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The mean field kinetic equation for interacting particle systems with non-Lipschitz force

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In this paper, we prove the global existence of the weak solution to the mean field kinetic equation derived from the N-particle Newtonian system. For $L^1 \cap L^\infty$ initial data, the solvability of the mean field kinetic equation can be obtained by using uniform estimates and compactness arguments while the difficulties arising from the nonlocal nonlinear interaction are tackled appropriately using the Aubin-Lions compact embedding theorem.

KEYWORDS
mean field limit, partial differential equations, Vlasov-like equations

MSC CLASSIFICATION
35Q83

1 | INTRODUCTION

In this paper, we investigate a two-dimensional kinetic mean field equation for the mass distribution $f(t, x, v)$ with position $x \in \mathbb{R}^2$ and velocity $v \in \mathbb{R}^2$ given by

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot [(F * f)f] + \nabla_v \cdot (Gf) = 0. \tag{1.1}$$

Equation (1.1) is motivated by several applications such as crowd dynamics[1,2] or material flow[3] and has been investigated from a numerical and theoretical point of view, see for example previous studies[4-6] for a general overview. Further extensions might be behavioral models including group dynamics,[7] minimal travel times,[8,9] or evacuation scenarios.[10,11] Model hierarchies for pedestrian and material flow applications have been introduced in Degond et al,[12] Etikyala et al,[13] and Göttlich et al,[3,14] where macroscopic equations are formally derived from a microscopic Newtonian system. Depending on the closure assumption, different nonlocal continuum models can occur, cf Colombo et al.[15] However, from an analytical point of view, there are still open problems that need to be thoroughly investigated as for instance the detailed derivation from the N-particle (pedestrian) Newtonian system to its mean field limit or Vlasov equation, see Chen et al.[16] Instead of the formal derivation with the help of the BBGKY hierarchy,[13,17] the kinetic description has been rigorously derived by a probabilistic method.[18-22]

In this paper, we now aim to prove the global existence of the weak solution to the mean field kinetic Equation (1.1). In the latter equation, $F(x, v)$ denotes the total interaction force and has the similar structure as $\frac{x}{|x|}$, ie,

$$F(x, v) = \nabla_x V(|x|, v) = \partial_r V(r, v)\frac{x}{|x|},$$

where $V(|x|, v)$ is some (regular) potential. More precisely, $F(x, v)$ can be a composition of the interaction force $F_{int}(x)$ and the dissipative force $F_{diss}(x, v)$, ie,

$$F(x, v) = (F_{int}(x) + F_{diss}(x, v))\mathcal{H}(x, v) \tag{1.2}$$

and $\mathcal{H}(x, v) := \mathcal{H}_{2R}(|x|) \cdot \widetilde{\mathcal{H}}_{2\tilde{R}}(|v|)$, where $\mathcal{H}_{2R}(|x|)$ and $\widetilde{\mathcal{H}}_{2\tilde{R}}(|v|)$ are smooth functions with compact support such that

$$\mathcal{H}_{2R}(|x|) = \begin{cases} 0, & |x| > 2R, \\ 1, & |x| < R, \end{cases} \quad \text{and} \quad \widetilde{\mathcal{H}}_{2\tilde{R}}(|v|) = \begin{cases} 0, & |v| > 2\tilde{R}, \\ 1, & |v| < \tilde{R}. \end{cases}$$

In order to cover a realistic behavior of moving crowds, the functions $\mathcal{H}_{2R}(|x|)$ and $\widetilde{\mathcal{H}}_{2\tilde{R}}(|v|)$ are used to express that the interaction force and the velocity of agents are of finite range. So the total force is considered on a bounded domain.

The other term $G(x, v)$ in Equation (1.1) represents the desired velocity and the direction acceleration and can be further written as

$$G(x, v) = g(x) - v, \tag{1.3}$$

where $\|g\|_{L^\infty}$ is bounded by some constant.

Apparently, the proposed model Equation (1.1) involves a singularity comparable with the Coulomb potential in 2-*d*, resulting from the total interaction force. That means that this singularity, or in other words the nonlocal term, needs extra care in the final limiting process. For more information about the Coulomb potential and the Vlasov-Poisson system, we refer to Pfaffelmoser,[23] Rein,[24] and Schaeffer. [25]

We now briefly explain our approach to obtain the existence of the weak solution. First, we consider an approximate problem (kinetic equation with cut-off) and show that the approximate problem has a weak solution, where the mean field characteristic flow is of great importance. Unlike the 3-*d* Vlasov-Poisson equation,[26,27] the nonlocal operator in (1.1) cannot be decoupled into an elliptic equation. Hence, the Calderón-Zygmund continuity theorem[28] for second order elliptic equations is not applicable in this case and we have to find an alternative way to fix the desired compactness arguments. The idea is to use the Aubin-Lions lemma[29,30] and to argue that because of that compact embedding theorem, we are able to pass the limit especially in the nonlocal term. We also remark that the result obtained in the present paper plays a crucial role in the proof of the rigorous derivation of the mean field equation in Chen et al.[16]

This article is organized as follows: In Section 2, we state our main result and further introduce some notations and preliminary work to show that the characteristic flow associated with the cut-off mean field equation admits a unique solution. We also prove the existence and uniqueness of the weak solution to the cut-off mean field equation. Section 3 is concerned with the compactness arguments that are needed to pass the limit and to obtain the desired weak formulation of the non-cut-off kinetic equation. However, the corresponding uniqueness can no longer be kept during the limiting procedure. Finally, we summarize our results.

2 | MEAN FIELD EQUATION WITH CUT-OFF

We start with the definition of a weak solution to the mean field Equation (1.1).

Definition 1. Let $f_0(x, v) \in L^1(\mathbb{R}^2 \times \mathbb{R}^2) \cap L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$. A function $f = f(t, x, v)$ is said to be a weak solution to the kinetic mean field Equation (1.1) with initial data f_0, if there holds

$$\begin{aligned}
&\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(t, x, v)\varphi(x, v)\,dxdv = \iint_{\mathbb{R}^2\times\mathbb{R}^2} f_0(x, v)\varphi(x, v)\,dxdv \\
&+ \int_0^t \iint_{\mathbb{R}^2\times\mathbb{R}^2} v f(s, x, v) \cdot \nabla_x \varphi(x, v)\,dxdvds \\
&+ \int_0^t \iint_{\mathbb{R}^2\times\mathbb{R}^2} (F(x, v) * f(s, x, v))\, f(s, x, v) \cdot \nabla_v \varphi(x, v)\,dxdvds \\
&+ \int_0^t \iint_{\mathbb{R}^2\times\mathbb{R}^2} G(x, v) f(s, x, v) \cdot \nabla_v \varphi(x, v)\,dxdvds
\end{aligned} \tag{2.1}$$

for all $\varphi(x, v) \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ and $t \in \mathbb{R}_+$.

Next, we present the main theorem of this paper. In the following, $G(x, v)$ is given by (1.3) while $F(x, v)$ is defined by (1.2).

Theorem 1. *For* $F(x,v) = \nabla_x V(|x|,v) = \partial_r V(r,v)\frac{x}{|x|}$ *and* $G(x,v) = g(x) - v$*, assume that* $\partial_r V(r,v), \nabla_v \partial_r V(r,v) \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ *and* $g \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$*. Let* $f_0(x,v)$ *be a nonnegative function in* $L^1(\mathbb{R}^2 \times \mathbb{R}^2) \cap L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$*,* $|x|^2 f_0(x,v) \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$*, and*

$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{1}{2}|v|^2 f_0(x,v)\,dxdv =: \mathcal{E}_0 < \infty.$$

Then, there exists a weak solution $f \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^2 \times \mathbb{R}^2))$ *to the mean field Equation (1.1) with initial data* f_0*. Moreover, this solution satisfies*

$$0 \le f(t,x,v) \le \|f_0\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)} e^{Ct}, \quad \text{for a.e.}\ (x,v) \in \mathbb{R}^2 \times \mathbb{R}^2,\ t \ge 0 \tag{2.2}$$

together with the mass conservation

$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(t,x,v)\,dxdv = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f_0(x,v)\,dxdv =: \mathcal{M}_0 \tag{2.3}$$

and the kinetic energy bound

$$\mathcal{E}(t) := \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{1}{2}|v|^2 f(t,x,v)\,dxdv \le C, \quad \forall t \ge 0, \tag{2.4}$$

where the constant C is independent of t.

Under the assumptions above, the interaction force is bounded but not Lipschitz continuous in x. We need to use the standard cut-off to overcome this difficulty. Another difficulty in this context is that the interaction force $F(x,v)$ not only depends on the position x but also on the velocity v. This leads to a totally different structure compared with the Vlasov-Poisson equation, where the $W^{2,p}$ theory for Poisson equations is generally used. The proof of Theorem 1 is therefore not as straightforward and intuitive as expected and therefore needs to be delicately handled step by step within the next sections. On the other hand, the self-generating force (or desired velocity and direction acceleration) $G(x,v)$ is not Lipschitz continuous, which requires an additional work of mollification.

We briefly recall essential assumptions and properties, cf Chen et al,[16] which are necessary for the existence proof.

2.1 | Notations and preliminary work

We consider the flow with cut-off of order $N^{-\theta}$ with arbitrary positive θ, ie,

$$F^N(x,v) = \begin{cases} V'(|x|,v)\frac{x}{|x|}\mathcal{H}(x,v), & |x| \ge N^{-\theta}, \\ N^\theta V'(|x|,v)x\mathcal{H}(x,v), & |x| < N^{-\theta}. \end{cases} \tag{2.5}$$

Then, the mean field cut-off equation becomes

$$\partial_t f^N + v \cdot \nabla_x f^N + \nabla_v \cdot [(F^N * f^N)f^N] + \nabla_v \cdot (G^N f^N) = 0, \tag{2.6}$$

where we also take the cut-off of $G(x,v)$ into consideration, ie,

$$G^N(x,v) = j_{\frac{1}{N}} * g(x) - v$$

with $j_{\frac{1}{N}}(x)$ being the standard mollifier.

We also point out several properties for the interaction force $F^N(x,v)$ and the acceleration $G^N(x,v)$, namely,

(1) $F^N(x,v)$ is bounded, ie, $|F^N(x,v)| \le C$.
(2) $F^N(x,v)$ satisfies

$$|F^N(x,v) - F^N(y,v)| \le q^N(x,v)|x-y|,$$

where q^N has compact support in $B_{2R} \times B_{2\tilde{R}}$ with

$$q^N(x,v) := \begin{cases} C \cdot \frac{1}{|x|} + C, & |x| \ge N^{-\theta}, \\ C \cdot N^\theta, & |x| < N^{-\theta}. \end{cases}$$

(3) $\nabla_v F^N(x, v)$ is uniformly bounded in N.
(4) $|G^N(x, v) - G^N(y, v)| \le C \cdot N \cdot |x - y|$.

Here, we use C as a universal constant that might depend on all the given constants $k_n, R, \tilde{R}, \gamma_n, \gamma_t$.
Furthermore, if there is a singularity in the velocity v in the interaction potential similar to property (2), it can be treated by using the same method as above and the results also apply.

2.2 | Mean field characteristic flow with cut-off

Before we start to prove the existence of the unique weak solution to the Equation (2.6), we need first the following definition.

Definition 2. Let (X_1, Σ_1) and (X_2, Σ_2) be measurable spaces (meaning that Σ_1 and Σ_2 are σ-algebras of the subsets of X_1 and X_2, respectively). Let $T : X_1 \to X_2$ be a (Σ_1, Σ_2)-measurable map and μ be a positive measure on (X_1, Σ_1). Then, the formula

$$\nu(B) := \mu(T^{-1}(B)), \quad \forall B \in \Sigma_2$$

defines a positive measure on (X_2, Σ_2), denoted by

$$\nu =: T\#\mu,$$

and is referred to as the push-forward of the measure μ under the map T.

The definition is often used when it comes to solving mean field characteristic flow. For more detailed information, we refer to Golse.[28] Because of the property of the transport equation, we know that solving Equation (2.6) is equivalent to investigating the corresponding characteristic system, ie,

$$\begin{cases} \frac{d}{dt} Z(t, z_0, \mu_0) = \int_{\mathbb{R}^4} K\left(Z(t, z_0), z'\right) \mu(t, dz'), \\ Z(0, z_0, \mu_0) = z_0, \end{cases} \tag{2.7}$$

where

$$K^N(z, z') = K^N(x, v, x', v') := \left(v, F^N(x - x', v - v') + G^N(x, v)\right)$$

and $\mu(t, \cdot)$ is the push-forward of the measure μ_0. Here, for the sake of convenience, we use $z = (x, v)$ and Z as the four-dimensional vector.
We denote $\mathcal{P}(\mathbb{R}^4)$ as the set of Borel probability measures on $\mathbb{R}^4$, and $\mathcal{P}_1(\mathbb{R}^4)$ is defined by

$$\mathcal{P}_1(\mathbb{R}^4) := \left\{ \mu \in \mathcal{P}(\mathbb{R}^4) \,\middle|\, \int_{\mathbb{R}^4} |v| \mu(dx, dv) < \infty \right\}.$$

Proposition 1. *Assume that the interaction kernel $K(z, z') \in C(\mathbb{R}^4 \times \mathbb{R}^4; \mathbb{R}^4)$ is Lipschitz continuous in z, uniformly in z' (and conversely), ie, there exists a constant $L > 0$ such that*

$$\sup_{z' \in \mathbb{R}^4} |K(z_1, z') - K(z_2, z')| \le L|z_1 - z_2|,$$
$$\sup_{z \in \mathbb{R}^4} |K(z, z_1) - K(z, z_2)| \le L|z_1 - z_2|.$$

For any given $z_0 = (x_0, v_0) \in \mathbb{R}^2 \times \mathbb{R}^2$ and Borel probability measure $\mu_0 \in \mathcal{P}_1(\mathbb{R}^4)$, there exists a unique C^1-solution, denoted by

$$\mathbb{R}_+ \ni t \mapsto Z(t, z_0, \mu_0) \in \mathbb{R}^4,$$

to the problem

$$\begin{cases} \frac{d}{dt} Z(t, z_0, \mu_0) = \int_{\mathbb{R}^4} K\left(Z(t, z_0), z'\right) \mu(t, dz'), \\ Z(0, z_0, \mu_0) = z_0, \end{cases} \tag{2.8}$$

where $\mu(t, \cdot)$ is the push-forward of the measure μ_0, ie, $\mu(t, \cdot) = Z(t, \cdot, \mu_0)\#\mu_0$.

This proposition is typically obtained via the standard argument using the Banach Fixed-Point Theorem, see Golse.[28]

With Proposition 1, we are now able to prove that there exists a unique weak solution to the Vlasov equation with cut-off (2.6).

Theorem 2. *Let F and G satisfy the same assumptions as in Theorem 1 and f_0^N be a nonnegative compactly supported function in $L^1(\mathbb{R}^2 \times \mathbb{R}^2) \cap L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ satisfying*

$$\|f_0^N\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)} = \mathcal{M}_0 \quad and \quad f_0^N(x,v) \le \|f_0\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)},$$

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{2}|v|^2 f_0^N(x,v)\,dxdv \le \mathcal{E}_0 < \infty,$$

and

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{2}|x|^2 f_0^N(x,v)\,dxdv \le \mathcal{M}_2 < \infty.$$

Then, there exists a unique weak solution $f^N \in C^1(\mathbb{R}_+; L^1(\mathbb{R}^2 \times \mathbb{R}^2))$ to the mean field cut-off Equation (2.6) with initial data f_0^N, ie, $f^N(t,x,v)$ satisfies

$$\begin{aligned}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \partial_t f^N(t,x,v)\varphi(x,v)\,dxdv &= \iint_{\mathbb{R}^2\times\mathbb{R}^2} v f^N(t,x,v)\cdot\nabla_x\varphi(x,v)\,dxdv \\ &+ \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left(F^N(x,v) * f^N(t,x,v)\right) f^N(s,x,v)\cdot\nabla_v\varphi(x,v)\,dxdv \\ &+ \iint_{\mathbb{R}^2\times\mathbb{R}^2} G^N(x,v) f^N(t,x,v)\cdot\nabla_v\varphi(x,v)\,dxdv\end{aligned} \tag{2.9}$$

for all $\varphi(x,v) \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$. Moreover, this solution satisfies

$$\lim_{t\to 0} f^N(t,x,v) = f_0^N(x,v), \quad \text{for a.e.}\,(x,v) \in \mathbb{R}^2\times\mathbb{R}^2,$$

$$0 \le f^N(t,x,v) \le \|f_0^N\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} e^{Ct}, \quad \text{for a.e.}\,(x,v) \in \mathbb{R}^2\times\mathbb{R}^2,\ t \ge 0 \tag{2.10}$$

together with the mass conservation

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} f^N(t,x,v)\,dxdv = \iint_{\mathbb{R}^2\times\mathbb{R}^2} f_0^N(x,v)\,dxdv =: \mathcal{M}_0, \tag{2.11}$$

the kinetic energy bound

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{2}|v|^2 f^N(t,x,v)\,dxdv \le C, \quad \forall t \ge 0, \tag{2.12}$$

and the bound of second moment

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{2}|x|^2 f^N(t,x,v)\,dxdv \le \mathcal{M}_2 e^{Ct}, \quad \forall t \ge 0, \tag{2.13}$$

where the constant C is independent of N and t.

Proof. Without loss of generality, we assume that $\mathcal{M}_0 = 1$. If we choose the interaction kernel K as

$$K^N(z,z') = K^N(x,v,x',v') := \left(v, F^N(x-x',v-v') + G^N(x,v)\right),$$

the mean field cut-off Eq. (2.6) can be put into the form

$$\partial_t f^N(t,z) + \mathrm{div}_z\left(f^N(t,z)\iint_{\mathbb{R}^2\times\mathbb{R}^2} K^N(z,z') f^N(t,z')\,dz'\right) = 0.$$

Notice that the nonlinear nonlocal dynamical system that appears in Proposition 1 is exactly the equation of characteristics for the mean field kinetic equation with cut-off (2.6), which we refer to as the mean field characteristic flow (with cut-off). The existence and uniqueness of the solution to (2.6) are therefore achieved as a direct result of the construction of the mean field characteristic flow. By Proposition 1, there exists a unique map

$$\mathbb{R}_+ \times \mathbb{R}^4 \times \mathcal{P}_1(\mathbb{R}^4) \ni (t, z_0, \mu_0) \mapsto Z^N(t, z_0, \mu_0) \in \mathbb{R}^4$$

such that $t \mapsto Z^N(t, z_0, \mu_0)$ is the integral curve of the vector field

$$z \mapsto \iint_{\mathbb{R}^2\times\mathbb{R}^2} K^N(z, z')\mu^N(t, dz')$$

passing through z_0 at time $t = 0$, where $\mu^N(t) := Z^N(t, \cdot, \mu_0)\#\mu_0$. For the given initial data f_0^N, letting $d\mu_0 = f_0^N dz$ results in

$$f^N(t, z) := f_0^N\left(Z^N(t, \cdot)^{-1}(z)\right) J(0, t, z), \quad \forall t \geq 0,$$

where $J(0, t, z)$ is the Jacobian, ie,

$$J(0, t, z) = \exp\left(\int_t^0 \operatorname{div}_v\left(F^N * f^N(s, Z^N(s, z)) + G^N(Z^N(s, z))\right) ds\right).$$

Then we have

$$\begin{aligned}
|f^N(t, z)| \leq& |f_0^N\left(Z^N(t, \cdot)^{-1}(z)\right) J(0, t, z)| \\
\leq& \|f_0^N\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \exp\left(\int_0^t \|\nabla_v F^N * f^N\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} ds + Ct\right) \\
\leq& \|f_0^N\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \exp\left(\int_0^t \|\nabla_v F^N\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}\|f^N\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)} ds + Ct\right) \\
\leq& \|f_0^N\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} e^{Ct},
\end{aligned}$$

where we have used the property of the acceleration $G^N(x, v)$, ie, $G^N(x, v) = j_{\frac{1}{N}} * g(x) - v$, where $j_{\frac{1}{N}} * g(x)$ is an L^∞-function. From the equation, (2.11) are straightforward. Property (2.12) is left to be proven. For the kinetic energy estimate, we will again use the property of the acceleration $G^N(x, v)$ and remark that v in $G^N(x, v)$ is critical in the estimate because it serves as a damping term. We now choose $\{\varphi_\eta(x)\phi_\eta(v)\}$ to be a smooth function which satisfies

$$\varphi_\eta(x) = \begin{cases} 0, & |x| > \frac{1}{\eta}, \\ 1, & |x| < \frac{1}{2\eta}, \end{cases} \quad \text{and} \quad \phi_\eta(v) = \begin{cases} 0, & |v| > \frac{1}{\eta}, \\ 1, & |v| < \frac{1}{2\eta}, \end{cases}$$

and

$$\left|\nabla_z\left(\varphi_\eta(x)\phi_\eta(v)\right)\right| \leq \eta\left|\varphi_\eta(x)\phi_\eta(v)\right|.$$

Since $\varphi_\eta(x)\phi_\eta(v)$ is monotone and converges to one for almost all x and v as η goes to 0, we have

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} v^2 f^N(t, x, v)\varphi_\eta(x)\phi_\eta(v)\, dxdv \to \iint_{\mathbb{R}^2\times\mathbb{R}^2} v^2 f^N(t, x, v)\, dxdv, \quad \text{as} \eta \to 0.$$

The compact support of f_0^N implies that $f^N(t,x,v)$ has compact support in (x,v) for any fixed time t. By the definition of weak solution for test functions $v^2\varphi_\eta(x)\phi_\eta(v)$, we have

$$
\begin{aligned}
&\frac{d}{dt}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\frac{1}{2}v^2 f^N(t,x,v)\varphi_\eta(x)\phi_\eta(v)\,dxdv\\
=&\frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} v f^N(t,x,v)\cdot\nabla_x\left(v^2\varphi_\eta(x)\phi_\eta(v)\right)\,dxdv\\
&+\frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\left(F^N(x,v)*f^N(t,x,v)\right)f^N(s,x,v)\cdot\nabla_v\left(v^2\varphi_\eta(x)\phi_\eta(v)\right)\,dxdv\\
&+\frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} G^N(x,v)f^N(t,x,v)\cdot\nabla_v\left(v^2\varphi_\eta(x)\phi_\eta(v)\right)\,dxdv
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} v^2 f^N(t,x,v)\phi_\eta(v)v\cdot\nabla_x\left(\varphi_\eta(x)\right)\,dxdv\\
&+\iint_{\mathbb{R}^2\times\mathbb{R}^2} v\left(F^N(x,v)*f^N(t,x,v)\right)f^N(t,x,v)\varphi_\eta(x)\phi_\eta(v)\,dxdv\\
&+\frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} v^2\left(F^N(x,v)*f^N(t,x,v)\right)f^N(s,x,v)\cdot\nabla_v\left(\varphi_\eta(x)\phi_\eta(v)\right)\,dxdv\\
&+\iint_{\mathbb{R}^2\times\mathbb{R}^2} v\cdot G^N(x,v)f^N(t,x,v)\varphi_\eta(x)\phi_\eta(v)\,dxdv\\
&+\frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} v^2 G^N(x,v)f^N(t,x,v)\cdot\nabla_v\left(\varphi_\eta(x)\phi_\eta(v)\right)\,dxdv\\
=:&\sum_{j=1}^{5} I_j.
\end{aligned}
$$

Next, we estimate the expressions $I_j, j=1,\ldots,5$ individually. It is easy to see

$$
\begin{aligned}
|I_1| &\le \frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\left|v^2 f^N(t,x,v)\phi_\eta(v)v\cdot\nabla_x\left(\varphi_\eta(x)\right)\right|\,dxdv\\
&\le \frac{1}{2}\eta\iint_{\mathbb{R}^2\times\mathbb{R}^2}|v|^3 f^N(t,x,v)|\phi_\eta(v)\varphi_\eta(x)|\,dxdv.
\end{aligned}
$$

Because of the fact that f_0^N is compactly supported, ie, f^N has also compact support for any finite time t, I_1 converges to zero as $\eta\to 0$ for fixed N. The same argument holds for I_3 and I_5, ie, I_3 and I_5 converge to zero as $\eta\to 0$:

$$
\begin{aligned}
|I_3| \le&\frac{1}{2}\cdot C\eta\|F^N * f^N\|_{L^\infty}\iint_{\mathbb{R}^2\times\mathbb{R}^2} v^2 f^N(t,x,v)\varphi_\eta(x)\phi_\eta(v)\,dxdv\\
\le&\frac{1}{2}\cdot C\eta\|F^N\|_{L^\infty}\|f^N\|_{L^1}\iint_{\mathbb{R}^2\times\mathbb{R}^2} v^2 f^N(t,x,v)\varphi_\eta(x)\phi_\eta(v)\,dxdv\\
I_5 \le&\frac{1}{2}\cdot\eta\|j_{\frac{1}{N}} * g\|_{L^\infty}\iint_{\mathbb{R}^2\times\mathbb{R}^2} v^2 f^N(t,x,v)\varphi_\eta(x)\phi_\eta(v)\,dxdv\\
&-\frac{1}{2}\eta\iint_{\mathbb{R}^2\times\mathbb{R}^2}|v|^3 f^N(t,x,v)\phi_\eta(v)\varphi_\eta(x)\,dxdv.
\end{aligned}
$$

However, for the other integral estimates, we need some extra calculations. Using the properties of the desired velocity and direction acceleration $G^N(x, v)$, we arrive at

$$\begin{aligned}
I_2 &\le \|F^N * f^N\|_{L^\infty} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left(\frac{1}{4\epsilon} + \epsilon v^2\right) f^N(t,x,v)\varphi_\eta(x)\phi_\eta(v)\,dxdv \\
&\le \|F^N\|_{L^\infty}\|f^N\|_{L^1} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left(\frac{1}{4\epsilon} + \epsilon v^2\right) f^N(t,x,v)\varphi_\eta(x)\phi_\eta(v)\,dxdv \\
I_4 &\le \|j_{\frac{1}{N}} * g\|_{L^\infty} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left(\frac{1}{4\epsilon} + \epsilon v^2\right) f^N(t,x,v)\varphi_\eta(x)\phi_\eta(v)\,dxdv \\
&\quad - \iint_{\mathbb{R}^2\times\mathbb{R}^2} v^2 f^N(t,x,v)\varphi_\eta(x)\phi_\eta(v)\,dxdv.
\end{aligned}$$

Combining all the five terms, taking η to zero in the inequality above and setting ϵ small enough such that

$$\epsilon < \frac{1}{2(\|F^N\|_{L^\infty}\|f^N\|_{L^1} + \|g\|_{L^\infty})},$$

where the fact that $\|j_{\frac{1}{N}} * g\|_{L^\infty} \le \|g\|_{L^\infty}$ has been used, we end up with

$$\frac{d}{dt}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{2}v^2 f^N(t,x,v)\,dxdv \le C - \iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{2}v^2 f^N(t,x,v)\,dxdv,$$

where C does not depend on N. A direct computation shows that the kinetic energy is bounded uniformly in t and N. The estimate for the second moment follows from

$$\begin{aligned}
\frac{d}{dt}\iint_{\mathbb{R}^2\times\mathbb{R}^2} |x|^2 f^N(t,x,v)\,dxdv &= \iint_{\mathbb{R}^2\times\mathbb{R}^2} |x|^2 \partial_t f^N(t,x,v)\,dxdv \\
&= \iint_{\mathbb{R}^2\times\mathbb{R}^2} x\cdot v f^N(t,x,v)\,dxdv \\
&\le \iint_{\mathbb{R}^2\times\mathbb{R}^2} (|x|^2 + |v|^2) f^N(t,x,v)\,dxdv \\
&\le \iint_{\mathbb{R}^2\times\mathbb{R}^2} |x|^2 f^N(t,x,v)\,dxdv + C.
\end{aligned}$$

□

3 | COMPACTNESS ARGUMENTS

In this section, we aim to achieve all the compactness arguments that are needed to pass the limit and to obtain the desired weak formulation of the non-cut-off kinetic equation, namely, to prove the main result Theorem 2.1.

For the given initial data f_0, let f_0^N be a sequence of functions with compact support which are w.l.o.g. assumed to be in B_N, ie, a ball of radius N centered at the origin. Furthermore, f_0^N satisfies

$$\|f_0^N - f_0\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \to 0, \text{as} N \to \infty.$$

Let $f^N(t, x, v)$ be the solution obtained from Theorem 2 with initial data $f_0^N(x, v)$. Then, we know

$$0 \le f^N(t,x,v) \le \|f_0\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} e^{Ct}, \quad \text{for a.e.} (x,v) \in \mathbb{R}^2\times\mathbb{R}^2,\ t \ge 0,$$

and for any fixed $T > 0$, there exists a subsequence of f^N, still denoted by f^N for simplicity, such that

$$f^N \overset{*}{\rightharpoonup} f \quad \text{in} L^\infty((0,T); L^\infty(\mathbb{R}^2\times\mathbb{R}^2)).$$

Because of the tightness in the variable x and v of the sequence f^N, implied from (2.12) and (2.13), we conclude that $f \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$. Moreover, we notice that the total mass is preserved, ie,

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)\,dxdv = \iint_{\mathbb{R}^2\times\mathbb{R}^2} f_0^N(x,v)\,dxdv =: \mathcal{M}_0.$$

By the definition of weak* convergence for characteristic functions $\chi_{|x|+|v|\le r} \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$, we have for each $a < b \in \mathbb{R}_+$

$$\begin{aligned}
&\int_a^b \iint_{\mathbb{R}^2\times\mathbb{R}^2} \chi_{|x|+|v|\le r} f(t,x,v)\,dxdvdt \\
&= \lim_{N\to\infty} \int_a^b \iint_{\mathbb{R}^2\times\mathbb{R}^2} \chi_{|x|+|v|\le r} f^N(t,x,v)\,dxdvdt \\
&\le \lim_{N\to\infty} \int_a^b \iint_{\mathbb{R}^2\times\mathbb{R}^2} f^N(t,x,v)\,dxdvdt = \mathcal{M}_0(b-a).
\end{aligned}$$

Letting $r \to \infty$ and applying Fatou's lemma yields

$$\begin{aligned}
&\int_a^b \iint_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)\,dxdvdt \\
&\le \lim_{r\to\infty} \int_a^b \iint_{\mathbb{R}^2\times\mathbb{R}^2} \chi_{|x|+|v|\le r} f(t,x,v)\,dxdvdt \\
&\le \lim_{N\to\infty} \int_a^b \iint_{\mathbb{R}^2\times\mathbb{R}^2} f^N(t,x,v)\,dxdvdt = \mathcal{M}_0(b-a).
\end{aligned}$$

By a similar argument for test functions of type $\chi_{|x|+|v|\le r}|v|^2$, we can show that

$$\int_a^b \iint_{\mathbb{R}^2\times\mathbb{R}^2} |v|^2 f(t,x,v)\,dxdvdt \le C(b-a)$$

by using

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{2}|v|^2 f^N(t,x,v)\,dxdv \le C(b-a), \quad \forall t \ge 0.$$

Since the above two inequalities hold for all $a < b \in \mathbb{R}_+$, they also hold for a.e. $t \in \mathbb{R}_+$.

Using all the estimates presented in Theorem 2, we are now ready to pass the limit in (2.6) to the desired weak formulation of the non-cut-off kinetic equation

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot [(F * f)f] + \nabla_v \cdot (Gf) = 0.$$

However, we need to take special care on the non-linear term, i.e., the consideration of the function $F^N * f^N$. In the following, we use the notation $L^p(L^q)$ to denote $L^p([0,T]; L^q(\mathbb{R}^2 \times \mathbb{R}^2))$, $1 \le p, q \le \infty$. It is obvious to see that

$$\begin{aligned}
&\|F^N * f^N\|_{L^\infty(L^1)} \\
&= \left\| \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left(\iint_{\mathbb{R}^2\times\mathbb{R}^2} F^N(x-y, v-w) f^N(t,y,w)\,dydw \right) dxdv \right\|_{L^\infty([0,T])} \\
&= \left\| \iint_{\mathbb{R}^2\times\mathbb{R}^2} f^N(t,y,w) \left(\iint_{\mathbb{R}^2\times\mathbb{R}^2} F^N(x-y, v-w)\,dxdv \right) dydw \right\|_{L^\infty([0,T])} \\
&\le C\left(\|F\|_{L^1}, \mathcal{M}_0, \bar{R} \right)
\end{aligned}$$

and

$$\|F^N * f^N\|_{L^\infty(L^\infty)} = \left\| \iint_{\mathbb{R}^2\times\mathbb{R}^2} F^N(x-y, v-w) f^N(t,y,w)\, dydw \right\|_{L^\infty(L^\infty)} \le C\left(\|F\|_{L^\infty}, \mathcal{M}_0\right).$$

Since $\nabla_v F^N$ is bounded uniformly in N, we get

$$\begin{aligned}
&\|\nabla_v\left(F^N * f^N\right)\|_{L^\infty(L^1)} \\
&= \left\| \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left(\iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla_v F^N(x-y, v-w) f^N(t,y,w)\, dydw \right) dxdv \right\|_{L^\infty(\mathbb{R}_+)} \\
&= \left\| \iint_{\mathbb{R}^2\times\mathbb{R}^2} f^N(t,y,w) \left(\iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla_v F^N(x-y, v-w)\, dxdv \right) dydw \right\|_{L^\infty(\mathbb{R}_+)} \\
&\le C\left(\|\nabla_v F\|_{L^1}, \mathcal{M}_0, \bar{R}\right)
\end{aligned}$$

and

$$\|\nabla_v\left(F^N * f^N\right)\|_{L^\infty(L^\infty)} = \left\| \iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla_v F^N(x-y, v-w) f^N(t,y,w)\, dydw \right\|_{L^\infty(L^\infty)} \le C\left(\|\nabla_v F\|_{L^\infty}, \mathcal{M}_0\right).$$

So far, we can conclude by interpolation that $F^N * f^N$ and $\nabla_v F^N * f^N$ are in $L^\infty(L^2)$. Furthermore, it holds

$$\|\nabla_x\left(F^N * f^N\right)\|_{L^\infty(L^2)} \le C \cdot \left\| \left(\chi_{\bar{R}} \cdot \frac{1}{|x|}\right) * f^N \right\|_{L^\infty(L^2)} \le \|f^N\|_{L^\infty(L^p)}, \quad \forall p > 1,$$

where $\chi_{\bar{R}} \cdot \frac{1}{|x|} \in L^r, \forall 1 < r < 2$, and the Young inequality has been used. Hence, we conclude that $F^N * f^N$ then belongs to $L^\infty(\mathbb{R}_+; W^{1,2}(\mathbb{R}^2\times\mathbb{R}^2))$. Since

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} \left(v f^N(t,x,v)\right)^2 dxdv \le \|f^N\|_{L^\infty} \|v^2 f^N\|_{L^\infty(L^1)} \le C(T),$$

we can get for every $\varphi \in C_0^\infty(\mathbb{R}^2\times\mathbb{R}^2)$ that

$$\begin{aligned}
&\left\| \iint_{\mathbb{R}^2\times\mathbb{R}^2} v f^N(t,x,v) \nabla_x \varphi(x,v)\, dxdv \right\|_{L^\infty(\mathbb{R}_+)} \\
&\le \|f^N\|_{L^\infty(L^\infty)}^{\frac{1}{2}} \cdot \|v^2 f^N\|_{L^\infty(L^1)}^{\frac{1}{2}} \cdot \|\nabla_x \varphi\|_{L^2} \\
&\le C(T)\|\nabla_x \varphi\|_{L^2}.
\end{aligned} \tag{3.1}$$

Moreover, we have

$$\begin{aligned}
&\left\| \iint_{\mathbb{R}^2\times\mathbb{R}^2} G^N(x,v) f^N(t,x,v) \nabla_v \varphi(x,v)\, dxdv \right\|_{L^\infty(\mathbb{R}_+)} \\
&\le \|j_{\frac{1}{N}} * g\|_{L^\infty} \cdot \|f^N\|_{L^\infty(L^\infty)}^{\frac{1}{2}} \cdot \|f^N\|_{L^\infty(L^1)}^{\frac{1}{2}} \cdot \|\nabla_v \varphi\|_{L^2} \\
&\quad + \|f^N\|_{L^\infty(L^\infty)}^{\frac{1}{2}} \cdot \|v^2 f^N\|_{L^\infty(L^1)}^{\frac{1}{2}} \cdot \|\nabla_v \varphi\|_{L^2} \\
&\le \|g\|_{L^\infty} \cdot \|f^N\|_{L^\infty(L^\infty)}^{\frac{1}{2}} \cdot \|f^N\|_{L^\infty(L^1)}^{\frac{1}{2}} \cdot \|\nabla_v \varphi\|_{L^2} \\
&\quad + \|f^N\|_{L^\infty(L^\infty)}^{\frac{1}{2}} \cdot \|v^2 f^N\|_{L^\infty(L^1)}^{\frac{1}{2}} \cdot \|\nabla_v \varphi\|_{L^2} \\
&\le C(T)\|\nabla_v \varphi\|_{L^2}.
\end{aligned} \tag{3.2}$$

On the other hand, we know

$$
\begin{aligned}
&\left\| \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left(F^N * f^N\right)(t,x,v)\cdot f^N(t,x,v)\nabla_v\varphi(x,v)\,dxdv \right\|_{L^\infty(\mathbb{R}_+)} \\
&\le \|F^N * f^N\|_{L^\infty(L^\infty)}\cdot \|f^N\|_{L^\infty(L^2)}\cdot\|\nabla_v\varphi\|_{L^2} \\
&\le C\|\nabla_v\varphi\|_{L^2}.
\end{aligned}
\tag{3.3}
$$

Combining (3.1)-(3.3), it holds for every $\varphi\in C_0^\infty\left(\mathbb{R}^2\times\mathbb{R}^2\right)$ that

$$
\begin{aligned}
&\left\| \iint_{\mathbb{R}^2\times\mathbb{R}^2} \partial_t f^N(t,x,v)\varphi(x,v)\,dxdv \right\|_{L^\infty(\mathbb{R}_+)} \\
\le &\left\| \iint_{\mathbb{R}^2\times\mathbb{R}^2} v f^N(t,x,v)\nabla_x\varphi(x,v)\,dxdv \right\|_{L^\infty(\mathbb{R}_+)} \\
&+\left\| \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left(F^N * f^N\right)(t,x,v)\cdot f^N(t,x,v)\nabla_v\varphi(x,v)\,dxdv \right\|_{L^\infty(\mathbb{R}_+)} \\
&+\left\| \iint_{\mathbb{R}^2\times\mathbb{R}^2} G^N(x,v) f^N(t,x,v)\nabla_v\varphi(x,v)\,dxdv \right\|_{L^\infty(\mathbb{R}_+)} \\
\le &C\|\varphi\|_{W^{1,2}},
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\left\| \iint_{\mathbb{R}^2\times\mathbb{R}^2} \partial_t\left((F^N * f^N)(t,x,v)\right)\varphi(x,v)\,dxdv \right\|_{L^\infty(\mathbb{R}_+)} \\
&= \left\| \iint_{\mathbb{R}^2\times\mathbb{R}^2} \partial_t f^N(t,x,v)(F^N * \varphi)(x,v)\,dxdv \right\|_{L^\infty(\mathbb{R}_+)} \\
&\le C\|F^N * \varphi\|_{W^{1,2}} \\
&= C\left\| \iint_{\mathbb{R}^2\times\mathbb{R}^2} F^N(y,w)\varphi(x-y,v-w)\,dydw \right\|_{W^{1,2}} \\
&\le C\|F^N\|_{L^\infty}\|\varphi\|_{W^{1,2}} \\
&\le C\|F\|_{L^\infty}\|\varphi\|_{W^{1,2}}
\end{aligned}
$$

or, in other words,

$$
\|\partial_t(F^N * f^N)\|_{L^\infty(W^{-1,2})} = \|F^N * \partial_t f^N\|_{L^\infty(W^{-1,2})} \le C.
$$

We then get $\forall\varphi\in C_0^\infty(\mathbb{R}^2\times\mathbb{R}^2)$

$$
F^N * f^N \in L^\infty([0,T];W^{1,2}(\Omega)),\quad \partial_t(F^N * f^N)\in L^\infty([0,T];W^{-1,2}(\Omega)),
$$

where $\Omega = \mathrm{supp}\varphi$. According to Aubin-Lions compact embedding theorem, eg,[29,30] there exists a subsequence and $h\in L^\infty([0,T];L^2(\Omega))$ such that

$$
F^N * f^N \to h \quad \text{in}\, L^\infty([0,T];L^2(\Omega)).
$$

It is not difficult to check that $h = F * f$. Therefore, we obtain the following estimates:

$$
\begin{aligned}
&\left|\int_0^t \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left(\left((F^N * f^N)f^N\right)(s,x,v)\nabla_v\varphi(x,v) - ((F*f)f)(s,x,v)\nabla_v\varphi(x,v)\right) dxdvds\right| \\
&= \left|\int_0^t \iint_{\Omega} \left(\left((F^N * f^N)f^N\right)(s,x,v)\nabla_v\varphi(x,v) - \left((F*f)f^N\right)(s,x,v)\nabla_v\varphi(x,v)\right.\right. \\
&\quad \left.\left. + \left((F*f)f^N\right)(s,x,v)\nabla_v\varphi(x,v) - ((F*f)f)(s,x,v)\nabla_v\varphi(x,v)\right) dxdvds\right| \\
&\le \left|\int_0^t \iint_{\Omega} \left(\left((F^N * f^N)f^N\right)(s,x,v)\nabla_v\varphi(x,v) - \left((F*f)f^N\right)(s,x,v)\nabla_v\varphi(x,v)\right) dxdvds\right| \\
&\quad + \left|\int_0^t \iint_{\Omega} \left(\left((F*f)f^N\right)(s,x,v)\nabla_v\varphi(x,v) - ((F*f)f)(s,x,v)\nabla_v\varphi(x,v)\right) dxdvds\right| \\
&=: J_1 + J_2.
\end{aligned}
$$

For the first term J_1, we have

$$
\lim_{N\to\infty} J_1 \le \lim_{N\to\infty} \|F^N * f^N - F * f\|_{L^\infty(L^2(\Omega))}\|f^N\|_{L^\infty(L^\infty)}\|\nabla_v\varphi\|_{L^2} = 0,
$$

while for the second term J_2, we use the fact that $f^N \overset{*}{\rightharpoonup} f$ in $L^\infty(\mathbb{R}_+; L^\infty(\mathbb{R}^2\times\mathbb{R}^2))$ for $F * f \cdot \nabla_v\varphi \in L^1(L^1)$, namely

$$
\lim_{N\to\infty} J_2 = 0.
$$

Finally, we have to examine the initial data. Since f^N is the weak solution to the cut-off mean field Equation (2.6), it obviously satisfies

$$
\begin{aligned}
\iint_{\mathbb{R}^2\times\mathbb{R}^2} f^N(t,x,v)\varphi(x,v)\,dxdv &= \iint_{\mathbb{R}^2\times\mathbb{R}^2} f_0^N(x,v)\varphi(x,v)\,dxdv \\
&+ \int_0^t \iint_{\mathbb{R}^2\times\mathbb{R}^2} v f^N(s,x,v)\cdot\nabla_x\varphi(x,v)\,dxdvds \\
&+ \int_0^t \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left(F^N(x,v) * f^N(s,x,v)\right) f^N(s,x,v)\cdot\nabla_v\varphi(x,v)\,dxdvds \\
&+ \int_0^t \iint_{\mathbb{R}^2\times\mathbb{R}^2} G^N(x,v) f^N(s,x,v)\cdot\nabla_v\varphi(x,v)\,dxdvds
\end{aligned}
$$

for any test function $\varphi(x,v) \in C_0^\infty(\mathbb{R}^2\times\mathbb{R}^2)$. We recall

$$
\|f_0^N - f_0\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \to 0, \text{as} N \to \infty,
$$

and that terms on the right (second till last) hand side are uniformly continuous in time t. Then, taking limit $t \to 0^+$ on both sides of the above equation verifies the initial data.

4 | SUMMARY

This paper deals with the core problem, which is to show existence of the $L^\infty((0,\infty); L^\infty(\mathbb{R}^2\times\mathbb{R}^2))$-solution to the mean field kinetic equation for interacting particle systems with non-Lipschitz force. Our main results, Theorem 1 and Theorem 2, state that there exists a weak solution to the mean field equation (or approximate equation with cut-off) to the interaction flow model. The solution is proven to satisfy the mass conservation and energy bounds, respectively. In particular, this paper addresses technical difficulties caused by the non-Lipschitz continuous interaction force and self-generating force.

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CONFLICT OF INTEREST

This work does not have any conflict of interest.

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