Journal of Integrable Systems (2020) **00**, 1–37
doi: 10.1093/integr/xyaa005

Properly embedded minimal annuli in $\mathbb{S}^2 \times \mathbb{R}$

L. HAUSWIRTH
Université Paris-Est, LAMA (UMR 8050), UPEMLV, UPEC, CNRS, F-77454, Marne-la-Vallée, France
hauswirth@univ-mlv.fr

M. KILIAN[†]
School of Mathematical Sciences, University College Cork, Cork, Ireland
[†]Corresponding author. Email: m.kilian@ucc.ie

AND

M. U. SCHMIDT
Institut für Mathematik, Universität Mannheim, B6, 28-29, 68131 Mannheim, Germany
schmidt@math.uni-mannheim.de

Communicated by: Prof. Alexander I. Bobenko

[Received on 15 January 2018; accepted on 3 July 2020]

We prove that every properly embedded minimal annulus in $\mathbb{S}^2 \times \mathbb{R}$ is foliated by circles. We show that such minimal annuli are given by periodic harmonic maps $\mathbb{C} \to \mathbb{S}^2$ of finite type. Such harmonic maps are parameterized by spectral data, and we show that continuous deformations of the spectral data preserve the embeddedness of the corresponding annuli. A curvature estimate of Meeks and Rosenberg is used to show that each connected component of spectral data of embedded minimal annuli contains a maximum of the flux of the third coordinate. A classification of these maxima allows us to identify the spectral data of properly embedded minimal annuli with the spectral data of minimal annuli foliated by circles.

Keywords: minimal surface, harmonic map, spectral curve

*M*athematics Subject Classification: 53A10, 37K10.

1. Introduction

In $\mathbb{S}^2 \times \mathbb{R}$, there is a two-parameter family of embedded minimal annuli foliated by horizontal constant curvature curves of $\mathbb{S}^2$. A member of this family is called an Abresch annulus. The simplest non-compact examples are the totally geodesic $\Gamma \times \mathbb{R}$, where Γ is a simple closed geodesic on $\mathbb{S}^2$. There exists a one-parameter family of periodic properly embedded annuli which are small graphs over $\Gamma \times \mathbb{R}$. These examples were described analytically by Pedrosa and Ritore [1] and they called them unduloids. They appear in the isoperimetric profile of $\mathbb{S}^2 \times \mathbb{S}^1$. These examples are rotational surfaces around vertical geodesics. A one-parameter helicoidal family, obtained by rotating a great circle on $\mathbb{S}^2$ at a constant rate in the third coordinate about an axis passing through a pair of antipodal points on the rotated great circle was constructed by Rosenberg [2].

A two-parameter family of deformations of previous examples was constructed by the first-named author [3], and by Meeks and Rosenberg (see [4, Section 2]) using variational arguments. This involves solving a Plateau problem with boundaries given by two geodesics Γ_1 and Γ_2 in parallel sections $\mathbb{S}^2 \times \{t_1\}$ and $\mathbb{S}^2 \times \{t_2\}$. The stable annulus bounded by these geodesics is foliated by horizontal constant curvature curves (see Theorem 2.4). Schwarz symmetry along boundary geodesics then gives a complete and properly embedded example. The two parameters (up to isometries of $\mathbb{S}^2 \times \mathbb{R}$) of such compact annuli are the distance between the two sections including the boundary, and the position of one geodesic in $\mathbb{S}^2 \times \{t\}$, keeping the other fixed. They are periodic in the third direction, foliated by constant curvature curves of $\mathbb{S}^2$ and have a vertical plane of symmetry.

Constant mean curvature (CMC) tori in $\mathbb{R}^3$ give rise to minimal annuli in $\mathbb{S}^2 \times \mathbb{R}$ under certain conditions as follows. The Gauß map of a CMC torus is a harmonic map $G : \mathbb{T}^2 \to \mathbb{S}^2$. Its holomorphic quadratic differential is

$$Q = \langle G_z, G_z\rangle (\mathrm{d}z)^2$$

where $z = x + \mathbb{i}y$ is a global holomorphic coordinate on the torus. Since $\langle G_z, G_z\rangle : \mathbb{T}^2 \to \mathbb{C}$ is holomorphic, it is a non-zero constant $\langle G_z, G_z\rangle \equiv c \in \mathbb{C}^\times$. After a linear change of coordinate, we can assume $c = \pm 1/4$. Then the map $X : \mathbb{C} \to \mathbb{S}^2 \times \mathbb{R}$,

$$X(z) = (G(z), \mathrm{Re}(-2\mathbb{i}\sqrt{c}z))$$

is conformal and harmonic, and thus locally a minimal surface in $\mathbb{S}^2 \times \mathbb{R}$ (possibly branched). We can choose the sign of $c \in \mathbb{R}$ in such a way that the large curvature line on the (CMC) torus corresponds to a horizontal curve in $\mathbb{S}^2 \times \{t\}$. In this case, X is an immersion. If the Gauß map G is periodic along this horizontal curve then, we have a minimal annulus.

The Gauß map of the flat CMC cylinder in $\mathbb{R}^3$ yields the totally geodesic annulus in $\mathbb{S}^2 \times \mathbb{R}$, and the Gauß maps of Delaunay surfaces yield the unduloids and the helicoids in $\mathbb{S}^2 \times \mathbb{R}$ under this correspondence. Supplementing these rotational examples, there is a two-parameter family of harmonic maps studied by Abresch [5]. To describe the equations of Wente tori in $\mathbb{R}^3$, Abresch studies conformal CMC $H = 1/2$ immersions with large lines or small lines of curvature contained in a plane. He studies constant mean curvature surfaces parametrized by $\mathbb{R}^2$, with the coordinate axes x and y yielding the lines of principal curvature, and solves closing conditions for the surface to obtain CMC tori. On these lines, the image of the Gauß map is a circle in $\mathbb{S}^2$.

In a conformal parametrization with $|c| = 1/4$, the metric of the minimal annulus in $\mathbb{S}^2 \times \mathbb{R}$ is given by $\mathrm{d}s^2 = \cosh^2\omega\, |\mathrm{d}z|^2$, where $\omega : \mathbb{C} \to \mathbb{R}$ is a solution of the sinh-Gordon equation

$$(**) \qquad \Delta\omega + \sinh\omega\cosh\omega = 0,$$

where Δ denotes the Laplacian of the flat metric $|\mathrm{d}z|^2$. The metric of the corresponding CMC $H = 1/2$ surface in $\mathbb{R}^3$ is given by $\mathrm{d}\tilde{s}^2 = e^{2\omega}|\mathrm{d}z|^2$. Abresch classified all real analytic solutions $\omega : \mathbb{C} \to \mathbb{R}$ of the system

$$(I) \qquad \begin{cases} \Delta\omega + \sinh\omega\cosh\omega = 0 \\ \sinh(\omega)(\omega_{xy}) - \cosh(\omega)(\omega_x)(\omega_y) = 0 \end{cases},$$

where the second equation is the condition that all small curvature lines are planar. He proves that this family contains CMC $H = 1/2$ tori. These tori have doubly periodic harmonic maps G and so yield minimal annuli by considering $X(z) = (G(z), x)$ (the map $X(z) = (G(z), y)$ is branched). The horizontal curves of this family have non-constant curvature. Abresch also studies solutions of the system

$$(II) \quad \begin{cases} \Delta\omega + \sinh\omega\cosh\omega = 0 \\ \cosh(\omega)(\omega_{xy}) - \sinh(\omega)(\omega_x)(\omega_y) = 0 \end{cases},$$

where the second equation is the condition that all large lines of curvature are planar. A solution ω induces a CMC immersion of $\mathbb{C}$ in $\mathbb{R}^3$ and a doubly periodic Gauß map $G : \mathbb{R}^2 \to \mathbb{S}^2$. The second equation is the condition that the immersion $X(z) = (G(z), y)$ has horizontal constant curvature curves and parameterizes the whole Abresch family. It was conjectured by Meeks–Rosenberg [4] that any properly embedded genus zero minimal surface in $\mathbb{S}^2 \times \mathbb{R}$ belongs to this family. In this direction, Hoffman–White [6] proved that if a properly embedded annulus of $\mathbb{S}^2 \times \mathbb{R}$ contains a vertical geodesic, then it is a helicoid type example. The first author in [3] characterized Abresch annuli as the only annuli which are foliated by horizontal curvature curves. Our main result confirms this conjecture:

Main theorem. A properly embedded minimal annulus in $\mathbb{S}^2 \times \mathbb{R}$ is an Abresch annulus.

The proof combines methods from geometric analysis with techniques of integrable systems. The technique is similar to the one used in the classification of Alexandrov embedded CMC tori of $\mathbb{S}^3$ given by the authors in [7, 8]. We use spectral curve theory and study the moduli space of properly embedded annuli of finite type. Locally CMC surfaces in $\mathbb{S}^3$ and minimal annuli in $\mathbb{S}^2 \times \mathbb{R}$ both give rise to solutions of the sinh-Gordon equation, but the global closing conditions are different in a subtle way. One of the differences is in the closing condition of the third coordinate function which give some additional real analytic constraint. Furthermore, in $\mathbb{S}^2 \times \mathbb{R}$ there exists a 2-parameter family of deformations of the totally geodesic annulus, while in $\mathbb{S}^3$, there is only a 1-parameter family of rotational Alexandrov embedded CMC annuli.

The first ingredient is a linear area growth and curvature estimate of Meeks–Rosenberg. Due to [9, Theorem 7.1] (see Theorem 3.2 below), the curvature K and thus the area growth of a properly embedded minimal annulus X in $\mathbb{S}^2 \times \mathbb{R}$ are bounded by constants depending on the flux of the third coordinate $h : \mathbb{C}/\tau\mathbb{Z} \to \mathbb{R}$ along horizontal sections. Properly immersed annuli in $\mathbb{S}^2 \times \mathbb{R}$ are parabolic (see Theorem 2.2) and the flux corresponds to the length of the period τ of the corresponding solution of the sinh-Gordon equation $(**)$ (see Lemma 3.1). If the flux $|\tau| \geq \epsilon_0$, there is a constant $C_1 > 0$ depending only on ϵ_0 such that

$$|K| \leq C_1(\epsilon_0).$$

We improve the linear area growth estimate of [9, Theorem 1.1] using parabolicity, and prove in Lemma 3.3 that there is a constant $C_2 > 0$ depending only on ϵ_0 such that for any $t > 0$,

$$\text{Area}(X \cap \mathbb{S}^2 \times [-t, t]) \leq C_2(\epsilon_0)t.$$

This estimate has two consequences. First, it implies that properly embedded minimal annuli in $\mathbb{S}^2 \times \mathbb{R}$ are of finite type. By the algebro-geometric correspondence such annuli are described by algebraic data, the so-called *spectral data* (a, b). It consists of two polynomials of degree $2g$, respectively $g + 1$ for some $g \in \mathbb{N}_0$. The polynomial a encodes a hyperelliptic Riemann surface called *spectral curve*. The genus of the spectral curve is called *spectral genus*. The other polynomial b encodes the closing conditions. One can deform a minimal annulus by deforming the corresponding spectral data. Starting with an embedded minimal annulus in $\mathbb{S}^2 \times \mathbb{R}$, the Whitham deformation allows us to deform the annulus preserving minimality, closing condition as well as embeddedness. Applying this to the totally geodesic annulus allows us to flow through the path-connected component of embedded minimal annuli. In this way, we are able to construct the whole family of Abresch annuli via Whitham deformation theory (see Section 8 and Appendix A). Applying this deformation to annuli gives us additional degrees of freedom for the deformation in contrast to the doubly periodic case.

The second consequence of the curvature estimate is in Lemma 10.1 the compactness of the space of spectral data of properly embedded minimal annuli with periods τ bounded away from zero. As a direct consequence each connected component of the space of such spectral data (a, b) contains a maximum of the length $|\tau|$ of the period τ, since connected components are closed.

We show in Theorem 9.4 that there always exists a Whitham deformation of (a, b) increasing $|\tau|$, if the polynomial a has non-unimodular roots. This implies that the spectral data of all local maxima of $|\tau|$ correspond to the unique totally geodesic annulus in $\mathbb{S}^2 \times \mathbb{R}$. These spectral data are not unique. Each connected component of spectral data (a, b) of Abresch annuli contains such spectral data. In particular, each connected component of spectral data of properly embedded minimal annuli contains spectral data of an Abresch annulus.

Helicoidal and rotational unduloids are of spectral genus one, while the Riemann's type examples are of spectral genus two. This family of Abresch annuli is characterized by an additional symmetry of the spectral data (a, b). In Theorem 11.3, we show that this symmetry cannot be broken along continuous deformations of these spectral data. Therefore the connected components of the Abresch annuli form the only connected components of properly embedded minimal annuli.

2. Minimal annuli and the sinh-Gordon equation

2.1 *Local parametrization.*

We consider $X = (G, h) : \mathbb{C} \to \mathbb{S}^2 \times \mathbb{R}$ a minimal surface conformally immersed in $\mathbb{S}^2 \times \mathbb{R}$ (see [10, Section 1]). As usual write $z = x + \mathbb{i}y$. The horizontal component $G : \mathbb{C} \to \mathbb{S}^2$ of the minimal immersion is a harmonic map. If we denote by $(\mathbb{C}, \sigma^2(u)|\mathrm{d}u|^2)$ the complex plane with metric induced by the stereographic projection of $\mathbb{S}^2$, the map G satisfies

$$G_{z\bar{z}} + 2(\log \sigma \circ G)_u G_z G_{\bar{z}} = 0.$$

The holomorphic *quadratic Hopf differential* associated to the harmonic map G is given by

$$Q(G) = (\sigma \circ G)^2 G_z \bar{G}_z (\mathrm{d}z)^2.$$

Conformality reads as $X(z) = (G(z), \mathrm{Re} \int -2\mathbb{i}\sqrt{Q})$ and the zeroes of Q are double. The unit normal vector n in $\mathbb{S}^2 \times \mathbb{R}$ has the third coordinate

$$\langle n, \tfrac{\partial}{\partial t} \rangle = n_3 = \frac{|g|^2 - 1}{|g|^2 + 1}, \quad \text{where} \quad g^2 := -\frac{G_z}{\overline{\overline{G_{\bar{z}}}}}.$$

We define the real function $\omega : \mathbb{C} \to \mathbb{R}$ by $n_3 := \tanh \omega$. The metric $\mathrm{d}s^2$ is given (see e.g. [11]) in a local coordinate z by $\mathrm{d}s^2 = 4|Q| \cosh^2 \omega$. We remark that the zeroes of Q correspond to the poles of ω, so that the immersion is well defined. Moreover the zeroes of Q are points, where the tangent plane is horizontal.

The Jacobi operator is

$$\mathcal{L} = \frac{1}{4|Q|\cosh^2\omega}\left(\partial_x^2 + \partial_y^2 + \mathrm{Ric}(n) + |\mathrm{d}n|^2\right) = \frac{1}{4|Q|\cosh^2\omega}\left(\partial_x^2 + \partial_y^2 + 4|Q| + \frac{2|\nabla\omega|^2}{\cosh^2\omega}\right).$$

Since $n_3 = \tanh\omega$ is a Jacobi field obtained by vertical translation in $\mathbb{S}^2 \times \mathbb{R}$, we have

$$\mathcal{L}\tanh\omega = 0 \qquad \Longleftrightarrow \qquad \Delta\omega + 4|Q|\sinh\omega\cosh\omega = 0,$$

where $\Delta = \partial_x^2 + \partial_y^2$ is the Laplacian of the flat metric.

Consider a minimal annulus X properly immersed in $\mathbb{S}^2 \times \mathbb{R}$. If X is tangent to a horizontal section $x_3 = 0$, the set $X \cap \{x_3 = 0\}$ bounds on X a compact component in some half-space $x_3 \geq 0$ or $x_3 \leq 0$ with boundary in $\mathbb{S}^2 \times \{0\}$, a contradiction to the maximum principle (see [10, p. 700]). Hence, the annulus is transverse to every horizontal section $\mathbb{S}^2 \times \{t\}$ and intersects the level section in one compact connected component, topologically a circle. The third coordinate map $h : X \to \mathbb{R}$ is a proper harmonic map on each end of X, with $\mathrm{d}h \neq 0$. Then each end of X is parabolic and the annulus can be conformally parameterized by $\mathbb{C}/\tau\mathbb{Z}$. We will consider in the following conformal minimal periodic immersions $X : \mathbb{C} \to \mathbb{S}^2 \times \mathbb{R}$ with $X(z + \tau) = X(z)$.

Since $\mathrm{d}h \neq 0$, the Hopf differential Q has no zeroes. If h^* is the harmonic conjugate of h, we can use the holomorphic map $\mathbb{i}(h + \mathbb{i}\,h^*) : \mathbb{C}^2 \to \mathbb{C}$ to parameterize the annulus by the conformal parameter $z = x + \mathbb{i}y$. In this parametrization, the period of the annulus is $\tau \in \mathbb{R}$ and

$$X(z) = (G(z),\, y) \text{ with } X(z + \tau) = X(z).$$

We say that we have parameterized the surface conformally by its *third coordinate*. We remark that $Q = \frac{1}{4}(\mathrm{d}z)^2$ and ω satisfies the sinh-Gordon equation $(**)$.

Remark 2.1 In this article, we will relax the condition $\tau \in \mathbb{R}$ into $\tau \in \mathbb{C}$, but we will parameterize our annuli conformally such that Q will be constant, independent of z and $4|Q| = 1$. This is a linear change in the conformal parameter $z \mapsto e^{\mathbb{i}\Theta}z$

In summary, we have

Theorem 2.2 [10, Theorem 1.2] A minimal annulus properly immersed is parabolic and $X : \mathbb{C}/\tau\mathbb{Z} \to \mathbb{S}^2 \times \mathbb{R}$ has conformal parametrization $X(z) = (G(z),\, h(z))$ with

(1) Harmonic map $G : \mathbb{C}/\tau\mathbb{Z} \to \mathbb{S}^2$, and $h(z) = \mathrm{Re}(-\mathbb{i}e^{\mathbb{i}\Theta/2}z)$.
(2) Constant Hopf differential $Q = \frac{1}{4}\exp(\mathbb{i}\Theta)\,\mathrm{d}z^2$.
(3) The metric of the immersion is $\mathrm{d}s^2 = \cosh^2\omega\,\mathrm{d}z \otimes \mathrm{d}\bar{z}$.
(4) The third coordinate of the unit normal vector is $n_3 = \tanh\omega$.
(5) The function $\omega : \mathbb{C}/\tau\mathbb{Z} \to \mathbb{R}$ is a solution of $(**)$.

In particular, the intersection of a properly immersed annulus with each horizontal section $\mathbb{S}^2 \times \{t\}$ is topologically a circle.

Conversely, if a minimal immersion $X : \mathbb{C}/\tau\mathbb{Z} \to \mathbb{S}^2 \times \mathbb{R}$ has a linear third coordinate like in (1), which has to be constant along the lines parallel to the period τ, then the pre-image $X^{-1}[K]$ of a compact $K \subset \mathbb{S}^2 \times \mathbb{R}$ is bounded in $\mathbb{C}/\tau\mathbb{Z}$ and X is proper.

2.2 *Annuli foliated by constant curvature curves.*

The function ω determines the geometry of the annulus. We are interested in a 2-parameter family of minimal annuli foliated by horizontal curves with constant geodesic curvature. For these reasons, we make the following definition:

DEFINITION 2.3 An Abresch annulus of $\mathbb{S}^2 \times \mathbb{R}$ is an embedded minimal annulus foliated by horizontal constant curvature curves.

For minimal surfaces in $\mathbb{R}^3$, Shiffman [12] characterized such surfaces by the vanishing of the Jacobi field $u = \cosh^2\omega\,(\partial_x k_g)$. In [3], this is done for $\mathbb{S}^2 \times \mathbb{R}$:

THEOREM 2.4 [3, Section 2] Let X be a minimal annulus immersed in $\mathbb{S}^2 \times \mathbb{R}$, transverse to every section of $\mathbb{S}^2 \times \{t\}$ and parameterized by the third coordinate. Then the geodesic curvature in $\mathbb{S}^2$ of the horizontal level curve $\gamma_h(t) = X \cap (\mathbb{S}^2 \times \{t\})$ is given by

$$k_g(\gamma_h) = \frac{-\omega_y}{\cosh\omega}. \tag{2.1}$$

The function $u = \cosh^2\omega\,(\partial_x k_g)$ is a Jacobi field, so that u is a solution of the elliptic equation

$$\mathcal{L}\,u = \Delta_g u + \mathrm{Ric}(n)u + |\mathrm{d}n|^2 u = 0.$$

Here, $\mathrm{Ric}(n)$ is the Ricci curvature of the two planes tangent to X, $|\mathrm{d}n|$ is the norm of the second fundamental form and $\Delta_g = \frac{1}{\cosh^2\omega}\Delta$.

Let X be a compact minimal annulus immersed in $\mathbb{S}^2 \times \mathbb{R}$, with $\mathrm{Index}(\mathcal{L}) \leq 1$. If X is bounded by two curves Γ_1 and Γ_2 with constant geodesic curvature, then u is identically zero and X is foliated by horizontal curves of constant curvature in $\mathbb{S}^2$.

This theorem proves the existence of a 2-parameter family of minimal surfaces foliated by horizontal constant geodesic curvature curves. They are similar to Riemann's minimal example of $\mathbb{R}^3$. Meeks–Rosenberg [13] prove the existence by solving a Plateau problem between two geodesics Γ_1 and Γ_2 contained in two horizontal sections $\mathbb{S}^2 \times \{t_1\}$ and $\mathbb{S}^2 \times \{t_2\}$ and then find a stable minimal annulus bounded by the geodesics. By the theorem above this annulus is foliated by horizontal circles and using symmetries along horizontal geodesics in $\mathbb{S}^2 \times \mathbb{R}$, one obtains a properly embedded minimal annulus in $\mathbb{S}^2 \times \mathbb{R}$. This annulus is periodic in the third direction.

PROPOSITION 2.5 [3, Section 3] A minimal annulus foliated by constant curvature horizontal curves admits a parametrization by the third coordinate where the metric $\mathrm{d}s^2 = \cosh^2\omega\,|\mathrm{d}z|^2$ satisfies the Abresch system

$$\begin{cases} \Delta\omega + \sinh\omega\cosh\omega = 0 \\ u = \cosh^2\omega\,(\partial_x k_g) = (\omega_{xy}) - \tanh(\omega)(\omega_x)(\omega_y) = 0. \end{cases}$$

Abresch [5] solved this system using elliptic functions and separation of variables

$$\partial_x \left(\frac{\omega_y}{\cosh \omega}\right) = \partial_y \left(\frac{\omega_x}{\cosh \omega}\right) = \frac{(\omega_{xy}) - \tanh(\omega)(\omega_x)(\omega_y)}{\cosh \omega} = 0.$$

The solution $\omega : \mathbb{C} \to \mathbb{R}$ yields the immersion up to isometry (see Section 5.1). The period closes in $\mathbb{C}$ because horizontal curves are circles. Abresch [5], proved that the real functions $x \mapsto f(x)$ and $y \mapsto g(y)$

$$f = \frac{-\omega_x}{\cosh \omega} \quad \text{and} \quad g = \frac{-\omega_y}{\cosh \omega}$$

depend only on one variable, and for $c \leq 0, d \leq 0$ satisfy the system

$$\begin{aligned} -(f_x)^2 &= f^4 + (1 + c - d)f^2 + c, & -f_{xx} &= 2f^3 + (1 + c - d)f, \\ -(g_y)^2 &= g^4 + (1 + d - c)g^2 + d, & -g_{yy} &= 2g^3 + (1 + d - c)g. \end{aligned} \tag{2.2}$$

Conversely, we can recover the solution ω from functions f and g by

$$\sinh \omega = (1 + f^2 + g^2)^{-1}(f_x + g_y)$$

There is a solution of the system if and only if $c \leq 0$ and $d \leq 0$, ω is doubly periodic and exists on the whole plane $\mathbb{R}^2$.

3. The curvature estimate of Meeks and Rosenberg

Meeks–Rosenberg [4] study properly embedded minimal annuli in $\mathbb{S}^2 \times \mathbb{R}$. They prove a bound on the curvature in terms of the third coordinate of the flux.

LEMMA 3.1 Let γ be a simple closed curve not homologous to zero on a properly embedded minimal annulus X, and let $\eta = J\gamma'/|J\gamma'|$ be a unit vector field tangent to X and orthogonal to γ' along γ. Consider $\eta_3 = \langle \eta, \frac{\partial}{\partial t} \rangle$ and the third coordinate of the flux map

$$F_3 = \int_\gamma \eta_3 \, \mathrm{d}s.$$

If X is conformally parameterized with $Q = \frac{1}{4} e^{\mathrm{i}\Theta} (\mathrm{d}z)^2$, and τ the period of X along γ, then $F_3 = |\tau|$.

Proof. After a conformal change of coordinate we may assume that the annulus is parameterized by its third coordinate with real period $|\tau|$. Along a horizontal curve $x \mapsto X(x, y_0)$ the co-normal is $\eta =$ sech (ω) $(\mathrm{G_y}, 1)$. Hence the third coordinate of the flux map is

$$F_3 = \int_0^{|\tau|} \eta_3 \, \mathrm{d}s = |\tau|. \qquad \square$$

Meeks and Rosenberg prove the following curvature estimate:

THEOREM 3.2 [13, Theorem 7.1] For any properly embedded minimal annulus X in $\mathbb{S}^2 \times \mathbb{R}$ with $|F_3| \geq \epsilon_0 > 0$, there exists a constant $C_1 > 0$ depending only on ϵ_0 such that $|K| \leq C_1(\epsilon_0)$.

They also prove a linear growth estimate for minimal surfaces embedded in general product spaces $M \times \mathbb{R}$, but in $\mathbb{S}^2 \times \mathbb{R}$ the annulus is parabolic and we can improve the result with a recent result of Mazet [14], and an estimate of Heintze–Karcher [15]. This implies that geometrically, an embedded annulus has a uniform tubular neighbourhood.

LEMMA 3.3 If $X : \mathbb{C}/\tau\mathbb{Z} \to \mathbb{S}^2 \times \mathbb{R}$ is a properly embedded minimal annulus, then X is the restriction of an ϵ_1-tubular embedded neighbourhood T_{ϵ_1} of the annulus i.e. there is $\epsilon_1 > 0$ such that

$$Y : (\mathbb{C}/\tau\mathbb{Z}) \times (-\epsilon_1, \epsilon_1) \to \mathbb{S}^2 \times \mathbb{R} \text{ with } Y(z, s) = \mathrm{Exp}_{X(z)}(s\, n(z))$$

is an embedded three-dimensional manifold $T_{\epsilon_1} = Y[(\mathbb{C}/\tau\mathbb{Z}) \times (-\epsilon_1, \epsilon_1)]$ into $\mathbb{S}^2 \times \mathbb{R}$. The constant ϵ_1 depends only on a lower bound of the flux $F_3 = |\tau| \geq \epsilon_0 > 0$. Thus for any $t > 0$, there is a constant $C_2 > 0$ which depends only on ϵ_0 such that

$$2t|\tau| \leq \mathrm{Area}(X \cap \mathbb{S}^2 \times [-t, t]) \leq C_2(\epsilon_0)t.$$

Proof. We denote the equidistant surface by $X(s) = Y(\mathbb{C}/\tau\mathbb{Z}, s)$ and its mean curvature by H_s. Following [15], the differential of the exponential map $Y : (z, s) \to \mathrm{Exp}_{X(z)}(sn(p))$ is uniformly bounded on $\mathbb{C} \times (-\epsilon_1, \epsilon_1)$ for $\epsilon_1 > 0$ depending only on the geometry of $\mathbb{S}^2 \times \mathbb{R}$, and the upper bound of the Gaußian curvature K of $X(0) = X[\mathbb{C}/\tau\mathbb{Z}]$. So ϵ_1 depends by Theorem 3.2 only on the lower bound ϵ_0 of the flux. Then the projection π_s along the geodesics of the equidistant surface $X(s)$ to $X(0)$ is a quasi-isometry. Consequently there exists a constant $K_1 > 0$ such that $K_1^{-1}(\epsilon_0)|v| \leq |\mathrm{d}\pi_s(v)| \leq K_1(\epsilon_0)|v|$ holds for any $v \in T_{Y(z,s)}X(s)$ and $s \in (-\epsilon_1, \epsilon_1)$.

Since the Ricci curvature is positive, we have $\frac{d}{ds}H_s = (\mathrm{Ric}(\partial_s) + |\mathrm{d}n_s|^2) \geq 0$ and the equidistant surface $X(s_0)$ has mean curvature vector pointing outside the tubular neighbourhood T_{s_0}.

Each equidistant surface $X(s)$ has a shape operator which satisfies a Riccati-type equation, hence by Karcher [16], the second fundamental form of $X(s)$ is uniformly bounded on $[-\epsilon_1, \epsilon_1]$.

We satisfy the hypothesis of Theorem 7 in Mazet [14]. If there is a parabolic annulus X such that $T_{\epsilon_1} = Y[(\mathbb{C}/\tau\mathbb{Z}) \times (\epsilon_1, \epsilon_1)]$ is not embedded, a subregion of X would produce a connected component S bounded or unbounded into the tubular neighbourhood T_{ϵ_1}, which contradicts the maximum principle (see [14]).

This uniform bound of the minimal width of the embedded tubular neighbourhood of the surface gives a linear area growth estimate. There is a constant C depending only on the geometry of $\mathbb{S}^2 \times \mathbb{R}$ and ϵ_0 such that

$$2\epsilon_1 t|\tau| \leq \epsilon_1 \mathrm{Area}(X \cap \mathbb{S}^2 \times [-t, t]) \leq C\mathrm{Vol}(T_{\epsilon_1} \cap [-t, t]) \leq 4C\pi t,$$

where the constant $C_2 = 4C\pi/\epsilon_1$ depends only on ϵ_0. The $\mathrm{Area}(X \cap \mathbb{S}^2 \times [-t, t])$ is at least $2t|\tau|$ since the metric is $\mathrm{d}s^2 = \cosh^2\omega\, |\mathrm{d}z|^2$. □

In the following, we will deform minimal annuli keeping F_3 bounded away from zero. Then the curvature of the annulus will remain uniformly bounded. As a corollary we derive a uniform estimate for ω, and hence for the third coordinate of the normal $n_3 = \tanh \omega$.

PROPOSITION 3.4 For $\epsilon_0 > 0$ there exists C_0 such that the solutions of the sinh-Gordon equation in Theorem 2.2 of all properly embedded minimal annuli X in $\mathbb{S}^2 \times \mathbb{R}$ with $|\tau| \geq \epsilon_0$ are uniformly bounded by $\sup_{z\in X} |\omega(z)| \leq C_0(\epsilon_0)$. Since $n_3 = \tanh \omega$ is bounded away from the value $n_3 = 1$, the intersection $T_{\epsilon_1} \cap (\mathbb{S}^2 \times \{t\})$ is a tubular neighbourhood in $\mathbb{S}^2$ of the level curve $X \cap (\mathbb{S}^2 \times \{t\})$ with a width $\epsilon > 0$ uniformly bounded above by a constant $c > 0$ depending only on ϵ_1 and C_0.

Proof. Assume on the contrary that there exists a sequence X_n of such annuli and a sequence of points $z_n \in X_n$ such that $\omega_n(z_n)$ goes to infinity. In the foregoing lemma, we have seen that the corresponding sequence τ_n is bounded by $\epsilon_0 \leq |\tau_n| \leq C_2(\epsilon_0)/2$. By passing to a subsequence, we may assume that τ_n converges. Consider a sequence of translations $t_n e_3$ such that $X_n + t_n e_3$ is a sequence of annuli with $z_n + t_n e_3 \in \mathbb{S}^2 \times \{0\}$. Then by the curvature estimate of Meeks and Rosenberg there is a subsequence converging locally to an embedded minimal surface X_0 in $\mathbb{S}^2 \times [-t, t]$. The area estimate shows that X_0 is an annulus, with the limiting flux $\lim |\tau_n|$. By our hypothesis this leads to a pole occurring at the height $t = 0$ since $|\omega| \to \infty$. The limit normal vector $n_3(z_n) = \tanh \omega_n(z_n) \to \pm 1$ and the annulus X_0 would be tangent to the height $\mathbb{S}^2 \times \{0\}$, a contradiction to the maximum principle (compare [10, p. 700]). □

Adapting an argument of Lockhart–McOwen [17], Meeks–Pérez-Ros [9] prove the following:

THEOREM 3.5 An elliptic operator $\mathcal{L} u = \Delta u + qu$ on a cylinder $\mathbb{S}^1 \times \mathbb{R}$ has for bounded and continuous q a finite dimensional kernel in the space of uniformly bounded C^2 functions on $\mathbb{S}^1 \times \mathbb{R}$.

4. Finite type theory of the sinh-Gordon equation

4.1 *Pinkall–Sterling induction.*

Suppose ω is a solution of the sinh-Gordon equation ($**$). There is an iteration of Pinkall–Sterling [18] to obtain an infinite hierarchy of solutions $u_0, u_1, \ldots$ of the linearized sinh-Gordon equation (LSG for short):

$$\mathcal{L} u_n = \Delta u_n + u_n \cosh(2\omega) = 0. \tag{4.1}$$

For given u_n they solve the system

$$\tau_{n;\bar{z}} = \tfrac{1}{2}\mathbb{i}e^{-2\omega} u_n, \qquad \tau_{n;z} = -2\mathbb{i}\, u_{n;zz} + 4\mathbb{i}\, \omega_z u_{n;z}$$

and then define

$$u_{n+1} = -2\mathbb{i}\tau_{n;z} - 4\mathbb{i}\omega_z \tau_n.$$

Each function constructed in this way is complex valued, and $u_n : \mathbb{C} \to \mathbb{C}$ has real and imaginary part that are both solutions of (4.1). Starting this iteration procedure with $u_{-1} = 0$ yields a sequence of Jacobi fields with first terms

$$u_{-1} = 0, \qquad u_0 = \omega_z, \qquad u_1 = \omega_{zzz} - 2\omega_z^3, \qquad u_2 = \omega_{zzzzz} - 10\omega_{zzz}\omega_z^2 - 10\omega_{zz}^2\omega_z + 6\omega_z^5, \ldots.$$

Finite type means (see [10, Definition 2.1 and Proposition 2.2]) that these solutions obey

$$\sum_{i=0}^{N} a_i u_i + b_i \bar{u}_i = 0 \qquad \text{for some } N \in \mathbb{N} \text{ and } a_0, \dots, a_N, b_0, \dots, b_N \in \mathbb{C}.$$

and span a finite dimensional vector space in the kernel of $\mathcal{L}$.

Now the combination of Proposition 3.4 and Theorem 3.5 implies that the solution ω of the sinh-Gordon equation of a properly embedded minimal annulus is of finite type:

THEOREM 4.1 A properly embedded minimal annulus in $\mathbb{S}^2 \times \mathbb{R}$ is of finite type.

Proof. By Theorem 2.2, the properly embedded annulus is parabolic and can be parameterized conformally by its third coordinate. The metric is given by $\mathrm{d}s^2 = \cosh^2\omega\, |\mathrm{d}z|^2$ and $n_3 = \tanh\omega$ is the third coordinate of the normal. By Proposition 3.4, the solution $\omega : \mathbb{C} \to \mathbb{R}$ of the sinh–Gordon equation $(**)$ is uniformly bounded. Schauder estimates apply and we have

$$|\omega|_{C^{k,\alpha}} \le C_0$$

for a constant C_0 depending on the annulus $X = X(\mathbb{C}/\tau\mathbb{Z})$. Now, we apply Theorem 3.5 which assures that the operator

$$\Delta + \cosh(2\omega) : C^{2,\alpha}(\mathbb{C}/\tau\mathbb{Z}) \to C^{0,\alpha}(\mathbb{C}/\tau\mathbb{Z})$$

has finite-dimensional kernel in the space of uniformly bounded C^2 functions on $\mathbb{C}/\tau\mathbb{Z}$. Hence the Pinkall–Sterling iteration spans a finite dimensional space and ω is of finite type. □

4.2 *Potentials*

Finite type solutions of the sinh–Gordon equation give rise to algebraic objects called potentials. We recall their definition and refer to [10] for details:

DEFINITION 4.2 The elements ξ_λ of the following open subset of a $3g + 1$ dimensional real vector space are called potentials:

$$\mathcal{P}_g = \left\{ \sum_{d=-1}^{g} \hat{\xi}_d \lambda^d \,\middle|\, \hat{\xi}_{-1} \in \left(\begin{smallmatrix} 0 & \mathrm{i}\mathbb{R}^+ \\ 0 & 0 \end{smallmatrix}\right), \mathrm{tr}(\hat{\xi}_{-1}\hat{\xi}_0) \neq 0,\ \hat{\xi}_d = -\bar{\hat{\xi}}^t_{g-1-d} \in \mathfrak{sl}_2(\mathbb{C}) \text{ for } d = -1, \dots, g \right\}.$$

The 'Symes method' [19–21] constructs solutions $\omega : \mathbb{C} \to \mathbb{R}$ of the sinh-Gordon equation in terms of these potentials $\xi_\lambda \in \mathcal{P}_g$. This construction is detailed in [10] and summarized below.

4.3 *From the potential to the solution of* sinh-*Gordon.*

Expanding a function $\zeta_\lambda : \mathbb{C} \to \mathcal{P}_g$ as

$$\zeta_\lambda = \begin{pmatrix} 0 & \beta_{-1} \\ 0 & 0 \end{pmatrix} \lambda^{-1} + \begin{pmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & -\alpha_0 \end{pmatrix} \lambda^0 + \dots + \begin{pmatrix} \alpha_g & \beta_g \\ \gamma_g & -\alpha_g \end{pmatrix} \lambda^g$$

we associate a matrix 1-form defined by

$$\alpha(\zeta_\lambda) = \begin{pmatrix} \alpha_0 & \beta_{-1}\lambda^{-1} \\ \gamma_0 & -\alpha_0 \end{pmatrix} \mathrm{d}z - \begin{pmatrix} \bar{\alpha}_0 & \bar{\gamma}_0 \\ \bar{\beta}_{-1}\lambda & -\bar{\alpha}_0 \end{pmatrix} \mathrm{d}\bar{z} \tag{4.2}$$

We recall a well-known existence and uniqueness result, see e.g. [19, Theorem 2.5] and [10, Proposition 3.2 and Remark 3.3]:

PROPOSITION 4.3 [10] For each $\xi_\lambda \in \mathcal{P}_g$ there is a unique solution $\zeta_\lambda : \mathbb{C} \to \mathcal{P}_g$ of

$$\mathrm{d}\zeta_\lambda = [\,\zeta_\lambda,\, \alpha(\zeta_\lambda)\,] \quad \text{with} \quad \zeta_\lambda(0) = \xi_\lambda. \tag{4.3}$$

If ξ_λ is normalized by $|\beta_{-1}\gamma_0| = \frac{1}{16}$, then the function $\omega : \mathbb{C} \to \mathbb{R}$ with $\mathbb{i}e^{\omega(z)} := 4\beta_{-1}(z)$ is a solution of the sinh-Gordon equation of finite type, and $\alpha(\zeta_\lambda(z)) = \alpha$ takes the following form with $|\gamma| = 1$:

$$\alpha := \frac{1}{4}\begin{pmatrix} 2\omega_z & \mathbb{i}\lambda^{-1}e^{\omega} \\ \mathbb{i}\gamma e^{-\omega} & -2\omega_z \end{pmatrix} \mathrm{d}z + \frac{1}{4}\begin{pmatrix} -2\omega_{\bar{z}} & \mathbb{i}\bar{\gamma}e^{-\omega} \\ \mathbb{i}\lambda\, e^{\omega} & 2\omega_{\bar{z}} \end{pmatrix} \mathrm{d}\bar{z}. \tag{4.4}$$

REMARK 4.4 The Lax equation (4.3) preserves $\beta_{-1}(z) \in \mathbb{i}\mathbb{R}^+$, and we can define a function $\omega : \mathbb{C} \to \mathbb{R}$ by setting $4\beta_{-1}(z) := \mathbb{i}e^{\omega}$. Now $\beta_{-1,z} = 2\alpha_0\beta_{-1}$ implies that $2\alpha_0 = \omega_z$. Since the trace of the right-hand side in (4.3) vanishes, the coefficients of $a(\lambda) = -\lambda \det \zeta_\lambda(z) = -\lambda \det \xi_\lambda$ do not depend on $z \in \mathbb{C}$. Therefore, $a(0) = \beta_{-1}\gamma_0$ does not depend on z, and γ_0 is equal to $\mathbb{i}\gamma e^{\omega(z)}$ with $\gamma \in \mathbb{C}$. The normalization $|a(0)| = \frac{1}{16}$ implies $|\gamma| = 1$. Since second derivatives commute $\partial_z\partial_{\bar{z}}\zeta_\lambda = \partial_{\bar{z}}\partial_z\zeta_\lambda$, the function ω solves the sinh-Gordon equation.

The polynomial $a(\lambda) := -\lambda \det \xi_\lambda$ satisfies the reality condition

$$\lambda^{2g}\overline{a(1/\bar{\lambda})} = a(\lambda). \tag{4.5}$$

Since $\chi_\lambda = \lambda^{\frac{1-g}{2}}\xi_\lambda$ is traceless and satisfies ${}^t\overline{\chi_{1/\bar{\lambda}}} = -\chi_\lambda$ for any $\xi_\lambda \in \mathcal{P}_g$ and for $\lambda \in \mathbb{S}^1$, the determinant is the square of a norm and we have $\det \chi_\lambda \geq 0$ for $\lambda \in \mathbb{S}^1$. Thus

$$\lambda^{-g}\, a(\lambda) \leq 0 \text{ for } \lambda \in \mathbb{S}^1 \tag{4.6}$$

The condition $\mathrm{tr}(\hat{\xi}_{-1}\hat{\xi}_0)$ in Definition 4.2 implies that $a(0) \neq 0$ and by symmetry the highest coefficient of a is non-zero. We denote (see also [10, (2.9)])

$$\mathcal{M}_g = \{a \in \mathbb{C}^{2g}[\lambda] \mid a = -\lambda \det \xi_\lambda \text{ with } \xi_\lambda \in \mathcal{P}_g\}. \tag{4.7}$$

4.4 *Spectral curve.*

The spectral curve is defined by the determinant of a polynomial Killing field ζ_λ. A property of the Lax equation is that $a(\lambda) = -\lambda \det \zeta_\lambda = -\lambda \det \xi_\lambda$ is independent of z. Following Bobenko [22], the polynomial a defines a hyperelliptic Riemann surface Σ:

Definition 4.5 For $\xi_\lambda \in \mathcal{P}_g$ set $a(\lambda) = -\lambda \det \xi_\lambda \in \mathcal{M}_g$. The associated spectral curve Σ of genus g is defined by adding $(\infty, 0)$ and (∞, ∞) as branch points in the compactification of

$$\Sigma^\times = \{(\nu, \lambda) \in \mathbb{C}^2 \mid \det(\nu\, \mathbb{1} - \zeta_\lambda) = 0\} = \{(\nu, \lambda) \in \mathbb{C}^2 \mid \nu^2 = \lambda^{-1} a(\lambda)\}. \tag{4.8}$$

Σ has three involutions (compare [10, (2.11)]):

$$\sigma : (\lambda, \nu) \mapsto (\lambda, -\nu), \qquad \rho : (\lambda, \nu) \mapsto (\bar{\lambda}^{-1}, -\bar{\lambda}^{1-g}\bar{\nu}), \qquad \eta : (\lambda, \nu) \mapsto (\bar{\lambda}^{-1}, \bar{\lambda}^{1-g}\bar{\nu}).$$

The involution σ is the hyperelliptic involution. The involution η has no fixed point while ρ fixes all points of the unit circle $|\lambda| = 1$. In particular, the roots of a are interchanged by $\lambda \mapsto \bar{\lambda}^{-1}$.

4.5 *Isospectral set.*

The following set of all potentials with the same spectral curve and the same off-diagonal product $a(0) = \beta_{-1}\gamma_0$ is called isospectral set:

$$\mathcal{I}(a) := \{\xi_\lambda \in \mathcal{P}_g \mid \lambda \det \xi_\lambda = -a(\lambda)\} \tag{4.9}$$

For a given potential $\xi_\lambda \in \mathcal{P}_g$ with corresponding solution ω of the sinh-Gordon equation the tangent space of $\mathcal{I}(a)$ at $\xi_\lambda \in \mathcal{P}_g$ is associated to the hierarchy of solutions $u_0, u_1, \dots$ of LSG (4.1). Each of these Jacobi fields can be integrated to a long time solution of the sinh-Gordon equation. These integrals fit together to a group action.

4.6 *Group action.*

In [10, Definition 4.2] a continuous group action is defined on $\mathcal{I}(a)$:

$$\pi : \mathbb{C}^g \times \mathcal{I}(a) \to \mathcal{I}(a). \tag{4.10}$$

It integrates the family of solutions of the linearized sinh-Gordon equation u_0, $u_1, \dots$ into deformations of the solutions ω of the sinh-Gordon equation. In particular, these flows exist for all time and are quasi-periodic. The group action of $(z, 0, ...0) \in \mathbb{C}^g$ integrates the first solution $u_0 = \omega_z$ and hence represents the annulus as a two-dimensional subgroup of the isospectral set. For $t_2 \in \mathrm{i}\mathbb{R}$ the normal variation of the group action of $(0, t_2, 0, \dots, 0)$ is the Shiffman Jacobi field. On minimal annuli of $\mathbb{S}^2 \times \mathbb{R}$ conformally parameterized by the third coordinate, this Jacobi field is given by $u = \cosh^2\omega\,(\partial_x k_g) = (\omega_{xy}) - \tanh(\omega)(\omega_x)(\omega_y)$. Here k_g is the geodesic curvature (see Theorem 2.4) of the horizontal curve.

Proposition 4.6 [10, Proposition 4.3] The group action (4.10) is commutative:

$$\pi(t')\pi(t)\xi_\lambda = \pi(t + t')\xi_\lambda = \pi(t)\pi(t')\xi_\lambda.$$

An important property of spectral curves without singularities (i.e. the polynomial a has only simple roots), is that the isospectral set $\mathcal{I}(a)$ has only one orbit diffeomorphic to a real g–dimensional torus. This property implies that all annuli with smooth spectral curves are immersed into the Jacobian $\simeq (\mathbb{S}^1)^g$ and have a quasi-periodic polynomial Killing field. Since this polynomial Killing field depends only on ω, this means that also the metric is quasi-periodic:

PROPOSITION 4.7 (1) For $a \in \mathcal{M}_g$ the isospectral set $\mathcal{I}(a)$ is compact and the corresponding solutions ω are uniformly bounded in terms of a bound on the coefficients of a.

(2) If $a \in \mathcal{M}_g$ has $2g$ pairwise distinct roots, then $\mathcal{I}(a)$ is a connected smooth g-dimensional manifold diffeomorphic to a g-dimensional real torus: $\mathcal{I}(a) \cong \left(\mathbb{S}^1\right)^g$.

Proof of (1): The proof of [10, Proposition 4.9] shows the compactness of $\mathcal{I}(a)$ and establishes a bound on all coefficients of $\xi_\lambda \in \mathcal{I}(a)$ in terms of a bound on the coefficients of a. Hence, the formulas (4.2) and (4.4) imply that the corresponding solutions ω of the sinh-Gordon equation are also uniformly bounded in terms of a bound on the coefficients of a.

(2) is proven in [10, Theorem 4.8]. □

5. Minimal annuli of finite type

5.1 *From the solution of* sinh-*Gordon to the immersion.*

We identify the sphere $\mathbb{S}^2$ with $\mathrm{SU}_2/\mathrm{U}(1)$. The map $g \mapsto g\sigma_3\bar{g}^t$ maps SU_2 into $\mathbb{S}^2 \subset \mathbb{R}^3$ with $\sigma_3 = \left(\begin{smallmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{smallmatrix}\right)$.

THEOREM 5.1 [10, Theorem 1.3] Let ξ_λ be a potential and $\zeta_\lambda : \mathbb{C} \to \mathcal{P}_g$ the polynomial Killing field (4.3) and let $F_\lambda : \mathbb{C} \to \mathrm{SL}_2(\mathbb{C})$ be the unique solution of

$$F_\lambda^{-1} dF_\lambda = \alpha(\zeta_\lambda) := \alpha(\omega) \qquad \text{with} \qquad F_\lambda(0) = \mathbb{1} \tag{5.1}$$

with $\alpha(\omega)$ as in (4.4) and ω as in Proposition 4.3. Then for any constant $\gamma \in \mathbb{S}^1$, the maps

$$X_\lambda(z) = (F_\lambda \sigma_3 F_\lambda^{-1}, \mathrm{Re}(-\mathrm{i}\sqrt{\gamma\lambda^{-1}}z))$$

with $\lambda \in \mathbb{S}^1$, define a 1-parameter isometric family of conformal minimal immersions $\mathbb{C} \to \mathbb{S}^2 \times \mathbb{R}$ of finite type with metric $\mathrm{d}s^2 = \cosh^2\omega\, |\mathrm{d}z|^2$. If $\lambda = \gamma = 1$, then $X_1(x, y) = (F_1\sigma_3F_1^{-1}, y)$ is an immersion conformally parameterized by its third coordinate.

REMARK 5.2 (Reality condition) From the relation $\bar{\alpha}^t\big|_{1/\bar{\lambda}} = -\alpha|_\lambda$, the solution of (5.1) satisfies $\bar{F}^t_{1/\bar{\lambda}} = F_\lambda^{-1}$. For $\lambda \in \mathbb{S}^1$, $\alpha(\omega)$ takes values in $\mathfrak{su}_2$ and F_λ takes values in SU_2. The values of F_λ at general $\lambda \in \mathbb{C}^\times$ belong to $\mathrm{SL}_2(\mathbb{C})$ since $\mathrm{tr}(\alpha(\omega)) = 0$. The variable λ is called spectral parameter, and the map F_λ the extended frame of the associated family X_λ.

5.2 *The Sym point and conformal parametrization.*

If F_λ is the extended frame of a minimal surface $X_\lambda : \mathbb{R}^2 \to \mathbb{S}^2 \times \mathbb{R}$ conformally parameterized by its third coordinate, then there is $\lambda_0 \in \mathbb{S}^1$, such that

$$X_{\lambda_0}(z) = (G_{\lambda_0}(z), y) = (F_{\lambda_0}(z)\sigma_3 F_{\lambda_0}^{-1}(z), \mathrm{Re}(-\mathrm{i}z)).$$

The Hopf differential is given by $Q_{\lambda_0}(z) = -4\beta_{-1}\gamma_0\lambda_0^{-1}(\mathrm{d}z)^2 = \frac{1}{4}(\mathrm{d}z)^2$ i.e. $\beta_{-1}\gamma_0 = -\frac{\lambda_0}{16}$. The value of $\lambda_0 = e^{\mathrm{i}\theta}$ associate to an immersion X_λ is called the Sym point. In the following, we will prefer a conformal parametrization which fixes the Sym point to $\lambda_0 = 1$. To do that, we make the conformal change $z \mapsto e^{\mathrm{i}(1-g)\theta/2}z$ and apply the Möbius transformation $\lambda \mapsto e^{\mathrm{i}\theta}\lambda$. Then $\widetilde{F}_\lambda(z) = F_{e^{\mathrm{i}\theta}\lambda}(e^{\mathrm{i}(1-g)\theta/2}z)$ is the extended frame obtained from the potential

$$\widetilde{\xi}_\lambda = e^{\mathrm{i}(1-g)\theta/2}\xi_{e^{\mathrm{i}\theta}\lambda}.$$

In particular, we have

$$\det \widetilde{\zeta}_\lambda(z) = \det \widetilde{\xi}_\lambda = -\lambda^{-1}\widetilde{a}(\lambda) = -\lambda^{-1}e^{-\mathrm{i}g\theta}a(e^{\mathrm{i}\theta}\lambda).$$

The immersion is locally given by

$$\widetilde{X}_1(z) = X_{\lambda_0}(e^{\mathrm{i}(1-g)\theta/2}z) = (\widetilde{F}_1(z)\sigma_3\widetilde{F}_1^{-1}(z), \operatorname{Re}(-\mathrm{i}e^{\mathrm{i}(1-g)\theta/2}z)). \tag{5.2}$$

DEFINITION 5.3 A finite type minimal immersion $X : \mathbb{R}^2 \to \mathbb{S}^2 \times \mathbb{R}$ is conformally parameterized by its Sym point if there is a polynomial Killing field

$$\zeta_\lambda : \mathbb{C} \to \left\{\xi_\lambda \in \mathcal{P}_g \mid \lambda \det \xi_\lambda = -a(\lambda) \text{ and } \beta_{-1}\gamma_0 = a(0) = -\tfrac{1}{16}\, e^{\mathrm{i}(1-g)\theta} := -\tfrac{1}{16}\, e^{\mathrm{i}\Theta}\right\}$$

which solves the Lax equation (4.3) with $a \in \mathcal{M}_g$. If F_λ is the frame (5.1) associated to ξ_λ, then in this parametrization the immersion is given by (5.2)

5.3 *Higher order roots of a.*

Different ξ_λ of different isospectral sets may give the same solution ω of sinh-Gordon and the same extended frame F_λ. This is the case if and only if one of the initial values ξ_λ has a root at some $\lambda = \alpha_0 \in \mathbb{C}^\times$. Then also the corresponding polynomial Killing field ζ_λ has a root at $\lambda = \alpha_0$ for all $z \in \mathbb{C}$. In this case, we may reduce the order of ξ_λ and ζ_λ without changing the corresponding extended frame F_λ. This configuration corresponds to a singular spectral curve, i.e., the polynomial a has a root of order at least two at α_0. We can remove this singularity without changing the surface. There is a polynomial p such that $\tilde{\xi}_\lambda = \xi_\lambda/p$ does not vanish at α_0 and is the initial value of a polynomial Killing field $\tilde{\zeta}_\lambda$ without zeroes at α_0. We show in [10, Proposition 4.4] that both polynomial Killing fields ζ_λ and ζ_λ/p induce congruent minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$:

PROPOSITION 5.4 [10, Proposition 4.4] If a polynomial Killing field ζ_λ with initial value $\xi_\lambda \in \mathcal{I}(a) \subset \mathcal{P}_g$ has zeroes in $\lambda \in \mathbb{C}^\times$, then there is a polynomial p with $|p(0)| = 1$ and the following properties:

(1) ξ_λ/p has no zeroes in $\lambda \in \mathbb{C}^\times$, and has degree $g - \deg p$.

(2) If F_λ and $\widetilde{F}_\lambda$ are the extended frames of ζ_λ respectively ζ_λ/p, then $\widetilde{F}_\lambda(p(0)z) = F_\lambda(z)$ for all z, and the induced immersions conformally parameterized by their Sym points are congruent.

Hence, among all polynomial Killing fields that give rise to a minimal surface of finite type there is a unique one of smallest possible degree (without roots in $\lambda \in \mathbb{C}^\times$).

PROPOSITION 5.5 [10, Lemma 4.7] Let $\mathcal{I}(a)$ be the isospectral set associated to $a \in \mathcal{M}_g$.

(1) If a has a double root α_0 with $|\alpha_0| = 1$, then $\mathcal{I}(a) = \{\xi_\lambda \in \mathcal{I}(a) \mid \xi_{\alpha_0} = 0\}$ and there is an isomorphism

$$\mathcal{I}(a) \longrightarrow \mathcal{I}(\alpha_0(\lambda-\alpha_0)^{-2}a) \text{ defined by } \xi_\lambda \mapsto \sqrt{\alpha_0}(\lambda-\alpha_0)^{-1}\,\xi_\lambda$$

(2) If a has double root α_0 with $|\alpha_0| \neq 1$ then $\mathcal{I}(a) = \{\xi_\lambda \in \mathcal{I}(a) \mid \xi_{\alpha_0} \neq 0\} \cup \{\xi_\lambda \in \mathcal{I}(a) \mid \xi_{\alpha_0} = 0\}$ and there is an isomorphism

$$\{\xi_\lambda \in \mathcal{I}(a) \mid \xi_{\alpha_0} = 0\} \longrightarrow \mathcal{I}(|\alpha_0|^2(\lambda-\alpha_0)^{-2}(1-\bar{\alpha}_0\lambda)^{-2}a) \text{ defined by } \xi_\lambda \mapsto |\alpha_0|(\lambda-\alpha_0)^{-1}(1-\bar{\alpha}_0\lambda)^{-1}\xi_\lambda.$$

Proof of (1): If a has a double root at α_0 with $|\alpha_0| = 1$, then for any $\xi_\lambda \in \mathcal{I}(a)$, we have $\xi_{\alpha_0} = 0$, because the determinant is a norm for any $\lambda \in \mathbb{S}^1$.
Proof of (2): If a has a double root at α_0 with $|\alpha_0| \neq 0$, then the isospectral set splits into the set of potentials with a zero at α_0 (which again can be removed), and the set of potentials non-vanishing at α_0. In the latter case ξ_{α_0} is nilpotent, and the surface is called a bubbleton. □

In general, the action (4.10) has several orbits. Two potentials belong to the same orbit, if and only if they have the same roots of the same order. In Proposition 7.2, we shall see that either all elements in the orbit of a potential correspond to embedded minimal annuli, or none.

6. Spectral data of minimal annuli of finite type

In this section, we characterize potentials which correspond to periodic minimal immersions. This property turns out to be a property of the polynomial a: For a given $a \in \mathcal{M}_g$ either all elements of the isospectral set $\mathcal{I}(a)$ (4.9) have this property or no element. We study the monodromy $M_\lambda(\tau) = F_\lambda(z+\tau)F_\lambda(z)^{-1}$ of the extended frame F_λ for a period τ. By construction, the monodromy takes values in SU_2 for $|\lambda| = 1$. The monodromy depends on the choice of base point z, but its conjugacy class and hence eigenvalues μ, μ^{-1} do not. The eigenspace of $M_\lambda(\tau)$ depends holomorphically on (μ, λ).

Let ζ_λ be a solution of the Lax equation (4.3) with initial value $\xi_\lambda \in \mathcal{P}_g$, with period τ so that $\zeta_\lambda(z+\tau) = \zeta_\lambda(z)$ for all $z \in \mathbb{C}$. Then for $z = 0$ we have

$$\xi_\lambda = \zeta_\lambda(0) = \zeta_\lambda(\tau) = F_\lambda^{-1}(\tau)\,\xi_\lambda\,F_\lambda(\tau) = M_\lambda^{-1}(\tau)\xi_\lambda\,M_\lambda(\tau)$$

and thus

$$[\,M_\lambda(\tau),\,\xi_\lambda\,] = 0.$$

Hence, the eigenvalues ν of ξ_λ and μ of $M_\lambda(\tau)$ are different functions on the same Riemann surface Σ, but the eigenspaces of $M_\lambda(\tau)$ and ξ_λ coincide point-wise. At $\lambda = 0$ and $\lambda = \infty$, the monodromy $M_\lambda(\tau) = F_\lambda(\tau)$ has essential singularities. The existence of a closed annulus depends on the existence of the function μ having the correct behaviour at $\lambda = 0$ and $\lambda = \infty$. Additionally, the condition $F(\tau) = \pm\mathbb{1}$ at the Sym point $\lambda = 1$ reformulate as $\mu(1) = \pm 1$. In the case where a has only simple roots, we have only to prove that μ is holomorphic on $\Sigma \setminus \{0, \infty\}$, which is a weaker condition than $\mu = f(\lambda)\nu + g(\lambda)$ with holomorphic functions f, g defined on $\mathbb{C}^\times$ in the general case (when ν has higher order roots, f

could have poles). In particular, there exits a polynomial b of degree $g+1$ such that the meromorphic differential takes the form:

$$\mathrm{d}\ln\mu = \frac{b(\lambda)\mathrm{d}\lambda}{\nu\lambda^2}. \tag{6.1}$$

Keeping this in mind we define spectral data of a minimal cylinder.

DEFINITION 6.1 [10, Definition 5.10] For each $g \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$ let $\mathcal{M}^g_{\text{ann}}$ denote the space of spectral data $(a,b) \in \mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda]$ of a minimal cylinder of finite type in $\mathbb{S}^2 \times \mathbb{R}$ with the following properties:

(i) $\lambda^{2g}\overline{a(\bar\lambda^{-1})} = a(\lambda)$ and $\lambda^{-g}a(\lambda) \le 0$ for all $\lambda \in \mathbb{S}^1$, and $a(0) = -\frac{1}{16}e^{\mathrm{i}\Theta}$.

(ii) $\lambda^{g+1}\overline{b(\bar\lambda^{-1})} = -b(\lambda)$.

(iii) $b(0) = \frac{\tau e^{\mathrm{i}\Theta}}{32} \in e^{\mathrm{i}\Theta/2}\mathbb{R}$.

(iv) $\mathrm{Re}\left(\int_{\alpha_i}^{1/\bar\alpha_i} \frac{b(\lambda)\mathrm{d}\lambda}{\nu\lambda^2}\right) = 0$ for all roots α_i of a where the integral is computed along the straight line segment from α_i to $\bar\alpha_i^{-1}$.

(v) The unique function $h : \tilde\Sigma \to \mathbb{C}$ where $\tilde\Sigma = \Sigma - \cup\gamma_i$ and γ_i are closed cycles over the straight lines connecting α_i and $\bar\alpha_i^{-1}$, such that

$$\sigma^* h(\lambda) = -h(\lambda) \text{ and } \mathrm{d}h = \frac{b(\lambda)\mathrm{d}\lambda}{\nu\lambda^2}$$

takes values in $\mathrm{i}\pi\mathbb{Z}$ at all roots of $\lambda \mapsto (\lambda-1)a(\lambda)$.

(vi) There are holomorphic functions f, g defined on $\mathbb{C}^\times$ with $\mu = e^h = f\,\nu + g$ (this follows from the other conditions unless a has higher order roots).

The disjoint union of all these sets is denoted by $\mathcal{M}_{\text{ann}} = \bigcup_{g\in\mathbb{N}_0} \mathcal{M}^g_{\text{ann}}$.
For all $(a,b) \in \mathcal{M}_{\text{ann}}$ let $\mathcal{A}(a,b)$ denote the corresponding set of minimal annuli.

REMARK 6.2 The normalization $|a(0)| = \frac{1}{16}$ in (i) is related to $4|Q| = 1$ in Theorem 2.2.

For potentials ξ_λ without roots (i.e. of minimal degree) the commuting monodromy can be written as $M_\lambda(\tau) = f\,\xi_\lambda + g\,\mathbb{1}$ with functions f, g on $\mathbb{C}^\times$. This implies condition (vi).

If $(a,b) \in \mathcal{M}^g_{\text{ann}}$, then also $(a,-b) \in \mathcal{M}^g_{\text{ann}}$. Since $\mathcal{A}(a,-b) = \mathcal{A}(a,b)$ we neglect this ambiguity.

THEOREM 6.3 [10, Corollary 5.9] If $\xi_\lambda \in \mathcal{P}_g$ corresponds to a periodic minimal immersion $X : \mathbb{C}/\tau\mathbb{Z} \to \mathbb{S}^2 \times \mathbb{R}$, then there exists a polynomial b, which obeys (i)-(vi) with $a(\lambda) = -\lambda\det(\xi_\lambda)$. If (a,b) obeys (i)-(vi), then all $\xi_\lambda \in \mathcal{I}(a)$ correspond to minimal annuli of finite type.

6.1 *Spectral data of the Abresch annuli.*

In the following proposition, we describe the spectral data $\mathcal{M}_{\text{Abr}} = \mathcal{M}^0_{\text{Abr}} \cup \mathcal{M}^1_{\text{Abr}} \cup \mathcal{M}^2_{\text{Abr}}$ of the Abresch annuli:

PROPOSITION 6.4 [10, Proposition 7.2]

(1) The spectral data $\mathcal{M}^0_{\text{Abr}}$ of Abresch annuli of genus 0 consists of the pair $a(\lambda) = \frac{-1}{16}$ and $b(\lambda) = \pm\frac{\pi}{16}(\lambda - 1)$, which is unique up to the sign of b.

(2) The spectral data $\mathcal{M}^1_{\text{Abr}}$ of Abresch annuli of genus 1 are 2 one-dimensional families:

i. $a(\lambda) = \frac{1}{16\alpha}(\lambda - \alpha)(\alpha\lambda - 1)$ and $b(\lambda) = \frac{b(0)}{\gamma}(\lambda - \gamma)(\gamma\lambda - 1)$ parameterized by $\alpha \in (0, 1]$ with $\gamma \in [\alpha, 1]$ and $b(0) \in \mathbb{iR}$ both determined by α.

ii. $a(\lambda) = \frac{-1}{16\beta}(\lambda + \beta)(\beta\lambda + 1)$ and $b(\lambda) = b(0)(1 - \lambda)(1 + \lambda)$ parameterized by $\beta \in (0, 1]$ with $b(0) \in \mathbb{R}$ determined by β.

(3) The spectral data $\mathcal{M}^2_{\text{Abr}}$ of Abresch annuli of genus 2 is a two-dimensional family:

$a(\lambda) = \frac{\lambda-\alpha}{16\beta\alpha}(\alpha\lambda - 1)(\lambda + \beta)(\beta\lambda + 1)$ and $b(\lambda) = \frac{b(0)}{\gamma}(1 + \lambda)(\lambda - \gamma)(\gamma\lambda - 1)$ parameterized by $(\alpha, \beta) \in (0, 1]^2$ with $\gamma \in [\alpha, 1]$ and $b(0) \in \mathbb{iR}$ determined by α and β.

Proof. The proof is given in [10]. The Abresch system (2.2) gives the relation $\omega_{zzz} - 2\omega_z^3 = -\frac{1}{4}\omega_{\bar{z}} + \frac{c-d}{2}\omega_z$. We apply the iteration of Pinkall–Sterling described in Section 4 and obtain a corresponding polynomial a. Finally, we construct b with $(a, b) \in \mathcal{M}_{\text{ann}}$. □

6.2 *Spectral data with higher order roots of a.*

We next characterize pairs (a, b), $(\tilde{a}, \tilde{b})$ of spectral data in $\mathcal{M}_{\text{ann}}$ for which $\tilde{a} = p^2 a$ and $\tilde{b} = pb$ (see Propositions 5.4 and 5.5). We decorate the objects corresponding to $(\tilde{a}, \tilde{b})$ with a tilde and set $\tilde{\lambda} = \lambda$. Suppose first $(\tilde{a}, \tilde{b}) \in \mathcal{M}^g_{\text{ann}}$. Choose any polynomial p such that p^2 divides $\tilde{a}$, and

$$\lambda^{\deg p}\overline{p(\bar{\lambda}^{-1})} = p(\lambda) \qquad |p(0)| = 1. \tag{6.2}$$

Due to Condition (vi) in Definition 6.1, $\tilde{h}$ is holomorphic on $\tilde{\Sigma}^\times$ and p divides $\tilde{b}$. For $a = \tilde{a}/p^2$ and $b = \tilde{b}/p$, we have $\mathrm{d}h = \mathrm{d}\tilde{h}$. Then (a, b) obeys conditions (i)–(vi) in Definition 6.1 with $f = p\tilde{f}$ and $g = \tilde{g}$.

Conversely, for $(a, b) \in \mathcal{M}^g_{\text{ann}}$ we set $\tilde{a} = p^2 a$ and $\tilde{b} = pb$. This implies $\tilde{h} = h$ with $\tilde{\lambda} = \lambda$. The relations $\sigma^* h = -h$ and $\sigma^*\nu = -\nu$ imply

$$g = \cosh(h) = \cosh(\tilde{h}) = \tilde{g}, \qquad \frac{f}{p} = \frac{\sinh(h)}{\nu p} = \frac{\sinh(\tilde{h})}{\tilde{\nu}} = \tilde{f}.$$

For $(\tilde{a}, \tilde{b})$ to satisfy condition (vi), p must divide $\frac{f(\lambda)}{\lambda-1} = \frac{\sinh(h)}{\nu(\lambda-1)}$. Thus $\sinh(h)$ vanishes at the roots of p. Differentiation gives that the following meromorphic function is either holomorphic or has first-order poles at the roots of p:

$$\frac{\mathrm{d}h/\mathrm{d}\lambda}{(\lambda - 1)\,\nu\, p(\lambda)} = \frac{b(\lambda)}{(\lambda - 1)\,\lambda^2\, a(\lambda)\, p(\lambda)}. \tag{6.3}$$

For such p, we have indeed $(\tilde{a}, \tilde{b}) \in \mathcal{M}^g_{\text{ann}}$. We summarize the discussion in the following

LEMMA 6.5 (Compare [8, Lemma 5.3]) For $(\tilde{a}, \tilde{b}) \in \mathcal{M}^g_{\text{ann}}$ choose any polynomial p obeying (6.2) such that p^2 divides $\tilde{a}$. Then p divides $\tilde{b}$ and $(a, b) \in \mathcal{M}^{g-\deg p}_{\text{ann}}$ with $\tilde{a} = p^2 a$ and $\tilde{b} = pb$.

Conversely, suppose $(a, b) \in \mathcal{M}^g_{\text{ann}}$ and p obeys (6.2). If in addition $\sinh(h)$ vanishes at the roots of p, and the function (6.3) has at the roots of p at worst simple poles, then $(p^2 a, pb) \in \mathcal{M}^{g+\deg p}_{\text{ann}}$. □

Due to Proposition 5.5 (compare [10, Sections 4 and 6]), the corresponding sets $\mathcal{A}(p^2 a, pb)$ and $\mathcal{A}(a, b)$ are related. For pairs $(a, b) \in \mathcal{M}^g_{\text{ann}}$ and $(p^2 a, pb) \in \mathcal{M}^{g+\deg p}_{\text{ann}}$ as in Lemma 6.5 we have $\mathcal{A}(a, b) \subset \mathcal{A}(p^2 a, pb)$ with equality if all roots of p are unimodular.

We interpret the supplementation of non-real singularities (away from $\mathbb{S}^1$) as an enrichment of the complexity and the removal as a reduction of the complexity. Geometrically, this corresponds to adding or removing bubbletons by a suitable Bianchi–Bäcklund transform. Adding or removing a unimodular singularity does not change the complexity. It will turn out that the enrichment of complexity destroys embeddedness, while the reduction of complexity preserves embeddedness. Finally, we determine for the totally geodesic annulus all possible higher order roots:

LEMMA 6.6 For $(a, b) \in \mathcal{M}^0_{\text{Abr}}$, the pair of polynomials $(p^2 a, pb)$ belongs to $\mathcal{M}^{\deg p}_{\text{ann}}$ if and only if the polynomial p obeys (6.2) and has simple roots in $\{2n^2 - 1 + 2n\sqrt{n^2 - 1} \mid n \in \mathbb{Z}\}$.

Proof. For $(a, b) \in \mathcal{M}^0_{\text{Abr}}$ the eigenvalue μ of the monodromy is calculated in [10, p. 733]:

$$h = \ln \mu = \frac{\pi \mathbb{i}}{2}(\lambda^{-1/2} + \lambda^{1/2}).$$

Therefore, $\sinh(h)$ vanishes at $\lambda^{-1/2} + \lambda^{1/2} \in 2\mathbb{Z} \iff \lambda \in \{2n^2 - 1 + 2n\sqrt{n^2 - 1} \mid n \in \mathbb{Z}\}$. Now the statement follows from Lemma 6.5. □

The spectral data in $\mathcal{M}^1_{\text{Abr}} \cup \mathcal{M}^2_{\text{Abr}}$ have higher order roots, if $\alpha = 1$ and $\gamma = 1$ or if $\beta = 1$. In these cases they are of the form $(\tilde{a}, \tilde{b}) = (p^2 a, pb)$ as in Lemma 6.5 with $(a, b) \in \mathcal{M}^0_{\text{Abr}} \cup \mathcal{M}^1_{\text{Abr}}$.

7. The spectral data of properly embedded minimal annuli

In this section, we first show that all potentials in the isospectral set $\mathcal{I}(a)$ of spectral data $(a, b) \in \mathcal{M}^g_{\text{ann}}$ correspond to proper minimal embeddings, if one potential $\xi_\lambda \in \mathcal{I}(a)$ without roots does so. This allows us to define the spectral data $\mathcal{M}^g_{\text{emb}} \subset \mathcal{M}^g_{\text{ann}}$ of properly embedded minimal annuli. In a second step, we show that these sets $\mathcal{M}^g_{\text{emb}}$ are open and closed subsets of $\mathcal{M}^g_{\text{ann}}$. In particular, all continuous deformations of spectral data preserve these subsets $\mathcal{M}^g_{\text{emb}}$. We remark that the arguments are similar to the arguments of [8, Section 7]. The main difference is the replacement of [8, Proposition 7.2] by the following lemma. In this lemma, we conceive minimal immersions $\mathbb{C}/\tau\mathbb{Z} \to \mathbb{S}^2 \times \mathbb{R}$ as maps on $\mathbb{C}$ which are periodic with period τ and compare two such immersions with different periods on bounded discs $B(w, r) \subset \mathbb{C}$. The diameter of these discs $B(w, r)$ will be larger than the length of the periods.

LEMMA 7.1 For $\epsilon_0 > 0$ there exists $r > 0$ and $\epsilon_2 > 0$ with the following property: a minimal annulus $\tilde{X} : \mathbb{C}/\tilde{\tau}\mathbb{Z} \to \mathbb{S}^2 \times \mathbb{R}$ properly immersed with $|\tilde{\tau}| \geq \epsilon_0$ is a proper embedding, if for all $w \in \mathbb{C}/\tilde{\tau}\mathbb{Z}$ there exists a proper minimal embedding $X_w : \mathbb{C}/\tau_w\mathbb{Z} \to \mathbb{S}^2 \times \mathbb{R}$ with $|\tau_w - \tilde{\tau}| < \epsilon_2$, $|\tau_w| \geq \epsilon_0$ which obeys on the discs $B(w, r) \subset \mathbb{C}$

$$\|\tilde{X}(z) - X_w(z)\|_{C^2(B(w,r),\mathbb{S}^2\times\mathbb{R})} < \epsilon_2. \tag{7.1}$$

Proof. Due to the linear area growth in Lemma 3.3 the length of the periods $|\tau_w|$ of the proper embeddings X_w are bounded by $C_2(\epsilon_0)/2$. For given $\epsilon_0 > 0$, choose r such that the diameter $2r$ of $B(w, r)$ is larger than $C_2(\epsilon_0)/2 + \epsilon_2 > \max\{|\tau_w|, |\tilde{\tau}|\}$. For sufficiently small ϵ_2, the C^2-bound (7.1) guarantees that the minimal immersion $\tilde{X}$ stays on $B(w, r)$ in the tubular neighbourhood T_{ϵ_1} of the proper minimal embedding X_w constructed in Lemma 3.3. Moreover, for sufficiently small ϵ_2 this part of $\tilde{X}$ is locally a normal graph over the corresponding part of X. If $|\tau_w - \tilde{\tau}|$ is small enough, then by unique continuation the whole image $\tilde{X}[B(w, r)]$ is a normal graph over the corresponding part of X and therefore embedded. Since all horizontal intersections are embedded circles $\tilde{X}$ is a proper embedding. □

We shall apply this lemma only to minimal annuli of finite type. Since the linear third coordinate of a minimal annulus $\tilde{X} : \mathbb{C}/\tilde{\tau}\mathbb{Z} \to \mathbb{S}^2 \times \mathbb{R}$ of finite type is constant along the lines parallel to the period $\tilde{\tau}$, the pre-image of a compact set in $\mathbb{S}^2 \times \mathbb{R}$ is bounded in $\mathbb{C}/\tilde{\tau}\mathbb{Z}$ and $\tilde{X}$ is proper. We verify the other assumptions of the lemma by considering the potentials $\tilde{\xi}_\lambda$ and $\xi_{w,\lambda}$ of both minimal annuli $\tilde{X}$ and X_w. More specifically, we assume that the corresponding spectral data $(\tilde{a}, \tilde{b}), (a, b) \in \mathcal{M}_{\text{ann}}$ with $\tilde{\xi}_\lambda \in \mathcal{I}(\tilde{a})$ and $\xi_{w,\lambda} \in \mathcal{I}(a)$ are sufficiently close in $\mathcal{M}^g_{\text{ann}}$ with $\epsilon_0 \le \min\{|\tilde{\tau}|, |\tau|\}$. Since on the compact isospectral sets $\mathcal{I}(\tilde{a})$ and $\mathcal{I}(a)$ all derivatives of the corresponding immersions are uniformly bounded, the C^2-bound (7.1) is satisfied if the potentials $\tilde{\xi}_\lambda$ and $\xi_{w,\lambda}$ translated by w are sufficiently close, i.e., $\|\pi(w)\tilde{\xi}_\lambda - \pi(w)\xi_{w,\lambda}\|$ is sufficiently small.

Let us now prepare the definition of spectral data of properly embedded annuli and prove that the property of being properly embedded depends only on the corresponding spectral data:

Proposition 7.2 (Compare [8, Proposition 7.3]) Let $\xi_\lambda \in \mathcal{P}_g$ have no roots in $\lambda \det \xi_\lambda \in \mathbb{C}^\times$ and correspond to a properly embedded minimal annulus $X : \mathbb{C}/\tau\mathbb{Z} \to \mathbb{S}^2 \times \mathbb{R}$. Then, we have:

(1) If $a(\lambda) = -\lambda \det \xi_\lambda$ has only simple roots, then $\{\pi(t)\xi_\lambda \mid t \in \mathbb{C}^g\} = \mathcal{I}(a)$ and all $\tilde{\xi}_\lambda \in \mathcal{I}(a)$ correspond to properly embedded minimal annuli.

(2) If $\tilde{a}(\lambda) = -\lambda \det \xi_\lambda$ has higher order roots, then $\mathcal{I}(\tilde{a})$ is the closure of $\{\pi(t)\xi_\lambda \mid t \in \mathbb{C}^g\}$ and all $\tilde{\xi}_\lambda \in \mathcal{I}(\tilde{a})$ correspond to properly embedded minimal annuli.

Proof of (1): Due to Theorem 6.3, all $\tilde{\xi}_\lambda \in \mathcal{I}(a)$ correspond to minimal immersions $\tilde{X} : \mathbb{C}/\tau\mathbb{Z} \to \mathbb{S}^2 \times \mathbb{R}$. The continuity and the commutativity of the group action (4.10) in Proposition 4.6

$$\pi(z + t)\xi_\lambda = \pi(z)\pi(t)\xi_\lambda = \pi(t)\pi(z)\xi_\lambda$$

and the compactness of $\mathcal{I}(a)$ implies that for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\|\pi(z)\pi(t)\tilde{\xi}_\lambda - \pi(z)\tilde{\xi}_\lambda\| = \|\pi(t)\pi(z)\tilde{\xi}_\lambda - \pi(z)\tilde{\xi}_\lambda\| \le \sup_{\xi_\lambda \in \mathcal{I}(a)} \|\pi(t)\xi_\lambda - \xi_\lambda\| \le \epsilon \text{ for } \tilde{\xi}_\lambda \in \mathcal{I}(a) \text{ and } |t| < \delta.$$

Hence, for $|t| < \delta$ the immersion $\tilde{X}_\lambda$ corresponding to $\pi(t)\xi_\lambda$ obeys (7.1) with suitable isometric copies X_w of the immersion corresponding to ξ_λ. Due to Lemma 7.1, there exists a $\delta > 0$, such that for all $t \in B(0, \delta)$ the minimal annuli corresponding to $\pi(t)\tilde{\xi}_\lambda$ are properly embedded, if $\tilde{\xi}_\lambda$ corresponds to a minimal proper embedding. Hence, the set of all $t \in \mathbb{C}^g$ such that $\pi(t)\xi_\lambda$ corresponds to a minimal proper embedding is $\mathbb{C}^g$. If a has only simple roots, then due to [10, Proposition 4.12] $\mathbb{C}^g$ acts transitively on $\mathcal{I}(a)$ and all $\tilde{\xi}_\lambda \in \mathcal{I}(a)$ correspond to minimal proper embeddings. This proves (1).

Proof of (2): Let $\tilde{a} = p^2 a$ with a polynomial p obeying (6.2). Since $\xi_\lambda \in \mathcal{I}(\tilde{a})$ would vanish at all roots of p on $\mathbb{S}^1$, p has no roots on $\mathbb{S}^1$ and $\deg p$ is even. For $\deg p = 2$, we parameterize in [10, Section 6] $\mathcal{I}(\tilde{a})$ by pairs (L, ξ_λ) of lines $L \in \mathbb{CP}^1$ together with $\xi_\lambda \in \mathcal{I}(a)$. The elements $\tilde{\xi}_\lambda \in \mathcal{I}(\tilde{a})$ without roots correspond to pairs such that $L^\perp$ is not an eigenline of the value of ξ_λ at a root of p [10, Proposition 6.6]. Such $\tilde{\xi}_\lambda$ forms a dense orbit in $\mathcal{I}(\tilde{a})$. By induction in $\frac{\deg p}{2}$, we conclude for general p without roots on $\mathbb{S}^1$ that $\{\pi(t)\tilde{\xi}_\lambda \mid t \in \mathbb{C}^{g+\deg p}\}$ is dense in $\mathcal{I}(\tilde{a})$, if $\tilde{\xi}_\lambda \in \mathcal{I}(\tilde{a})$ has no roots. The first assertion together with Lemma 7.1 implies (2). □

Let us now define the spectral data of properly embedded minimal annuli. Any such annulus corresponds to many $(a, b) \in \mathcal{M}_{\mathrm{ann}}$. If we replace $(a, b) \in \mathcal{M}^g_{\mathrm{ann}}$ by $(p^2 a, pb) \in \mathcal{M}^{g+\deg p}_{\mathrm{ann}}$ as in Lemma 6.5, then $\mathcal{A}(a, b) \subset \mathcal{A}(p^2 a, pb)$. This is the only ambiguity of (a, b). Indeed all properly embedded minimal annuli correspond, up to the sign of b, uniquely to $(a, b) \in \mathcal{M}_{\mathrm{ann}}$ and $\xi_\lambda \in \mathcal{I}(a)$ without roots such that all $(\tilde{a}, \tilde{b}) \in \mathcal{M}_{\mathrm{ann}}$ with $X \in \mathcal{A}(\tilde{a}, \tilde{b})$ are of the form $(\tilde{a}, \tilde{b}) = (p^2 a, pb)$ in Lemma 6.5. Proposition 7.2 shows that for all properly embedded minimal annuli, these minimal sets $\mathcal{A}(a, b)$ contain only properly embedded minimal annuli. Therefore any properly embedded minimal annulus is contained in the set $\mathcal{A}(a, b)$ of an element $(a, b) \in \mathcal{M}_{\mathrm{emb}} \subset \mathcal{M}_{\mathrm{ann}}$ defined as follows:

Definition 7.3 Let $\mathcal{M}^g_{\mathrm{emb}}$ denote the space of $(a, b) \in \mathcal{M}^g_{\mathrm{ann}}$ whose $\mathcal{A}(a, b)$ contain only properly embedded minimal annuli. The union $\bigcup_{g \in \mathbb{N}_0} \mathcal{M}^g_{\mathrm{emb}}$ is denoted by $\mathcal{M}_{\mathrm{emb}}$.

For the second application of Lemma 7.1, we will utilize the openness and properness of the map $\xi_\lambda \mapsto -\lambda \det \xi_\lambda$.

Lemma 7.4 (compare [8, Lemma 3.4]) The following map is open and proper:

$$A : \mathcal{P}_g \to \mathcal{M}_g, \quad \xi_\lambda \mapsto -\lambda \det \xi_\lambda \tag{7.2}$$

Proof. In [10, Proposition 4.4] the properness is proven. Due to [10, Proposition 4.12 and Theorem 6.8] the orbits of the group action (4.10) are the subsets of $\mathcal{I}(a)$ of all elements ξ_λ with the same roots on $\mathbb{C}^\times$ counted with multiplicities. For any $a \in \mathcal{M}_g$, an off-diagonal potential

$$\xi_\lambda = \begin{pmatrix} 0 & \lambda^{-1}\beta(\lambda) \\ \gamma(\lambda) & 0 \end{pmatrix}$$

belongs to $\mathcal{I}(a)$, if and only if the polynomials β and γ of degree g obey $\beta(\lambda)\gamma(\lambda) = a(\lambda)$ and $\gamma(\lambda) = -\lambda^g \overline{\beta(\bar{\lambda}^{-1})}$. The roots of β are g roots of a, which are mapped by $\lambda \mapsto \bar{\lambda}^{-1}$ onto the remaining g roots of a. For any choice of such roots, β and γ are determined up to multiplication by inverse unimodular numbers. At higher order roots $\alpha \in \mathbb{C}^\times \setminus \mathbb{S}^1$ of a, we can choose the multiplicity of the root of β at α between zero and the multiplicity of the root α of a. The sum of the multiplicities of the roots of β at α and at $\bar{\alpha}^{-1}$ has to be equal to the multiplicity of the root of a at α. Therefore, there exists in every orbit of the isospectral group action (4.10) at least one off-diagonal ξ_λ. Due to the relation between the polynomials a and β and γ, the map A (7.2) is open at off-diagonal ξ_λ. Since the isospectral action (4.10) acts by diffeomorphisms on $\mathcal{P}_g$ and preserves the fibres of the map A, this map is (globally) open. □

Now we show that $\mathcal{M}^g_{\mathrm{emb}}$ is open and closed in $\mathcal{M}^g_{\mathrm{ann}}$ (compare [8, Proposition 7.5]):

PROPOSITION 7.5 For all $g \in \mathbb{N}_0$ the space $\mathcal{M}^g_{\mathrm{emb}}$ is an open and closed subset of $\mathcal{M}^g_{\mathrm{ann}}$.

Proof. We combine Lemmas 7.1 and 7.4 and show that $\mathcal{M}^g_{\mathrm{emb}}$ is open and closed in $\mathcal{M}^g_{\mathrm{ann}}$. For both arguments, we choose a compact neighbourhood $N \subset \mathcal{M}^g_{\mathrm{ann}}$ of some $(a, b) \in \mathcal{M}^g_{\mathrm{ann}}$. The continuous function $(a, b) \mapsto |\tau|$ takes a minimum $\epsilon_0 > 0$ on N with constants $\epsilon_2 > 0$ and $r > 0$ in Lemma 7.1. The image $M \subset \mathcal{M}_g$ of N with respect to the projection $(a, b) \mapsto a$ is compact.

First, we use the properness of (7.2) to prove that $\mathcal{M}^g_{\mathrm{emb}}$ is open in $\mathcal{M}_{\mathrm{ann}}$, so let $(a, b) \in \mathcal{M}^g_{\mathrm{emb}}$. Therefore, any $\xi_\lambda \in \mathcal{I}(a)$ corresponds to a proper minimal embedding X and has in $A^{-1}[M]$ an open neighbourhood, whose immersions $\tilde{X}$ obey (7.1) with $w = 0$, $X_w = X$ and the constants ϵ_2 and r. The union U of these open neighbourhoods is in $A^{-1}[M]$ an open neighbourhood of the compact subset $A^{-1}[\{a\}]$. Due to Lemma 7.1, the following set O is contained in $\mathcal{M}^g_{\mathrm{emb}}$:

$$O = \{(\tilde{a}, \tilde{b}) \in \mathcal{M}^g_{\mathrm{ann}} \mid \tilde{\xi}_\lambda \in U \text{ for all } \tilde{\xi}_\lambda \in \mathcal{I}(\tilde{a})\}.$$

We claim that O is an open neighbourhood of (a, b) in N. Let a sequence $(a_n, b_n)_{n\in\mathbb{N}}$ in $N \setminus O$ converge to $(\tilde{a}, \tilde{b}) \in \mathcal{M}^g_{\mathrm{ann}}$. Then, there exists a sequence $(\xi_n)_{n\in\mathbb{N}}$ in $A^{-1}[\{a_n\}] \setminus U$. The set

$$A^{-1}[\{a_n \mid n \in \mathbb{N}\} \cup \{\tilde{a}\}]$$

is compact, since A is proper. A subsequence of $(\xi_n)_{n\in\mathbb{N}}$ converges to $\tilde{\xi} \in A^{-1}[\{\tilde{a}\}]$. If $\tilde{a} \notin M$, then $(\tilde{a}, \tilde{b}) \notin O$. Otherwise, the subsequence of $(\xi_n)_{n\in\mathbb{N}}$ is mapped by A to a convergent sequence in M. This subsequence stays in the compact subset $A^{-1}[M] \setminus U$ of $A^{-1}[M]$ and has limit $\tilde{\xi}_\lambda \notin U$. This again implies $(\tilde{a}, \tilde{b}) \notin O$. Therefore $N \setminus O$ is closed and $\mathcal{M}^g_{\mathrm{emb}} \supset O$ is open in $\mathcal{M}^g_{\mathrm{ann}}$.

Now we use the openness of A to show that $\mathcal{M}^g_{\mathrm{emb}}$ is closed in $\mathcal{M}^g_{\mathrm{ann}}$. Let (a_n, b_n) be a sequence in $\mathcal{M}^g_{\mathrm{emb}}$ converging in $\mathcal{M}^g_{\mathrm{ann}}$ to the aforementioned (a, b). We have to show that any $\tilde{\xi}_\lambda \in A^{-1}[\{a\}]$ corresponds to an embedded annulus $\tilde{X}$. Since A is open every neighbourhood of $\tilde{\xi}_\lambda$ contains elements of $A^{-1}[\{a_n\}]$ for sufficiently large n. Therefore $\tilde{\xi}_\lambda$ is the limit of a sequence $\xi_{\lambda,n} \in A^{-1}[\{a_n\}]$. Then $\tilde{X}$ fulfils the condition of Lemma 7.1 and is embedded. □

We summarize the results of this section in the following theorem (compare [8, Theorem 3]):

THEOREM 7.6 The subsets $\mathcal{M}^g_{\mathrm{emb}} \subset \mathcal{M}^g_{\mathrm{ann}}$ have the following properties:

(1) $\mathcal{M}^0_{\mathrm{emb}} = \mathcal{M}^0_{\mathrm{Abr}}$.

(2) Let $(\tilde{a}, \tilde{b}) = (p^2 a, pb)$, $(a, b) \in \mathcal{M}_{\mathrm{ann}}$ and p be as in Lemma 6.5.

- If $(\tilde{a}, \tilde{b}) \in \mathcal{M}^{g+\deg p}_{\mathrm{emb}}$, then $(a, b) \in \mathcal{M}^g_{\mathrm{emb}}$.
- If $(a, b) \in \mathcal{M}^g_{\mathrm{emb}}$ and all roots of p belong to $\mathbb{S}^1$, then $(\tilde{a}, \tilde{b}) \in \mathcal{M}^{g+\deg p}_{\mathrm{emb}}$.

(3) For all $g \in \mathbb{N} \cup \{0\}$, the subset $\mathcal{M}^g_{\mathrm{emb}}$ is closed and open in $\mathcal{M}^g_{\mathrm{ann}}$.

Proof. The only solution of the sinh-Gordon equation for $g = 0$ is trivial $\omega = 0$. Therefore, $\mathcal{M}^0_{\mathrm{emb}}$ is equal to $\mathcal{M}^0_{\mathrm{Abr}}$ which contains the spectral data of the unique totally geodesic annulus.

Property (2) follows from the properties of $\mathcal{A}(a, b)$ and $\mathcal{A}(p^2 a, pb)$ in the situation of Lemma 6.5; Due to Propositions 5.4 and 5.5, we have in all cases $\mathcal{A}(a, b) \subset \mathcal{A}(p^2 a, pb)$ with equality, if all roots of p are unimodular. Property (3) is proven in Proposition 7.5. □

8. A smooth parametrization of $\mathcal{M}^g_{\text{ann}}$

In this section, we adapt two methods of the theory of moduli spaces of spectral curves to the present situation. The first method is a parametrization of the moduli space by the values of the function h in property (v) of Definition 6.1 at the roots of b. This parametrization was introduced by Marchenko–Ostrowskii [23] in the context of periodic solutions of Hill's equation. The other method are the Whitham deformations, which were introduced by Krichever [24] in the study of moduli space theory. We next describe the Whitham deformations of the spectral curves of the sinh-Gordon equation.

We construct vector fields on the space of spectral data. We conceive $\ln \mu$ as a function depending on λ and t. This function has on Σ simple poles at $\lambda = 0$ and $\lambda = \infty$. Let us take a covering $O_1, O_2, \ldots, O_{2g}$ of open subsets of Σ, such that each O_i contains at most one branch point α_i and O_{2g+1} is an open neighbourhood of $(\infty, 0)$, and O_{2g+2} an open neighbourhood of (∞, ∞). We can locally express the meromorphic function on Σ by

$$\ln \mu = \begin{cases} f_i(\lambda)\nu + \pi \mathrm{i}\, n_i & \text{on } O_i, 1 \leq i \leq 2g \\ \nu f_{2g+1}(\lambda) + \pi \mathrm{i}\, n_{2g+1} & \text{on } O_{2g+1} \\ \nu f_{2g+2}(\lambda) + \pi \mathrm{i}\, n_{2g+2} & \text{on } O_{2g+2}. \end{cases}$$

We can write locally on the open set O_i,

$$\partial_t \ln \mu = \partial_t f_i(\lambda)\nu - \frac{\dot{a}(\lambda) f_i(\lambda)}{2\lambda\nu}.$$

We remark that at each branch point $\partial_t \ln \mu$ has a first-order pole on Σ. Since the branches of $\ln \mu$ differ from each other by an integer multiple of $2\pi \mathrm{i}$, then $\partial_t \ln \mu$ is single valued on Σ and can have poles only at the branch points of Σ, or equivalently at the zeroes of a and at $\lambda = 0$ or $\lambda = \infty$. Collecting all these conditions, we can write $\partial_t \ln \mu$ globally on Σ by

$$\partial_t \ln \mu = \frac{c(\lambda)}{\nu\lambda} \tag{8.1}$$

with a real polynomial c of degree at most $g + 1$ which satisfies the reality condition

$$\lambda^{g+1}\overline{c(\bar{\lambda}^{-1})} = c(\lambda). \tag{8.2}$$

The abelian differential $\mathrm{d} \ln \mu$ of the second kind is of the form (6.1), where b is a real polynomial of degree $g + 1$ which satisfies the reality condition (ii) in Definition 6.1. We differentiate (6.1) with respect to t, and (8.1) with respect to λ and obtain

$$\partial^2_{t\lambda} \ln \mu = \partial_\lambda \frac{c}{\nu\lambda} = \frac{c'}{\nu\lambda} - \frac{c}{\nu\lambda^2} - \frac{c\nu'}{\nu^2\lambda} = \frac{2\nu^2\lambda^2 c' - 2\lambda\nu^2 c - ca'\lambda + ca}{2\nu^3\lambda^3},$$
$$\partial^2_{\lambda t} \ln \mu = \partial_t \frac{b}{\nu\lambda^2} = \frac{\dot{b}}{\nu\lambda^2} - \frac{b\dot{\nu}}{\nu^2\lambda^2} = \frac{2\nu^2\lambda\dot{b} - b\dot{a}}{2\nu^3\lambda^3}.$$

Since λ does not depend on t both second derivatives coincide:

$$-2\dot{b}a + b\dot{a} = -2\lambda a c' + ac + \lambda a' c. \tag{8.3}$$

Both sides in the last formula are polynomials of degree at most $3g+1$ which satisfy a reality condition. This corresponds to $3g+2$ real equations. Choosing a polynomial c which satisfies the reality condition (8.2), we thus obtain a vector field on the space of $(a,b) \in \mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda]$. In the case where a and b have only simple roots α_i respectively β_i, this vector field is equal to

$$\dot{a}(\alpha_i) = \frac{\alpha_i a'(\alpha_i)\, c(\alpha_i)}{b(\alpha_i)} \qquad \dot{b}(\beta_i) = \frac{2\beta_i\, a(\beta_i)\, c'(\beta_i) - a(\beta_i)\, c(\beta_i) - \beta_i\, a'(\beta_i)\, c(\beta_i)}{2\, a(\beta_i)}. \tag{8.4}$$

When a and b have no common roots, then equation (8.3) uniquely determines the singular parts of $\frac{\dot{a}}{a}$ at the roots of a and the singular parts of $\frac{\dot{b}}{b}$ at the roots of b. Condition (i) in Definition 6.1 uniquely determines a in terms of the roots of a and $\dot{a}$ in terms of the singular parts of $\frac{\dot{a}}{a}$. Therefore, (8.3) defines a smooth vector field on the space of pairs $(a,b) \in \mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda]$ with $\operatorname{resultant}(a,b) \neq 0$ and with properties (i)–(ii) in Definition 6.1. These vector fields extend to meromorphic vector fields on the space of (a,b) with properties (i)–(ii) in Definition 6.1. Let us now determine those vector fields which preserve $\mathcal{M}^g_{\text{ann}}$. Their values at (a,b) with $\operatorname{resultant}(a,b) \neq 0$ span the tangent space of $\mathcal{M}^g_{\text{ann}}$ which is locally at (a,b) a submanifold of $\mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda]$.

Lemma 8.1 At all $(a,b) \in \mathcal{M}^g_{\text{ann}}$ with $\operatorname{resultant}(a,b) \neq 0$ the moduli space $\mathcal{M}^g_{\text{ann}}$ is a submanifold of $(a,b) \in \mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda]$. The tangent space is the image of the isomorphism of the following subspace of $c \in \mathbb{C}^{g+1}[\lambda]$ onto the corresponding solutions $(\dot{a},\dot{b})$ of (8.3):

$$T_{(a,b)}\mathcal{M}^g_{\text{ann}} \simeq \{c \in \mathbb{C}^{g+1}[\lambda] \mid c \text{ obeys } (8.2), c(1) = 0 \text{ and } \mathrm{Im(c(0)/b(0))} = 0\}. \tag{8.5}$$

If $\operatorname{Re}(c(0)/b(0)) < 0$, then $|\tau|$ is increasing.

Proof. We shall use the Implicit Function Theorem. The space of polynomials $(a,b) \in \mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda]$, which obey the first equation in condition (i) and condition (ii) in Definition 6.1 form a real $(3g+3)$-dimensional vector space. If we impose in addition, the third equation in condition (i) the space becomes a real $(3g+2)$-dimensional affine space. Due to condition (vi) higher order roots of a are common roots of a and b, which are excluded by $\operatorname{resultant}(a,b) \neq 0$. So the inequality in condition (i) and condition (vi) are locally preserved. Condition (iii) is equivalent to the vanishing of one real function and the two conditions (iv)–(v) are equivalent to the values of h at the $2g+1$ roots of $a(\lambda)(\lambda-1)$ being constant. Using the transformation properties of h with respect to the involutions, we see that $\mathcal{M}^g_{\text{ann}}$ is at (a,b) with $\operatorname{resultant}(a,b) \neq 0$ locally the level set of $2g+2$ real functions. If the kernel of the derivatives of all these functions has real dimension g, then these derivatives are linearly independent. Moreover, the corresponding level sets are real g-dimensional submanifolds of $\mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda]$. Finally, the tangent space of this submanifold is equal to the kernel of the derivatives of these functions.

Let $(\dot{a},\dot{b})$ be an element in this kernel. Then all periods of $d\dot{h}$ vanish and this 1-form is exact. By the transformation properties of dh there exists a polynomial $c \in \mathbb{C}^{g+1}[\lambda]$ obeying (8.2), such that $d\dot{h}$ is equal to the exterior derivative of (8.1). Since we conceive $\ln\mu$ locally as a function depending on λ and t, the polynomials $(\dot{a},\dot{b})$ obey (8.3). The antisymmetry of (8.1) guarantees that the values of h at the $2g$ roots of $a(\lambda)$ are preserved. The value of h at $\lambda = 1$ is preserved if and only if $c(1) = 0$. It remains to preserve property (iii) such that $b(0) = \tau e^{\mathrm{i}\Theta}/32$ takes values in $e^{\mathrm{i}\Theta/2}\mathbb{R}$, where Θ is defined by $a(0) = \frac{-e^{\mathrm{i}\Theta}}{16}$. We insert $\lambda = 0$ into (8.3) and obtain

$$\frac{b(0)}{\sqrt{-a(0)}} = \frac{\tau e^{\mathrm{i}\Theta/2}}{8}, \qquad \partial_t \ln \frac{b(0)}{\sqrt{-a(0)}} = \frac{\dot{b}(0)}{b(0)} - \frac{\dot{a}(0)}{2a(0)} = \frac{-c(0)}{2b(0)}. \tag{8.6}$$

Hence property (iii) is preserved if and only if $\mathrm{Im}(\mathrm{c}(0)/\mathrm{b}(0)) = 0$, and $|\tau|$ is increasing for $\mathrm{Re}(\mathrm{c}(0)/\mathrm{b}(0)) < 0$. So the kernel of the derivatives of the functions, whose level set is locally $\mathcal{M}^g_{\mathrm{ann}}$, is the image of the isomorphism of the space (8.5) onto the corresponding solutions $(\dot{a}, \dot{b})$ of (8.3). This subspace (8.5) of $c \in \mathbb{C}^{g+1}[\lambda]$ has real dimension g and the proof is complete. □

The polynomials (a, b) describe the dependence of $d \ln \mu$ and λ on each other. Let us now use locally the function $\ln \mu$ instead of its derivative $d \ln \mu$. This allows us to extend the foregoing lemma to all $(a, b) \in \mathcal{M}^g_{\mathrm{ann}}$ with no common roots of a and the function f. Here f is the first of the two holomorphic functions $f, g : \mathbb{C}^\times \to \mathbb{C}$ in condition (iv) of Definition 6.1 with $\mu = f\nu + g$. This condition that a and f have no common roots is equivalent to the condition that $\ln \mu$ and λ generate the same functions as λ and ν. Due to $g^2 - f^2\nu^2 = 1$, the roots of f are points where $\mu^2 = 1$. They are possible roots of p, such that $(\tilde{a}, \tilde{b}) = (p^2 a, pb)$ belongs to $\mathcal{M}^g_{\mathrm{ann}}$ as in Lemma 6.5. By an appropriate choice of p, we can always achieve in Lemma 6.6 that the corresponding function $\tilde{f}$ has no common roots with $\tilde{a}$. Moreover, the set of roots of such p contains not only all the common roots of f and a, but in addition can also contain finitely many of the infinitely many roots of f.

For the proof of the following proposition, it is convenient to slightly enlarge $\mathcal{M}^g_{\mathrm{ann}}$.

Definition 8.2 For all $g \in \mathbb{N}_0$, let $\mathcal{M}^g_{\mathrm{per}}$ denote the space of all $(a, b) \in \mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda]$ which obey conditions (ii)–(iv),(vi) in Definition 6.1 and slightly weaker conditions (i) and (v): In condition (i) we remove the inequality $\lambda^{-g} a(\lambda) \le 0$ for unimodular λ and in condition (v) we restrict the values of h only at all roots of a and not at $\lambda = 1$.

Let us now fix an element $(a, b) \in \mathcal{M}^g_{\mathrm{ann}}$ such that the corresponding f has no common root with a. We choose simply connected neighbourhoods $V_1, \ldots, V_M$ in $\mathbb{C}^\times$ at all roots of b including the common roots with a. Let $U_1, \ldots, U_M$ denote the pre-images in Σ^* of $V_1, \ldots, V_M$ under the map λ. For $m = 1, \ldots, M$, we choose on U_m a branch of the function $\ln \mu$. On U_m, the function $\frac{1}{2\pi\mathbb{i}}(\ln \mu + \sigma^* \ln \mu)$ is equal to a constant integer n_m at the branch point. These branches obey

$$(\ln \mu - n_m \mathbb{i}\pi)^2 = A_m \quad \text{for} \quad m = 1, \ldots, M, \tag{8.7}$$

with holomorphic functions A_m on V_m which vanish at the roots of a. Since $\sigma^*(\ln \mu - n_m \mathbb{i}\pi) = -(\ln \mu - n_m \mathbb{i}\pi)$, the function A_m depends only on λ (see Theorem 8.2 in [25]). If we choose U_m and V_m pairwise disjoint, then the derivative of A_m has no roots besides the corresponding root of b ($d \ln \mu$ vanishes at roots of b). The roots of b are exactly the roots of the derivative of A_m. For small enough U_m and V_m, there exists a biholomorphic map $\lambda \mapsto z_m(\lambda)$ from V_m to a simply connected open neighbourhood W_m of $0 \in \mathbb{C}$, such that A_m coincides with

$$A_m(\lambda) = z_m^{d_m}(\lambda) + A_{m,d_m}. \tag{8.8}$$

At a root of b, which is not a root of a the constant $A_{m,d_m} \neq 0$, and $d_m - 1$ is the order of the root of b. At a common root of a and b, the constant $A_{m,d_m} = 0$ and d_m is an odd integer in the case of common roots of a and b, and an even integer in the case of double points.

We describe spectral data $(\hat{a}, \hat{b}) \in \mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda]$ in a neighbourhood of the given spectral data (a, b) by small perturbations $\hat{A}_1, \ldots, \hat{A}_M$ of the polynomials $A_1, \ldots, A_M$. More precisely, we consider polynomials $\hat{A}_1, \ldots, \hat{A}_M$ of the form

$$\hat{A}_m(z_m) = z_m^{d_m} + \hat{A}_{m,2} z_m^{d_m-2} + \hat{A}_{m,3} z_m^{d_m-3} + \ldots + \hat{A}_{m,d_m} \tag{8.9}$$

with coefficients $\hat{A} = ((\hat{A}_{1,2}, \ldots, \hat{A}_{1,d_1}), \ldots, (\hat{A}_{M,2}, \ldots, \hat{A}_{M,d_M})) \in \mathbb{C}^{d_1-1} \times \ldots \times \mathbb{C}^{d_M-1} = \mathbb{C}^{g+1}$. Here, the centre of the local parameter z_m is chosen in such a way, that the sum of the roots of $\hat{A}_m$ is zero. Let $A = ((A_{1,2}, \ldots, A_{1,d_1}), \ldots, (A_{M,2}, \ldots, A_{M,d_M}))$ be the corresponding coefficients of the polynomials $A_1, \ldots, A_M$ (8.8). For sufficiently small supremum norm $\|\hat{A} - A\|_\infty$, we glue each W_m of the sets $W_1, \ldots, W_M$ to $\mathbb{C}\mathrm{P}^1 \setminus (\mathrm{V}_1 \cup \ldots \cup \mathrm{V}_\mathrm{M})$ along the boundary of V_m in such a way that for all $m = 1, \ldots, M$ the polynomial $\hat{A}_m$ coincides with the unperturbed function A_m in a tubular neighbourhood of the boundary ∂W_m. We obtain a new copy of $\mathbb{C}\mathrm{P}^1$. By uniformization, there exists a new global parameter $\hat{\lambda}$, which is equal to 0 and ∞ at the two points corresponding to $\lambda = 0$ and $\lambda = \infty$, respectively. This new parameter is unique up to multiplication with elements of $\mathbb{C}^\times$. There exists a biholomorphic map $\hat{\lambda} = \phi(\lambda)$ which changes the parameter $\lambda \in \mathbb{C}\mathrm{P}^1 \setminus (\mathrm{V}_1 \cup \ldots \cup \mathrm{V}_\mathrm{M})$ in the global parameter $\hat{\lambda}$. Furthermore, for each $m = 1, \ldots, M$ there is a biholomorphic map $\hat{\lambda} = \phi_m(z_m)$ which changes the local parameter $z_m \in W_m$ into $\hat{\lambda}$. Let $\hat{\lambda} \mapsto \hat{a}(\hat{\lambda})$ be the polynomial whose roots (counted with multiplicities) coincide with the roots of $\hat{A}_1(\hat{\lambda}), ..., \hat{A}_M(\hat{\lambda})$ and the roots of $\hat{\lambda} \mapsto a \circ \phi^{-1}(\hat{\lambda})$ on $\mathbb{C}^\times \setminus (V_1 \cup \ldots \cup V_M)$. Now $\hat{\Sigma} = \{(\hat{\nu}, \hat{\lambda}) \in \mathbb{C}^2 \mid \nu^2 = \hat{\lambda}^{-1}\hat{a}(\hat{\lambda})\}$ yields a new hyperelliptic curve. The equations

$$(\ln \mu - n_m \pi \mathbb{i})^2 = \hat{A}_m(\hat{\lambda}) = \hat{A}_m \circ \phi_m^{-1}(\hat{\lambda}) = \hat{A}_m(z_m) \quad \text{for} \quad m = 1, \ldots, M \tag{8.10}$$

define a function μ on the pre-image of $\phi_m(W_m) \cap \mathbb{C}\mathrm{P}^1$ by the map $\hat{\lambda}$ into $\hat{\Sigma}$. The function μ extends to the pre-image of $\mathbb{C}^* \setminus (V_1 \cup \ldots \cup V_M)$ by $\hat{\lambda} = \phi(\lambda)$ and coincides with the unperturbed μ on this set. On $\hat{\Sigma}$ the differential $d \ln \mu$ is meromorphic and takes the form $d \ln \mu = \frac{\hat{b}\, d\hat{\lambda}}{\hat{\nu}\hat{\lambda}^2}$ with a unique polynomial $\hat{b}$. By taking the derivative of (8.10), we have

$$2(\ln \mu - n_m \pi \mathbb{i})\, \partial_{\hat{\lambda}} \ln \mu = \hat{A}'_m(z_m(\hat{\lambda}))\, z'_m(\hat{\lambda}).$$

The roots of $\hat{b}$ are the roots of the derivatives of $\hat{A}_1, ..., \hat{A}_M$. Let us now impose a reality condition on these coefficients, such that $(\hat{a}, \hat{b})$ obey the reality conditions (i)–(ii) in Definition 6.1. The involution ρ interchanges the roots of b. Since $m \in \{1, \ldots, M\}$ labels these roots, the involution ρ also acts on $\{1, \ldots, M\}$. We denote this action by $m \mapsto \rho m$. We may choose the open sets $V_1, \ldots, V_M$ in such way, that the map $\lambda \mapsto \bar{\lambda}^{-1}$ maps V_m onto $V_{\rho m}$ and η maps U_m onto $U_{\rho m}$. Furthermore, we choose the local parameters $z_1, \ldots, z_M$ such that

$$z_{\rho m}(\lambda) = \bar{z}_m(\bar{\lambda}^{-1}) \quad \text{for all} \quad m = 1, \ldots, M.$$

Now, we impose on the coefficients $\hat{A}$ the following reality condition:

$$(\hat{A}_{\rho m,1}, \ldots, \hat{A}_{\rho m, d_{\rho m}}) = (\overline{\hat{A}_{m,1}}, \ldots, \overline{\hat{A}_{m,d_m}}) \text{ for } m = 1, \ldots, M. \tag{8.11}$$

These conditions ensure that both anti-holomorphic involutions ρ and η of the hyperelliptic spectral curve Σ of (a, b) extend to corresponding involutions of $\hat{\Sigma}$ defined by $\hat{a}$. Since the involutions ρ and η interchanges $\lambda = 0$ and $\lambda = \infty$, we may choose the new parameter $\hat{\lambda}$ in such a way, that both ivolutions act as $\hat{\lambda} \mapsto \bar{\hat{\lambda}}^{-1}$. This condition determines the new spectral parameter $\hat{\lambda}$ uniquely up to $\hat{\lambda} \mapsto e^{\mathbb{i}\theta}\hat{\lambda}$. In order to satisfy condition (iii) in Definition 6.1, we impose the condition that the derivative of $\lambda \mapsto \hat{\lambda} = \phi(\lambda)$ has at $\lambda = \infty$ the form $\lambda \mapsto C\lambda$ for some $C > 0$. In this way, the parameter $\hat{\lambda}$ is uniquely determined and the spectral data $(\hat{a}, \hat{b})$ obey conditions (ii)–(iii) in Definition 6.1 and the weaker form of condition (i) in Definition 7.3. So far, we have constructed for sufficiently small ϵ on the set

$$\mathcal{A}_\epsilon = \Big\{\hat{A} \in \mathbb{C}^{g+1} \mid \|\hat{A} - A\|_\infty \leq \epsilon \text{ and } \hat{A} \text{ satisfies (8.11)}\Big\}. \tag{8.12}$$

the map Ψ from the coefficients $\hat{A}$ onto the corresponding pair $(\hat{a}, \hat{b})$ of spectral data:

$$\Psi : \mathcal{A}_\epsilon \to \mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda], \qquad \hat{A} \mapsto \Psi(\hat{A}) = (\hat{a}, \hat{b}). \tag{8.13}$$

PROPOSITION 8.3 Let the first polynomial of the pair $(a, b) \in \mathcal{M}^g_{\text{ann}}$ have no common root with the corresponding function f in condition (vi) of Definition 6.1. For small ϵ, the map Ψ in (8.13) is an embedding and $\mathcal{M}^g_{\text{per}}$ is locally at (a, b) the real submanifold $\Psi[\mathcal{A}_\epsilon]$ of $\mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda]$ with

$$T_{(a,b)}\mathcal{M}^g_{\text{per}} \simeq \{c \in \mathbb{C}^{g+1}[\lambda] \mid c \text{ obeys (8.2) and } \mathrm{Im}(c(0)/b(0)) = 0\}. \tag{8.14}$$

Proof. We adapt the proof of Lemma 8.1 to the present situation. Since we removed in condition (i) the inequality $\lambda^{-g}a(\lambda) \leq 0$ for unimodular λ, the space of all polynomials a which satisfy this modified condition (i) constitute an affine real $2g$-dimensional subset of $\mathbb{C}^{2g}[\lambda]$. Furthermore, in condition (iii) we restrict the value of one real function and in condition (v) the values of the function h at the roots of a. By construction of the map Ψ the functions λ and $\ln\mu$ (8.8) generate locally the same functions as λ and ν. This ensures condition (vi). Therefore $\mathcal{M}^g_{\text{per}}$ is the level set of $2g + 1$ smooth real functions on an open subset of $\mathbb{R}^{3g+2}$. So it suffices to show that the kernel of these functions is $(g + 1)$-dimensional. By the same arguments as in the proof of Lemma 8.1, this kernel is described by meromorphic functions (8.1) with polynomials $c \in \mathbb{C}^{g+1}[\lambda]$ in the space (8.14). It remains to show that for $c = 0$ there is no non-trivial $(\dot{a}, \dot{b})$. For $c = 0$ equation (8.3) implies that $\dot{a}$ vanishes at all roots of a which are no roots of b. For the common roots, we use the map Ψ in (8.13) instead of equation (8.3).

First, we remark that the polynomials $\hat{A}_1, \ldots, \hat{A}_M$ (8.9) are uniquely determined by the values of $\ln\mu$ and finitely many derivatives of $\ln\mu$ at the roots of $d\ln\mu$. This implies that Ψ is bijective. Furthermore, Cauchy's Integral Formula implies that the inverse map Ψ^{-1} is smooth.

In a second step, we show that $\dot{A}(0)$ vanishes for a smooth family $(-\epsilon, \epsilon) \to \mathcal{A}_\epsilon$, $t \mapsto \hat{A}(t)$, if the corresponding $\partial_t \ln\mu$ (8.1) vanishes. If the polynomial A_m changes, then also the biholomorphic map $\lambda \mapsto z_m(\lambda)$ changes. If we differentiate (8.10) with $z_m(\hat{\lambda}) = \phi_m^{-1}(\hat{\lambda})$, we obtain

$$2(\ln\mu - n_m\pi\mathbb{i})\,\partial_t \ln\mu = \dot{\hat{A}}_m(z_m(\hat{\lambda})) + \hat{A}'_m(z_m(\hat{\lambda}))\dot{z}_m(\hat{\lambda}).$$

The equations (8.1) and (6.1) imply

$$\frac{\hat{\lambda}\, c(\hat{\lambda})}{\hat{b}(\hat{\lambda})} = \frac{\dot{\hat{A}}_m(z_m(\hat{\lambda}))}{\hat{A}'_m(z_m(\hat{\lambda}))\, z'_m(\hat{\lambda})} + \frac{\dot{z}_m(\hat{\lambda})}{z'_m(\hat{\lambda})}. \tag{8.15}$$

This implies $\dot{A}(0) = 0$ for $c = 0$. Since common roots of a and b are roots of some $\hat{A}_m$, $\dot{a}$ vanishes at all common roots of a and b. By equation (8.3) $\dot{a}$ vanishes at all other roots of a, so $\dot{a} = 0$ for $c = 0$. Now (8.3) implies $\dot{b} = 0$. Since (8.14) has dimension $g + 1$ the proof is complete. □

THEOREM 8.4 Let $(a, b) \in \mathcal{M}^g_{\text{ann}}$ have the following two properties:

(i) the function f in condition (vi) of Definition 6.1 has no common root with a.

(ii) a has no unimodular root.

Then $\mathcal{M}^g_{\text{ann}}$ is locally at (a, b) a real submanifold of $\mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda]$ with tangent space (8.5).

Proof. Due to the foregoing proposition $\mathcal{M}^g_{\text{per}}$ is locally at (a, b) a manifold. Condition (ii) guarantees that the inequality in condition (i) of Definition 6.1 is satisfied on a small neighbourhood of $(a, b) \in \mathcal{M}^g_{\text{per}}$. The function h in condition (v) takes imaginary values at unimodular λ. Therefore $\mathcal{M}^g_{\text{ann}}$ is locally at (a, b) the level set of the smooth function $\Lambda : \mathcal{A}_\epsilon \to \mathbb{R}$, which maps the spectral data to the values of the corresponding function $\operatorname{Im} h = \operatorname{Im} \ln \mu$ at $\lambda = 1$. Since there exists a polynomial c in the space (8.14) with $c(1) \neq 0$, the derivative $\Lambda'(0)$ is non-zero. Now the theorem follows from Proposition 8.3 and the Implicit Function Theorem. □

9. Local maxima of $|\tau|$ on $\mathcal{M}^g_{\text{ann}}$

In this section, we classify the local maxima of the function $(a, b) \mapsto |\tau|$ on $\mathcal{M}^g_{\text{ann}}$. For this purpose, we need to consider possible singularities of $\mathcal{M}^g_{\text{ann}}$ at some $(a, b) \in \mathcal{M}^g_{\text{ann}}$. We shall see that it suffices to consider singularities of $\mathcal{M}^g_{\text{ann}}$ at (a, b) with a having only simple roots. We shall restrict to these cases. But our methods apply to more general situations. By Lemma 8.1, $\mathcal{M}^g_{\text{ann}}$ can only have singularities at (a, b) with resultant$(a, b) = 0$ and by Theorem 8.4 only if there exists a common root of a, b and f (the function defined in condition (vi) of Definition 6.1). If a has only simple roots, then a root α of a is also a root of b if and only if it is a root of f. We prove that in this case $\mathcal{M}^g_{\text{ann}}$ is locally at (a, b) homeomorphic to the level set of a smooth function $\Lambda : \mathcal{B}_\epsilon \to \mathbb{R}$ on some open subset $\mathcal{B}_\epsilon \ni B$ of a real $(g+1)$-dimensional subspace of $\mathbb{C}^{g+1}$. Moreover, at the point $B \in \mathcal{B}_\epsilon$ which corresponds to (a, b) the Hessian $\Lambda''(B)$ is neither positive nor negative semi-definite. For such level sets the tangent cone spans the tangent space.

Let $(a, b) \in \mathcal{M}^g_{\text{ann}}$ be such an element with a having only simple roots and resultant$(a, b) = 0$. Due to Lemma 6.6, there exists a unique polynomial p whose roots are contained in the roots of a, such that $(\tilde{a}, \tilde{b}) = (p^2 a, pb) \in \mathcal{M}^{g+\deg p}_{\text{ann}}$ satisfy the assumptions of Theorem 8.4. We apply the smooth parametrization described in Section 8 to this pair $(\tilde{a}, \tilde{b})$. To simplify notation, we do not decorate most of the corresponding objects by a tilde. So let $V_1, \ldots, V_M$ be the pairwise disjoint open neighbourhoods of the roots of $\tilde{b}$ and let $\tilde{\Psi} : \mathcal{A}_\epsilon \to \mathcal{M}^{g+\deg p}_{\text{per}}$ be the corresponding embedding (8.13) in Proposition 8.3 which maps the coefficients $\hat{A}$ of the polynomials (8.9) onto an open neighbourhood of $(\tilde{a}, \tilde{b})$ in $\mathcal{M}^{g+\deg p}_{\text{per}}$. The geometric genus of the spectral curve $\hat{\Sigma}$ corresponding to $\tilde{\Psi}(\hat{A})$ is g, if and only if the polynomial $\hat{A}_m$ (8.9) has one odd order root for all m in the following subset of $\{1, \ldots, M\}$:

$$\mathcal{N} := \{m \in \{1, \ldots, M\} \mid V_m \text{ contains a common root of } \tilde{a} \text{ and } \tilde{b}\}.$$

For $m \in \mathcal{N}$, $d_m = 2\ell_m + 1$ is odd and $\hat{A}_m$ has one odd order root if and only if it is of the form

$$\hat{A}_m(z_m) = (z_m - 2\hat{B}_{m,1}) p_m^2(z_m) \quad \text{with} \quad p_m(z_m) = z_m^{\ell_m} + \hat{B}_{m,1} z^{\ell_m - 1} + \ldots + \hat{B}_{m,\ell_m}. \tag{9.1}$$

We supplement the new coefficients $(\hat{B}_{m,1} \ldots, \hat{B}_{m,\ell_m})_{m \in \mathcal{N}}$ by the old ones:

$$\hat{B}_{m,l} = \hat{A}_{m,l} \quad \text{for} \quad m \in \{1, \ldots, M\} \setminus \mathcal{N} \text{ and } 2 \leq l \leq d_m. \tag{9.2}$$

So the index set of the new coefficients $\hat{B}_{m,l}$ is

$$\{(m,l)\mid m\in\mathcal{N}, 1\leq l\leq \ell_m\}\cup\{m\in\{1,\ldots,M\}\setminus\mathcal{N}, 1\leq l\leq d_m\}.$$

It has $g+1$ elements and $\hat{B}=(\hat{B}_{m,l})$ takes values in $\mathbb{C}^{g+1}$. Let $\Phi:\mathbb{C}^{g+1}\to\mathbb{C}^{g+\deg p+1}$ be the polynomial injection, which maps $\hat{B}$ onto the corresponding $\hat{A}$ with (9.1) for $m\in\mathcal{N}$ and with (9.2) for $m\notin\mathcal{N}$. Then

$$\Phi:\mathcal{B}_\epsilon=\Phi^{-1}[\mathcal{A}_\epsilon]\hookrightarrow\mathcal{A}_\epsilon, \qquad \hat{B}\mapsto\Phi(\hat{B}). \tag{9.3}$$

is a smooth injection. The elements of $\mathcal{B}_\epsilon$ satisfy (8.11) for $m\in\{1,\ldots,M\}\setminus\mathcal{N}$ and

$$(\hat{B}_{\rho m,1},\ldots,\hat{B}_{\rho m,\ell_m})=(\overline{\hat{B}_{m,1}},\ldots,\overline{\hat{B}_{m,\ell_m}})\quad\text{for}\quad m\in\mathcal{N}. \tag{9.4}$$

So $\mathcal{B}_\epsilon$ is an open subset of a real $(g+1)$-dimensional subspace of $\mathbb{C}^{g+1}$. Let $B\in\mathcal{B}_\epsilon$ be the element whose image $A=\Phi(B)$ corresponds to $(\tilde{a},\tilde{b})$. For $m\in\mathcal{N}$ the coefficients $B_{m,l}$ vanish, and for $m\notin\mathcal{N}$ only B_{m,d_m} does not vanish. By definition of Φ there exists for all $(\hat{a},\hat{b})\in\tilde{\Psi}[\Phi[\mathcal{B}_\epsilon]]\subset\mathcal{M}^{g+\deg p}_{\mathrm{per}}$ a unique polynomial $\hat{p}$ which obeys (6.2) and has the same degree as p such that $(\hat{a}/\hat{p}^2,\hat{b}/\hat{p})$ belongs to $\mathcal{M}^g_{\mathrm{per}}$. We define

$$\Psi:\mathcal{B}_\epsilon\to\mathcal{M}^g_{\mathrm{per}},\qquad \hat{B}\mapsto\Psi(\hat{B})=(\hat{a}/\hat{p}^2,\hat{b}/\hat{p})\qquad\text{with}\qquad(\hat{a},\hat{b})=\tilde{\Psi}(\Phi(\hat{B})). \tag{9.5}$$

On a neighbourhood of a common root of a and b the roots of μ^2-1 coincide with the roots of $\tilde{a}$, since $\tilde{f}$ does not vanish there. By Cauchy's Argument Principle the number of roots of μ^2-1 is locally preserved in $\mathcal{M}^g_{\mathrm{per}}$. We conclude that $\Psi[\mathcal{B}_\epsilon]$ is open in $\mathcal{M}^g_{\mathrm{per}}$.

LEMMA 9.1 Let $(a,b)\in\mathcal{M}^g_{\mathrm{ann}}$ with $\mathrm{resultant}(a,b)=0$ such that a has only simple roots. For sufficiently small $\epsilon>0$ a neighbourhood of (a,b) in $\mathcal{M}^g_{\mathrm{ann}}\subset\mathcal{M}^g_{\mathrm{per}}$ is parameterized by the level set of a smooth function $\Lambda:\mathcal{B}_\epsilon\to\mathbb{R}$. If $\Lambda'(B)=0$, then there exists a two-dimensional subspace of $T_B\mathcal{B}_\epsilon$ on which the Hessian $\Lambda''(B)$ is neither positive nor negative semi-definite.

Proof. We already proved that $\Psi:\mathcal{B}_\epsilon\hookrightarrow\mathcal{M}^g_{\mathrm{per}}$ is a parametrization of an open neighbourhood of $(a,b)=\Psi(B)$ in $\mathcal{M}^g_{\mathrm{per}}$. Let $\tilde{\Lambda}:\mathcal{A}_\epsilon\to\mathbb{R}$ be the smooth function in the proof of Theorem 8.4 whose level set parameterizes a neighbourhood of $(\tilde{a},\tilde{b})=\tilde{\Psi}(A)$ in $\mathcal{M}^{g+\deg p}_{\mathrm{ann}}$ with $A=\Phi(B)$. The level set of $\Lambda=\tilde{\Lambda}\circ\Phi:\mathcal{B}_\epsilon\to\mathbb{R}$ parameterizes an open neighbourhood of (a,b) in $\mathcal{M}^g_{\mathrm{ann}}$. In $\mathcal{N}$, there is no fixed point of ρ, since a has only simple roots. For each $\mathfrak{m}\in\mathcal{N}$ let $\mathcal{V}_{\mathfrak{m}}$ denote

$$\mathcal{V}_{\mathfrak{m}}=\left\{\dot{\hat{B}}\in T_B\mathcal{B}_\epsilon\mid \dot{\hat{B}}_{m,l}=0\text{ for all }(m,l)\notin\{(\mathfrak{m},1),(\rho\mathfrak{m},1)\}\right\}. \tag{9.6}$$

First consider the case that $\tilde{b}(1)=0$. This is equivalent to $b(1)=0$, since the roots of p are contained in the set of non-unimodular roots of a. In this case V_m contains $\lambda=1$ for a unique $m\in\{1,\ldots,M\}\setminus\mathcal{N}$. The corresponding A_{m,d_m} does not vanish. By (8.15) the partial derivative of Λ with respect to the real coefficient $\hat{B}_{m,d_m}$ does not vanish, and there is nothing to prove.

Now we assume $\tilde{b}(1) \neq 0$. Clearly, it suffices to show that $\Lambda'(B)$ vanishes on $\mathcal{V}_{\mathfrak{m}}$ and $\Lambda''(B)$ is on $\mathcal{V}_{\mathfrak{m}}$ neither positive nor negative semi-definite. For $B_{\mathfrak{m},2} = 0, \ldots, B_{\mathfrak{m},\ell_{\mathfrak{m}}} = 0$ only two coefficients of $\hat{A}_{\mathfrak{m}}(z_m)$ depend on $\hat{B}_{\mathfrak{m},1}$:

$$\hat{A}_{\mathfrak{m},2} = -3\hat{B}^2_{\mathfrak{m},1}, \qquad \hat{A}_{\mathfrak{m},3} = -2\hat{B}^3_{\mathfrak{m},1}.$$

Hence $\Lambda'(B)$ vanishes on $\mathcal{V}_{\mathfrak{m}}$ and $\Lambda''(B)|_{\mathcal{V}_{\mathfrak{m}}}$ is determined by the restriction of $\tilde{\Lambda}''(A)$ to

$$\left\{\dot{\hat{A}} \in T_A\mathcal{A}_\epsilon \mid \dot{\hat{A}}_{m,l} = 0 \text{ for all } (m,l) \notin \{(\mathfrak{m},2),(\mathfrak{m},2)\}\right\}. \tag{9.7}$$

In Proposition 8.3, we identified $T_A\mathcal{A}_\epsilon$ with a subspace of $c \in \mathbb{C}^{g+\deg p}[\lambda]$. Let $\alpha_{\mathfrak{m}}$ denote the value of λ at the common root in $V_{\mathfrak{m}}$. Due to (8.15), the space (9.7) is identified with

$$\left\{c(\lambda) = \hat{c}(\lambda)\tfrac{\tilde{b}(\lambda)}{(\lambda-\alpha_{\mathfrak{m}})(\bar{\alpha}_{\mathfrak{m}}\lambda-1)}\,\middle|\,\hat{c} \in \mathbb{C}^2[\lambda],\ \lambda^2\overline{\hat{c}(\bar{\lambda}^{-1})} = -\hat{c}(\lambda) \text{ and } \operatorname{Im}(\hat{c}(0)/\alpha_{\mathfrak{m}}) = 0\right\}. \tag{9.8}$$

With $\hat{c}(\lambda) = \mathbb{i}\lambda$ this space contains an element with $c(1) \neq 0$. So $\tilde{\Lambda}'(A)$ does not vanish on (9.7), and $\Lambda''(B)$ has on $\mathcal{V}_{\mathfrak{m}}$ a non-zero eigenvalue. The multiplication of $\hat{B}_{\mathfrak{m},1}$ by $\mathbb{i}$ and of $\hat{B}_{\rho\mathfrak{m},1}$ by $-\mathbb{i}$ preserves (9.4) and switches the sign of the Hessian $\Lambda''(B)$. This completes the proof. □

LEMMA 9.2 Let $\Lambda \in \mathbb{C}^\infty(\Omega, \mathbb{R})$ on an open subset $\Omega \ni 0$ of $\mathbb{R}^{g+1}$ obey $\Lambda(0) = 0 = \Lambda'(0)$. Then

$$\begin{aligned}\left\{v \in \mathbb{R}^{g+1} \mid \Lambda''(0)(v,v) = 0 \text{ and there exists } w \in \mathbb{R}^{g+1} \text{ with } \Lambda''(0)(v,w) \neq 0\right\} \subset \\ \subset \{\dot{\gamma}(0) \mid \gamma \in C^\infty((-\epsilon,\epsilon),\Omega) \text{ with } \gamma(0) = 0 \text{ and } \Lambda(\gamma(t)) = 0 \text{ for } t \in (-\epsilon,\epsilon)\}.\end{aligned} \tag{9.9}$$

Proof. Let $v, w \in \mathbb{R}^{g+1}$ with $\Lambda''(0)(v,v) = 0$ and $\Lambda''(0)(v,w) \neq 0$. Then

$$\Upsilon : (-\delta,\delta) \times (-\delta,\delta) \to \mathbb{R} \qquad (x,y) \mapsto \begin{cases} 2x^{-2}\Lambda(x(v+yw)) & \text{for } x \neq 0 \\ \Lambda''(0)(v+yw, v+yw) & \text{for } x = 0 \end{cases}$$

is for small $\delta > 0$ smooth with $\Upsilon(0,0) = 0$ and $\frac{\partial\Upsilon(0,0)}{\partial y} = 2\Lambda''(0)(v,w) \neq 0$. Due to the Implicit Function Theorem, there exists a smooth function $x \mapsto y(x)$ on a small interval $x \in (-\epsilon,\epsilon)$ with $y(0) = 0$ and $\Upsilon(x,y(x)) = 0 = \Lambda(x(v+y(x)w))$. The derivative v of $x \mapsto x(v+y(x)w)$ at $x = 0$ belongs to the tangent cone. □

LEMMA 9.3 Let $(a,b) \in \mathcal{M}^g_{\text{ann}}$ be a local maximum of the function $(a,b) \mapsto |\tau|$ on $\mathcal{M}^g_{\text{ann}}$ such that a has only simple roots. Then $g = 0$.

Proof. For (a,b) with resultant$(a,b) \neq 0$, we apply Lemma 8.1. For a polynomial c in the space, (8.5) with $\operatorname{Re}(c(0)/b(0)) \neq 0$ the derivative of $|\tau|$ (8.6) does not vanish. For $g > 0$, the space (8.5) contains $c(\lambda) = (\lambda - 1)(\bar{b}(0)\lambda^g - b(0))$ with these properties. So g has to vanish.

For resultant$(a,b) = 0$, we apply Lemma 9.1 and use the local parametrization of $\mathcal{M}^g_{\text{ann}}$ by the level set of $\Lambda : \mathcal{B}_\epsilon \to \mathbb{R}$. We choose $\mathfrak{m} \in \mathcal{N}$. As in the proof of Lemma 9.1, we distinguish between $\tilde{b}(1) = 0$ and $\tilde{b}(1) \neq 0$. In the first case, we apply Lemma 9.2. For $\hat{c}(\lambda) = (\lambda - 1)(\bar{\alpha}_{\mathfrak{m}}\lambda + \alpha_{\mathfrak{m}})$ the element c

of (9.8) vanishes at $\lambda = 1$ and $c(0)/\tilde{b}(0)$ is a non-zero real number. Along the corresponding element of (9.7) the derivative $\tilde{\Lambda}'(A)$ vanishes in contrast to the derivative of $|\tau|$. The multiplication of $\hat{B}_{\mathfrak{m},1}$ by $\mathbb{i}$ and of $\hat{B}_{\rho\mathfrak{m},1}$ by $-\mathbb{i}$ preserves (9.4) and switches the sign of $\Lambda''(B)$. Therefore, $\mathcal{V}_{\mathfrak{m}}$ (9.6) has for each $\mathfrak{m} \in \mathcal{N}$ a base v_1, v_2 with $\Lambda''(B)(v_1, v_1) = 0 = \Lambda''(B)(v_2, v_2)$. Since $\Lambda''(B)$ does not vanish on $\mathcal{V}_{\mathfrak{m}}$, they obey $\Lambda''(B)(v_1, v_2) \neq 0$. By Lemma 9.2, there exist smooth functions $y_1, y_2 : (-\epsilon, \epsilon) \to \mathbb{R}$ such that

$$(-\epsilon,\epsilon) \to \mathbb{R}, x \mapsto x(v_1 + y_1(x)v_2) \qquad \text{and} \qquad (-\epsilon,\epsilon) \to \mathbb{R}, x \mapsto x(v_2 + y_2(x)v_1)$$

stay in the level set of Λ. Along both paths $|\tau|$ has a critical point at $x = 0$. Since $c(0)/\tilde{b}(0) \neq 0$ the second derivative of $|\tau|$ at $x = 0$ does not vanish and has opposite signs on both paths. Along one of both paths $|\tau|$ is not a local maximum.

We extend this argument to the case $\tilde{b}(1) \neq 0$. In this case the proof of Lemma 9.1 shows that the level set of Λ is at B a smooth submanifold of $\mathcal{B}_\epsilon$. Its tangent space contains $\mathcal{V}_{\mathfrak{m}}$ for all $\mathfrak{m} \in \mathcal{N}$. We choose any vector $w \in T_B\mathcal{B}_\epsilon$, with $\Lambda'(B)(w) \neq 0$. By the Implicit Function Theorem there exists for any non–trivial $v \in \mathcal{V}_{\mathfrak{m}}$ a smooth function $y : (-\epsilon, \epsilon) \to \mathbb{R}$ with $y(0) = 0 = y'(0)$ such that $(-\epsilon, \epsilon) \to \mathcal{B}_\epsilon$, $x \mapsto xv + y(x)w$ stays in the level set of Λ. Along this path the first derivative of $|\tau|$ at $x = 0$ vanishes. Since (9.8) contains an element with $c(0)/\tilde{b}(0) \in \mathbb{R}^\times$, for some $v \in V$ the second derivative of $|\tau|$ at $x = 0$ does not vanish. Furthermore, for some $v \in V$, $|\tau|$ has no local maximum at $x = 0$. Hence (a, b) is no local maximum of $|\tau|$ for $g > 0$. □

THEOREM 9.4 If a pair $(\tilde{a}, \tilde{b}) \in \mathcal{M}^g_{\mathrm{ann}}$ is a local maximum of the function $(a, b) \mapsto |\tau|$ on $\mathcal{M}^g_{\mathrm{ann}}$, then all roots of $\tilde{a}$ are unimodular. In particular there exist $(a, b) = (-\frac{1}{16}, b) \in \mathcal{M}^0_{\mathrm{ann}}$ and a polynomial p as in Lemma 6.5 with $(\tilde{a}, \tilde{b}) = (p^2 a, pb)$.

Proof. Due to Lemma 6.5, there exists for every $(\tilde{a}, \tilde{b}) \in \mathcal{M}^g_{\mathrm{ann}}$ a unique p satisfying (6.2) with maximal degree whose square divides $\tilde{a}$. This p divides $\tilde{b}$ and $(a, b) = (\tilde{a}/p^2, \tilde{b}/p) \in \mathcal{M}^{g-\deg p}_{\mathrm{ann}}$ with a having only simple roots. By Cauchy's Argument Principle the number of roots of the holomorphic function $\mu^2 - 1$ is locally preserved in $\mathcal{M}^g_{\mathrm{ann}}$. This implies that the mapping $(a, b) \mapsto (p^2 a, pb)$ extends to an embedding of a neighbourhood of (a, b) in $\mathcal{M}^{g-\deg p}_{\mathrm{ann}}$ into $\mathcal{M}^g_{\mathrm{ann}}$. Consequently, (a, b) is a local maximum of $|\tau|$ in $\mathcal{M}^{g-\deg p}_{\mathrm{ann}}$ if $(\tilde{a}, \tilde{b})$ is a local maximum of $|\tau|$ in $\mathcal{M}^g_{\mathrm{ann}}$. The foregoing lemma implies $g = \deg p$. In particular, all roots of $\tilde{a}$ are even order roots and the corresponding spectral curve has geometric genus zero. Since b has only one unimodular root, condition (vi) in Definition 6.1 implies that all non-unimodular roots of $\tilde{a}$ are double roots.

Now let p^2 contain all unimodular roots of $\tilde{a}$. Again $(a, b) = (\tilde{a}/p^2, \tilde{b}/p) \in \mathcal{M}^{g-\deg p}_{\mathrm{ann}}$ and a has only non-unimodular double roots. This implies that (a, b) satisfies the assumptions of Theorem 8.4. Again by Cauchy's Argument Principle a neighbourhood of (a, b) in $\mathcal{M}^{g-\deg p}_{\mathrm{ann}}$ is embedded into $\mathcal{M}^g_{\mathrm{ann}}$, and (a, b) is a local maximum of $|\tau|$, if $(\tilde{a}, \tilde{b})$ is. If $g > \deg p$, then $c(\lambda) = (\lambda - 1)(\bar{b}(0)\lambda^{g-p} - b(0))$ belongs to (8.5) and (a, b) is no critical point of $|\tau|$. This implies that all roots of $\tilde{a}$ are unimodular. □

10. Connected components of $\mathcal{M}^g_{\mathrm{emb}}$

In this section, we show that $\mathcal{M}^g_{\mathrm{emb}}$ is empty for $g > 2$ and has for $g \leq 2$ at most the same number of connected components as $\mathcal{M}^g_{\mathrm{Abr}}$, i.e. up to the sign of b one for $g = 0$ and $g = 2$, and two for $g = 1$. We first derive from the curvature estimate that each connected component of $\mathcal{M}^g_{\mathrm{emb}}$ contains a maximum of $|\tau|$ and then show that this maximum belongs to $\mathcal{M}^g_{\mathrm{Abr}}$.

LEMMA 10.1 For $g \in \mathbb{N}_0$ and $\epsilon_0 > 0$ the following sets are compact:

$$\{(a,b) \in \mathcal{M}^g_{\text{emb}} \mid |\tau(a,b)| \geq \epsilon_0\}. \tag{10.1}$$

Proof. Let us first prove that the coefficients of a are bounded, if $|\tau| \geq \epsilon_0$. Due to Proposition 3.4 the solutions ω of the sinh-Gordon equation corresponding to all $\xi_\lambda \in \mathcal{I}(a)$ are bounded by $|\omega| \leq C_0(\epsilon_0)$. For any roots $\alpha_1, \ldots, \alpha_g$ of a, such that $\bar{\alpha}_1^{-1}, \ldots, \bar{\alpha}_g^{-1}$ are the remaining roots of a, there exist an off-diagonal $\xi_\lambda = \left(\begin{smallmatrix} 0 & \beta \\ \gamma & 0 \end{smallmatrix}\right) \in \mathcal{I}(a)$ with

$$\beta = \frac{\mathbb{i}}{4\lambda\sqrt{\prod_d |\alpha_d|}} \prod_d (1 - \bar{\alpha}_d \lambda) \qquad \gamma = \frac{\mathbb{i}}{4\sqrt{\prod_d |\alpha_d|}} \prod_d (\lambda - \alpha_d).$$

The corresponding ω at $z = 0$ is due to Proposition 4.3 and Remark 4.4 equal to $\omega(0) = -\frac{1}{2}\sum_d \ln|\alpha_d|$ and $\nabla\omega(0) = 0$. Since $|\omega(0)| \leq C_0(\epsilon_0)$ all roots of a are bounded away from ∞ and 0 and the coefficients of a are bounded.

Next, we shall show that the coefficients of b are bounded. Lemma 3.3 gives a bound on $|\tau|$ from above. Hence, it suffices to show that the polynomials b with the properties (ii)–(iv) in Definition 6.1 are uniquely determined by τ and a and depend continuously on $(\tau, a) \in \mathbb{C} \times \mathcal{M}_g$. We first consider a in the subspace $\mathcal{M}^1_g$ of polynomials $a \in \mathcal{M}_g$ with pairwise different roots. The corresponding hyperelliptic compact Riemann surfaces Σ have a canonical base of A and B cycles, such that twice the integrals in property (iv) are the A-periods of the 1-form (6.1). These cycles extend to open subsets of $\mathcal{M}^1_g$ and the A-periods depend continuously on (a,b). The polynomials b with the properties (ii)–(iii) in Definition 6.1 for $\tau = 0$ correspond to the holomorphic 1-forms on this compact Riemann surface. The A-periods define an isomorphism from the g–dimensional space of (real) holomorphic 1-forms to $\mathbb{R}^g$. Hence the basis of such holomorphic 1-forms dual to the A-cycles also depends continuously on a. This implies that the polynomials b with the properties (ii)–(iv) in Definition 6.1 are uniquely determined by (τ, a) and depend continuously on $(\tau, a) \in \mathbb{C} \times \mathcal{M}^1_g$.

Now we assume that for a sequence $(\tau_n, a_n) \in \mathbb{C} \times \mathcal{M}^1_g$ with limit $(\tau, a) \in \mathbb{C} \times \mathcal{M}_g$ the corresponding sequence b_n with properties (ii)-(iv) in Definition 6.1 is unbounded. Let p be the up to sign unique polynomial (6.2) such that p^2 divides a and $\tilde{a} = a/p$ has pairwise different roots. We choose a norm on $\mathbb{C}^{g+1}[\lambda]$ and pass to a subsequence such that $(b_n/\|b_n\|)$ converges. The corresponding sequence of functions $h_n/\|b_n\|$ defined in Definition 6.1 (v) are multivalued and meromorphic on the corresponding sequence Σ_n of compact hyperelliptic Riemann surfaces. The real part of $h_n/\|b_n\|$ is single valued and harmonic on $\Sigma_n^\times$ (4.8). Due to the maximum principle of harmonic functions they are bounded outside small discs around $\lambda = 0$ and $\lambda = \infty$. Therefore the real parts of $h_n/\|b_n\|$ converge to a harmonic function on

$$\Sigma^\times = \{(\lambda, \nu) \in \mathbb{C}^2 \mid \nu^2 = \lambda^{-1}\tilde{a}(\lambda)\}.$$

In particular, p divides the limit of $b_n/\|b_n\|$. The arguments for $a \in \mathcal{M}^1_g$ show that this limit is now uniquely determined by the limit of $\tau_n/\|b_n\|$ and $\tilde{a}$. Since τ_n is bounded and $\|b_n\|$ is unbounded, the limit of $\tau_n/\|b_n\|$ vanishes together with the limit of $b_n/\|b_n\|$. This contradicts the unboundedness of b_n and proves that (10.1) are bounded subsets of $\mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda]$.

It remains to prove that (10.1) is also closed in $\mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda]$. For sequences (a_n, b_n) in $\mathcal{M}^g_{\text{emb}}$, which converge in $\mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda]$ the limit (a,b) clearly has properties (i)-(iii). By the Maximum

Modulus Theorem the corresponding functions f_n and g_n in Definition 6.1 (vi) converge on compact subsets of $\mathbb{C}^\times$. Therefore the limit also has the remaining properties (iv)-(vi) and belongs to $\mathcal{M}^g_{\mathrm{ann}}$. Since $\mathcal{M}^g_{\mathrm{emb}}$ is closed in $\mathcal{M}^g_{\mathrm{ann}}$ (Proposition 7.5) the limit belongs also to $\mathcal{M}^g_{\mathrm{emb}}$. □

In particular, each connected component of $\mathcal{M}^g_{\mathrm{emb}}$ contains a maximum of $(a,b) \mapsto |\tau|$. Next we apply the characterization of the local maxima of $|\tau|$ in the foregoing section.

THEOREM 10.2 For $g \in \mathbb{N}_0$ each connected component of $\mathcal{M}^g_{\mathrm{emb}}$ contains a maximum $(\tilde{a},\tilde{b}) \in \mathcal{M}^g_{\mathrm{Abr}}$ of $|\tau|$. In particular, $\mathcal{M}^g_{\mathrm{emb}}$ has at most as many connected components as $\mathcal{M}^g_{\mathrm{Abr}}$, i.e., up to the sign of $\tilde{b}$ one for $g=0$ and $g=2$, two for $g=1$ and none for $g>2$.

Proof. Due to Lemma 10.1, the continuous function $(a,b) \mapsto |\tau|$ has in every connected component of $\mathcal{M}^g_{\mathrm{emb}}$ a maximum $(\tilde{a},\tilde{b})$. Theorem 9.4 shows that $\tilde{a}$ has only unimodular roots. By Theorem 7.6 all roots of $\tilde{a}$ are roots of the function f in condition (vi) of Definition 6.1 of the unique $(a,b) \in \mathcal{M}^0_{\mathrm{Abr}}$. The corresponding elements of the form $(\tilde{a},\tilde{b}) = (p^2 a, pb) \in \mathcal{M}^{\deg p}_{\mathrm{ann}}$ are determined in Lemma 6.6. Up to sign of $\tilde{b}$, there are four such elements with p having only unimodular roots. Since they belong to $\mathcal{M}^g_{\mathrm{Abr}}$, we finally obtain $(\tilde{a},\tilde{b}) \in \mathcal{M}^g_{\mathrm{Abr}}$. This shows that every connected component of $\mathcal{M}^g_{\mathrm{emb}}$ contains an element of $\mathcal{M}^g_{\mathrm{Abr}}$, and $\mathcal{M}^g_{\mathrm{emb}}$ has at most as many connected components as $\mathcal{M}^g_{\mathrm{Abr}}$. □

11. Isolated property of the Abresch family

In this section, we prove that for all $g = 0, 1, 2$ the space $\mathcal{M}^g_{\mathrm{Abr}}$ is open and closed in $\mathcal{M}^g_{\mathrm{ann}}$. Our proof is based on the Four-Vertex Theorem. There exists another proof which uses the smooth parametrization of spectral data in Section 8 and the Inverse Function Theorem.

Consider the solution $\omega : \mathbb{C}/\tau\mathbb{Z} \to \mathbb{R}$ of the sinh-Gordon equation corresponding to spectral data $(a,b) \in \mathcal{M}^g_{\mathrm{Abr}}$. By Section 2, there exist two elliptic functions $x \mapsto f(x) = \frac{-\omega_x}{\cosh\omega}$ and $y \mapsto g(y) = \frac{-\omega_y}{\cosh\omega}$. The Jacobi operator on $\mathbb{C}/\tau\mathbb{Z}$ is given by

$$\mathcal{L} = \frac{1}{\cosh^2\omega}\left(\partial_x^2 + \partial_y^2 + 1 + \frac{2|\nabla\omega|^2}{\cosh^2\omega}\right) = \frac{1}{\cosh^2\omega}\left(\partial_x^2 + \partial_y^2 + 1 + 2f^2(x) + 2g^2(y)\right).$$

We use Fourier analysis. We define the set of periodic eigenfunctions $\{e_n\}$ associate to eigenvalues $\lambda_0 < \lambda_1 \leq \lambda_2 \ldots.$ repeated with multiplicity,

$$\partial_x^2 e_n(x) + 2f^2(x)e_n(x) = -\lambda_n e_n(x). \tag{11.1}$$

where f is an elliptic function which satisfies

$$-(f_x)^2 = f^4 + (1+c-d)f^2 + c = (f^2 - \delta_1)(f^2 - \delta_2).$$

If τ denotes the period of the annulus in the x-direction, which coincides with a period of f, then the set $\{e_n\}_{n\in\mathbb{N}}$ span the Hilbert space $L^2(\mathbb{R}/\tau\mathbb{Z})$. A bounded solution of $\mathcal{L}u = 0$ decomposes into

$$u(x,y) = \sum_{n\geq 0} u_n(y)e_n(x),$$

where $u_n : \mathbb{R} \to \mathbb{R}$ are uniformly bounded functions.

LEMMA 11.1 Let $u : \mathbb{C}/\tau\mathbb{Z} \to \mathbb{R}$ be a bounded solution of $\mathcal{L}u = 0$, where $x \mapsto f(x)$ and $y \mapsto g(y)$ are the functions defined in Section 2. Then u cannot have more than two zeroes on horizontal sections unless it vanishes identically.

Proof. The following elliptic differential equations define functions $x \mapsto f(x)$ and $y \mapsto g(y)$

$$-(f_x)^2 = f^4 + (1 + c - d)f^2 + c = (f^2 - \delta_1)(f^2 - \delta_2)$$

$$-(g_y)^2 = g^4 + (1 + d - c)g^2 + d = (g^2 - \beta_1)(g^2 - \beta_2)$$

with roots $2\delta_1 = -(1 + c - d) + \sqrt{\Delta}$, $2\delta_2 = -(1 + c - d) - \sqrt{\Delta}$, $2\beta_1 = -(1 + d - c) + \sqrt{\Delta}$, $2\beta_2 = -(1 + d - c) - \sqrt{\Delta}$ and $\Delta = (1 + c - d)^2 - 4c = (1 + d - c)^2 - 4d$, we see that f is oscillating around zero between $-\sqrt{\delta_1}$ and $\sqrt{\delta_1}$ since $\delta_2 < 0$ and g is oscillating between $-\sqrt{\beta_1}$ and $\sqrt{\beta_1}$.

We solve the equation (11.1) on $[0, \tau/2]$. The function $\wp := \alpha - f^2$ (with $3\alpha = -(1 + c - d)$) is the Weierstrass $\wp$-function which satisfies the elliptic equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

for constants g_2, g_3 depending only on constants c and d. The equation (11.1) transforms into the Lamé equation

$$\partial_x^2 e_n - 2\wp e_n = -\mu_n e_n$$

On $[0, \tau/2]$, the functions $e_0 = \sqrt{f^2 - \delta_2}, e_1 = f, e_2 = \sqrt{\delta_1 - f^2}$ are known as Lamé functions of degree one of the first kind (see [26, Chapter XXIII]). These functions extend to $[0, \tau]$ by symmetry and they are the first three eigenfunctions of the Lamé operator.

The function $e_0 = \sqrt{f^2 - \delta_2} > 0$ is an eigenfunction associate to the eigenvalue $\lambda_0 = -\delta_1$ with boundary data $e_0(0) = e_0(\tau/2), \partial_x e_0(0) = \partial_x e_0(\tau/2) = 0$. The second eigenfunction $e_1 = f$ is associate to eigenvalue $\lambda_1 = 1 + c - d$ with $e_1(0) = -e_1(\tau/2), \partial_x e_0(0) = \partial_x e_0(\tau/2) = 0$. The third function is $e_2 = \sqrt{\delta_1 - f^2}$ associate to eigenvalue $\lambda_2 = -\delta_2$ with $e_2(0) = e_2(\tau/2) = 0$ and $\partial_x e_0(0) = -\partial_x e_0(\tau/2)$. These eigenfunctions extend by symmetry to $\mathbb{R}/\tau\mathbb{Z}$. They are the first three eigenfunctions of the spectrum with $\lambda_0 < \lambda_1 < \lambda_2$ and have at most two zeroes on each horizontal curve. If e_k is an eigenfunction having strictly more than two zeroes on a period $[0, \tau]$, then the associated eigenvalues $\lambda_k > \lambda_2$. If not, one can argue by contradiction and compute $W = e_k(\partial_x e_2) - (\partial_x e_k)e_2$. Then $W' = (\lambda_k - \lambda_2)e_k e_2$ and by studying the behaviour of W between two consecutive zeroes of e_k, the function e_2 has to change sign. Thus e_2 would have at least four zeroes, a contradiction.

Now we consider a bounded Jacobi field u on $\mathbb{C}/\tau\mathbb{Z}$. By Fourier expansion, we decompose u as

$$u(x, y) = \sum_{n \geq 0} u_n(y)e_n(x).$$

Since u is bounded on $\mathbb{C}/\tau\mathbb{Z}$ then u_n is bounded on $\mathbb{R}$. Inserting u in the equation $\mathcal{L}u = 0$ we obtain a countable set of equations for $n \in \mathbb{N}$:

$$\partial_y^2 u_n(y) + 2g^2(y)u_n(y) + (1 - \lambda_n)u_n(y) = 0.$$

For $n \geq 2$, we have $(1-\lambda_n) < 1-\lambda_2 = 1+\delta_2 = \frac{1}{2}(1+d-c+\sqrt{\Delta}) = -\beta_1$. But as we remarked for equation (11.1), the function $\sqrt{g^2-\beta_2} > 0$ is the first periodic eigenfunction associated with the first eigenvalue $\mu_0 = -\beta_1$ of

$$\partial_y^2 v(y) + 2g^2(y)\, v(y) = -\mu\, v(y) \tag{11.2}$$

It is a well known fact (see [27] for example) that for $\mu < \mu_0$ the equation (11.2) cannot have bounded solutions on $\mathbb{R}$. Then $u_n = 0$ for $n \geq 2$. The function u is a linear combination of e_0, e_1, e_2 and we obtain a contradiction with the following lemma. $\square$

LEMMA 11.2 For any real constants $\alpha_0, \alpha_1, \alpha_2$, the function $\alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2$ has at most two roots on $\mathbb{R}/\tau\mathbb{Z}$.

Proof. The functions e_0, e_1 and e_2 obey $e_0^2 + \delta_2 = e_1^2 = \delta_1 - e_2^2$. Therefore, the expression

$$(\alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2)(\alpha_0 e_0 + \alpha_1 e_1 - \alpha_2 e_2)(\alpha_0 e_0 - \alpha_1 e_1 + \alpha_2 e_2)(\alpha_0 e_0 - \alpha_1 e_1 - \alpha_2 e_2)$$

is an even polynomial $p(e_1)$ of degree four with respect to e_1 with real coefficients not depending on e_0 and e_1. Along the period τ the function e_1 takes all values in $(-\sqrt{\delta_1}, \sqrt{\delta_1})$ exactly twice and $\pm\sqrt{\delta_1}$ exactly once. At two points in the pre-image of one value of e_1 in $(-\sqrt{\delta_1}, \sqrt{\delta_1})$ the function e_0 takes the same value and e_2 takes values with opposite sign. Therefore every root of $p(e_1)$ corresponds to at most one root of $\alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2$. For non-vanishing values e_1 at roots of $\alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2$, the negative $-e_1$ is the value at a root of $\alpha_0 e_0 - \alpha_1 e_1 + \alpha_2 e_2$ and not of a root of $\alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2$. Therefore at most two of the four roots of $p(e_1)$ are the values of e_1 at one root of $\alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2$. $\square$

Now we prove the main theorem of this section.

THEOREM 11.3 For $g = 0, 1, 2$ the set $\mathcal{M}^g_{\mathrm{Abr}}$ is open and closed in $\mathcal{M}^g_{\mathrm{emb}}$.

Proof. Since $\mathcal{M}^g_{\mathrm{Abr}}$ is closed in $\mathcal{M}^g_{\mathrm{ann}}$, it is also closed in $\mathcal{M}^g_{\mathrm{emb}}$. To prove openness we show that a sequence (a_n, b_n) in $\mathcal{M}^g_{\mathrm{emb}} \setminus \mathcal{M}^g_{\mathrm{Abr}}$ cannot converge to $(a, b) \in \mathcal{M}^g_{\mathrm{Abr}}$. Let us assume to the contrary that a sequence (a_n, b_n) in $\mathcal{M}^g_{\mathrm{emb}} \setminus \mathcal{M}^g_{\mathrm{Abr}}$ converges to $(a, b) \in \mathcal{M}^g_{\mathrm{Abr}}$. Due to Lemma 7.4 the map A (7.2) is proper. After passing to a subsequence there exists a sequence $\xi_{\lambda,n} \in \mathcal{I}(a_n)$ which converges to $\xi_\lambda \in \mathcal{I}(a)$. This sequence $\xi_{n,\lambda}$ corresponds to a sequence of properly embedded minimal annuli X_n and the limit ξ_λ to an Abresch annulus X. Since $(a_n, b_n) \notin \mathcal{M}^g_{\mathrm{Abr}}$ the proper minimal embeddings X_n are not foliated by constant curvature lines and have non–trivial Shiffman's Jacobi fields u_n on X_n. We shall now use u_n and construct a non–zero bounded Jacobi field v on X. Using the Four-Vertex Theorem on embedded horizontal curves we conclude that v has at least four zeroes and then $v \equiv 0$, a contradiction to Lemma 11.1.

On compact sets u_n converges to zero. Since ω_n is bounded in $C^{k,\alpha}$ norm by Proposition 3.4 on the whole annulus, the Jacobi field $u_n = (\omega_{n,xy}) - \tanh(\omega_n)(\omega_{n,x})(\omega_{n,y})$ is bounded on X_n. We translate the sequence of annuli X_n, such that $|u_n(0)|$ is the maximum of u_n and renormalize

$$v_n := \frac{u_n}{|u_n(0)|}.$$

Then there is a subsequence which converges by Arzela–Ascoli's Theorem to a bounded function v on X with $|v(0)| = 1$. We denote by $\mathrm{d}s_n = \cosh^2\omega_n\,|\mathrm{d}z|^2$ and $\mathrm{d}s = \cosh^2\omega\,|\mathrm{d}z|^2$ the associated metrics with $\omega_n \to \omega$ uniformly. The Jacobi operators are given by

$$\mathcal{L}_n = \frac{1}{\cosh^2\omega_n}\left(\partial_x^2 + \partial_y^2 + 1 + 2\frac{|\nabla\omega_n|^2}{\cosh^2\omega_n}\right).$$

Since ω_n is bounded by Proposition 3.4, the Jacobi field $u_n = (\omega_{n,xy}) - \tanh(\omega_n)(\omega_{n,x})(\omega_{n,y})$ is bounded on X_n and v_n converges by Arzela–Ascoli's Theorem to a bounded solution of $\mathcal{L}\,v = 0$, a bounded Jacobi function on X. Since $u_n = \cosh^2\omega_n\,(\partial_x k_g)$ (see Theorem 2.4), it has at least four zeroes on any level horizontal curve by the Four-Vertex Theorem (see [28]). The set of curves $\Gamma = \{v_n^{-1}(0)\} = \{u_n^{-1}(0)\}$ separate at least four nodal domains intersecting every horizontal curve (see Theorem 2.4). By counting the number of zeroes of v_n on each horizontal section $x \mapsto X(x, y_0)$, we deduce that v cannot have generically two zeroes on horizontal curves and we argue as follows to get a contradiction. Otherwise, it means that two or three zeroes of v_n coalesce in the limit. We find an open interval $y \in (t_1, t_2)$, such that on every curve $\gamma(t) = A \cap \mathbb{S}^2 \times \{t\}$ the zeroes of v_n coalesce in the limit at two zeroes. A coalescing of zeroes will produce a new nodal curve $\Gamma_0 = v_0^{-1}(0)$ generically transverse to horizontal sections of $X \cap \mathbb{S}^2 \times [t_1, t_2]$. We can find a horizontal section transverse to Γ_0. Since Γ_0 is a limit of several nodal curves collapsing together at the limit, v will not change sign along $\gamma(t)$ crossing Γ_0, or v will change sign but with $\partial_x v = 0$ on Γ_0. This contradicts a Theorem of Cheng [29] on the singularity of nodal curves for the solution of an elliptic operator which are isolated and describing equiangular curves at the singularity.

In summary, the Four-Vertex Theorem implies that v_0 has at least four zeroes generically on each horizontal section. Now the analysis of the Jacobi operator on $\mathbb{C}/\tau\mathbb{Z}$ in Lemma 11.1 gives a contradiction. We conclude that such a Jacobi field cannot exist on an Abresch annulus X. □

12. Proof of the main theorem

Due to Proposition 7.2, the unique potential ξ_λ without roots of a properly embedded minimal annulus is contained in the isospectral set $\mathcal{I}(a)$ of spectral data $(a, b) \in \mathcal{M}^g_{\mathrm{emb}}$. Therefore, it suffices to show the equality $\mathcal{M}^g_{\mathrm{emb}} = \mathcal{M}^g_{\mathrm{Abr}}$ for all $g \in \mathbb{N}_0$. Due to Theorem 10.2, every connected component of $\mathcal{M}^g_{\mathrm{emb}}$ contains a connected component of $\mathcal{M}^g_{\mathrm{Abr}}$. Finally, Theorem 11.3 shows that all connected components of $\mathcal{M}^g_{\mathrm{Abr}}$ coincide with connected components of $\mathcal{M}^g_{\mathrm{emb}}$. This implies that the connected components of $\mathcal{M}^g_{\mathrm{emb}}$ coincide with the connected components of $\mathcal{M}^g_{\mathrm{Abr}}$ and proves $\mathcal{M}^g_{\mathrm{emb}} = \mathcal{M}^g_{\mathrm{Abr}}$ for all $g \in \mathbb{N}_0$.

A. Whitham deformation of spectral genus 0, 1 and 2

We apply the Whitham flow described in Section 10 to the Abresch family. The whole Abresch family of Riemann's type examples turns out to be a deformation of the totally geodesic annulus.

In the section on the Whitham flow, we defined the vector field (8.3) on the space of pairs of polynomials $(a, b) \in \mathcal{R}^g \subset \mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda]$ in terms of a third polynomial c of degree $g + 1$. All three polynomials a, b and c obey reality conditions (i)-(ii) in the definition of $\mathcal{R}^g$ and (8.2). In Lemma 8.1 we characterize those polynomials c, whose vector fields preserve $\mathcal{M}^g_{\mathrm{ann}}$.

We start with the spectral data of the totally geodesic annulus as described in Proposition 6.4. They are given by $a(\lambda) = -1/16$ and $b(\lambda) = \pm\frac{\pi}{16}(\lambda - 1)$. Due to Proposition 5.5 (1), the spectral data $(\tilde{a}, \tilde{b}) = (p^2 a, pb)$ described in Lemma 6.6 with polynomials p having roots contained in $\mathbb{S}^1$ correspond to the same annulus. It suffices to consider the case $p = (\lambda - 1)(\lambda + 1)$ since the other cases can be obtained from this case by removing higher order roots. So, we look for all deformations of $a(\lambda) = \frac{1}{16}(\lambda-1)^2(\lambda+1)^2$ and $b(\lambda) = \pm\frac{\pi\mathbb{i}}{16}(\lambda-1)^2(\lambda+1)$ in $\mathcal{M}^2_{\text{ann}}$ preserving embeddedness. The corresponding c's have to obey $c(1) = 0$ and $\operatorname{Re} c(0) = 0$. The solution space is the two-dimensional space spanned by $\mathbb{i}(\lambda^3-1)$ and $\mathbb{i}(\lambda^2-\lambda)$. Therefore they obey

$$\lambda^3 c(1/\lambda) = -c(\lambda).$$

This implies that all of them preserve the symmetry

$$\lambda^4 a(\lambda^{-1}) = a(\lambda) \qquad \lambda^3 b(1/\lambda) = b(\lambda) \qquad \lambda^3 c(1/\lambda) = -c(\lambda).$$

The flow induced by the corresponding vector fields (8.3) is integrated in [10, Section 7]. It gives a two-dimensional family parameterized by $(\alpha, \beta) \in (0, 1] \times (0, 1]$

$$a(\lambda) = \frac{1}{\beta\alpha}(\lambda-\alpha)(\alpha\lambda-1)(\lambda+\beta)(\beta\lambda+1) \qquad b(\lambda) = \frac{b(0)}{\gamma}(1+\lambda)(\lambda-\gamma)(\gamma\lambda-1)$$

with $b(0) \in \mathbb{i}\mathbb{R}$ and $\gamma \in [\alpha, 1]$ determined by α and β. For $\alpha = 1 = \beta$, the polynomial a has two double roots. By removing them we obtain the element of $\mathcal{M}^0_{\text{Abr}}$. For $\alpha = 1$ and $\beta \in (0, 1]$ the polynomial a has a double root at $\lambda = 1$ and for $\alpha \in (0, 1]$ and $\beta = 1$ at $\lambda = -1$. By removing them, we obtain the two families in $\mathcal{M}^1_{\text{Abr}}$. In this way, we sweep out all of $\mathcal{M}_{\text{Abr}}$.

References

1. Pedrosa, R. H. L. & Ritoré, M. (1999) Isoperimetric domains in the Riemannian product of a circle with a simply connected space form and applications to free boundary problems. *Indiana Univ. Math. J.*, **48**, 1357–1394.
2. Rosenberg, H. (2002) Minimal surfaces in $M^2 \times \mathbb{R}$. *Illinois J. Math.*, **46**, 1177–1195.
3. Hauswirth, L. (2006) Minimal surfaces of Riemann type in three-dimensional product manifolds. *Pac. J. Math.*, **224**, 91–117.
4. Meeks III, W. H. & Rosenberg, H. (2004) Stable minimal surfaces in $M \times \mathbb{R}$, *J. Differential Geom.* **68**, 515–534.
5. Abresch, U. (1987) Constant mean curvature tori in terms of elliptic functions, *J. Reine U. Angew Math.*, **374**, 169–192.
6. Hoffman, D. & White, B. (2011) Axial minimal surfaces in $S^2 \times \mathbb{R}$ are helicoidal, *J. Differential Geom.*, **87**, 515–523.
7. Hauswirth, L., Kilian, M. & Schmidt, M. U. (2015) On mean-convex Alexandrov embedded surfaces in the 3-sphere, *Math. Z.* **281**, 483–499.
8. Hauswirth, L., Kilian, M. & Schmidt, M. U. (2016) Mean-convex Alexandrov embedded constant mean curvature tori in the 3-sphere, *Proc. Lond. Math. Soc.*, **112**, 588–622.
9. Meeks III, W. H., Pérez, J. & Ros, A. (2015) Properly embedded minimal planar domains, *Ann. Math.*, **181**, 473–546.
10. Hauswirth, L., Kilian, M. & Schmidt, M. U. (2013) Finite type minimal annuli in $\mathbb{S}^2 \times \mathbb{R}$, *Illinois J. Math.*, **57**, 697–741.
11. Hauswirth, L., Sa Earp, R. & Toubiana, E. (2008) Associate and conjugate minimal immersions in $M \times \mathbb{R}$, *Tohoku Math. J.*, **60**, 267–286.
12. Shiffman, M. (1956) On surfaces of stationary area bounded by two circles, or convex curves, in parallel planes, *Ann. of Math.*, **63**, 77–90.
13. Meeks III, W. H. & Rosenberg, H. (2005) The theory of minimal surfaces in $M \times \mathbb{R}$, *Comment. Math. Helv.*, **80**, 811–858.

14. Mazet, L. (2013) A general half-space theorem for constant mean curvature surfaces, *Am. J. Math.*, **135**, 801–834.
15. Heintze, E. & Karcher, H. (1978) A general comparison theorem with applications to volume estimates for submanifolds, *Ann. Sci. École Norm. Sup.*, **11**, 451–470.
16. Karcher, H. (1989) Riemannian comparison constructions, Global differential geometry, MAA Stud. Math., vol. 27, Math. Assoc. America, Washington, DC, pp. 170–222.
17. Lockhart, R. B. & McOwen, R. C. (1985) Elliptic differential operators on noncompact manifolds, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **12**, 409–447.
18. Pinkall, U. & Sterling, I. (1989) On the classification of constant mean curvature tori, *Ann. Math.*, **130**, 407–451.
19. Burstall, F. E. & Pedit, F. (1994) Harmonic maps via Adler-Kostant-Symes theory. *Harmonic Maps and Integrable Systems*. Aspects of Mathematics, vol. E23. Wiesbaden: Vieweg.
20. Burstall, F. E. & Pedit, F. (1995) Dressing orbits of harmonic maps, *Duke Math. J.*, **80**, 353–382.
21. Symes, W. W. (1980) Systems of Toda type, inverse spectral problems, and representation theory, *Invent. Math.*, **59**, 13–51.
22. Bobenko, A. I. (1991) All constant mean curvature tori in $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$ in terms of theta-functions, *Math. Ann.*, **290**, 209–245.
23. Marčenko, V. A. & Ostrovs′kiĭ, I. V. A. (1975) Characterization of the spectrum of the Hill operator, Mat. Sb. (N.S.) **97(139)**, 540–606, 633–634.
24. Krichever, I. M. (1995) Algebraic-geometrical methods in the theory of integrable equations and their perturbations, *Acta Appl. Math.*, **39**, 93–125.
25. Forster, O. (1991) *Lectures on Riemann Surfaces, GTM*, vol. 81. New York: Springer.
26. Whittaker, E. T. & Watson, G. N. (1996) *A Course of Modern Analysis*. Cambridge: Cambridge Mathematical Library, Cambridge University Press.
27. Eastham, M. S. P. (1975) *Results and Problems in the Spectral Theory of Periodic Differential Equations*. Berlin: Springer.
28. Jackson, S. B. (1945) The four-vertex theorem for surfaces of constant curvature, *Amer. J. Math.*, **67**, 563–582.
29. Cheng, S.-Y. (1976) Eigenfunctions and nodal sets, *Comment. Math. Helv.*, **51**, 43–55.