_{n,i+1}}{\sqrt{8\hat{\eta}} \left|\widetilde{Z}_{n,i+1}\right|^{3/4}} \right| - \left| \frac{\overline{Z}_{n,i} - \widetilde{Z}_{n,i+1}}{\sqrt{8\eta}\sigma_{i\alpha_n h_n}^{3/2}} \right| \right| \xrightarrow{\mathbb{P}} 0\,.$$

Proof of Proposition 3.18.
We have to bound the term

$$\begin{aligned}
&\max_i \sqrt{\alpha_n \log(n)} \left| \left(\overline{Z}_{n,i} - \widetilde{Z}_{n,i+1}\right) \left(\frac{1}{\sqrt{8\hat{\eta}} \left|\widetilde{Z}_{n,i+1}\right|^{3/4}} - \frac{1}{\sqrt{8\eta}\sigma_{i\alpha_n h_h}^{3/2}} \right) \right| \\
&= \max_i \sqrt{\alpha_n \log(n)} \left|\overline{Z}_{n,i} - \widetilde{Z}_{n,i+1}\right| \max_i \left| \frac{1}{\sqrt{8\hat{\eta}} \left|\widetilde{Z}_{n,i+1}\right|^{3/4}} - \frac{1}{\sqrt{8\eta}\sigma_{i\alpha_n h_h}^{3/2}} \right| .
\end{aligned}$$

We will employ a 2-dimensional Taylor expansion of order 1 with respect to the second term. We set $\psi(x,y) = x^{-1/2}y^{-3/4}$ and expand around the point $(a,b) = (\eta, \sigma^2_{i\alpha_n h_n})$. Therefore, we have to bound the term

$$\frac{\partial \psi(\eta, \sigma^2_{i\alpha_n h_n})}{\partial x} (\hat{\eta} - \eta) + \frac{\partial \psi(\eta, \sigma^2_{i\alpha_n h_n})}{\partial y} \left(\left|\widetilde{Z}_{n,i+1}\right| - \sigma^2_{i\alpha_n h_n} \right) + o_{\mathbb{P}}(1). \tag{3.51}$$

Since $(\sigma_t^2)_{t\in[0,1]}$ can be bounded globally, we get the following uniform bounds in i:

$$\max\left(\frac{\partial \psi(\eta, \sigma^2_{i\alpha_n h_n})}{\partial x}, \frac{\partial \psi(\eta, \sigma^2_{i\alpha_n h_n})}{\partial y} \right) = \mathcal{O}_{\mathbb{P}}(1)\,.$$

The first summand in (3.51) with $(\hat{\eta} - \eta)$ can be handled easily, using

$$(\hat{\eta} - \eta) = \mathcal{O}_{\mathbb{P}}(n^{-1/2})\,.$$

This implies

$$\max_i \frac{\partial \psi(\eta, \sigma^2_{i\alpha_n h_n})}{\partial x} (\hat{\eta} - \eta) = o_{\mathbb{P}}(1).$$

Proceeding with the second term in (3.51), we need bounds for the uniform error. Such bounds are given in Bibinger and Reiß [12] on page 10 for the estimators in (3.7) with $J = 1$. Such a uniform bound readily extends to the more sophisticated estimators in the same way. Since $\alpha_n \propto h_n^{\frac{-2\mathfrak{a}}{2\mathfrak{a}+1}}$ is the rate-optimal choice, we get with the upper bound from Bibinger and Reiß [12] that

$$\max_i \frac{\partial\psi(\eta,\sigma^2_{i\alpha_n h_n})}{\partial y}\Big(\Big|\widetilde{Z}_{n,i+1}\Big| - \sigma^2_{i\alpha_n h_n}\Big) = \mathcal{O}_{\mathbb{P}}\left(h_n^{\frac{\mathfrak{a}}{2\mathfrak{a}+1}}\left(\log(h_n^{-1})\right)^{\frac{\mathfrak{a}}{2\mathfrak{a}+1}}\right) = o_{\mathbb{P}}(1)\,.$$

Proceeding with the term

$$\max_i \sqrt{\alpha_n \log(n)}\,\Big|\overline{Z}_{n,i} - \widetilde{Z}_{n,i+1}\Big|$$

we conclude similarly with the triangle inequality,

$$\begin{aligned}&\max_i \sqrt{\alpha_n \log(n)}\,\Big|\overline{Z}_{n,i} - \widetilde{Z}_{n,i+1}\Big|\\ &\le \max_i \sqrt{\alpha_n \log(n)}\,\left|\overline{Z}_{n,i} - \sigma^2_{i\alpha_n h_n}\right| + \max_i \sqrt{\alpha_n \log(n)}\,\left|\widetilde{Z}_{n,i+1} - \sigma^2_{i\alpha_n h_n}\right|\,,\end{aligned}$$

the uniform bound applied to each summand and the regularity of $\left(\sigma_t^2\right)_{t\in[0,1]}$ under the null hypothesis (H_0-$\mathfrak{a}$). This implies

$$\max_i \sqrt{\alpha_n \log(n)}\,\Big|\overline{Z}_{n,i} - \widetilde{Z}_{n,i+1}\Big| = \mathcal{O}_{\mathbb{P}}(1)\,,$$

such that the convergence in Proposition 3.18 follows. □

It is worth to mention that the optimal choice of α_n, $\alpha_n \propto h_n^{\frac{-2\mathfrak{a}}{2\mathfrak{a}+1}}$, together with Theorem 3.8 yields a similar uniform bound as we have used above. In order to conclude the convergence for the adaptive statistics in Theorem 3.8, we have to show that replacing the oracle versions by the adaptive statistics does not affect the limit. It is sufficient to show the following for the *fourth step* to complete the proof of the approximation steps mentioned in the outline of the proof section.

Proposition 3.19. *Given the assumptions of Theorem 3.8, it holds under (H_0-$\mathfrak{a}$) that*

$$\sqrt{\alpha_n \log\left(h_n^{-1}\right)}\max_i \left|\overline{RV}^{ad}_{n,i} - \overline{RV}_{n,i}\right| \xrightarrow{\mathbb{P}} 0\,.$$

Proof of Proposition 3.19.
As we have argued above, η^2 can be replaced by the $\sqrt{n}$-rate consistent estimator (3.1) without affecting the limit behaviour of the statistics. Therefore it is sufficient to consider the plug-in estimation of the spot volatility in the weights $(w_{ij\ell})$. First of all, taking into account that the asymptotic order of the weights (3.40) does not depend on i,ℓ, we may consider them as a function $w_j = w_j(\sigma^2)$ of the spot volatility.

Calculating the first derivative, w_j' as pursued on page 40 in Altmeyer and Bibinger [5], we get the upper bound

$$w'(\sigma^2) = \mathcal{O}_{\mathbb{P}}(w_j(\sigma^2)\log^2(n))\,. \tag{3.52}$$

In order to bound $\max_i |\overline{RV}^{ad}_{n,i} - \overline{RV}_{n,i}|$, take into account that $\sum_j w_j(x) = 1$ for every x. So it is sufficient to consider the term

$$\max_i \frac{1}{\alpha_n} \sum_{j=1}^{\lfloor nh_n\rfloor -1} (w_j(\sigma^2_{i\alpha_n h_n}) - w_j(\hat{\sigma}^2_{i\alpha_n h_n})) \sum_{\ell=1}^{\alpha_n} (S^2_{ij\ell}(Y) - \mathbb{E}[S^2_{ij\ell}(Y)])\,.$$

The only difference compared with Bibinger and Winkelmann [13] and Altmeyer and Bibinger [5], is to replace the point-wise L^1 bound for $|\hat{\sigma}^2_{i\alpha_n h_n} - \sigma^2_{i\alpha_n h_n}|$ by the uniform bound from Proposition 3.18, with that the bound

$$|w_j(\sigma^2_{i\alpha_n h_n}) - w_j(\hat{\sigma}^2_{i\alpha_n h_n})| = \mathcal{O}_{\mathbb{P}}(w_j(\sigma^2)\log^2(n) h_n^{\frac{\mathfrak{a}}{2\mathfrak{a}+1}} \log(n)^{\frac{\mathfrak{a}}{2\mathfrak{a}+1}})$$

follows, using the mean value theorem and (3.52). □

The key, proving the last conclusion is to apply strong invariance principles presented in Section 2.4. First of all we have to take into account that the rescaling factors in $\widetilde{U}'/\sqrt{8\eta}$ provide only an asymptotically distribution-free limit. So it is more adequate for our purpose to rescale with the exact finite-sample standard deviation, that is

$$2\Bigg(\sum_{m=1}^{\lfloor nh_n\rfloor -1} \bigg(\sigma^2_{i\alpha_n h_n} + \frac{\eta^2}{n}[\varphi_{mk},\varphi_{mk}]_n\bigg)^{-2}\Bigg)^{-1}.$$

Using a Taylor approximation and the convergence of the above variances to $8\sigma^3_{i\alpha_n h_n}\eta$, presented in Section 6.2. of Altmeyer and Bibinger [5], it is clear that the approximation holds.
Let $I_{i,\nu}$ and $\widetilde{I}_{i,\nu}$ be the exact finite-sample variances and define $\mathbb{L}^{(n)}_{i,\nu}$ and $\widetilde{\mathbb{L}}^{(n)}_{i,\nu}$ given by

$$\mathbb{L}^{(n)}_{i,\nu} = \frac{\sum_{j=1}^{\lfloor nh_n\rfloor -1} w_{ij\nu}\left((\sigma_{i\alpha_n h_n} S_{j\nu}(W) + S_{j\nu}(\varepsilon))^2 - \mu_{i,\nu,j}\right)}{\sqrt{I_{i,\nu}}}\,,$$

$$\widetilde{\mathbb{L}}^{(n)}_{i,\nu} = \frac{\sum_{j=1}^{\lfloor nh_n\rfloor -1} w_{ij\nu}\left(\left(\sigma_{(i-1)\alpha_n h_n} S_{j\nu}(W) + S_{j\nu}(\varepsilon)\right)^2 - \mu_{i,\nu,j}\right)}{\sqrt{\widetilde{I}_{i,\nu}}}\,.$$

The distributions of $\left(\mathbb{L}^{(n)}_{i,\nu}\right)_\nu$ and $\left(\widetilde{\mathbb{L}}^{(n)}_{i+1,\nu}\right)_\nu$ do not depend on the volatility. Therefore, and due to the independence of Brownian increments, the latter are two independent families. Furthermore, the independence of Brownian increments also yields that each family itself forms an independent family in ν. Taking into account the remark in Komlós et al. [42] below Theorem 4, we can proceed as follows. Since we want to ensure the existence of a properly approximating independent Gaussian family $(Z_i)_i$, according to Theorem 4 in Komlós et al. [42], we have to pick a function H such that

$$\frac{H(x)}{x^{3+\delta}} \text{ is increasing for some } \delta > 0\,, \tag{3.53}$$

$$\frac{\log\left(H\left(|x|\right)\right)}{x} \text{ is decreasing and} \tag{3.54}$$

$$\int H\left(|x|\right)\, d\mathbb{P}_{\mathbb{L}_{i,\nu}^{(n)}} < \infty\,. \tag{3.55}$$

We pick a power function H and set $H(x) = |x|^p$ with some $p \geq 4$ such that (3.53) and (3.54) are fulfilled. For the latter condition (3.55), by Jensen's inequality and Rosenthal's inequality, we require at this point (3.4) up to $m = 8$. In order to control the remainder term in the approximation, we take into account that

$$\max_i \left| \sum_{\nu=1}^{(i+1)\alpha_n} \left(\mathbb{L}_{i,\nu}^{(n)} - Z_\nu\right) - \sum_{\nu=1}^{i\alpha_n} \left(\mathbb{L}_{i,\nu}^{(n)} - Z_\nu\right) \right| \leq 4 \cdot \max_i \left| \sum_{\nu=1}^{(i+1)\alpha_n} \left(\mathbb{L}_{i,\nu}^{(n)} - Z_\nu\right) \right|.$$

Furthermore, the triangle inequality and the Markov inequality yield

$$\begin{aligned}
&\mathbb{P}\left[\max_i \left| \sum_{\nu=1}^{(i+1)\alpha_n} \left(\mathbb{L}_{i,\nu}^{(n)} - Z_\nu\right) \right| \geq x_n \right] \\
&\leq \mathbb{P}\left[\max_i \sum_{\nu=1}^{(i+1)\alpha_n} \left|\mathbb{L}_{i,\nu}^{(n)} - Z_\nu\right| \geq x_n \right] \leq \sum_{\nu=1}^{h_n^{-1}} \mathbb{P}\left[\left|\mathbb{L}_{i,\nu}^{(n)} - Z_\nu\right| \geq x_n\right] \\
&\leq \sum_{\nu=1}^{h_n^{-1}} x_n^{-p} \mathbb{E}\left[\left|\mathbb{L}_{i,\nu}^{(n)} - Z_\nu\right|^p\right].
\end{aligned}$$

Applying (1.6) in Sakhanenko [56], we get

$$\mathbb{P}\left[\max_i \left| \sum_{\nu=1}^{(i+1)\alpha_n} \left(\mathbb{L}_{i,\nu}^{(n)} - Z_\nu\right) \right| \geq x_n \right] \leq \sum_{\nu=1}^{h_n^{-1}} x_n^{-p} \mathbb{E}\left[\left|\mathbb{L}_{i,\nu}^{(n)}\right|^p\right] \leq C h_n^{-1} x_n^{-p},$$

where $C > 0$ is the positive constant given in (1.6) in Sakhanenko [56]. We set

$$x_n = \sqrt{\alpha_n}\left(\log\left(h_n^{-1}\right)\right)^{-1/2}.$$

Since there are more bins than big blocks, the conditions of Theorem 4 in Komlós et al. [42] are fulfilled. Furthermore, we can choose p by (3.5) such that

$$\alpha_n^{-p/2} h_n^{-1} = \mathcal{O}\left(\left(\log\left(h_n^{-1}\right)\right)^{-p/2}\right).$$

So the remainder term fulfills

$$\max_i \left| \sum_{\nu=1}^{(i+1)\alpha_n} \left(\mathbb{L}_{i,\nu}^{(n)} - Z_\nu \right) - \sum_{\nu=1}^{i\alpha_n} \left(\mathbb{L}_{i,\nu}^{(n)} - Z_\nu \right) \right| = \mathcal{O}_{\mathbb{P}} \left(\sqrt{\alpha_n} \left(\log \left(h_n^{-1} \right) \right)^{-1/2} \right) .$$

Let $\mathbb{B}$ be the Brownian Motion in the invariance principle. This implies that the family $(Z_i)_i$ defined as

$$Z_i := \alpha_n^{-1/2} (\mathbb{B}((i+1)\alpha_n) - \mathbb{B}(i\alpha_n))$$

are i.i.d. standard normal variables. We set

$$\eta_i := \frac{1}{\sqrt{\alpha_n}} \left(\mathbb{B}\left((i+1)\,\alpha_n \right) - \mathbb{B}\left(i\alpha_n \right) + \mathcal{O}_{\mathbb{P}} \left(\left(\log \left(n \right) \right)^{-1/2} \right) \right) .$$

The scaling properties of Brownian motion and the upper bound given for the remainder term give the desired result using Lemma 1 in Wu and Zhao [61] applied to $(\eta_i)_i$.

Proof of Corollary 3.9

The proof of Corollary 3.9 works along the same lines as the one of Theorem 3.8. More precisely,

(a) in a first step, we have to show that the overlapping versions $\overline{RV}_{n,i}^{ov}$ can be replaced by $\overline{Z}_{n,i}^{ov}$. In a second step, we

(b) have to do a shift in the volatility and proceed

(c) by showing that the estimated asymptotic standard deviations can be replaced by their limits and that

(d) the difference between oracle and adaptive versions is asymptotically negligible, where the final step is to

(e) use a limit theorem for extreme value statistics similar to Lemma 1 in Wu and Zhao [61]. The appropriate tool for the overlapping versions is given by Lemma 2 in Wu and Zhao [61], which can be directly applied choosing H as the rectangular kernel. The latter works, since even if the big blocks may intersect, it is crucial that the bins remain to be disjoint.

Starting with (a) we will argue that the estimates provided in the proof of Theorem 3.8 are sufficient to conclude the limit for the overlapping statistics. We have to show that

$$\max_{i=\alpha_n,\dots,h_n^{-1}-\alpha_n} \left| \left| \frac{\overline{RV}_{n,i}^{ov} - \overline{RV}_{n,i+1}^{ov}}{|\overline{RV}_{n,i+1}^{ov}|^{3/4}} \right| - \left| \frac{\overline{Z}_{n,i}^{ov} - \overline{Z}_{n,i+1}^{ov}}{|\overline{Z}_{n,i+1}^{ov}|^{3/4}} \right| \right| = \mathcal{O}_{\mathbb{P}}(\alpha_n^{-1/2} \log(n)^{-1/2}) .$$

The triangle inequality, the decomposition (3.25) and a Taylor expansion yield that it is sufficient to prove

$$\max_{i=\alpha_n,\dots,h_n^{-1}-1} \sqrt{\alpha_n \log(n)}\, \big|\overline{RV}^{ov}_{n,i} - \overline{Z}^{ov}_{n,i}\big| \overset{\mathbb{P}}{\longrightarrow} 0\,.$$

The key step proving this is to consider the term corresponding to A_n in (3.26). It is basically sufficient to translate the terms $A_n^{1,1}$ and $A_n^{1,2}$ to the overlapping case. Starting with $A_n^{1,1}$ we have to take into account the fact that in the overlapping case, the index set $i \in \{\alpha_n, \dots, h_n^{-1}-\alpha_n\}$ is a factor α_n times larger than the index set for the non-overlapping case. But since we can adapt the exponent r in the Markov inequality by (3.4), we get a similar upper bound for $A_n^{1,1}$. Considering the corresponding part to the term $A_n^{1,2}$ we proceed as follows using Assumption (H_0-$\mathfrak{a}$) and (3.31):

$$\begin{aligned}
&\frac{\sqrt{\log(n)}}{\sqrt{\alpha_n}} \sum_{\ell=1}^{\alpha_n} \sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{ij\ell} \int_{(\ell+i-1)h_n}^{(\ell+i-1+\alpha_n)h_n} \Big(\xi_{ij\ell}^{(n)}(\tau)\Big)^2 \big|\sigma_\tau^2 - \sigma_{ih_n}^2\big|\, d\tau \\
&\le L_n \frac{\sqrt{\log(n)}}{\sqrt{\alpha_n} h_n} \sum_{\ell=1}^{\alpha_n} (\ell h_n)^{\mathfrak{a}}\, h_n \le L_n \sqrt{\log(n)} \sqrt{\alpha_n}\, (\alpha_n h_n)^{\mathfrak{a}} \longrightarrow 0, \text{ as } n \to \infty.
\end{aligned}$$

Concerning (b) we have to show that

$$\max_{i=\alpha_n,\dots,h_n^{-1}-\alpha_n} \left\| \frac{\overline{Z}^{ov}_{n,i} - \overline{Z}^{ov}_{n,i+1}}{|\overline{Z}^{ov}_{n,i+1}|^{3/4}} \right| - \left| \frac{\overline{Z}^{ov}_{n,i} - \widetilde{Z}^{ov}_{n,i+1}}{|\widetilde{Z}^{ov}_{n,i+1}|^{3/4}} \right\| = \mathcal{O}_{\mathbb{P}}(\alpha_n^{-1/2} \log(n)^{-1/2})\,.$$

Again, after a proper decomposition of the terms and a Taylor expansion, it is sufficient to show that

$$\max_{i=\alpha_n,\dots,h_n^{-1}-\alpha_n} |\overline{Z}^{ov}_{n,i} - \widetilde{Z}^{ov}_{n,i}| = \mathcal{O}_{\mathbb{P}}(\alpha_n^{-1/2} \log(n)^{-1/2})\,.$$

The discussion of this term works very similar as in the non-overlapping case. Using (H_0-$\mathfrak{a}$) and the central limit theorem, as presented above, we can conclude the desired asymptotic behaviour by adapting the exponent r in the Markov inequality. The third and fourth steps (c) and (d) are analogues of Propositions 3.18 and 3.19. Since the upper bound, which is presented in Bibinger and Reiß [12], is not affected for overlapping big blocks, we omit the details. Concerning (e) let us only mention that an additional tool which is necessary is Lévy's modulus of continuity theorem in order to control the discretization error. Then, the limit (3.13) in Corollary 3.9 is an immediate consequence of Lemma 2 in Wu and Zhao [61].

Proof of Proposition 3.10

We decompose the process $Y_t = C_t + J_t + \varepsilon_t$ with the continuous semimartingale part

$$C_t = X_0 + \int_0^t a_s\, ds + \int_0^t \sigma_s\, dW_s$$

and write $\overline{RV}^{ad}_{n,i}(C+\varepsilon)$ for the statistics (3.8) applied to observations of a process where the jump part $(J_t)_{t\in[0,1]}$ is eliminated. We begin with some preliminaries for the proof. Throughout this proof, K is a generic constant that may change from line to line. For $N^n(v_n)$ a sequence of counting processes with $N^n_t(v_n) = \int_0^t\int_{\mathbb{R}} \mathbb{1}_{\{\gamma(x)>v_n\}}\mu(ds,dx)$, with $\gamma(x)$ from Assumption 3.2, we have by (13.1.14) from Jacod and Protter [35] that

$$\mathbb{P}\Big(N^n_{h_n}(v_n)\ge l\Big)\le Kh_n^l v_n^{-rl} \tag{3.56}$$

with r from (3.3). We may restrict to the more difficult result for $\overline{V}^{ov,\tau}_n$ with overlapping statistics. With an analogous decomposition as in (3.25), the proof reduces to

$$\max_{i=\alpha_n,\ldots,h_n^{-1}}\big|\overline{RV}^{tr}_{n,i}-\overline{RV}^{ad}_{n,i}(C+\varepsilon)\big| = \mathcal{O}_{\mathbb{P}}\Big(\big(\log(n)\alpha_n\big)^{-1/2}\Big)\,. \tag{3.57}$$

We separate bins on that truncations occur from (most) other bins

$$\begin{aligned}
&\max_{i=\alpha_n,\ldots,h_n^{-1}}\big|\overline{RV}^{tr}_{n,i}-\overline{RV}^{ad}_{n,i}(C+\varepsilon)\big|\\
&=\alpha_n^{-1}\max_i\Big|\sum_{\ell=i-\alpha_n+1}^{i}\Big(\hat\sigma^{2,ad}_{(\ell-1)h_n}\mathbb{1}_{\{|\hat\sigma^{2,ad}_{(\ell-1)h_n}|\le h_n^{\tau-1}\}}-\hat\sigma^{2,ad}_{(\ell-1)h_n}(C+\varepsilon)\Big)\Big|\\
&\le\alpha_n^{-1}\max_i\Big|\sum_{\ell=i-\alpha_n+1}^{i}\mathbb{1}_{\{|\hat\sigma^{2,ad}_{(\ell-1)h_n}|>h_n^{\tau-1}\}}\hat\sigma^{2,ad}_{(\ell-1)h_n}(C+\varepsilon)\Big|\\
&+\alpha_n^{-1}\max_i\Big|\sum_{\ell=i-\alpha_n+1}^{i}\mathbb{1}_{\{|\hat\sigma^{2,ad}_{(\ell-1)h_n}|\le h_n^{\tau-1}\}}\sum_{j=1}^{\lfloor nh_n\rfloor-1}\hat w_{j\ell}\Big(S^2_{j\ell}(J)+2S_{j\ell}(J)S_{j\ell}(\varepsilon)+2S_{j\ell}(J)S_{j\ell}(C)\Big)\Big|,
\end{aligned}$$

and consider the two terms separately. For the second maximum the term with $S^2_{j\ell}(J)$ is the most involved one and we prove that

$$\max_i\Big|\sum_{\ell=i-\alpha_n+1}^{i}\mathbb{1}_{\{|\hat\sigma^{2,ad}_{(\ell-1)h_n}|\le h_n^{\tau-1}\}}\sum_{j=1}^{\lfloor nh_n\rfloor-1}\hat w_{j\ell}\,S^2_{j\ell}(J)\Big| = \mathcal{O}_{\mathbb{P}}\left(\sqrt{\frac{\alpha_n}{\log(n)}}\right)\,. \tag{3.58}$$

With some $c,\tilde c\in(0,1)$ the relation

$$\mathbb{1}_{\{|\hat\sigma^{2,ad}_{(\ell-1)h_n}|\le h_n^{\tau-1}\}}\le\mathbb{1}_{\{\sum_{j=1}^{\lfloor nh_n\rfloor-1}\hat w_{j\ell}\,S^2_{j\ell}(J)\le c\,h_n^{\tau-1}\}}+\mathbb{1}_{\{|\hat\sigma^{2,ad}_{(\ell-1)h_n}(C+\varepsilon)|>\tilde c\,h_n^{\tau-1}\}}$$

can be used to decompose the term in two addends. First, we prove that

$$\max_i\Big|\sum_{\ell=i-\alpha_n+1}^{i}\mathbb{1}_{\{\sum_{j=1}^{\lfloor nh_n\rfloor-1}\hat w_{j\ell}\,S^2_{j\ell}(J)\le c\,h_n^{\tau-1}\}}\sum_{j=1}^{\lfloor nh_n\rfloor-1}\hat w_{j\ell}\,S^2_{j\ell}(J)\Big| = \mathcal{O}_{\mathbb{P}}\left(\sqrt{\frac{\alpha_n}{\log(n)}}\right)\,. \tag{3.59}$$

Using the elementary estimate $|\Phi_j(t)| \leq \sqrt{2}h_n^{-1/2}$ and that $\sum_{j\geq 1}\hat{w}_{j\ell} = 1$, we obtain the bound

$$\sum_{j=1}^{\lfloor nh_n\rfloor -1}\hat{w}_{j\ell}\,S_{j\ell}^2(J) = \sum_{j=1}^{\lfloor nh_n\rfloor -1}\hat{w}_{j\ell}\Big(\sum_{i=1}^{n}\Delta_i^n J\Phi_{j\ell}\left(\frac{i}{n}\right)\Big)^2 \leq 2h_n^{-1}\Big(\sum_{i=\lfloor(\ell-1)nh_n\rfloor}^{\lfloor \ell nh_n\rfloor}|\Delta_i^n J|\Big)^2. \tag{3.60}$$

We deduce the upper bound

$$\begin{aligned}
&\max_i\Big|\sum_{\ell=i-\alpha_n+1}^{i}\mathbb{1}_{\{\sum_{j=1}^{\lfloor nh_n\rfloor-1}\hat{w}_{j\ell}\,S_{j\ell}^2(J)\leq c\,h_n^{\tau-1}\}}\sum_{j=1}^{\lfloor nh_n\rfloor -1}\hat{w}_{j\ell}\,S_{j\ell}^2(J)\Big|\\
&\leq \max_i\Big|\sum_{\ell=i-\alpha_n+1}^{i}\Big(\Big(\sum_{j=1}^{\lfloor nh_n\rfloor -1}\hat{w}_{j\ell}\,S_{j\ell}^2(J)\Big)\wedge c\,h_n^{\tau-1}\Big)\Big|\\
&\leq \max_i\Big|\sum_{\ell=i-\alpha_n+1}^{i}\Big(\Big(2h_n^{-1}\Big(\sum_{i=\lfloor(\ell-1)nh_n\rfloor}^{\lfloor \ell nh_n\rfloor}|\Delta_i^n J|\Big)^2\Big)\wedge c\,h_n^{\tau-1}\Big)\Big|.
\end{aligned}$$

We decompose this term as follows

$$\begin{aligned}
&\max_i\Big|\sum_{\ell=i-\alpha_n+1}^{i}\Big(\Big(2h_n^{-1}\Big(\sum_{i=\lfloor(\ell-1)nh_n\rfloor}^{\lfloor \ell nh_n\rfloor}|\Delta_i^n J|\Big)^2\Big)\wedge c\,h_n^{\tau-1}\Big)\Big|\\
&\leq \max_i\Big|\sum_{\ell=i-\alpha_n+1}^{i}2h_n^{-1}\Big(\sum_{i=\lfloor(\ell-1)nh_n\rfloor}^{\lfloor \ell nh_n\rfloor}|\Delta_i^n J|\Big)^2\mathbb{1}_{\big\{\sum_{i=\lfloor(\ell-1)nh_n\rfloor}^{\lfloor \ell nh_n\rfloor}|\Delta_i^n J|\leq\sqrt{c/2}h_n^{\tau/2}\big\}}\Big|\\
&+\max_i\Big|\sum_{\ell=i-\alpha_n+1}^{i}ch_n^{\tau-1}\mathbb{1}_{\big\{\sum_{i=\lfloor(\ell-1)nh_n\rfloor}^{\lfloor \ell nh_n\rfloor}|\Delta_i^n J|>\sqrt{c/2}h_n^{\tau/2}\big\}}\Big|\\
&\leq \max_i\Big|\sum_{\ell=i-\alpha_n+1}^{i}2h_n^{-1}\big|J_{\ell h_n}-J_{(\ell-1)h_n}\big|^2\,\mathbb{1}_{\{|J_{\ell h_n}-J_{(\ell-1)h_n}|\leq\sqrt{c/2}h_n^{\tau/2}\}}\Big|\\
&+\max_i\Big|\sum_{\ell=i-\alpha_n+1}^{i}2h_n^{-1}\Big(\big|J_{\ell h_n}-J_{(\ell-1)h_n}\big|^2\\
&\qquad-\Big(\sum_{i=\lfloor(\ell-1)nh_n\rfloor}^{\lfloor \ell nh_n\rfloor}|\Delta_i^n J|\Big)^2\Big)\mathbb{1}_{\big\{\sum_{i=\lfloor(\ell-1)nh_n\rfloor}^{\lfloor \ell nh_n\rfloor}|\Delta_i^n J|\leq\sqrt{c/2}h_n^{\tau/2}\big\}}\Big|\\
&+\max_i\Big|\sum_{\ell=i-\alpha_n+1}^{i}ch_n^{\tau-1}\mathbb{1}_{\big\{\sum_{i=\lfloor(\ell-1)nh_n\rfloor}^{\lfloor \ell nh_n\rfloor}|\Delta_i^n J|>\sqrt{c/2}h_n^{\tau/2}\big\}}\Big|\\
&=\Gamma_1+\Gamma_2+\Gamma_3\,,
\end{aligned}$$

where we use the triangle inequality, that $\mathbb{1}_{\{B\le C\}} \le \mathbb{1}_{\{A\le C\}}$ if $A \le B$, and elementary inequalities for the maximum. We begin with Γ_2. It holds for $\varpi > 0$ arbitrarily small that

$$\Gamma_2 \le 4\max_i \Big| \sum_{\ell=i-\alpha_n+1}^{i} h_n^{-1}\Big(\sum_{i=\lfloor(\ell-1)nh_n\rfloor}^{\lfloor \ell nh_n\rfloor} |\Delta_i^n J|\Big)^2 \mathbb{1}_{\big\{\sum_{i=\lfloor(\ell-1)nh_n\rfloor}^{\lfloor \ell nh_n\rfloor} |\Delta_i^n J| \le h_n^{2/3+\varpi}\big\}}\Big|$$
$$+ 2\max_i \Big| \sum_{\ell=i-\alpha_n+1}^{i} h_n^{-1}\Big(\big|J_{\ell h_n} - J_{(\ell-1)h_n}\big|^2 - \Big(\sum_{i=\lfloor(\ell-1)nh_n\rfloor}^{\lfloor \ell nh_n\rfloor} |\Delta_i^n J|\Big)^2\Big)$$
$$\times \mathbb{1}_{\{N^n_{\ell h_n}(h_n^{2/3+\varpi}) - N^n_{(\ell-1)h_n}(h_n^{2/3+\varpi}) \ge 2\}}\,\mathbb{1}_{\big\{h_n^{2/3+\varpi} \le \sum_{i=\lfloor(\ell-1)nh_n\rfloor}^{\lfloor \ell nh_n\rfloor} |\Delta_i^n J| \le \sqrt{c/2}h_n^{\tau/2}\big\}}\Big|,$$

with $N_t^n(v_n)$ from (3.56). The additional indicator function in the last addend may be added, since $|J_{\ell h_n} - J_{(\ell-1)h_n}| = \sum_{i=\lfloor(\ell-1)nh_n\rfloor}^{\lfloor \ell nh_n\rfloor} |\Delta_i^n J|$ when there is at most one jump on the bin. By (3.56) with $v_n = h_n^{2/3+\varpi}$ and $l = 2$, we obtain for the Poisson process $N_t^n(h_n^{2/3+\varpi})$:

$$\mathbb{P}\Big(N^n_{\ell h_n}(h_n^{2/3+\varpi}) - N^n_{(\ell-1)h_n}(h_n^{2/3+\varpi}) \ge 2\Big) \le h_n^2 h_n^{-2r(2/3+\varpi)},$$

for all ℓ, and we infer that

$$\begin{aligned}\Gamma_2 &= \mathcal{O}\big(\alpha_n h_n^{-1} h_n^{4/3+2\varpi}\big) + \mathcal{O}_{\mathbb{P}}\big(\alpha_n \log(\alpha_n) h_n^{1+\tau} h_n^{-2r(2/3+\varpi)}\big)\\ &= \scriptstyle{\mathcal{O}}\big(\alpha_n^{1/2}(\log(n))^{-1/2}\big) + \scriptstyle{\mathcal{O}}_{\mathbb{P}}\big(\alpha_n^{1/2}(\log(n))^{-1/2}\big).\end{aligned}$$

We used that $\alpha_n \le h_n^{2/3}$ by (3.5), since $\mathfrak{a} \le 1$, for the first term and that by Condition (3.15):

$$\alpha_n^{1/2}\log(\alpha_n)\sqrt{\log(n)}\,h_n^{1+\tau} h_n^{-2r(2/3+\varpi)} \to 0\,. \tag{3.61}$$

Define the sequence of random variables

$$\mathcal{Z}_\ell = \Big(\big(J_{\ell h_n} - J_{(\ell-1)h_n}\big)\mathbb{1}_{\{|J_{\ell h_n} - J_{(\ell-1)h_n}| \le \sqrt{c/2}h_n^{\tau/2}\}}\Big)^2, \quad \ell = 1, \ldots, h_n^{-1}.$$

We have that

$$\Gamma_1 = \max_{i=\alpha_n,\ldots,h_n^{-1}} \sum_{\ell=i-\alpha_n+1}^{i} 2h_n^{-1}\mathcal{Z}_\ell\,.$$

From equation (54) of Aït-Sahalia and Jacod [2], we obtain the bounds

$$\mathbb{E}\Big[\Big(\big|J_{\ell h_n} - J_{(\ell-1)h_n}\big| \wedge \sqrt{c/2}\,h_n^{\tau/2}\Big)^2\Big] \le K h_n h_n^{\tau(1-r/2)}, \tag{3.62a}$$

$$\mathbb{V}\mathrm{ar}\Big(\Big(\big|J_{\ell h_n} - J_{(\ell-1)h_n}\big| \wedge \sqrt{c/2}\,h_n^{\tau/2}\Big)^2\Big) \le K h_n h_n^{2\tau - r\tau/2}. \tag{3.62b}$$

Observe that

$$\Gamma_1 = \max_{i=\alpha_n,\ldots,h_n^{-1}} 2h_n^{-1}\Big(\sum_{\ell=1}^{i}\mathcal{Z}_\ell - \sum_{\ell=1}^{i-\alpha_n}\mathcal{Z}_\ell\Big) \le 2h_n^{-1}\max_{i=1,\ldots,h_n^{-1}}\sum_{\ell=1}^{i}\mathcal{Z}_\ell\,.$$

Since $\int_0^t\int_{\mathbb{R}}\delta(s,x)\mathbb{1}_{\{\gamma(x)\le\sqrt{c/2}\,h_n^{\tau/2}\}}(\mu-\nu)(dx,dx)$ is a martingale, $\big(\sum_{\ell=1}^{i}2h_n^{-1}\mathcal{Z}_\ell\big)_{1\le i\le h_n^{-1}}$ is a submartingale as the martingale increments are uncorrelated and a squared martingale is always a submartingale. We apply Doob's submartingale maximal inequality which yields

$$\lambda\,\mathbb{P}\Big(\max_{i=\alpha_n,\ldots,h_n^{-1}}\sum_{\ell=1}^{i}2h_n^{-1}\mathcal{Z}_\ell\ge\lambda\Big)\le 2h_n^{-1}\mathbb{E}\Big[\sum_{\ell=1}^{h_n^{-1}}\mathcal{Z}_\ell\Big]\propto h_n^{\tau(1-r/2)}\,, \tag{3.63}$$

such that $\mathbb{P}\Big(\max_{i=\alpha_n,\ldots,h_n^{-1}}\sum_{\ell=1}^{i}2h_n^{-1}\mathcal{Z}_\ell\ge\lambda\Big)\to 0$ for $\lambda^{-1}=\scriptstyle{\mathcal{O}}(h_n^{\tau(r/2-1)})$. Thus, Γ_1 is negligible as long as

$$\alpha_n^{1/2}\sqrt{\log(n)}h_n^{\tau(1-r/2)}\to 0\,. \tag{3.64}$$

From Condition (3.15) we have that $r<2-\tau^{-1}\beta$ and it follows that $\beta<\tau(2-r)$, what ensures the above relation. Under this condition, Γ_3 becomes negligible as well, since with (3.56) for $l=1$ and $v_n=h_n^{\tau/2}$, we obtain that

$$\Gamma_3=\mathcal{O}_{\mathbb{P}}\Big(\alpha_n\log(\alpha_n)h_n^{\tau(1-r/2)}\Big)=\scriptstyle{\mathcal{O}}_{\mathbb{P}}\left(\sqrt{\frac{\alpha_n}{\log(n)}}\right)\,.$$

We have proved (3.59). Finally, we show that

$$\max_i\Big|\sum_{\ell=i-\alpha_n+1}^{i}\mathbb{1}_{\{|\hat\sigma^{2,ad}_{(\ell-1)h_n}|>h_n^{\tau-1}\}}\hat\sigma^{2,ad}_{(\ell-1)h_n}(C+\varepsilon)\Big|=\scriptstyle{\mathcal{O}}_{\mathbb{P}}\left(\sqrt{\frac{\alpha_n}{\log(n)}}\right)\,, \tag{3.65}$$

and discuss the similar remaining second term for (3.58). With some $c,\tilde c\in(0,1)$, we use the relation

$$\mathbb{1}_{\{|\hat\sigma^{2,ad}_{(\ell-1)h_n}|>h_n^{\tau-1}\}}\le\mathbb{1}_{\{\sum_{j=1}^{\lfloor nh_n\rfloor-1}\hat w_{j\ell}\,S^2_{j\ell}(J)>c\,h_n^{\tau-1}\}}+\mathbb{1}_{\{|\hat\sigma^{2,ad}_{(\ell-1)h_n}(C+\varepsilon)|>\tilde c\,h_n^{\tau-1}\}}\,.$$

With Markov's inequality, we obtain that

$$\mathbb{P}\Big(\max_{k=1,\ldots,h_n^{-1}}\big|\hat\sigma^{2,ad}_{(k-1)h_n}(C+\varepsilon)\big|>\lambda v_n\Big)\le h_n^{-1}\mathbb{P}\Big(\big|\hat\sigma^{2,ad}_{h_n}(C+\varepsilon)\big|>\lambda v_n\Big) \tag{3.66}$$

$$\le h_n^{-1}K\frac{\mathbb{E}\Big[\big|\hat\sigma^{2,ad}_{h_n}(C+\varepsilon)\big|^p\Big]}{(\lambda v_n)^p}$$

$$=\mathcal{O}\big(h_n^{-1}\log(n)v_n^{-p}\big)\,, \tag{3.67}$$

using moment bounds from Lemma 2 of Bibinger and Winkelmann [13] under condition (3.4). We derive that

$$\max_{k=1,\dots,h_n^{-1}} \big|\hat\sigma^{2,ad}_{(k-1)h_n}(C+\varepsilon)\big| = \mathcal{O}_{\mathbb{P}}\big(h_n^{-\varpi}\big) \tag{3.68}$$

for arbitrary $\varpi > 0$. In particular, for ϖ from (3.5)

$$\max_{k=1,\dots,h_n^{-1}} \big|\hat\sigma^{2,ad}_{(k-1)h_n}(C+\varepsilon)\big| = \mathcal{O}_{\mathbb{P}}\big(h_n^{-\varpi/2}\big) = \scriptstyle{\mathcal{O}}_{\mathbb{P}}\left(\sqrt{\frac{\alpha_n}{\log(n)}}\right)\,.$$

With (3.60), we obtain the estimate

$$\begin{aligned}
&\max_i \Big|\sum_{\ell=i-\alpha_n+1}^{i} \mathbb{1}_{\{\sum_{j=1}^{\lfloor nh_n\rfloor -1}\hat w_{j\ell}\,S^2_{j\ell}(J)>c\,h_n^{\tau-1}\}}\hat\sigma^{2,ad}_{(\ell-1)h_n}(C+\varepsilon)\Big|\\
&\le \max_i \Big|\sum_{\ell=i-\alpha_n+1}^{i} \mathbb{1}_{\big\{\sum_{i=\lfloor(\ell-1)nh_n\rfloor}^{\lfloor \ell nh_n\rfloor}|\Delta_i^n J|>\sqrt{c/2}h_n^{\tau/2}\big\}}\hat\sigma^{2,ad}_{(\ell-1)h_n}(C+\varepsilon)\Big|\,.
\end{aligned}$$

Since for $r\tau<1$, we have by (3.56) that

$$\mathbb{P}\Big(\bigcup_{\ell=1}^{h_n^{-1}}\big\{\big(N^n_{\ell h_n}\big(\sqrt{c/2}h_n^{\tau/2}\big)-N^n_{(\ell-1)h_n}\big(\sqrt{c/2}h_n^{\tau/2}\big)\big)\ge 2\big\}\Big)\le K h_n^{-1}h_n^2 h_n^{-\tau r}\to 0\,,$$

we may consider instead

$$\max_i \Big|\sum_{\ell=i-\alpha_n+1}^{i}\mathbb{1}_{\{|J_{\ell h_n}-J_{(\ell-1)h_n}|>\sqrt{c/2}\,h_n^{\tau/2}\}}\hat\sigma^{2,ad}_{(\ell-1)h_n}(C+\varepsilon)\Big|\,.$$

It is thus sufficient to show that

$$\mathbb{P}\Big(\bigcup_{k=\alpha_n}^{h_n^{-1}}\big\{\big(N^n_{(k-1)h_n}\big(\sqrt{c/2}h_n^{\tau/2}\big)-N^n_{(k-\alpha_n)h_n}\big(\sqrt{c/2}h_n^{\tau/2}\big)\big)\ge l\big\}\Big)\le K h_n^{-1+l}\alpha_n^l h_n^{-\tau r l/2}\to 0\,,$$

for some $l<\infty$ where we have applied an inequality analogous to (3.56). This holds true, since

$$h_n^{-1+l}\alpha_n^l h_n^{-\tau r l/2}\le K\,h_n^{-1}\big(h_n^{1-\beta-r\tau/2}\big)^l$$

and $r\tau < 2(1-\beta)$ by Condition (3.15). Using (3.60), (3.66) and that the squared jumps are summable, we obtain that for $\tau < 1-(3-2\varpi)/p$ with $p\in\mathbb{N}$

$$\begin{aligned}
&\max_i \Big|\sum_{\ell=i-\alpha_n+1}^{i}\mathbb{1}_{\{|\hat\sigma^{2,ad}_{(\ell-1)h_n}(C+\varepsilon)|>\tilde c\,h_n^{\tau-1}\}}\sum_{j=1}^{\lfloor nh_n\rfloor-1}\hat w_{j\ell}\,S^2_{j\ell}(J)\Big|\\
&\le h_n^{-1}\mathbb{1}_{\{\max_\ell|\hat\sigma^{2,ad}_{(\ell-1)h_n}(C+\varepsilon)|>\tilde c\,h_n^{\tau-1}\}}\\
&=\mathcal{O}_{\mathbb{P}}\big(h_n^{-3/2+p/2(1-\tau)}\big)=\scriptstyle{\mathcal{O}}_{\mathbb{P}}\left(\sqrt{\frac{\alpha_n}{\log(n)}}\right)\,.
\end{aligned}\tag{3.69}$$

This is sufficient for

$$\max_i \left| \sum_{\ell=i-\alpha_n+1}^{i} \mathbb{1}_{\{|\hat\sigma^{2,ad}_{(\ell-1)h_n}(C+\varepsilon)|>\tilde c\, h_n^{\tau-1}\}} \hat\sigma^{2,ad}_{(\ell-1)h_n}(C+\varepsilon)\right| = \mathcal{O}_{\mathbb{P}}\left(\sqrt{\frac{\alpha_n}{\log(n)}}\right). \qquad (3.70)$$

Equations (3.59) and (3.69) imply Equation (3.58). Maxima of the terms with cross terms $S_{j\ell}(\varepsilon)S_{j\ell}(J)$ and $S_{j\ell}(\varepsilon)S_{j\ell}(C)$ can be handled similarly (or with Cauchy-Schwarz) and are of smaller order. Equations (3.58) and (3.65) imply Equation (3.57) what finishes the proof of Proposition 3.10. □

Proof of Theorem 3.12

We have to show that (3.17), (3.18), (3.19) and (3.20) yield asymptotically tests with power 1. Concerning (3.17), that is, under (H_1-$\mathfrak{a}$),

$$\mathbb{P}\left[\overline{V}_n \ge \alpha_n^{-1/2}\left((\log(m_n))^{-1/2} c_\alpha + \gamma_{m_n}\right)\right] \longrightarrow 1 \quad \text{as } n \to +\infty. \qquad (3.71)$$

We set

$$\hat V_{n,i} = \left| \frac{\zeta_i^n - \zeta_{i+1}^n}{\sqrt{8\hat\eta}\left|\overline{RV}^{ad}_{n,i+1}\right|^{3/4}} \right|, \quad i = 0, \ldots, \lfloor(\alpha_n h_n)^{-1}\rfloor - 2 \qquad (3.72)$$

with

$$\zeta_i^n := \alpha_n^{-1} \sum_{\ell=1}^{\alpha_n}\left(\hat\sigma^{2,ad}_{h_n(i\alpha_n+(\ell-1))} - \mathbb{E}\left[\hat\sigma^{2,ad}_{h_n(i\alpha_n+(\ell-1))}\right]\right).$$

For $\theta - \lfloor(\alpha_n h_n)^{-1}\theta\rfloor\alpha_n h_n > \alpha_n h_n/2$, set $i^* = \lfloor(\alpha_n h_n)^{-1}\theta\rfloor$. For $\theta - \lfloor(\alpha_n h_n)^{-1}\theta\rfloor\alpha_n h_n \le \alpha_n h_n/2$, set $i^* = \lfloor(\alpha_n h_n)^{-1}\theta\rfloor - 1$. Since $\theta \in (0,1)$, $i^* \ge 0$ for n sufficiently large. By the reverse triangle inequality we get:

$$\overline{V}_n \ge -\hat V_{n,i^*} + \left| \frac{\sum_{\ell=1}^{\alpha_n}\mathbb{E}\left[\hat\sigma^{2,ad}_{i^*\alpha_n h_n+(\ell-1)h_n}\right] - \sum_{\ell=1}^{\alpha_n}\mathbb{E}\left[\hat\sigma^{2,ad}_{(i^*+1)\alpha_n h_n+(\ell-1)h_n}\right]}{\alpha_n\sqrt{8\hat\eta}\left|\overline{RV}^{ad}_{n,i+1}\right|^{3/4}} \right|.$$

First of all, we can conclude by Theorem 3.8 that for all i:

$$\hat V_{n,i} = \mathcal{O}_{\mathbb{P}}\left(\alpha_n^{-1/2}\right).$$

Then we take into account that the sum over j is convex and $\hat\sigma^{2,ad}_{h_n(i\alpha_n+(\ell-1))}$ is already bias corrected with respect to the noise part. Furthermore, bounding the volatility from below, using the Itô isometry and

$$\sum_{\ell=1}^{\alpha_n}\sum_{j=1}^{nh_n-1} w_{ij\ell}\int_{h_n(i\alpha_n+(\ell-1))-n^{-1}}^{i\alpha_n h_n+\ell h_n}\left(\xi^{(n)}_{ij\ell}(s)\right)^2\sigma_s^2\,ds \propto \frac{1}{h_n}\int_{i\alpha_n h_n}^{(i+1)\alpha_n h_n}\sigma_s^2\,ds\,,$$

we obtain that with a constant $c>0$:

$$\overline{V}_n \geq -\mathcal{O}_{\mathbb{P}}\left(\frac{1}{\sqrt{\alpha_n}}\right) + c\left|\varsigma\left(i^*,n\right)-\varsigma\left(i^*+1,n\right)\right|\left(1-\scriptstyle{\mathcal{O}}_{\mathbb{P}}\left(1\right)\right),$$

with

$$\varsigma\left(i,n\right) := \frac{1}{\alpha_n h_n}\int_{i\alpha_n h_n}^{(i+1)\alpha_n h_n}\sigma_s^2\,ds\,.$$

Note that the denominator in (3.72) can be 'absorbed' by the constant c. We give a lower bound on $|\varsigma(i^*,n)-\varsigma(i^*+1,n)|$. Under the alternative hypothesis (H_1-$\mathfrak{a}$) we have for the continuous volatility part that

$$\left|\int_{i^*\alpha_n h_n}^{(i^*+1)\alpha_n h_n}\left(\sigma_s^{2,(c)}-\sigma_{s+\alpha_n h_n}^{2,(c)}\right)ds\right| \leq \alpha_n h_n L_n\left(\alpha_n h_n\right)^{\mathfrak{a}}\,.$$

The jump component of the volatility is most difficult to handle for $r=2$. If it satisfies (3.3) with some $r\geq 1$, we derive for some constant K_p dependent on p the bound

$$\begin{aligned}\forall s,t\geq 0:\ \mathbb{E}\left[\left|\sigma_t^{2,(j)}-\sigma_s^{2,(j)}\right|^p\Big|\mathcal{F}_s\right] &\leq K_p\,\mathbb{E}\Big[\Big(\int_s^t\int_{\mathbb{R}}(\gamma^r(x)\wedge 1)\lambda(dx)ds\Big)^{p/r}\Big]\\ &\leq K_p|t-s|^{((p/r)\wedge 1)}\,. \qquad (3.73)\end{aligned}$$

With $r=2$ and for $p=1$, we thus obtain for $i^*=\lfloor(\alpha_n h_n)^{-1}\theta\rfloor$ that

$$\begin{aligned}\left|\int_{i^*\alpha_n h_n}^{(i^*+1)\alpha_n h_n}\left(\sigma_s^{2,(j)}-\sigma_{s+\alpha_n h_n}^{2,(j)}-\Delta\sigma_\theta^2\mathbb{1}_{[0,\theta)}(s)\right)ds\right| &= \mathcal{O}_{\mathbb{P}}\Big(\int_{i^*\alpha_n h_n}^{(i^*+1)\alpha_n h_n}|\alpha_n h_n|^{1/2}\,ds\Big)\\ &= \mathcal{O}_{\mathbb{P}}((\alpha_n h_n)^{3/2})\,,\end{aligned}$$

and an analogous bound for $i^*=\lfloor(\alpha_n h_n)^{-1}\theta\rfloor-1$. Thus, we obtain that

$$\begin{aligned}&|\varsigma(i^*,n)-\varsigma(i^*+1,n)| = (\alpha_n h_n)^{-1}\left|\int_{i^*\alpha_n h_n}^{(i^*+1)\alpha_n h_n}\left(\sigma_s^2-\sigma_{s+\alpha_n h_n}^2\right)ds\right|\\ &\geq (\alpha_n h_n)^{-1}\left(\left|\int_{i^*\alpha_n h_n}^{(i^*+1)\alpha_n h_n}\left(\sigma_s^{2,(j)}-\sigma_{s+\alpha_n h_n}^{2,(j)}\right)ds\right|-\left|\int_{i^*\alpha_n h_n}^{(i^*+1)\alpha_n h_n}\left(\sigma_s^{2,(c)}-\sigma_{s+\alpha_n h_n}^{2,(c)}\right)ds\right|\right)\\ &\geq (\alpha_n h_n)^{-1}\min\left(\Big|\int_{i^*\alpha_n h_n}^{\theta}\Delta\sigma_\theta^2\,ds\Big|,\Big|\int_\theta^{(i^*+2)\alpha_n h_n}\Delta\sigma_\theta^2\,ds\Big|\right)\\ &\qquad -\mathcal{O}_{\mathbb{P}}((\alpha_n h_n)^{1/2})-L_n\left(\alpha_n h_n\right)^{\mathfrak{a}}\\ &\geq \frac{1}{2}\Delta\sigma_\theta^2-\mathcal{O}_{\mathbb{P}}((\alpha_n h_n)^{1/2})-L_n\left(\alpha_n h_n\right)^{\mathfrak{a}}\,,\end{aligned}$$

where we have applied the reverse triangle inequality. This implies (3.71). In the non-overlapping case two neighboring differences $|\varsigma(i,n)-\varsigma(i+1,n)|$ incorporate the volatility jump. Our above definition of i^* ensures that we consider the most affected one for the lower bound. A corresponding lower bound for $\overline{V}_n^{ov}$ in the overlapping case becomes simpler as we always include statistics over two neighboring blocks, such that θ is close to the end-point between the two blocks. Proving that

$$\mathbb{P}\left[\overline{V}_n \leq \alpha_n^{-1/2}\left((\log(m_n))^{-1/2}\, c_\alpha + \gamma_{m_n}\right)\right] \longrightarrow 1-\alpha \quad \text{as } n \to +\infty$$

under (H_0-$\mathfrak{a}$), is an immediate consequence of Theorem 3.8. This completes the proof for (3.17). We omit further details concerning (3.18), (3.19) and (3.20), since the estimates we have presented above can be readily adapted. □

Proof of Proposition 3.14

We adopt the following elementary lemma, related to Lemma B.1 in Aue et al. [6] and Lemma D.1. in Bibinger et al. [10].

Lemma 3.20. *Let $f(t)$ and $g(t)$ be functions on $[0,\theta]$ such that $f(t)$ is non-negative and increasing. As long as $f(\theta)-f(\theta-\gamma) > 2\sup_{0\le t\le\theta}|g(t)|$ for some $\gamma\in[0,\theta]$, we have that*

$$\operatorname{argmax}_{0\le t\le\theta}\big(f(t)+g(t)\big) \geq \theta-\gamma.$$

An analogous result holds, if $f(t)$ and $g(t)$ are functions on $[\theta,1]$ and $f(t)$ is decreasing.

Proof of Lemma 3.20.
Since

$$\sup_{0\le t<\theta-\gamma}|g(t)| - g(\theta) \leq 2\sup_{0\le t\le\theta}|g(t)| < f(\theta)-f(\theta-\gamma)\,,$$

we derive that

$$\begin{aligned}\max_{0\le t<\theta-\gamma}\big(f(t)+g(t)\big) &\leq \sup_{0\le t<\theta-\gamma}\big(f(t)\big) + \sup_{0\le t<\theta-\gamma}\big|g(t)\big| \\ &\leq f(\theta-\gamma) + \sup_{0\le t<\theta-\gamma}\big|g(t)\big| < f(\theta)+g(\theta)\,,\end{aligned}$$

such that $\operatorname{argmax}_{0\le t\le\theta}\big(f(t)+g(t)\big) \geq \theta-\gamma$. □

Let $\theta\in(0,1)$ be the change point, that is, the jump time of the volatility. Without loss of generality $\delta = \Delta\sigma_\theta^2 > 0$. Define $(i^*-1) = \lceil\theta h_n^{-1}\rceil$, the smallest integer such that $(i^*-1)h_n \geq \theta$ holds. We use the following decomposition of $\overline{V}_{n,i}^{\diamond}$ for $i=\alpha_n,\ldots,h_n^{-1}-\alpha_n$:

$$\overline{V}_{n,i}^{\diamond} = \alpha_n^{-1/2}\left|A_{n,i}+B_{n,i}+C_{n,i}+D_{n,i}\right|\,,$$

where

$$A_{n,i} = \sum_{\ell=i-\alpha_n+1}^{i} \big(\hat\sigma^{2,ad}_{(\ell-1)h_n} - \mathbb{E}[\hat\sigma^{2,ad}_{(\ell-1)h_n}]\big) - \sum_{\ell=i+1}^{i+\alpha_n} \big(\hat\sigma^{2,ad}_{(\ell-1)h_n} - \mathbb{E}[\hat\sigma^{2,ad}_{(\ell-1)h_n}]\big)\,,$$

$$B_{n,i} = \sum_{\ell=i-\alpha_n+1}^{i} \big(\mathbb{E}[\hat\sigma^{2,ad}_{(\ell-1)h_n}] - \sigma^2_{(\ell-1)h_n}\big) - \sum_{\ell=i+1}^{i+\alpha_n} \big(\mathbb{E}[\hat\sigma^{2,ad}_{(\ell-1)h_n}] - \sigma^2_{(\ell-1)h_n}\big)\,,$$

$$C_{n,i} = \sum_{\ell=i-\alpha_n+1}^{i} \widetilde\sigma^2_{(\ell-1)h_n} - \sum_{\ell=i+1}^{i+\alpha_n} \widetilde\sigma^2_{(\ell-1)h_n}\,,$$

$$D_{n,i} = \sum_{\ell=i-\alpha_n+1}^{i} \delta \mathbb{1}_{\{\ell\ge i^*\}} - \sum_{\ell=i+1}^{i+\alpha_n} \delta \mathbb{1}_{\{\ell\ge i^*\}}\,,$$

with $\big(\widetilde\sigma^2_t\big)_{t\in[0,1]}$ the path of the volatility from that the jump is eliminated:

$$\sigma^2_{(\ell-1)h_n} = \widetilde\sigma^2_{(\ell-1)h_n} + \delta \mathbb{1}_{\{\ell\ge i^*\}}\,.$$

By definition, $\big(\widetilde\sigma^2_t\big)_{t\in[0,1]}$ then fulfills the regularity properties on (H_0-$\mathfrak{a}$). This implies that

$$\begin{aligned}|C_{n,i}| &= \left| \sum_{\ell=i-\alpha_n+1}^{i} \big(\widetilde\sigma^2_{(\ell-1)h_n} - \widetilde\sigma^2_{(i-1)h_n}\big) - \sum_{\ell=i+1}^{i+\alpha_n} \big(\widetilde\sigma^2_{(\ell-1)h_n} - \widetilde\sigma^2_{(i-1)h_n}\big) \right| \\ &\le 2\max\left(\left| \sum_{\ell=i-\alpha_n+1}^{i} \big(\widetilde\sigma^2_{(\ell-1)h_n} - \widetilde\sigma^2_{(i-1)h_n}\big) \right|, \left| \sum_{\ell=i+1}^{i+\alpha_n} \big(\widetilde\sigma^2_{(\ell-1)h_n} - \widetilde\sigma^2_{(i-1)h_n}\big) \right| \right).\end{aligned}$$

Under (H_0-$\mathfrak{a}$), we obtain uniformly in i that almost surely

$$\left| \sum_{\ell=i-\alpha_n+1}^{i} \big(\widetilde\sigma^2_{(\ell-1)h_n} - \widetilde\sigma^2_{(i-1)h_n}\big) \right| \le \sum_{\ell=i-\alpha_n+1}^{i} |(\ell-1)h_n - (i-1)h_n|^{\mathfrak{a}} \le \alpha_n(\alpha_n h_n)^{\mathfrak{a}}\,.$$

This is sufficient for

$$\max_i |C_{n,i}| = \mathcal{O}_{\mathbb{P}}\big(\sqrt{\alpha_n \log((\alpha_n h_n)^{-1})}\big)\,.$$

From the proof of Theorem 3.8 we can thus conclude the following bound:

$$\max_i \Big(|A_{n,i}| + |B_{n,i}| \Big) = \mathcal{O}_{\mathbb{P}}\big(\sqrt{\alpha_n \log((\alpha_n h_n)^{-1})}\big)\,.$$

Next we consider a step-wise defined function $(g(t))_{t\in[0,1]}$ given by

$$g(ih_n) = \alpha_n^{-1/2}(A_{n,i} + B_{n,i} + C_{n,i})$$

and $(f(t))_{t\in[0,1]}$ being step-wise defined via

$$f(ih_n) = \begin{cases} 0, & \text{for } i + \alpha_n < i^*\,, \\ \delta\alpha_n^{-1/2}(i - i^* + \alpha_n + 1), & \text{for } i^* - \alpha_n \le i \le i^* - 1\,, \\ \delta\alpha_n^{-1/2}(\alpha_n - i + i^* - 1), & \text{for } i^* - 1 \le i \le i^* + \alpha_n - 1\,, \\ 0, & \text{for } i > i^* + \alpha_n - 1\,. \end{cases}$$

The function f fulfills

- $f\big|_{[0,\theta]}$ is monotonically increasing and
- $f\big|_{[\theta,1]}$ is monotonically decreasing.

We get the following representation of $\overline{V}^{\diamond}_{n,i}$:

$$\overline{V}^{\diamond}_{n,i} = |g(ih_n) - f(ih_n)|\ .$$

The calculations above imply that

$$\sup_{t\in[0,\theta]} |g(t)| = \mathcal{O}_{\mathbb{P}}(\sqrt{\log((\alpha_n h_n)^{-1})})\,. \tag{3.74}$$

Furthermore, for $i^* - c\alpha_n \le i \le i^* + c\alpha_n$, with any $0 < c < 1$, it holds that

$$f(ih_n) > |g(ih_n)| > 0\,,$$

with a probability tending to one as $n \to +\infty$. Therefore,

$$\overline{V}^{\diamond}_{n,i} = f(ih_n) - g(ih_n)\,,$$

for those i with a probability tending to one as $n \to +\infty$. For a sequence γ_n, with $\gamma_n \in [0, \alpha_n h_n]$, it holds that

$$f((i^* - 1)h_n - \gamma_n) = \delta\alpha_n^{-1/2}(-\lfloor \gamma_n h_n^{-1} \rfloor + \alpha_n)$$

and

$$f((i^* - 1)h_n) - f((i^* - 1)h_n - \gamma_n) = \lfloor \gamma_n h_n^{-1} \rfloor \delta\alpha_n^{-1/2}\,.$$

When we set

$$\gamma_n = h_n\delta^{-1}\sqrt{\alpha_n \log(n)} \le \alpha_n h_n\,,$$

we derive with (3.74) that almost surely for n sufficiently large:

$$f((i^* - 1)h_n) - f((i^* - 1)h_n - \gamma_n) \ge 2 \sup_{t\in[0,\theta]} |g(t)|\ .$$

Therefore, $f\big|_{[0,\theta]}$ satisfies the conditions of Lemma 3.20. This implies that

$$(i^*-1)h_n \geq \operatorname{argmax}_{i=\alpha_n,\dots,h_n^{-1}-\alpha_n} \overline{V}^{\diamond}_{n,i} h_n \geq (i^*-1)h_n - \gamma_n \,.$$

An analogous procedure applied to the function $f\big|_{[\theta,1]}$ yields that

$$(i^*-1)h_n \leq \operatorname{argmax}_{i=\alpha_n,\dots,h_n^{-1}-\alpha_n} \overline{V}^{\diamond}_{n,i} h_n \leq (i^*-1)h_n + \gamma_n \,.$$

Overall this yields

$$\left|\hat{\theta}_n - (i^*-1)h_n\right| = \mathcal{O}_{\mathbb{P}}(\gamma_n) = o_{\mathbb{P}}(1)\,,$$

which completes the proof of Proposition 3.14. □

4. Uniform spot volatility estimation

This chapter is devoted to present limit theorems allowing for the construction of *confidence bands* for the spot volatility process $\left(\sigma_t^2\right)_{t\in[0,1]}$.
It is organized as follows. We start with a short discussion on literature on this topic and afterwards the construction of the key statistic is presented. We will present the asymptotic theory and conclude the chapter with the proofs of the presented limit theorems.

4.1. The state of the art

Before presenting the main results, we will briefly discuss the state of the art on uniform spot volatility estimation. Starting with Kristensen [43], the author considers spot volatility estimation within the model

$$dX_t = \mu_t \, dt + \sigma_t \, dW_t \, ,$$

with a standard Brownian Motion W. The results on spot volatility estimation seem to be quite restrictive as the author assumes the processes μ, σ and W to be mutually independent. Based on these assumptions, the author proves a uniform consistency result in Theorem 2 therein and a point-wise central limit theorem for the kernel based spot volatility estimator.
Another recent work, considering some uniform inference on spot volatility is given by Kanaya and Kristensen [40], which only address a very narrow setup and not the general situation we are interested in.
Let us conclude this subsection with a more detailed discussion of the result presented in Fan and Wang [23]. The main Theorem 2 in Fan and Wang [23] directly tackles those questions, which we have raised in the introduction of this chapter. More precisely, for a data generating *continuous* process $(X_t)_{t\in[0,1]}$ they provide a Gumbel type approximation, which allows for uniform inference concerning the spot volatility process $\left(\sigma_t^2\right)_{t\in[0,1]}$. Unfortunately, the limit in Theorem 2 has been proved under seriously restrictive assumptions. In addition to other restrictions they assume

(F1) $(\sigma_t)_{t\in[0,1]}$ to be a stationary process and

(F2) $\sup\left\{|\sigma_s - \sigma_t| : s,t \in [0,1]\,, |s-t| \leq a\right\} = \mathcal{O}_{\mathbb{P}}(a^{1/2}\,|\log(a)|^{1/2})$.

From an application and theoretical point of view it is desirable to relax both restrictions.

Example 4.1 (Diffusion stochastic volatility models)**.** In their seminal paper Hull and White [30] consider the stochastic volatility model given by the stochastic differential equation

$$d\sigma_t^2 = \alpha(\sigma_t^2)dt + \beta(\sigma_t^2)dB_t$$

with a standard Brownian motion B. In general, a solution $\left(\sigma_t^2\right)_{t\in[0,1]}$ is a *diffusion*, i.e. in general the latter is a non-stationary process. Thus, this well known and prominent stochastic volatility model does not fulfill condition (F1) and therefore, the asymptotic theory in Fan and Wang [23] is not applicable.

Example 4.2 (Rough and long memory stochastic volatility models)**.** There is econometric evidence for the fact that, in contrast to the classical stochastic volatility models, the volatility process has the property of long range dependence effects. The latter is not reflected within the classical framework due to the uncorrelated increments of the standard Brownian motion, driving the stochastic differential equation determining the dynamics of the volatility process. We refer to F. Engle and G.J. Lee [22] for further discussions on these long memory effects. In order to address these empirical facts, a lot of work has been provided in order to generalize the stochastic volatility models. For continuous time modeling of spot volatility the prominent model due to Comte and Renault [18] is given by the stochastic differential equation

$$d\sigma_t^2 = \left(m - \sigma_t^2\right)dt + \beta dB_t^H\,,$$

with a fractional Brownian motion B^H.
Whereas condition (F2) doesn't exclude these type of stochastic volatility models, the theory presented in Fan and Wang [23] yields suboptimal convergence rates. Uniform confidence is, however, of interest for the best feasible rate. Otherwise, the confidence bands are far from being sharp and unreasonably conservative. More recent research results due to Gatheral et al. [25] give plausibility arguments for rough stochastic volatility models. Rough stochastic volatility models, e.g. processes with a small Hurst index H, i.e. $H \ll \frac{1}{2}$, are excluded in Fan and Wang [23] due to condition (F2).

Remark 4.3 (Discontinuous price processes)**.** The question whether we allow for discontinuous price processes or not, is quite crucial from a statistical point of view, since the existence of jumps requires a significant modification of the involved statistics in order to ensure jump robustness. Nevertheless, due to econometric evidence, it seems to be necessary to allow for non-trivial jump components. We refer to Jiang and Oomen [39] for an extensive discussion of these aspects. Obviously, the asymptotic theory presented in Fan and Wang [23] is not robust with respect to a non-trivial jump component, even if the latter only exhibits finite activity.

Remark 4.4 (Microstructure noise)**.** As we have already argued in the previous chapter, microstructure noise in high-frequency data is a commonly accepted fact. The theory presented in Fan and Wang [23] is clearly not noise robust. The extension of their theory to noise robust estimators is postponed to the next chapter.

In the very recent independent work Koike [41] general approximation results for maxima of Wiener functionals, based on Malliavin Calculus and Stein's method, are presented. Uniform confidence results for spot volatility as an application of the main Theorem 2.1 given therein are considered as a corollary of the general asymptotic theory. In contrast to this work we will consider Gumbel type approximations and also allow for a more general model allowing for non-trivial drift and jump-component for X.

4.2. The estimator and some assumptions

In order to tackle the estimation problem described in the introduction of this chapter, we will employ the *Nadaraya-Watson* type estimator $\widehat{\Gamma}_t^n$, c.f. Tsybakov [59], given by

$$\widehat{\Gamma}_t^n = \frac{1}{b_n} \sum_{i=1}^{n} K_{b_n}(t - i/n) \left(\Delta_i^n X\right)^2 , \tag{4.1}$$

with a kernel function K,

$$K_x(z) = K\left(z/x\right) ,$$

and bandwidth b_n. The estimator (4.1) will be used for the continuous Itô semimartingale case as it is not jump robust. We will postpone the jump robust version based on truncated realized power variation to the end of this subsection. Similar assumptions as in Chapter 3 with respect to the spot volatility process $(\Gamma_t)_{t\in[0,1]}$, i.e. $\Gamma_t = \sigma_t^2$ and the drift process are necessary.

Assumption 4.5 (The coefficient processes)**.** *The coefficient processes a and σ are assumed to fulfill the following properties.*

(1) The processes a and σ are locally bounded. σ is almost surely strictly positive, that is, $\inf_{t\in[0,1]} \sigma_t^2 \geq K^- > 0$.

(2) The modulus of continuity

$$w_\delta(\sigma)_t = \sup_{s,r\leq t} \{|\sigma_s - \sigma_r| : |s - r| < \delta\}$$

is locally bounded in the sense that there exists $\mathfrak{a} > 0$ and a sequence of stopping times $T_n \to \infty$, such that $w_\delta(\sigma)_{(T_n\wedge 1)} \leq L_n \delta^{\mathfrak{a}}$, for some $\mathfrak{a} > 0$ and some (almost surely finite) random variables L_n.

Remark 4.6. The conditions imposed in Assumption 4.5 are exactly those which were used for the asymptotic theory in the previous chapter. In particular, using the arguments which have already been presented, we can assume the sequence L_n to grow arbitrarily slowly. The latter ensures that the upper bounds presented in the proof section can be shown to be negligible.

We need further assumptions on the bandwidth $(b_n)_{n\in\mathbb{N}}$.

Assumption 4.7 (The bandwidth b_n). *The sequence $(b_n)_{n\in\mathbb{N}}$ is assumed to fulfill*

(1) $b_n \longrightarrow 0$ *and* $nb_n \longrightarrow \infty$,

(2) $\sqrt{nb_n \log(n)} b_n^{\mathfrak{a}} \longrightarrow 0$ *and*

(3) $\frac{\log(n)\sqrt{\log(n)}}{\sqrt{nb_n}} \longrightarrow 0$,

as $n \longrightarrow \infty$.

Let us briefly discuss the assumptions imposed on the bandwidth $(b_n)_{n\in\mathbb{N}}$. The second property (2) in Assumption 4.7 is addressed to modulate the roughness of the volatility process $(\Gamma_t)_{t\in[0,1]}$, such that the rougher the volatility paths are, i.e. $\mathfrak{a}$ being small, the smaller the bandwidth b_n has to be chosen. Assumption (3) in (4.7) is due to strong invariance principles ensuring the strong approximation. The latter serves as the key technique in the final step of the proof. This assumption and its necessity will be clarified in the proof section. Let us emphasize that in contrast to the analogue assumption in Wu and Zhao [61],

$$\frac{\log^3(n)}{b_n\sqrt{n}} \longrightarrow 0 \text{ as } n \longrightarrow +\infty\,,$$

we can relax this assumption significantly. The reason is twofold. On the one hand, the authors intend to impose as less assumptions as possible on the moments of their error terms, namely the existence of the fourth moment. On the other hand, they also allow for dependence in the error term, which implies a slower rate of convergence in the strong approximation. In our case, we will end up with *independent* and shifted χ^2 distributed random variables, such that the existence of the moment generating function is ensured and we can apply better approximation results presented in Chapter 2.
Further assumptions on the kernel function K are necessary.

Assumption 4.8 (The kernel function K). *The kernel function K is assumed to fulfill*

(1) K is a continuous, bounded, symmetric function with compact support such that supp $K = [-1,1]$,

(2) it holds that $\int_{\mathbb{R}} \Psi_{K,\delta}(u)du = \mathcal{O}(\delta)$ *as* $\delta \longrightarrow 0$, *with*

$$\Psi_{K,\delta}(u) = \sup\{|K(x)-K(y)| : x,y\,[u-\delta,u+\delta]\}$$

and

(3) the limit $D_{K,\alpha} = \lim_{\delta\to 0}|\delta|^{-\alpha}\int_{\mathbb{R}}(K(x+\delta)-K(x))^2\,dx$ *exists for some* $1 \le \alpha \le 2$.

Remark 4.9. The class of kernels K which are allowed for our asymptotic theory include those in the existing literature. In contrast to Fan and Wang [23] we neither impose differentiability nor some kind of Lipschitz regularity. Let us emphasize that the compact support property or the symmetry assumption could also be relaxed as they are only assumed to reduce notation and more technical considerations. Prominent kernel functions as the Gaussian kernel also allow for the asymptotic theory. The crucial property is that every kernel K exhibits a certain decay behaviour with respect to its tails. The symmetry assumption can simply be relaxed by considering the symmetrization $\widetilde{K}$, given by $\widetilde{K}(x) = \frac{1}{2}(K(x) + K(-x))$.

We will proceed with a jump robust version of Theorem 4.12. We fix a truncation exponent $\tau \in (0, 1/2)$ and define the truncating sequence $(u_n)_{n\in\mathbb{N}}$, such that $u_n \propto n^{-\tau}$. A jump robust version of (4.1) is given by

$$\widehat{\Gamma}^n_{t,\tau} = \frac{1}{b_n} \sum_{i=1}^{n} K_{b_n}(t - i/n) \left(\Delta_i^n X\right)^2 \mathbb{1}_{\left\{|\Delta_i^n X| \leq u_n\right\}} . \tag{4.2}$$

The general discontinuous case will provide further error terms due to the non-trivial jump component. Therefore, we need further conditions on the bandwidth b_n, ensuring the additional terms to be negligible.

Assumption 4.10 (The bandwidth b_n II)**.** *For some p with $1 < p < (2r\tau)^{-1}$ and some $\ell < \infty$, the sequence $(b_n)_{n\in\mathbb{N}}$ is assumed to fulfill*

(1) $\frac{\log^{3/2}(n)}{\sqrt{nb_n}} b_n^{\ell} n^{\tau r p \ell} = \mathcal{O}(1)$ *and*

(2) $\sqrt{nb_n \log(n)} n^{\tau(r-2)} = \mathcal{O}(1)$,

as $n \longrightarrow \infty$.

4.3. Uniform limit theorem I: The continuous case

We will start presenting our asymptotic theory for the continuous case. More precisely, we will prove a limit theorem for quantities of the form

$$\sup_{t\in[0,1]} \left| \widehat{\Gamma}^n_t - \Gamma_t \right| . \tag{4.3}$$

The Gumbel type approximation, which we will prove in the main theorem of this subsection needs a further rescaling of the statistic (4.3), since otherwise we could only ensure a limit with a conditional rescaled Gumbel distribution. Therefore, we will replace (4.3) by the rescaled version

$$\sup_{t\in[0,1]} \frac{\left| \widehat{\Gamma}^n_t - \Gamma_t \right|}{\sqrt{2}\widehat{\Gamma}^n_t} , \tag{4.4}$$

which is more appropriate for further statistical inference, e.g. testing and confidence sets, as it will provide a distribution free limit.

Remark 4.11. Due to the missing rescaling in Theorem 2 in Fan and Wang [23], we think that the limit presented therein only holds for $\sigma \equiv 1$. Of course, the extension to $\sigma \equiv c$ for some constant c is straightforward, but needs a modification of the limit distribution as it is a different rescaled Gumbel distribution.

In order to formulate the main theorem of this section, we need further notation. For $m_n \geq 3$, we define the sequence $(d_n^{\alpha,K})_{n\geq 3}$, given by

$$d_n^{\alpha,K} = \sqrt{2\log(n)} + \frac{1}{2\sqrt{\log(n)}}\left[\frac{2-\alpha}{\alpha}\log(\log(n)) + \log\left(\frac{C_{K,\alpha}H_\alpha 2^{1/\alpha}}{2\sqrt{\pi}}\right)\right]$$

with

$$C_{K,\alpha} = D_{K,\alpha}/2\lambda_K \quad \text{and} \quad \lambda_K = \int_{\mathbb{R}} K^2(s)\,ds\,,$$

and H_α being Pickand's constant, c.f. Bickel and Rosenblatt [14] or (2.27) in Theorem 2.27. We set

$$M_n = \sup_{t\in[0,1]} M_n(t)\,,$$

with

$$M_n(t) = \frac{\left|\widehat{\Gamma}_t^n - \Gamma_t\right|}{\sqrt{2}\widehat{\Gamma}_t^n}\,.$$

Theorem 4.12 (Uniform estimation of spot volatility)**.** *Under the Assumptions 4.5, 4.7, 4.8 it holds with $m_n = 1/b_n$ that for all x*

$$\mathbb{P}\left[(2\log(m_n))^{1/2}\left(\frac{\sqrt{nb_n}}{\sqrt{\lambda_K}}M_n - d_{m_n}^{\alpha,K}\right) \leq x\right] \longrightarrow \exp(-2\exp(-x))\,. \tag{4.5}$$

Using Theorem 4.12 we can construct asymptotic confidence sets. Choosing $b_n \asymp n^{-\frac{1}{2\mathfrak{a}+1}}$ optimally under Assumption 4.7 yields confidence bands at almost optimal rate. We set $z_\beta = -\log(\log(1-\beta))^{-1/2}$ and define the set C_β given by

$$C_\beta = \left[\widehat{\Gamma}_{t,\ell,\beta}^{n,\mathfrak{a}}, \widehat{\Gamma}_{t,u,\beta}^{n,\mathfrak{a}}\right]\,,$$

with

$$\widehat{\Gamma}_{t,\ell,\beta}^{n,\mathfrak{a}} = \widehat{\Gamma}_t^n - \left(\frac{z_\beta}{\sqrt{2\log(m_n)}} + d_n^{\alpha,K}\right)\frac{\sqrt{2\lambda_K}\widehat{\Gamma}_t^n}{n^{\frac{\mathfrak{a}}{2\mathfrak{a}+1}}}\,,$$

$$\widehat{\Gamma}_{t,u,\beta}^{n,\mathfrak{a}} = \widehat{\Gamma}_t^n + \left(\frac{z_\beta}{\sqrt{2\log(m_n)}} + d_n^{\alpha,K}\right)\frac{\sqrt{2\lambda_K}\widehat{\Gamma}_t^n}{n^{\frac{\mathfrak{a}}{2\mathfrak{a}+1}}}\,.$$

Corollary 4.13 (Confidence band for spot volatility)**.** *The set C_β is a $(1-\beta)$ simultaneous confidence band for the unknown spot volatility process $(\Gamma_t)_{t\in[0,1]}$.*

Proof of Corollary 4.13.
This is an immediate consequence of Theorem 4.12. □

Remark 4.14. For $\alpha \in \{1,2\}$ the values of Pickand's constant H_α are known and $H_1 = 1$ and $H_2 = \pi^{-1/2}$, i.e. this enables uniform confidence for a large class of kernels K. Beyond this well known cases Theorem 4.12 is only of probabilistic interest.

4.4. Uniform limit theorem II: The general case

In this section we will extend the asymptotic theory developed in the previous section to the general case. That is, we will prove a limit theorem for quantities of the form

$$\sup_{t\in[0,1]} \left|\widehat{\Gamma}^n_{t,\tau} - \Gamma_t\right| . \tag{4.6}$$

We further set

$$M_{n,\tau} = \sup_{t\in[0,1]} M_{n,\tau}(t)\,,$$

with

$$M_{n,\tau}(t) = \frac{\left|\widehat{\Gamma}^n_{t,\tau} - \Gamma_t\right|}{\sqrt{2}\widehat{\Gamma}^n_{t,\tau}} .$$

Theorem 4.15 (Uniform estimation of spot volatility)**.** *Under the Assumptions 3.2, 4.5, 4.7, 4.8 and 4.10 it holds that for all x*

$$\mathbb{P}\left[(2\log(m_n))^{1/2}\left(\frac{\sqrt{nb_n}}{\sqrt{\lambda_K}}M_{n,\tau} - d^{\alpha,K}_{m_n}\right) \le x\right] \longrightarrow \exp(-2\exp(-x))\,. \tag{4.7}$$

Using Theorem 4.15 we can construct asymptotic confidence sets. We define the set $C_{\beta,\tau}$, given by

$$C_{\beta,\tau} = \left[\widehat{\Gamma}^{n,\mathfrak{a}}_{t,\tau,\ell,\beta}, \widehat{\Gamma}^{n,\mathfrak{a}}_{t,\tau,u,\beta}\right] ,$$

with

$$\widehat{\Gamma}^{n,\mathfrak{a}}_{t,\tau,\ell,\beta} = \widehat{\Gamma}^n_{t,\tau} - \left(\frac{z_\beta}{\sqrt{2\log(m_n)}} + d^{\alpha,K}_n\right)\frac{\sqrt{2\lambda_K}\widehat{\Gamma}^n_{t,\tau}}{n^{\frac{\mathfrak{a}}{2\mathfrak{a}+1}}} ,$$

$$\widehat{\Gamma}^{n,\mathfrak{a}}_{t,\tau,u,\beta} = \widehat{\Gamma}^n_{t,\tau} + \left(\frac{z_\beta}{\sqrt{2\log(m_n)}} + d^{\alpha,K}_n\right)\frac{\sqrt{2\lambda_K}\widehat{\Gamma}^n_{t,\tau}}{n^{\frac{\mathfrak{a}}{2\mathfrak{a}+1}}} .$$

Corollary 4.16 (Confidence band for spot volatility)**.** *The set $C_{\beta,\tau}$ is a $(1-\beta)$ simultaneous confidence band for the unknown spot volatility process $(\Gamma_t)_{t\in[0,1]}$.*

Proof of Corollary 4.16.
This is an immediate consequence of Theorem 4.15. □

4.5. Proofs

This section is intended to present the proofs of the main theorems. We will use an universal constant C which may change from line to line. Furthermore, we will use the notation C_p to indicate that the constant depends on an external parameter p. The constant will never depend on n.

Proof of Theorem 4.12

The proof of Theorem 4.12 is quite lengthy and will be split into several parts. The crucial step of the whole proof is to replace the true price process increments $\Delta_i^n X$ by (properly rescaled) Brownian increments $\Delta_i^n W$. First of all, we assume $nb_n, b_n^{-1} \in \mathbb{N}$, such that there is some $k \in \mathbb{N}$, fulfilling $k/n = b_n$. This assumption is only due to notational convenience. The general case can be concluded very similarly.
Furthermore, we define some repeatedly used abbreviations. We set

$$K_{t,n}^{i,\ell} = K_{b_n}(t - (ib_n + \ell/n)),$$

i.e. $K_{t,n}^{0,\ell} = K_{b_n}(t - \ell/n)$. We also set, for any stochastic process (L_t),

$$\Delta_{\ell,i}^n L = L_{ib_n + \frac{\ell}{n}} - L_{ib_n + \frac{\ell-1}{n}}.$$

We will use a generic constant $c > 0$ which may change from line to line and the sequence ν_n given by

$$\nu_n = \frac{\sqrt{n \log(n)}}{b_n}.$$

Finally, we define indices $t_n^{i,\ell}$ given by

$$t_n^{i,\ell} = ib_n + \frac{\ell}{n}.$$

The approximation described above can be pursued via

$$\overline{\Gamma}_t^n = \frac{1}{b_n} \sum_{i=0}^{b_n^{-1}-1} \sigma_{ib_n}^2 \sum_{\ell=1}^{nb_n} K_{t,n}^{i,\ell} \left(\Delta_{\ell,i}^n W\right)^2.$$

Proposition 4.17. *Under the assumptions of Theorem 4.12 it holds that*

$$\sqrt{nb_n \log(n)} \sup_{t\in[0,1]} \left| \left| \frac{\widehat{\Gamma}_t^n - \Gamma_t}{\widehat{\Gamma}_t^n} \right| - \left| \frac{\overline{\Gamma}_t^n - \Gamma_t}{\overline{\Gamma}_t^n} \right| \right| \xrightarrow{\mathbb{P}} 0. \tag{4.8}$$

Proof of Proposition 4.17.
The following decomposition holds:

$$\begin{aligned} &\frac{\widehat{\Gamma}_t^n - \Gamma_t}{\widehat{\Gamma}_t^n} - \frac{\overline{\Gamma}_t^n - \Gamma_t}{\overline{\Gamma}_t^n} \\ &= \widehat{\Gamma}_t^n \left(\frac{1}{\widehat{\Gamma}_t^n} - \frac{1}{\overline{\Gamma}_t^n} \right) + \frac{\widehat{\Gamma}_t^n - \overline{\Gamma}_t^n}{\overline{\Gamma}_t^n} + \Gamma_t \left(\frac{1}{\overline{\Gamma}_t^n} - \frac{1}{\widehat{\Gamma}_t^n} \right). \end{aligned}$$

This yields the following upper bound via the reverse triangle inequality and elementary properties of the $\sup_{t\in[0,1]}$

$$
\begin{aligned}
&\leq \sup_{t\in[0,1]} \widehat{\Gamma}_t^n \left(\frac{1}{\widehat{\Gamma}_t^n} - \frac{1}{\overline{\Gamma}_t^n}\right) + \sup_{t\in[0,1]} \frac{\widehat{\Gamma}_t^n - \overline{\Gamma}_t^n}{\overline{\Gamma}_t^n} + \sup_{t\in[0,1]} \Gamma_t \left(\frac{1}{\overline{\Gamma}_t^n} - \frac{1}{\widehat{\Gamma}_t^n}\right) \\
&:= (\mathbf{I}) + (\mathbf{II}) + (\mathbf{III})\,.
\end{aligned}
$$

We start with the term $(\mathbf{II})$. It holds that for every $\delta > 0$

$$
\begin{aligned}
&\mathbb{P}\left[\sup_{t\in[0,1]} \sqrt{nb_n\log(n)}\frac{\left|\widehat{\Gamma}_t^n - \overline{\Gamma}_t^n\right|}{\overline{\Gamma}_t^n} > \delta\right] \\
&= \mathbb{P}\left[\sup_{t\in[0,1]} \sqrt{nb_n\log(n)}\frac{\left|\widehat{\Gamma}_t^n - \overline{\Gamma}_t^n\right|}{\overline{\Gamma}_t^n} > \delta,\ \inf_{t\in[0,1]} \overline{\Gamma}_t^n \geq K^- - \kappa^-\right] \\
&+ \mathbb{P}\left[\sup_{t\in[0,1]} \sqrt{nb_n\log(n)}\frac{\left|\widehat{\Gamma}_t^n - \overline{\Gamma}_t^n\right|}{\overline{\Gamma}_t^n} > \delta,\ \inf_{t\in[0,1]} \overline{\Gamma}_t^n < K^- - \kappa^-\right] \\
&\leq \mathbb{P}\left[\sup_{t\in[0,1]} \sqrt{nb_n\log(n)}\left|\widehat{\Gamma}_t^n - \overline{\Gamma}_t^n\right| > \delta\left(K^- - \kappa^-\right)\right] \\
&\qquad + \mathbb{P}\left[\inf_{t\in[0,1]} \overline{\Gamma}_t^n < K^- - \kappa^-\right] \\
&=: A_n + B_n\,.
\end{aligned}
$$

We start with A_n.
It holds that

$$
\widehat{\Gamma}_t^n - \overline{\Gamma}_t^n = \sum_{i=0}^{b_n^{-1}-1}\left[\sum_{\ell=1}^{nb_n} K_{t,n}^{i,\ell}\left(\Delta_{\ell,i}^n X\right)^2 - \sigma_{ib_n}^2 \sum_{\ell=1}^{nb_n} K_{t,n}^{i,\ell}\left(\Delta_{\ell,i}^n W\right)^2\right].
$$

Using Itô's formula, it holds that

$$
\begin{aligned}
\left(\Delta_{\ell,i}^n X\right)^2 &= 2\int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}} (X_\tau - X_{t_n^{i,\ell-1}})\sigma_\tau\, dW_\tau + 2\int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}} (X_\tau - X_{t_n^{i,\ell-1}})a_\tau\, d\tau + \int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}} \sigma_\tau^2\, d\tau \\
\left(\Delta_{\ell,i}^n W\right)^2 &= 2\int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}} (W_\tau - W_{t_n^{i,\ell-1}})\, dW_\tau + \int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}} d\tau\,.
\end{aligned}
$$

This implies the following decomposition with respect to $\widehat{\Gamma}_t^n - \overline{\Gamma}_t^n$

$$\begin{aligned}\widehat{\Gamma}_t^n - \overline{\Gamma}_t^n = \frac{2}{b_n} \sum_{i=0}^{b_n^{-1}-1} \sum_{\ell=1}^{nb_n} K_{t,n}^{i,\ell} \int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}} (X_\tau - X_{t_n^{i,\ell-1}})\sigma_\tau - \sigma_{ib_n}^2 (W_\tau - W_{t_n^{i,\ell-1}})\, d\tau \\ + \frac{1}{b_n} \sum_{i=0}^{b_n^{-1}-1} \sum_{\ell=1}^{nb_n} K_{t,n}^{i,\ell} \int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}} (\sigma_\tau^2 - \sigma_{ib_n}^2)\, d\tau \\ + \frac{2}{b_n} \sum_{i=0}^{b_n^{-1}-1} \sum_{\ell=1}^{nb_n} K_{t,n}^{i,\ell} \int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}} \left(X_\tau - X_{t_n^{i,\ell-1}}\right) a_\tau\, d\tau\,.\end{aligned}$$

Thus, we can decompose A_n via

$$A_n \leq A_n^1 + A_n^2 + A_n^3\,,$$

such that A_n^1 is given by

$$\mathbb{P}\left[\sup_{t\in[0,1]} \nu_n \left| \sum_{i=0}^{b_n^{-1}-1} \sum_{\ell=1}^{nb_n} K_{t,n}^{i,\ell} \int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}} (X_\tau - X_{t_n^{i,\ell-1}})\sigma_\tau - \sigma_{ib_n}^2 (W_\tau - W_{t_n^{i,\ell-1}})\, d\tau \right| > c\right]\,,$$

A_n^2 is given by

$$\mathbb{P}\left[\sup_{t\in[0,1]} \nu_n \left| \sum_{i=0}^{b_n^{-1}-1} \sum_{\ell=1}^{nb_n} K_{t,n}^{i,\ell} \int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}} (\sigma_\tau^2 - \sigma_{ib_n}^2)\, d\tau \right| > c\right]$$

and A_n^3 is given by

$$\mathbb{P}\left[\sup_{t\in[0,1]} \nu_n \left| \sum_{i=0}^{b_n^{-1}-1} \sum_{\ell=1}^{nb_n} K_{t,n}^{i,\ell} \int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}} \left(X_\tau - X_{t_n^{i,\ell-1}}\right) a_\tau\, d\tau \right| > c\right]\,.$$

We start with A_n^1 and employ a further decomposition

$$\begin{aligned}&\big(X_\tau - X_{t_n^{i,\ell-1}}\big)\sigma_\tau - \big(W_\tau - W_{t_n^{i,\ell-1}}\big)\sigma_{ib_n}^2 \\ &= \sigma_\tau \int_{t_n^{i,\ell-1}}^{\tau} a_u\, du + (\sigma_\tau - \sigma_{ib_n}) \int_{t_n^{i,\ell-1}}^{\tau} \sigma_u\, dW_u \\ &\qquad + \sigma_{ib_n} \int_{t_n^{i,\ell-1}}^{\tau} (\sigma_u - \sigma_{ib_n})\, dW_u\,.\end{aligned}$$

This yields the decomposition

$$A_n^1 \leq A_n^{1,1} + A_n^{1,2} + A_n^{1,3}\,,$$

where $A_n^{1,1}$ is given by

$$\mathbb{P}\left[\sup_{t\in[0,1]} \nu_n \left| \sum_{i=0}^{b_n^{-1}-1} \sum_{\ell=1}^{nb_n} K_{t,n}^{i,\ell} \int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}} \sigma_\tau \int_{t_n^{i,\ell-1}}^{\tau} a_u \, du dW_\tau \right| > c \right],$$

$A_n^{1,2}$ is given by

$$\mathbb{P}\left[\sup_{t\in[0,1]} \nu_n \left| \sum_{i=0}^{b_n^{-1}-1} \sum_{\ell=1}^{nb_n} K_{t,n}^{i,\ell} \int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}} (\sigma_\tau - \sigma_{ib_n}) \int_{t_n^{i,\ell-1}}^{\tau} \sigma_u \, dW_u dW_\tau \right| > c \right]$$

and finally $A_n^{1,3}$ is given by

$$\mathbb{P}\left[\sup_{t\in[0,1]} \nu_n \left| \sum_{i=0}^{b_n^{-1}-1} \sum_{\ell=1}^{nb_n} K_{t,n}^{i,\ell} \int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}} \sigma_{ib_n} \int_{t_n^{i,\ell-1}}^{\tau} (\sigma_u - \sigma_{ib_n}) \, dW_u dW_\tau \right| > c \right].$$

We start with $A_n^{1,1}$ and set

$$q_{n,i}^{t,1}(\tau) = \sum_{\ell=1}^{nb_n} K_{t,n}^{i,\ell} \sigma_\tau \int_{t_n^{i,\ell-1}}^{\tau} a_u \, du \mathbb{1}_{[t_n^{i,\ell-1}, t_n^{i,\ell})}(\tau).$$

This implies the representation

$$A_n^{1,1} = \mathbb{P}\left[\sup_{t\in[0,1]} \nu_n \left| \sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} q_{n,t}^{t,1}(\tau) \, dW_\tau \right| > c \right].$$

We need a further decomposition:
Set $v_n = n^2$ and $t_j = j/v_n$ for $j = 1, \ldots, v_n$. Note the difference between the indices $t_n^{i,\ell}$ and t_j. Then the following decomposition holds

$$\begin{aligned}
&\mathbb{P}\left[\sup_{t\in[0,1]} \nu_n \left| \sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} q_{n,t}^{t,1}(\tau) \, dW_\tau \right| > c \right] \\
&\leq \mathbb{P}\left[\max_{1\leq j\leq v_n} \sup_{t\in[t_{j-1},t_j]} \nu_n \left| \sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} \left(q_{n,i}^{t,1}(\tau) - q_{n,i}^{t_j,1}(\tau)\right) dW_\tau \right| > c \right] \\
&+ \mathbb{P}\left[\max_{1\leq j\leq v_n} \nu_n \left| \sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} q_{n,i}^{t_j,1}(\tau) \, dW_\tau \right| > c \right],
\end{aligned}$$

i.e. there is a decomposition

$$A_n^{1,1} \leq A_n^{1,1,1} + A_n^{1,1,2}.$$

Starting with $A_n^{1,1,1}$ it holds that,

$$q_{n,i}^{t,1}(\tau) - q_{n,i}^{t_j,1}(\tau) = \sum_{i=0}^{b_n^{-1}-1} (K_{t,n}^{i,\ell} - K_{t_j,n}^{i,\ell})\sigma_\tau \int_{t_n^{i,\ell-1}}^{\tau} a_u \, du \mathbb{1}_{[t_n^{i,\ell-1}, t_n^{i,\ell})}(\tau) \, .$$

In order to exploit the regularity of the kernel function K, we have to rewrite the sum as an integral using the floor function $\lfloor\rfloor$. This yields

$$\left|K_{t,n}^{i,\ell} - K_{t_j,n}^{i,\ell}\right| \le \sup\left\{\left|K(z) - K(z')\right| \, : z, z' \in \left[\frac{t-(ib_n+\ell/n)}{b_n}, \frac{t_j-(ib_n+\ell/n)}{b_n}\right]\right\}$$

and

$$\begin{aligned}\sum_{\ell=1}^{nb_n}(K_{t,n}^{i,\ell} - K_{t_j,n}^{i,\ell}) &= \int_0^{nb_n} (K_{t,n}^{i,\lfloor u\rfloor} - K_{t_j,n}^{i,\lfloor u\rfloor}) \, du \\ &= \mathcal{O}\left(\int_{\mathbb{R}} \Psi_{K,\varrho}(u) \, du\right) \\ &= \mathcal{O}\left(\frac{|t-t_j|}{b_n}\right) ,\end{aligned}$$

i.e.

$$\varrho = \frac{|t-t_j|}{b_n} \, .$$

Burkholder's inequality and $\tau \in [t_n^{i,\ell-1}, t_n^{i,\ell}]$ yield

$$\left|\int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}} \sigma_\tau \int_{t_n^{i,\ell-1}}^{\tau} a_u \, du \, dW_\tau\right| = \mathcal{O}_{\mathbb{P}}\left(\left(\int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}} \left(\sigma_\tau \int_{t_n^{i,\ell-1}}^{\tau} a_u \, du\right)^2 d\tau\right)^{\frac{1}{2}}\right) .$$

Since $\tau \in [t_n^{i,\ell-1}, t_n^{i,\ell}]$ it holds that

$$\left|\sigma_\tau \int_{t_n^{i,\ell-1}}^{\tau} a_u \, du\right| = \mathcal{O}_{\mathbb{P}}(t_n^{i,\ell} - t_n^{i,\ell-1}) = \mathcal{O}_{\mathbb{P}}(n^{-1}) \, .$$

Furthermore, $t \in [t_{j-1}, t_j]$, i.e.

$$\mathcal{O}\left(\frac{|t-t_j|}{b_n}\right) = \mathcal{O}\left(n^{-2} b_n^{-1}\right) \, .$$

Note that

$$\#\left\{(i,\ell) \in \mathbb{N}^2 : \frac{1}{b_n}(t-(ib_n+\ell/n)) \in \mathsf{supp}\ K\right\} = \mathcal{O}(nb_n) \qquad (4.9)$$

and

$$1 \leq \ell \leq nb_n \, ,$$

which imply, with Cauchy-Schwarz and Burkholder inequality, the bound

$$\begin{aligned}
&\max_{1\leq j\leq v_n} \sup_{t\in[t_{j-1},t_j]} \frac{\sqrt{n\log(n)}}{\sqrt{b_n}} \left| \sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} \left(q_{n,i}^{t,1}(\tau) - q_{n,i}^{t_j,1}(\tau)\right) dW_\tau \right| \\
&= \mathcal{O}_{\mathbb{P}}\left(\sqrt{n\log(n)} b_n^{-1/2} b_n^{1/2} n^{-2} b_n^{-1} n^{-1}\right) \\
&= \scriptstyle{\mathcal{O}}_{\mathbb{P}}(1) \, .
\end{aligned}$$

We can proceed with $A_n^{1,1,2}$. We use Markov's inequality applied to the function $z \mapsto |z|^{2m}$ with $m > 0$ and $m \in \mathbb{N}$. Then we have

$$\begin{aligned}
&\mathbb{P}\left[\max_{1\leq j\leq v_n} \nu_n \left| \sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} q_{n,i}^{t_j,1}(\tau)\, dW_\tau \right| > c\right] \\
&\leq \sum_{j=1}^{v_n} \mathbb{P}\left[\nu_n \left| \sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} q_{n,i}^{t_j,1}(\tau)\, dW_\tau \right| > c\right] \\
&\leq C_m \frac{(n\log(n))^m}{b_n^m} \sum_{j=1}^{v_n} \mathbb{E}\left[\left| \sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} q_{n,i}^{t_j,1}(\tau)\, dW_\tau \right|^{2m}\right] .
\end{aligned}$$

Considering the expectation, we employ (4.9), the triangle inequality, Burkholder's inequality and Jensen's inequality, such that it is sufficient to bound

$$\begin{aligned}
\mathbb{E}\left[\left| \int_{ib_n}^{(i+1)b_n} q_{n,i}^{t_j,1}(\tau)\, dW_\tau \right|^{2m}\right] &\leq \mathbb{E}\left[\left(\int_{ib_n}^{(i+1)b_n} \left(q_{n,i}^{t_j,1}(\tau)\right)^2 d\tau \right)^m\right] \\
&\leq \left(\int_{ib_n}^{(i+1)b_n} \mathbb{E}\left[\left(q_{n,i}^{t_j,1}(\tau)\right)^{2m}\right]^{1/m} d\tau \right)^m .
\end{aligned}$$

Since $\tau \in [t_n^{i,\ell-1}, t_n^{i,\ell})$, it holds that

$$\begin{aligned}
&\mathbb{E}\left[\left(q_{n,i}^{t_j,1}(\tau)\right)^{2m}\right] \\
&= \mathbb{E}\left[\left(\sum_{\ell=1}^{nb_n} K_{t_j,n}^{t,\ell} \sigma_\tau \int_{t_n^{i,\ell-1}}^{\tau} a_u \, du \mathbb{1}_{[t_n^{i,\ell-1}, t_n^{i,\ell})}(\tau) \right)^{2m}\right] \\
&= \mathbb{E}\left[\sum_{\ell=1}^{nb_n} \left(K_{t_j,n}^{i,\ell}\right)^{2m} \sigma_\tau^{2m} \left(\int_{t_n^{i,\ell-1}}^{\tau} a_u \, du \right)^{2m} \mathbb{1}_{[t_n^{i,\ell-1}, t_n^{i,\ell})}(\tau) \right] \\
&= \mathcal{O}(n^{-2m}) \, .
\end{aligned}$$

Due to Assumptions 4.5, 4.8 and Burkholder's inequality it holds that

$$\frac{(n\log(n))^m}{b_n^m}\left(\int_{ib_n}^{(i+1)b_n}\mathbb{E}\left[\left(q_{n,i}^{t_j,1}(\tau)\right)^{2m}\right]^{1/m}d\tau\right)^m = \mathcal{O}\left((n\log(n))^m b_n^{-m} b_n^m n^{-2m}\right)$$
$$= \mathcal{O}(n^{-m}\log^m(n))$$
$$= o(v_n^{-1}),$$

if m is sufficiently large. This implies $A_n^{1,1,2} = o(1)$ and so $A_n^{1,1} = o(1)$.
We can proceed with $A_n^{1,2}$ and define

$$q_{n,i}^{t,2}(\tau) = \sum_{\ell=1}^{nb_n} K_{t,n}^{i,\ell}(\sigma_\tau - \sigma_{ib_n})\int_{t_n^{i,\ell-1}}^{\tau}\sigma_u\,dW_u\mathbb{1}_{[t_n^{i,\ell-1},t_n^{i,\ell})}(\tau)\,.$$

This implies the representation

$$A_n^{1,2} = \mathbb{P}\left[\sup_{t\in[0,1]}\nu_n\left|\sum_{i=0}^{b_n^{-1}-1}\int_{ib_n}^{(i+1)b_n} q_{n,i}^{t,2}(\tau)\,dW_\tau\right| > c\right].$$

We need a further decomposition

$$A_n^{1,2} \le A_n^{1,2,1} + A_n^{1,2,2}\,,$$

with $A_n^{1,2,1}$ given by

$$\mathbb{P}\left[\max_{1\le j\le v_n}\sup_{t\in[t_{j-1},t_j]}\nu_n\left|\sum_{i=0}^{b_n^{-1}-1}\int_{ib_n}^{(i+1)b_n}\left(q_{n,i}^{t,2}(\tau) - q_{n,i}^{t_j,2}(\tau)\right)dW_\tau\right| > c\right]$$

and $A_n^{1,2,2}$ given by

$$\mathbb{P}\left[\max_{1\le j\le v_n}\nu_n\left|\sum_{i=0}^{b_n^{-1}-1}\int_{ib_n}^{(i+1)b_n} q_{n,i}^{t_j,2}(\tau)\,dW_\tau\right| > c\right].$$

We omit the details verifying $A_n^{1,2,1} = o(1)$, since it works similar as for the term $A_n^{1,1,1}$.
We proceed with $A_n^{1,2,2}$,

$$\mathbb{P}\left[\max_{1\le j\le v_n}\nu_n\left|\sum_{i=0}^{b_n^{-1}-1}\int_{ib_n}^{(i+1)b_n} q_{n,i}^{t_j,2}(\tau)\,dW_\tau\right| > c\right]$$
$$\le \sum_{j=1}^{v_n}\mathbb{P}\left[\nu_n\left|\sum_{i=0}^{b_n^{-1}-1}\int_{ib_n}^{(i+1)b_n} q_{n,i}^{t_j,2}(\tau)\,dW_\tau\right| > c\right].$$

Markov's inequality applied with $z \mapsto |z|^{2m}$, $m > 0$ and $m \in \mathbb{N}$, yields

$$\begin{aligned}
&\mathbb{P}\left[\nu_n \left|\sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} q_{n,i}^{t_j,2}(\tau)\, dW_\tau\right| > c\right] \\
&\leq C_m \frac{(n\log(n))^m}{b_n^m} \mathbb{E}\left[\left|\sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} q_{n,i}^{t_j,2}(\tau)\, dW_\tau\right|^{2m}\right],
\end{aligned}$$

whereas Burkholder's inequality, triangle inequality, (4.9) and the generalized Minkowski inequality yield

$$\begin{aligned}
\mathbb{E}\left[\left|\sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} q_{n,i}^{t_j,2}(\tau)\, dW_\tau\right|^{2m}\right] &\leq C_m \mathbb{E}\left[\left|\int_{ib_n}^{(i+1)b_n} \left(q_{n,i}^{t,2}(\tau)\right)^2 d\tau\right|^m\right] \\
&\leq C_m \left(\int_{ib_n}^{(i+1)b_n} \mathbb{E}\left[\left(q_{n,i}^{t,2}(\tau)\right)^{2m}\right]^{1/m} d\tau\right)^m .
\end{aligned}$$

For a fixed $\tau \in [t_n^{i,\ell-1}, t_n^{i,\ell})$ it holds that

$$\begin{aligned}
\mathbb{E}\left[\left(q_{n,i}^{t,2}(\tau)\right)^{2m}\right] = \sum_{\ell=1}^{nb_n} \left(K_{t_j,n}^{i,\ell}\right)^{2m} \mathbb{E}\left[(\sigma_\tau - \sigma_{ib_n})^{2m} \left(\int_{t_n^{i,\ell-1}}^{\tau} \sigma_u\, dW_u\right)^{2m}\right] \\
\times \mathbb{1}_{[t_n^{i,\ell-1}, t_n^{i,\ell})}(\tau) = \mathcal{O}(b_n^{2m\mathfrak{a}} n^{-m}) .
\end{aligned}$$

The boundedness of the coefficient processes, the regularity of $(\sigma_t)_{t\in[0,1]}$ and Burkholder's inequality yield

$$\begin{aligned}
\frac{(n\log(n))^m}{b_n^m} \mathbb{E}\left[\left|\sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} q_{n,i}^{t,2}(\tau)\, dW_\tau\right|^{2m}\right] &= \mathcal{O}\left((n\log(n))^m b_n^{-m} b_n^{2m\mathfrak{a}} b_n^m n^{-m}\right) \\
&= \mathcal{O}\left(\log^m(n) b_n^{2m\mathfrak{a}}\right) \\
&= \mathcal{o}(v_n^{-1}) ,
\end{aligned}$$

due to Assumptions 4.5 and 4.7 for a m being sufficiently large. This completes the term $A_n^{1,2,2}$ and so $A_n^{1,2}$.
We can proceed with $A_n^{1,3}$. We define the step functions $q_{n,i}^{t,3}$ given by

$$q_{n,i}^{t,3}(\tau) = \sum_{\ell=1}^{nb_n} K_{t,n}^{i,\ell} \sigma_{ib_n} \int_{t_n^{i,\ell-1}}^{\tau} (\sigma_u - \sigma_{ib_n})\, dW_u \mathbb{1}_{[t_n^{i,\ell-1}, t_n^{i,\ell})}(\tau)$$

and use the representation

$$A_n^{1,3} = \mathbb{P}\left[\sup_{t\in[0,1]} \nu_n \left|\sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} q_{n,i}^{t,3}(\tau)\, dW_\tau\right| > c\right] .$$

We have a decomposition similar to the previous ones,

$$A_n^{1,3} \leq A_n^{1,3,1} + A_n^{1,3,2}\,,$$

with $A_n^{1,3,1}$ given by

$$\mathbb{P}\left[\max_{1\leq j\leq v_n}\sup_{t\in[t_{j-1},t_j]}\nu_n\left|\sum_{i=0}^{b_n^{-1}-1}\int_{ib_n}^{(i+1)b_n}\left(q_{n,i}^{t,3}(\tau)-q_{n,i}^{t_j,3}(\tau)\right)dW_\tau\right|>c\right]$$

and $A_n^{1,3,2}$ given by

$$\mathbb{P}\left[\max_{1\leq j\leq v_n}\nu_n\left|\sum_{i=0}^{b_n^{-1}-1}\int_{ib_n}^{(i+1)b_n}q_{n,i}^{t_j,3}(\tau)\,dW_\tau\right|>c\right]\,.$$

We omit the details on $A_n^{1,3,1}$ and proceed with $A_n^{1,3,2}$. It holds that

$$\begin{aligned}&\mathbb{P}\left[\max_{1\leq j\leq v_n}\nu_n\left|\sum_{i=0}^{b_n^{-1}-1}\int_{ib_n}^{(i+1)b_n}q_{n,i}^{t_j,3}(\tau)\,dW_\tau\right|>c\right]\\&\leq\sum_{j=1}^{v_n}\mathbb{P}\left[\nu_n\left|\sum_{i=0}^{b_n^{-1}-1}\int_{ib_n}^{(i+1)b_n}q_{n,i}^{t_j,3}(\tau)\,dW_\tau\right|>c\right]\,.\end{aligned}$$

Markov's inequality with $z\mapsto|z|^{2m}$, $m>0$ and $m\in\mathbb{N}$, Burkholder's inequality, Minkowski inequality, triangle inequality and (4.9) yield

$$\begin{aligned}&\mathbb{P}\left[\nu_n\left|\sum_{i=0}^{b_n^{-1}-1}\int_{ib_n}^{(i+1)b_n}q_{n,i}^{t_j,3}(\tau)\,dW_\tau\right|>c\right]\\&\leq C_m\frac{(n\log(n))^m}{b_n^m}\mathbb{E}\left[\left|\int_{ib_n}^{(i+1)b_n}q_{n,i}^{t_j,3}(\tau)\,dW_\tau\right|^{2m}\right]\\&\leq C_m\frac{(n\log(n))^m}{b_n^m}\left(\int_{ib_n}^{(i+1)b_n}\mathbb{E}\left[\left(q_{n,i}^{t_j,3}(\tau)\right)^{2m}\right]^{1/m}d\tau\right)^m\,.\end{aligned}$$

Again, for a fixed $\tau\in[t_n^{i,\ell-1},t_n^{i,\ell})$, the boundedness of K, the Itô isometry, standard Lebesgue upper bounds and the $\mathfrak{a}$-regularity of the volatility yield

$$\begin{aligned}\mathbb{E}\left[\left(q_{n,i}^{t_j,3}(\tau)\right)^{2m}\right]=\sum_{\ell=1}^{nb_n}\left(K_{t_j,n}^{i,\ell}\right)^{2m}\mathbb{E}\left[\left(\int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}}(\sigma_u-\sigma_{ib_n})^2\,du\right)^m\right]\\\times\mathbb{1}_{[t_n^{i,\ell-1},t_n^{i,\ell})}(\tau)=\mathcal{O}(n^{-m}b_n^{2m\mathfrak{a}})\,.\end{aligned}$$

Overall, via (4.9) and triangle inequality, this yields the bound

$$\begin{aligned}
&\frac{(n\log(n))^m}{b_n^m}\mathbb{E}\left[\left|\sum_{i=0}^{b_n^{-1}-1}\int_{ib_n}^{(i+1)b_n} q_{n,i}^{t_j,3}(\tau)\,dW_\tau\right|^{2m}\right] \\
&= \mathcal{O}\left((n\log(n))^m b_n^{-m} b_n^m b_n^{2m\mathfrak{a}} n^{-m}\right) \\
&= \mathcal{O}\left(\log^m(n) b_n^{2m\mathfrak{a}}\right) = \mathcal{o}(v_n^{-1})
\end{aligned}$$

if $m \in \mathbb{N}$ is sufficiently large and due to Assumption 4.7. This completes the term $A_n^{1,3,2}$ and so $A_n^{1,3}$. This yields $A_n^1 = \mathcal{o}(1)$.
We can proceed with A_n^2. Therefore, we define functions $k_{n,t}^i$ given by

$$k_{n,t}^i(\tau) = \sum_{\ell=1}^{nb_n} K_{t,n}^{\ell,i}\mathbb{1}_{[t_n^{i,\ell-1},t_n^{i,\ell})}(\tau)\,.$$

Then it holds that

$$\begin{aligned}
&\mathbb{P}\left[\sup_{t\in[0,1]}\frac{\sqrt{n\log(n)}}{\sqrt{b_n}}\left|\sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{nb_n} K_{t,n}^{i,\ell}\int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}}(\sigma_\tau^2-\sigma_{ib_n}^2)\,d\tau\right| > c\right] \\
&= \mathbb{P}\left[\sup_{t\in[0,1]}\frac{\sqrt{n\log(n)}}{\sqrt{b_n}}\left|\sum_{i=0}^{b_n^{-1}-1}\int_{ib_n}^{(i+1)b_n} k_{n,t}^i(\tau)(\sigma_\tau^2-\sigma_{ib_n}^2)\,d\tau\right| > c\right]\,.
\end{aligned}$$

Proceeding with the triangle inequality and Assumption 4.5, it holds that

$$\begin{aligned}
&\sup_{t\in[0,1]}\frac{\sqrt{n\log(n)}}{\sqrt{b_n}}\left|\sum_{i=0}^{b_n^{-1}-1}\int_{ib_n}^{(i+1)b_n} k_{n,t}^i(\tau)(\sigma_\tau^2-\sigma_{ib_n}^2)\,d\tau\right| \\
&\leq \sup_{t\in[0,1]}\frac{\sqrt{n\log(n)}}{\sqrt{b_n}}\sum_{i=0}^{b_n^{-1}-1}\left|\int_{ib_n}^{(i+1)b_n} k_{n,t}^i(\tau)(\sigma_\tau^2-\sigma_{ib_n}^2)\,d\tau\right| \\
&\leq \sup_{t\in[0,1]}\frac{\sqrt{n\log(n)}}{\sqrt{b_n}}\sum_{i=0}^{b_n^{-1}-1}\int_{ib_n}^{(i+1)b_n} k_{n,t}^i(\tau)\left|\sigma_\tau^2-\sigma_{ib_n}^2\right|\,d\tau \\
&= \mathcal{O}_{\mathbb{P}}\left(\sqrt{nb_n\log(n)}b_n^{\mathfrak{a}}\right) = \mathcal{o}_{\mathbb{P}}(1)\,,
\end{aligned}$$

due to (4.9), triangle inequality, the boundedness of K and Assumption 4.7. We have completed A_n^2 and can proceed with A_n^3.
Therefore, we use a further decomposition

$$A_n^3 \leq A_n^{3,1} + A_n^{3,2}\,,$$

with $A_n^{3,1}$ given by

$$\mathbb{P}\left[\max_{1\le j\le v_n}\sup_{t\in[t_{j-1},t_j]}\nu_n\left|\sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{nb_n}(K_{t,n}^{i,\ell}-K_{t_j,n}^{i,\ell})\int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}}(X_\tau-X_{t_n^{i,\ell-1}})a_\tau\,d\tau\right|>c\right]$$

and $A_n^{3,2}$ given by

$$\mathbb{P}\left[\max_{1\le j\le v_n}\nu_n\left|\sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{nb_n}K_{t_j,n}^{i,\ell}\int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}}(X_\tau-X_{t_n^{i,\ell-1}})a_\tau\,d\tau\right|>c\right].$$

We start with $A_n^{3,1}$. First of all, the regularity of K and a proper integral representation imply

$$\begin{aligned}\sum_{\ell=1}^{nb_n}(K_{t,n}^{i,\ell}-K_{t_j,n}^{i,\ell})&=\mathcal{O}\left(\int_{\mathbb{R}}\Psi_{K,\varrho}(u)\,du\right)\\&=\mathcal{O}(n^{-2}b_n)\end{aligned}$$

with $\varrho=|t-t_j|/2b_n$ and $t\in[t_{j-1},t_j]$.
Burkholder's inequality, the boundedness of the coefficient processes and the standard Lebesgue bound imply

$$\begin{aligned}\int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}}(X_\tau-X_{t_n^{i,\ell-1}})a_\tau\,d\tau&=\mathcal{O}_{\mathbb{P}}\left(\int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}}\left(\int_{t_n^{i,\ell-1}}^{\tau}\sigma_u^2\,du\right)^{1/2}d\tau\right)\\&=\mathcal{O}_{\mathbb{P}}(n^{-1}n^{-1/2}).\end{aligned}$$

The Cauchy Schwarz inequality and (4.9) imply

$$\begin{aligned}\max_{1\le j\le v_n}\sup_{t\in[t_{j-1},t_j]}\frac{\sqrt{n\log(n)}}{\sqrt{b_n}}&\left|\sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{nb_n}(K_{t,n}^{i,\ell}-K_{t_j,n}^{i,\ell})\int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}}(X_\tau-X_{t_n^{i,\ell-1}})a_\tau\,d\tau\right|\\&=\mathcal{O}_{\mathbb{P}}\left(\sqrt{n\log(n)}n^{-1/2}b_n^{-1/2}n^{-2}b_n^{-1}n^{-1/2}\right)\\&=o_{\mathbb{P}}(1).\end{aligned}$$

We can proceed with $A_n^{3,2}$ via

$$\begin{aligned}&\mathbb{P}\left[\max_{1\le j\le v_n}\frac{\sqrt{n\log(n)}}{\sqrt{b_n}}\left|\sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{nb_n}\int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}}(X_\tau-X_{t_n^{i,\ell-1}})a_\tau\,d\tau\right|>c\right]\\&\le\sum_{j=1}^{v_n}\mathbb{P}\left[\frac{\sqrt{n\log(n)}}{\sqrt{b_n}}\left|\sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{nb_n}\int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}}(X_\tau-X_{t_n^{i,\ell-1}})a_\tau\,d\tau\right|>c\right].\end{aligned}$$

Furthermore, with Markov's inequality applied to $z \mapsto |z|^r$, $r > 0$, (4.9), the triangle inequality, the generalized Minkowski inequality, Burkholder's inequality, the boundedness of the coefficient processes and the boundedness of K, we can proceed as follows:

$$
\begin{aligned}
&\mathbb{P}\left[\frac{\sqrt{n\log(n)}}{\sqrt{b_n}}\left|\sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{nb_n} K_{t_j,n}^{i,\ell}\int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}}(X_\tau - X_{t_n^{i,\ell-1}})a_\tau\,d\tau\right| > c\right] \\
&\leq C_r\frac{(n\log(n))^{r/2}}{b_n^{r/2}}\mathbb{E}\left[\left|\sum_{\ell=1}^{nb_n} K_{t_j,n}^{i,\ell}\int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}}(X_\tau - X_{t_n^{i,\ell-1}})a_\tau\,d\tau\right|^r\right] \\
&\leq C_r n^{3r/2}(b_n\log(n))^{r/2}\mathbb{E}\left[\left|\int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}}(X_\tau - X_{t_n^{i,\ell-1}})a_\tau\,d\tau\right|^r\right] \\
&\leq C_r n^{3r/2}(b_n\log(n))^{r/2}\left(\int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}}\mathbb{E}\left[\left|X_\tau - X_{t_n^{i,\ell-1}}\right|^r\right]^{1/r}d\tau\right)^r \\
&= \mathcal{O}(n^{3r/2}(b_n\log(n))^{r/2}n^{-r}n^{-r/2}) \\
&= \mathcal{O}((b_n\log(n))^{r/2}) = \mathcal{o}(v_n^{-1})\,,
\end{aligned}
$$

if r is sufficiently large. This completes the term $A_n^{3,2}$ and so A_n^3. Overall, we have completed A_n.
We can proceed with B_n. We have to bound the probability

$$
\begin{aligned}
&\mathbb{P}\left[\inf_{t\in[0,1]}\overline{\Gamma}_t^n < K^- - \kappa^-\right] \\
&= \mathbb{P}\left[\inf_{t\in[0,1]}\frac{1}{b_n}\sum_{i=0}^{b_n^{-1}-1}\sigma_{ib_n}^2\sum_{\ell=1}^{nb_n}K_{t,n}^{i,\ell}(\Delta_{\ell,i}^n W)^2 < K^- - \kappa^-\right].
\end{aligned}
$$

Using a Riemann sum approximation $\int K = 1$ and $K^- < \inf_{t\in[0,1]}\sigma_t^2$ we can proceed as follows, using

$$
\sum_{i=0}^{b_n^{-1}-1}\sigma_{ib_n}^2\sum_{\ell=1}^{nb_n}K_{t,n}^{i,\ell} > \sum_{i=0}^{b_n^{-1}-1}K^-\sum_{\ell=1}^{nb_n}K_{t,n}^{i,\ell}\,. \tag{4.10}
$$

It holds that

$$
\begin{aligned}
&\mathbb{P}\left[\inf_{t\in[0,1]}\frac{1}{b_n}\sum_{i=0}^{b_n^{-1}-1}\sigma_{ib_n}^2\sum_{\ell=1}^{nb_n}K_{t,n}^{i,\ell}(\Delta_{\ell,i}^n W)^2 < K^- - \kappa^-\right] \\
&= \mathbb{P}\left[\inf_{t\in[0,1]}\frac{1}{nb_n}\sum_{i=0}^{b_n^{-1}-1}\sigma_{ib_n}^2\sum_{\ell=1}^{nb_n}K_{t,n}^{i,\ell}(\sqrt{n}\Delta_{\ell,i}^n W)^2 < K^- - \kappa^-\right]
\end{aligned}
$$

$$\leq \mathbb{P}\left[\inf_{t\in[0,1]} \frac{1}{nb_n}\sum_{i=0}^{b_n^{-1}-1}\sigma_{ib_n}^2\sum_{\ell=1}^{nb_n}K_{t,n}^{i,\ell}((\sqrt{n}\Delta_{\ell,i}^n W)^2-1) < -\kappa^-\right]$$
$$= \mathbb{P}\left[\sup_{t\in[0,1]} \frac{-1}{nb_n}\sum_{i=0}^{b_n^{-1}-1}\sigma_{ib_n}^2\sum_{\ell=1}^{nb_n}K_{t,n}^{i,\ell}((\sqrt{n}\Delta_{\ell,i}^n W)^2-1) > \kappa^-\right],$$

where we have subtracted $\sum_{i=0}^{b_n^{-1}-1}\sigma_{ib_n}^2\sum_{\ell=1}^{nb_n}K_{t,n}^{i,\ell}$ on both sides and applied (4.10). The last equality is a consequence of the elementary identity $-\inf(S)=\sup(-S)$ for any set S. Thus, we can split B_n via

$$B_n \leq B_n^1 + B_n^2 ,$$

where B_n^1 is given by

$$\mathbb{P}\left[\max_{1\leq j\leq v_n}\sup_{t\in[t_{j-1},t_j]}\frac{1}{nb_n}\left|\sum_{i=0}^{b_n^{-1}-1}\sigma_{ib_n}^2\sum_{i=0}^{b_n^{-1}-1}(K_{t,n}^{i,\ell}-K_{t_j,n}^{i,\ell})((\sqrt{n}\Delta_{\ell,i}^n W)^2-1)\right| > \frac{\kappa^-}{2}\right]$$

and B_n^2 is given by

$$\mathbb{P}\left[\max_{1\leq j\leq v_n}\frac{1}{nb_n}\left|\sum_{i=0}^{b_n^{-1}-1}\sigma_{ib_n}^2\sum_{\ell=1}^{nb_n}K_{t_j,n}^{i,\ell}((\sqrt{n}\Delta_{\ell,i}^n W)^2-1)\right| > \frac{\kappa^-}{2}\right].$$

We start with the first probability and use that $(\sqrt{n}\Delta_{\ell,i}^n W)^2-1=\mathcal{O}_{\mathbb{P}}(1)$ via Burkholder's inequality, the regularity properties of the kernel K, (4.9) and triangle inequality it holds that

$$\max_{1\leq j\leq v_n}\sup_{t\in[t_{j-1},t_j]}\frac{1}{nb_n}\left|\sum_{i=0}^{b_n^{-1}-1}\sigma_{ib_n}^2\sum_{\ell=1}^{nb_n}(K_{t,n}^{i,\ell}-K_{t_j,n}^{i,\ell})((\sqrt{n}\Delta_{\ell,i}^n W)^2-1)\right|$$
$$=\mathcal{O}_{\mathbb{P}}\left(\int_{\mathbb{R}}\Psi_{K,\varrho}(u)\,du\right)=\mathcal{O}_{\mathbb{P}}(\varrho),$$

with $\varrho=|t-t_j|/2b_n$. Therefore, it is sufficient to consider the probability

$$\mathbb{P}\left[\max_{1\leq j\leq v_n}\frac{1}{nb_n}\left|\sum_{i=0}^{b_n^{-1}-1}\sigma_{ib_n}^2\sum_{\ell=1}^{nb_n}K_{t_j,n}^{i,\ell}((\sqrt{n}\Delta_{\ell,i}^n W)^2-1))\right| > \frac{\kappa^-}{2}\right].$$

Note that for every i $((\sqrt{n}\Delta_{\ell,i}^n W)^2 - 1))_\ell$ is an i.i.d. family of centered random variables. Thus, we can argue with the classical central limit theorem which yields

$$\begin{aligned}
&\mathbb{P}\left[\max_{1\le j\le v_n} \frac{1}{nb_n}\left|\sum_{i=0}^{b_n^{-1}-1}\sigma_{ib_n}^2\sum_{\ell=1}^{nb_n}K_{t_j,n}^{i,\ell}((\sqrt{n}\Delta_{\ell,i}^n W)^2-1))\right| > \frac{\kappa^-}{2}\right]\\
&\le \sum_{j=1}^{v_n}\mathbb{P}\left[\frac{1}{nb_n}\left|\sum_{i=0}^{b_n^{-1}-1}\sigma_{ib_n}^2\sum_{\ell=1}^{nb_n}K_{t_j,n}^{i,\ell}((\sqrt{n}\Delta_{\ell,i}^n W)^2-1))\right| > \frac{\kappa^-}{2}\right]\\
&\le C_r\sum_{j=1}^{v_n}\mathbb{E}\left[\left|\frac{1}{nb_n}\sum_{i=0}^{b_n^{-1}-1}\sigma_{ib_n}^2\sum_{\ell=1}^{nb_n}K_{t_j,n}^{i,\ell}((\sqrt{n}\Delta_{\ell,i}^n W)^2-1))\right|^r\right]\\
&= \mathcal{O}(v_n(nb_n)^{-r/2}),
\end{aligned}$$

where we have used the Markov inequality with an exponent $r > 0$, the (4.9), the triangle inequality, the boundedness of K and the boundedness of $(\sigma_t)_{t\in[0,1]}$. For $r > 0$ sufficiently large, it holds that $\mathcal{O}(v_n(nb_n)^{-r/2})) = o(1)$ which completes B_n^2 and so B_n. Finally, the term (**II**) has been completed.

We can proceed with (**I**). For every $\delta > 0$ we have

$$\begin{aligned}
&\mathbb{P}\left[\sup_{t\in[0,1]}\sqrt{nb_n\log(n)}\widehat{\Gamma}_t^n\left(\frac{1}{\widehat{\Gamma}_t^n}-\frac{1}{\overline{\Gamma}_t^n}\right) > \delta\right]\\
&= \mathbb{P}\left[\sup_{t\in[0,1]}\sqrt{nb_n\log(n)}\widehat{\Gamma}_t^n\left(\frac{\overline{\Gamma}_t^n-\widehat{\Gamma}_t^n}{\widehat{\Gamma}_t^n\overline{\Gamma}_t^n}\right) > \delta\right]\\
&= \mathbb{P}\left[\sup_{t\in[0,1]}\sqrt{nb_n\log(n)}\widehat{\Gamma}_t^n\left(\frac{\overline{\Gamma}_t^n-\widehat{\Gamma}_t^n}{\widehat{\Gamma}_t^n\overline{\Gamma}_t^n}\right) > \delta,\ \inf_{t\in[0,1]}\widehat{\Gamma}_t^n\overline{\Gamma}_t^n \ge \frac{(K^- - \kappa^-)^2}{4}\right]\\
&+ \mathbb{P}\left[\sup_{t\in[0,1]}\sqrt{nb_n\log(n)}\widehat{\Gamma}_t^n\left(\frac{\overline{\Gamma}_t^n-\widehat{\Gamma}_t^n}{\widehat{\Gamma}_t^n\overline{\Gamma}_t^n}\right) > \delta,\ \inf_{t\in[0,1]}\widehat{\Gamma}_t^n\overline{\Gamma}_t^n < \frac{(K^- - \kappa^-)^2}{4}\right]\\
&\le \mathbb{P}\left[\sup_{t\in[0,1]}\sqrt{nb_n\log(n)}\widehat{\Gamma}_t^n\left(\overline{\Gamma}_t^n-\widehat{\Gamma}_t^n\right) > \frac{\delta(K^- - \kappa^-)^2}{4}\right] \qquad (4.11)\\
&+ \mathbb{P}\left[\inf_{t\in[0,1]}\widehat{\Gamma}_t^n\overline{\Gamma}_t^n < \frac{(K^- - \kappa^-)^2}{4}\right]. \qquad (4.12)
\end{aligned}$$

We start with (4.12):

$$\begin{aligned}
&\mathbb{P}\left[\inf_{t\in[0,1]}\widehat{\Gamma}_t^n\overline{\Gamma}_t^n < \frac{(K^- - \kappa^-)^2}{4}\right]\\
&\le \mathbb{P}\left[\inf_{t\in[0,1]}\widehat{\Gamma}_t^n < \frac{K^- - \kappa^-}{2}\right] + \mathbb{P}\left[\inf_{t\in[0,1]}\overline{\Gamma}_t^n < \frac{K^- - \kappa^-}{2}\right]. \qquad (4.13)
\end{aligned}$$

The second probability in (4.13) has already been considered in B_n, where we exploited the boundedness of the volatility from below. We proceed with the first probability. It holds that

$$\begin{aligned}
&\mathbb{P}\left[\inf_{t\in[0,1]} \widehat{\Gamma}_t^n < \frac{K^- - \kappa^-}{2}\right] \\
&= \mathbb{P}\left[\inf_{t\in[0,1]} \widehat{\Gamma}_t^n < \frac{K^- - \kappa^-}{2}, \sup_{t\in[0,1]} \left|\widehat{\Gamma}_t^n - \overline{\Gamma}_t^n\right| \leq \frac{K^- - \kappa^-}{2}\right] \\
&+ \mathbb{P}\left[\inf_{t\in[0,1]} \widehat{\Gamma}_t^n < \frac{K^- - \kappa^-}{2}, \sup_{t\in[0,1]} \left|\widehat{\Gamma}_t^n - \overline{\Gamma}_t^n\right| > \frac{K^- - \kappa^-}{2}\right] \\
&\leq \mathbb{P}\left[\inf_{t\in[0,1]} \overline{\Gamma}_t^n < K^- - \kappa^-\right] + \mathbb{P}\left[\sup_{t\in[0,1]} \left|\widehat{\Gamma}_t^n - \overline{\Gamma}_t^n\right| > \frac{K^- - \kappa^-}{2}\right].
\end{aligned}$$

Both probabilities have already been considered. We can proceed with (4.11). We use the decomposition

$$\begin{aligned}
&\mathbb{P}\left[\sup_{t\in[0,1]} \sqrt{nb_n \log(n)}\widehat{\Gamma}_t^n \left(\overline{\Gamma}_t^n - \widehat{\Gamma}_t^n\right) > \frac{\delta(K^- - \kappa^-)^2}{4}\right] \\
&= \mathbb{P}\left[\sup_{t\in[0,1]} \sqrt{nb_n \log(n)}\widehat{\Gamma}_t^n \left(\overline{\Gamma}_t^n - \widehat{\Gamma}_t^n\right) > \frac{\delta(K^- - \kappa^-)^2}{4}, \sup_{t\in[0,1]} \widehat{\Gamma}_t^n \leq 2(K^+ + \kappa^+)\right] && (4.14) \\
&+ \mathbb{P}\left[\sup_{t\in[0,1]} \sqrt{nb_n \log(n)}\widehat{\Gamma}_t^n \left(\overline{\Gamma}_t^n - \widehat{\Gamma}_t^n\right) > \frac{\delta(K^- - \kappa^-)^2}{4}, \sup_{t\in[0,1]} \widehat{\Gamma}_t^n > 2(K^+ + \kappa^+)\right]. && (4.15)
\end{aligned}$$

We start with (4.14). It holds that

$$\begin{aligned}
&\mathbb{P}\left[\sup_{t\in[0,1]} \sqrt{nb_n \log(n)}\widehat{\Gamma}_t^n \left(\overline{\Gamma}_t^n - \widehat{\Gamma}_t^n\right) > \frac{\delta(K^- - \kappa^-)^2}{4}, \sup_{t\in[0,1]} \widehat{\Gamma}_t^n \leq 2(K^+ + \kappa^+)\right] \\
&\leq \mathbb{P}\left[\sup_{t\in[0,1]} \sqrt{nb_n \log(n)} \left(\overline{\Gamma}_t^n - \widehat{\Gamma}_t^n\right) > \frac{\delta(K^- - \kappa^-)^2}{8(K^+ + \kappa^+)}\right],
\end{aligned}$$

i.e. this follows from A_n. We can proceed with (4.15):

$$\begin{aligned}
&\mathbb{P}\left[\sup_{t\in[0,1]} \sqrt{nb_n \log(n)}\widehat{\Gamma}_t^n \left(\overline{\Gamma}_t^n - \widehat{\Gamma}_t^n\right) > \frac{\delta(K^- - \kappa^-)^2}{4}, \sup_{t\in[0,1]} \widehat{\Gamma}_t^n > 2(K^+ + \kappa^+)\right] \\
&\leq \mathbb{P}\left[\sup_{t\in[0,1]} \widehat{\Gamma}_t^n > 2(K^+ + \kappa^+)\right] \\
&\leq \mathbb{P}\left[\sup_{t\in[0,1]} \overline{\Gamma}_t^n > K^+ + \kappa^+\right] + \mathbb{P}\left[\sup_{t\in[0,1]} \left|\widehat{\Gamma}_t^n - \overline{\Gamma}_t^n\right| > K^+ + \kappa^+\right].
\end{aligned}$$

Only the first probability has to be considered. In contrast to B_n, in this case, we have to bound the probability from above via K^+. A Riemann sum approximation, $\int K = 1$ and

$$\sum_{i=0}^{b_n^{-1}-1} \sigma_{ib_n}^2 \sum_{\ell=1}^{nb_n} K_{t,n}^{i,\ell} < \sum_{i=0}^{b_n^{-1}-1} K^+ \sum_{\ell=1}^{nb_n} K_{t,n}^{i,\ell}\,,$$

yield the following bound:

$$\begin{aligned}
&\mathbb{P}\left[\sup_{t\in[0,1]} \frac{1}{b_n} \sum_{i=0}^{b_n^{-1}-1} \sigma_{ib_n}^2 \sum_{\ell=1}^{nb_n} K_{t,n}^{i,\ell}(\Delta_{\ell,i}^n W)^2 > K^+ + \kappa^+\right] \\
&= \mathbb{P}\left[\sup_{t\in[0,1]} \frac{1}{nb_n} \sum_{i=0}^{b_n^{-1}-1} \sigma_{ib_n}^2 \sum_{\ell=1}^{nb_n} K_{t,n}^{i,\ell}(\sqrt{n}\Delta_{\ell,i}^n W)^2 > K^+ + \kappa^+\right] \\
&\leq \mathbb{P}\left[\sup_{t\in[0,1]} \frac{1}{nb_n} \sum_{i=0}^{b_n^{-1}-1} \sigma_{ib_n}^2 \sum_{\ell=1}^{nb_n} K_{t,n}^{i,\ell}(\left(\sqrt{n}\Delta_{\ell,i}^n W\right)^2 - 1) > \kappa^+\right].
\end{aligned}$$

Due to the same argument and decomposition applied to B_n, it is sufficient to consider the probability

$$\begin{aligned}
&\mathbb{P}\left[\max_{1\leq j\leq v_n} \frac{1}{nb_n} \sum_{i=0}^{b_n^{-1}-1} \sigma_{ib_n}^2 \sum_{\ell=1}^{nb_n} K_{t,n}^{i,\ell}((\Delta_{\ell,i}^n W)^2 - 1) > \kappa^+\right] \\
&\leq C_r \sum_{j=1}^{v_n} \mathbb{E}\left[\left|\frac{1}{nb_n} \sum_{i=0}^{b_n^{-1}-1} \sigma_{ib_n}^2 \sum_{\ell=1}^{nb_n} K_{t_j,n}^{i,\ell}((\Delta_{\ell,i}^n W)^2 - 1)\right|^r\right] \\
&= \mathcal{O}(v_n(nb_n)^{-r/2}) \\
&= \mathcal{o}(1)\,,
\end{aligned}$$

if r is sufficiently large via the central limit theorem, (4.9) and the triangle inequality. This completes (4.15) and (4.11), i.e. overall, the term (**I**) has been shown to be negligible. We omit the details on (**II**), since the procedure is exactly the same, bounding the volatility from above. Thus, Proposition 4.17 is shown. □

In order to apply limit theorems stated in Section 2.4, we need two further approximations. Therefore, we define the quantities

$$\widetilde{\Gamma}_t^n = \frac{1}{b_n} \sum_{i=0}^{b_n^{-1}-1} \sum_{\ell=1}^{nb_n} K_{t,n}^{i,\ell} \int_{t_n^{i,\ell-1}}^{t_n^{i,\ell}} \sigma_u^2\, du$$

and

$$\Gamma'^n_t = \frac{1}{nb_n} \sum_{i=0}^{b_n^{-1}-1} \sigma^2_{ib_n} \sum_{\ell=1}^{nb_n} K^{i,\ell}_{t,n} \,.$$

We have to show the following convergences in probability. The first one is given in

Proposition 4.18. *Under the assumptions of Theorem 4.12 it holds that*

$$\sup_{t\in[0,1]} \left|\widetilde{\Gamma}^n_t - \Gamma'^n_t\right| = \scriptstyle{\mathcal{O}}_{\mathbb{P}}((nb_n \log(n))^{-1/2}) \,.$$

Proof of Proposition 4.18.
It holds that

$$\begin{aligned}
&\sup_{t\in[0,1]} \left|\widetilde{\Gamma}^n_t - \Gamma'^n_t\right| \\
&= \sup_{t\in[0,1]} \left| \frac{1}{b_n} \sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} k^i_{n,t}(\tau)(\sigma^2_\tau - \sigma^2_{ib_n})\, d\tau \right| \\
&= \mathcal{O}_{\mathbb{P}}(b_n^{\mathfrak{a}}) = \scriptstyle{\mathcal{O}}_{\mathbb{P}}((nb_n \log(n))^{-1/2}) \,,
\end{aligned}$$

where we have used Assumption 4.5, the triangle inequality and (4.9). □

The second convergence in probability is

Proposition 4.19. *Under the assumptions of Theorem 4.12 it holds that*

$$\sup_{t\in[0,1]} \left|\widetilde{\Gamma}^n_t - \Gamma_t\right| = \scriptstyle{\mathcal{O}}_{\mathbb{P}}((nb_n \log(n))^{-1/2}) \,.$$

Proof of Proposition 4.19.
For a proof of this convergence we refer to Section A.1 in Kanaya and Kristensen [40]. Therein the term R_5 contains the proof of the approximation above. The idea is to define proper step functions and to exploit the $\mathfrak{a}$-regularity of the volatility process proving the bound $\mathcal{O}_{\mathbb{P}}(b_n^{\mathfrak{a}})$. Note that the setting and assumptions imposed therein, especially the non-standard assumption $\mathbb{K}$.**1** are compatible with our assumptions, since we assume our kernel to have compact support and the volatility to be uniformly bounded and fulfilling the $\mathfrak{a}$-regularity due to Assumption 4.5. □

Using Proposition 4.17-4.19 it is sufficient to consider the term

$$\frac{1}{nb_n} \sum_{i=1}^{n} K_{b_n}(t - i/n)\, \eta_i \,.$$

The family of random variables $(\eta_i)_{i\in\mathbb{N}}$, with

$$\eta_i = \frac{\left(\sqrt{n}\Delta^n_i W\right)^2 - 1}{\sqrt{2}} \,,$$

are

- independent, centered and normalized with
- shifted χ^2 distributed random variables.

The latter ensures the existence of the moment generating function. Hence, we can apply the approximation explained in Remark 2.25. We define the partial sums $S_\ell = \sum_{i=1}^{\ell} \eta_i$ and $T_\ell = \sum_{i=1}^{\ell} Z_i$ with standard normal random variables Z_i. It holds that

$$\max_{\ell \le n} |S_\ell - T_\ell| = \mathcal{O}_{\mathbb{P}}(\log(n)) .$$

It is sufficient to consider the sequence of stochastic processes U^n given by

$$U_t^n = \frac{1}{\sqrt{nb_n}} \sum_{i=1}^{n} K_{b_n}(t - i/n)\, \eta_i .$$

Furthermore, we define another sequence W^n given by

$$W_t^n = \frac{1}{\sqrt{nb_n}} \sum_{i=1}^{n} K_{b_n}(t - i/n)\, Z_i .$$

We will use a Brownian motion $\mathbb{B}$ from the invariance principles such that stochastic analysis tools are applicable. Therefore, we use the stochastic integral representation of W^n,

$$W_t^n = \frac{1}{\sqrt{nb_n}} \int_0^n K_{b_n}(t - \lfloor u+1 \rfloor / n)\; d\mathbb{B}_u .$$

Abel's partial summation and the boundedness of K yield

$$\begin{aligned}
&|U_t^n - W_t^n| \\
&= \left| \frac{1}{\sqrt{nb_n}} \sum_{i=1}^{n} (\eta_i - Z_i) K_{t,n}^{0,n} + \sum_{\ell=1}^{n-1} \left(\sum_{i=1}^{\ell} (Z_i - \eta_i) \right) (K_{t,n}^{0,\ell} - K_{t,n}^{0,\ell+1}) \right| \\
&= \mathcal{O}(R_n) \left[1 + \int_{\mathbb{R}} \Psi_{K,\varrho} \left(\frac{t - u/n}{b_n} \right) du \right] ,
\end{aligned}$$

where the latter bound holds uniformly in t and

$$\begin{aligned}
\varrho &= \frac{1}{nb_n} , \\
R_n &= \max_{\ell \le n} |S_\ell - T_\ell| / \sqrt{nb_n} .
\end{aligned}$$

We have to bound R_n properly. Therefore, take into account that

$$\begin{aligned}
&\max_{\ell\le n}\sqrt{\log(n)}\frac{|S_\ell-\mathbb{B}_\ell|}{\sqrt{nb_n}}\\
=&\max_{\ell\le n}\sqrt{\log(n)}\frac{|S_\ell-\mathbb{B}_\ell|}{\sqrt{nb_n}}\frac{\log(n)}{\log(n)}\\
=&\max_{\ell\le n}\frac{|S_\ell-\mathbb{B}_\ell|}{\log(n)}\frac{\log(n)\sqrt{\log(n)}}{\sqrt{nb_n}}\\
=&\mathcal{O}_{a.s.}(1)\mathcal{o}(1)\,.
\end{aligned}$$

The scaling invariance of the Brownian motion and a change of variables imply

$$(W^n_{sb_n})_{0\le s\le b_n^{-1}}\overset{d}{=}(\widetilde{Y}^n_s)_{0\le s\le b_n^{-1}}\,,$$

with

$$\widetilde{Y}^n_t=\int_0^{\frac{1}{b_n}}K\left(t-\frac{\lfloor 1+nb_nv\rfloor}{nb_n}\right)d\mathbb{B}_v\,.$$

The key step is to pass from the processes $\widetilde{Y}^n$ to another sequence of processes Y^n, being *stationary and Gaussian*. This sophisticated construction has already been presented in Wu and Zhao [61], where Y^n is given by

$$Y^n_t=\int_{\mathbb{R}}K(t-u)\,d\mathbb{B}_u\cdot\mathbb{1}_{[0,b_n^{-1}]}(t)\,,$$

being stationary and Gaussian. The approximation $\sup_{0\le t\le b_n^{-1}}\left|\widetilde{Y}^n_t-Y^n_t\right|=\mathcal{O}_{\mathbb{P}}(r_n)=\mathcal{o}_{\mathbb{P}}(\log(n)^{-1/2})$ has already been pursued on page 406 in Wu and Zhao [61]. Therefore we omit the details. An application of Theorem 2.27 with $T=b_n^{-1}$ yields the desired limit theorem, such that the proof of Theorem 4.12 is completed.

Proof of Theorem 4.15

We have to show that

$$\sqrt{nb_n\log(n)}\sup_{t\in[0,1]}\left|\widehat{\Gamma}^n_{t,\tau}-\widehat{\Gamma}^n_t\right|\xrightarrow{\mathbb{P}}0 \tag{4.16}$$

holds. Theorem 4.15 then immediately follows from Theorem 4.12. We use the decomposition

$$X_t=X_0+X^c_t+J_t\,,$$

with

$$X^c_t=\int_0^t\widetilde{a}_s\,ds+\int_0^t\sigma_s\,dW_s\,.$$

The purely discontinuous local martingale $(J_t)_{t\in[0,1]}$ is given by

$$J_t = \int_0^t \int_{\mathbb{R}} \delta(s,x)(\mu-\nu)\,(ds,dx)$$

and $\widetilde{a}_t = a_t + \int_{\mathbb{R}} \overline{\kappa}(\delta(s,x))\,\lambda(dx)$. It holds that

$$\widehat{\Gamma}^n_{t,\tau} - \widehat{\Gamma}^n_t = \frac{1}{b_n}\sum_{i=1}^n K_{b_n}(t-i/n)\left((\Delta_i^n X)^2 \mathbb{1}_{\{|\Delta_i^n X|\le u_n\}} - (\Delta_i^n X^c)^2\right)$$

and

$$\begin{aligned}
&(\Delta_i^n X)^2 \mathbb{1}_{\{|\Delta_i^n X|\le u_n\}} - (\Delta_i^n X^c)^2 \\
&= (\Delta_i^n X^c + \Delta_i^n J)^2 \mathbb{1}_{\{|\Delta_i^n X|\le u_n\}} - (\Delta_i^n X^c)^2 \\
&= ((\Delta_i^n X^c)^2 + 2\Delta_i^n X^c \Delta_i^n J + (\Delta_i^n J)^2) \mathbb{1}_{\{|\Delta_i^n X|\le u_n\}} - (\Delta_i^n X^c)^2 \\
&= (\Delta_i^n X^c)^2 \mathbb{1}_{\{|\Delta_i^n X|> u_n\}} + 2\Delta_i^n X^c \Delta_i^n J \mathbb{1}_{\{|\Delta_i^n X|\le u_n\}} + (\Delta_i^n J)^2 \mathbb{1}_{\{|\Delta_i^n X|\le u_n\}}\,.
\end{aligned}$$

Thus, we have to bound the following three terms:

$$\sup_{t\in[0,1]} \frac{\sqrt{n\log(n)}}{\sqrt{b_n}} \sum_{i=1}^n K_{b_n}(t-i/n)\,(\Delta_i^n X^c)^2 \mathbb{1}_{\{|\Delta_i^n X|> u_n\}}\,, \tag{4.17}$$

$$\sup_{t\in[0,1]} \frac{\sqrt{n\log(n)}}{\sqrt{b_n}} \sum_{i=1}^n K_{b_n}(t-i/n)\,\Delta_i^n X^c \Delta_i^n J \mathbb{1}_{\{|\Delta_i^n X|\le u_n\}}\,, \tag{4.18}$$

$$\sup_{t\in[0,1]} \frac{\sqrt{n\log(n)}}{\sqrt{b_n}} \sum_{i=1}^n K_{b_n}(t-i/n)\,(\Delta_i^n J)^2 \mathbb{1}_{\{|\Delta_i^n X|\le u_n\}}\,. \tag{4.19}$$

It is sufficient to consider (4.17) and (4.19) only, since (4.18) is a simple consequence of (4.17) and (4.19) via Cauchy Schwarz inequality. We start with (4.17). Using the main theorem in Fischer and Nappo [24], it holds that $\max_i (\Delta_i^n X^c)^2 = \mathcal{O}_{\mathbb{P}}(n^{-1}\log(n))$ and $\max_i |\Delta_i^n X^c| = o_{a.s.}(u_n)$. We will stick to the ideas presented in Bibinger et al. [10] and Chapter 13 in Jacod and Protter [35].
We use the decomposition

$$X = X'^n + X''^n\,,$$

with

$$X''^n_t = \int_0^t \int_{\mathbb{R}} \delta(s,x) \mathbb{1}_{\{\gamma(x)>u_n^p\}}\,\mu(ds,dx)$$

and

$$X'^n_t = X_t - X''^n_t\,.$$

We define the sets $A_j^n = \{|\Delta_j^n X'^n| \leq u_n/2\}$ and the counting process N^n, given by

$$N_t^n = \int_0^t \int_{\mathbb{R}} \mathbb{1}_{\{\gamma(x) > u_n^p\}} \, \mu(ds, dx) \, .$$

We know from (13.1.10) in Jacod and Protter [35] that

$$\mathbb{E}\left[\sum_{i=1}^n \mathbb{1}_{\{|\Delta_i^n X| > u_n\}} \mathbb{1}_{\{(A_j^n)^{\mathsf{c}}\}}\right] \leq \sum_{i=1}^n \mathbb{P}\left[(A_j^n)^{\mathsf{c}}\right] \longrightarrow 0 \, ,$$

for all such p with $1 < p < (2r\tau)^{-1}$. Using the inequality

$$\mathbb{1}_{\{|\Delta_j^n X| > u_n\}} \mathbb{1}_{A_j^n} \leq \mathbb{1}_{\{|\Delta_j^n X''^n| > u_n/2\}}$$

we can proceed as follows. We define the sets $\mathcal{I}_{n,t}$ given by

$$\mathcal{I}_{n,t} = [nt - nb_n, nt + nb_n] \cap \mathbb{Z} \, .$$

It obviously holds that $|\mathcal{I}_{n,t}| \leq n$. Since N^n is a Poisson process with parameter

$$\int_{\mathbb{R}} \mathbb{1}_{\{\gamma(x) > u_n^p\}} \, \lambda(dx) = \mathcal{O}(u_n^{-rp}) \, ,$$

it holds that for some $\ell < \infty$, with $i^* = \max \mathcal{I}_{n,t}$ and $i_* = \min \mathcal{I}_{n,t}$,

$$\begin{aligned} \mathbb{P}\left[N_{i^*/n}^n - N_{i_*/n}^n > \ell\right] &= \mathbb{P}\left[N_{(i^* - i_*)/n}^n \geq \ell\right] \\ &= \mathcal{O}(b_n^\ell u_n^{-rp\ell}) \, . \end{aligned}$$

This yields the bound

$$\begin{aligned} &\mathcal{O}_{\mathbb{P}}(\log^{3/2}(n)(nb_n)^{-1/2} b_n^\ell n^{\tau rp\ell}) \\ &= \mathcal{o}_{\mathbb{P}}(1) \, , \end{aligned}$$

via the boundedness of K and Assumption 4.10. This completes the proof of the term (4.17). We can proceed with (4.19). Starting with the finite activity case, we get the bound $\sqrt{n \log(n)} b_n^{-1/2} u_n^2 = \mathcal{o}(1)$. Therefore, we can proceed with the general infinite activity case. It is sufficient to bound

$$\begin{aligned} &\sup_{t \in [0,1]} \frac{\sqrt{n \log(n)}}{\sqrt{b_n}} \sum_{i=1}^n K_{b_n}(t - i/n) \, (\Delta_i^n J)^2 \mathbb{1}_{\{(\Delta_i^n J)^2 \leq u_n\}} \\ &= \sup_{t \in [0,1]} \frac{\sqrt{n \log(n)}}{\sqrt{b_n}} \sum_{i=1}^n K_{b_n}(t - i/n) \left(\Delta_i^n J \mathbb{1}_{\{|\Delta_i^n J| \leq u_n^{1/2}\}}\right)^2 \, . \end{aligned}$$

Using the inequality (3.62a) and defining the process $\mathcal{Z}_\ell$ given by

$$\mathcal{Z}_\ell = \left(\Delta_\ell^n J \mathbb{1}_{\{|\Delta_\ell^n J| \leq u_n^{1/2}\}}\right)^2$$

we get the upper bound

$$
\begin{aligned}
&\lambda \mathbb{P}\left[\sup_{t\in[0,1]} \max_{\ell\in\mathcal{I}_{n,t}} \frac{\sqrt{n\log(n)}}{\sqrt{b_n}} \sum_{i=1}^{\ell} K_{b_n}(t-i/n)\, \mathcal{Z}_i \geq \lambda\right] \\
&\leq \sqrt{n\log(n)} b_n^{-1/2} \mathbb{E}\left[\max_{\ell\in\mathcal{I}_{n,t}} \sum_{i=1}^{\ell} \mathcal{Z}_i\right] \\
&= \mathcal{O}(\sqrt{n\log(n)} b_n^{-1/2} n^{-1} u_n^{2-r} |\mathcal{I}_{n,t}|) \\
&= \scriptstyle{\mathcal{O}}(1)
\end{aligned}
$$

for $\lambda^{-1} = \scriptstyle{\mathcal{O}}(u_n^{r-2} b_n^{-1} n^{-1/2} \log^{-1/2}(n))$ via Doob's submartingale inequality and Assumption 4.10.

5. Uniform spot volatility estimation for noisy Itô semimartingales

In this final chapter of this thesis we will present a further extension of the model and the methods presented in Chapter 4. This chapter is organized as follows. Starting with the asymptotic theory in the continuous case we will proceed with a version being jump and noise robust. We will construct confidence sets and will conclude the chapter with the proof section.

5.1. Construction of the main statistics and asymptotic theory

We will start with the continuous price process X, i.e. we assume $\mu \equiv 0$ in (2.4). As in the previous chapter we will use a Nadaraya-Watson type estimator. More precisely, an adaptive version $\hat{\Gamma}^{\varepsilon,ad}_{t,n}$ of our estimator is given by

$$\hat{\Gamma}^{\varepsilon,ad}_{t,n} = \frac{h_n}{b_n} \sum_{k=1}^{h_n^{-1}} K_{b_n}(t - kh_n)\, \zeta_k^{ad}(Y)\,, \tag{5.1}$$

with

$$\zeta_k^{ad}(Y) = \sum_{j=1}^{\lfloor nh_n \rfloor - 1} \hat{w}_{jk}(S_{jk}^2(Y) - [\varphi_{jk}, \varphi_{jk}]_n \frac{\hat{\eta}^2}{n})$$

and the notation introduced in the Chapters 3 and 4.
In contrast to Chapter 4 we need modified assumptions on the bandwidth $(b_n)_{n\in\mathbb{N}}$.

Assumption 5.1 (The bandwidth b_n). *The sequence $(b_n)_{n\in\mathbb{N}}$ is assumed to fulfill*

(1) $b_n \longrightarrow 0$ *and* $b_n/h_n \longrightarrow \infty$,

(2) $\sqrt{b_n \log(h_n^{-1})/h_n} b_n^{\mathfrak{a}} \longrightarrow 0$ *and*

(3) $\frac{h_n^{-c}\sqrt{\log(h_n^{-1})}}{\sqrt{b_n/h_n}} \longrightarrow 0$ *for an arbitrarily small* $c \in (0,1)$,

as $n \longrightarrow \infty$.

Remark 5.2. (1) Similar to (2) in Assumption 4.7 the second assumption above is necessary in order to modulate the roughness of the volatility paths controlled by the index $\mathfrak{a} \in (0,1)$.

(2) The third condition above is seemingly different to the analogous one in Assumption 4.7. This is mainly due to the fact that we wanted to impose as less restrictions as possible with respect to the noise process structure, e.g. we do not assume the existence of the moment generating function of the noise distribution.

(3) The choice of the exponent $c \in (0,1)$ crucially depends on the highest order of existing moments of the noise process. More precisely, if

$$m = \max\left\{k \in \mathbb{N} : \mathbb{E}\left[|\varepsilon_t|^k\right] < \infty\right\},$$

then this would implies the choice $c \approx 1/m$, i.e. the more moments exist the less restrictive is the choice of the bandwidth.

(4) The magnitude of the parameter $c \in (0,1)$ is also directly linked to the regularity of the volatility paths $(\sigma_t^2)_{t\in[0,1]}$ controlled via $\mathfrak{a} \in (0,1)$. The latter holds in a sense that, the smaller $\mathfrak{a} \in (0,1)$, i.e. the rougher the paths are, the smaller $c \in (0,1)$ has to be chosen, i.e. the more restrictive is the choice of the bandwidth $(b_n)_{n\in\mathbb{N}}$.

(5) Finally, an existing moment generating function of $(\varepsilon_t)_{t\in[0,1]}$ would allow to replace (3) in Assumption 5.1 by

$$\frac{\log^{3/2}(h_n^{-1})h_n^{1/2}}{b_n^{1/2}} = \mathcal{O}(1).$$

The conditions on the noise process, which are necessary for the asymptotic theory of this chapter are exactly the same as formulated in Chapter 3 in Assumption 3.4. The properties on the coefficient processes of the semimartingale X are exactly the same as formulated in Assumption 4.5. Finally, we will also stick to Assumption 4.8 concerning the kernel function K. We set

$$\overline{M}_n(t) = \frac{\left|\hat{\Gamma}_{t,n}^{\varepsilon,ad} - \Gamma_t\right|}{\sqrt{8\hat{\eta}}\left|\hat{\Gamma}_{t,n}^{\varepsilon,ad}\right|^{3/4}}$$

and

$$\overline{M}_n = \sup_{t\in[0,1]} \overline{M}_n(t).$$

Theorem 5.3. *Under the Assumptions 3.4, 4.5, 4.8, 5.1, with the notation introduced in Theorem 4.12, it holds that for all x*

$$\mathbb{P}\left[\sqrt{2\log(m_n)}\left(\frac{\sqrt{b_n/h_n}}{\sqrt{\lambda_K}}\overline{M}_n - d_{m_n}^{\alpha,K}\right) \le x\right] \longrightarrow \exp(-2\exp(-x)).$$

Using Theorem 5.3 we can construct asymptotic confidence sets. Choosing b_n optimally, i.e. $b_n \propto h_n^{\frac{1}{2\mathfrak{a}+1}}$ yields the following confidence bands. Thereto, let z_β be the $(1-\beta)$- quantile of $\exp(-2\exp(-x))\,dx$ and define the sets $\overline{C}_\beta$ given by

$$\overline{C}_\beta = \left[\widehat{\Gamma}^{\varepsilon,ad,\mathfrak{a}}_{t,n,\ell,\beta}, \widehat{\Gamma}^{\varepsilon,ad,\mathfrak{a}}_{t,n,u,\beta}\right],$$

with

$$\widehat{\Gamma}^{\varepsilon,ad,\mathfrak{a}}_{t,n,\ell,\beta} = \widehat{\Gamma}^{\varepsilon,ad}_{t,n} - \left(\frac{z_\beta}{\sqrt{2\log(m_n)}} + d^{\alpha,K}_{m_n}\right)\frac{\sqrt{8\hat{\eta}}\left|\hat{\Gamma}^{\varepsilon,ad}_{t,n}\right|^{3/4}}{h_n^{\frac{-\mathfrak{a}}{2\mathfrak{a}+1}}}$$

$$\widehat{\Gamma}^{\varepsilon,ad,\mathfrak{a}}_{t,n,u,\beta} = \widehat{\Gamma}^{\varepsilon,ad}_{t,n} + \left(\frac{z_\beta}{\sqrt{2\log(m_n)}} + d^{\alpha,K}_{m_n}\right)\frac{\sqrt{8\hat{\eta}}\left|\hat{\Gamma}^{\varepsilon,ad}_{t,n}\right|^{3/4}}{h_n^{\frac{-\mathfrak{a}}{2\mathfrak{a}+1}}}.$$

Corollary 5.4 (Confidence band for spot volatility)**.** *The set* $\overline{C}_\beta$ *is a* $(1-\beta)$ *simultaneous confidence band for the unknown spot volatility process* $(\Gamma_t)_{t\in[0,1]}$.

Proof of Corollary 5.4.
This is an immediate consequence of Theorem 5.3. □

We can proceed with the general Itô semimartingale case with non-trivial jump measure μ. Therefore we define a jump and noise robust version $\hat{\Gamma}^{\varepsilon,\tau,ad}_{t,n}$ of $\hat{\Gamma}^{\varepsilon,ad}_{t,n}$ given by

$$\widehat{\Gamma}^{\varepsilon,\tau,ad}_{t,n} = \frac{h_n}{b_n}\sum_{k=1}^{h_n^{-1}} K_{b_n}(t-kh_n)\,\zeta^{ad}_k(Y)\mathbb{1}_{\left\{h_n|\zeta^{ad}_k(Y)|\le u_n\right\}}, \tag{5.2}$$

with $u_n \propto h_n^\tau$ and $\tau \in (0,1)$.
As in the previous chapter the additional jump components will lead to further error terms. We need further conditions on $(b_n)_{n\in\mathbb{N}}$ being necessary to ensure that those terms are also negligible. Note that we introduced the index r in Assumption (3.2).

Assumption 5.5 (The bandwidth b_n II)**.** *For some* $\varpi > 0$ *the sequence* $(b_n)_{n\in\mathbb{N}}$ *is assumed to fulfill*

(1) $\log^{1/2}(n) b_n^{1/2} h_n^{-1/2} h_n^{1/3+\varpi} = \mathcal{O}(1)$,

(2) $\log^{1/2}(n) h_n^{-1/2} b_n^{1/2} h_n^{\tau(1-r/2)} = \mathcal{O}(1)$ *and*

(3) $\log^{1/2}(n) b_n^{1/2} h_n^{\tau+1/2} h_n^{-2r(2/3+\varpi)} = \mathcal{O}(1)$

as $n \longrightarrow \infty$.

In fact, considering the details of the proof, we will observe that there are more error terms. Nevertheless, the three conditions above are sufficient as they imply a relation between ϖ, τ, r and b_n ensuring the additional mixed error terms to be negligible. We set

$$\overline{M}_{n,\tau}(t) = \frac{\left|\hat{\Gamma}^{\varepsilon,\tau,ad}_{t,n} - \Gamma_t\right|}{\sqrt{8\hat{\eta}}\left|\hat{\Gamma}^{\varepsilon,\tau,ad}_{t,n}\right|^{3/4}}$$

and

$$\overline{M}_{n,\tau} = \sup_{t\in[0,1]} \overline{M}_{n,\tau}(t)\,.$$

Theorem 5.6 (Uniform confidence: The general case)**.** *Under the Assumptions 3.2, 3.4, 4.5, 4.8, 5.1,5.5, with the notation introduced in Theorem 4.12, it holds that for all x*

$$\mathbb{P}\left[\sqrt{2\log(m_n)}\left(\frac{\sqrt{b_n/h_n}}{\sqrt{\lambda_K}}\overline{M}_{n,\tau} - d^{\alpha,K}_{m_n}\right) \le x\right] \longrightarrow \exp(-2\exp(-x))\,.$$

Remark 5.7. Based on Theorem 5.6 one can construct uniform confidence bands in the general model analogous to Corollary 5.4.

5.2. Proofs

In this section we will give the proofs of the main Theorems 5.3 and 5.6. As in the previous chapter we will use an universal constant C, which may change from line to line. Furthermore, we will use the notation C_p to indicate that the constant depends on an external parameter p. The constant will never depend on n.

Proof of Theorem 5.3

The proof of Theorem 5.3 is quite lengthy and will be split into several parts. As in the proof of Theorem 4.12 the main challenge is to show that replacing the true price process increments $\Delta_i^n X$ by properly rescaled Brownian increments $\Delta_i^n W$ does not affect the limit. The terms, which have to be considered, are more involved compared to Theorem 4.12, since the existence of noise imply more error terms, which have to be controlled. The final step of the proof is to conclude the extreme value limit via Theorem 2.27. Before starting with the proof we need some minor assumptions and further notation. First of all, we assume $b_n/h_n, b_n^{-1} \in \mathbb{N}$, which ensures the existence of some $k \in \mathbb{N}$, such that $kh_n = b_n$. Due to localization we can assume that all coefficient processes in X are in fact globally bounded. Finally, we introduce some further notation. Due to notational brevity we introduce the sequence $\overline{\nu}_n$ given by

$$\overline{\nu}_n = \frac{\sqrt{h_n\log(n)}}{\sqrt{b_n}}\,,$$

and define indices $\bar{t}_n^{i,\ell}$ given by

$$\bar{t}_n^{i,\ell} = ib_n + \ell h_n,$$

and fix the repeatedly used abbreviation $\overline{K}_{t,n}^{i,\ell}$ defined by

$$\overline{K}_{t,n}^{i,\ell} = K_{b_n}(t - \bar{t}_n^{i,\ell}),$$

i.e. $\overline{K}_{t,n}^{0,\ell} = K_{b_n}(t - \ell h_n)$.
We also set

$$\Phi_{ij\ell}(t) = \Phi_j(t - \bar{t}_n^{i,\ell-1})$$

and

$$\varphi_{ij\ell}(t) = \varphi_j(t - \bar{t}_n^{i,\ell-1})\,.$$

Finally, we use the notation $S_{ij\ell}(L)$ for any stochastic process L for the associated spectral statistics. We will start proving an oracle version of Theorem 5.3 replacing the adaptive statistic $\widehat{\Gamma}_{t,n}^{\varepsilon,ad}$ by $\widehat{\Gamma}_{t,n}^{\varepsilon}$. Finally we will show that this replacement does not affect the limit behaviour.
The first approximation outlined above will be pursued via $\overline{\Gamma}_{t,n}^{\varepsilon}$ given by

$$\overline{\Gamma}_{t,n}^{\varepsilon} = \frac{h_n}{b_n} \sum_{i=0}^{b_n^{-1}-1} \sum_{\ell=1}^{b_n/h_n} \overline{K}_{t,n}^{i,\ell} \sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{ij\ell}((\sigma_{ib_n} S_{ij\ell}(W) + S_{ij\ell}(\varepsilon))^2 - \mu_{ij\ell})\,,$$

with weights $w_{ij\ell}$ given by

$$w_{ij\ell} = \frac{\left(\sigma^2_{\bar{t}_n^{i,\ell-1}} + \frac{\eta^2}{n}\left[\varphi_{ij\ell}, \varphi_{ij\ell}\right]_n\right)^{-2}}{\sum_{j=1}^{\lfloor nh_n \rfloor -1}\left(\sigma^2_{\bar{t}_n^{i,\ell-1}} + \frac{\eta^2}{n}\left[\varphi_{ij\ell}, \varphi_{ij\ell}\right]_n\right)^{-2}}\,.$$

Proposition 5.8. *Under the assumptions of Theorem 5.3 it holds that*

$$\sqrt{b_n/h_n \log(n)}\left|\left|\frac{\widehat{\Gamma}_{t,n}^{\varepsilon} - \Gamma_t}{\left|\widehat{\Gamma}_{t,n}^{\varepsilon}\right|^{3/4}}\right| - \left|\frac{\overline{\Gamma}_{t,n}^{\varepsilon} - \Gamma_t}{\left|\overline{\Gamma}_{t,n}^{\varepsilon}\right|^{3/4}}\right|\right| \xrightarrow{\mathbb{P}} 0\,.$$

Proof of Proposition 5.8.
The following decomposition holds

$$\begin{aligned}
&\frac{\widehat{\Gamma}_{t,n}^{\varepsilon} - \Gamma_t}{\left|\widehat{\Gamma}_{t,n}^{\varepsilon}\right|^{3/4}} - \frac{\overline{\Gamma}_{t,n}^{\varepsilon} - \Gamma_t}{\left|\overline{\Gamma}_{t,n}^{\varepsilon}\right|^{3/4}} \\
&= \widehat{\Gamma}_{t,n}^{\varepsilon}\left(\frac{1}{\left|\widehat{\Gamma}_{t,n}^{\varepsilon}\right|^{3/4}} - \frac{1}{\left|\overline{\Gamma}_{t,n}^{\varepsilon}\right|^{3/4}}\right) + \frac{\widehat{\Gamma}_{t,n}^{\varepsilon} - \overline{\Gamma}_{t,n}^{\varepsilon}}{\overline{\Gamma}_{t,n}^{\varepsilon}} + \Gamma_t\left(\frac{1}{\left|\overline{\Gamma}_{t,n}^{\varepsilon}\right|^{3/4}} - \frac{1}{\left|\widehat{\Gamma}_{t,n}^{\varepsilon}\right|^{3/4}}\right)\,.
\end{aligned}$$

This yields the following decomposition via the reverse triangle inequality and elementary properties of the $\sup_{t\in[0,1]}$.

$$\begin{aligned}&\sup_{t\in[0,1]} \widehat{\Gamma}^{\varepsilon}_{t,n}\left(\frac{1}{\left|\widehat{\Gamma}^{\varepsilon}_{t,n}\right|^{3/4}}-\frac{1}{\left|\overline{\Gamma}^{\varepsilon}_{t,n}\right|^{3/4}}\right)+\sup_{t\in[0,1]}\frac{\widehat{\Gamma}^{\varepsilon}_{t,n}-\overline{\Gamma}^{\varepsilon}_{t,n}}{\overline{\Gamma}^{\varepsilon}_{t,n}}\\ &\qquad+\sup_{t\in[0,1]}\Gamma_t\left(\frac{1}{\left|\overline{\Gamma}^{\varepsilon}_{t,n}\right|^{3/4}}-\frac{1}{\left|\widehat{\Gamma}^{\varepsilon}_{t,n}\right|^{3/4}}\right)\\ &=(\mathbf{I})+(\mathbf{II})+(\mathbf{III})\,.\end{aligned}$$

We start with term $(\mathbf{II})$. It holds that for every $\delta>0$

$$\begin{aligned}&\mathbb{P}\left[\sup_{t\in[0,1]}\sqrt{b_n/h_n\log(n)}\frac{\left|\widehat{\Gamma}^{\varepsilon}_{t,n}-\overline{\Gamma}^{\varepsilon}_{t,n}\right|}{\left|\overline{\Gamma}^{\varepsilon}_{t,n}\right|^{3/4}}>\delta\right]\\ &=\mathbb{P}\left[\sup_{t\in[0,1]}\sqrt{b_n/h_n\log(n)}\frac{\left|\widehat{\Gamma}^{\varepsilon}_{t,n}-\overline{\Gamma}^{\varepsilon}_{t,n}\right|}{\left|\overline{\Gamma}^{\varepsilon}_{t,n}\right|^{3/4}}>\delta,\ \inf_{t\in[0,1]}\overline{\Gamma}^{\varepsilon}_{t,n}\geq K^{-}-\kappa^{-}\right]\\ &+\mathbb{P}\left[\sup_{t\in[0,1]}\sqrt{b_n/h_n\log(n)}\frac{\left|\widehat{\Gamma}^{\varepsilon}_{t,n}-\overline{\Gamma}^{\varepsilon}_{t,n}\right|}{\left|\overline{\Gamma}^{\varepsilon}_{t,n}\right|^{3/4}}>\delta,\ \inf_{t\in[0,1]}\overline{\Gamma}^{\varepsilon}_{t,n}< K^{-}-\kappa^{-}\right]\\ &\leq\mathbb{P}\left[\sup_{t\in[0,1]}\sqrt{b_n/h_n\log(n)}\left|\widehat{\Gamma}^{\varepsilon}_{t,n}-\overline{\Gamma}^{\varepsilon}_{t,n}\right|>\delta(K^{-}-\kappa^{-})^{3/4}\right]+\mathbb{P}\left[\inf_{t\in[0,1]}\overline{\Gamma}^{\varepsilon}_{t,n}<K^{-}-\kappa^{-}\right]\\ &=:A_n+B_n\,.\end{aligned}$$

We start with A_n. Note the following decomposition of $\widehat{\Gamma}^{\varepsilon}_{t,n}$, given by

$$\widehat{\Gamma}^{\varepsilon}_{t,n}=\frac{h_n}{b_n}\sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{b_n/h_n}\overline{K}^{i,\ell}_{t,n}\sum_{j=1}^{\lfloor nh_n\rfloor-1}w_{ij\ell}(S^2_{ij\ell}(X)-\mu_{ij\ell})\,.$$

Thus, we get the decomposition

$$\begin{aligned}\widehat{\Gamma}^{\varepsilon}_{t,n}-\overline{\Gamma}^{\varepsilon}_{t,n}&=\frac{h_n}{b_n}\sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{b_n/h_n}\overline{K}^{i,\ell}_{t,n}\sum_{j=1}^{\lfloor nh_n\rfloor-1}w_{ij\ell}(S^2_{ij\ell}(X)-\sigma^2_{ib_n}S^2_{ij\ell}(W))\\ &=\frac{2h_n}{b_n}\sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{b_n/h_n}\overline{K}^{i,\ell}_{t,n}\sum_{j=1}^{\lfloor nh_n\rfloor-1}w_{ij\ell}S_{ij\ell}\left(\varepsilon\right)\left(S_{ij\ell}\left(X\right)-\sigma_{ib_n}S_{ij\ell}\left(W\right)\right).\end{aligned}$$

This yields the decomposition

$$A_n\leq A_n^1+A_n^2\,,$$

where with a generic constant $c > 0$, A_n^1 is given by

$$\mathbb{P}\left[\sup_{t\in[0,1]} \overline{\nu}_n \left|\sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{b_n/h_n} \overline{K}_{t,n}^{i,\ell} \sum_{j=1}^{\lfloor nh_n\rfloor -1} w_{ij\ell}(S_{ij\ell}^2(X) - \sigma_{ib_n}^2 S_{ij\ell}^2(W))\right| > c\right]$$

and A_n^2 is given by

$$\mathbb{P}\left[\sup_{t\in[0,1]} \overline{\nu}_n \left|\sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{b_n/h_n} \overline{K}_{t,n}^{i,\ell} \sum_{j=1}^{\lfloor nh_n\rfloor -1} w_{ij\ell} S_{ij\ell}\left(\varepsilon\right)\left(S_{ij\ell}\left(X\right) - \sigma_{ib_n} S_{ij\ell}\left(W\right)\right)\right| > c\right] .$$

Starting with A_n^1, we use the notation $\overline{t}_{n_-}^{i,\ell} = \overline{t}_n^{i,\ell} - n^{-1}$ and define step functions $\xi_{ij\ell}^{(n)}$ given by

$$\xi_{ij\ell}^{(n)}(t) = \sum_{\nu=1}^{n} \Phi_{ij\ell}\left(\frac{\nu}{n}\right) \mathbb{1}_{\left(\frac{\nu-1}{n},\frac{\nu}{n}\right]}(t) ,$$

which yields

$$S_{ij\ell}(L) = \int_{\overline{t}_{n_-}^{i,\ell-1}}^{\overline{t}_n^{i,\ell}} \xi_{ij\ell}^{(n)}(s)\, dL_s$$

for any semimartingale $(L_t)_{t\in[0,1]}$. The Itô process structure provides the following decomposition

$$\int_{\overline{t}_{n_-}^{i,\ell-1}}^{\overline{t}_n^{i,\ell}} \xi_{ij\ell}^{(n)}(s)\, dX_s = \int_{\overline{t}_{n_-}^{i,\ell}}^{\overline{t}_n^{i,\ell}} \xi_{ij\ell}^{(n)}(s) a_s\, ds + \int_{\overline{t}_{n_-}^{i,\ell}}^{\overline{t}_n^{i,\ell}} \xi_{ij\ell}^{(n)}(s)\sigma_s\, dW_s .$$

Using Itô's formula we get the following decompositions:

$$\begin{aligned} S_{ij\ell}^2(X) &= 2\int_{\overline{t}_{n_-}^{i,\ell-1}}^{\overline{t}_n^{i,\ell}} (\widetilde{X}_\tau - \widetilde{X}_{\overline{t}_{n_-}^{i,\ell-1}})\xi_{ij\ell}^{(n)}(\tau)\sigma_\tau\, dW_\tau \\ &\quad + 2\int_{\overline{t}_{n_-}^{i,\ell-1}}^{\overline{t}_n^{i,\ell}} (\widetilde{X}_\tau - \widetilde{X}_{\overline{t}_{n_-}^{i,\ell-1}})\xi_{ij\ell}^{(n)}(\tau) a_\tau\, d\tau \\ &\quad + \int_{\overline{t}_{n_-}^{i,\ell-1}}^{\overline{t}_n^{i,\ell}} \left(\xi_{ij\ell}^{(n)}(\tau)\right)^2 \sigma_\tau^2\, d\tau \end{aligned}$$

and

$$S_{ij\ell}^2(W) = 2\int_{\overline{t}_{n_-}^{i,\ell-1}}^{\overline{t}_n^{i,\ell}} \widetilde{W}_\tau \xi_{ij\ell}^{(n)}(\tau)\, dW_\tau + \int_{\overline{t}_{n_-}^{i,\ell-1}}^{\overline{t}_n^{i,\ell}} \left(\xi_{ij\ell}^{(n)}(\tau)\right)^2 d\tau ,$$

with

$$\begin{aligned}\widetilde{X}_t &= X_0 + \int_0^t \xi_{ij\ell}^{(n)}(s) a_s \, ds + \int_0^t \xi_{ij\ell}^{(n)}(s)\sigma_s \, dW_s \,, \\ \widetilde{W}_t &= \int_0^t \xi_{ij\ell}^{(n)}(s)\, dW_s \,.\end{aligned}$$

Using the representations above we can pursue a further decomposition

$$A_n^1 \leq A_n^{1,1} + A_n^{1,2} + A_n^{1,3} \,,$$

where $A_n^{1,1}$ is given by

$$\mathbb{P}\Big[\sup_{t\in[0,1]} \overline{\nu}_n \Big| \sum_{i=0}^{b_n^{-1}-1} \sum_{\ell=1}^{b_n/h_n} \overline{K}_{t,n}^{i,\ell} \sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{ij\ell} \int_{\overline{t}_{n_-}^{i,\ell-1}}^{\overline{t}_n^{i,\ell}} (\widetilde{X}_\tau - \widetilde{X}_{\overline{t}_{n_-}^{i,\ell-1}}) \xi_{ij\ell}^{(n)}(\tau) a_\tau \, d\tau \Big| > c\Big] \,,$$

$A_n^{1,2}$ is given by

$$\mathbb{P}\Big[\sup_{t\in[0,1]} \overline{\nu}_n \Big| \sum_{i=0}^{b_n^{-1}-1} \sum_{\ell=1}^{b_n/h_n} \overline{K}_{t,n}^{i,\ell} \sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{ij\ell} \int_{\overline{t}_{n_-}^{i,\ell-1}}^{\overline{t}_n^{i,\ell}} \big(\xi_{ij\ell}^{(n)}(\tau)\big)^2 (\sigma_\tau^2 - \sigma_{ib_n}^2) \, d\tau \Big| > c\Big]$$

and $A_n^{1,3}$ is given by

$$\begin{aligned}\mathbb{P}\Big[\sup_{t\in[0,1]} \overline{\nu}_n \Big| \sum_{i=0}^{b_n^{-1}-1} \sum_{\ell=1}^{b_n/h_n} \overline{K}_{t,n}^{i,\ell} \sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{ij\ell} \int_{\overline{t}_{n_-}^{i,\ell-1}}^{\overline{t}_n^{i,\ell}} \xi_{ij\ell}^{(n)}(\tau) \\ \times \Big((\widetilde{X}_\tau - \widetilde{X}_{\overline{t}_{n_-}^{i,\ell-1}})\sigma_\tau - \sigma_{ib_n}^2 \widetilde{W}_\tau \Big) \, dW_\tau \Big| > c\Big] \,.\end{aligned}$$

Starting with $A_n^{1,1}$ we need a further decomposition. Therefore, we set $\overline{t}_d = dh_n^2$ and $\overline{v}_n = h_n^{-2}$ for $d = 1, \ldots, \overline{v}_n$. Note that $h_n^{-1} \in \mathbb{N}$, s.t. $\overline{v}_n$ is well defined. We have a further decomposition

$$A_n^{1,1} \leq A_n^{1,1,1} + A_n^{1,1,2} \,,$$

with $A_n^{1,1,1}$ given by

$$\begin{aligned}\mathbb{P}\Big[\max_{1\leq d\leq \bar{v}_n} \sup_{t\in[\overline{t}_{d-1},\overline{t}_d]} \overline{\nu}_n \Big| \sum_{i=0}^{b_n^{-1}-1} \sum_{\ell=1}^{b_n/h_n} (\overline{K}_{t,n}^{i,\ell} - \overline{K}_{\overline{t}_d,n}^{i,\ell}) \sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{ij\ell} \\ \times \int_{\overline{t}_{n_-}^{i,\ell-1}}^{\overline{t}_n^{i,\ell}} (\widetilde{X}_\tau - \widetilde{X}_{\overline{t}_{n_-}^{i,\ell-1}}) \xi_{ij\ell}^{(n)}(\tau) a_\tau \, d\tau \Big| > c\Big]\end{aligned}$$

and $A_n^{1,1,2}$ given by

$$\mathbb{P}\left[\max_{1\le d\le\tilde{v}_n}\overline{\nu}_n\left|\sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{b_n/h_n}\overline{K}^{i,\ell}_{\overline{t}_d,n}\sum_{j=1}^{\lfloor nh_n\rfloor-1}w_{ij\ell}\int_{\overline{t}_{n_-}^{i,\ell-1}}^{\overline{t}_n^{i,\ell}}(\widetilde{X}_\tau-\widetilde{X}_{\overline{t}_{n_-}^{i,\ell-1}})\xi_{ij\ell}^{(n)}(\tau)a_\tau\,d\tau\right|>c\right].$$

Starting with $A_n^{1,1,1}$ we have to exploit the regularity properties of the kernel K. Therefore, note

$$\left|\overline{K}^{i,\ell}_{t,n}-\overline{K}^{i,\ell}_{\overline{t}_d,n}\right|\le\sup\left\{\left|K(z)-K(z')\right|:z,z'\in\left[\frac{t-\overline{t}_n^{i,\ell}}{b_n},\frac{\overline{t}_d-\overline{t}_n^{i,\ell}}{b_n}\right]\right\}$$

and it holds that

$$\begin{aligned}\sum_{\ell=1}^{b_n/h_n}(\overline{K}^{i,\ell}_{t,n}-\overline{K}^{i,\ell}_{\overline{t}_d,n})&=\int_0^{b_n/h_n}(\overline{K}^{i,\lfloor u\rfloor}_{t,n}-\overline{K}^{i,\lfloor u\rfloor}_{\overline{t}_d,n})\,du\\&=\mathcal{O}\left(\int_{\mathbb{R}}\Psi_{K,\varrho}(u)\,du\right)=\mathcal{O}\left(\frac{\left|t-\overline{t}_d\right|}{b_n}\right),\end{aligned}$$

i.e.

$$\varrho=\frac{\left|t-\overline{t}_d\right|}{b_n}.$$

Furthermore, Jensen's inequality for convex linear combinations, the triangle inequality for integrals, Burkholder's inequality and the boundedness of the coefficient processes imply

$$\begin{aligned}&\left|\sum_{j=1}^{\lfloor nh_n\rfloor-1}w_{ij\ell}\int_{\overline{t}_{n_-}^{i,\ell-1}}^{\overline{t}_n^{i,\ell}}(\widetilde{X}_\tau-\widetilde{X}_{\overline{t}_{n_-}^{i,\ell-1}})\xi_{ij\ell}^{(n)}(\tau)a_\tau\,d\tau\right|\\&=\mathcal{O}_{\mathbb{P}}\left(\sum_{j=1}^{\lfloor nh_n\rfloor-1}w_{ij\ell}\int_{\overline{t}_{n_-}^{i,\ell-1}}^{\overline{t}_n^{i,\ell}}\left(\int_{\overline{t}_{n_-}^{i,\ell-1}}^{\tau}(\xi_{ij\ell}^{(n)}(s))^2\sigma_s^2\,ds\right)^{1/2}|\xi_{ij\ell}^{(n)}(\tau)|d\tau\right)\\&=\mathcal{O}_{\mathbb{P}}(h_n^{1/2}).\end{aligned}$$

Overall using $t\in[\overline{t}_{d-1},\overline{t}_d]$, the bounds calculated above and the Cauchy Schwarz inequality, we get the bound $\mathcal{O}(h_n^{5/2}b_n)=o_{\mathbb{P}}(1)$ concerning $A_n^{1,1,2}$ since

$$\#\left\{(i,\ell)\in\mathbb{N}^2:\frac{1}{b_n}(t-(ib_n+\ell h_n))\in\mathsf{supp}\ K\right\}=\mathcal{O}(b_n/h_n)\tag{5.3}$$

and

$$1\le\ell\le b_n/h_n,$$

such that the $i-$ sum can be simply bounded via triangle inequality in a way that the term still remains negligible as $n \longrightarrow \infty$. We can proceed with $A_n^{1,1,2}$. It holds that

$$\mathbb{P}\left[\max_{1\le d\le \tilde{v}_n} \overline{\nu}_n \left|\sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{b_n/h_n} \overline{K}_{\bar{t}_d,n}^{i,\ell} \sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\int_{\bar{t}_{n_-}^{i,\ell-1}}^{\bar{t}_n^{i,\ell}} (\widetilde{X}_\tau - \widetilde{X}_{\bar{t}_{n_-}^{i,\ell-1}})\xi_{ij\ell}^{(n)}(\tau)a_\tau\,d\tau\right| > c\right]$$
$$\le \sum_{d=1}^{\tilde{v}_n}\mathbb{P}\left[\overline{\nu}_n \left|\sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{b_n/h_n} \overline{K}_{\bar{t}_d,n}^{i,\ell} \sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\int_{\bar{t}_{n_-}^{i,\ell-1}}^{\bar{t}_n^{i,\ell}} (\widetilde{X}_\tau - \widetilde{X}_{\bar{t}_{n_-}^{i,\ell-1}})\xi_{ij\ell}^{(n)}(\tau)a_\tau\,d\tau\right| > c\right].$$

With Markov's inequality applied to $z \mapsto |z|^r$ with $r > 0$ we get

$$\mathbb{P}\left[\overline{\nu}_n \left|\sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{b_n/h_n} \overline{K}_{\bar{t}_d,n}^{i,\ell} \sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\int_{\bar{t}_{n_-}^{i,\ell-1}}^{\bar{t}_n^{i,\ell}} (\widetilde{X}_\tau - \widetilde{X}_{\bar{t}_{n_-}^{i,\ell-1}})\xi_{ij\ell}^{(n)}(\tau)a_\tau\,d\tau\right| > c\right]$$
$$\le C_r\mathbb{E}\left[\left|\overline{\nu}_n \sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{b_n/h_n} \overline{K}_{\bar{t}_d,n}^{i,\ell} \sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\int_{\bar{t}_{n_-}^{i,\ell-1}}^{\bar{t}_n^{i,\ell}} (\widetilde{X}_\tau - \widetilde{X}_{\bar{t}_{n_-}^{i,\ell-1}})\xi_{ij\ell}^{(n)}(\tau)a_\tau\,d\tau\right|^r\right].$$

Using (5.3) and the triangle inequality concerning the $i-$ sum, Jensen's inequality concerning the $\ell-$ sum, the boundedness of the coefficient processes, the boundedness of K and Jensen's inequality applied to the $j-$ sum we can proceed as follows:

$$\mathbb{E}\left[\left|\overline{\nu}_n \sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{b_n/h_n} \overline{K}_{\bar{t}_d,n}^{i,\ell} \sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\int_{\bar{t}_{n_-}^{i,\ell-1}}^{\bar{t}_n^{i,\ell}} (\widetilde{X}_\tau - \widetilde{X}_{\bar{t}_{n_-}^{i,\ell-1}})\xi_{ij\ell}^{(n)}(\tau)a_\tau\,d\tau\right|^r\right]$$
$$\le C_r\overline{\nu}_n^r\mathbb{E}\left[\left|\sum_{\ell=1}^{b_n/h_n} \overline{K}_{\bar{t}_d,n}^{i,\ell} \sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\int_{\bar{t}_{n_-}^{i,\ell-1}}^{\bar{t}_n^{i,\ell}} (\widetilde{X}_\tau - \widetilde{X}_{\bar{t}_{n_-}^{i,\ell-1}})\xi_{ij\ell}^{(n)}(\tau)a_\tau\,d\tau\right|^r\right]$$
$$\le C_r\overline{\nu}_n^r(b_n/h_n)^r\mathbb{E}\left[\left|\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\int_{\bar{t}_{n_-}^{i,\ell-1}}^{\bar{t}_n^{i,\ell}} (\widetilde{X}_\tau - \widetilde{X}_{\bar{t}_{n_-}^{i,\ell-1}})\xi_{ij\ell}^{(n)}(\tau)a_\tau\,d\tau\right|^r\right]$$
$$\le C_r\overline{\nu}_n^r(b_n/h_n)^r\mathbb{E}\left[\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\left|\int_{\bar{t}_{n_-}^{i,\ell-1}}^{\bar{t}_n^{i,\ell}} (\widetilde{X}_\tau - \widetilde{X}_{\bar{t}_{n_-}^{i,\ell-1}})\xi_{ij\ell}^{(n)}(\tau)a_\tau\,d\tau\right|^r\right].$$

Proceeding with the generalized Minkowski inequality and Burkholder inequality we calculate the bound

$$C_r\overline{\nu}_n^r(b_n/h_n)^r \sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\mathbb{E}\left[\left|\int_{\bar{t}_{n_-}^{i,\ell-1}}^{\bar{t}_n^{i,\ell}} (\widetilde{X}_\tau - \widetilde{X}_{\bar{t}_{n_-}^{i,\ell-1}})\xi_{ij\ell}^{(n)}(\tau)a_\tau\,d\tau\right|^r\right]$$
$$\le C_r\overline{\nu}_n^r(b_n/h_n)^r \sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\left(\int_{\bar{t}_{n_-}^{i,\ell-1}}^{\bar{t}_n^{i,\ell}} \mathbb{E}\left[\left|(\widetilde{X}_\tau - \widetilde{X}_{\bar{t}_{n_-}^{i,\ell-1}})\xi_{ij\ell}^{(n)}(\tau)a_\tau\right|^r\right]^{1/r} d\tau\right)^r$$

$$\leq C_r \overline{\nu}_n^r (b_n/h_n)^r \sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{ij\ell} \left(\int_{\overline{t}_{n-}^{i,\ell-1}}^{\overline{t}_n^{i,\ell}} |\xi_{ij\ell}^{(n)}(\tau)| \mathbb{E} \left[\left| \int_{\overline{t}_{n-}^{i,\ell-1}}^{\tau} (\xi_{ij\ell}^{(n)}(s))^2 \sigma_s^2 \, ds \right|^{r/2} \right]^{1/r} d\tau \right)^r$$
$$= \mathcal{O}(\overline{\nu}_n^r (b_n/h_n)^r h_n^{-r/2} h_n^r) = \mathcal{o}(\overline{\nu}_n^{-1}) \,,$$

with r being sufficiently large. This completes the term $A_n^{1,1,2}$ and so $A_n^{1,1}$. We can proceed with $A_n^{1,2}$. Therefore, we define functions $\overline{k}_{n,t}^i$ given by

$$\overline{k}_{n,t}^i(\tau) = \sum_{\ell=1}^{b_n/h_n} \overline{K}_{t,n}^{i,\ell} \sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{ij\ell} \mathbb{1}_{[\overline{t}_{n-}^{i,\ell-1}, \overline{t}_n^{i,\ell})}(\tau) \,.$$

Then the identity

$$\mathbb{P}\left[\sup_{t\in[0,1]} \overline{\nu}_n \left| \sum_{i=0}^{b_n^{-1}-1} \sum_{\ell=1}^{b_n/h_n} \overline{K}_{t,n}^{i,\ell} \sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{ij\ell} \int_{\overline{t}_{n-}^{i,\ell-1}}^{\overline{t}_n^{i,\ell}} \big(\xi_{ij\ell}^{(n)}(\tau)\big)^2 (\sigma_\tau^2 - \sigma_{ib_n}^2) \, d\tau \right| > c \right]$$
$$= \mathbb{P}\left[\sup_{t\in[0,1]} \overline{\nu}_n \left| \sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} \overline{k}_{n,t}^i(\tau) (\xi_{ij\ell}^{(n)}(\tau))^2 (\sigma_\tau^2 - \sigma_{ib_n}^2) \, d\tau \right| > c \right]$$

holds. Proceeding with the triangle inequality, Minkowski inequality and Assumption 4.5 it holds that

$$\sup_{t\in[0,1]} \overline{\nu}_n \left| \sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} \overline{k}_{n,t}^i(\tau) (\xi_{ij\ell}^{(n)}(\tau))^2 (\sigma_\tau^2 - \sigma_{ib_n}^2) \, d\tau \right|$$
$$\leq \sup_{t\in[0,1]} \overline{\nu}_n \sum_{i=0}^{b_n^{-1}-1} \left| \int_{ib_n}^{(i+1)b_n} \overline{k}_{n,t}^i(\tau) (\xi_{ij\ell}^{(n)}(\tau))^2 (\sigma_\tau^2 - \sigma_{ib_n}^2) \, d\tau \right|$$
$$\leq \sup_{t\in[0,1]} \overline{\nu}_n \sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} \left| \overline{k}_{n,t}^i(\tau) (\xi_{ij\ell}^{(n)}(\tau))^2 (\sigma_\tau^2 - \sigma_{ib_n}^2) \right| \, d\tau$$
$$= \mathcal{O}_{\mathbb{P}}(\overline{\nu}_n b_n b_n^{\mathfrak{a}}) = \mathcal{o}_{\mathbb{P}}(1) \,,$$

due to (5.3), the triangle inequality, the boundedness of K, Jensen's inequality,(3.31) and Assumption 5.1. We have completed $A_n^{1,2}$ and proceed with $A_n^{1,3}$. We need further decompositions.

$$(\widetilde{X}_\tau - \widetilde{X}_{\overline{t}_{n-}^{i,\ell-1}}) \sigma_\tau - \sigma_{ib_n}^2 \widetilde{W}_\tau$$
$$= \sigma_\tau \int_{\overline{t}_{n-}^{i,\ell-1}}^{\tau} \xi_{ij\ell}^{(n)}(s) a_s \, ds$$
$$+ \sigma_\tau \int_{\overline{t}_{n-}^{i,\ell-1}}^{\tau} \xi_{ij\ell}^{(n)}(s) \sigma_s \, dW_s - \sigma_{ib_n} \int_{\overline{t}_{n-}^{i,\ell-1}}^{\tau} \xi_{ij\ell}^{(n)}(s) \sigma_s \, dW_s$$

$$
\begin{aligned}
&+\sigma_{ib_n}\int_{\bar{t}_{n_-}^{i,\ell-1}}^{\tau}\xi_{ij\ell}^{(n)}(s)\sigma_s\,dW_s-\sigma_{ib_n}\int_{\bar{t}_{n_-}^{i,\ell-1}}^{\tau}\xi_{ij\ell}^{(n)}(s)\sigma_{ib_n}\,dW_s\\
&=\sigma_\tau\int_{\bar{t}_{n_-}^{i,\ell-1}}^{\tau}\xi_{ij\ell}^{(n)}(s)a_s\,ds+(\sigma_\tau-\sigma_{ib_n})\int_{\bar{t}_{n_-}^{i,\ell-1}}^{\tau}\xi_{ij\ell}^{(n)}(s)\sigma_s\,dW_s\\
&+\sigma_{ib_n}\int_{\bar{t}_{n_-}^{i,\ell-1}}^{\tau}\xi_{ij\ell}^{(n)}(s)(\sigma_s-\sigma_{ib_n})\,dW_s\,.
\end{aligned}
$$

That yields a further decomposition

$$
A_n^{1,3}\le A_n^{1,3,1}+A_n^{1,3,2}+A_n^{1,3,3}\,,
$$

with $A_n^{1,3,1}$ given by

$$
\mathbb{P}\left[\sup_{t\in[0,1]}\overline{\nu}_n\left|\sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{b_n/h_n}\overline{K}_{t,n}^{i,\ell}\sum_{j=1}^{\lfloor nh_n\rfloor-1}w_{ij\ell}\int_{\bar{t}_{n_-}^{i,\ell-1}}^{\bar{t}_n^{i,\ell}}\xi_{ij\ell}^{(n)}(\tau)\sigma_\tau\int_{\bar{t}_{n_-}^{i,\ell-1}}^{\tau}\xi_{ij\ell}^{(n)}(s)a_s\,ds\,dW_\tau\right|>c\right]\,,
$$

$A_n^{1,3,2}$ given by

$$
\begin{aligned}
\mathbb{P}\Bigg[\sup_{t\in[0,1]}\overline{\nu}_n\Bigg|&\sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{b_n/h_n}\overline{K}_{t,n}^{i,\ell}\sum_{j=1}^{\lfloor nh_n\rfloor-1}w_{ij\ell}\int_{\bar{t}_{n_-}^{i,\ell-1}}^{\bar{t}_n^{i,\ell}}\xi_{ij\ell}^{(n)}(\tau)\\
&\times(\sigma_\tau-\sigma_{ib_n})\int_{\bar{t}_{n_-}^{i,\ell-1}}^{\tau}\xi_{ij\ell}^{(n)}(s)\sigma_s\,dW_s\,dW_\tau\Bigg|>c\Bigg]
\end{aligned}
$$

and $A_n^{1,3,3}$ given by

$$
\begin{aligned}
\mathbb{P}\Bigg[\sup_{t\in[0,1]}\overline{\nu}_n\Bigg|&\sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{b_n/h_n}\overline{K}_{t,n}^{i,\ell}\sum_{j=1}^{\lfloor nh_n\rfloor-1}w_{ij\ell}\int_{\bar{t}_{n_-}^{i,\ell-1}}^{\bar{t}_n^{i,\ell}}\xi_{ij\ell}^{(n)}(\tau)\sigma_{ib_n}\\
&\times\int_{\bar{t}_{n_-}^{i,\ell-1}}^{\tau}\xi_{ij\ell}^{(n)}(s)(\sigma_s-\sigma_{ib_n})\,dW_s\,dW_\tau\Bigg|>c\Bigg]\,.
\end{aligned}
$$

We start with $A_n^{1,3,1}$ and define

$$
\overline{c}_{n,i}^{t,1}(\tau)=\sum_{\ell=1}^{b_n/h_n}\overline{K}_{t,n}^{i,\ell}\sum_{j=1}^{\lfloor nh_n\rfloor-1}w_{ij\ell}\xi_{ij\ell}^{(n)}(\tau)\int_{\bar{t}_{n_-}^{i,\ell-1}}^{\tau}\xi_{ij\ell}^{(n)}(s)a_s\,ds\mathbb{1}_{[\bar{t}_{n_-}^{i,\ell-1},\bar{t}_n^{i,\ell})}(\tau)\,.
$$

With respect to $A_n^{1,3,1}$ this yields the representation

$$
\mathbb{P}\left[\sup_{t\in[0,1]}\overline{\nu}_n\left|\sum_{i=0}^{b_n^{-1}-1}\int_{ib_n}^{(i+1)b_n}\overline{c}_{n,i}^{t,1}(\tau)\,dW_\tau\right|>c\right]\,.
$$

We employ a further decomposition

$$\begin{aligned}
&\mathbb{P}\left[\sup_{t\in[0,1]} \overline{\nu}_n \left|\sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} \overline{c}_{n,i}^{t,1}(\tau)\, dW_\tau\right| > c\right] \\
&\leq \mathbb{P}\left[\max_{1\leq d\leq \tilde{v}_n} \sup_{t\in[\bar{t}_{d-1},\bar{t}_d]} \overline{\nu}_n \left|\sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} (\overline{c}_{n,i}^{\bar{t}_d,1}(\tau) - \overline{c}_{n,i}^{t,1}(\tau))\, dW_\tau\right| > c\right] \\
&\qquad + \mathbb{P}\left[\max_{1\leq d\leq \tilde{v}_n} \overline{\nu}_n \left|\sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} \overline{c}_{n,i}^{\bar{t}_d,1}(\tau)\, dW_\tau\right| > c\right] \\
&= A_n^{1,3,1,1} + A_n^{1,3,1,2}\,.
\end{aligned}$$

Proceeding with $A_n^{1,3,1,1}$ it holds that

$$\begin{aligned}
&\overline{c}_{n,i}^{t,1}(\tau) - \overline{c}_{n,i}^{\bar{t}_d,1}(\tau) \\
&= \sum_{\ell=1}^{b_n/h_n} (\overline{K}_{t,n}^{i,\ell} - \overline{K}_{\bar{t}_d,n}^{i,\ell}) \sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\xi_{ij\ell}^{(n)}(\tau) \int_{\bar{t}_{n_-}^{i,\ell-1}}^{\tau} \xi_{ij\ell}^{(n)}(s) a_s\, ds \mathbb{1}_{[\bar{t}_{n_-}^{i,\ell-1},\bar{t}_n^{i,\ell})}(\tau)\,.
\end{aligned}$$

As we have already argued in $A_n^{1,1,1}$ it holds that

$$\sum_{\ell=1}^{b_n/h_n} (\overline{K}_{t,n}^{i,\ell} - \overline{K}_{\bar{t}_d,n}^{i,\ell}) = \mathcal{O}(\varrho)\,,$$

with

$$\varrho = \frac{|t-\bar{t}_d|}{b_n} \leq \frac{|\bar{t}_{d-1}-\bar{t}_d|}{b_n} = \mathcal{O}(h_n^2 b_n^{-1})\,.$$

Furthermore, due to Jensen's inequality, (3.31), standard Lebesgue upper bounds and the boundedness of the coefficient process, it holds that

$$\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\xi_{ij\ell}^{(n)}(\tau) \int_{\bar{t}_{n_-}^{i,\ell-1}}^{\tau} \xi_{ij\ell}^{(n)}(s) a_s\, ds \mathbb{1}_{(\bar{t}_{n_-}^{i,\ell-1},\bar{t}_n^{i,\ell}]}(\tau) = \mathcal{O}_{\mathbb{P}}(1)\,.$$

Due to (5.3), the triangle inequality, Burkholder's inequality and the bounds calculated above we get

$$\begin{aligned}
&\max_{1\leq d\leq \tilde{v}_n} \sup_{t\in[\bar{t}_{d-1},\bar{t}_d]} \overline{\nu}_n \left|\sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} (\overline{c}_{n,i}^{\bar{t}_d,1}(\tau) - \overline{c}_{n,i}^{t,1}(\tau))\, dW_\tau\right| \\
&\leq \max_{1\leq d\leq \tilde{v}_n} \sup_{t\in[\bar{t}_{d-1},\bar{t}_d]} \overline{\nu}_n \sum_{i=0}^{b_n^{-1}-1} \left|\int_{ib_n}^{(i+1)b_n} (\overline{c}_{n,i}^{\bar{t}_d,1}(\tau) - \overline{c}_{n,i}^{t,1}(\tau))^2\, d\tau\right|^{1/2} \\
&= \mathcal{O}_{\mathbb{P}}(\overline{\nu}_n b_n^{-1/2} h_n^2) = \mathcal{o}_{\mathbb{P}}(1)\,.
\end{aligned}$$

We can proceed with $A_n^{1,3,1,2}$. With Markov's inequality applied to the function $z \mapsto |z|^{2m}$, $m \in \mathbb{N}$ and $m > 0$, it holds that

$$\begin{aligned}
&\mathbb{P}\left[\max_{1\le d\le \tilde{v}_n} \overline{\nu}_n \left| \sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} \overline{c}_{n,i}^{\overline{t}_d,1}(\tau)\, dW_\tau \right| > c\right] \\
&\le \sum_{d=1}^{\overline{v}_n} \mathbb{P}\left[\overline{\nu}_n \left| \sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} \overline{c}_{n,i}^{\overline{t}_d,1}(\tau)\, dW_\tau \right| > c\right] \\
&\le C_r \sum_{d=1}^{\overline{v}_n} \mathbb{E}\left[\left|\overline{\nu}_n \sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} \overline{c}_{n,i}^{\overline{t}_d,1}(\tau)\, dW_\tau \right|^{2m}\right] .
\end{aligned}$$

Due to (5.3) and triangle inequality, it is sufficient to bound the following term via Itô isometry and the generalized Minkowski inequality, such that

$$\begin{aligned}
\mathbb{E}\left[\left|\int_{ib_n}^{(i+1)b_n} \overline{c}_{n,i}^{\overline{t}_d,1}(\tau)\, dW_\tau\right|^{2m}\right] &= \mathbb{E}\left[\left|\int_{ib_n}^{(i+1)b_n} (\overline{c}_{n,i}^{\overline{t}_d,1}(\tau))^2\, d\tau\right|^{m}\right] \\
&\le \left(\int_{ib_n}^{(i+1)b_n} \mathbb{E}\left[(\overline{c}_{n,i}^{\overline{t}_d,1}(\tau))^{2m}\right]^{1/m} d\tau\right)^m .
\end{aligned}$$

For a fixed $\tau \in [\overline{t}_{n-}^{i,\ell-1}, \overline{t}_n^{i,\ell})$ we can conclude the following bound with Jensen's inequality

$$\begin{aligned}
\mathbb{E}\left[(\overline{c}_{n,i}^{\overline{t}_d,1}(\tau))^{2m}\right] &= \mathbb{E}\left[\sum_{\ell=1}^{b_n/h_n} (\overline{K}_{\overline{t}_d,n}^{i,\ell})^{2m} \left(\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\xi_{ij\ell}^{(n)}(\tau) \int_{\overline{t}_{n-}^{i,\ell-1}}^{\tau} \xi_{ij\ell}^{(n)}(s) a_s\, ds\right)^{2m} \right.\\
&\qquad\left. \times \mathbb{1}_{[\overline{t}_{n-}^{i,\ell-1}, \overline{t}_n^{i,\ell})}(\tau)\right] \\
&= \mathbb{E}\left[\sum_{\ell=1}^{b_n/h_n} (\overline{K}_{\overline{t}_d,n}^{i,\ell})^{2m} \sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell} \left(\xi_{ij\ell}^{(n)}(\tau) \int_{\overline{t}_{n-}^{i,\ell-1}}^{\tau} \xi_{ij\ell}^{(n)}(s) a_s\, ds\right)^{2m} \right.\\
&\qquad\left. \times \mathbb{1}_{[\overline{t}_{n-}^{i,\ell-1}, \overline{t}_n^{i,\ell})}(\tau)\right] \\
&= \mathcal{O}(1) .
\end{aligned}$$

That yields the bound

$$\mathbb{P}\left[\overline{\nu}_n \left| \sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} \overline{c}_{n,i}^{\overline{t}_d,1}(\tau)\, dW_\tau \right| > c\right] = \mathcal{O}(b_n^m \overline{\nu}_n^{2m}) = \mathcal{O}(\overline{v}_n^{-1}) ,$$

for m being sufficiently large.This completes the term $A_n^{1,3,1,2}$ and so $A_n^{1,3,1}$.

We can proceed with $A_n^{1,3,2}$. Therefore, we define functions $\overline{c}_{n,i}^{t,2}$ given by

$$\overline{c}_{n,i}^{t,2}(\tau) = \sum_{\ell=1}^{b_n/h_n} \overline{K}_{t,n}^{i,\ell} \sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{ij\ell} \xi_{ij\ell}^{(n)}(\tau)(\sigma_\tau - \sigma_{ib_n}) \int_{\overline{t}_{n_-}^{i,\ell-1}}^{\tau} \xi_{ij\ell}^{(n)}(s) \sigma_s \, dW_s \mathbb{1}_{[\overline{t}_{n_-}^{i,\ell-1}, \overline{t}_n^{i,\ell})}(\tau) \, ,$$

which yields the following representation and decomposition with respect to $A_n^{1,3,2}$

$$\begin{aligned}
&\mathbb{P}\left[\sup_{t\in[0,1]} \overline{\nu}_n \left| \sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} \overline{c}_{n,i}^{t,2}(\tau) \, dW_\tau \right| > c \right] \\
&\leq \mathbb{P}\left[\max_{1\leq d\leq \tilde{v}_n} \sup_{t\in[\overline{t}_{d-1},\overline{t}_d]} \overline{\nu}_n \left| \sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} (\overline{c}_{n,i}^{\overline{t}_d,2}(\tau) - \overline{c}_{n,i}^{t,2}(\tau)) \, dW_\tau \right| > c \right] \\
&\qquad + \mathbb{P}\left[\max_{1\leq d\leq \tilde{v}_n} \overline{\nu}_n \left| \sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} \overline{c}_{n,i}^{\overline{t}_d,2}(\tau) \, dW_\tau \right| > c \right] \\
&= A_n^{1,3,2,1} + A_n^{1,3,2,2} \, .
\end{aligned}$$

We omit the details concerning $A_n^{1,3,2,1}$ and refer to $A_n^{1,3,1,1}$. We proceed with $A_n^{1,3,2,2}$. The triangle inequality, and Markov's inequality applied to the function $z \mapsto |z|^{2m}$ with $m > 0$ and $m \in \mathbb{N}$ yield

$$\begin{aligned}
&\mathbb{P}\left[\max_{1\leq d\leq \tilde{v}_n} \overline{\nu}_n \left| \sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} \overline{c}_{n,i}^{\overline{t}_d,2}(\tau) \, dW_\tau \right| > c \right] \\
&\leq \sum_{d=1}^{\overline{v}_n} \mathbb{P}\left[\overline{\nu}_n \left| \sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} \overline{c}_{n,i}^{\overline{t}_d,2}(\tau) \, dW_\tau \right| > c \right] \\
&\leq C_m \sum_{d=1}^{\overline{v}_n} \mathbb{E}\left[\left| \overline{\nu}_n \sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} \overline{c}_{n,i}^{\overline{t}_d,2}(\tau) \, dW_\tau \right|^{2m} \right] .
\end{aligned}$$

Again, due to (5.3) and triangle inequality, it is sufficient to bound the following term via Itô isometry and the generalized Minkowski

$$\begin{aligned}
\mathbb{E}\left[\left| \int_{ib_n}^{(i+1)b_n} \overline{c}_{n,i}^{\overline{t}_d,2}(\tau) \, dW_\tau \right|^{2m} \right] &= \mathbb{E}\left[\left| \int_{ib_n}^{(i+1)b_n} (\overline{c}_{n,i}^{\overline{t}_d,2}(\tau))^2 \, d\tau \right|^{m} \right] \\
&\leq \left(\int_{ib_n}^{(i+1)b_n} \mathbb{E}\left[(\overline{c}_{n,i}^{\overline{t}_d,2}(\tau))^{2m} \right]^{1/m} d\tau \right)^m .
\end{aligned}$$

For a fixed $\tau \in [\bar{t}_{n_-}^{i,\ell-1}, \bar{t}_n^{i,\ell})$ we can conclude the following bound with Jensen's inequality, (3.31) and Assumption 4.5

$$\begin{aligned}
&\mathbb{E}\Big[(\bar{c}_{n,i}^{\bar{t}_d,2}(\tau))^{2m}\Big] \\
&= \mathbb{E}\left[\sum_{\ell=1}^{b_n/h_n} (\overline{K}_{\bar{t}_d,n}^{i,\ell})^{2m} \left(\sum_{j=1}^{\lfloor nh_n\rfloor -1} w_{ij\ell}\xi_{ij\ell}^{(n)}(\tau)(\sigma_\tau - \sigma_{ib_n}) \int_{\bar{t}_{n_-}^{i,\ell-1}}^{\tau} \xi_{ij\ell}^{(n)}(s)\sigma_s\, dW_s\right)^{2m} \right. \\
&\qquad\qquad\qquad\qquad \left. \times \mathbb{1}_{[\bar{t}_{n_-}^{i,\ell-1}, \bar{t}_n^{i,\ell})}(\tau)\right] \\
&\le \mathbb{E}\left[\sum_{\ell=1}^{b_n/h_n} (\overline{K}_{\bar{t}_d,n}^{i,\ell})^{2m} \sum_{j=1}^{\lfloor nh_n\rfloor -1} w_{ij\ell}(\xi_{ij\ell}^{(n)}(\tau))^{2m} b_n^{2m\mathfrak{a}} \left(\int_{\bar{t}_{n_-}^{i,\ell-1}}^{\tau} \xi_{ij\ell}^{(n)}(s)\sigma_s\, dW_s\right)^{2m} \right. \\
&\qquad\qquad\qquad\qquad \left. \times \mathbb{1}_{[\bar{t}_{n_-}^{i,\ell-1}, \bar{t}_n^{i,\ell})}(\tau)\right] \\
&\le \mathbb{E}\left[\sum_{\ell=1}^{b_n/h_n} (\overline{K}_{\bar{t}_d,n}^{i,\ell})^{2m} \sum_{j=1}^{\lfloor nh_n\rfloor -1} w_{ij\ell}(\xi_{ij\ell}^{(n)}(\tau))^{2m} b_n^{2m\mathfrak{a}} \left(\int_{\bar{t}_{n_-}^{i,\ell-1}}^{\tau} (\xi_{ij\ell}^{(n)}(s))^2\sigma_s^2\, ds\right)^{m} \right. \\
&\qquad\qquad\qquad\qquad \left. \times \mathbb{1}_{[\bar{t}_{n_-}^{i,\ell-1}, \bar{t}_n^{i,\ell})}(\tau)\right] \\
&= \mathcal{O}(h_n^{-m} b_n^{2m\mathfrak{a}} b_n^m)\,.
\end{aligned}$$

Overall, we get the bound

$$\mathbb{P}\left[\overline{\nu}_n \left|\sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} \bar{c}_{n,i}^{\bar{t}_d,2}(\tau)\, dW_\tau\right| > c\right] = \mathcal{O}(h_n^{-m} b_n^m b_n^{2m\mathfrak{a}} \overline{\nu}_n^{2m}) = \scriptstyle{\mathcal{O}}(\overline{v}_n^{-1})\,,$$

for m being sufficiently large due to Assumption 5.1. We have completed $A_n^{1,3,2,2}$ and so $A_n^{1,3,2}$. We can proceed with $A_n^{1,3,3}$. Therefore, we define functions $\bar{c}_{n,i}^{t,3}$ given by

$$\bar{c}_{n,i}^{t,3}(\tau) = \sum_{\ell=1}^{b_n/h_n} \overline{K}_{t,n}^{i,\ell} \sum_{j=1}^{\lfloor nh_n\rfloor -1} w_{ij\ell}\xi_{ij\ell}^{(n)}(\tau)\sigma_{ib_n} \int_{\bar{t}_{n_-}^{i,\ell-1}}^{\tau} \xi_{ij\ell}^{(n)}(s)(\sigma_s - \sigma_{ib_n})\, dW_s \mathbb{1}_{[\bar{t}_{n_-}^{i,\ell-1}, \bar{t}_n^{i,\ell})}(\tau)\,.$$

This yields the following representation and decomposition with respect to $A_n^{1,3,3}$

$$\begin{aligned}
&\mathbb{P}\left[\sup_{t\in[0,1]} \overline{\nu}_n \left|\sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} \bar{c}_{n,i}^{t,3}(\tau)\, dW_\tau\right| > c\right] \\
&\le \mathbb{P}\left[\max_{1\le d\le \tilde{v}_n} \sup_{t\in[\bar{t}_{d-1},\bar{t}_d]} \overline{\nu}_n \left|\sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} (\bar{c}_{n,i}^{\bar{t}_d,3}(\tau) - \bar{c}_{n,i}^{t,2}(\tau))\, dW_\tau\right| > c\right] \\
&\qquad + \mathbb{P}\left[\max_{1\le d\le \tilde{v}_n} \overline{\nu}_n \left|\sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} \bar{c}_{n,i}^{\bar{t}_d,3}(\tau)\, dW_\tau\right| > c\right] \\
&= A_n^{1,3,3,1} + A_n^{1,3,3,2}\,.
\end{aligned}$$

We omit the details concerning $A_n^{1,3,3,1}$ and refer to $A_n^{1,3,1,1}$. We proceed with $A_n^{1,3,3,2}$. The triangle inequality, and Markov's inequality applied to the function $z \mapsto |z|^{2m}$ with $m > 0$ and $m \in \mathbb{N}$, yield

$$\begin{aligned}
&\mathbb{P}\left[\max_{1\le d\le \bar{v}_n} \overline{\nu}_n \left|\sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} \bar{c}_{n,i}^{\bar{t}_d,3}(\tau)\, dW_\tau\right| > c\right] \\
&\quad \le \sum_{d=1}^{\bar{v}_n} \mathbb{P}\left[\overline{\nu}_n \left|\sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} \bar{c}_{n,i}^{\bar{t}_d,3}(\tau)\, dW_\tau\right| > c\right] \\
&\quad \le C_r \sum_{d=1}^{\bar{v}_n} \mathbb{E}\left[\left|\overline{\nu}_n \sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} \bar{c}_{n,i}^{\bar{t}_d,3}(\tau)\, dW_\tau\right|^{2m}\right].
\end{aligned}$$

Again, due to (5.3) and triangle inequality, it is sufficient to bound the following term via Itô isometry and the generalized Minkowski

$$\begin{aligned}
\mathbb{E}\left[\left|\int_{ib_n}^{(i+1)b_n} \bar{c}_{n,i}^{\bar{t}_d,3}(\tau)\, dW_\tau\right|^{2m}\right] &= \mathbb{E}\left[\left|\int_{ib_n}^{(i+1)b_n} (\bar{c}_{n,i}^{\bar{t}_d,3}(\tau))^2\, d\tau\right|^{m}\right] \\
&\le \left(\int_{ib_n}^{(i+1)b_n} \mathbb{E}\left[(\bar{c}_{n,i}^{\bar{t}_d,3}(\tau))^{2m}\right]^{1/m} d\tau\right)^m .
\end{aligned}$$

For a fixed $\tau \in [\bar{t}_{n_-}^{i,\ell-1}, \bar{t}_n^{i,\ell})$ we can conclude the following bound with Jensen's inequality, (3.31), Burkholder's inequality and Assumption 4.5

$$\begin{aligned}
&\mathbb{E}\left[(\bar{c}_{n,i}^{\bar{t}_d,3}(\tau))^{2m}\right] \\
&= \mathbb{E}\left[\sum_{\ell=1}^{b_n/h_n} (\overline{K}_{\bar{t}_d,n}^{i,\ell})^{2m} \left(\sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}\xi_{ij\ell}^{(n)}(\tau)\sigma_{ib_n} \int_{\bar{t}_{n_-}^{i,\ell-1}}^{\tau} \xi_{ij\ell}^{(n)}(s)(\sigma_s - \sigma_{ib_n})\, dW_s\right)^{2m} \right. \\
&\qquad\qquad \left. \times \mathbb{1}_{[\bar{t}_{n_-}^{i,\ell-1}, \bar{t}_n^{i,\ell})}(\tau)\right] \\
&\le \mathbb{E}\left[\sum_{\ell=1}^{b_n/h_n} (\overline{K}_{\bar{t}_d,n}^{i,\ell})^{2m} \sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}(\xi_{ij\ell}^{(n)}(\tau)\sigma_{ib_n})^{2m} \left(\int_{\bar{t}_{n_-}^{i,\ell-1}}^{\tau} \xi_{ij\ell}^{(n)}(s)(\sigma_s - \sigma_{ib_n})\, dW_s\right)^{2m} \right. \\
&\qquad\qquad \left. \times \mathbb{1}_{[\bar{t}_{n_-}^{i,\ell-1}, \bar{t}_n^{i,\ell})}(\tau)\right] \\
&\le \mathbb{E}\left[\sum_{\ell=1}^{b_n/h_n} (\overline{K}_{\bar{t}_d,n}^{i,\ell})^{2m} \sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}(\xi_{ij\ell}^{(n)}(\tau)\sigma_{ib_n})^{2m} \left(\int_{\bar{t}_{n_-}^{i,\ell-1}}^{\tau} (\xi_{ij\ell}^{(n)}(s))^2(\sigma_s - \sigma_{ib_n})^2\, ds\right)^{m} \right. \\
&\qquad\qquad \left. \times \mathbb{1}_{[\bar{t}_{n_-}^{i,\ell-1}, \bar{t}_n^{i,\ell})}(\tau)\right] \\
&= \mathcal{O}(b_n^{2m\mathfrak{a}} h_n^{-m}) .
\end{aligned}$$

Overall, we can conclude the bound

$$\mathbb{P}\left[\overline{\nu}_n\left|\sum_{i=0}^{b_n^{-1}-1}\int_{ib_n}^{(i+1)b_n}\overline{c}_{n,i}^{\bar{t}_d,3}(\tau)\,dW_\tau\right|>c\right]=\mathcal{O}(h_n^{-m}b_n^m b_n^{2m\mathfrak{a}}\overline{\nu}_n^{2m})=\mathcal{O}(\overline{v}_n^{-1})\,,$$

for m being sufficiently large due to Assumption 5.1. We have completed the term $A_n^{1,3,3,2}$ and so $A_n^{1,3,3}$. Finally, we have completed the term $A_n^{1,3}$, i.e. A_n^1 has also been shown to be negligible. We can proceed with A_n^2. We need a further decomposition and take into account that

$$\begin{aligned}
&S_{ij\ell}\,(X)-\sigma_{ib_n}S_{ij\ell}\,(W)\\
&=\int_{\overline{t}_{n_-}^{i,\ell-1}}^{\overline{t}_n^{i,\ell}}\xi_{ij\ell}^{(n)}(s)\,dX_s-\sigma_{ib_n}\int_{\overline{t}_{n_-}^{i,\ell-1}}^{\overline{t}_n^{i,\ell}}\xi_{ij\ell}^{(n)}(s)\,dW_s\\
&=\int_{\overline{t}_{n_-}^{i,\ell-1}}^{\overline{t}_n^{i,\ell}}\xi_{ij\ell}^{(n)}(s)(\sigma_s-\sigma_{ib_n})\,dW_s+\int_{\overline{t}_{n_-}^{i,\ell-1}}^{\overline{t}_n^{i,\ell}}\xi_{ij\ell}^{(n)}(s)a_s\,ds\,.
\end{aligned}$$

That yields the decomposition

$$\begin{aligned}
&\mathbb{P}\left[\sup_{t\in[0,1]}\overline{\nu}_n\left|\sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{b_n/h_n}\overline{K}_{t,n}^{i,\ell}\sum_{j=1}^{\lfloor nh_n\rfloor-1}w_{ij\ell}S_{ij\ell}\,(\varepsilon)\left(S_{ij\ell}\,(X)-\sigma_{ib_n}S_{ij\ell}\,(W)\right)\right|>c\right]\\
&\le\mathbb{P}\left[\sup_{t\in[0,1]}\overline{\nu}_n\left|\sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{b_n/h_n}\overline{K}_{t,n}^{i,\ell}\sum_{j=1}^{\lfloor nh_n\rfloor-1}w_{ij\ell}S_{ij\ell}\,(\varepsilon)\int_{\overline{t}_{n_-}^{i,\ell-1}}^{\overline{t}_n^{i,\ell}}\xi_{ij\ell}^{(n)}(s)(\sigma_s-\sigma_{ib_n})\,dW_s\right|>c\right]\\
&\le\mathbb{P}\left[\sup_{t\in[0,1]}\overline{\nu}_n\left|\sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{b_n/h_n}\overline{K}_{t,n}^{i,\ell}\sum_{j=1}^{\lfloor nh_n\rfloor-1}w_{ij\ell}S_{ij\ell}\,(\varepsilon)\int_{\overline{t}_{n_-}^{i,\ell-1}}^{\overline{t}_n^{i,\ell}}\xi_{ij\ell}^{(n)}(s)a_s\,ds\right|>c\right]\\
&=A_n^{2,1}+A_n^{2,2}\,.
\end{aligned}$$

Starting with $A_n^{2,1}$ we use the order of the weights w_{jk},

$$w_{jk}\propto\begin{cases}1, & \text{for } j\le\sqrt{n}h_n\\ j^{-4}n^2h_n^4, & \text{for } j>\sqrt{n}h_n\,,\end{cases}\tag{5.4}$$

in order to use a further decomposition

$$\begin{aligned}
&\mathbb{P}\left[\sup_{t\in[0,1]}\overline{\nu}_n\left|\sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{b_n/h_n}\overline{K}_{t,n}^{i,\ell}\sum_{j=1}^{\lfloor nh_n\rfloor-1}w_{ij\ell}S_{ij\ell}\,(\varepsilon)\int_{\overline{t}_{n_-}^{i,\ell-1}}^{\overline{t}_n^{i,\ell}}\xi_{ij\ell}^{(n)}(s)(\sigma_s-\sigma_{ib_n})\,dW_s\right|>c\right]\\
&\le\mathbb{P}\left[\sup_{t\in[0,1]}\overline{\nu}_n\left|\sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{b_n/h_n}\overline{K}_{t,n}^{i,\ell}\sum_{j=1}^{\sqrt{n}h_n}S_{ij\ell}\,(\varepsilon)\int_{\overline{t}_{n_-}^{i,\ell-1}}^{\overline{t}_n^{i,\ell}}\xi_{ij\ell}^{(n)}(s)(\sigma_s-\sigma_{ib_n})\,dW_s\right|>c\right]
\end{aligned}$$

$$+\mathbb{P}\left[\sup_{t\in[0,1]}\overline{\nu}_n\left|\sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{b_n/h_n}\overline{K}_{t,n}^{i,\ell}\sum_{j=\sqrt{n}h_n+1}^{\lfloor nh_n\rfloor-1}j^{-4}nh_nS_{ij\ell}\left(\varepsilon\right)\int_{\overline{t}_{n_-}^{i,\ell-1}}^{\overline{t}_n^{i,\ell}}\xi_{ij\ell}^{(n)}(s)(\sigma_s-\sigma_{ib_n})\,dW_s\right|>c\right]$$
$$=A_n^{2,1,1}+A_n^{2,1,2}\,.$$

We start with $A_n^{2,1,1}$ an define functions $\overline{c}_{n,i}^{t,4}$ given by

$$\overline{c}_{n,i}^{t,4}(\tau)=\frac{1}{h_n}\sum_{\ell=1}^{b_n/h_n}\overline{K}_{t,n}^{i,\ell}\sum_{j=1}^{\sqrt{n}h_n}S_{ij\ell}\left(\varepsilon\right)\int_{\overline{t}_{n_-}^{i,\ell-1}}^{\tau}\xi_{ij\ell}^{(n)}(s)(\sigma_s-\sigma_{ib_n})\,dW_s\mathbb{1}_{(\overline{t}_{n_-}^{i,\ell-1},\overline{t}_n^{i,\ell}]}(\tau)$$

and infer the representation as well as the decomposition

$$\mathbb{P}\left[\sup_{t\in[0,1]}\overline{\nu}_n\left|\sum_{i=0}^{b_n^{-1}-1}\int_{ib_n}^{(i+1)b_n}\overline{c}_{n,i}^{t,4}(\tau)\,d\tau\right|>c\right]$$
$$\leq\mathbb{P}\left[\max_{1\leq d\leq\tilde{v}_n}\sup_{t\in[\overline{t}_{d-1},\overline{t}_d]}\overline{\nu}_n\left|\sum_{i=0}^{b_n^{-1}-1}\int_{ib_n}^{(i+1)b_n}(\overline{c}_{n,i}^{\overline{t}_d,4}(\tau)-\overline{c}_{n,i}^{t,4}(\tau))\,d\tau\right|>c\right]$$
$$+\mathbb{P}\left[\max_{1\leq d\leq\tilde{v}_n}\overline{\nu}_n\left|\sum_{i=0}^{b_n^{-1}-1}\int_{ib_n}^{(i+1)b_n}\overline{c}_{n,i}^{\overline{t}_d,4}(\tau)\,d\tau\right|>c\right]$$
$$=A_n^{2,1,1,1}+A_n^{2,1,1,2}$$

We will only sketch the upper bound for $A_n^{2,1,1,1}$ and refer to similar calculations in preceding parts of this work. Using the bound in Lemma 2 in Bibinger and Winkelmann [13], the independence of $(\varepsilon_t)_{t\in[0,1]}$, the regularity of the kernel K and Burkholder's inequality, we infer the bound

$$\max_{1\leq d\leq\tilde{v}_n}\sup_{t\in[\overline{t}_{d-1},\overline{t}_d]}\overline{\nu}_n\left|\sum_{i=0}^{b_n^{-1}-1}\int_{ib_n}^{(i+1)b_n}(\overline{c}_{n,i}^{\overline{t}_d,4}(\tau)-\overline{c}_{n,i}^{t,4}(\tau))\,d\tau\right|$$
$$=\mathcal{O}_{\mathbb{P}}(h_n^2\log(n)b_n^{\mathfrak{a}}\overline{\nu}_n)=\mathcal{O}_{\mathbb{P}}(1)\,.$$

We can proceed with $A_n^{2,1,1,2}$. We have the bound

$$\mathbb{P}\left[\max_{1\leq d\leq\tilde{v}_n}\overline{\nu}_n\left|\sum_{i=0}^{b_n^{-1}-1}\int_{ib_n}^{(i+1)b_n}\overline{c}_{n,i}^{\overline{t}_d,4}(\tau)\,dW\tau\right|>c\right]$$
$$\leq\sum_{d=1}^{\overline{v}_n}\mathbb{P}\left[\overline{\nu}_n\left|\sum_{i=0}^{b_n^{-1}-1}\int_{ib_n}^{(i+1)b_n}\overline{c}_{n,i}^{\overline{t}_d,4}(\tau)\,d\tau\right|>c\right]$$
$$\leq C_m\sum_{d=1}^{\overline{v}_n}\mathbb{E}\left[\left|\overline{\nu}_n\sum_{i=0}^{b_n^{-1}-1}\int_{ib_n}^{(i+1)b_n}\overline{c}_{n,i}^{\overline{t}_d,4}(\tau)\,d\tau\right|^{2m}\right]\,.$$

We proceed with the generalized Minkowski inequality which yields

$$\mathbb{E}\left[\left|\sum_{i=0}^{b_n^{-1}-1}\int_{ib_n}^{(i+1)b_n}\bar{c}_{n,i}^{\bar{t}_d,4}(\tau)\,d\tau\right|^{2m}\right]\le\left(\int_{ib_n}^{(i+1)b_n}\mathbb{E}[(\bar{c}_{n,i}^{\bar{t}_d,4}(\tau))^{2m}]^{1/2m}\,d\tau\right)^{2m}.$$

Lemma 2 in Bibinger and Winkelmann [13], Burkholder's inequality, Assumption 4.5,(3.31) and a fixed $\tau\in[\bar{t}_{n_-}^{i,\ell-1},\bar{t}_n^{i,\ell})$ yield the bound

$$\mathbb{E}[(\bar{c}_{n,i}^{\bar{t}_d,4}(\tau))^{2m}]=\mathcal{O}(h_n^{-2m}\log(n)b_n^{2\mathfrak{a}}).$$

Overall using (5.3), the triangle inequality and standard Lebesgue integral upper bounds we conclude

$$\begin{aligned}&\mathbb{P}\left[\overline{\nu}_n\left|\sum_{i=0}^{b_n^{-1}-1}\int_{ib_n}^{(i+1)b_n}\bar{c}_{n,i}^{\bar{t}_d,4}(\tau)\,d\tau\right|>c\right]\\&=\mathcal{O}(\overline{\nu}_n^{2m}h_n^{-2m}b_n^{2m\mathfrak{a}}b_n^{2m}\log(n))=\mathcal{o}(\overline{v}_n^{-1}),\end{aligned}$$

due to Assumption 4.5 for m being sufficiently large. We have completed the probability $A_n^{2,1,1,2}$ and so $A_n^{2,1,1}$. We can proceed with $A_n^{2,1,2}$. Therefore, it is sufficient to note that

$$j^{-4}n^2h_n^4=\mathcal{O}(1)\quad\text{for}\quad\sqrt{n}h_n\le nh_n.$$

We omit the details, since the procedure is very similar to $A_n^{2,1,1}$. Hence, we have completed the term $A_n^{2,1,2}$ and so $A_n^{2,1}$. The probability $A_n^{2,2}$ is in fact easier than $A_n^{2,1}$ as it only includes a (finite variation) Lebesgue integral, such that we can use standard Lebesgue integral upper bounds instead of Burkholder's inequality. We will therefore omit the details. Finally, the probability A_n^2 has been completed and so A_n, i.e. we can proceed with B_n. We have to bound the probability

$$\begin{aligned}&\mathbb{P}\left[\inf_{t\in[0,1]}\overline{\Gamma}_{t,n}^{\varepsilon}<K^{-}-\kappa^{-}\right]\\&=\mathbb{P}\left[\inf_{t\in[0,1]}\frac{h_n}{b_n}\sum_{i=0}^{b_n^{-1}-1}\sum_{\ell=1}^{b_n/h_n}\overline{K}_{t,n}^{i,\ell}\sum_{j=1}^{\lfloor nh_n\rfloor-1}w_{ij\ell}((\sigma_{ib_n}S_{ij\ell}(W)+S_{ij\ell}(\varepsilon))^2-\mu_{ij\ell})<K^{-}-\kappa^{-}\right].\end{aligned}$$

First of all, using a Riemann sum approximation, $\int K=1$, $K^{-}<\inf_{t\in[0,1]}\sigma_t^2$, the convexity of the $j-$ sum and

$$\sum_{i=0}^{b_n^{-1}-1}\sigma_{ib_n}^2\sum_{\ell=1}^{b_n/h_n}\overline{K}_{t,n}^{i,\ell}>\sum_{i=0}^{b_n^{-1}-1}K^{-}\sum_{\ell=1}^{b_n/h_n}\overline{K}_{t,n}^{i,\ell},$$

we can proceed as follows. Using a similar argument applied for B_n in the previous chapter we have the following decomposition

$$\begin{aligned}
&\mathbb{P}\left[\inf_{t\in[0,1]} \overline{\Gamma}^{\varepsilon}_{t,n} < K^{-} - \kappa^{-}\right] \\
&\leq \mathbb{P}\left[\sup_{t\in[0,1]} \frac{h_n}{b_n}\left|\sum_{i=0}^{b_n^{-1}-1} \sigma^2_{ib_n} \sum_{\ell=1}^{b_n/h_n} \overline{K}^{i,\ell}_{t,n} \sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}(S^2_{ij\ell}(W)-1)\right| > \frac{\kappa^-}{3}\right] \\
&+ \mathbb{P}\left[\sup_{t\in[0,1]} \frac{h_n}{b_n}\left|\sum_{i=0}^{b_n^{-1}-1} \sigma_{ib_n} \sum_{\ell=1}^{b_n/h_n} \overline{K}^{i,\ell}_{t,n} \sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell} S_{ij\ell}(W)\, S_{ij\ell}(\varepsilon)\right| > \frac{\kappa^-}{6}\right] \\
&+ \mathbb{P}\left[\sup_{t\in[0,1]} \frac{h_n}{b_n}\left|\sum_{i=0}^{b_n^{-1}-1} \sum_{\ell=1}^{b_n/h_n} \overline{K}^{i,\ell}_{t,n} \sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}(S^2_{ij\ell}(\varepsilon)-\mu_{ij\ell})\right| > \frac{\kappa^-}{3}\right] \\
&= B_n^1 + B_n^2 + B_n^3\,.
\end{aligned}$$

We will only give a more detailed argument for the term B_n^1, as the other terms can be handled very similarly, c.f. Chapter 3.

$$\begin{aligned}
&\mathbb{P}\left[\sup_{t\in[0,1]} \frac{h_n}{b_n}\left|\sum_{i=0}^{b_n^{-1}-1} \sigma^2_{ib_n} \sum_{\ell=1}^{b_n/h_n} \overline{K}^{i,\ell}_{t,n} \sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}(S^2_{ij\ell}(W)-1)\right| > \frac{\kappa^-}{3}\right] \\
\leq&\mathbb{P}\left[\max_{1\leq d\leq \tilde v_n}\sup_{t\in[\bar t_{d-1},\bar t_d]} \frac{h_n}{b_n}\left|\sum_{i=0}^{b_n^{-1}-1} \sigma^2_{ib_n} \sum_{\ell=1}^{b_n/h_n} (\overline{K}^{i,\ell}_{t,n} - \overline{K}^{i,\ell}_{\bar t_d,n}) \sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}(S^2_{ij\ell}(W)-1)\right| > \frac{\kappa^-}{3}\right] \\
+&\mathbb{P}\left[\max_{1\leq d\leq \tilde v_n} \frac{h_n}{b_n}\left|\sum_{i=0}^{b_n^{-1}-1} \sigma^2_{ib_n} \sum_{\ell=1}^{b_n/h_n} \overline{K}^{i,\ell}_{\bar t_d,n} \sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}(S^2_{ij\ell}(W)-1)\right| > \frac{\kappa^-}{3}\right] \\
&= B_n^{1,1} + B_n^{1,2}\,.
\end{aligned}$$

The term $B_n^{1,1}$ can be handled analogous to several terms, which we have considered previously, exploiting the regularity of the kernel K. We omit the details. We proceed with $B_n^{1,2}$ using (5.3), the triangle inequality, the Markov inequality with an exponent $r > 0$, the boundedness of the volatility process and a classical central limit theorem argument such that

$$\begin{aligned}
&\mathbb{P}\left[\frac{h_n}{b_n}\left|\sum_{i=0}^{b_n^{-1}-1} \sigma^2_{ib_n} \sum_{\ell=1}^{b_n/h_n} \overline{K}^{i,\ell}_{\bar t_d,n} \sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}(S^2_{ij\ell}(W)-1)\right| > \frac{\kappa^-}{3}\right] \\
&\leq C_r \mathbb{E}\left[\left|\frac{h_n}{b_n} \sum_{\ell=1}^{b_n/h_n} \overline{K}^{i,\ell}_{\bar t_d,n} \sum_{j=1}^{\lfloor nh_n\rfloor-1} w_{ij\ell}(S^2_{ij\ell}(W)-1)\right|^r\right] \\
&= \mathcal{O}(h_n^{r/2} b_n^{-r/2}) = \mathcal{o}(\overline{v}_n^{-1})\,,
\end{aligned}$$

with m being sufficiently large. That completes $B_n^{1,1}$ and so B_n^1. That implies B_n to be negligible. Finally we have completed the term (**II**). We can proceed with (**I**). For every $\delta > 0$ and a generic constant $c > 0$ (it may change from line to line) we have

$$\begin{aligned}
&\mathbb{P}\left[\sup_{t\in[0,1]} \sqrt{b_n/h_n \log(n)}\widehat{\Gamma}^{\varepsilon}_{t,n}\left(\frac{1}{|\widehat{\Gamma}^{\varepsilon}_{t,n}|^{3/4}} - \frac{1}{|\overline{\Gamma}^{\varepsilon}_{t,n}|^{3/4}}\right) > \delta\right] \\
&= \mathbb{P}\left[\sup_{t\in[0,1]} \sqrt{b_n/h_n \log(n)}\widehat{\Gamma}^{\varepsilon}_{t,n}\left(\frac{|\overline{\Gamma}^{\varepsilon}_{t,n}|^{3/4} - |\widehat{\Gamma}^{\varepsilon}_{t,n}|^{3/4}}{|\widehat{\Gamma}^{\varepsilon}_{t,n}\overline{\Gamma}^{\varepsilon}_{t,n}|^{3/4}}\right) > \delta\right] \\
&= \mathbb{P}\left[\sup_{t\in[0,1]} \sqrt{b_n/h_n \log(n)}\widehat{\Gamma}^{\varepsilon}_{t,n}\left(\frac{|\overline{\Gamma}^{\varepsilon}_{t,n}|^{3/4} - |\widehat{\Gamma}^{\varepsilon}_{t,n}|^{3/4}}{|\widehat{\Gamma}^{\varepsilon}_{t,n}\overline{\Gamma}^{\varepsilon}_{t,n}|^{3/4}}\right) > \delta,\ \inf_{t\in[0,1]} \widehat{\Gamma}^{\varepsilon}_{t,n}\overline{\Gamma}^{\varepsilon}_{t,n} \geq \frac{(K^- - \kappa^-)^2}{4}\right] \\
&+ \mathbb{P}\left[\sup_{t\in[0,1]} \sqrt{b_n/h_n \log(n)}\widehat{\Gamma}^{\varepsilon}_{t,n}\left(\frac{|\overline{\Gamma}^{\varepsilon}_{t,n}|^{3/4} - |\widehat{\Gamma}^{\varepsilon}_{t,n}|^{3/4}}{|\widehat{\Gamma}^{\varepsilon}_{t,n}\overline{\Gamma}^{\varepsilon}_{t,n}|^{3/4}}\right) > \delta,\ \inf_{t\in[0,1]} \widehat{\Gamma}^{\varepsilon}_{t,n}\overline{\Gamma}^{\varepsilon}_{t,n} < \frac{(K^- - \kappa^-)^2}{4}\right] \\
&\leq \mathbb{P}\left[\sup_{t\in[0,1]} \sqrt{b_n/h_n \log(n)}\widehat{\Gamma}^{\varepsilon}_{t,n}(|\overline{\Gamma}^{\varepsilon}_{t,n}|^{3/4} - |\widehat{\Gamma}^{\varepsilon}_{t,n}|^{3/4}) > c\right] \qquad (5.5) \\
&+ \mathbb{P}\left[\inf_{t\in[0,1]} \widehat{\Gamma}^{\varepsilon}_{t,n}\overline{\Gamma}^{\varepsilon}_{t,n} < \frac{(K^- - \kappa^-)^2}{4}\right]. \qquad (5.6)
\end{aligned}$$

Starting with probability (5.5) it holds that

$$\begin{aligned}
&\mathbb{P}\left[\sup_{t\in[0,1]} \sqrt{b_n/h_n \log(n)}\widehat{\Gamma}^{\varepsilon}_{t,n}(|\overline{\Gamma}^{\varepsilon}_{t,n}|^{3/4} - |\widehat{\Gamma}^{\varepsilon}_{t,n}|^{3/4}) > c\right] \\
&= \mathbb{P}\left[\sup_{t\in[0,1]} \sqrt{b_n/h_n \log(n)}\widehat{\Gamma}^{\varepsilon}_{t,n}(|\overline{\Gamma}^{\varepsilon}_{t,n}|^{3/4} - |\widehat{\Gamma}^{\varepsilon}_{t,n}|^{3/4}) > c,\ \sup_{t\in[0,1]} \widehat{\Gamma}^{\varepsilon}_{t,n} \leq 2(K^+ + \kappa^+)\right] \qquad (5.7) \\
&+ \mathbb{P}\left[\sup_{t\in[0,1]} \sqrt{b_n/h_n \log(n)}\widehat{\Gamma}^{\varepsilon}_{t,n}(|\overline{\Gamma}^{\varepsilon}_{t,n}|^{3/4} - |\widehat{\Gamma}^{\varepsilon}_{t,n}|^{3/4}) > c,\ \sup_{t\in[0,1]} \widehat{\Gamma}^{\varepsilon}_{t,n} > 2(K^+ + \kappa^+)\right]. \qquad (5.8)
\end{aligned}$$

Considering (5.7) we have

$$\begin{aligned}
&\mathbb{P}\left[\sup_{t\in[0,1]} \sqrt{b_n/h_n \log(n)}\widehat{\Gamma}^{\varepsilon}_{t,n}(|\overline{\Gamma}^{\varepsilon}_{t,n}|^{3/4} - |\widehat{\Gamma}^{\varepsilon}_{t,n}|^{3/4}) > c,\ \sup_{t\in[0,1]} \widehat{\Gamma}^{\varepsilon}_{t,n} \leq 2(K^+ + \kappa^+)\right] \\
&\leq \mathbb{P}\left[\sup_{t\in[0,1]} \sqrt{b_n/h_n \log(n)}(|\overline{\Gamma}^{\varepsilon}_{t,n}|^{3/4} - |\widehat{\Gamma}^{\varepsilon}_{t,n}|^{3/4}) > c\right],
\end{aligned}$$

i.e. it can be traced back to A_n using a Taylor expansion of the function $x \mapsto x^{3/4}$

around the point $\overline{\Gamma}^{\varepsilon}_{t,n}$. Proceeding with (5.8) it holds that

$$
\begin{aligned}
&\mathbb{P}\left[\sup_{t\in[0,1]} \sqrt{b_n/h_n \log(n)}\widehat{\Gamma}^{\varepsilon}_{t,n}(|\overline{\Gamma}^{\varepsilon}_{t,n}|^{3/4} - |\widehat{\Gamma}^{\varepsilon}_{t,n}|^{3/4}) > c,\ \sup_{t\in[0,1]} \widehat{\Gamma}^{\varepsilon}_{t,n}\, 2(K^+ + \kappa^+)\right] \\
&\leq \mathbb{P}\left[\sup_{t\in[0,1]} \widehat{\Gamma}^{\varepsilon}_{t,n} > 2(K^+ + \kappa^+)\right] \\
&\leq \mathbb{P}\left[\sup_{t\in[0,1]} \widehat{\Gamma}^{\varepsilon}_{t,n} > K^+ + \kappa^+\right] + \mathbb{P}\left[\sup_{t\in[0,1]} |\widehat{\Gamma}^{\varepsilon}_{t,n} - \overline{\Gamma}^{\varepsilon}_{t,n}| > K^+ + \kappa^+\right] .
\end{aligned}
$$

Only the first probability has to be considered. Since we have already presented similar arguments we will only indicate the proof. Note that we can bound the volatility from above, such that

$$
\sum_{i=0}^{b_n^{-1}-1} \sigma^2_{ib_n} \sum_{\ell=1}^{b_n/h_n} \overline{K}^{i,\ell}_{t,n} < \sum_{i=0}^{b_n^{-1}-1} K^+ \sum_{\ell=1}^{b_n/h_n} \overline{K}^{i,\ell}_{t,n} .
$$

Afterwards, we can split the probability into three terms including $\sup_{t\in[0,1]}$. Then we can split each probability via $\max_{1\leq d\leq \tilde{v}_n} \sup_{t\in[\bar{t}_{d-1},\bar{t}_d]}$ and $\max_{1\leq d\leq \tilde{v}_n}$. The former can be handled via the kernel regularity, the latter can be handled using the Markov inequality combined with the central limit theorem which has already been presented above. Concerning the probability 5.6 we refer to 4.12 and omit the details. We have completed the term (**I**). We skip the term (**III**), since bounding the volatility from above it is a direct consequence of (**I**). We have completed the proof of Proposition 5.8. □

In order to apply the result in Theorem 2.27 we need further approximation steps. Therefore, we define the quantities

$$
\widetilde{\Gamma}^{\varepsilon}_{t,n} = \frac{1}{b_n} \sum_{i=0}^{b_n^{-1}-1} \sum_{\ell=1}^{b_n/h_n} \overline{K}^{i,\ell}_{t,n} \sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{ij\ell} \int_{\bar{t}^{i,\ell-1}_{n_-}}^{\bar{t}^{i,\ell}_{n_-}} \sigma^2_u \, du
$$

and

$$
\Gamma'^{\varepsilon}_{t,n} = \frac{h_n}{b_n} \sum_{i=0}^{b_n^{-1}-1} \sigma^2_{ib_n} \sum_{\ell=1}^{b_n/h_n} \overline{K}^{i,\ell}_{t,n} \sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{ij\ell} .
$$

Then we have the following approximation.

Proposition 5.9. *Under the assumptions of Theorem 5.3 it holds that*

$$
\sup_{t\in[0,1]} |\widetilde{\Gamma}^{\varepsilon}_{t,n} - \Gamma'^{\varepsilon}_{t,n}| = \mathcal{O}_{\mathbb{P}}((b_n/h_n \log(n))^{-1/2}) .
$$

Proof of Proposition 5.9.
We will use Jensen's inequality, (5.3), the triangle inequality and Assumption 4.5. This yields

$$\begin{aligned}\sup_{t\in[0,1]} |\widetilde{\Gamma}^{\varepsilon}_{t,n} - \Gamma'^{\varepsilon}_{t,n}| &= \sup_{t\in[0,1]} \left| \frac{1}{b_n} \sum_{i=0}^{b_n^{-1}-1} \int_{ib_n}^{(i+1)b_n} \overline{k}^{i}_{n,t}(\tau)(\sigma_\tau^2 - \sigma_{ib_n}^2)\, d\tau \right| \\ &= \mathcal{O}_{\mathbb{P}}(b_n^{\mathfrak{a}}) = \scriptstyle{\mathcal{O}}_{\mathbb{P}}((b_n/h_n \log(n))^{-1/2})\,.\end{aligned}$$

□

Proposition 5.10. *Under the assumptions of Theorem 5.3 it holds that*

$$\sup_{t\in[0,1]} |\widetilde{\Gamma}^{\varepsilon}_{t,n} - \Gamma_t| = \scriptstyle{\mathcal{O}}_{\mathbb{P}}((b_n/h_n \log(n))^{-1/2})\,.$$

Proof of Proposition 5.9.
We will omit the details and refer to the argument presented in 4.19 combined with the fact that the $j-$ sum is convex. □

Due to the approximations pursued above,

$$\hat{\eta}^2 = \eta^2 + \mathcal{O}_{\mathbb{P}}(n^{-1/2})$$

a two-dimensional Taylor expansion of the function $(x,y) \mapsto x^{1/2}y^{3/4}$ around the point $|(\eta, \Gamma_t)|$, and a Riemann sum approximation of $\int K = 1$ it is sufficient to consider the sequence of stochastic processes $\overline{U}_t^n$ given by

$$\overline{U}_t^n = \frac{1}{\sqrt{b_n/h_n}} \sum_{\ell=1}^{h_n^{-1}} K_{b_n}(t - \ell h_n)\overline{\rho}_\ell\,,$$

with a family of random variables $(\overline{\rho}_\ell)_{1\le\ell\le h_n^{-1}}$ given by

$$\overline{\rho}_\ell = \frac{\sum_{j=1}^{\lfloor nh_n\rfloor -1} w_{j\ell}(\sigma_t S_{j\ell}(W)) + S_{j\ell}(\varepsilon))^2 - (\sigma_t^2 + [\varphi_{j\ell}, \varphi_{j\ell}]_n \frac{\eta^2}{n}))}{\sqrt{I^{\ell}_{n,t}}}\,.$$

Note that $\overline{\rho}_\ell \in L^p$ for every p and ℓ, i.e. we can use Remark 2.22. Furthermore, taking into account the calculations for the final parts of the proof of Theorem 4.12, it is sufficient to proceed as follows: With $\overline{S}_\ell = \sum_{i=1}^{\ell} \overline{\rho}_i$ it holds with an arbitrarily small $c > 0$ that

$$\begin{aligned}&\max_{\ell\le h_n^{-1}} \frac{|\overline{S}_\ell - \mathbb{B}_\ell|}{\sqrt{b_n/h_n}} \\ &= \max_{\ell\le h_n^{-1}} \sqrt{\log(n)} \frac{|\overline{S}_\ell - \mathbb{B}_\ell|}{\sqrt{b_n/h_n}} \frac{h_n^{-c}}{h_n^{-c}} \\ &= \max_{\ell\le h_n^{-1}} \frac{|\overline{S}_\ell - \mathbb{B}_\ell|}{h_n^{-c}} \frac{h_n^{-c}\sqrt{\log(n)}}{\sqrt{b_n/h_n}} \\ &= \scriptstyle{\mathcal{O}}_{a.s}(1)\mathcal{O}(1)\,.\end{aligned}$$

Now, we have to pass to a sequence of stochastic processes $\overline{Y}_t^n$ being Gaussian and stationary. Again, we refer to the final parts of the proof of Theorem 4.12, as the procedure is exactly the same. To finalize the proof of Theorem 5.3 we have to show that the limit is not affected, if we replace the oracle version $\widehat{\Gamma}_{t,n}^{\varepsilon}$ by the adaptive version $\hat{\Gamma}_{t,n}^{\varepsilon,ad}$.

Proposition 5.11. *Under the assumptions of Theorem 5.3 it holds that*

$$\sup_{t\in[0,1]} |\widehat{\Gamma}_{t,n}^{\varepsilon} - \hat{\Gamma}_{t,n}^{\varepsilon,ad}| = o_{\mathbb{P}}((b_n/h_n \log(n))^{-1/2}) .$$

Proof of Proposition 5.11.
The proof can be traced back to the proof of Proposition 3.19, exploiting the uniform bound given therein and the compact support of the kernel K. We will therefore omit the details. □

Finally, the proof of Theorem 5.3 has been completed.

Proof of Theorem 5.6

Due to Proposition 5.11 it is sufficient to show that

$$\sup_{t\in[0,1]} \sqrt{b_n/h_n \log(n)} |\widehat{\Gamma}_{t,n}^{\varepsilon,\tau} - \widehat{\Gamma}_{t,n}^{\varepsilon}| \xrightarrow{\mathbb{P}} 0 .$$

The procedure is very closely based on the proof of Proposition 3.10. Therefore, we will keep the exposition very short, referring to the above proof. We use the notation $\overline{K}_{t,n}^{0,k} = \overline{K}_{t,n}^{k}$. We have to bound the terms

$$\sup_{t\in[0,1]} \overline{\nu}_n \sum_{k=1}^{h_n^{-1}} \overline{K}_{t,n}^{k} \zeta_k(X^c+\varepsilon) \mathbb{1}_{\{h_n|\zeta_k(Y)|>u_n\}} , \tag{5.9}$$

$$\sup_{t\in[0,1]} \overline{\nu}_n \sum_{k=1}^{h_n^{-1}} \overline{K}_{t,n}^{k} \sum_{j=1}^{\lfloor nh_n\rfloor -1} w_{jk} S_{jk}^2(J) \mathbb{1}_{\{h_n|\zeta_k(Y)|\le u_n\}} , \tag{5.10}$$

$$\sup_{t\in[0,1]} \overline{\nu}_n \sum_{k=1}^{h_n^{-1}} \overline{K}_{t,n}^{k} \sum_{j=1}^{\lfloor nh_n\rfloor -1} w_{jk} S_{jk}(J) S_{jk}(X^c) \mathbb{1}_{\{h_n|\zeta_k(Y)|\le u_n\}} , \tag{5.11}$$

$$\sup_{t\in[0,1]} \overline{\nu}_n \sum_{k=1}^{h_n^{-1}} \overline{K}_{t,n}^{k} \sum_{j=1}^{\lfloor nh_n\rfloor -1} w_{jk} S_{jk}(J) S_{jk}(\varepsilon) \mathbb{1}_{\{h_n|\zeta_k(Y)|\le u_n\}} . \tag{5.12}$$

We will only discuss the quadratic terms (5.9) and (5.10). The remaining mixed terms can be handled via Cauchy Schwarz inequality. Starting with (5.10) we fix some

$c, \overline{c} \in (0,1)$ and get the decomposition

$$\sup_{t\in[0,1]} \overline{\nu}_n \left| \sum_{k=1}^{h_n^{-1}} \overline{K}_{t,n}^k \sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{jk} S_{jk}^2(J) \mathbb{1}_{\{h_n |\zeta_k(Y)| \le u_n\}} \right|$$

$$\le \sup_{t\in[0,1]} \overline{\nu}_n \left| \sum_{k=1}^{h_n^{-1}} \overline{K}_{t,n}^k \sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{jk} S_{jk}^2(J) \mathbb{1}_{\left\{\sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{jk} S_{jk}^2(J) \le c h_n^{\tau-1}\right\}} \right| \tag{5.13}$$

$$+ \sup_{t\in[0,1]} \overline{\nu}_n \left| \sum_{k=1}^{h_n^{-1}} \overline{K}_{t,n}^k \sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{jk} S_{jk}^2(J) \mathbb{1}_{\left\{|\zeta_k(X^c+\varepsilon)| > \overline{c} h_n^{\tau-1}\right\}} \right| . \tag{5.14}$$

We will only consider (5.13) and infer the bound

$$\sup_{t\in[0,1]} \overline{\nu}_n \left| \sum_{k=1}^{h_n^{-1}} \overline{K}_{t,n}^k \sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{jk} S_{jk}^2(J) \mathbb{1}_{\left\{\sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{jk} S_{jk}^2(J) \le c h_n^{\tau-1}\right\}} \right|$$

$$\le 2 \sup_{t\in[0,1]} \overline{\nu}_n \left| \sum_{k=1}^{h_n^{-1}} \overline{K}_{t,n}^k h_n^{-1} |J_{kh_n} - J_{(k-1)h_n}|^2 \mathbb{1}_{\left\{|J_{kh_n} - J_{(k-1)h_n}| \le \sqrt{c/2} h_n^{\tau/2}\right\}} \right| \tag{5.15}$$

$$+ 2 \sup_{t\in[0,1]} \overline{\nu}_n \left| \sum_{k=1}^{h_n^{-1}} \overline{K}_{t,n}^k h_n^{-1} \Big(|J_{kh_n} - J_{(k-1)h_n}|^2 - \big(\sum_{i=\lfloor (k-1)nh_n \rfloor}^{\lfloor knh_n \rfloor} |\Delta_i^n J| \big)^2 \Big) \right|$$

$$\times \mathbb{1}_{\left\{\sum_{i=\lfloor (k-1)nh_n \rfloor}^{\lfloor knh_n \rfloor} |\Delta_i^n J| \le \sqrt{c/2} h_n^{\tau/2}\right\}} \tag{5.16}$$

$$+ \sup_{t\in[0,1]} \overline{\nu}_n \left| \sum_{k=1}^{h_n^{-1}} \overline{K}_{t,n}^k c h_n^{\tau-1} \mathbb{1}_{\left\{\sum_{i=\lfloor (k-1)nh_n \rfloor}^{\lfloor knh_n \rfloor} |\Delta_i^n J| \le \sqrt{c/2} h_n^{\tau/2}\right\}} \right| . \tag{5.17}$$

Proceeding with (5.16) we have

$$2 \sup_{t\in[0,1]} \overline{\nu}_n \left| \sum_{k=1}^{h_n^{-1}} \overline{K}_{t,n}^k h_n^{-1} \Big(|J_{kh_n} - J_{(k-1)h_n}|^2 - \big(\sum_{i=\lfloor (k-1)nh_n \rfloor}^{\lfloor knh_n \rfloor} |\Delta_i^n J| \big)^2 \Big) \right|$$

$$\times \mathbb{1}_{\left\{\sum_{i=\lfloor (k-1)nh_n \rfloor}^{\lfloor knh_n \rfloor} |\Delta_i^n J| \le \sqrt{c/2} h_n^{\tau/2}\right\}}$$

$$\le 4 \sup_{t\in[0,1]} \overline{\nu}_n \left| \sum_{k=1}^{h_n^{-1}} \overline{K}_{t,n}^k h_n^{-1} \big(\sum_{i=\lfloor (k-1)nh_n \rfloor}^{\lfloor knh_n \rfloor} |\Delta_i^n J| \big)^2 \mathbb{1}_{\left\{\sum_{i=\lfloor (k-1)nh_n \rfloor}^{\lfloor knh_n \rfloor} |\Delta_i^n J| \le h_n^{2/3+\varpi}\right\}} \right|$$

$$+ 2 \sup_{t\in[0,1]} \overline{\nu}_n \left| \sum_{k=1}^{h_n^{-1}} \overline{K}_{t,n}^k h_n^{-1} \Big(|J_{kh_n} - J_{(k-1)h_n}|^2 - \big(\sum_{i=\lfloor (k-1)nh_n \rfloor}^{\lfloor knh_n \rfloor} |\Delta_i^n J| \big)^2 \Big) \right|$$

$$\times \mathbb{1}_{\left\{N_{kh_n}(h_n^{2/3}+\varpi) - N_{(k-1)h_n}(h_n^{2/3+\varpi}) \ge 2\right\}} \mathbb{1}_{\left\{h_n^{2/3+\varpi} \le \sum_{i=\lfloor (k-1)nh_n \rfloor}^{\lfloor knh_n \rfloor} |\Delta_i^n J| \le \sqrt{c/2} h_n^{\tau/2}\right\}}$$

$$= \mathcal{O}_{\mathbb{P}}((\log(n))^{1/2} b_n^{1/2} h_n^{-1/2} h_n^{1/3+2\varpi}) + \mathcal{O}_{\mathbb{P}}((\log(n))^{1/2} b_n^{1/2} h_n^{\tau+1/2} h_n^{-2r(2/3+\varpi)}) ,$$

which is negligible, due to Assumption 5.5. Proceeding with (5.15) we define the sequence $(\mathcal{Z}_\ell)_{\ell=1,\dots,h_n^{-1}}$ given by

$$\mathcal{Z}_k = \left(\left(J_{kh_n} - J_{(k-1)h_n}\right) \mathbb{1}_{\left\{|J_{kh_n} - J_{(k-1)h_n}| \le \sqrt{c/2} h_n^{\tau/2}\right\}} \right)^2 .$$

We use Theorem 2.5, the bound (3.62a) and set $\overline{I}_{n,t} = [-b_n/h_n + t/h_n, t/h_n + b_n/h_n] \cap \mathbb{Z}$. This yields

$$\begin{aligned}
\lambda \mathbb{P}\left[\sup_{t\in[0,1]} \max_{\ell \in \overline{I}_{n,t}} \overline{\nu}_n \sum_{k=1}^{\ell} \overline{K}_{t,n}^k h_n^{-1} \mathcal{Z}_k \ge \lambda \right] &\le \overline{\nu}_n h_n^{-1} \mathbb{E}\left[\max_{\ell \in \overline{I}_{n,t}} \sum_{k=1}^{\ell} \mathcal{Z}_k \right] \\
&= \mathcal{O}(\overline{\nu}_n b_n / h_n h_n^{\tau(1-r/2)}) \\
&= \mathcal{O}((\log(n))^{1/2} h_n^{-1/2} b_n^{1/2} h_n^{\tau(1-r/2)}) \,,
\end{aligned}$$

which is negligible if $\lambda^{-1} = \scriptstyle\mathcal{O}((\log(n))^{-1/2} h_n^{\tau(r/2-1)} b_n^{-1/2} h_n^{1/2})$. We omit the details on (5.17) as the bound can be concluded similarly. Considering the term (5.9), it is sufficient to bound

$$\begin{aligned}
&\sup_{t\in[0,1]} \overline{\nu}_n \left| \sum_{k=1}^{h_n^{-1}} \overline{K}_{t,n}^k \zeta_k (X^c + \varepsilon) \mathbb{1}_{\left\{\sum_{j=1}^{\lfloor nh_n \rfloor -1} w_{jk} S_{jk}^2(J) > c h_n^{\tau-1}\right\}} \right| \\
&\le \sup_{t\in[0,1]} \overline{\nu}_n \left| \sum_{k=1}^{h_n^{-1}} \overline{K}_{t,n}^k \zeta_k (X^c + \varepsilon) \mathbb{1}_{\left\{\sum_{\lfloor (k-1) nh_n \rfloor}^{\lfloor knh_n \rfloor} |\Delta_i^n J| > \sqrt{c/2} h_n^{\tau/2}\right\}} \right| .
\end{aligned}$$

A decomposition similar to (5.16) completes the term using the bounds in Lemma 2 in Bibinger and Winkelmann [13].

6. Conclusion and further questions

6.1. Conclusion and summary

In this thesis we have provided new results for inference on stochastic volatility models deepening and extending existing literature. In Chapter 3 we have extended the results in Bibinger et al. [10]. We have provided weak limit theorems for noise and jump robust statistics and constructed a consistent change-point test. Finally we have proved a consistency result for the change-point estimator.
In Chapter 4 we have crucially improved and generalized the results presented in Fan and Wang [23]. Our results allow for more general price process models including infinite activity jumps as well as for more general stochastic volatility models. In particular, we were able to show that serious restrictions, e.g. stationarity of the spot volatility process imposed in Fan and Wang [23] are unnecessary.
In Chapter 5 we have provided an even more general method being noise and jump robust. We have proved limit theorems for extreme values and explained how to construct confidence bands.

6.2. Further questions

Though the results we have presented are quite general, there are still several open questions which we will address in this section.
Concerning Chapter 3 there are still open question. Starting with the global change-point problem, that is, constructing a consistent testing procedure, to detect changes in the regularity index $\mathfrak{a}$ of the volatility process $(\sigma_t^2)_{t\in[0,1]}$. Therefore, similar functional stable limit theorems as presented in Theorem 5.4 Bibinger et al. [10], are key tools tackling this testing problem. A second open question, which we have already mentioned in Remark 3.13, is a formal proof of the conjecture that our test yields an asymptotic minimax-optimal decision rule.
A further challenging but very interesting question is concerning the asymptotic distribution of the change-point estimator $\hat{\theta}_n$ in (3.21). That is, to investigate the limit distribution of the random variable T given by

$$\frac{\delta}{h_n\sqrt{\alpha_n\log(n)}}|\hat{\theta}_n-\theta| \xrightarrow{d} T\,.$$

The limit distribution of the change-point remains unknown in the case with microstructure noise as well as in the pure semimartingale case.
Though, the models which have been considered in Chapter 4 and Chapter 5 are quite

general, it might be useful to pass to a more general class of processes $(X_t)_{t\in[0,1]}$, if we intend to model within different areas of applications beyond financial mathematics. Typical fields of applications are turbulence modeling, hydrology and electricity markets. From an application point of view it is desirable to be able to reflect phenomena which exhibit long and short range dependence. Therefore, we could consider X_t given by

$$X_t = \int_0^t \sigma_s \, dB_s^H \,,$$

where the standard Brownian motion W has been replaced by a fractional Brownian motion B^H, allowing for more flexibility in modeling. We refer to Manuel Corcuera et al. [45] for high-frequency statistics with respect to X_t. Furthermore, the model above also allows for very interesting and possibly challenging theoretical questions. Some of these possible questions are

(1) constructing uniform confidence bands for $(\sigma_t^2)_{t\in[0,1]}$,

(2) change-point inference for $(\sigma_t^2)_{t\in[0,1]}$ and

(3) change-point inference for the Hurst index $H \in (0,1)$, extending the recent work Bibinger [8].

Due to the fact that X_t is beyond the semimartingale framework almost every classical result in stochastic calculus is not applicable within this model. Therefore, these questions offer exciting challenges linked to other areas of stochastic analysis such as Malliavin calculus and rough path theory.

Appendix A. Some useful inequalities

In this appendix we will present two inequalities, which we have used in the main part of this dissertation.

A.1. Rosenthal's inequality

In Chapter 3, we have used Rosenthal's inequality for sums of i.i.d. variables.

Theorem A.1 (Rosenthal's inequality). *Let $p \geq 1$. Suppose that $X, X_1, \dots, X_n$ are independent, identically distributed random variables with mean 0 and $X \in L^p$. Set $S_n = \sum_{k=1}^n X_k$, $n \geq 1$. Then it holds that*

$$\mathbb{E}[|S_n|^p] \leq \begin{cases} C_p n \mathbb{E}[|X|^p], & \text{if } p \in [1,2] \\ C_p n^{p/2} \mathbb{E}[|X|^p], & \text{if } p \geq 2\,, \end{cases} \tag{A.1}$$

with an universal constant C_p.

For a proof we refer to Gut [27].

A.2. Sakhanenko's inequality

Controlling the error term in the proof of Theorem 3.8, we have used the following inequality.

Theorem A.2 (Sakhanenko's inequality). *Let $\xi_1, \dots, \xi_n$ be independent random variables and Z be a standard normal random variable constructed on some probability space. We set*

$$\delta = |S_n/c - Z|$$

with $S_n = \sum_{j=1}^n \xi_j$ and $c^2 = \sum_{j=1}^n \mathbb{V}ar(\xi_j)$. Then

$$\mathbb{E}[\delta^\alpha] \leq (C\alpha)^\alpha \sum_{j=1}^n \mathbb{E}[\min\left\{|\xi_j/c|^{\alpha+1}, |\xi/c|^\alpha\right\}]$$

holds for $\alpha \geq 2$ with an universal constant C.

For a proof we refer to Sakhanenko [56].

Selbstständigkeitserklärung
Hiermit erkläre ich, dass ich die vorliegende Arbeit mit dem Titel "Change points and uniform confidence for spot volatility" selbstständig angefertigt und keine anderen als die angegebenen Hilfsmittel verwendet habe.

Mannheim, den 30.09.2019

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Mehmet Madensoy

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