



Research article

The global classical solution to compressible Euler system with velocity alignment

Lining Tong¹, Li Chen^{2,*}, Simone Göttlich² and Shu Wang^{3,4}

¹ Department of Mathematics, Shanghai University, 200244, P. R. China

² Department of Mathematics, University of Mannheim, 68131 Mannheim, Germany

³ School of Mathematics and Information Sciences, Guangzhou University, 510006, P. R. China

⁴ School of Mathematics, Faculty of Science, Beijing University of Technology, 100124, P. R. China

* **Correspondence:** Email: chen@math.uni-mannheim.de; Tel: +496211812779;
Fax: +496211812498.

Abstract: In this paper, the compressible Euler system with velocity alignment and damping is considered, where the influence matrix of velocity alignment is not positive definite. Sound speed is used to reformulate the system into symmetric hyperbolic type. The global existence and uniqueness of smooth solution for small initial data is provided.

Keywords: non-local velocity alignment; global existence; nonlinear pressure

Mathematics Subject Classification: 35L65, 35Q70

1. Introduction

In this paper, we study the following Cauchy problem

$$\partial_t \rho + \nabla \cdot (\rho u) = 0, \tag{1.1}$$

$$\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p(\rho) = -\frac{1}{\tau} \rho u - \rho \int_{\mathbb{R}^N} \Gamma(x-y)(u(x) - u(y)) \rho(y) dy, \tag{1.2}$$

in $(0, \infty) \times \mathbb{R}^N$ and the initial conditions

$$\rho|_{t=0} = \rho_0(x), \quad u|_{t=0} = u_0(x), \quad x \in \mathbb{R}^N, \tag{1.3}$$

where ρ and u are the unknown density and velocity, and the pressure $p(\rho) = A\rho^\gamma$. The matrix is $\Gamma(x) \in L^1(\mathbb{R}^N)$. The constants $A, \gamma \geq 1, \tau > 0$ are given. For simplicity, it is assumed that $A = 1$. This compressible Euler type system can be formally derived, for example in [11], from mean field type

interacting many particle system where the particle velocities are also involved in the interacting force. In the intermediate step, from mesoscopic kinetic model to Fluid dynamic model, one can choose different closure Ansatz of the probability density to obtain the system with or without pressure term. For example, by taking the density to be a localized Gaussian function, one obtains a linear pressure in [7]. In this paper, we consider the case with nonlinear pressure, which can be obtained by taking the probability density as an indicator function in the velocity space.

For quasi-linear hyperbolic systems, intensive studies have been carried out in numerous literature, such as [17, 19, 21, 23], and etc. It has been proved in general that for symmetrizable hyperbolic systems, smooth solutions exist locally in time and shock waves formation will breakdown the smoothness of the solutions in finite time even for the scalar case, see [21]. However, if some damping terms are taken into account, shock waves can be avoided for small perturbation of the diffusion waves [10, 15]. Moreover, with the help of velocity damping, the existence and uniqueness of the global classical solution can be obtained, see for example [16, 24–27].

The pressureless Euler system with non-local forces has been studied recently and very limited results have been done. The local existence of classical solutions of the complex material flow dynamics which has been derived in [11], under structural condition for the interaction force has been obtained in [6]. In [3], for the 1-D model with damping and non-local interaction, a critical threshold for the existence of classical solution by using the characteristic method is presented. 1-D entropy weak solution for Cucker-Smale type interaction has been obtained with the help of the compensated compactness argument in [12]. The global existence of smooth solutions with small initial data for the model with velocity alignment can be found in [2, 9, 13, 14, 18]. In these references, the influence function of velocity alignment is $\Gamma(x) = \phi(|x|)\mathbf{I}_{N \times N}$, where $\phi(|x|)$ has a positive lower bound and $\mathbf{I}_{N \times N}$ is the identity matrix, with which the formation of shock can be prevented. In case that additional pressure and viscosity are added, for restrictive interaction potentials, the global weak solution and its long time behavior are obtained in [5]. By adapting the convex integration method, it has been shown in [4] that infinitely many weak solutions exist. Recently, the global existence of classical solutions for the hydrodynamic model with linear pressure term and non-local velocity alignment was given in [7], where the shock wave was prevented by velocity alignment.

In many models, the communication weight matrices have different structures. Many of them do not need to be positive definite, as for example in the material flow model that has been proposed in [11], the interaction force includes $\Gamma = \frac{x}{|x|} \otimes \frac{x}{|x|}$ as the weight of velocity alignment.

In our model, the influence function of velocity alignment Γ is a matrix which corresponds to a linear projection of the velocity field. Furthermore it is non-constant and not positive definite, which reflects the anisotropic non-local interaction within the system. Therefore, the velocity alignment alone can not prevent the formation of shock wave. In order to obtain the global existence of smooth solutions, the additional damping effect in the system is necessary. Additionally, the symmetry of the coefficients plays an important role in the analysis of the existence of smooth solutions when using the method of standard energy estimates. We will use sound speed to reconstruct Eqs (1.1) and (1.2) into symmetric hyperbolic equations. It should be pointed out that this method will make the term of velocity alignment more complicated. After a detailed analysis of the relationship between velocity alignment and damping, the anisotropic non-local interaction is overcome by using damping, and the existence of the global classical solution of problem (1.1)–(1.2) is obtained.

Here, we introduce several notations used throughout the paper. For a function $u = u(x)$, $\|u\|_{L^p}$

denotes the usual $L^p(\mathbb{R}^N)$ -norm. We also set C as a generic positive constant independent of t . For any non-negative integer k , $H^k := H^k(\mathbb{R}^N)$ denotes the Sobolev space $= W^{k,2}(\mathbb{R}^N)$, and $C^k(I; E)$ is the space of k -times continuously differentiable functions from an interval $I \subset \mathbb{R}$ into a Banach space E . ∇^k denotes any partial derivative ∂^α with multi-index α , where $|\alpha| = k$. For simplicity, we write $\int f dx := \int_{\mathbb{R}^N} f dx$.

This paper is structured as follows: In Section 2, we reformulate the Cauchy problem (1.1)–(1.3) into a symmetric hyperbolic system and present our main result. In Section 3, we demonstrate the local existence under uniqueness of the classical solution for the reconstructed system. Finally, we establish the priori estimates to prove the global existence result.

2. Reformulation and main result

2.1. Reformulation of the problem

In this subsection, we will reformulate the Cauchy problem of the compressible Euler system (1.1)–(1.3) as in [24]. The main point is to obtain a symmetric system. We consider the case $\gamma > 1$ in this paper and introduce the sound speed:

$$\kappa(\rho) = \sqrt{p'(\rho)},$$

where $\bar{\kappa} = \kappa(\bar{\rho})$ is set to the sound speed at a background density $\bar{\rho} > 0$. The symmetrization in the case of $\gamma = 1$ can be done similarly with a new variable $\ln \rho$.

Define

$$\sigma(\rho) = \nu(\kappa(\rho) - \bar{\kappa}) \text{ with } \nu = \frac{2}{\gamma - 1}.$$

Then the Eqs (1.1) and (1.2) are transformed into the following system:

$$\partial_t \sigma + \bar{\kappa} \nabla \cdot u = -u \cdot \nabla \sigma - \frac{1}{\nu} \sigma \nabla \cdot u, \quad (2.1)$$

$$\partial_t u + \bar{\kappa} \nabla \sigma + \frac{1}{\tau} u = -u \cdot \nabla u - \frac{1}{\nu} \sigma \nabla \cdot \sigma - a \int \Gamma(x - y)(u(x) - u(y)) \left(\frac{1}{\nu} \sigma(y) + \bar{\kappa} \right) dy, \quad (2.2)$$

where the constants are $a = \gamma^{-1/(\gamma-1)} > 0$. The initial condition (1.3) becomes

$$(\sigma, u)|_{t=0} = (\sigma_0(x), u_0(x)) \quad (2.3)$$

with $\sigma_0 = \nu(\kappa(\rho_0) - \bar{\kappa})$. Note that as we did at the formal level, we can find the relation between the classical solutions (ρ, u) and (σ, u) to the systems (1.1)–(1.2) and (2.1)–(2.2), respectively, in the following two lemmas. The proofs can be obtained by taking the similar strategy as in [24].

Lemma 1. *For any $T > 0$, if $(\rho, u) \in C^1(\mathbb{R}^N \times [0, T])$ is a solution of system (1.1)–(1.2) with $\rho > 0$, then $(\sigma, u) \in C^1(\mathbb{R}^N \times [0, T])$ is a solution for the system (2.1)–(2.2) with $(\frac{1}{\nu} \sigma + \bar{\kappa})^\nu > 0$. Conversely, if $(\sigma, u) \in C^1(\mathbb{R}^N \times [0, T])$ is a solution for the system (2.1)–(2.2) with $(\frac{1}{\nu} \sigma + \bar{\kappa})^\nu > 0$, then $(\rho, u) \in C^1(\mathbb{R}^N \times [0, T])$ is a solution of system (1.1)–(1.2) with $\rho > 0$.*

Lemma 2. *For any $T > 0$, if $(\rho, u) \in C^1(\mathbb{R}^N \times [0, T])$ is a uniformly bounded solution of system (1.1)–(1.2) with $\rho_0 > 0$, then $\rho > 0$ on $\mathbb{R}^N \times [0, T]$. Conversely, if $(\sigma, u) \in C^1(\mathbb{R}^N \times [0, T])$ is a uniformly bounded solution of system (2.1)–(2.2) with $(\frac{1}{\nu} \sigma_0 + \bar{\kappa})^\nu > 0$, then $(\frac{1}{\nu} \sigma + \bar{\kappa})^\nu > 0$ on $\mathbb{R}^N \times [0, T]$.*

2.2. Main result

In the subsection 2.1, we have presented the equivalent reconstruction system (2.1)–(2.2) of the problem (1.1)–(1.2) and the equivalence relation between them. Next, we will study the reconstructed system (2.1)–(2.2) and provide the following results.

Theorem 1. (Local-in-time existence) *For $s > \frac{N}{2} + 1$, assume the initial values $(\sigma_0(x), u_0(x)) \in H^s(\mathbb{R}^N)$. Then there exist a unique classical solution (σ, u) of the Cauchy problem (2.1)–(2.3) satisfying*

$$(\sigma, u) \in C([0, T], H^s(\mathbb{R}^N)) \cap C^1([0, T], H^2(\mathbb{R}^N)) \quad (2.4)$$

for some finite $T > 0$.

Theorem 2. (Global-in-time existence) *Suppose background sound speed $\bar{\kappa}$ satisfying $2a\bar{\kappa}^\nu \|\Gamma\|_{L^1} < \frac{1}{\tau}$. If $\|\sigma_0\|_{H^s} + \|u_0\|_{H^s} \leq \delta_0$ with sufficiently small $\delta_0 > 0$, then the Cauchy problem (2.1)–(2.3) has a unique global classical solution.*

Remark 1. Since the matrix $\Gamma(x)$ is not positive definite, the damping coefficient needs to be large enough to make the damping term restrain the self-acceleration effect caused by velocity alignment to get the global well-posedness. The definition of condition $2a\bar{\kappa}^\nu \|\Gamma\|_{L^1} < \frac{1}{\tau}$ is therefore natural.

3. Local existence and uniqueness

In this section, we demonstrate the local existence and uniqueness of the classical solutions to (2.1)–(2.3). We will present a successive iteration scheme to construct approximate solutions and to obtain the energy estimates. Then we show that approximate solutions are convergent in Sobolev spaces using the contraction mapping principle and prove that the limit function is the local solution.

3.1. Approximate solutions

We construct approximate solution by the following iterative method:

- the zeroth approximation: $(\sigma^0, u^0)(x, t) = (\sigma_0, u_0)$;
- Suppose that the k th approximation $(\sigma^k, u^k)(x, t)$, $k \geq 1$ is given. Then define the $(k + 1)$ th approximation $(\sigma^{k+1}, u^{k+1})(x, t)$ as a solution of the linear system

$$\partial_t \sigma^{k+1} + \bar{\kappa} \nabla \cdot u^{k+1} = -u^k \cdot \nabla \sigma^{k+1} - \frac{1}{\nu} \sigma^k \nabla \cdot u^{k+1}, \quad (3.1)$$

$$\begin{aligned} \partial_t u^{k+1} + \bar{\kappa} \nabla \sigma^{k+1} + \frac{1}{\tau} u^{k+1} &= -u^k \cdot \nabla u^{k+1} - \frac{1}{\nu} \sigma^k \nabla \cdot \sigma^{k+1} \\ &\quad - a \int \Gamma(x-y) (u^k(x) - u^k(y)) \left(\frac{1}{\nu} \sigma^k(y) + \bar{\kappa} \right)^\nu dy \end{aligned} \quad (3.2)$$

with the initial data

$$(\sigma^{k+1}, u^{k+1})|_{t=0} = (\sigma_0(x), u_0(x)) \in H^s(\mathbb{R}^N). \quad (3.3)$$

The local existence of the solutions (σ^{k+1}, u^{k+1}) in Sobolev spaces can be obtained by applying the linear theory of the multi-dimensional hyperbolic equations in [1].

3.2. *Priori estimates*

We first set up several constants:

$$M = \sqrt{\|\sigma_0\|_{H^s}^2 + \|u_0\|_{H^s}^2 + 1}, \quad (3.4)$$

and choose $T_0 > 0$ so that

$$(e^{C(M, \nu, \bar{\kappa})T_0} - 1)(\|\sigma_0\|_{H^s}^2 + \|u_0\|_{H^s}^2) + e^{C(M, \nu, \bar{\kappa})T_0} C(M, \nu, \bar{\kappa})T_0 \leq 1, \quad (3.5)$$

where $C(M, \nu, \bar{\kappa})$ is given in the proof of Lemma 3 below.

Lemma 3. *Let (σ^k, u^k) be a sequence of the approximate solutions generated by (3.1)–(3.2) together with the initial step $(\sigma^0, u^0) = (\sigma_0, u_0)$. Then the following estimate holds*

$$\sup_{0 \leq t \leq T_0} \|\sigma^k\|_{H^s} + \sup_{0 \leq t \leq T_0} \|u^k\|_{H^s} \leq M, \quad \text{for all } k \geq 0, \quad (3.6)$$

where $s > \frac{N}{2} + 1$, M and T_0 are given in (3.4) and (3.5).

Proof. We use the method of induction to prove the Lemma.

Step 1. (Initial step) Because we choose $(\sigma^0, u^0) = (\sigma_0, u_0)$, together with the choice of M , $T_0 > 0$ in (3.4) and (3.5), it is easy to check that

$$\sup_{0 \leq t \leq T_0} (\|\sigma^0\|_{H^s} + \|u^0\|_{H^s}) \leq M.$$

Step 2. (Inductive step) Suppose that

$$\sup_{0 \leq t \leq T_0} (\|\sigma^k\|_{H^s} + \|u^k\|_{H^s}) \leq M, \quad (3.7)$$

where T_0 , M are positive constants determined in (3.4) and (3.5). We will prove that

$$\sup_{0 \leq t \leq T_0} (\|\sigma^{k+1}\|_{H^s} + \|u^{k+1}\|_{H^s}) \leq M.$$

First, multiplying σ^{k+1} , u^{k+1} on both sides of (3.1), (3.2) respectively, summing up and integrating over \mathbb{R}^N , we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\sigma^{k+1}\|_{L^2}^2 + \|u^{k+1}\|_{L^2}^2) + \frac{1}{\tau} \|u^{k+1}\|_{L^2}^2 \\ &= - \int (u^k \cdot \nabla \sigma^{k+1} \sigma^{k+1} + u^k \cdot \nabla u^{k+1} \cdot u^{k+1}) dx - \frac{1}{\nu} \int (\sigma^k \nabla \cdot u^{k+1} \sigma^{k+1} + \sigma^k \nabla \sigma^{k+1} \cdot u^{k+1}) dx \\ & \quad - a \int \int \Gamma(x-y) (u^k(x) - u^k(y)) \left(\frac{1}{\nu} \sigma^k(y) + \bar{\kappa} \right)^\nu dy \cdot u^{k+1}(x) dx \\ &= I_1 + I_2 + I_3. \end{aligned} \quad (3.8)$$

We shall estimate the terms on the right-hand side of (3.8). Thanks to the Sobolev embedding theorem and the inductive assumption (3.7), using integration by parts, we obtain

$$\begin{aligned}
I_1 &= - \int (u^k \cdot \nabla \sigma^{k+1} \sigma^{k+1} + u^k \cdot \nabla u^{k+1} \cdot u^{k+1}) dx = \frac{1}{2} \int \nabla \cdot u^k (|\sigma^{k+1}|^2 + |u^{k+1}|^2) dx \\
&\leq \|\nabla \cdot u^k\|_{L^\infty} (\|\sigma^{k+1}\|_{L^2}^2 + \|u^{k+1}\|_{L^2}^2) \leq CM (\|\sigma^{k+1}\|_{L^2}^2 + \|u^{k+1}\|_{L^2}^2); \\
I_2 &= -\frac{1}{\nu} \int (\sigma^k \nabla \cdot u^{k+1} \sigma^{k+1} + \sigma^k \nabla \sigma^{k+1} \cdot u^{k+1}) dx = \frac{1}{\nu} \int \nabla \sigma^k \cdot (u^{k+1} \sigma^{k+1}) dx \\
&\leq C \|\nabla \sigma^k\|_{L^\infty} \|\sigma^{k+1}\|_{L^2} \|u^{k+1}\|_{L^2} \leq CM (\|\sigma^{k+1}\|_{L^2}^2 + \|u^{k+1}\|_{L^2}^2); \\
I_3 &= -a \int \int \Gamma(x-y) (u^k(x) - u^k(y)) \left(\frac{1}{\nu} \sigma^k(y) + \bar{\kappa}\right)^\nu \cdot u^{k+1}(x) dy dx \\
&\leq C \left\| \left(\frac{1}{\nu} \sigma^k + \bar{\kappa}\right)^\nu \right\|_{L^\infty} \|\Gamma\|_{L^1} \|u^k\|_{L^2} \|u^{k+1}\|_{L^2} \\
&\leq C(M, \nu, \bar{\kappa}) (\|u^{k+1}\|_{L^2}^2 + 1).
\end{aligned}$$

Combining the estimate of I_i , $i = 1, 2, 3$, we obtain

$$\frac{d}{dt} (\|\sigma^{k+1}\|_{L^2}^2 + \|u^{k+1}\|_{L^2}^2) + \frac{2}{\tau} \|u^{k+1}\|_{L^2}^2 \leq C(M, \nu, \bar{\kappa}) (\|\sigma^{k+1}\|_{L^2}^2 + \|u^{k+1}\|_{L^2}^2 + 1). \quad (3.9)$$

Next we will get the higher order estimate of (σ^{k+1}, u^{k+1}) .

Taking ∇^r , $1 \leq r \leq s$ with respect to x on both sides of (3.1)–(3.2), and then multiplying the resulting identities by $\nabla^r \sigma^{k+1}$, $\nabla^r u^{k+1}$ respectively, summing up and integrating over \mathbb{R}^N , we obtain

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} (\|\nabla^r \sigma^{k+1}\|_{L^2}^2 + \|\nabla^r u^{k+1}\|_{L^2}^2) + \frac{1}{\tau} \|\nabla^r u^{k+1}\|_{L^2}^2 \\
&= - \int (\nabla^r (u^k \cdot \nabla \sigma^{k+1}) \nabla^r \sigma^{k+1} + \nabla^r (u^k \cdot \nabla u^{k+1}) \cdot \nabla^r u^{k+1}) dx \\
&\quad - \frac{1}{\nu} \int (\nabla^r (\sigma^k \nabla \cdot u^{k+1}) \nabla^r \sigma^{k+1} + \nabla^r (\sigma^k \nabla \sigma^{k+1}) \cdot \nabla^r u^{k+1}) dx \quad (3.10)
\end{aligned}$$

$$\begin{aligned}
&- a \int \nabla_x^r \left(\int \Gamma(x-y) u^k(x) \left(\frac{1}{\nu} \sigma^k(y) + \bar{\kappa}\right)^\nu dy \right) \cdot \nabla^r u^{k+1}(x) dx \\
&+ a \int \nabla_x^r \left(\int \Gamma(x-y) u^k(y) \left(\frac{1}{\nu} \sigma^k(y) + \bar{\kappa}\right)^\nu dy \right) \cdot \nabla^r u^{k+1}(x) dx \\
&= \sum_{i=1}^4 I_i. \quad (3.11)
\end{aligned}$$

In the following we will estimate I_i term by term. Using the Sobolev embedding theorem and Moser type inequality, we obtain

$$\begin{aligned}
I_1 &= - \int (\nabla^r (u^k \cdot \nabla \sigma^{k+1}) \nabla^r \sigma^{k+1} + \nabla^r (u^k \cdot \nabla u^{k+1}) \cdot \nabla^r u^{k+1}) dx \\
&= - \int u^k \cdot \nabla \nabla^r \sigma^{k+1} \nabla^r \sigma^{k+1} dx - \int (\nabla^r (u^k \cdot \nabla \sigma^{k+1}) - u^k \cdot \nabla \nabla^r \sigma^{k+1}) \nabla^r \sigma^{k+1} dx \\
&\quad - \int u^k \cdot \nabla \nabla^r u^{k+1} \nabla^r u^{k+1} dx - \int (\nabla^r (u^k \cdot \nabla u^{k+1}) - u^k \cdot \nabla \nabla^r u^{k+1}) \nabla^r u^{k+1} dx
\end{aligned}$$

$$\begin{aligned}
&\leq C\|\nabla \cdot u^k\|_{L^\infty}(\|\nabla^r \sigma^{k+1}\|_{L^2}^2 + \|\nabla^r u^{k+1}\|_{L^2}^2) \\
&\quad + \|\nabla^r \sigma^{k+1}\|_{L^2}(\|\nabla^r u^k\|_{L^2}\|\nabla \sigma^{k+1}\|_{L^\infty} + \|\nabla^r \sigma^{k+1}\|_{L^2}\|\nabla u^k\|_{L^\infty}) \\
&\quad + \|\nabla^r u^{k+1}\|_{L^2}(\|\nabla^r u^k\|_{L^2}\|\nabla u^{k+1}\|_{L^\infty} + \|\nabla^r u^{k+1}\|_{L^2}\|\nabla u^k\|_{L^\infty}) \\
&\leq CM(\|\nabla^r \sigma^{k+1}\|_{L^2}^2 + \|\nabla^r u^{k+1}\|_{L^2}^2) + CM(\|\nabla \sigma^{k+1}\|_{H^{s-1}}^2 + \|\nabla u^{k+1}\|_{H^{s-1}}^2); \tag{3.12} \\
I_2 &= -\frac{1}{\nu} \int (\nabla^r(\sigma^k \nabla \cdot u^{k+1}) \nabla^r \sigma^{k+1} + \nabla^r(\sigma^k \nabla \sigma^{k+1}) \cdot \nabla^r u^{k+1}) dx \\
&= -\frac{1}{\nu} \int \sigma^k \nabla \cdot \nabla^r u^{k+1} \nabla^r \sigma^{k+1} dx - \frac{1}{\nu} \int_{\mathbb{R}^N} (\nabla^r(\sigma^k \nabla \cdot u^{k+1}) - \sigma^k \nabla \cdot \nabla^r u^{k+1}) \nabla^r \sigma^{k+1} dx \\
&\quad - \frac{1}{\nu} \int \sigma^k \nabla \nabla^r \sigma^{k+1} \nabla^r u^{k+1} dx - \frac{1}{\nu} \int (\nabla^r(\sigma^k \nabla \sigma^{k+1}) - \sigma^k \nabla \nabla^r \sigma^{k+1}) \nabla^r u^{k+1} dx \\
&\leq C\|\nabla \sigma^k\|_{L^\infty}(\|\nabla^r \sigma^{k+1}\|_{L^2}^2 + \|\nabla^r u^{k+1}\|_{L^2}^2) \\
&\quad + C(\|\nabla^r \sigma^k\|_{L^2}\|\nabla \cdot u^{k+1}\|_{L^\infty} + \|\nabla^r u^{k+1}\|_{L^2}\|\nabla \sigma^k\|_{L^\infty})\|\nabla^r \sigma^{k+1}\|_{L^2} \\
&\quad + C(\|\nabla^r \sigma^k\|_{L^2}\|\nabla \sigma^{k+1}\|_{L^\infty} + \|\nabla^r \sigma^{k+1}\|_{L^2}\|\nabla \sigma^k\|_{L^\infty})\|\nabla^r u^{k+1}\|_{L^2} \\
&\leq CM(\|\nabla^r \sigma^{k+1}\|_{L^2}^2 + \|\nabla^r u^{k+1}\|_{L^2}^2) + CM(\|\nabla \sigma^{k+1}\|_{H^{s-1}}^2 + \|\nabla u^{k+1}\|_{H^{s-1}}^2), \tag{3.13}
\end{aligned}$$

where we have used $\|\nabla u^{k+1}\|_{L^\infty} \leq \|\nabla u^{k+1}\|_{H^{s-1}}$ and the inductive assumption (3.7).

Next, we estimate the I_3 . Using Young's inequality and Moser type inequality, we have

$$\begin{aligned}
I_3 &= -a \int \nabla_x^r(u^k(x) \int \Gamma(x-y)(\frac{1}{\nu}\sigma^k(y) + \bar{\kappa})^\nu dy) \cdot \nabla^r u^{k+1}(x) dx \\
&\leq C\|\nabla^r u^{k+1}\|_{L^2}\|\nabla^r(u^k \Gamma * (\frac{1}{\nu}\sigma^k + \bar{\kappa})^\nu)\|_{L^2} \\
&\leq C\|\nabla^r u^{k+1}\|_{L^2}\|\nabla^r u^k\|_{L^2}\|\Gamma * (\frac{1}{\nu}\sigma^k + \bar{\kappa})^\nu\|_{L^\infty} + C\|\nabla^r u^{k+1}\|_{L^2}\|u^k\|_{L^\infty}\|\nabla^r \Gamma * (\frac{1}{\nu}\sigma^k + \bar{\kappa})^\nu\|_{L^2}. \tag{3.14}
\end{aligned}$$

Applying the Sobolev embedding theorem and the inductive assumption (3.7), direct calculation shows

$$\begin{aligned}
\|\Gamma * (\frac{1}{\nu}\sigma^k + \bar{\kappa})^\nu\|_{L^\infty} &= \|\int \Gamma(x-y)(\frac{1}{\nu}\sigma^k(y) + \bar{\kappa})^\nu dy\|_{L^\infty} \leq \|\Gamma\|_{L^1}\|(\frac{1}{\nu}\sigma^k + \bar{\kappa})^\nu\|_{L^\infty} \\
&\leq C(\|\sigma^k\|_{L^\infty}, \nu, \bar{\kappa}) \leq C(M, \nu, \bar{\kappa}). \tag{3.15}
\end{aligned}$$

$$\begin{aligned}
\|\nabla^r \Gamma * (\frac{1}{\nu}\sigma^k + \bar{\kappa})^\nu\|_{L^2} &= \|\nabla_x^r \int \Gamma(x-y)(\frac{1}{\nu}\sigma^k(y) + \bar{\kappa})^\nu dy\|_{L^2} = \|\int \Gamma(x-y) \nabla_y^r (\frac{1}{\nu}\sigma^k(y) + \bar{\kappa})^\nu dy\|_{L^2} \\
&\leq C\|\Gamma\|_{L^1}\|\nabla^r (\frac{1}{\nu}\sigma^k + \bar{\kappa})^\nu\|_{L^2} \leq C(\|\sigma^k\|_{L^\infty}, \nu, \bar{\kappa})\|\Gamma\|_{L^1}\|\sigma^k\|_{H^r} \\
&\leq C(M, \nu, \bar{\kappa}), \tag{3.16}
\end{aligned}$$

where $C(M, \nu, \bar{\kappa})$ is non-decreasing in M .

Then, we obtain that

$$I_3 \leq C(M, \nu, \bar{\kappa})(\|\nabla^r u^{k+1}\|_{L^2}^2 + 1). \tag{3.17}$$

Finally, we provide the estimate of I_4 . By applying the Moser type inequality and Young's inequality, we have

$$I_4 = a \int \nabla^r u^{k+1}(x) dx \nabla_x^r \left(\int \Gamma(x-y) u^k(y) (\frac{1}{\nu}\sigma^k(y) + \bar{\kappa})^\nu dy \right)$$

$$\begin{aligned}
&= a \int \nabla^r u^{k+1}(x) dx \int \Gamma(x-y) \nabla_y^r \left(u^k(y) \left(\frac{1}{\nu} \sigma^k(y) + \bar{\kappa} \right)^{\nu} \right) dy \\
&\leq C \|\Gamma\|_{L^1} \|\nabla^r u^{k+1}\|_{L^2} \|\nabla^r \left(u^k(y) \left(\frac{1}{\nu} \sigma^k(y) + \bar{\kappa} \right)^{\nu} \right)\|_{L^2} \\
&\leq C \|\Gamma\|_{L^1} \|\nabla^r u^{k+1}\|_{L^2} \left(\|\nabla^r u^k\|_{L^2} \left\| \left(\frac{1}{\nu} \sigma^k + \bar{\kappa} \right)^{\nu} \right\|_{L^{\infty}} + \|\nabla^r \left(\frac{1}{\nu} \sigma^k + \bar{\kappa} \right)^{\nu}\|_{L^2} \|u^k\|_{L^{\infty}} \right) \\
&\leq C(M, \nu, \bar{\kappa}) (\|\nabla^r u^{k+1}\|_{L^2}^2 + 1).
\end{aligned} \tag{3.18}$$

Here, we used (3.16) and the inductive assumption (3.7).

Collecting all estimates of I_i from 1 to 4, we obtain that

$$\begin{aligned}
&\frac{d}{dt} \left(\|\nabla^r \sigma^{k+1}\|_{L^2}^2 + \|\nabla^r u^{k+1}\|_{L^2}^2 \right) + \frac{2}{\tau} \|\nabla^r u^{k+1}\|_{L^2}^2 \\
&\leq C(M, \nu, \bar{\kappa}) (\|\nabla^r u^{k+1}\|_{L^2}^2 + \|\nabla^r \sigma^{k+1}\|_{L^2}^2 + 1) + CM (\|\nabla u^{k+1}\|_{H^{s-1}}^2 + \|\nabla \sigma^{k+1}\|_{H^{s-1}}^2).
\end{aligned} \tag{3.19}$$

We can sum (3.19) over $1 \leq r \leq s$ and combine (3.9) to obtain

$$\frac{d}{dt} \left(\|\sigma^{k+1}\|_{H^s}^2 + \|u^{k+1}\|_{H^s}^2 \right) + \frac{2}{\tau} \|\nabla^r u^{k+1}\|_{L^2}^2 \leq C(M, \nu, \bar{\kappa}) (\|u^{k+1}\|_{H^s}^2 + \|\sigma^{k+1}\|_{H^s}^2) + C(M, \nu, \bar{\kappa}). \tag{3.20}$$

This yields

$$\begin{aligned}
&\sup_{0 \leq t \leq T_0} \|\sigma^{k+1}\|_{H^s}^2 + \|u^{k+1}\|_{H^s}^2 + \int_0^{T_0} \|\nabla^r u^{k+1}\|_{L^2}^2 dt \\
&\leq e^{C(M, \nu, \bar{\kappa})T_0} (\|\sigma_0\|_{H^s}^2 + \|u_0\|_{H^s}^2) + e^{C(M, \nu, \bar{\kappa})T_0} C(M, \nu, \bar{\kappa})T_0.
\end{aligned}$$

By the choice of M and T_0 as in (3.4) and (3.5), we can easily check that

$$e^{C(M, \nu, \bar{\kappa})T_0} (\|\sigma_0\|_{H^s}^2 + \|u_0\|_{H^s}^2) + e^{C(M, \nu, \bar{\kappa})T_0} C(M, \nu, \bar{\kappa})T_0 \leq M^2.$$

So, we obtain

$$\|\sigma^{k+1}\|_{H^s} + \|u^{k+1}\|_{H^s} \leq M \tag{3.21}$$

which completes the induction process. \square

3.3. Convergence in lower-order norm

In this subsection, we will show that the $\{\sigma^k, u^k\}_{k=1}^{\infty}$ are convergent in some lower-order Sobolev spaces using the contraction mapping principle.

Let

$$\bar{n}^{k+1} = \sigma^{k+1} - \sigma^k, \quad \bar{u}^{k+1} = u^{k+1} - u^k.$$

Note that (σ^{k+1}, u^{k+1}) and (σ^k, u^k) satisfy

$$\begin{aligned}
\partial_t \sigma^{k+1} + \bar{\kappa} \nabla \cdot u^{k+1} &= -u^k \cdot \nabla \sigma^{k+1} - \frac{1}{\nu} \sigma^k \nabla \cdot u^{k+1}, \\
\partial_t u^{k+1} + \bar{\kappa} \nabla \sigma^{k+1} + \frac{1}{\tau} u^{k+1} &= -u^k \cdot \nabla u^{k+1} - \frac{1}{\nu} \sigma^k \nabla \sigma^{k+1}
\end{aligned} \tag{3.22}$$

$$- a \int \Gamma(x-y)(u^k(x) - u^k(y))\left(\frac{1}{\nu}\sigma^k(y) + \bar{\kappa}\right)^\nu dy$$

as well as

$$\begin{aligned} \partial_t \sigma^k + \bar{\kappa} \nabla \cdot u^k + \frac{1}{\tau} u^k &= -u^{k-1} \cdot \nabla \sigma^k - \frac{1}{\nu} \sigma^{k-1} \nabla \cdot u^k, \\ \partial_t u^k + \bar{\kappa} \nabla \sigma^k &= -u^{k-1} \cdot \nabla u^k - \frac{1}{\nu} \sigma^{k-1} \nabla \sigma^k \\ &- a \int_{\mathbb{R}^N} \Gamma(x-y)(u^{k-1}(x) - u^{k-1}(y))\left(\frac{1}{\nu}\sigma^{k-1}(y) + \bar{\kappa}\right)^\nu dy \end{aligned} \quad (3.23)$$

subject to the same initial data

$$(\sigma^{k+1}, u^{k+1}) = (\sigma^k, u^k) = (n_0, u_0) \in H^s. \quad (3.24)$$

It follows from (3.22) and (3.23) that

$$\begin{aligned} \partial_t(\sigma^{k+1} - \sigma^k) + \bar{\kappa} \nabla \cdot (u^{k+1} - u^k) &= -(\sigma^k - \sigma^{k-1}) \nabla \sigma^{k+1} \\ &- u^{k-1} \nabla(\sigma^{k+1} - \sigma^k) - \frac{1}{\nu}(\sigma^k - \sigma^{k-1}) \nabla \cdot u^{k+1} - \frac{1}{\nu} \sigma^{k-1} \nabla \cdot (u^{k+1} - u^k) \end{aligned} \quad (3.25)$$

$$\begin{aligned} \partial_t(u^{k+1} - u^k) + \bar{\kappa} \nabla(\sigma^{k+1} - \sigma^k) + \frac{1}{\tau}(u^{k+1} - u^k) \\ = -(u^k - u^{k-1}) \nabla \cdot u^{k+1} - u^{k-1} \nabla \cdot (u^{k+1} - u^k) - \frac{1}{\nu}(\sigma^k - \sigma^{k-1}) \nabla \sigma^{k+1} - \frac{1}{\nu} \sigma^{k-1} \nabla(\sigma^{k+1} - \sigma^k) \\ - a \int \Gamma(x-y)(u^k(x) - u^k(y))\left(\frac{1}{\nu}\sigma^k(y) + \bar{\kappa}\right)^\nu - \left(\frac{1}{\nu}\sigma^{k-1}(y) + \bar{\kappa}\right)^\nu dy \\ - a \int \Gamma(x-y)(u^k(x) - u^{k-1}(x))\left(\frac{1}{\nu}\sigma^{k-1}(y) + \bar{\kappa}\right)^\nu dy \\ - a \int \Gamma(x-y)(u^k(y) - u^{k-1}(y))\left(\frac{1}{\nu}\sigma^{k-1}(y) + \bar{\kappa}\right)^\nu dy. \end{aligned} \quad (3.26)$$

Multiplying (3.25) and (3.26) by $(\sigma^{k+1} - \sigma^k)$, $(u^{k+1} - u^k)$ respectively, summing up and integrating over \mathbb{R}^N , similar to the estimate in subsection 3.2, we obtain

$$\begin{aligned} \frac{d}{dt} (|\sigma^{k+1} - \sigma^k|_{L^2}^2 + |u^{k+1} - u^k|_{L^2}^2) + \frac{2}{\tau} \|u^{k+1} - u^k\|_{L^2}^2 \\ = - \int (u^k - u^{k-1}) \nabla \sigma^{k+1} (\sigma^{k+1} - \sigma^k) + (u^k - u^{k-1}) \nabla \cdot u^{k+1} \cdot (u^{k+1} - u^k) dx \\ - \int u^{k-1} \nabla(\sigma^{k+1} - \sigma^k) (\sigma^{k+1} - \sigma^k) + u^{k-1} \nabla \cdot (u^{k+1} - u^k) \cdot (u^{k+1} - u^k) dx \\ - \frac{1}{\nu} \int (\sigma^k - \sigma^{k-1}) \nabla \cdot u^{k+1} (\sigma^{k+1} - \sigma^k) + (\sigma^k - \sigma^{k-1}) \nabla \sigma^{k+1} (u^{k+1} - u^k) dx \\ - \frac{1}{\nu} \int \sigma^{k-1} \nabla \cdot (u^{k+1} - u^k) (\sigma^{k+1} - \sigma^k) + \sigma^{k-1} \nabla(\sigma^{k+1} - \sigma^k) (u^{k+1} - u^k) dx \\ - a \int (u^{k+1} - u^k) dx \int \Gamma(x-y)(u^k(x) - u^k(y))\left(\frac{1}{\nu}\sigma^k(y) + \bar{\kappa}\right)^\nu - \left(\frac{1}{\nu}\sigma^{k-1}(y) + \bar{\kappa}\right)^\nu dy \end{aligned}$$

$$\begin{aligned}
& - a \int (u^{k+1} - u^k) dx \int \Gamma(x-y)(u^k(x) - u^{k-1}(x)) \left(\frac{1}{\nu} \sigma^{k-1}(y) + \bar{\kappa}\right)^\nu dy \\
& - a \int (u^{k+1} - u^k) dx \int \Gamma(x-y)(u^k(y) - u^{k-1}(y)) \left(\frac{1}{\nu} \sigma^{k-1}(y) + \bar{\kappa}\right)^\nu dy \\
& \leq C(M, \nu, \bar{\kappa}) \left(\|\sigma^{k+1} - \sigma^k\|_{L^2}^2 + \|u^{k+1} - u^k\|_{L^2}^2 \right) + C(M, \nu, \bar{\kappa}) \left(\|\sigma^k - \sigma^{k-1}\|_{L^2}^2 + \|u^k - u^{k-1}\|_{L^2}^2 \right), \quad (3.27)
\end{aligned}$$

where we use the following estimate, for $a.e. (t, x) \in (0, +\infty) \times \mathbb{R}^N$

$$\begin{aligned}
\left| \left(\frac{1}{\nu} \sigma^k + \bar{\kappa}\right)^\nu - \left(\frac{1}{\nu} \sigma^{k-1} + \bar{\kappa}\right)^\nu \right| &= \left| \int_0^1 \left(\frac{s}{\nu} \sigma^k + \frac{(1-s)}{\nu} \sigma^{k-1} + \bar{\kappa} \right)^{\nu-1} (\sigma^k - \sigma^{k-1}) ds \right| \\
&\leq C(M, \nu, \bar{\kappa}) |\sigma^k - \sigma^{k-1}|. \quad (3.28)
\end{aligned}$$

We can integrate (3.27) over $(0, t)$ to obtain

$$\begin{aligned}
\sup_{0 \leq \tilde{t} \leq t} \|\sigma^{k+1} - \sigma^k\|_{L^2}^2 + \|u^{k+1} - u^k\|_{L^2}^2 &\leq C \int_0^t \|\sigma^{k+1}(\tilde{t}) - \sigma^k(\tilde{t})\|_{L^2}^2 + \|u^{k+1}(\tilde{t}) - u^k(\tilde{t})\|_{L^2}^2 d\tilde{t} \\
&+ C \int_0^t \|\sigma^k(\tilde{t}) - \sigma^{k-1}(\tilde{t})\|_{L^2}^2 + \|u^k(\tilde{t}) - u^{k-1}(\tilde{t})\|_{L^2}^2 d\tilde{t}. \quad (3.29)
\end{aligned}$$

The we sum up for $k = 1, 2, \dots$ together with the Gronwall's inequality to obtain

$$\sum_{k=1}^{\infty} \|\sigma^{k+1} - \sigma^k\|_{L^2}^2 + \|u^{k+1} - u^k\|_{L^2}^2 \leq C, \quad \text{for } t \leq T_0. \quad (3.30)$$

This implies that σ^k and u^k are Cauchy sequences in $C([0, T_0]; L^2)$.

3.4. The proof of Theorem 1

In this subsection, we will prove the local well-posedness of the system (2.1)–(2.2) given in Theorem 2.1. First, we prove the existence of classical solutions.

By the Gagliardo-Nirenberg inequality together with the uniform bound of (σ^k, u^k) and the convergence result (3.30), we can conclude that, for $s > \frac{N}{2} + 1$,

$$\sigma^k \rightarrow \sigma \in C([0, T_0]; H^{s-1}) \text{ and } u^k \rightarrow u \in C([0, T_0]; H^{s-1}). \quad (3.31)$$

It easily follows from (3.31) that limit function (σ, u) is a solution to (2.1)–(2.2) in a distributional sense. Using a similar argument as in [13], we can obtain the regularity of (σ, u) :

$$(\sigma, u) \in C([0, T_0]; H^s). \quad (3.32)$$

Applying Sobolev's embedding theorem, we prove $(\sigma, u) \in C^1([0, T_0] \times \mathbb{R}^N)$ is a classical solution.

Next, we prove the uniqueness. Let (σ, u) and $(\tilde{\sigma}, \tilde{u})$ be the two classical solutions of (2.1)–(2.2) corresponding the same initial data (σ_0, u_0) . We set

$$U(t) = \|\sigma - \tilde{\sigma}\|_{L^2}^2 + \|u - \tilde{u}\|_{L^2}^2.$$

Then, by the same argument as in subsection 4.1, $U(t)$ satisfies Gronwall's inequality:

$$U(t) \leq C \int_0^t U(t)dt, \quad U(0) = 0.$$

This yields that

$$\sigma \equiv \tilde{\sigma}, \quad u \equiv \tilde{u} \in C([0, T]; L^2(\mathbb{R}^N)).$$

So, we complete the proof of theorem.

4. Global existence of classical solution

In this section, we discuss the global existence of the classical solution on the basis of the local existence results in Section 3. According to Remark 1, we assume that the background density and the bottom viscous damping satisfy

$$2a\bar{\kappa}^\nu \|\Gamma\|_{L^1} < \frac{1}{\tau}. \quad (4.1)$$

4.1. A priori estimates

In this subsection, we will provide the a priori estimates for the Cauchy problem (2.1)–(2.3). Hence, we assume a priori assumption that for $s > \frac{N}{2} + 1$ and a sufficiently small $\delta > 0$,

$$\sup_{0 \leq t \leq T_0} (\|\sigma\|_{H^s} + \|u\|_{H^s}) \leq \delta. \quad (4.2)$$

We show the L^2 -norm estimates which contains the dissipation estimate for u . It should be noticed that there is no dissipation estimate of $\|\sigma\|_{L^2}$.

Lemma 4. Assume (σ, u) are classical solution of (2.1)–(2.2) and (4.1), (4.2) hold, then we have

$$\frac{1}{2} \frac{d}{dt} (\|\sigma\|_{L^2}^2 + \|u\|_{L^2}^2) + \left(\frac{1}{\tau} - 2a\bar{\kappa}^\nu \|\Gamma\|_{L^1}\right) \|u\|_{L^2}^2 \leq C\delta (\|u\|_{L^2}^2 + \|\nabla\sigma\|_{L^2}^2). \quad (4.3)$$

Proof. We multiply (2.1) and (2.2) by σ and u respectively, sum up and integrate over \mathbb{R}^N , we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int (|\sigma|^2 + |u|^2) dx + \frac{1}{\tau} \|u\|_{L^2}^2 \\ &= - \int (u \cdot \nabla\sigma\sigma + u \cdot \nabla u \cdot u) dx - \frac{1}{\nu} \int (\sigma \nabla \cdot u\sigma + \sigma \nabla\sigma \cdot u) dx \\ & \quad - a \int \int \Gamma(x-y)(u(x) - u(y)) \left(\frac{1}{\nu} \sigma(y) + \bar{\kappa}\right)^\nu dy \cdot u(x) dx \\ &= I_1 + I_2 + I_3. \end{aligned} \quad (4.4)$$

We estimate I_i item by item. Using Young's inequality, we have

$$I_1 = - \int (u \cdot \nabla\sigma\sigma + u \cdot \nabla u \cdot u) dx = - \int u \cdot \nabla\sigma\sigma dx + \frac{1}{2} \int |u|^2 \nabla \cdot u dx$$

$$\begin{aligned} &\leq C\|\sigma\|_{L^\infty}\|u\|_{L^2}\|\nabla\sigma\|_{L^2} + C\|\nabla\cdot u\|_{L^\infty}\|u\|_{L^2}^2 \\ &\leq C\delta(\|\nabla\sigma\|_{L^2}^2 + \|u\|_{L^2}^2), \end{aligned} \quad (4.5)$$

$$\begin{aligned} I_2 &= -\frac{1}{\nu} \int (\sigma\nabla\cdot u\sigma + \sigma\nabla\sigma\cdot u)dx = \frac{1}{\nu} \int \sigma u\cdot\nabla\sigma dx \leq C\|\sigma\|_{L^\infty}\|u\|_{L^2}\|\nabla\sigma\|_{L^2} \\ &\leq C\delta(\|\nabla\sigma\|_{L^2}^2 + \|u\|_{L^2}^2); \end{aligned} \quad (4.6)$$

$$\begin{aligned} I_3 &= -a \int u(x)dx \int \Gamma(x-y)(u(x)-u(y))\left(\frac{1}{\nu}\sigma(y)+\bar{\kappa}\right)^\nu dy \\ &= -a \int u^2(x) \int \Gamma(x-y)\left(\left(\frac{1}{\nu}\sigma(y)+\bar{\kappa}\right)^\nu - \bar{\kappa}^\nu\right) dy dx \\ &\quad + a \int u(x) \int \Gamma(x-y)u(y)\left(\left(\frac{1}{\nu}\sigma(y)+\bar{\kappa}\right)^\nu - \bar{\kappa}^\nu\right) dy dx \\ &\quad - a\bar{\kappa}^\nu \int u(x)dx \int \Gamma(x-y)(u(x)-u(y)) dy \\ &\leq 2a\|\Gamma\|_{L^1}\left\|\left(\frac{1}{\nu}\sigma+\bar{\kappa}\right)^\nu - \bar{\kappa}^\nu\right\|_{L^\infty}\|u\|_{L^2}^2 + 2a\kappa^\nu\|\Gamma\|_{L^1}\|u\|_{L^2}^2. \end{aligned}$$

Similar to (3.28), we can get

$$\left\|\left(\frac{1}{\nu}\sigma+\bar{\kappa}\right)^\nu - \bar{\kappa}^\nu\right\|_{L^\infty} \leq C(\|\sigma\|_{H^{s-1}}, \nu, \bar{\kappa})\|\sigma\|_{L^\infty} \leq C(\|\sigma\|_{H^{s-1}}, \nu, \bar{\kappa})\delta, \quad (4.7)$$

with the help of the Sobolev embedding theorem. Then, we obtain that

$$I_3 \leq C\delta\|u\|_{L^2}^2 + 2a\bar{\kappa}^\nu\|\Gamma\|_{L^1}\|u\|_{L^2}^2. \quad (4.8)$$

Collecting estimates (4.5)–(4.8) into (4.4), we obtain (4.3). \square

Next, we provide the high order energy estimates which contains the dissipation estimate for u .

Lemma 5. Assume $1 \leq r \leq s$ and (4.1), (4.2) hold, then for $s > \frac{N}{2} + 1$, we have

$$\frac{1}{2} \frac{d}{dt} (\|\nabla^r \sigma\|_{L^2}^2 + \|\nabla^r u\|_{L^2}^2) + \left(\frac{1}{\tau} - 2a\bar{\kappa}^\nu\|\Gamma\|_{L^1}\right)\|\nabla^r u\|_{L^2}^2 \leq C\delta(\|\nabla^r u\|_{L^2}^2 + \|\nabla^r \sigma\|_{L^2}^2) + C\delta\|u\|_{H^{s-1}}^2. \quad (4.9)$$

Proof. For $1 \leq r \leq s$, we apply ∇^r to (2.1), (2.2), and multiply the resulting identities by $\nabla^r \sigma$, $\nabla^r u$ respectively, sum up and integrating over \mathbb{R}^N to obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int (|\nabla^r \sigma|^2 + |\nabla^r u|^2) dx + \frac{1}{\tau} \|\nabla^r u\|_{L^2}^2 \\ &= - \int (\nabla^r(u \cdot \nabla \sigma) \nabla^r \sigma + \nabla^r(u \cdot \nabla u) \cdot \nabla^r u) dx \\ &\quad - \frac{1}{\nu} \int (\nabla^r(\sigma \nabla \cdot u) \nabla^r \sigma + \nabla^r(\sigma \nabla \sigma) \cdot \nabla^r u) dx \\ &\quad - a \int \nabla_x^r \left(\int \Gamma(x-y)u(x)\left(\frac{1}{\nu}\sigma(y)+\bar{\kappa}\right)^\nu dy \right) \cdot \nabla^r u(x) dx \\ &\quad + a \int \nabla_x^r \left(\int \Gamma(x-y)u(y)\left(\frac{1}{\nu}\sigma(y)+\bar{\kappa}\right)^\nu dy \right) \cdot \nabla^r u(x) dx \end{aligned}$$

$$= \sum_{i=1}^4 I_i. \quad (4.10)$$

Similar to the estimate of (3.12) and (3.13) in Section 3, by Hölder's inequality and Moser type inequality, we have

$$I_1 = - \int \nabla^r(u \cdot \nabla \sigma) \nabla^r \sigma + \nabla^r(u \cdot \nabla u) \cdot \nabla^r u dx \leq C \delta (\|\nabla^r \sigma\|_{L^2}^2 + \|\nabla^r u\|_{L^2}^2), \quad (4.11)$$

$$I_2 = -\frac{1}{\nu} \int \nabla^r(\sigma \nabla \cdot u) \nabla^r \sigma + \nabla^r(\sigma \nabla \sigma) \cdot \nabla^r u dx \leq C \delta (\|\nabla^r \sigma\|_{L^2}^2 + \|\nabla^r u\|_{L^2}^2). \quad (4.12)$$

Next, we estimate I_3 . Applying Moser type inequality and the Hölder inequality we have

$$\begin{aligned} I_3 &= -a \int \nabla^r u(x) \nabla_x^r \left(\int \Gamma(x-y) u(x) \left(\frac{1}{\nu} \sigma(y) + \bar{\kappa} \right)^{\nu} dy \right) \cdot dx \\ &= -a \int \nabla^r u(x) \nabla^r \left(u(x) \Gamma * \left(\frac{1}{\nu} \sigma + \bar{\kappa} \right)^{\nu}(x) \right) dx \\ &= -a \int \nabla^r u(x) \Gamma * \left(\frac{1}{\nu} \sigma + \bar{\kappa} \right)^{\nu}(x) \nabla^r u(x) dx \\ &\quad - a \int \nabla^r u(x) \left(\nabla^r \left(u(x) \Gamma * \left(\frac{1}{\nu} \sigma + \bar{\kappa} \right)^{\nu}(x) \right) - \Gamma * \left(\frac{1}{\nu} \sigma + \bar{\kappa} \right)^{\nu}(x) \nabla^r u(x) \right) dx \\ &\leq a \|\Gamma * \left(\frac{1}{\nu} \sigma + \bar{\kappa} \right)^{\nu}\|_{L^{\infty}} \|\nabla^r u\|_{L^2}^2 \\ &\quad + C \|\nabla^r u\|_{L^2} \left(\|u\|_{L^{\infty}} \|\nabla^r \Gamma * \left(\frac{1}{\nu} \sigma + \bar{\kappa} \right)^{\nu}\|_{L^2} + \|\nabla \Gamma * \left(\frac{1}{\nu} \sigma + \bar{\kappa} \right)^{\nu}\|_{L^{\infty}} \|\nabla^{r-1} u\|_{L^2} \right). \end{aligned} \quad (4.13)$$

To deal with the dissipation of u , we need the following estimates. Similar to (3.28), we can get

$$\begin{aligned} \|\Gamma * \left(\frac{1}{\nu} \sigma + \bar{\kappa} \right)^{\nu}\|_{L^{\infty}} &= \left\| \int \Gamma(x-y) \left(\frac{1}{\nu} \sigma(y) + \bar{\kappa} \right)^{\nu} dy \right\|_{L^{\infty}} \\ &\leq \left\| \int \Gamma(x-y) \left(\left(\frac{1}{\nu} \sigma + \bar{\kappa} \right)^{\nu} - \bar{\kappa}^{\nu} \right) dy \right\|_{L^{\infty}} + \left\| \int \bar{\kappa}^{\nu} \Gamma(x-y) dy \right\|_{L^{\infty}} \\ &\leq C (\|\sigma\|_{L^{\infty}}, \nu, \bar{\kappa}) \|\Gamma\|_{L^1} \|\sigma\|_{L^{\infty}} + \bar{\kappa}^{\nu} \|\Gamma\|_{L^1}. \end{aligned} \quad (4.14)$$

Using the differential properties of the convolution and the Sobolev embedding theorem, we can compute

$$\begin{aligned} \|\nabla \Gamma * (\nu \sigma + \bar{\kappa})^{\nu}\|_{L^{\infty}} &= \|\nabla_x \int \Gamma(x-y) \left(\frac{1}{\nu} \sigma(y) + \bar{\kappa} \right)^{\nu} dy\|_{L^{\infty}} = \left\| \int \Gamma(x-y) \nabla_y \left(\frac{1}{\nu} \sigma(y) + \bar{\kappa} \right)^{\nu} dy \right\|_{L^{\infty}} \\ &\leq \|\Gamma\|_{L^1} \left\| \frac{1}{\nu} \sigma(y) + \bar{\kappa} \right)^{\nu-1} \|_{L^{\infty}} \|\nabla \sigma\|_{L^{\infty}} \\ &\leq C (\|\sigma\|_{L^{\infty}}, \nu, \bar{\kappa}) \|\Gamma\|_{L^1} \|\nabla \sigma\|_{H^{s-1}}, \end{aligned} \quad (4.15)$$

and similar to (3.16), we have

$$\|\nabla^r \Gamma * (\nu \sigma + \bar{\kappa})^{\nu}\|_{L^2} \leq C (\|\sigma\|_{L^{\infty}}, \nu, \bar{\kappa}) \|\Gamma\|_{L^1} \|\sigma\|_{H^r}, \quad (4.16)$$

where $C(\|\sigma\|_{L^\infty}, \nu, \bar{\kappa})$ is non-decreasing in $\|\sigma\|_{L^\infty}$.

Substituting (4.14)–(4.16) for (4.13), we obtain that

$$\begin{aligned} I_3 &\leq C\left(\|\sigma\|_{L^\infty}\|\nabla^r u\|_{L^2}^2 + \|\sigma\|_{H^r}\|u\|_{H^{s-1}}\|\nabla^r u\|_{L^2} + \|\nabla\sigma\|_{H^{s-1}}\|\nabla^{r-1}u\|_{L^2}\|\nabla^r u\|_{L^2}\right) \\ &\quad + a\bar{\kappa}^\nu\|\mathbf{\Gamma}\|_{L^1}\|\nabla^r u\|_{L^2}^2 \\ &\leq C\delta(\|\nabla^r u\|_{L^2}^2 + \|u\|_{H^{s-1}}^2) + a\bar{\kappa}^\nu\|\mathbf{\Gamma}\|_{L^1}\|\nabla^r u\|_{L^2}^2, \end{aligned} \quad (4.17)$$

where the Sobolev embedding theorem is used.

Similar to estimate of I_3 , we can deduce that

$$\begin{aligned} I_4 &= a \int \nabla^r u(x) \nabla_x^r \left(\int \mathbf{\Gamma}(x-y)u(y) \left(\frac{1}{\nu}\sigma(y) + \bar{\kappa}\right)^\nu dy \right) dx \\ &= a \int \nabla^r u(x) \int \mathbf{\Gamma}(x-y) \nabla_y^r \left(u(y) \left(\frac{1}{\nu}\sigma(y) + \bar{\kappa}\right)^\nu \right) dy dx \\ &= a \int \nabla^r u(x) \int \mathbf{\Gamma}(x-y) \nabla_y^r \left(u(y) \left(\frac{1}{\nu}\sigma(y) + \bar{\kappa}\right)^\nu \right) dy dx \\ &= a \int \nabla^r u(x) \int \mathbf{\Gamma}(x-y) \nabla_y^r u(y) \left(\frac{1}{\nu}\sigma(y) + \bar{\kappa}\right)^\nu dy dx \\ &\quad + a \int \nabla^r u(x) \int \mathbf{\Gamma}(x-y) (\nabla_y(u(y) \left(\frac{1}{\nu}\sigma(y) + \bar{\kappa}\right)^\nu) - \nabla_y^r u(y) \left(\frac{1}{\nu}\sigma(y) + \bar{\kappa}\right)^\nu) dy dx \\ &\leq C\delta(\|\nabla^r u\|_{L^2}^2 + \|u\|_{H^{s-1}}^2) + a\bar{\kappa}^\nu\|\mathbf{\Gamma}\|_{L^1}\|\nabla^r u\|_{L^2}^2. \end{aligned} \quad (4.18)$$

Collecting estimates (4.11), (4.12), (4.17), (4.18) and put them into (4.10), we obtain that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\nabla^r \sigma\|_{L^2}^2 + \|\nabla^r u\|_{L^2}^2) + \left(\frac{1}{\tau} - 2a\bar{\kappa}^\nu\|\mathbf{\Gamma}\|_{L^1}\right) \|\nabla^r u\|_{L^2}^2 \\ \leq C\delta(\|\nabla^r u\|_{L^2}^2 + \|u\|_{H^{s-1}}^2) + C\delta\|\nabla^r \sigma\|_{L^2}^2. \end{aligned} \quad (4.19)$$

□

Now, we will bring forward the dissipation estimate for σ .

Lemma 6. For $1 \leq r \leq s$,

$$\frac{d}{dt} \int \nabla^{r-1} u \nabla^r \sigma dx + \frac{\bar{\kappa}}{4} \|\nabla^r \sigma\|_{L^2}^2 \leq C\|u\|_{H^s}^2 + C\delta(\|\nabla^r \sigma\|_{L^2}^2 + \|\nabla\sigma\|_{H^{s-1}}^2). \quad (4.20)$$

Proof. First, we can directly calculate to obtain

$$\frac{d}{dt} \int \nabla^{r-1} u \nabla^r \sigma dx = \int \nabla^{r-1} u \nabla^r \sigma_t dx + \int \nabla^{r-1} u_t \nabla^r \sigma dx. \quad (4.21)$$

Next, we will estimate the right-hand two terms of the upper equation. Let $1 \leq r \leq s$, applying ∇^r to (2.1), multiplying it by $\nabla^{r-1} u$ and integrating over \mathbb{R}^N we obtain

$$\int \nabla^{r-1} u \nabla^r \sigma_t dx = -\bar{\kappa} \int \nabla \cdot \nabla^r u \cdot \nabla^{r-1} u dx$$

$$\begin{aligned}
& - \int \nabla^r(u \cdot \nabla \sigma) \cdot \nabla^{r-1} u dx - \frac{1}{\nu} \int \nabla^r(\sigma \nabla \cdot u) \cdot \nabla^{s-1} u dx \\
& \leq C \|\nabla^r u\|_{L^2}^2 + C \|\nabla^r u\|_{L^2} \|\nabla^{r-1}(u \cdot \nabla \sigma)\|_{L^2} + C \|\nabla^r u\|_{L^2} \|\nabla^{r-1}(\sigma \nabla \cdot u)\|_{L^2}.
\end{aligned} \tag{4.22}$$

In order to get the estimate of (4.22), we estimate $\|\nabla^{r-1}(u \cdot \nabla \sigma)\|_{L^2}$ and $\|\nabla^{r-1}(\sigma \cdot \nabla u)\|_{L^2}$. By Moser type inequality and Sobolev embedding theorem, we have

$$\begin{aligned}
\|\nabla^{r-1}(u \cdot \nabla \sigma)\|_{L^2} & \leq C \|u\|_{L^\infty} \|\nabla^r \sigma\|_{L^2} + C \|\nabla \sigma\|_{L^\infty} \|\nabla^{r-1} u\|_{L^2} \\
& \leq C \|u\|_{H^{s-1}} \|\nabla^r \sigma\|_{L^2} + C \|\nabla \sigma\|_{H^{s-1}} \|\nabla^{r-1} u\|_{L^2} \\
& \leq C \delta (\|\nabla^r \sigma\|_{L^2} + \|\nabla \sigma\|_{H^{s-1}}).
\end{aligned} \tag{4.23}$$

$$\begin{aligned}
\|\nabla^{r-1}(\sigma \cdot \nabla u)\|_{L^2} & \leq \|\sigma\|_{L^\infty} \|\nabla^r u\|_{L^2} + \|\nabla u\|_{L^\infty} \|\nabla^{r-1} \sigma\|_{L^2} \\
& \leq C \|\sigma\|_{H^{s-1}} \|\nabla^r u\|_{L^2} + C \|\nabla u\|_{H^{s-1}} \|\nabla^{r-1} \sigma\|_{L^2} \\
& \leq C \delta (\|\nabla^r u\|_{L^2} + \|\nabla u\|_{H^{s-1}})
\end{aligned} \tag{4.24}$$

Then, substituting (4.23) and (4.24) for (4.22) and applying Young's inequality, we can deduce that

$$\int \nabla^{r-1} u \nabla^r \sigma dx \leq C (\|\nabla^r u\|_{L^2}^2 + \|\nabla u\|_{H^{s-1}}^2) + C \delta (\|\nabla^r \sigma\|_{L^2}^2 + \|\nabla \sigma\|_{H^{s-1}}^2). \tag{4.25}$$

Now, we estimate the second item on the right side of (4.21). Let $1 \leq r \leq s$, applying ∇^{r-1} to (2.2), multiplying it by $\nabla^r \sigma$ and integrating over \mathbb{R}^N we obtain

$$\begin{aligned}
& \int \nabla^{r-1} u_t \nabla^r \sigma dx + \bar{\kappa} \|\nabla^r \sigma\|_{L^2}^2 = -\frac{1}{\tau} \int \nabla^{r-1} u \nabla^r \sigma dx \\
& - \int \nabla^{r-1}(u \cdot \nabla u) \nabla^r \sigma dx - \frac{1}{\nu} \int \nabla^{r-1}(\sigma \nabla \sigma) \nabla^r \sigma dx \\
& - a \int \nabla^{r-1} \left(\int \Gamma(x-y) \left(\frac{1}{\nu} \sigma(y) + \bar{\kappa} \right) u(x) dy \right) \nabla^r \sigma(x) dx \\
& + a \int \nabla^{r-1} \left(\int \Gamma(x-y) \left(\frac{1}{\nu} \sigma(y) + \bar{\kappa} \right) u(y) dy \right) \nabla^r \sigma(x) dx = \sum_{i=1}^5 I_i.
\end{aligned} \tag{4.26}$$

Using Young's and Holder's inequality, we have

$$I_1 = -\frac{1}{\tau} \int \nabla^{r-1} u \nabla^r \sigma dx \leq C \|\nabla^{r-1} u\|_{L^2}^2 + \frac{\bar{\kappa}}{4} \|\nabla^r \sigma\|_{L^2}^2. \tag{4.27}$$

By a method similar to the estimate for $\|\nabla^{r-1}(u \cdot \nabla \sigma)\|_{L^2}$ and $\|\nabla^{r-1}(\sigma \cdot \nabla u)\|_{L^2}$, we have

$$I_2 = - \int \nabla^{r-1}(u \cdot \nabla u) \nabla^r \sigma dx \leq C (\|\nabla^r u\|_{L^2}^2 + \|\nabla u\|_{H^{s-1}}^2) + C \delta \|\nabla^r \sigma\|_{L^2}^2, \tag{4.28}$$

$$I_3 = -\frac{1}{\nu} \int \nabla^{r-1}(\sigma \nabla \sigma) \nabla^r \sigma dx \leq C \delta (\|\nabla^r \sigma\|_{L^2}^2 + \|\nabla \sigma\|_{H^{s-1}}^2). \tag{4.29}$$

Next, we estimate I_4 . Similar to (3.15) and (3.16), using Young's inequality and Moser type inequality, we have

$$\begin{aligned}
 I_4 &= -a \int \nabla^r \sigma(x) \nabla_x^{r-1} \left(u(x) \int \Gamma(x-y) \left(\frac{1}{\nu} \sigma(y) + \bar{\kappa} \right)^\nu dy \right) dx \\
 &\leq \|\nabla^r \sigma(x)\|_{L^2} \|\nabla^{r-1} \left(u(x) \Gamma * \left(\frac{1}{\nu} \sigma + \bar{\kappa} \right)^\nu(x) \right)\|_{L^2} \\
 &\leq \|\nabla^r \sigma\|_{L^2} \|\nabla^{r-1} u\|_{L^2} \|\Gamma * \left(\frac{1}{\nu} \sigma + \bar{\kappa} \right)^\nu\|_{L^\infty} + \|\nabla^r \sigma\|_{L^2} \|u\|_{L^\infty} \|\Gamma * \left(\frac{1}{\nu} \sigma + \bar{\kappa} \right)^\nu\|_{L^2} \\
 &\leq C \|\Gamma\|_{L^1} \|\nabla^r \sigma\|_{L^2} \left(\|\nabla^{r-1} u\|_{L^2} + \|u\|_{H^{s-1}} \|\sigma\|_{H^{r-1}} \right) \\
 &\leq C \|u\|_{H^{s-1}}^2 + \frac{\bar{\kappa}}{4} \|\nabla^r \sigma\|_{L^2}^2.
 \end{aligned} \tag{4.30}$$

Applying the same method as I_4 , we can get

$$\begin{aligned}
 I_5 &= a \int \nabla_x^{r-1} \left(\int \Gamma(x-y) \left(\frac{1}{\nu} \sigma(y) + \bar{\kappa} \right)^\nu u(y) dy \right) \nabla^r \sigma(x) dx \\
 &= a \int \int \Gamma(x-y) \nabla_y^{r-1} \left(\left(\frac{1}{\nu} \sigma(y) + \bar{\kappa} \right)^\nu u(y) \right) dy \nabla^r \sigma(x) dx \\
 &\leq C \|u\|_{H^{s-1}}^2 + \frac{\bar{\kappa}}{4} \|\nabla^r \sigma\|_{L^2}^2.
 \end{aligned} \tag{4.31}$$

Collecting all estimates of I_i , we have

$$\int \nabla^{r-1} u_i \nabla^r \sigma dx + \frac{\bar{\kappa}}{4} \|\nabla^r \sigma\|_{L^2}^2 \leq C \|u\|_{H^{s-1}}^2 + C \delta (\|\nabla^r \sigma\|_{L^2}^2 + \|\nabla \sigma\|_{H^{s-1}}^2). \tag{4.32}$$

Combining (4.25) and (4.32) to obtain

$$\frac{d}{dt} \int \nabla^{r-1} u \nabla^r \sigma dx + \frac{\bar{\kappa}}{4} \|\nabla^r \sigma\|_{L^2}^2 \leq C \|u\|_{H^s}^2 + C \delta (\|\nabla^r \sigma\|_{L^2}^2 + \|\nabla \sigma\|_{H^{s-1}}^2). \tag{4.33}$$

□

4.2. The proof of global existence

In this subsection, we construct the global-in-time solution by combining the local existence theory.

We sum up the estimate (4.9) in Lemma 5 from $r = 1$ to s , and then add the estimate (4.3) in Lemma 4, since δ is small and conditions (4.1), we can deduce that there exists $\varepsilon_1 > 0$, $C_1 > 0$ such that

$$\frac{d}{dt} (\|\sigma\|_{H^s}^2 + \|u\|_{H^s}^2) + \varepsilon_1 \|u\|_{H^s}^2 \leq C_1 \delta \sum_{r=1}^s \|\nabla^r \sigma\|_{L^2}^2. \tag{4.34}$$

Summing up the estimates (4.20) in Lemma 6 from $r = 1$ to s , for sufficient small δ , there exist $\varepsilon_2 > 0$, $C_2 > 0$, such that

$$\frac{d}{dt} \sum_{r=1}^s \int \nabla^{r-1} u \nabla^r \sigma dx + \varepsilon_2 \|\nabla \sigma\|_{H^{s-1}}^2 \leq C_2 \|u\|_{H^s}^2. \tag{4.35}$$

Multiplying (4.35) by $\frac{2C_1\delta}{\varepsilon_2}$, adding it to (4.34), since δ is small, then there exists a constant $\varepsilon_3 > 0$ such that

$$\frac{d}{dt}(\|\sigma\|_{H^s}^2 + \|u\|_{H^s}^2 + \frac{2C_1\delta}{\varepsilon_2} \sum_{r=1}^s \int \nabla^{r-1} u \nabla^r \sigma dx) + \varepsilon_3(\|u\|_{H^s}^2 + \|\nabla \sigma\|_{H^{s-1}}^2) \leq 0. \quad (4.36)$$

Note that there exist constant $C_3 > 0$ such that

$$C_3^{-1}(\|\sigma\|_{H^s}^2 + \|u\|_{H^s}^2) \leq (\|\sigma\|_{H^s}^2 + \|u\|_{H^s}^2 + \frac{2C_1\delta}{\varepsilon_2} \sum_{r=1}^s \int \nabla^{r-1} u \nabla^r \sigma dx) \leq C_3(\|\sigma\|_{H^s}^2 + \|u\|_{H^s}^2). \quad (4.37)$$

Integrating directly in time, with help of (4.37), we obtain

$$\sup_{0 \leq \tilde{t} \leq t} (\|\sigma\|_{H^s}^2(\tilde{t}) + \|u\|_{H^s}^2(\tilde{t})) + \int_0^t (\|u\|_{H^s}^2(\tilde{t}) + \|\nabla \sigma\|_{H^{s-1}}^2(\tilde{t})) d\tilde{t} \leq C_3^2(\|\sigma_0\|_{H^s}^2 + \|u_0\|_{H^s}^2). \quad (4.38)$$

Finally, we can use Theorem 1 and (4.38) to prove the global existence of classical solutions for (2.1)–(2.2). Applying the local existence of the classical solution in Theorem 1, we deduce that for any positive constant $\delta_0 < \delta$, there is a positive constant T_0 depending only on δ_0 , δ such that if $\|\sigma_0\|_{H^s} + \|u_0\|_{H^s} < \delta_0$, then the solution of the Cauchy problem (2.1)–(2.3) satisfies

$$\sup_{0 \leq t \leq T_0} \|\sigma\|_{H^s} + \|u\|_{H^s} < \delta. \quad (4.39)$$

Then, we choose that

$$\delta_0 = \frac{\delta}{\sqrt{2(1 + C_3^2)}}, \quad (4.40)$$

where δ and C_3^2 are given in (4.2) and (4.38), respectively. Let us define the maximal existence time $T_{max} > 0$ of the system (2.1)–(2.2) by

$$T_{max} := \sup\{t \geq 0 : \sup_{0 \leq \tilde{t} \leq T_0} \|\sigma\|_{H^s} + \|u\|_{H^s} < \delta\}. \quad (4.41)$$

Suppose $T_{max} < \infty$, then we can use the continuation argument and (4.38) to get

$$\delta^2 = \sup_{0 \leq t \leq T_{max}} \|\sigma\|_{H^s}^2 + \|u\|_{H^s}^2 \leq C_3^2(\|\sigma_0\|_{H^s}^2 + \|u_0\|_{H^s}^2) < C_3^2 \frac{\delta^2}{2(1 + C_3^2)} < \frac{\delta^2}{2}. \quad (4.42)$$

This is a contradiction, hence, we can conclude that $T_{max} = \infty$.

In summary, we have completed the proof of Theorem 2.2.

Acknowledgments

The publication of this article was funded by the Ministry of Science, Research and the Arts Baden-Württemberg and the University of Mannheim.

Lining Tong and Shu Wang would like to thank the University of Mannheim for the hospitality during their scientific visiting, where the main part of this article has been obtained.

Lining Tong is supported by NSFC(No. 11771274, 11901379). Simone Göttlich is supported by the German Research Foundation, DFG grant GO 1920/7-1. Shu Wang is supported by NSFC(No. 11831003, 11771031, 11531010).

Conflict of interest

The authors declare no conflict of interest.

References

1. S. Benzoni-Gavage, D. Serre, *Multi-Dimensional Hyperbolic Partial Differential Equations: First-Order Systems and Applications*, Oxford University Press, 2007.
2. J. A. Carrillo, Y. P. Choi, E. Tadmor, et al. *Critical thresholds in 1D Euler equations with non-local forces*, Math. Mod. Meth. Appl. S., **26** (2016), 185–206.
3. J. A. Carrillo, Y. P. Choi, E. Zatorska, *On the pressureless damped Euler-Poisson equations with quadratic confinement: Critical thresholds and large-time behavior*, Math. Mod. Meth. Appl. S., **26** (2016), 2311–2340.
4. J. A. Carrillo, E. Feireisl, P. Gwiazda, et al. *Weak solutions for Euler systems with non-local interactions*, J. Lond. Math. Soc., **95** (2017), 705–724.
5. J. A. Carrillo, A. Wróblewska-Kamińska, E. Zatorska, *On long-time asymptotics for viscous hydrodynamic models of collective behavior with damping and nonlocal interactions*, Math. Mod. Meth. Appl. S., **29** (2019), 31–63.
6. J. Che, L. Chen, S. Göttlich, et al. *Existence of a classical solution to complex material flow problems*, Math. Method. Appl. Sci., **39** (2016), 4069–4081.
7. Y. P. Choi, *The global Cauchy problem for compressible Euler equations with a nonlocal dissipation*, Math. Mod. Meth. Appl. S., **29** (2019), 185–207.
8. E. DiBenedetto, *Real Analysis*, Birkhäuser, 2002.
9. T. Do, A. Kiselev, L. Ryzhik, et al. *Global regularity for the fractional Euler alignment system*, Arch. Ration. Mech. Anal., **228** (2018), 1–37.
10. I. Gasser, L. Hsiao, H. Li, *Large time behavior of solutions of the bipolar hydrodynamical model for semiconductors*, J. Differ. Equations, **192** (2003), 326–335.
11. S. Göttlich, A. Klar, S. Tiwari, *Complex material flow problems: A multi-scale model hierarchy and particle methods*, J. Eng. Math., **92** (2015), 15–29.

12. S. Y. Ha, F. Huang, Y. Wang, *A global unique solvability of entropic weak solution to the one-dimensional pressureless Euler system with a flocking dissipation*, *J. Differ. Equations*, **257** (2014), 1333–1371.
13. S. Y. Ha, M. J. Kang, B. Kwon, *A hydrodynamic model for the interaction of Cucker-Smale particles and incompressible fluid*, *Math. Mod. Meth. Appl. S.*, **24** (2014), 2311–2359.
14. S. Y. Ha, M. J. Kang, B. Kwon, *Emergent Dynamics for the Hydrodynamic Cucker-Smale System in a Moving Domain*, *SIAM J. Math. Anal.*, **47** (2015), 3813–3831.
15. L. Hsiao, T. P. Liu, *Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping*, *Commun. Math. Phys.*, **143** (1992), 599–605.
16. F. Huang, P. Marcati, R. Pan, *Convergence to the Barenblatt solution for the compressible Euler equations with damping and vacuum*, *Arch. Ration. Mech. Anal.*, **176** (2005), 1–24.
17. T. Kato, *The Cauchy problem for quasi-linear symmetric hyperbolic systems*, *Arch. Ration. Mech. Anal.*, **58** (1975), 181–205.
18. A. Kiselev, C. Tan, *Global regularity for 1D Eulerian dynamics with singular interaction forces*, *SIAM J. Math. Anal.*, **50** (2018), 6208–6229.
19. P. D. Lax, *Hyperbolic systems of conservation laws and the mathematical theory of shock waves*, In: *Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics*, Society for Industrial and Applied Mathematics, Philadelphia, 1973, 1–48.
20. E. H. Lieb, M. Loss, *Analysis*, American Mathematical Society, Providence, 2001.
21. A. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*, Springer-Verlag, New York, 1984.
22. A. Majda, A. L. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge University Press, 2002.
23. A. Matsumura, T. Nishida, *The initial value problem for the equations of motion of viscous and heat-conductive gases*, *J. Math. Kyoto U.*, **20** (1980), 67–104.
24. T. C. Sideris, B. Thomases, D. Wang, *Long time behavior of solutions to the 3D compressible Euler equations with damping*, *Commun. Part. Diff. Eq.*, **28** (2003), 795–816.
25. Z. Tan, Y. Wang, *Global solution and large-time behavior of the 3D compressible Euler equations with damping*, *J. Differ. Equations*, **254** (2013), 1686–1704.
26. W. Wang, T. Yang, *The pointwise estimates of solutions for Euler equations with damping in multi-dimensions*, *J. Differ. Equations*, **173** (2001), 410–450.
27. T. Yang, C. Zhu, H. Zhao, *Global smooth solutions for a class of quasilinear hyperbolic systems with dissipative terms*, *P. Roy. Soc. Edinb. A*, **127** (1997), 1311–1324.

Appendix

In the appendix, we present several lemmas used in the existence proof in Sections 3 and 4.

Lemma 7. (Moser type inequality) *Let $s \geq 1$ and $1 \leq r \leq s$, then for any pair of functions $f, g \in H^s \cap L^\infty$, we have*

$$\|\nabla^r(fg)\|_{L^2} \leq C(\|\nabla^s f\|_{L^2}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|\nabla^s g\|_{L^2}). \quad (4.43)$$

Furthermore if $\nabla f \in L^\infty(\mathbb{R}^N)$ we have

$$\|\nabla^r(fg) - f\nabla^r g\|_{L^2} \leq C(\|\nabla f\|_{L^\infty}\|\nabla^{s-1}g\|_{L^2} + \|g\|_{L^\infty}\|\nabla^s f\|_{L^2}). \quad (4.44)$$

Proof. See Lemma 3.4 in [22]. □

Lemma 8. (Young's inequality) *Let $p, q, r \geq 1$ and $1/p + 1/q + 1/r = 2$. Let $f \in L^p, g \in L^q$ and $h \in L^r$. Then*

$$\begin{aligned} \int f(x)(g * h)(x)dx &= \int \int f(x)g(x-y)h(x)dx dy \\ &\leq C_{p,q,r,N}\|f\|_{L^p}\|g\|_{L^q}\|h\|_{L^r}. \end{aligned} \quad (4.45)$$

Proof. See Theorem 4.2 in [20]. □

Lemma 9. (Gagliardo-Nirenberg inequality) *Let $f \in W_0^{1,q} \cap L^r$ for some $r \leq 1$. There exists a constant C depending upon N, p, r such that*

$$\|u\|_{L^p} \leq C\|\nabla u\|_{L^q}^\theta \|u\|_{L^r}^{1-\theta}, \quad (4.46)$$

where $\theta \in [0, 1]$ and $p, q \leq 1$ are linked by

$$\theta = \left(\frac{1}{r} - \frac{1}{p}\right)\left(\frac{1}{N} - \frac{1}{q} + \frac{1}{r}\right)^{-1}. \quad (4.47)$$

Proof. See Theorem 1.1 of the Chapter 10 in [8]. □



AIMS Press

©2020 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/4.0>)