

Optimal inflow control in transport systems with uncertain demands - A comparison of undersupply penalties

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We address the challenging task of setting up an optimal production plan taking into account uncertain demand. The transport process is represented by the linear advection equation and the uncertain demand stream is captured by an Ornstein-Uhlenbeck process (OUP). With a model predictive control approach, we determine the optimal inflow. We use two types of undersupply penalties and compare the average undersupply as well as the number of undersupply cases in a numerical simulation.

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1 Stochastic optimal control problem

Accounting for uncertainty is crucial in the modeling of many real-world processes. One example is the increasing interest in solutions accounting for large fluctuations in renewable energy generation (see e.g. [1]). Another source of uncertainty in the production context arises in terms of the unknown demand for goods (see e.g. [2]). A reliable supply is crucial to avoid high costs of short-term external purchase and for the company's reputation. We focus on the optimal inflow control in transport systems avoiding undersupply. We use the stochastic optimal control framework set up in [3].

1.1 Supply system and demand dynamics

We consider a *supply system* where goods are transported from left ($x = 0$) to right ($x = 1$) with a constant velocity λ , i.e. goods feeded into the system need $1/\lambda$ time units to pass from left to right. Mathematically, this transportation process is described by the linear advection equation with initial and boundary condition given by

$$z_t + \lambda z_x = 0, \quad x \in (0, 1), \quad t \in [0, T], \quad z(x, 0) = 0, \quad z(0, t) = u(t), \quad t \in [0, T - 1/\lambda], \quad (1)$$

where $u \in L^2([0, T - 1/\lambda])$ is the inflow control. We set $u(0) = 0$. The output of the system is $y(t) = z(1, t)$. We aim at determining the inflow control $u(t)$ such that the resulting supply $y(t)$ optimally matches the uncertain demand $(Y_t)_{t \in [0, T]}$.

The *demand dynamics* are described by an Ornstein-Uhlenbeck process (OUP) being the unique strong solution of

$$dY_t = \kappa (\mu(t) - Y_t) dt + \sigma dW_t, \quad Y_0 = y_0. \quad (2)$$

W_t is a one-dimensional Brownian motion, $\sigma > 0$, $\kappa > 0$ are constants, and y_0 is the initial demand. The OUP is a popular stochastic process in demand modeling (see e.g. [4] for electricity demand modeling). Its mean reverting property allows the interpretation as random fluctuations around a given mean demand level $\mu(t)$ possibly obtained from historical demand data. For further details and an extension of it including the possibility of jumps, please see [3].

The resulting *stochastic optimal control (SOC) problem* can be stated as

$$\min_{u \in L^2([0, T - 1/\lambda])} \int_{1/\lambda}^T OF(Y_s, t_0, y_{t_0}, y(s)) ds \text{ subject to (1) and (2).} \quad (3)$$

Here, we assume $u(t)$ to be $\mathcal{F}_{\hat{t}_i}$ -measurable for $t \in [\hat{t}_i, \hat{t}_{i+1}]$, where $\hat{t}_i = i \cdot \Delta t_{up}$, $i \in \{0, 1, \dots, T - 1/\lambda / \Delta t_{up}\}$, and $\Delta t_{up} \in [0, T - 1/\lambda]$ is the update frequency. Then, $0 = \hat{t}_0 < \hat{t}_1 < \dots < \hat{t}_n \leq T - 1/\lambda$ specifies the grid of update times, where the current demand at the market is observed. Accordingly, we subdivide our optimization horizon $[0, T]$ into subintervals $[\hat{t}_i, \hat{t}_{i+1}]$ and solve the SOC problem of type (3) thereon with updated state of the supply system and updated initial demand (see control method CM2 in [3]).

1.2 Choice of cost function

Our focus is the choice of the cost function for the optimization. We will compare different types of undersupply penalties in terms of different cost functions $OF(Y_s, t_0, y_{t_0}, y(s))$. The first one is taken from [5]:

$$OF_{penC}(Y_s, t_0, y_{t_0}, y(s)) = \underbrace{\mathbb{E} [(Y_s - y(s))^2 | Y_{t_0} = y_{t_0}]}_I + \alpha \cdot \underbrace{\mathbb{E} [(Y_s - y(s))^2 | Y_s > y(s) \wedge Y_{t_0} = y_{t_0}]}_{II}. \quad (4)$$

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I is of tracking type and II represents a multiple α of the expected quadratic undersupply given the information of being in an undersupply case. The multiple α gives the possibility to control the intensity of penalization. For the effect of different intensities on the optimal supply, we refer the reader to [5]. We formulate an alternative penalization of undersupply as follows:

$$OF_{\text{penI}}(Y_s, t_0, y_{t_0}, y(s)) = \underbrace{\mathbb{E} [(Y_s - y(s))^2 | Y_{t_0} = y_{t_0}]}_I + \alpha \cdot \underbrace{\mathbb{E} [(Y_s - y(s))^2 \mathbb{1}_{Y_s > y(s)} | Y_{t_0} = y_{t_0}]}_{III}. \quad (5)$$

In contrast to (4), it might be of greater practical relevance for a producer to also account for the probability that an undersupply occurs. Note that the difference between the two objective functions is determined by the latter probability $P(Y_s > y(s))$:

$$(OF_{\text{penC}} - OF_{\text{penI}})(Y_s, t_0, y_{t_0}, y(s)) = \alpha \cdot \mathbb{E} [(Y_s - y(s))^2 | Y_s > y(s) \wedge Y_{t_0} = y_{t_0}] \cdot (1 - P(Y_s > y(s))) \geq 0.$$

We always have $OF_{\text{penC}}(Y_s, t_0, y_{t_0}, y(s)) \geq OF_{\text{penI}}(Y_s, t_0, y_{t_0}, y(s))$ with equality if $P(Y_s > y(s)) = 1$, i.e. undersupply is penalized less for OF_{penI} . That goes along with two intuitive hypothesis numerically analyzed in the Section 2:

(H1) The average undersupply $\mathbb{E} [(y^*(s) - Y_s) \mathbb{1}_{Y_s > y(s)} | Y_{t_0} = y_{t_0}]$ for OF_{penI} is higher than that for OF_{penC} .

(H2) The number of undersupply cases is expected to be higher for OF_{penI} .

2 Numerical results

With only a slight modification of the reformulation of equation (5) in [5], we again have a deterministic reformulation of the SOC problem (3). We use the numerical procedure set up in [5] adapted to the alternative cost function OF_{penI} to test our hypothesis (H1) and (H2) numerically. We take 10^3 Monte Carlo repetitions for the following parameter setting: $T = 1$, $\lambda = 4$, $\mu(t) = 2 + 3 \cdot \sin(2\pi t)$, $\kappa = 3$, $\sigma = 2$, $u_0 = 1$.

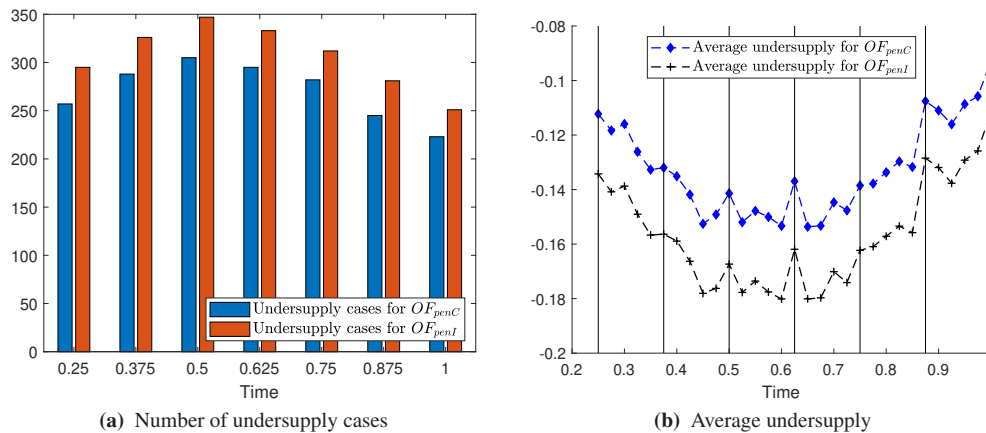


Fig. 1: Comparison of OF_{penI} with OF_{penC}

In Figure 1, we see that the number of undersupply cases is higher for OF_{penI} . The average undersupply (negative value) resulting from OF_{penI} lies clearly below the one based on the optimization with respect to OF_{penC} . Both the binary measure of undersupply as well as the quantitative height of the average undersupply show the higher undersupply penalization by OF_{penC} , and numerically confirm our hypotheses (H1) and (H2). However, the increasing or decreasing tendency is the same for both measures indicating that the use of either one is reasonable.

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References

- [1] D. Bienstock, M. Chertkov, and S. Harnett, Chance-constrained optimal power flow: risk-aware network control under uncertainty, *SIAM Rev.* **56**(3), 461–495 (2014).
- [2] J. L. Hight and K. G. Kempf, Production planning under supply and demand uncertainty: a stochastic programming approach, in: *Stochastic programming*, *Internat. Ser. Oper. Res. Management Sci.*, Vol. 150 (Springer, New York, 2011), pp. 297–315.
- [3] S. Göttlich, R. Korn, and K. Lux, Optimal control of electricity input given an uncertain demand, arXiv preprint: 1810.05480, 2018.
- [4] M. T. Barlow A diffusion model for electricity prices, *Math. Finance* **12**(4), 287–298 (2002).
- [5] S. Göttlich, R. Korn, and K. Lux, Optimal inflow control penalizing undersupply in transport systems with uncertain demands, arXiv preprint: 1901.09653, 2019.