Path Integrals of Standard Markov Processes

Inauguraldissertation
zur Erlangung des akademischen Grades
eines Doktors der Naturwissenschaften
der Universität Mannheim

vorgelegt von

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aus Cambridge, Großbritannien

Mannheim, 2020

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Tag der mündlichen Prüfung: 16. Oktober 2020

Abstract

We consider the law of the perpetual integral of a standard Markov process, deriving sufficient and necessary conditions for finiteness both with positive probability and probability one, and prove a zero-one law. The proofs involve defining the class of super-finite sets and proving several facts about them. Using these results we prove a collection of similar theorems for finite-time path integrals over transient standard Markov processes. We also prove two related results for Lévy processes with local times.

Our theorems on path integrals have significance for weak stable SDE solutions, and we construct this connection explicitly, highlighting the way that the solution process behaviour is influenced by the law of the path integrals. We lastly develop a characterisation of avoidable sets for stable processes in the form of a summation test, expanding upon an existing potential theoretic result.

Zusammenfassung

Wir betrachten die Verteilung des Perpetual Integrals eines Standard-Markov-Prozesses, leiten ausreichende und notwendige Bedingungen für die Endlichkeit sowohl mit positiver Wahrscheinlichkeit als auch mit der Wahrscheinlichkeit eins ab und beweisen ein Zero-One Law. Bei den Beweisen geht es darum, die Klasse der super-finite Mengen zu definieren und mehrere Fakten über sie zu beweisen. Anhand dieser Ergebnisse beweisen wir eine Sammlung ähnlicher Sätze für endliche Path Integrals über transiente Standard-Markov-Prozesse. Wir beweisen auch zwei verwandte Ergebnisse für Lévy-Prozesse mit Local Times.

Unsere Sätze über Path Integrals haben Bedeutung für schwach stabile SDE-Lösungen, und wir konstruieren diese Verbindung explizit, indem wir die Art und Weise hervorheben, wie das Verhalten des Lösungsprozesses durch der Path Integrals beeinflusst wird. Zuletzt entwickeln wir eine Charakterisierung vermeidbarer Mengen für stabile Prozesse in Form eines Summationstests, wobei wir auf ein vorhandenes potenzielles theoretisches Ergebnis aufbauen.

Acknowledgements

First and greatest thanks go to Prof. Dr. Leif Döring, whose expertise, enthusiasm, and patience made this thesis possible. I count myself very fortunate to have worked with him these past four years, and his confidence in my work has been exceptionally valuable to me. Thanks also go to Prof. Dr. Andreas Kyprianou for fruitful and enjoyable discussions during my stay in Bath.

Special thanks go to Dr. Philip Weißmann and Dr. Stefan Lindt, whom I count not just as talented colleagues but as friends, and who inspired and encouraged me while we shared an office. My thanks also to all colleagues at Universität Mannheim, in particular Dr. Quan Shi and Dr. Helmut Pitters, for frequent mathematical discussions and for cultivating a positive and relaxed atmosphere.

My deepest heartfelt thanks to my family and to friends near and far for their support, confidence, and for walking every step of this journey with me. Their importance to me cannot be overstated.

Finally I would like to thank the Heidelberg-Mannheim RTG 1953 funded by the Deutsche Forschungsgesellschaft for supporting my research financially and for providing an outstanding professional environment in which to work.

Enough with exposition!
Get to the heart of it, no matter
what you started with.

Contents

1 Introduction

The greatest achievement in probability theory of the last century is the development of a rigorous mathematical framework for studying what remain essentially intuitive concepts. In the world of stochastic processes this joining of intuition and rigour is typified in the Brownian motion, an astounding object whose properties and behaviour have had a hand in the development of almost every part of modern stochastic process research.

In this thesis two classes of stochastic process will be studied, both of which count the Brownian motion as a member. The first are Markov processes, some examples of which were already studied in the 19th century, and which formalise an intuitive property of 'forgetfulness' or homogeneity in time. The second are stable processes, a family of Markov processes which share many properties with the Brownian motion, including stationary and independent increments and a form of self-similarity, but which can have discontinuous paths.

For X a stochastic process, and f a nonnegative measurable function, the path integrals over $f(X)$ are defined as

$$\int_0^t f(X_s)\,\mathrm{d}s, \qquad t \in [0,\infty].$$

The collection of path integrals is a real-valued stochastic process in its own right, and its behaviour can be studied in relation to that of X and f. The terminal value of the path integral process is given by

$$\int_0^\infty f(X_s)\,\mathrm{d}s,$$

and is called the perpetual integral over $f(X)$. While path integrals have been extensively studied for the Brownian motion they are less well understood for stable processes, and for general Markov processes there is still much to learn. Most research on path and perpetual integrals relies on assumptions ensuring their behaviour adheres to a zero-one law, and one aim of this thesis is to present general theorems on those objects which don't fall into such a strict regime. Perpetual integrals are major objects in probabilistic potential theory, appearing in the definition of the potential of measurable f,

$$Uf(x) = \mathbb{E}_x\Big[\int_0^\infty f(X_s)\,\mathrm{d}s\Big],$$

and in all parts of this thesis we shall make heavy use of the varied and powerful theory of potentials for Markov process.

An object related to path integrals is the stochastic integral

$$\int_0^t f(Z_{s-})\,\mathrm{d}X_s, \qquad t \in [0,\infty],$$

where Z is another stochastic process. When X is a stable process, stochastic integrals and path integrals are very closely linked, and in this thesis we shall exploit that connection to develop some theorems that bridge both worlds.

1.1 Outline

Chapter 2: Preliminaries

We introduce the families of Markov, Lévy, and stable processes, make clear and precise the tools we shall use, and establish some basic properties. We then lay some foundational potential theory for Markov processes, following the lead of Blumenthal and Getoor [5], and present some more specialised results for Lévy processes as given in Bertoin [2] and Sato [39], paying particular attention to the strange world of capacities.

Chapter 3: Perpetual Integral Tests

This chapter is mostly dedicated to proving a characterisation of the perpetual integral of a standard Markov process under the action of a general measurable function, which unlike previous results is not a zero-one law. To do so we develop the theory of superfinite sets, prove a series of technical lemmas that describe the properties of these sets, and thereby construct the main theorem. From that theorem we deduce two similar results, one for the almost sure case and the other a zero-one law.

Chapter 4: Path Integral Tests

The results of this chapter are chipped from the block of Chapter 3 and provide a counterpoint to - and extension of - the well-known zero-one law for path integrals of the Brownian motion, which was proven by Engelbert and Schmidt [15] in 1981. We present sufficient and necessary conditions for the probability

$$\mathbb{P}_x\Big(\int_0^t f(X_s)\,\mathrm{d}s < \infty \text{ for every } t < \zeta\Big)$$

to be either positive or equal one in the cases that X is either a transient standard Markov process or a Lévy process on $\mathbb{R}$ with local times, and give more specific result in the case that X is a transient stable process.

Chapter 5: Stable SDEs

We prove general existence criteria for weak solutions to the driftless stable SDE

$$\mathrm{d}Z_t = \sigma(Z_{t-})\,\mathrm{d}X_t, \qquad Z_0 = z,$$

building upon the work of Zanzotto [44, 45, 46] and Kallenberg [27], and draw a connection to the stable process results of Chapter 4.

Chapter 6: Avoidable Sets

We prove a summation test which provides a necessary and sufficient condition for a set to be avoidable for the d-dimensional isotropic transient stable process. This work is based on the Wiener Criterion for thin sets, and is proven here using only probabilistic methods.

1.2 Collaborative Work

The results of §3.2 and §3.5 were developed and refined in collaboration with Leif Döring of Universität Mannheim and Andreas Kyprianou of Bath University. The results for standard Markov processes in Chapter 4 are collaborative work with Leif Döring and Quan Shi of Universität Mannheim. The results of §4.3 and Chapter 6, and much of the discussion in §2.6, would not have been possible without the invaluable input of Mateusz Kwaśnicki of Wrocław University of Science and Technology.

2 Preliminaries

Stochastic processes are mathematical models of random phenomena which move through time, and their uses are rich and extensive. The central rôle of this thesis is played by a particular class of continuous-time stochastic processes called strong Markov processes, which have their origins in the works of A. Markov in the first decade of the twentieth century, and for whom they are also named.

The first section of this background will define a Markov process in a general setting, closely following the definition of Blumenthal and Getoor [5], and discuss the strong Markov property. The second section will introduce a broad family of Markov processes called Lévy processes, and the third will narrow the focus to those satisfying a particular scaling property, which are called stable processes. The fourth section will establish some objects from potential theory, an area of analysis that has long had a fruitful relationship with Markov processes, beginning in its modern form with the works of Doob and Hunt from the 1950s, and expertly presented in Blumenthal and Getoor [5]. The fifth and sixth sections will look more closely at the potential theory of Lévy processes, following the presentation in Bertoin [2] and Sato [39].

2.1 Markov Processes

A Markov process is a stochastic process that formally describes the intuitive Markov property:

(Markov property) *If the present state of a system is known, then the conditional future behaviour of the system is independent of any additional information about the past.*

Markov processes are well suited to modelling systems which are largely 'memoryless'. Take for example the flight of a dandelion seed through the air. If the current position of the seed were known, and its movement could be accurately and precisely modelled, then any additional information about where the seed had previously been - mathematically, the past of the process - would be irrelevent to predicting its future, since the seed has no way of changing its behaviour based on past experience. Many physical phenomena also have this memoryless property, but any system with longer-term memory - for example, a model of human behaviour - is not Markovian.

There is a delicate point hidden in the intutive Markov property: whether or not the 'present state' of a system includes the information of how long it has been running. Markov processes in which the observer also knows the run-time are called *time inhomogeneous*, because the behaviour of the system can change with time while preserving the Markov property, whereas processes in which the run-time is hidden from the observer are called *time homogenous*.

First Definitions

We shall need the following ingredients to build the definition of a Markov process:

(i) A measurable space $(E, \mathcal{E})$, and Δ an additional point not in E. Let $E_\Delta = E \cup \{\Delta\}$ denote the augmentation of E with the additional point, and let $\mathcal{E}_\Delta$ be the σ-algebra in E_Δ generated by $\mathcal{E}$, that is, the smallest σ-algebra in E_Δ containing $\mathcal{E}$.

(ii) The space Ω of paths $\omega : [0, \infty] \to E_\Delta$ such that $\omega(\infty) = \Delta$, and if $\omega(t) = \Delta$ then $\omega(s) = \Delta$ for all $s \geq t$. In addition an element ω_Δ of Ω satisfying $\omega_\Delta(t) = \Delta$ for all t.

(iii) A family $(X_t,\, t \in [0, \infty])$ of coordinate maps $X_t : \Omega \to E_\Delta$, that satisfy

$$X_t(\omega) = \omega_t \text{ for all } t \in [0, \infty].$$

(iv) The canonical filtration $\mathcal{F}_t = \sigma(X_s,\, 0 \leq s \leq t)$ and $\mathcal{F} = \sigma(X_s,\, s \in [0, \infty])$ of X satisfying the natural conditions.

(v) A family $(\theta_t,\, t \in [0, \infty])$ of translation maps $\theta_t : \Omega \to \Omega : \omega \mapsto (\omega_{t+s},\, s \geq 0)$.

(vi) A family of probability measures $(\mathbb{P}_x,\, x \in E_\Delta)$ on $(\Omega, \mathcal{F})$.

Definition 2.1.1. *The collection* $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbb{P}_x)$ *is called a (time homogeneous)* canonical Markov process *with state space* $(E, \mathcal{E})$ *and cemetary state* Δ *if the following two conditions simultaneously hold.*

Condition 1 (Regularity)

(a) *For each* $t \in [0, \infty)$ *and each* $B \in \mathcal{E}$*, the map*

$$: E \to [0, 1], \qquad x \mapsto \mathbb{P}_x(X_t \in B)$$

is $\mathcal{E}$*-measurable.*[1]

(b) $\mathbb{P}_\Delta(X_0 = \Delta) = 1$.

Condition 2 (Markov property)
For all $x \in E_\Delta$*,* $t, s \in [0, \infty]$*, and* $f \in b\mathcal{E}_\Delta$*,*

$$\mathbb{E}_x[f(X_{t+s}) | \mathcal{F}_t] = \mathbb{E}_{X_t}[f(X_s)] \qquad \mathbb{P}_x\text{-almost surely.}$$

[1]When the σ-algebra on the second space is not explicitly named, it is the Borel σ-algebra on $[-\infty, \infty]$.

In particular, for all $B \in \mathcal{E}_\Delta$,

$$\mathbb{P}_x(X_{t+s} \in B | \mathcal{F}_t) = \mathbb{P}_{X_t}(X_s \in B) \qquad \mathbb{P}_x\text{-almost surely.}$$

Here is an intuitive way of thinking about Definition 2.1.1 and the ingredients **(i)** - **(vi)**. The function

$$t \mapsto X_t$$

is the random trajectory of a particle moving through the space E. Under the law $\mathbb{P}_x$ the particle is issued from the point x, and at some random time in $[0, \infty]$ it leaves E and attains a cemetary state Δ, where it remains for all time. The random time

$$\begin{aligned} \zeta &: \Omega \to [0, \infty] \\ &: \omega \mapsto \inf\{t > 0 : X_t(\omega) = \Delta\} \end{aligned}$$

is called the *lifetime* of X. The translation operators θ_t have the effect of shifting the ω in time, and for all $s, t \in [0, \infty]$, $B \in \mathcal{E}_\Delta$,

$$\mathbb{P}_x(X_{t+s} \in B) = \mathbb{P}_x(X_t \circ \theta_s \in B).$$

The regularity condition is a technical necessity which also has the following far more useful form, from Blumenthal and Getoor [5] Theorem I(3.6)(a): the map $x \mapsto \mathbb{E}_x[Y]$ is $\mathcal{E}_\Delta$-measurable for all $Y \in b\mathcal{F}$. This in particular means that for any $Y \in \mathcal{F}$ and $B \in \mathcal{B}(\mathbb{R})$,

$$x \mapsto \mathbb{P}_x(Y \in B) \tag{2.1}$$

is $\mathcal{E}_\Delta$-measurable.

It is worth now taking a moment to broaden the definition of a Markov process, because although the canonical process allows for a nice intuitive interpretation, it can bloat proofs and obscure other properties of the process. Blumenthal and Getoor [5] Theorem I(4.3) shows that the following coincides with the usual definition of a Markov process, given in Definition I(3.1) of the same book.

Take **(i)** - **(v)** as above, and let $(A, \mathcal{A})$ be a measurable space with a distinguished point a_Δ. Let $(Y_t,\, t \in [0, \infty])$ be a family of maps $Y_t : A \to E_\Delta$ such that

(i) $Y_t(a_\Delta) = \Delta$ for all t;

(ii) for all $a \in A$, if $Y_t(a) = \Delta$ then $Y_s(a) = \Delta$ for all $s \geq t$;

(iii) for all $a \in A$, $Y_\infty(a) = \Delta$.

Let $(\mathbb{P}_y,\, y \in E_\Delta)$ be a family of probability measures on $(A, \mathcal{A})$, and let

$$\begin{aligned} \pi &: A \to \Omega \\ &: a \mapsto \big(Y_t(a), t \in [0, \infty]\big). \end{aligned}$$

Define a new family of measures $(\tilde{\mathbb{P}}_y,\, y \in E_\Delta)$ on $(\Omega, \mathcal{F})$ by $\tilde{\mathbb{P}}_y = \mathbb{P}_y \circ \pi^{-1}$.

Definition 2.1.2. *We say that Y is a* Markov process *with law $(\mathbb{P}_y,\ y \in E_\Delta)$, state space $(E, \mathcal{E})$ and cemetary state Δ if $(\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, X_t, \tilde{\mathbb{P}}_y)$ is a canonical Markov process with state space $(E, \mathcal{E})$ and cemetary state Δ.*

If Y is a Markov process with law $(\mathbb{P}_y,\ y \in E_\Delta)$ then for $x \in E_\Delta$, $t, s \in [0, \infty]$, and $f \in b\mathcal{E}_\Delta$,

$$\mathbb{E}_x[f(Y_{t+s})|\mathcal{G}_t] = \mathbb{E}_{Y_t}[f(Y_s)] \qquad \mathbb{P}_x\text{-almost surely,}$$

where $\mathcal{G}_t = \sigma(Y_s,\ 0 \le s \le t)$.

From this moment on the term 'canonical' will no longer be used, but it should be understood that every Markov process Y defined on a measurable space $(A, \mathcal{A})$ and a family of probability measures $(\mathbb{P}_x,\ x \in E_\Delta)$ can be thought of as a canonical Markov process via the correspondence above.

In the next section we shall introduce Markov processes for which the entire family $(\mathbb{P}_x,\ x \in E_\Delta)$ can be defined by a single measure $\mathbb{P}_y$ for a fixed $y \in E$. One reason for Definition 2.1.2 will be to allow several of these Markov processes to be defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P}_y)$. This is not possible for canonical processes since in that case the behaviour of the process is entirely defined by the laws $\mathbb{P}_y$.

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbb{P}_x)$ be a Markov process. The family of probability laws $(\mathbb{P}_x,\ x \in E_\Delta)$ on $(\Omega, \mathcal{F})$ has some interesting properties. Let us introduce a slight change in notation. Define

$$P_t(x, A) = \mathbb{P}_x(X_t \in A), \qquad t \in [0, \infty],\ x \in E_\Delta,\ A \in \mathcal{E}_\Delta. \tag{2.2}$$

Then for $t, s \in [0, \infty]$,

$$P_{t+s}(x, A) = \mathbb{E}_x[\mathbb{P}_x(X_{t+s} \in A|\mathcal{F}_t)] = \mathbb{E}_x[\mathbb{P}_{X_t}(X_s \in A)] = \int P_s(y, A) P_t(x, \mathrm{d}y).$$

This relationship is called the *Chapman-Kolmogorov equation*, and is the centrepiece of the following definition.

Definition 2.1.3. *Let $(M, \mathcal{M})$ be a measurable space. Then a function $P_t(x, A)$ defined for $t \in [0, \infty)$, $x \in M$, and $A \in \mathcal{M}$ is called a* Markovian semigroup on $(M, \mathcal{M})$ *if*

(i) *$A \mapsto P_t(x, A)$ is a probability measure on $\mathcal{M}$ for all t, x;*

(ii) *$x \mapsto P_t(x, A)$ is $\mathcal{M}$-measurable for all t, A;*

(iii) *$P_t(x, A)$ satisfies the Chapman-Kolmogorov equation*

$$P_{t+s}(x, A) = \int P_s(y, A) P_t(x, \mathrm{d}y)$$

for all $t, s \in [0, \infty)$, $x \in M$, and $A \in \mathcal{M}$.

If **(i)** is replaced by $A \mapsto P_t(x, A)$ being a *sub*-probability measure, then $P_t(x, A)$ is instead called a sub-Markovian semigroup.

Lemma 2.1.4. *Let X be a Markov process on $(E, \mathcal{E})$. Then* (2.2) *defines a Markovian semigroup on $(E_\Delta, \mathcal{E}_\Delta)$. Restricting this function to $(E, \mathcal{E})$ we obtain a sub-Markovian semigroup*

$$P_t(x, A) = \mathbb{P}_x(X_t \in A), \qquad t \in [0, \infty),\ x \in E,\ A \in \mathcal{E}.$$

This sub-Markovian semigroup $P_t(x, A)$ is called the transition function *of X.*

Two Markov processes on the same state space $(E, \mathcal{E})$ are said to be *equivalent* if they have the same transition function.

Strong Markov Processes

We are now ready to extend our foundations a little further, by introducing a new class of random variables which will be a key component of many of the proofs that follow later.

Definition 2.1.5. *A random variable $T : \Omega \to [0, \infty]$ is called a* stopping time (for X) *if $\{T \leq t\} \in \mathcal{F}_t$ for all $t \in [0, \infty)$.*

Intuitively a stopping time is a random time which announces its arrival: the observer of a process can see that a stopping time has been reached *at that stopping time.* The classic example of a random time which is not a stopping time is the final time a process attains a certain value x, because in most cases one cannot know whether a time t at which $X_t = x$ was the final time until reaching time ζ.

Stopping times are useful for us because they motivate a family of Markov processes which retain the Markov property when restarted at stopping times, called strong Markov processes. These processes are fundamental to the work of this thesis.

Definition 2.1.6. *A Markov process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbb{P}_x)$ on $(E, \mathcal{E})$ is called a* strong Markov process *if for each stopping time T and $f \in b\mathcal{E}_\Delta$,*

(i) *X_T is $\mathcal{F}_T$-measurable relative to $\mathcal{E}^*_\Delta$;*

(ii) *for all $x \in E_\Delta$, and $s \in [0, \infty]$*

$$\mathbb{E}_x[f(X_{T+s})|\mathcal{F}_T] = \mathbb{E}_{X_T}[f(X_s)] \qquad \mathbb{P}_x\text{-almost surely.}$$

In particular, for all $B \in \mathcal{E}_\Delta$,

$$\mathbb{P}_x(X_{T+s} \in B|\mathcal{F}_T) = \mathbb{P}_{X_T}(X_s \in B) \qquad \mathbb{P}_x\text{-almost surely.}$$

The form of the strong Markov property below is far more versatile.

Lemma 2.1.7 ([5] Corollary I(8.6))**.** *Let X be strong Markov. Then for every $Y \in b\mathcal{F}$ one has*

$$\mathbb{E}_x[Y \circ \theta_T|\mathcal{F}_T] = \mathbb{E}_{X_T}[Y] \qquad \mathbb{P}_x\text{-almost surely.}$$

for all stopping times T and all $x \in E$.

The strong Markov property turns out to be not only remarkably powerful but also automatic for a great many Markov processes, and for that reason is almost ubiquitous. According to Blumenthal and Getoor [5] in early work it was not even realised that a distinction was necessary, and the strong Markov property was more or less tacitly assumed. But this ubiquity can be dangerous if it leads to sloppiness when using the property, and for that reason the following corollary explicitly proves one application which will appear often later in this thesis.

Corollary 2.1.8. *Let $f : E_\Delta \to [0,\infty]$ be a $\mathcal{E}_\Delta$-measurable function, and $(f_n,\ n \in \mathbb{N})$ be a pointwise increasing sequence of functions in $b\mathcal{E}_\Delta$ with pointwise limit f. Then for fixed $t \in [0,\infty)$, $n \in \mathbb{N}$, the map*

$$: \Omega \to [0,\infty], \qquad \omega \mapsto \int_0^t f_n(\omega_s)\,\mathrm{d}s$$

is in $b\mathcal{F}$. So Lemma 2.1.7 and the (conditional) monotone convergence theorem together yield

$$\begin{aligned}\mathbb{E}_x\Big[\int_0^t f(X_{T+s})\,\mathrm{d}s\Big|\mathcal{F}_T\Big] &= \lim_{n\to\infty}\mathbb{E}_x\Big[\int_0^t f_n(X_{T+s})\,\mathrm{d}s\Big|\mathcal{F}_T\Big]\\ &= \lim_{n\to\infty}\mathbb{E}_{X_T}\Big[\int_0^t f_n(X_s)\,\mathrm{d}s\Big] = \mathbb{E}_{X_T}\Big[\int_0^t f(X_s)\,\mathrm{d}s\Big].\end{aligned}$$

From an additional application of monotone convergence we see

$$\begin{aligned}\mathbb{E}_x\Big[\int_0^\infty f(X_{T+s})\,\mathrm{d}s\Big|\mathcal{F}_T\Big] &= \lim_{t\to\infty}\mathbb{E}_x\Big[\int_0^t f(X_{T+s})\,\mathrm{d}s\Big|\mathcal{F}_T\Big]\\ &= \lim_{t\to\infty}\mathbb{E}_{X_T}\Big[\int_0^t f(X_s)\,\mathrm{d}s\Big] = \mathbb{E}_{X_T}\Big[\int_0^\infty f(X_s)\,\mathrm{d}s\Big].\end{aligned}$$

Standard Markov Processes

All this ground work leads us to define a family of Markov processes in the form they shall be used in this thesis. This definition establishes a collection of sensible or intuitive behaviours and properties which are not a part of the definition of a Markov process but which, like the strong Markov property, are almost always assumed.

Definition 2.1.9. *A Markov process $X = (\Omega,\mathcal{F},\mathcal{F}_t,X_t,\theta_t,\mathbb{P}_x)$ with state space $(E,\mathcal{E})$ and cemetary state Δ is called a* standard Markov process *if the following conditions hold.*

- ***(i)*** *E is a locally compact Hausdorff space with a countable base, and Δ is adjoined to E as the point at infinity if E is non-compact, and as an isolated point if E is compact. Furthermore $\mathcal{E}$ is the Borel σ-algebra on E.*
- ***(ii)*** ***(càdlàg paths)*** *The path functions $t \mapsto X_t(\omega)$ are right continuous on $[0,\infty)$ and have left limits on $[0,\zeta)$ almost surely.*

(iii) *X is a strong Markov process.*

(iv) ***(quasi-left-continuity)*** *For any sequence T_n of $\mathcal{F}_t$-stopping times with limit T, it holds that $X_{T_n} \to X_T$ almost surely on $\{T < \zeta\}$.*

(v) ***(normality)*** *$\{x\} \in \mathcal{E}$ and $\mathbb{P}_x(X_0 = x) = 1$ for all $x \in E$.*

Where it is clear from context we may not specifically state the state space on which X moves. It is worth noting that $\mathcal{E}_\Delta$ is the Borel σ-algebra on E_Δ.

Together, normality and right-continuity of paths imply that $\mathbb{P}_x(\zeta > 0) = 1$ for all $x \in E$. A standard process which is quasi-left-continuous on $\{T < \infty\}$ is called a *Hunt process*. In particular, standard processes with almost surely infinite lifetime are Hunt processes. Sharpe [40] offers the intuitive explanation that for a Hunt process the cemetary state Δ is essentially just a trapping point of the state space.

The assumption that E is Hausdorff ensures that compact subsets of E are closed, and therefore $\mathcal{E}$-measurable. It might be that for some of the results of this thesis this condition can be relaxed, since many of the compact sets we shall work with are *nearly* Borel - in the sense of Blumenthal and Getoor [5] Definition I(10.21) - and a great deal of Markov potential theory works just as well for nearly Borel sets as for Borel. However, we shall sacrifice this generality at the gain of far greater clarity of presentation.

Many works consider a more general class of Markov processes called (Borel) right processes, which need not be quasi-left-continuous. That generalisation is also a step too far for this thesis, though many results for standard Markov processes have direct analogues for right processes. Sharpe's book [40] is a comprehensive exposition of that theory, see in particular §8 for the definition and §47 for the connection to standard processes.

The following fact is extremely useful.

Lemma 2.1.10 (Blumenthal's zero-one law)**.** *Let X be a standard Markov process. If $A \in \mathcal{F}_0$ then for every $x \in E$, $\mathbb{P}_x(A) = 0$ or 1.*

Here are three additional lemmas which will be used often later.

Lemma 2.1.11 ([5] Theorem I(10.7))**.** *Let X be a standard Markov process, and for a set $B \in \mathcal{E}_\Delta$ define random times*

$$\begin{aligned} D_B &\coloneqq \inf\{t \geq 0 : X_t \in B\} \\ T_B &\coloneqq \inf\{t > 0 : X_t \in B\} \\ L_B &\coloneqq \sup\{t \geq 0 : X_t \in B\}. \end{aligned}$$

Then D_B and T_B are both stopping times, and are called the first entry time *and* first hitting time *of B respectively. L_B is not in general a stopping time, and is called the* last exit time *of B.*

It is conventional to set $\inf \emptyset = \infty$ *and* $\sup \emptyset = 0$.

Lemma 2.1.12 ([5] Theorem I(10.19)). *Let X be a standard Markov process. Take $B \in \mathcal{E}_\Delta$ and μ a probability measure on $(E, \mathcal{E})$. Then there exists an increasing nested sequence (K_n) of compact subsets of B such that $T_{K_n} \downarrow T_B$ $\mathbb{P}_\mu$-almost surely.*

For a set $B \in E_\Delta$, we shall call a point $x \in E_\Delta$ *regular for B* provided

$$\mathbb{P}_x(T_B = 0) = 1,$$

and *irregular* otherwise. From Lemma 2.1.10 we see that this probability is in $\{0, 1\}$. The set of points $x \in E_\Delta$ which are regular for B is denoted by B^r. Right-continuity of paths ensures that $B^\circ \subseteq B^r \subseteq \bar{B}$, and in particular that $X_{T_K} \in K$ almost surely for compact K.

Lemma 2.1.13 ([5] Theorem I(11.4)). *Let X be a standard Markov process and $B \in \mathcal{E}_\Delta$. Then*

- ***(i)*** *$X_{T_B} \in B \cup B^r$ almost surely on $\{T_B < \infty\}$;*
- ***(ii)*** *for each $x \in E_\Delta$, the measure $\mathbb{P}_x(X_{T_B} \in \cdot\,)$ is concentrated on $B \cup B^r$.*

The above lemma also holds for D_B in place of T_B, since a necessary requirement for $D_B \neq T_B$ to hold is that $X_{D_B} = X_0 \in B$.

2.2 Lévy Processes

A general Markov process may have vastly different behaviour as it moves through different parts of its state space E. Lévy processes are the class of Markov processes which have homogeneous behaviour in space as well as time. They are named for Paul Lévy, who did significant work developing their foundations in the first half of the twentieth century.

Definition 2.2.1. *A standard Markov process X on $\mathbb{R}^d$ is called a* Lévy process *if for all $x \in \mathbb{R}^d$, $t \in [0, \infty)$, $B \in \mathcal{B}(\mathbb{R}^d)$,*

$$\mathbb{P}_x(X_t \in B) = \mathbb{P}_0(x + X_t \in B) \tag{2.3}$$

and $\mathbb{P}_x(\zeta < \infty) = 0$.

It follows from (2.3) that knowing $\mathbb{P}_x$ for some fixed $x \in E$ is enough to determine the entire family $\mathbb{P}_x$, $x \in E_\Delta$. This in combination with Definition 2.1.2 allows us to speak without ambiguity of a Lévy process X as a family of random variables $X_t : \Omega \to \mathbb{R}^d$, $t \in [0, \infty]$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where conventionally $\mathbb{P} = \mathbb{P}_0$.

Basic Properties

A Lévy process X has the following properties:

(a) (Stationary increments) For any $x, y \in \mathbb{R}^d$, $0 \le s \le t$, $B \in \mathcal{B}(\mathbb{R}^d)$,

$$\begin{aligned}\mathbb{P}_x(X_t - X_s \in B) &= \mathbb{E}_x[\mathbb{P}_x(X_t - X_s \in B | \mathcal{F}_s)] \\ &= \mathbb{E}_x[\mathbb{P}_{X_s}(X_{t-s} - X_0 \in B)] \\ &= \int \mathbb{P}_z(X_{t-s} - z \in B)\mathbb{P}_x(X_s \in \mathrm{d}z)\end{aligned}$$

and applying (2.3) yields

$$= \int \mathbb{P}_y(X_{t-s} - y \in B)\mathbb{P}_x(X_s \in \mathrm{d}z) = \mathbb{P}_y(X_{t-s} - X_0 \in B).$$

In particular for any $0 \le r \le s \le t$, $x \in \mathbb{R}^d$, $B \in \mathcal{B}(\mathbb{R}^d)$,

$$\mathbb{P}_x(X_{t-r} - X_{s-r} \in B) = \mathbb{P}_x(X_{t-s} - x \in B) = \mathbb{P}_x(X_t - X_s \in B).$$

(b) (Independent increments) For any $x \in \mathbb{R}^d$, Borel sets $A, B \in \mathcal{B}(\mathbb{R}^d)$, and times $0 \le s_0 < s_1 \le t_0 < t_1$,

$$\begin{aligned}&\mathbb{P}_x(X_{t_1} - X_{t_0} \in A,\, X_{s_1} - X_{s_2} \in B) \\ &\quad= \mathbb{E}_x[\mathbb{P}_x(X_{t_1} - X_{t_0} \in A | \mathcal{F}_{s_1}); X_{s_1} - X_{s_0} \in B] \\ &\quad= \mathbb{E}_x[\mathbb{P}_{X_{s_1}}(X_{t_1 - s_1} - X_{t_0 - s_1} \in A); X_{s_1} - X_{s_0} \in B] \\ &\quad= \mathbb{P}_x(X_{t_1} - X_{t_0} \in A)\mathbb{P}_x(X_{s_1} - X_{s_2} \in B). \qquad \text{(by stat. incr.)}\end{aligned}$$

(c) (Infinite divisibility) For any $t > 0$ and $n \in \mathbb{N}$, the random variable $X_t - X_0$ can be written as the sum of n independent, identically distributed random variables,

$$X_t = X_0 + \sum_{i=1}^{n} \big(X_{it/n} - X_{(i-1)t/n}\big).$$

The characteristic exponent of X_t for $t \ge 0$ is defined by

$$\Psi_t(\theta) := -\log \mathbb{E}_0\big[\mathrm{e}^{\mathrm{i}\langle \theta, X_t \rangle}\big], \qquad \theta \in \mathbb{R}^d.$$

Due to infinite divisibility, $t\Psi_1(\theta) = \Psi_t(\theta)$ for any $t > 0$. This motivates the definition of the *characteristic exponent of a Lévy process* X as $\Psi(\theta) := \Psi_1(\theta)$.

Characterisations

Lévy processes are most often characterised by one of the two following classical theorems. The first gives $\Psi(\theta)$ a precise expression built from a group of three objects called a Lévy triplet.

Definition 2.2.2. *A* Lévy triplet *or* characteristic triplet *is a collection* (γ, σ, π) *of* $\gamma \in \mathbb{R}^d$*, a symmetric nonnegative-definite matrix* $\sigma \in \mathbb{R}^{d\times d}$*, and a measure* π *on* $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ *satisfying*

$$\pi(\{0\}) = 0, \qquad \int_{\mathbb{R}^d} (1 \wedge |x|^2)\pi(\mathrm{d}x) < \infty.$$

Theorem 2.2.3 (Lévy-Khintchine formula)**.** *Let* X *be a Lévy process. Then there exists a Lévy triplet* (γ, σ, π) *such that*

$$\Psi(\theta) = \mathrm{i}\langle\theta, \gamma\rangle + \frac{1}{2}\langle\theta, \sigma\theta\rangle + \int_{\mathbb{R}^d} \Big(1 - \mathrm{e}^{\mathrm{i}\langle\theta,x\rangle} + \mathrm{i}\langle\theta, x\rangle \mathbf{1}_{(|x|\le 1)}\Big)\pi(\mathrm{d}x). \tag{2.4}$$

On the other hand, for any Lévy triplet (γ, σ, π) *there exists a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *on which there is a Lévy process* X *with characteristic exponent given by* Ψ *in* (2.4)*.*

The second classical theorem says that any Lévy process X can be decomposed into a sum of three independent Lévy processes, which have a particular form. In this sense these three process classes are the building blocks of all Lévy processes. The theorem can either be stated by constructing the processes themselves or by giving their characteristic exponent, which Theorem 2.2.3 tells us is equivalent.

In order to state that theorem we must first give the definition of a random measure. The presentation here is inspired by that in Çinlar [9]. Jacod and Shiryaev [26] comprehensively lay out the theory of random measures from a semimartingale perspective, which is quite different but essentially equivalent. Let $(E, \mathcal{E})$ be a measurable space and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. A *random measure* on $(E, \mathcal{E})$ is a map $\mu : \Omega \times \mathcal{E} \to [0, \infty]$ such that

(i) $\omega \mapsto \mu(\omega, B)$ is a random variable for every $B \in \mathcal{E}$;

(ii) $B \mapsto \mu(\omega, B)$ is a measure on $(E, \mathcal{E})$ for every $\omega \in \Omega$.

The *mean* of μ is a deterministic measure μ^p on $(E, \mathcal{E})$ defined by $\mu^p(B) := \mathbb{E}[\mu(B)]$. The names *intensity* or *compensator*[2] are also used for μ^p.

When discussing random measures the probability space is not usually mentioned unless it has particular significance. A well-behaved type of random measure is the Poisson random measure. Let $(E, \mathcal{E}, \nu)$ be a measure space. A random measure μ on $(E, \mathcal{E})$ is called a *Poisson random measure with mean* ν if

(i) the random variable $\mu(B)$ is Poisson distributed with mean $\nu(B)$ for every $B \in \mathcal{E}$;

(ii) if $B_1, \dots, B_n$ is a finite collection of disjoint elements of $\mathcal{E}$ then the random variables $\mu(B_1), \dots, \mu(B_n)$ are independent.

[2]The compensator of a random measure is also commonly defined as the unique (in a particular sense) *random* measure such that $\mathbb{E}[\mu^p(B)] = \mathbb{E}[\mu(B)]$ for all $B \in \mathcal{E}$, see for example Theorem II.1.6 of Jacod and Shiryaev [26], and this fits more naturally to the study of semimartingales. The presentation here instead follows that of Çinlar [9] §VI, who is mostly concerned with Poisson random measures, in which case the two definitions coincide.

Theorem 2.2.4 (Lévy-Itô decomposition)**.** *Let (γ, σ, π) be a Lévy triplet, and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. Then a stochastic process X on $(\Omega, \mathcal{F}, \mathbb{P})$ with state space $\mathbb{R}^d$ is a Lévy process with triplet (γ, σ, π) if and only if there exist a matrix $C \in \mathbb{R}^{d \times k}$, a standard k-dimensional Brownian motion B and, independent of it, a Poisson random measure μ on $[0, \infty) \times \mathbb{R}^d$ with mean Leb $\times \pi$, both on $(\Omega, \mathcal{F}, \mathbb{P})$, such that*

$$X_t = \gamma t + CB_t + \int_{[0,t] \times \mathbb{R}^d} x \big[\mu(\mathrm{d}s, \mathrm{d}x) - \mathbf{1}_{(|x|<1)}\, \mathrm{d}s\, \pi(\mathrm{d}x)\big], \qquad t \in [0, \infty),$$

and $\sigma = C^\top C$. We can equivalently write $X = X^{(1)} + X^{(2)} + X^{(3)}$, where $X^{(1)}$, $X^{(2)}$, and $X^{(3)}$ are independent Lévy processes on $(\Omega, \mathcal{F}, \mathbb{P})$ given by

(i) $X_t^{(1)} = \gamma t + CB_t$, *a d-dimensional Brownian motion with drift γ;*

(ii) $X_t^{(2)} = \displaystyle\int_{[0,t] \times \mathbb{R}^d} x \mathbf{1}_{(|x| \geq 1)} \mu(\mathrm{d}s, \mathrm{d}x)$, *a compound Poisson process;*

(iii) $X_t^{(3)} = \displaystyle\int_{[0,t] \times \mathbb{R}^d} x \mathbf{1}_{(|x|<1)} \big[\mu(\mathrm{d}s, \mathrm{d}x) - \mathrm{d}s\, \pi(\mathrm{d}x)\big]$, *a square-integrable martingale.*

2.3 Stable Processes

The grandfather of all continuous-time stochastic processes is the Brownian motion. It is the canonical example of a martingale, a Markov process, and a self-similar process. It was first comprehensively studied by Einstein in 1905, more than two decades before Kolmogorov's *Grundbegriffe*, and over a century later remains an absolutely fundamental object in any study of stochastic processes.

The Lévy-Itô decomposition of Theorem 2.2.4 establishes that any Lévy process is, in a particular sense, extended from a Brownian motion by the addition of jumps. But one class of Lévy processes is related to Brownian motion in a different way: they share the self-similarity property, which allows that a rescaling in time and space can result in a process with the same law as the unscaled process.

Definition 2.3.1. *A standard Markov process X on $\mathbb{R}^d$ is called* self-similar *if there exists an $\alpha > 0$ such that*

$$(cX_{c^{-\alpha}t}, t \geq 0) \text{ under } \mathbb{P}_x \quad \stackrel{(d)}{=} \quad (X_t, t \geq 0) \text{ under } \mathbb{P}_{cx} \tag{2.5}$$

for all $c > 0$, $x \in \mathbb{R}^d$. We call α the self-similarity index *or simply* index *of X. A self-similar Lévy process with index α is called a (strictly) α-*stable process.

The class of positive self-similar Markov processes are well known for their correspondence to stationary processes and Lévy processes via one or other of the surprising and powerful Lamperti transforms, see [34] and [35]. Stable processes themselves have many interesting properties.

Index: For a general self-similar Markov process, α has the run of the entire real line. Stable process however can take only $\alpha \in (0, 2]$. The stable process with index $\alpha = 2$ is the Brownian motion, and is one of only two stable processes with continuous paths, the other being the linear drift $X_t = t$. All stable processes of index $\alpha \in (0, 1) \cup (1, 2)$ are pure jump processes.

Lévy measure: For $f \in b\mathcal{B}(\mathbb{R}^d)$ the Lévy measure π of a stable process satisfies

$$\pi f = \int_{\mathbb{R}^d} f(x)\, \pi(\mathrm{d}x) = \int_{\mathbb{R}} c|v|^{-\alpha-1} \int_S f(vu)\, \sigma(\mathrm{d}u)\, \mathrm{d}v$$

where S is the unit sphere in $\mathbb{R}^d$, $c > 0$ is a constant, and σ is a finite measure on S. If X has state space $\mathbb{R}$ then π is absolutely continuous with respect to the Lebesgue measure, and has density

$$c_+ x^{-\alpha-1}\mathbf{1}_{(x>0)} + c_- x^{-\alpha-1}\mathbf{1}_{(x<0)}$$

for some constants c_-, $c_+ \geq 0$. A Lévy process is called *isotropic* if its law is invariant under all orthogonal transformations of $\mathbb{R}^d$, and an isotropic stable process on $\mathbb{R}^d$ has Lévy measure

$$\pi(\mathrm{d}x) = c|x|^{-\alpha-d}\, \mathrm{d}x \tag{2.6}$$

for some constant $c > 0$.

Characteristic function: Self-similarity immediately yields that the characteristic exponent of a stable process satisfies

$$\Psi(\theta) = c^{-\alpha}\Psi(c\theta)$$

for $\theta \in \mathbb{R}^d$, $c > 0$. If X has state space $\mathbb{R}$ then

$$\Psi(\theta) = \begin{cases} c|\theta|^2 & \text{if } \alpha = 2 \\ c|\theta|^\alpha\Big(1 - \mathrm{i}\beta\tan\left(\frac{\pi\alpha}{2}\right)\mathrm{sgn}(\theta)\Big) & \text{if } \alpha \in (0,1)\cup(1,2) \\ c|\theta| + \mathrm{i}\gamma\theta & \text{if } \alpha = 1, \end{cases}$$

for some $c > 0$, $\beta \in [-1, 1]$ and $\gamma \in \mathbb{R}$. See Sato [39] Theorem 14.15 for the derivation, and [39] Proposition 14.9 for the d-dimensional case.

A sum of independent α-stable processes is again α-stable. Any linear drift is a 1-stable process, and the driftless 1-stable process is called the *Cauchy process*.

2.4 Potential Theory for Markov Processes

Potential theory existed first as a series of precise physical problems which had a convenient mathematical expression. As the subject grew it left its practical roots behind and became a branch of analysis in its own right, called 'classical potential theory'.

The relationship between Markov processes and classical potential theory was first studied in the 1930's, and came into its own as a probabilistic subject with the works of Doob on Brownian motion. In 1956-7 the view was greatly widened by three papers written by Hunt [24], and although the style of presentation has changed somewhat, his work remains central to the discipline of modern probabilistic potential theory.

Potential Operators

Let X be a standard Markov process. The operators introduced in the following definition are the bread and butter of any work on probabilistic potential theory.

Definition 2.4.1. *For $t \geq 0$, $q \geq 0$, the* q-potential *of X is given by*

$$U^q(x, B) = \int_0^\infty \mathrm{e}^{-qt} P_t(x, B)\, \mathrm{d}t = \mathbb{E}_x\Big[\int_0^\infty \mathrm{e}^{-qt} \mathbf{1}_{(X_t \in B)}\, \mathrm{d}t\Big] \qquad \text{for } x \in E,\ B \in \mathcal{E}^*$$

Like the transition function P_t of X, the potential is both a measure with respect to B and a universally measurable function with respect to x. We define in addition the transition operator P_t*, the* q-potential operator U^q*, and the* q-balayage operator P_B^q *by*

$$P_t f(x) = \int_E f(y) P_t(x, \mathrm{d}y) = \mathbb{E}_x[f(X_t)], \qquad U^q f(x) = \mathbb{E}_x\Big[\int_0^\infty \mathrm{e}^{-qt} f(X_t)\, \mathrm{d}t\Big],$$
$$P_B^q f(x) = \mathbb{E}_x[\mathrm{e}^{-qT_B} f(X_{T_B}); T_B < \zeta]$$

for $B \in \mathcal{E}_\Delta$ and $f \in \mathcal{E}_+^*$. When $q = 0$ it is dropped from notation.

The 0-potential (or simply *potential*) $U(x, B)$ of $B \in \mathcal{E}^*$ is the expected amount of time the process issued from a point x spends in B, and in this sense it is a valuable tool for solving problems relating to the range of X. But whereas the q-potential of $B \in \mathcal{E}^*$ always exists, there are plenty of interesting processes for which the potential of B is infinite.

Now let $f \in \mathcal{E}_+^*$ and T be a stopping time. Then

(i) *(semigroup property)* $P_t P_s f = P_{t+s} f$ for all $t, s \geq 0$;

(ii) *(resolvent equation)* for all $0 \leq q < r$,

$$U^q - U^r = (r - q) U^r U^q = (r - q) U^q U^r;$$

(iii) $U^q f(x) = \mathbb{E}_x\Big[\int_0^T \mathrm{e}^{-qt} f(X_t)\, \mathrm{d}t\Big] + \mathbb{E}_x[\mathrm{e}^{-qT} U^q f(X_T)]$ for all $q \geq 0$.

It follows immediately from the resolvent equation **(ii)** that $U^q f$ is monotone increasing as q decreases. So by monotone convergence we can define $Uf := \lim_{q \downarrow 0} U^q f$ for $f \in \mathcal{E}_+^*$, regardless of whether $U(x, B)$ is finite or infinite.

One aspect of Hunt's potential theory for Markov processes was the idea of a close reciprocal relationship between two Markov processes called duality. Blumenthal and Getoor

[5] are characteristically understated when they say that "probabilistic descriptions of the relationship between the process and its dual are complicated". But duality is the correct setting for introducing potential densities, which will play a role later in this thesis, and the situation for Lévy processes simplifies dramatically.

Definition 2.4.2. *Let $\hat{X}$ be a standard Markov processes on the same state space $(E, \mathcal{E})$ as X, with potential operator $\hat{U}^q$. We say X and $\hat{X}$ are in* weak duality *with respect to a σ-finite measure ξ on $\mathcal{E}^*$ if for all $q > 0$ and all $f, g \in \mathcal{E}^*_+$,*

$$\int_E f(x) U^q g(x) \xi(\mathrm{d}x) = \int_E \hat{U}^q f(x) g(x) \xi(\mathrm{d}x).$$

If in addition the measures $U^q(x, \cdot)$ and $\hat{U}^q(x, \cdot)$ are absolutely continuous with respect to ξ for all $q > 0$, $x \in E$ then X and $\hat{X}$ are said to be in strong duality *with respect to ξ.*

The process $\hat{X}$ is called the dual of X. We will most often consider processes in strong duality, as the densities of U^q and $\hat{U}^q$ have some useful properties. One in particular inspires a class of functions called excessive functions.

Excessive Functions

Definition 2.4.3. *Let $0 \leq q < \infty$. A function $f \in \mathcal{E}^*_+$ is called* q-excessive *if*

- ***(i)*** $\mathrm{e}^{-qt} P_t f \to f$ *pointwise as $t \downarrow 0$;*
- ***(ii)*** $\mathrm{e}^{-qt} P_t f \leq f$ *for every $t \geq 0$.*

If X and $\hat{X}$ are in weak duality with respect to a measure ξ and $g \in \mathcal{E}^*_+$ is q-excessive for $\hat{X}$ then g is called q-co-excessive.

The class of q-excessive functions is sometimes denoted $\mathscr{S}^q$. When $q = 0$ a q-excessive function is simply called *excessive*.

Constant nonnegative functions are q-excessive for every $q \geq 0$. Excessive functions have interesting interactions with the operators from Definition 2.4.1. For example, for $B \in \mathcal{E}^n$, if f is q-excessive then [5] Proposition II(2.8) yields that

$$P^q_B f(x) = \mathbb{E}_x[\mathrm{e}^{-qT_B} f(X_{T_B}); T_B < \zeta] \leq f(x) \tag{2.7}$$

for all $x \in E$, and in addition that $P^q_B f$ is itself q-excessive. When $f \equiv 1$ this function is denoted by

$$\Phi^q_B(x) = \mathbb{E}_x[\mathrm{e}^{-qT_B}; T_B < \zeta]. \tag{2.8}$$

In particular, taking $q = 0$, $\Phi_B(x) = \mathbb{P}_x(T_B < \zeta) = \mathbb{P}_x(L_B > 0)$ is excessive as a function in x. The next lemma concerns how excessive functions behave at the regular points B^r of a set $B \in \mathcal{E}^n$.

Lemma 2.4.4 ([5] Proposition II(2.10))**.** *Let $B \in \mathcal{E}^n$ and $f \in \mathscr{S}^q$. Then for $x \in B^r$,*

$$\inf\{f(y) : y \in B\} \leq f(x) \leq \sup\{f(y) : y \in B\}.$$

Suppose now that f is a q-excessive function such that $f \geq 1$ on some $B \in \mathcal{E}$. By Lemma 2.4.4, $f \geq 1$ on $B \cup B^r$, and in particular $f(X_{T_B}) \geq 1$ almost surely. Thus looking again at (2.7) we see that

$$f(x) \geq \mathbb{E}_x[\mathrm{e}^{-qT_B} f(X_{T_B}); T_B < \zeta] \geq \Phi_B^q(x) \tag{2.9}$$

for all $x \in E$. This interesting property of Φ_B^q is related to an aspect of classical potential theory called *balayage*. In this setting Φ_B^q would be called the balayage of 1 onto B. If Φ_B^q is lower semi-continuous it can be written

$$\Phi_B^q = \inf\{q\text{-excessive } f : f \geq 1 \text{ on } B\},$$

and is called the *réduite* of 1 on B. We shall see settings below in which Φ_B^q, and indeed all q-excessive functions, are lower semi-continuous. On of the best-known references for an introduction to balayage is §7.3 of Helms [23], and a probabilistic interpretation is given in §VI.3 of Bliedtner and Hansen [4].

Potential Densities

The next lemma is a useful characterisation of the densities of the q-potential measures U^q and $\hat{U}^q$.

Lemma 2.4.5. *Let X, $\hat{X}$ be standard Markov processes in strong duality with respect to ξ, so that by assumption $U^q(x,\cdot)$ and $\hat{U}^q(x,\cdot)$ are absolutely continuous with respect to ξ for every $q > 0$. The densities of U^q, $q > 0$ with respect to ξ are non-negative functions $u^q \in \mathcal{E}^* \times \mathcal{E}^*$ such that*

- ***(i)*** *$u^q(\cdot, y)$ is q-excessive;*
- ***(ii)*** *$u^q(x, \cdot)$ is q-co-excessive;*
- ***(iii)*** *$U^q f(x) = \int f(y)u^q(x,y)\xi(\mathrm{d}y)$ and $\hat{U}^q f(y) = \int f(x)u^q(x,y)\xi(\mathrm{d}x)$ for all $f \in b\mathcal{E}^*$, all $x, y \in E$.*

*For $q > 0$ the function u^q is called the q-*potential density *of X. The function $\hat{u}^q$ defined by $\hat{u}^q(x,y) = u^q(y,x)$ is the density of $\hat{U}^q(x,\cdot)$ with respect to ξ and is called the* co-q-potential density*.*

Existence of a potential density u for $q = 0$ is not immediate. But in any case it is shown in Blumenthal and Getoor [5] VI(1.5) that for any $0 < q \leq r$,

$$u^q(x,y) = u^r(x,y) + (r-q)U^q u^r(x,y),$$

and therefore that the functions u^q are monotone increasing as q decreases. So we define $u := \lim_{q\downarrow 0} u^q$, which is excessive in the first variable, co-excessive in the second, and which by monotone convergence satisfies

$$Uf(x) = \mathbb{E}_x\Big[\int_0^\infty f(X_t)\,\mathrm{d}t\Big] = \int f(y)u(x,y)\,\xi(\mathrm{d}y)$$

for $f \in \mathcal{E}_+^*$. It may be true that u is identically infinity. But if a set $B \in \mathcal{E}^*$ has finite potential $U(x,B)$, then $u(x,y) < \infty$ for ξ-almost every $y \in B$. An extension of this reasoning leads to the following concepts.

Transience and Recurrence

It can be interesting to consider how a given standard Markov process behaves at large times. Two behaviours in particular have been studied a great deal: transience and recurrence. There is no single way to define either concept, and different but related definitions are used in different settings. In several common cases, for example when X is either a random walk or a Lévy processes, the two form a dichotomy.

Definition 2.4.6. *Let X, $\hat{X}$ be standard Markov processes in strong duality with respect to ξ. X is called* weakly transient *if there exists a strictly positive function $h \in \mathcal{E}^*$ such that Uh is finite everywhere. In that case, for every $x \in E$, $u(x,y) < \infty$ for ξ-almost every y.*

Weak transience is more often simply called transience in the Markov process literature, but we call it weak to distinguish it from the other stronger types of transience which we shall see soon, and in particular that which is usually defined for Lévy processes.

Denote by (LSC) the condition that there exists a $q > 0$ such that all q-excessive functions are lower-semicontinuous. Getoor [18] showed that under (LSC) following conditions are equivalent:

(i) X is weakly transient

(ii) $U(\cdot, K)$ is bounded for all compact K;

(iii) $U(\cdot, K) < \infty$ for all compact K; (2.10)

(iv) $L_K < \infty$ $\mathbb{P}_x$-almost surely for all compact K, all $x \in E$.

A comprehensive discussion of weak transience, and the relationships of (i)-(iv) of (2.10) without the assumption of (LSC), can be found in §3.7 of Chung and Walsh [10]. They show, for example, that in general either (iii) or (iv) imply (i).

It will be useful when we focus on Lévy processes to know a little more about transience in the following specific case. X is said to have *strong Feller resolvents* if for all $q > 0$, $U^q f$ is continuous for all $f \in b\mathcal{E}_+$ having compact support. If X has strong Feller resolvents then it is given in [5] Exercise II(2.16) that for all $q \geq 0$, any q-excessive function is lower-semicontinuous, and thus (LSC) holds.

Even if (LSC) does not hold, an alternative characterisation of transience also exists, given by condition **(iv)** above: we shall call X *transient* if $\mathbb{P}_x(L_K < \infty) = 1$ for all compact $K \in \mathcal{E}$, all $x \in E$, and *strongly transient* if $\mathbb{P}_x(L_K < \zeta) = 1$ for all compact $K \in \mathcal{E}$, all $x \in E$.[3] Transience of X implies weak transience, which can be seen from Getoor [18] Proposition 2.2(v), and clearly strong transience implies transience. If X is transient and has strong Feller resolvents then [5] II(4.24) yields that

$$\lim_{t\to\infty} X_t = \Delta \qquad \text{almost surely.}$$

If the state space of X is $\mathbb{R}^d$ then transience is equivalent to

$$\liminf_{t\to\infty} |X_t| = \infty \quad \mathbb{P}_x\text{-a.s., for all } x \in E.$$

Properties of the range of X can also be studied in terms of the subsets of $\mathcal{E}$. For a fixed $x \in E$, a set $B \in \mathcal{E}$ is called

(i) $\mathbb{P}_x$-*transient* if $\mathbb{P}_x(L_B < \infty) = 1$, *strongly* $\mathbb{P}_x$-*transient* if $\mathbb{P}_x(L_B < \zeta) = 1$, and $\mathbb{P}_x$-*recurrent* if $\mathbb{P}_x(L_B < \infty) = 0$. There can be sets which are neither transient nor recurrent. It is given in §3.7 of Chung and Walsh [10] that weak transience of X is equivalent to the condition that the state space E is a union of transient sets.

(ii) $\mathbb{P}_x$-*thin* or *thin at* x if $\mathbb{P}_x(T_B = 0) = 0$. If B is $\mathbb{P}_x$-thin at every $x \in E$, it is simply called *thin*. Blumenthal's zero-one law gives that for any $B \in \mathcal{E}$, $\mathbb{P}_x(T_B = 0)$ is equal 0 or 1.

(iii) *finely open* if $E \setminus B$ is $\mathbb{P}_x$-thin for every $x \in B$. The finely open subsets of E form a topology on E called the fine topology, which has many interesting properties relating to excessive functions. A set $A \in \mathcal{E}$ is called *finely closed* if $E \setminus A$ is finely open. Exercise II(4.9) of Blumenthal and Getoor [5] yields that A is finely closed if and only if it contains its regular points. The same authors also explain that by right-continuity of paths every open set is finely open, and therefore that every closed set is finely closed.

(iv) $\mathbb{P}_x$-*avoidable* if $\mathbb{P}_x(D_B < \infty) < 1$. In this case $M = E \setminus B$ is called $\mathbb{P}_x$-*supportive*, and satisfies $\mathbb{P}_x(X_t \in M \text{ for all } t \in [0,\zeta)) > 0$.

(v) $\mathbb{P}_x$-*polar* if $\mathbb{P}_x(T_B < \infty) = 0$. If B is $\mathbb{P}_x$-polar at every $x \in E$, it is simply called *polar*.

(vi) *semipolar* if B is contained in a countable union of thin sets.

(vii) *essentially polar* if $E = \mathbb{R}^d$ and B is $\mathbb{P}_x$-*polar* at almost every $x \in \mathbb{R}^d$.

Clearly every polar set is thin, and every thin set is semipolar. For some processes all three of these concepts are equivalent, and we shall discuss this in a little more detail

[3] There are other forms of transience, for example m-transience or left-transience of sets as in [19, 20] and others. But for our purposes the definitions we have given suffice.

with respect to Lévy processes in §2.6. Subadditivity of measure yields that a finite union of $\mathbb{P}_x$-thin/-polar sets is again $\mathbb{P}_x$-thin/-polar.

The companion concept to transience is called *recurrence*, and is defined as follows.

Definition 2.4.7 (Getoor [18] Proposition 2.4)**.** *Suppose E has at least two points. X is called* recurrent *if it satisfies the following equivalent conditions.*

(i) *For each $B \in \mathcal{E}^*$, either $U(\cdot, B) = 0$ or $U(\cdot, B) = \infty$;*

(ii) *if $B \in \mathcal{E}^n$ is not polar, then $\mathbb{P}_x(T_B < \zeta) = 1$ for all $x \in E$;*

(iii) *each excessive function is constant;*

(iv) *if $B \in \mathcal{E}^n$ is not polar, then $L_B = \infty$ almost surely;*

Two interesting facts can be immediately shown to hold for recurrent standard Markov processes. The first is that if X is recurrent then it follows from **(iii)**, and the fact that $\mathbb{E}_x[1 - \mathrm{e}^{-\zeta}]$ is excessive, that $\mathbb{P}_x(\zeta < \infty) = 0$ for all $x \in E$. That is, recurrent standard Markov processes have almost surely infinite lifetime when issued from any point of the state space. The second interesting fact is that if M is a $\mathbb{P}_x$-supportive set of recurrent X then the excessive map $x \mapsto \mathbb{P}_x(T_{E\setminus M} < \infty)$ is constant $\leq \mathbb{P}_x(D_{E\setminus M} < \infty) < 1$, and thus by Blumenthal's zero-one law the map $x \mapsto \mathbb{P}_x(T_{E\setminus M} = 0) \leq \mathbb{P}_x(T_{E\setminus M} < \infty) < 1$ is constant zero, and it therefore follows that $E \setminus M$ is thin.

2.5 Potential Theory for Lévy Processes

Potential theory for Lévy processes is in many ways simpler than that for general Markov processes. The most significant work particular to Lévy processes remains Hawkes' 1979 paper [21], and Chapter II of Bertoin [2] gives a thorough modern account of the theory.

Potential Operators

Let X be a Lévy process on $\mathbb{R}^d$ with characteristic exponent Ψ. Then X is automatically in weak duality with $-X$ with respect to Lebesgue measure, that is,

$$\int_{\mathbb{R}^d} f(x) U^q g(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \hat{U}^q f(x) g(x)\,\mathrm{d}x$$

for all $q > 0$ and $f, g \in \mathcal{B}(\mathbb{R}^d)^*_+$, where

$$\hat{U}^q f(x) = \int_{\mathbb{R}^d} f(y) \hat{U}^q(x, \mathrm{d}y) = \int_{\mathbb{R}^d} f(-y) U^q(-x, \mathrm{d}y).$$

Let (ACP) denote the condition that X and $\hat{X}$ are in strong duality with respect to Lebesgue measure, so that for each $q > 0$, each $x \in \mathbb{R}^d$, the potential $U^q(x, \cdot)$ is absolutely continuous with respect to Lebesgue measure. In that case the space-homogeneity of Lévy processes allows us to define $u^q := u^q(0, \cdot)$ and see that

$$u^q(x, y) = u^q(y - x)$$

for all $x, y \in \mathbb{R}^d$. It immediately follows that $\hat{u}^q(x) = u^q(-x)$ for all $q > 0$, and $\hat{u}(x) = u(-x)$ if X is transient. Hawkes proved several equivalent conditions for (ACP), some of which also appear in Sato [39] Theorem 41.15, and a combination of both are presented below.

Theorem 2.5.1 (Hawkes [21] Theorem 2.1 and Sato [39] Theorem 41.15)**.** *The following are equivalent.*

(i) *condition (ACP) holds;*

(ii) *X has strong Feller resolvents;*

(iii) *for every $q \geq 0$ every q-excessive function is lower-semicontinuous;*

(iv) *if f is bounded and universally measurable, then for $q > 0$, $U^q f$ is continuous;*

(v) *all essentially polar sets are polar.*

For general Markov processes, strong Feller resolvents is a stronger assumption than strong duality, but for Lévy processes they are equivalent. Not all Lévy processes have strong Feller resolvents - see the example directly following Theorem 2.1 in Hawkes [21], or Examples 41.21-3 in Sato [39]. But many do, including all Lévy processes with local times, see Theorem 2.6.5 for more details.

Hawkes [21] also gives a sufficient condition for the conditions of Theorem 2.5.1 to hold. X is said to have a *strong Feller semigroup* if for all $t > 0$, $P_t f$ is continuous for all $f \in b\mathcal{E}_+$ having compact support. This is well-defined for all Markov processes, but for Lévy processes in particular the following holds.

Theorem 2.5.2 (Hawkes [21] Theorem 2.2 and Lemma 2.2)**.** *X has a strong Feller semigroup if and only if $P_t(x, \cdot)$ is absolutely continuous with respect to Lebesgue measure for each $t > 0$. In that case the density p_t of P_t is lower-semicontinuous for each $t > 0$, (ACP) holds, and*

$$u^q(x) = \int_0^\infty e^{-qt} p_t(x)\, dt$$

for all $q > 0$.

Sato [39] calls the existence of a transition density p_t condition (ACT), see [39] Definition 41.11 and Remark 41.20. Hawkes' theorem tells us that all Lévy processes satisfying (ACT) have strong Feller resolvents, including all stable processes on $\mathbb{R}^d$.

Transience and Recurrence

For Lévy processes all concepts of transience that we have seen so far coincide, and form a dichotomy with a particular type of recurrence, which is close to that of Definition 2.4.7, but not exactly the same.

Definition 2.5.3 (Sato [39] Theorem 35.4)**.** *The following are equivalent.*

(i) *X is transient;*

(ii) *X is strongly transient;*

(iii) $\lim_{t\to\infty} |X_t| = \infty$ *almost surely;*

(iv) $U(0, K) < \infty$ *for every compact K;*

(v) $\int_0^\infty \mathbf{1}_K(X_s)\,\mathrm{d}s < \infty$ *a.s. for every compact K.*

On the other hand, if X is not transient then the following equivalent conditions hold, and X is called weakly recurrent*.*

(i) $\liminf_{t\to\infty} |X_t| = 0$ *a.s.;*

(ii) $U(0, K) = \infty$ *for every compact K;*

(iii) $\int_0^\infty \mathbf{1}_K(X_s)\,\mathrm{d}s = \infty$ *a.s. for every compact K.*

If Lévy X is weakly transient it is not weakly recurrent, and the dichotomy above tells us that X is therefore strongly transient. What is also interesting is that for Lévy processes the conditions of (2.10) are equivalent, without assuming anything on the q-excessive functions. This allows us to talk of 'transient Lévy processes' without ambiguity.[4]

According to Sato [39] Exercise 44.10, a Lévy process is recurrent if and only if it is weakly recurrent and satisfies condition (ACP). A lot of literature just uses the term recurrence for Lévy processes, but it is worthwhile for us to make the distinction.

At the same time Sato [39] makes a further interesting distinction with respect to recurrence of Lévy processes, which will be of use to us, and also introduces no confusion of terminology. A weakly recurrent Lévy process is called *point recurrent* if

$$\limsup_{t\to\infty} \mathbf{1}_{\{0\}}(X_t) = 1.$$

If the above holds for $\mathbf{1}_A$ instead of $\mathbf{1}_{\{0\}}$ for any open A containing 0 then the process is called *set recurrent*, a property which is clearly implied by point recurrence. Point recurrence is the stronger form of a related property: a Lévy process *hits points* if $\{0\}$ is not essentially polar.[5] Hitting points is not particular to either transient or recurrent Lévy processes. For example, in one dimension both the standard Brownian motion and the Brownian motion with a positive linear drift are Lévy processes which hit points, but the first is point recurrent while the second is transient.

[4] It is worth noting that the distinction made by Sato [39] at the end of Chapter 37 between weakly and strongly transient Lévy processes is something else, and does not feature in this work.

[5] In other words, $\mathbb{P}_x(T_{\{0\}} < \infty) > 0$ for a Lebesgue-positive set of $x \in \mathbb{R}^d$.

If a Lévy process is weakly recurrent, satisfies (ACP), and hits points, then it is recurrent and by **(iv)** of Definition 2.4.7, the process is point recurrent. But even if X is not assumed to be weakly recurrent, the following equivalence holds.

Lemma 2.5.4 (Sato [39] Theorem 43.5)**.** *Let $q > 0$. The following are equivalent:*

(i) *X hits points and $\{0\}$ is regular for itself;*

(ii) *(ACP) holds, and u^q is bounded, continuous, and positive on $\mathbb{R}^d$.*

If X is transient, the same holds for $q = 0$.

2.6 Capacity

It is possible to define much of what follows for Borel right processes in weak duality, and indeed that is the main idea of Getoor [19] and also partially addressed in Getoor and Sharpe [20], but the generalisation is not easy. This author is also not aware of any modern comprehensive review of capacity theory for Markov processes in weak duality, whereas there are excellent discussions for Lévy processes in both Bertoin [2] and Sato [39]. Since this thesis uses capacity theory exclusively for Lévy processes, nothing is lost by presenting it in that setting.

Let X be a Lévy process on $\mathbb{R}^d$, in weak duality with $\hat{X} = -X$. For a measure μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, we denote by μU^q and $\mu \hat{U}^q$ the measures

$$\mu U^q(B) = \int U^q(x, B)\, \mu(\mathrm{d}x), \qquad \mu \hat{U}^q(B) = \int \hat{U}^q(x, B)\, \mu(\mathrm{d}x), \tag{2.11}$$

respectively called the *q-resolvent measure* and *q-co-resolvent measure* of μ. If μ is a Radon measure, it is uniquely determined by either μU^q or $\mu \hat{U}^q$. If X satisfies (ACP) then we define the function

$$U^q\mu(x) = \int u^q(x, y)\, \mu(\mathrm{d}y) = \int u^q(y - x)\, \mu(\mathrm{d}y), \qquad x \in \mathbb{R}^d, \tag{2.12}$$

called the *potential of* μ, and the equivalent for $\hat{U}^q\mu$. Because $u^q(x, y)$ is q-excessive as a function in x, $U^q\mu$ is also q-excessive, and in a similar way $\hat{U}^q\mu$ is q-co-excessive. It is important to note the distinction between μU^q, which is a measure, and $U^q\mu$, which is a function.

Potentials of measures play a role in the classical Riesz Decomposition Theorem, which is given in its traditional form in Helms [23] §6.1, see in particular Corollary 6.19, and from a Markov perspective in Blumenthal and Getoor [5] Theorem VI(2.11), who were building upon the work of Hunt [24]. The following lemmas are closely related to the Riesz Decomposition Theorem, and are useful to us because they will allow us to explicitly prove existence of a particular measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, associated to a fixed Borel set B, called the equilibrium measure of B.

First, suppose that X satisfies (ACP). Because this implies that X has strong Feller resolvents it ensures that the following condition holds, corresponding to condition VI(2.1) of Blumenthal and Getoor [5], see also Theorem 41.15 of Sato [39].

(C1) If $q > 0$ and $f \in b\mathcal{B}(\mathbb{R}^d)^*$ has compact support then $y \mapsto \hat{U}^q f(y)$ is continuous and bounded.

Under this assumption, the following two lemmas can be proven.

Lemma 2.6.1 ([5] Corollary VI(2.6))**.** *Suppose that X is a Lévy process satisfying condition (C1), and suppose $q > 0$. Let (μ_n) be a sequence of measures such that each $U^q\mu_n$ is locally integrable, the sequence $(U^q\mu_n)$ is increasing, and $f = \lim U^q\mu_n$ is also locally integrable. Suppose further that either the supports of all the μ_n are contained in one compact set $K \subseteq \mathbb{R}^d$, or that $\lim_{K_m \uparrow \mathbb{R}^d} P^q_{\mathbb{R}^d \setminus K_m} f = 0$ almost everywhere, where the K_m are compact and increasing to $\mathbb{R}^d$. Then (μ_n) converges weakly to a measure μ, and $f = U^q\mu$.*

Lemma 2.6.2 ([5] Proposition VI(2.10))**.** *Suppose that X is a Lévy process satisfying condition (C1), and suppose $q > 0$. A locally integrable q-excessive function f is the q-potential $U^q\mu$ of a measure μ if and only if*

$$\lim_{K_m \uparrow \mathbb{R}^d} P^q_{\mathbb{R}^d \setminus K_m} f(x) = \lim_{K_m \uparrow \mathbb{R}^d} \mathbb{E}_x[\mathrm{e}^{-qT_B} f(X_{T_{\mathbb{R}^d \setminus K_m}}); T_{\mathbb{R}^d \setminus K_m} < \infty] = 0$$

for almost every $x \in \mathbb{R}^d$, where $K_m \subseteq \mathbb{R}^d$ are compact.

Moreover the following lemma ensures that if it exists, the representation $f = U^q\mu$ of f in Lemmas 2.6.1 and 2.6.2 is unique.

Lemma 2.6.3 ([5] Propositions VI(1.15) and VI(2.3))**.** *Let μ be a measure and $q \geq 0$. If $U^q\mu$ is locally integrable, then $U^q\mu$ determines μ.*

Now in addition suppose that X is transient and satisfies (ACT), which implies[6] the following condition:

(C2) If $f \in b\mathcal{B}(\mathbb{R}^d)^*$ has compact support then Uf and $\hat{U}^q f$ are bounded, and $\hat{U}f$ is continuous.

This is condition VI(2.2) of Blumenthal and Getoor [5]. Under the assumption of (C2), Lemmas 2.6.1 and 2.6.2 hold for $q = 0$.

The Capacitary Measures

We shall now introduce a representation as in Lemmas 2.6.1 and 2.6.2 for the q-excessive function

$$\Phi^q_B(x) = \mathbb{E}_x\left[\mathrm{e}^{-qT_B}\right], \qquad B \in \mathcal{B}(\mathbb{R}^d).$$

Most of what follows here appears in §42 of Sato [39], see in particular Theorems 42.5, 42.8, and Proposition 42.13, but it is also partially inspired by §VI.4 of Blumenthal and

[6] See Sato [39] Exercise 44.5. The proof of the result is almost identical to that of [39] Theorem 41.15(3).

Getoor [5]. Let $B \in \mathcal{B}(\mathbb{R}^d)$. For every $q > 0$ there exists a unique Radon measure m_B^q carried by $B \cup B^r$ having q-co-resolvent measure

$$m_B^q \hat{U}^q(A) = \int_A \mathbb{E}_x[\mathrm{e}^{-qT_B}]\,\mathrm{d}x, \qquad A \in \mathcal{B}(\mathbb{R}^d).$$

The measure m_B^q is called the *q-capacitary measure* of B, and is given by

$$m_B^q(A) = q\int_{\mathbb{R}^d} \hat{\mathbb{E}}_x[\mathrm{e}^{-qT_B};\ X_{T_B} \in A]\,\mathrm{d}x, \qquad A \in \mathcal{B}(\mathbb{R}^d). \tag{2.13}$$

The *q-capacity* and *capacity* of B are defined as

$$C^q(B) := m_B^q(\mathbb{R}^d), \qquad C(B) := \lim_{q \downarrow 0} C^q(B).$$

A general set may have infinite capacity. But if X is transient and $B \in \mathcal{B}(\mathbb{R}^d)$ is bounded there exists a unique finite measure m_B carried by $B \cup B^r$ such that $m_B(\mathbb{R}^d) = C(B)$ and

$$m_B \hat{U}(A) = \int_A \mathbb{P}_x(T_B < \infty)\,\mathrm{d}x, \qquad A \in \mathcal{B}(\mathbb{R}^d).$$

If X satisfies (ACP) then for any $B \in \mathcal{B}(\mathbb{R}^d)$, this m_B^q is the unique measure with q-co-resolvent measure given by

$$\hat{U}^q m_B^q(x) = \int_{\mathbb{R}^d} \hat{u}^q(y-x)\,m_B^q(\mathrm{d}y) = \mathbb{E}_x[\mathrm{e}^{-qT_B}], \qquad x \in \mathbb{R}^d. \tag{2.14}$$

If X is transient and $B \in \mathcal{B}(\mathbb{R}^d)$ is bounded then (2.14) holds for $q = 0$ as

$$\hat{U} m_B(x) = \int_{\mathbb{R}^d} \hat{u}(y-x)\,m_B(\mathrm{d}y) = \mathbb{P}_x(T_B < \infty), \qquad x \in \mathbb{R}^d.$$

The measure m_B is interchangeably called the capacitary measure or equilibrium measure of B. Every Borel set has a q-capacitary measure, but a priori only bounded Borel sets have an equilibrium measure. Whenever m_B exists, $C(B) = m_B(\mathbb{R}^d)$.

Because of the uniqueness given in Lemma 2.6.3, any measure μ which satisfies $\hat{U}\mu = \mathbb{P}_x(T_B < \infty)$ is called the equilibrium measure of B, and Lemma 2.6.2 suggests that in some situations unbounded B may also have a well-defined equilibrium measure. More will be said on that in Chapter 6, see also the discussion following Definition VI(4.5) in Blumenthal and Getoor [5].

The following lemma gives some additional properties of capacities for Lévy processes. Take $q > 0$ and $A, B \in \mathcal{B}(\mathbb{R}^d)$.

Lemma 2.6.4 (Sato [39] Proposition 42.12)**.**

(i) *If B is bounded, then $C^q(B) < \infty$.*

(ii) *If $A \subseteq B$ then $C^q(A) \leq C^q(B)$.*

(iii) *$C^q(A \cup B) + C^q(A \cap B) \leq C^q(A) + C^q(B)$.*

(iv) *$C^q(B) = \inf\{C^q(D) : D$ is open and $B \subseteq D\}$.*

(v) *If B_n, $n \in \mathbb{N}$ are increasing and $\bigcup_n B_n = B$ then $C^q(B_n) \uparrow C^q(B)$.*

(vi) *For $x \in \mathbb{R}^d$, $C^q(x + B) = C^q(B) = C^q(-B)$.*

Properties **(ii)-(iv)** are those of a Choquet capacity. From **(vi)** it follows that for every $q > 0$ one can unambiguously define the q-capacity of a single point $C^q := C^q(\{0\}) = C^q(\{x\})$ for all $x \in \mathbb{R}^d$.

Local Times

For $t \in (0, \infty)$ the *occupation measure* on $[0, t]$ is the measure μ_t on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ satisfying

$$\int_{\mathbb{R}^d} f(x)\,\mu_t(\mathrm{d}x) = \int_0^t f(X_s)\,\mathrm{d}s$$

for $f \in \mathcal{E}_+$. We say that X has *local times* if, for every $t \in (0, \infty)$, μ_t is absolutely continuous with respect to Lebesgue measure. The density of each μ_t is denoted by $L(\cdot, t)$, and the function L on $\mathbb{R}^d \times [0, \infty)$ defined by those densities is called the *local time* of X. $L(x, \cdot\,)$ is increasing for every $x \in \mathbb{R}^d$, and increases only when $X = x$. That final fact means that $x \mapsto L(x, t)$ has compact support for every $t > 0$ almost surely, and that $x \mapsto L(x, T_B-)$ has support on $\mathbb{R}^d \setminus B$. This in turn implies that μ_{T_B} has support on $\mathbb{R}^d \setminus B$, because the path integral is left-continuous.

The following theorem characterises local times with respect to capacity.

Theorem 2.6.5 (Bertoin [2] Theorems II.16 and V.1)**.** *Let X be a Lévy process on $\mathbb{R}$. The following are equivalent.*

(i) *X has local times;*

(ii) *$C^q > 0$ for all $q > 0$, where C^q is the capacity of a point;*

(iii) *condition (ACP) holds, and u^q is bounded for all $q > 0$;*

(iv) *singleton sets $\{x\}$ are not essentially polar that is, X hits points.*

If X is transient the above holds for $q \geq 0$. Other valuable perspectives on this theorem can be found in Sato [39] Theorem 43.3 and Kyprianou [31] Theorem 7.12. If a Lévy process on $\mathbb{R}$ has local times then (ACP) holds and (2.14) admits that

$$\hat{u}^q(-x) = u^q(x) = \frac{1}{C^q}\mathbb{E}_x[\mathrm{e}^{-qT_{\{0\}}}], \qquad x \in \mathbb{R}. \tag{2.15}$$

Since X hits points, we see that $\mathbb{P}_x(T_{\{0\}} < \infty) > 0$ for all $x \in \mathbb{R}$, and it follows from (2.15) that u^q is positive everywhere. Since (ACP) holds we have from Theorem 2.5.1

that u^q is lower semi-continuous, and combined with positivity this yields that u^q is bounded away from zero on compact sets.

Theorem II.19 of Bertoin [2] shows that if 0 is regular for $\{0\}$ then each u^q is continuous, and in addition in Proposition V.2 that $t \mapsto L(t,\cdot)$ is almost surely continuous. This contrasts with the approach in Blumenthal and Getoor [5] Definition V(3.12) and Sato [39] Remark 43.27, in which continuity of $t \mapsto L(t,\cdot)$ is included in its definition. Furthermore there exists a condition for *joint* continuity of L, given as follows, taken here from §V.3 of Bertoin [2] but which also appears in Theorem 6 at the end of Hawkes [22].

Theorem 2.6.6 (Bertoin [2] Theorem V.15)**.** *Let X be a Lévy process on $\mathbb{R}$. If*

$$\int_{0+} \sqrt{-\log(\eta(u))}\,\mathrm{d}u < \infty,$$

where

$$\eta(u) = \mathrm{Leb}\Big(\Big\{x \in \mathbb{R} : \frac{1}{\pi}\int_{-\infty}^{\infty}(1-\cos(x\theta))\mathrm{Re}\Big(\frac{1}{\Psi(\theta)}\Big)\,\mathrm{d}\theta < u\Big\}\Big)$$

then the map $(t,x) \mapsto L(t,x)$ is almost surely continuous.

Any Lévy process with local times such that $(t,x) \mapsto L(t,x)$ is almost surely continuous is said to have *jointly continuous local times.* Stable processes on $\mathbb{R}$ with index $\alpha \in (1,2]$ are one class of Lévy process with jointly continuous local times.

Hunt's Condition

The condition that every semipolar set is polar is called (H) or Hunt's condition, and Getoor posed the problem, which remains open, of which Lévy processes satisfy (H). Hunt's condition is known to hold for symmetric Lévy processes, and for all non-trivial stable processes on $\mathbb{R}$.[7] Although at first sight it appears to be quite a technical assumption, interesting and powerful results can be proven for processes satisfying (H). Here are two examples, the second of which will be of use to us in Chapter 5.

Lemma 2.6.7. *Let X be a standard Markov process on $(E,\mathcal{E})$, recurrent in the sense of Definition 2.4.7, and satisfying (H). Then for any $x,y \in E$ and any $\mathbb{P}_x$-supportive set $M \in \mathcal{E}$,*

$$\mathbb{P}_y(X_t \in M \text{ for all } t > 0) = 1.$$

That is, the complement of M is a polar set.

Proof. In the brief discussion below Definition 2.4.7 we saw that the $E \setminus M$ is thin, and thus semipolar. If in addition X satisfies (H) then $E \setminus M$ is polar. □

[7] This fact is often stated, but rarely proved. For a comprehensive but slightly outdated approach see Blumenthal and Getoor [6] - in particular the very last paragraph!

We have seen that if X is a weakly recurrent Lévy process satisfying (ACP) then it is recurrent in the sense of Definition 2.4.7, and in that case it follows immediately from Lemma 2.6.7 that any supportive set is equal to the support of X, as defined in Sato [39] Definition 24.13.

This next result was originally proven by Hunt [24], and to state it we first need to know that a locally integrable q-excessive function f is called *regular* if, with probability one, the mapping $t \mapsto f(X_t)$ is continuous whenever $t \mapsto X_t$ is continuous on $[0, \zeta)$. Getoor [17] states that regularity of f implies the condition that whenever $\{T_n\}$ is an increasing sequence of stopping times with limit T,

$$f(X_{T_n}) \to f(X_T) \qquad \text{almost surely on } \{T < \zeta\}. \tag{2.16}$$

Proposition IV(5.9) of Blumenthal and Getoor [5] gives that the two are equivalent if X is what they term a *special standard process*.[8] A valuable discussion of regularity can also be found below Proposition VI(2.7) of [5].

Lemma 2.6.8. *Let X be a Lévy process on $\mathbb{R}^d$ satisfying (ACP). Then the following are equivalent.*

(i) *X satisfies condition (H).*

(ii) *For all $q > 0$, all locally integrable q-excessive functions are regular.*

If X satisfies (ACT) then the same holds for $q = 0$, and in addition the statements above are equivalent to the following for $q \geq 0$.

(iii) *Let μ be a finite measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with compact support K. Then $U\mu$ is continuous if it is bounded and its restriction to K is continuous.*

(iv) *If f is a locally integrable q-excessive function, $q \geq 0$, and $\varepsilon > 0$, then there exists an open set G with $C(G) < \varepsilon$ such that f restricted to $\mathbb{R}^d \setminus G$ is finite and continuous.*

Proof. This result is built from some more general results in Chapter VI of Blumenthal and Getoor [5]. Condition (ACP) implies conditions (2.1) and (4.1) of that chapter, and the first part of the lemma follows from [5] Theorem VI(4.9); condition (ACT) implies conditions (2.2) and (4.2), which combined with (v) of Theorem 2.5.1 satisfies the conditions of [5] Theorem VI(4.12), and the rest of the lemma follows. More detail on the relationship of (ACP) and (ACT) with the conditions of Blumenthal and Getoor was given above, see (C1) and (C2). □

An alternative characterisation of Lemma 2.6.8 can be found in Chung and Walsh [10] Theorem 13.80. Chung and Walsh intentionally offer a different perspective on condition

[8] The precise definition of a special standard process is given in Blumenthal and Getoor [5] IV(4.1), but we make no use of it here. We can at least note that the class of special standard processes is not contrived: all Hunt processes are special standard (see [5]§IV4), as are all standard processes satisfying (LSC) (see Getoor [6] (4.7))

(H), emphasising symmetry of the process X and its dual, see their Definition 13.81 and Remark 13.82.

Any standard Markov process X satisfies (H) if and only if the killed process X^q satisfies (H) for some $q > 0$, and thus (iii) of Lemma 2.6.8 is equivalent to the same condition but on $U^q\mu$ for all $q \geq 0$.

Stable Processes

Now we can say a little more about stable processes on $\mathbb{R}^d$. Stable processes are quite neatly divided into three very distinct groups, each having common properties. First let us compare those with index $\alpha < d$ and $\alpha > d$, which for processes on $\mathbb{R}$ corresponds to $\alpha \in (0,1)$ and $\alpha \in (1,2]$.

$\boldsymbol{\alpha < d}$	$\boldsymbol{\alpha > d}$
(i) *Transient.*	**(i)** *Point recurrent.*
(ii) *Points $\{x\}$ are polar.*	**(ii)** *Hits points.*
(iii) *No local times.*	**(iii)** *Has jointly cts. local times $L(x,t)$.*

The slightly awkward middle child is $\boldsymbol{\alpha = d}$, which on $\mathbb{R}^2$ is the Brownian motion. In that case X is set recurrent, doesn't hit points, and doesn't have local times.

An isotropic stable process of index $\alpha < d$ on $\mathbb{R}^d$ has potential density

$$u(x) = \mathrm{const}|x|^{\alpha-d}, \tag{2.17}$$

see Example 37.19(ii) of Sato [39]. In potential theory functions of this form are called Riesz potentials, and a discussion of Riesz potentials in that setting can be found in Bliedtner and Hansen [4] §V.4. One final result worth mentioning is the isoperimetric inequality, which relates capacities of compact sets to capacities of balls.

Lemma 2.6.9 (Isoperimetric inequality, Betsakos [3] Theorem 1)**.** *Let X be a symmetric stable process of index $\alpha \in (0,2)$, and $K \subseteq \mathbb{R}^d$ be a compact set with Lebesgue measure $\lambda(K)$. If $\mathbf{B}$ is a ball in $\mathbb{R}^d$ with $\lambda(\mathbf{B}) = \lambda(K)$ then*

$$C(K) \geq C(\mathbf{B}).$$

Let X be a stable process on $\mathbb{R}^d$. Example 42.17 of Sato [39] gives that if $\alpha < d$ and X is genuinely d-dimensional then for any ball $\mathbf{B} \subseteq \mathbb{R}^d$ and any $a > 0$,

$$C(a\mathbf{B}) = a^{d-\alpha} C(\mathbf{B}). \tag{2.18}$$

In particular, if $K \subseteq \mathbb{R}^d$ is compact and we denote by $\mathbf{B}_r$ the ball about 0 of radius r, then it follows from the isoperimetric inequality and the formula for the volume of a d-dimensional sphere that

$$C(K) \geq \Big(\frac{\lambda(K)\Gamma(\frac{d}{2}+1)}{\pi^{\frac{d}{2}}}\Big)^{\frac{d-\alpha}{d}} C(\mathbf{B}_1),$$

which in 1-dimension is the far nicer

$$C(K) \geq \Big(\frac{\lambda(K)}{2}\Big)^{1-\alpha} C(\mathbf{B}_1). \tag{2.19}$$

In Example V.4.16(2), Bliedtner and Hansen [4] calculate that for $\alpha \in (0,2)$ and $\alpha < d$,

$$C(\mathbf{B}_1) = \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{\alpha}{2})\Gamma((\frac{d-\alpha}{2})+1)}. \tag{2.20}$$

This also holds for the Brownian motion when $d \geq 3$, see Sato [39] Exercise 44.12.

As a final point, Lemma 2.1.12 yields that for any set $B \in \mathcal{B}(\mathbb{R}^d)$ with an equilibrium measure m_B there exists an increasing sequence of compact subsets K_n of B such that

$$\hat{U}m_{K_n}(x) = \mathbb{P}_x(T_{K_n} < \infty) \uparrow \mathbb{P}_x(T_B < \infty) = \hat{U}m_B(x), \qquad x \in \mathbb{R}^d.$$

Suppose now that X is stable with index $\alpha < d$, and therefore transient. It follows from Lemma 2.6.1 - applied with $q = 0$, which is valid because X satisfies (ACT) and thus condition (C2) - that m_{K_n} converges weakly to m_B, and therefore that $C(K_n) \uparrow C(B)$. In particular, this shows that Lemma 2.6.9 and (2.19) hold for all bounded $B \in \mathcal{B}(\mathbb{R})$ in place of compact K.

3 Perpetual Integral Tests

The *perpetual integral* over a Markov process X on state space $(E, \mathcal{E})$ and nonnegative $\mathcal{E}_\Delta$-measurable function $f : E_\Delta \to [0, \infty]$ is the infinite time integral

$$\int_0^\infty f(X_s)\,\mathrm{d}s.$$

Since the behaviour of a perpetual integral can depend on the process at every point of its trajectory, their study requires that we understand which parts of the state space can be reached, and how long the process is liable to stay there, which can be done via potentials.

In what follows we shall often make use of variations of the following argument, which relies on of the strong Markov property from Corollary 2.1.7. For T a stopping time and $x \in E$,

$$\begin{aligned}\mathbb{P}_x\Big(\int_T^\infty f(X_s)\,\mathrm{d}s < \infty;\ T < \infty\Big) &= \mathbb{E}_x\big[\mathbb{E}_x\big[\mathbf{1}_{(\int_T^\infty f(X_s)\,\mathrm{d}s<\infty)}\mathbf{1}_{(T<\infty)}|\mathcal{F}_T\big]\big] \\ &= \mathbb{E}_x\Big[\mathbb{P}_{X_T}\Big(\int_0^\infty f(X_s)\,\mathrm{d}s < \infty\Big);\ T < \infty\Big].\end{aligned}$$

Additionally we shall often make use of the simple fact that for $s \le t$,

$$\mathbb{P}_x\Big(\int_0^s f(X_u)\,\mathrm{d}u < \infty\Big) \ge \mathbb{P}_x\Big(\int_0^t f(X_u)\,\mathrm{d}u < \infty\Big).$$

From here onwards the notation $f \in \overline{\mathcal{E}}_+$ will refer to a function $f : E \to [0, \infty]$, $f \in \mathcal{E}$. Such functions are extended to $(E_\Delta, \mathcal{E}_\Delta)$ by setting $f(\Delta) = 0$, unless explicitly stated otherwise. This assumption is extremely common in Markov process literature, and one convenient side-effect when studying perpetual integrals is that

$$\int_0^\zeta f(X_s)\,\mathrm{d}s = \int_0^\infty f(X_s)\,\mathrm{d}s \qquad \text{almost surely.}$$

For finite-time path integrals we introduce the notation

$$I_t^f = \int_0^t f(X_s)\,\mathrm{d}s, \quad t \in [0, \infty), \qquad I_\infty^f := \lim_{t\to\infty} I_t^f = \int_0^\infty f(X_s)\,\mathrm{d}s. \tag{3.1}$$

We define the right-continuous inverse of $(I^f_t,\, t \geq 0)$ by

$$\begin{aligned}\varphi^f_t &:= \Big(\int_0^{\cdot} f(X_u)\,\mathrm{d}u \Big)^{-1}(t) = \inf \Big\{ s > 0 : \int_0^s f(X_u)\,\mathrm{d}u > t \Big\}, \quad t \in [0,\infty), \\ \varphi^f_\infty &:= \lim_{t\to\infty} \varphi^f_t.\end{aligned} \tag{3.2}$$

Each φ^f_t is a stopping time for X, because the filtration $(\mathcal{F}_s)$ is right-continuous and $\{\varphi^f_t \leq s\} \in \mathcal{F}_{s+\varepsilon}$ for each $s \geq 0$, $\varepsilon > 0$. When f is unambiguous it will be dropped from notation.

We will also be interested in the quantities

$$\bar{I}^f := \sup_{t\in[0,\infty):I^f_t<\infty} I^f_t, \qquad \bar{\varphi}^f := \sup_{t\in[0,\infty):\varphi^f_t<\infty} \varphi^f_t.$$

The connection between them is that

$$\varphi^f(\bar{I}^f) = \varphi^f_\infty \qquad \text{and} \qquad I^f(\varphi_\infty) = I^f(\bar{\varphi}^f) = \bar{I}^f. \tag{3.3}$$

The map $t \mapsto I_t$ is non-decreasing, left-continuous everywhere, continuous for $t \in [0,\varphi_\infty)$, and constant equal to I^f_∞ for $t > \varphi^f_\infty$, though that constant may be $+\infty$. The map $s \mapsto \varphi_s$ is right-continuous everywhere, and continuity of I yields that it is strictly increasing for $s \in [0,\bar{I}^f)$, and constant equal to φ^f_∞ afterwards, though that constant might be $+\infty$. But I is not necessarily strictly increasing, and therefore φ is not continuous. All of these hold for every $\omega \in \Omega$, rather than almost surely, because they come from properties of the Lebesgue integral.

We shall be interested in the time-changed process $(Y_t,\, t \in [0,\infty])$ given by

$$Y_t = X_{\varphi^f_t} \text{ for } t \in [0,\infty), \qquad Y_\infty = \Delta, \tag{3.4}$$

which moves on the same state space E as X. It will often be convenient to use notation $X_\varphi = Y$ for the time-changed process.

It is not immediately clear how many of the properties of X are inherited by X_φ, for example whether it is even Markovian. In 1958 Volkonskii [41] proved that X_φ is a strong Markov process if $\varphi_t(\omega)$ is non-decreasing and right-continuous for every $\omega \in \Omega$ and

$$\varphi_{\tau+t} - \varphi_\tau = \varphi_t \circ \theta_{\varphi_\tau} \tag{3.5}$$

for all $t > 0$ and all stopping times τ. We have just seen that φ is non-decreasing and right-continuous. We can also verify (3.5), but only under the condition that almost surely,

$$t \mapsto I_t \text{ is continuous on the whole of } [0,\infty), \tag{3.6}$$

which is equivalent to assuming $\bar{I} = I_\infty$. It makes sense that this condition is needed, since without it X_φ can become 'stuck' at a point forever - corresponding to I jumping to infinity - even if it already visited that point previously, and that behaviour is incompatible with the strong Markov property. For fixed $\omega \in \Omega$ satisfying (3.6),

$$\begin{aligned}\varphi_{\tau+t}(\omega) &= \inf\Big\{s > 0 : \int_0^s f(\omega_u)\,\mathrm{d}u > \tau(\omega) + t\Big\}\\ &= \inf\Big\{s > \varphi_\tau(\omega) : \int_0^s f(\omega_u)\,\mathrm{d}u > \tau(\omega) + t\Big\}.\end{aligned}$$

By continuity of I, we have $\tau(\omega) = I(\varphi_\tau(\omega))$, and

$$\begin{aligned}&= \inf\Big\{s > \varphi_\tau(\omega) : \int_0^s f(\omega_u)\,\mathrm{d}u - \int_0^{\varphi_\tau(\omega)} f(\omega_u)\,\mathrm{d}u > t\Big\}\\ &= \inf\Big\{s > \varphi_\tau(\omega) : \int_{\varphi_\tau(\omega)}^s f(\omega_u)\,\mathrm{d}u > t\Big\}\\ &= \inf\Big\{s > \varphi_\tau(\omega) : \int_0^{s-\varphi_\tau(\omega)} f(\tilde{\omega}_r)\,\mathrm{d}r > t\Big\}\end{aligned}$$

where $\tilde{\omega} = \theta_{\varphi_\tau(\omega)} \circ \omega$,

$$= \inf\Big\{s > 0 : \int_0^s f(\tilde{\omega}_r)\,\mathrm{d}r > t\Big\} + \varphi_\tau(\omega) = \varphi_t \circ \theta_{\varphi_\tau(\omega)}(\omega) + \varphi_\tau(\omega).$$

So (3.5) holds for every $\omega \in \Omega$ satisfying (3.6). Thus if (3.6) holds for every ω then X_φ is a strong Markov process, and weakening that to (3.6) holding only almost surely doesn't sacrifice the strong Markov property.

Moreover, assuming (3.6) almost surely, we can relate perpetual integrals over X_φ and X. First let $f \in \overline{\mathcal{E}}_+$, $\sigma \in b\mathcal{E}$ and fix a time $u \in [0, \bar{I}^f)$. Then we can use the substitution $t = \varphi^f_s$, which implies $I^f_t = s$ for all $s \in [0, u]$, to see

$$\int_0^u \sigma(X_{\varphi^f_s})\,\mathrm{d}s = \int_0^{\varphi^f_u} \sigma(X_t) f(X_t)\,\mathrm{d}t.$$

On the other hand, for any $s \in [\bar{I}^f, \infty)$ it holds that $\varphi^f_s = \infty$,[1] and therefore that $\sigma(X(\varphi^f_s)) = \sigma(\Delta) = 0$. So for $u \in [\bar{I}^f, \infty)$ we see

$$\int_0^u \sigma(X_{\varphi^f_s})\,\mathrm{d}s = \int_0^{\bar{I}^f} \sigma(X_{\varphi^f_s})\,\mathrm{d}s = \int_0^{\varphi^f_\infty} \sigma(X_t) f(X_t)\,\mathrm{d}t,$$

So either way taking limits as $u \to \infty$ yields

$$\int_0^\infty \sigma(X_{\varphi^f_s})\,\mathrm{d}s = \int_0^{\varphi^f_\infty} \sigma(X_t) f(X_t)\,\mathrm{d}t \quad \text{almost surely.} \tag{3.7}$$

[1]This is only true under (3.6). If I^f can jump to ∞ then φ^f could be a finite constant after time $\bar{I}^f$.

A sufficient condition for (3.6) to hold almost surely, though not necessary, is that

$$I_t < \infty \text{ for all } t \in [0,\infty) \text{ almost surely.} \tag{3.8}$$

This implies (indeed is equivalent to) $\varphi^f_\infty = \infty$ almost surely, and from (3.7) we get

$$\int_0^\infty \sigma(X_{\varphi^f_s})\,\mathrm{d}s = \int_0^\infty \sigma(X_t) f(X_t)\,\mathrm{d}t \quad \text{almost surely.} \tag{3.9}$$

This in particular will be of use to us later in this chapter.

3.1 Main Result

Döring and Kyprianou [11] proved that for X a transient Lévy process on $\mathbb{R}$, $f \in \mathcal{B}(\mathbb{R})$ positive and locally-integrable, and $x \in \mathbb{R}$,

$$\mathbb{P}_x\Big(\int_0^\infty f(X_s)\,\mathrm{d}s < \infty\Big) = 1 \qquad \Longleftrightarrow \qquad Uf(x) = \mathbb{E}_x\Big[\int_0^\infty f(X_s)\,\mathrm{d}s\Big] < \infty$$

and $\mathbb{P}_x\big(\int_0^\infty f(X_s)\,\mathrm{d}s < \infty\big) = 0$ otherwise. This result was extended by Kolb and Savov [30], who found that for $f \in \mathcal{B}(\mathbb{R})_+$ either continuous or ultimately non-increasing,

$$\begin{aligned}&\mathbb{P}_x\Big(\int_0^\infty f(X_s)\,\mathrm{d}s < \infty\Big) = 1\\ \Longleftrightarrow \quad &\text{there exists a } \mathbb{P}_x\text{-transient set } B \text{ such that } \int_{\mathbb{R}\setminus B} f(y)\,U(x,\mathrm{d}y) < \infty,\end{aligned}$$

and $\mathbb{P}_x\big(\int_0^\infty f(X_s)\,\mathrm{d}s < \infty\big) = 0$ otherwise. The central result of this section is the following theorem, which is in the same spirit as the two above but is more general.

Theorem 3.1.1. *Let X be a standard Markov process on state space E, and take $f \in \overline{\mathcal{E}}_+$. Then for fixed $x \in E$, the following are equivalent.*

(i) $\mathbb{P}_x\Big(\int_0^\infty f(X_s)\,\mathrm{d}s < \infty\Big) > 0;$

(ii) *There exists a $\mathbb{P}_x$-supportive set M such that*

$$\int_M f(y)\,U(x,\mathrm{d}y) < \infty.$$

The integral in **(ii)** is equal

$$\mathbb{E}_x\Big[\int_0^\infty f(X_s)\mathbf{1}_M(X_s)\,\mathrm{d}s\Big].$$

The proof of Theorem 3.1.1 is technical, and is laid out in detail in the following two sections. But the result itself is quite intuitive: the supportive set M, which with positive probability contains the entire path of the process, describes the 'safe points' of the state space E, in the sense that $x \in M$ are exactly the issuing points of the state space from which the perpetual integral does not attain value ∞ in finite time with positive probability. If X is recurrent then Definition 2.4.7 gives that Borel sets have either potential zero or infinity, and the integral test is trivial. Most applications of Theorem 3.1.1 in later sections will be for transient standard Markov processes, but it is worth remembering that there is no dichotomy for standard Markov processes, it is possible for them to be neither recurrent nor transient.

3.2 Super-Finite Sets

The key to proving Theorem 3.1.1 is to define the class of '*super-finite*' sets, which uniformly bound I_t^f in a particular way, and prove properties of those sets. We will then draw a connection between super-finite sets and supportive sets, which will be used to complete the proof. In all of this section X is a standard Markov process on state space $(E, \mathcal{E})$, and $f \in \overline{\mathcal{E}}_+$.

Definition 3.2.1. *A* super-finite *set for (X, f) is defined in relation to an $n \in \mathbb{N}$ and $c \in (0,1)$ by*

$$M = \Big\{y \in E : \mathbb{P}_y\Big(\int_0^\infty f(X_s)\,\mathrm{d}s \le n\Big) > c\Big\}.$$

The discussion around (2.1) allows that the pullback of (c, ∞) by $x \mapsto \mathbb{P}_x(I_\infty^f \le n)$ is a set in $\mathcal{E}_\Delta$, and it is trivial to remove the Δ point and see that $M \in \mathcal{E}$.

This first lemma shows that if there exists some super-finite set for (X, f) then it is possible to choose another $g \in \overline{\mathcal{E}}_+$ such that the whole state space E is super-finite for (X, g).

Lemma 3.2.2. *Suppose*

$$M = \Big\{y \in E : \mathbb{P}_y\Big(\int_0^\infty f(X_s)\,\mathrm{d}s \le n\Big) > c\Big\}$$

is a non-empty super-finite set for (X, f). Let $g = \mathbf{1}_M \cdot f$. Then

$$\mathbb{P}_y\Big(\int_0^\infty g(X_s)\,\mathrm{d}s \le 2n\Big) > c^2, \quad \forall y \in E.$$

That is, the entire state space E is super-finite for (X, g).

Proof. Since $g \le f$, it also holds that $I_t^g \le I_t^f$ for all $t \in [0, \infty]$. Then the result holds immediately for $y \in M$, since by definition of that set

$$\mathbb{P}_y(I_\infty^g \le 2n) \ge \mathbb{P}_y(I_\infty^f \le 2n) \ge \mathbb{P}_y(I_\infty^f \le n) > c > c^2.$$

We now prove the result for $y \in M^r$. Lemma 2.1.12 gives a nested increasing sequence of compact sets $K_m \subseteq M$, $m \in \mathbb{N}$, such that, $\mathbb{P}_y$-almost surely, $T_{K_m} \downarrow T_M$ as $m \to \infty$. Since $P_y(T_M = 0) = 1$ it then follows that

$$\begin{aligned}
&\mathbb{P}_y\Big(\int_0^{T_{K_1}} g(X_s)\,\mathrm{d}s \le n\Big) \\
&= \mathbb{P}_y\Big(\lim_{m\to\infty}\int_{T_{K_m}}^{T_{K_1}} g(X_s)\,\mathrm{d}s \le n\Big) \\
&= \mathbb{P}_y\Big(\bigcap_{m=1}^{\infty}\Big\{\int_{T_{K_m}}^{T_{K_1}} g(X_s)\,\mathrm{d}s \le n\Big\}\Big) \\
&= \lim_{m\to\infty}\mathbb{P}_y\Big(\int_{T_{K_m}}^{T_{K_1}} g(X_s)\,\mathrm{d}s \le n\Big) \\
&\ge \lim_{m\to\infty}\mathbb{P}_y\Big(\int_{T_{K_m}}^{T_{K_1}} g(X_s)\,\mathrm{d}s \le n;\, T_{K_m} < \infty\Big) \\
&= \lim_{m\to\infty}\int_{K_m}\mathbb{P}_a\Big(\int_0^{T_{K_1}} g(X_s)\,\mathrm{d}s \le n\Big)\mathbb{P}_y(X_{T_{K_m}} \in \mathrm{d}a;\, T_{K_m} < \infty) \\
&\ge \lim_{m\to\infty}\int_{K_m}\mathbb{P}_a\Big(\int_0^{\infty} f(X_s)\,\mathrm{d}s \le n\Big)\mathbb{P}_y(X_{T_{K_m}} \in \mathrm{d}a;\, T_{K_m} < \infty).
\end{aligned}$$

From Lemma 2.1.13 we have that $X_{T_{K_m}} \in K_m \subseteq M$ almost surely, and thus

$$\begin{aligned}
&> c \cdot \lim_{m\to\infty}\mathbb{P}_y(T_{K_m} < \infty) \\
&= c\,\mathbb{P}_y(T_M < \infty) \\
&= c.
\end{aligned}$$

In addition, since T_{K_m} converges to $T_M = 0$ $\mathbb{P}_y$-almost surely,

$$\mathbb{P}_y(A; T_{K_m} < \infty) \uparrow \mathbb{P}_y(A)$$

for any $A \in \mathcal{F}$. This, in combination with the inequality above, gives that there exists a choice of $j \in \mathbb{N}$ such that

$$\mathbb{P}_y\Big(\int_0^{T_{K_1}} g(X_s)\,\mathrm{d}s \le n;\, T_{K_j} < \infty\Big) > c.$$

Monotonicity of the sequence of sets $(K_m,\, m \in \mathbb{N})$ yields $T_{K_1} \ge T_{K_j}$, and thus

$$\mathbb{P}_y\Big(\int_0^{T_{K_j}} g(X_s)\,\mathrm{d}s \le n;\, T_{K_j} < \infty\Big) > c. \tag{3.10}$$

Fix this j. Next, using the strong Markov property, we get

$$\mathbb{P}_y\Big(\int_0^{\infty} g(X_s)\,\mathrm{d}s \le 2n;\, T_{K_j} < \infty\Big)$$

$$\geq \mathbb{P}_y\Big(\int_0^{T_{K_j}} g(X_s)\,\mathrm{d}s \leq n;\ \int_{T_{K_j}}^\infty g(X_s)\,\mathrm{d}s \leq n;\ T_{K_j} < \infty\Big)$$
$$= \mathbb{E}_y\Big[\mathbf{1}_{\big(\int_0^{T_{K_j}} g(X_s)\,\mathrm{d}s \leq n\big)}\mathbf{1}_{(T_{K_j}<\infty)}\mathbb{E}_y\Big[\mathbf{1}_{\big(\int_{T_{K_j}}^\infty g(X_s)\,\mathrm{d}s\leq n\big)}\Big|\mathcal{F}_{T_{K_j}}\Big]\Big]$$
$$= \mathbb{E}_y\Big[\mathbf{1}_{\big(\int_0^{T_{K_j}} g(X_s)\,\mathrm{d}s \leq n\big)}\mathbf{1}_{(T_{K_j}<\infty)}\mathbb{P}_{X_{T_{K_j}}}\Big(\int_0^\infty g(X_s)\,\mathrm{d}s \leq n\Big)\Big],$$

and now since $I_\infty^g \leq I_\infty^f$,

$$\geq \mathbb{E}_y\Big[\mathbf{1}_{\big(\int_0^{T_{K_j}} g(X_s)\,\mathrm{d}s \leq n\big)}\mathbf{1}_{(T_{K_j}<\infty)}\mathbb{P}_{X_{T_{K_j}}}\Big(\int_0^\infty f(X_s)\,\mathrm{d}s \leq n\Big)\Big].$$

Since $K_j \subseteq M$ the inner probability is bounded below by c. This holds even in the extreme case $K_j = M$, because then M is compact and $X_{T_{K_j}} \in M$ almost surely. The remaining expectation can also be bounded from below by c using (3.10). In total this leads to

$$\mathbb{P}_y\Big(\int_0^\infty g(X_s)\,\mathrm{d}s \leq 2n\Big) \geq \mathbb{P}_y\Big(\int_0^\infty g(X_s)\,\mathrm{d}s \leq 2n;\ T_{K_j} < \infty\Big) > c^2.$$

And so the lemma is proven for $y \in M^r$. What now remains is to extend it to all $y \in E$. In this case,

$$\begin{aligned}
&\mathbb{P}_y\Big(\int_0^\infty g(X_s)\,\mathrm{d}s \leq 2n\Big)\\
&\quad= \mathbb{P}_y\Big(\int_0^\infty g(X_s)\,\mathrm{d}s \leq 2n;\ T_M < \infty\Big) + \mathbb{P}_y\Big(\int_0^\infty g(X_s)\,\mathrm{d}s \leq 2n;\ T_M = \infty\Big)\\
&\quad= \mathbb{P}_y\Big(\int_{T_M}^\infty g(X_s)\,\mathrm{d}s \leq 2n;\ T_M < \infty\Big) + \mathbb{P}_y(T_M = \infty),
\end{aligned}$$

equality because g has support in M,

$$= \int_E \mathbb{P}_a\Big(\int_0^\infty g(X_s)\,\mathrm{d}s \leq 2n\Big)\mathbb{P}_y(X_{T_M} \in \mathrm{d}a;\ T_M < \infty) + \mathbb{P}_y(T_M = \infty).$$

Lemma 2.1.13 tells us that the integrating measure is concentrated on $M \cup M^r$, and thus we can use what we have already proven for elements of $M \cup M^r$ to conclude that

$$\begin{aligned}
\mathbb{P}_y\Big(\int_0^\infty g(X_s)\,\mathrm{d}s \leq 2n\Big) &> c^2\mathbb{P}_y(T_M < \infty) + \mathbb{P}_y(T_M = \infty)\\
&\geq c^2\big(\mathbb{P}_y(T_M < \infty) + \mathbb{P}_y(T_M = \infty)\big) = c^2.
\end{aligned}$$

□

The next lemma prepares for the proposition that follows it. It shows that restricting f to a super-finite set ensures that the path integral does not hit ∞ in finite time, which is exactly condition (3.8). Its use in the next proposition will be related to the time-changed process.

Lemma 3.2.3. *Suppose*

$$M = \Big\{ y \in E : \mathbb{P}_y\Big(\int_0^\infty f(X_s)\,\mathrm{d}s \le n \Big) > c \Big\}$$

is a non-empty super-finite set for (X, f)*. Let* $g = \mathbf{1}_M \cdot f$*. Then*

$$\mathbb{P}_y\Big(\exists\, t \in (0,\infty) : \int_0^t g(X_s)\,\mathrm{d}s = \infty \Big) = 0, \quad \forall y \in E.$$

Proof. Fix $y \in E$ and recall the stopping times

$$\varphi_n^g = \inf\Big\{ t > 0 : I_t^g > n \Big\},\, n \in \mathbb{N}, \qquad \varphi_\infty^g = \lim_{n\to\infty} \varphi_n^g.$$

Since

$$\varphi_n^g < \infty \quad \Leftrightarrow \quad \int_0^\infty g(X_s)\,\mathrm{d}s > n \quad \Leftrightarrow \quad \exists\, t \in (0,\infty) : \int_0^t g(X_s)\,\mathrm{d}s > n, \tag{3.11}$$

we obtain from Lemma 3.2.2 that for all $y \in E$,

$$\mathbb{P}_y(\varphi_{2n} < \infty) \le 1 - c^2. \tag{3.12}$$

We then see for $k, n \in \mathbb{N}$ that

$$\begin{aligned}
\mathbb{P}_y(\varphi_{2kn}^g < \infty) &= \mathbb{P}_y\Big(\exists t > 0 : \int_0^t g(X_s)\,\mathrm{d}s > 2kn \Big) \\
&= \mathbb{P}_y\Big(\varphi_{2n(k-1)}^g < \infty\,;\, \exists t' > 0 : \int_{\varphi_{2n(k-1)}^g}^{\varphi_{2n(k-1)}^g + t'} g(X_s)\,\mathrm{d}s > 2n \Big) \\
&= \mathbb{E}_y\Big[\mathbf{1}_{(\varphi_{2n(k-1)}^g < \infty)} \mathbb{E}_y\Big[\mathbf{1}_{\left(\exists t' > 0 : \int_{\varphi_{2n(k-1)}^g}^{\varphi_{2n(k-1)}^g + t'} g(X_s)\,\mathrm{d}s > 2n\right)} \Big| \mathcal{F}_{\varphi_{2n(k-1)}^g} \Big]\Big] \\
&= \mathbb{E}_y\Big[\mathbf{1}_{(\varphi_{2n(k-1)}^g < \infty)} \mathbb{P}_{X_{\varphi_{2n(k-1)}^g}} \Big(\exists t' > 0 : \int_0^{t'} g(X_s)\,\mathrm{d}s > 2n \Big)\Big] \\
&= \mathbb{E}_y\Big[\mathbf{1}_{(\varphi_{2n(k-1)}^g < \infty)} \mathbb{P}_{X_{\varphi_{2n(k-1)}^g}} \Big(\varphi_{2n}^g < \infty \Big)\Big],
\end{aligned}$$

and by (3.12),

$$\begin{aligned}
&\le (1 - c^2)\mathbb{P}_y(\varphi_{2n(k-1)}^g < \infty) \\
&\qquad \vdots \\
&\le (1 - c^2)^k \to 0, \quad \text{as } k \to \infty.
\end{aligned}$$

Continuity of measures yields $\mathbb{P}_y(\varphi_\infty^g < \infty) = 0$, which via (3.11) implies the claim. $\square$

The next proposition is partially motivated by ideas from Getoor [18], in particular the proof of Lemma (3.1), used in a different fashion. It proves that super-finite sets for (X, f) have finite f-potential.

Proposition 3.2.4. *Suppose for $n \in \mathbb{N}$, $p \in (0,1)$ that*

$$M = \Big\{ y \in E : \mathbb{P}_y\Big(\int_0^\infty f(X_s)\,\mathrm{d}s \leq \frac{n}{2} \Big) > p \Big\}$$

is a non-empty super-finite set for (X, f). Then

$$\int_M f(x) U(y, \mathrm{d}x) \leq \frac{n}{p^2} \qquad \textit{for all } y \in E.$$

Proof. With $g = \mathbf{1}_M \cdot f$ we introduce the time-changed process $Y_t = X_{\varphi_t^g}$ from (3.4), where

$$\varphi_t^g = \inf \Big\{ s > 0 : \int_0^s g(X_u)\,\mathrm{d}u > t \Big\}, \quad t \in [0, \infty).$$

Lemma 3.2.3 tells us that (3.8) and thus (3.6) are fulfilled almost surely and that therefore, as per the discussion at the start of the chapter concerning the work of Volkonskii [41], Y is a strong Markov process. For this proof we will use the notation

$$h_n(x) = \mathbb{P}_x(I_\infty^g \leq n), \quad x \in E, n \in \mathbb{N}. \tag{3.13}$$

By Lemma 3.2.2 h_n is bounded below by p^2 on E.

We denote by (P_t) the transition operator of Y and by U_Y the corresponding potential operator, to distinguish it from U_X the potential operator of X. For each n the function h_n from (3.13) is extended to a function h on E_Δ by $h = h_n$ on E and $h(\Delta) = 0$. We then see that

$$\begin{aligned}
P_n \mathbf{1}_E(x) &= P_n(x, E) \\
&= \mathbb{P}_x(Y_n \in E) \\
&= \mathbb{P}_x(X_{\varphi_n} \in E) \\
&= \mathbb{P}_x(\varphi_n < \zeta_X) \\
&= \mathbb{P}_x\Big(\exists\, s < \zeta_X \text{ such that } \int_0^s g(X_u)\,\mathrm{d}u > n \Big) \\
&\leq \mathbb{P}_x\Big(\int_0^{\zeta_X} g(X_u)\,\mathrm{d}u > n \Big) \\
&= \mathbf{1}_E(x) - h(x)
\end{aligned}$$

for all $x \in E_\Delta$. The main part of the proof is showing that $U_Y h$ is bounded above, which coincidentally implies Y is weakly transient. Using the definition of the potential, the form of $P_n \mathbf{1}_E$ above, and the semigroup property yields

$$
\begin{aligned}
U_Y h(x) &= \lim_{t\to\infty} \int_0^t P_s h(x)\,\mathrm{d}s \\
&\le \lim_{t\to\infty} \int_0^t \big(P_s \mathbf{1}_E(x) - P_s P_n \mathbf{1}_E(x)\big)\,\mathrm{d}s \\
&= \lim_{t\to\infty} \Big(\int_0^t P_s \mathbf{1}_E(x)\,\mathrm{d}s - \int_0^t P_{s+n} \mathbf{1}_E(x)\,\mathrm{d}s\Big) \\
&= \lim_{t\to\infty} \Big(\int_0^t P_s \mathbf{1}_E(x)\,\mathrm{d}s - \int_n^{t+n} P_s \mathbf{1}_E(x)\,\mathrm{d}s\Big) \\
&= \lim_{t\to\infty} \Big(\int_0^n P_s \mathbf{1}_E(x)\,\mathrm{d}s - \int_t^{t+n} P_s \mathbf{1}_E(x)\,\mathrm{d}s\Big) \\
&\le \int_0^n P_s \mathbf{1}_E(x)\,\mathrm{d}s \\
&\le n,
\end{aligned}
\tag{3.14}
$$

for all $x \in E_\Delta$. It has already been noted that h is bounded below by $p^2 > 0$ on E. Then by monotonicity of the potential operator,

$$
U_Y \mathbf{1}_E(y) \le \frac{1}{p^2} U_Y h(y) \quad \text{for all } y \in E. \tag{3.15}
$$

Combining (3.14) and (3.15) implies

$$
U_Y \mathbf{1}_E(y) \le \frac{1}{p^2} U_Y h(y) \le \frac{n}{p^2} < \infty \quad \text{for all } y \in E.
$$

What is left to show is $\int_M f(x) U_X(y, \mathrm{d}x) = \int_E g(x) U_X(y, \mathrm{d}x) = U_Y \mathbf{1}_E(y)$. This follows directly from change of variables: for bounded measurable $\sigma \in b\mathcal{E}$,

$$
\begin{aligned}
U_Y \sigma(y) &= \mathbb{E}_y \Big[\int_0^\infty \sigma(Y_s)\,\mathrm{d}s \Big] \\
&= \mathbb{E}_y \Big[\int_0^\infty \sigma(X_{\varphi_s^g})\,\mathrm{d}s \Big],
\end{aligned}
$$

and we saw in (3.9) that a substitution here of $t = \varphi_s^g$ yields

$$
\begin{aligned}
&= \mathbb{E}_y \Big[\int_0^\infty \sigma(X_t) g(X_t)\,\mathrm{d}t \Big] \\
&= \int_M \sigma(x) f(x) U_X(y, \mathrm{d}x).
\end{aligned}
$$

Thus in particular

$$
\int_M f(x) U_X(y, \mathrm{d}x) = U_Y \mathbf{1}_E(y) < \frac{n}{p^2} \quad \text{for all } y \in E.
$$

□

Lemma 3.2.4 is a strong result for super-finite sets. All that remains to be done in order to prove Theorem 3.1.1 is to prove that super-finite sets are also supportive. Before doing so, another technical lemma is needed.

Lemma 3.2.5. *For $n \in \mathbb{N}, c \in (0,1)$, the set*

$$B = \Big\{ x \in E : \mathbb{P}_x\Big(\int_0^\infty f(X_s)\,\mathrm{d}s \le n \Big) \le c \Big\}$$

contains its regular points, and is therefore also finely closed.

Proof. The definition of B is exactly that of the complement in E of some super-finite set for (X, f). Let $K \subseteq B$ be a compact set, so that, by right-continuity of paths, $X_{T_K} \in K$ almost surely. Then for any $a \in E$ we obtain

$$\begin{aligned}
\mathbb{P}_a\big(I^f_\infty \le n\big) &= \mathbb{P}_a\big(I^f_\infty \le n;\, T_K < \zeta\big) + \mathbb{P}_a\big(I^f_\infty \le n;\, T_K \ge \zeta\big) \\
&\le \mathbb{P}_a\Big(\int_{T_K}^\infty f(X_s)\,\mathrm{d}s \le n;\, T_K < \zeta \Big) + \mathbb{P}_a\big(I^f_\infty \le n;\, T_K \ge \zeta\big) \\
&= \int \mathbb{P}_y\big(I^f_\infty \le n\big)\mathbb{P}_a(X_{T_K} \in \mathrm{d}y;\, T_K < \zeta) + \mathbb{P}_a\big(I^f_\infty \le n;\, T_K \ge \zeta\big) \\
&\le c + \mathbb{P}_a(T_K \ge \zeta),
\end{aligned} \tag{3.16}$$

as $K \subseteq B$. Since (3.16) is true for all compact sets $K \subseteq B$, it then holds for all $a \in E$ that

$$\mathbb{P}_a\Big(\int_0^\infty f(X_s)\,\mathrm{d}s \le n \Big) \le c + \inf_{K \subseteq B} \mathbb{P}_a(T_K \ge \zeta). \tag{3.17}$$

We now argue that the second summand of the righthand side equals 0 if a is regular for B. From Lemma 2.1.12 it follows that there exists an increasing sequence (K_n) of compact subsets of B such that $\mathbb{P}_a(T_{K_n} \downarrow T_B) = 1$. The fact that the stopping times T_{K_n} are decreasing as $n \to \infty$ yields

$$\inf_{n \in \mathbb{N}} \mathbb{P}_a(T_{K_n} \ge \zeta) \le \limsup_{n \to \infty} \mathbb{E}_a\big[\mathbf{1}_{(T_{K_n} \ge \zeta)}\big],$$

and hence, we can apply the reverse Fatou lemma to see that for the regular points $a \in B^r$,

$$\begin{aligned}
\inf_{n \in \mathbb{N}} \mathbb{P}_a(T_{K_n} \ge \zeta) &\le \mathbb{E}_a\big[\limsup_{n \to \infty} \mathbf{1}_{(T_{K_n} \ge \zeta)}\big] \\
&= \mathbb{P}_a\big(\limsup_{n \to \infty} T_{K_n} \ge \zeta\big) \\
&= \mathbb{P}_a(T_B \ge \zeta) = 0
\end{aligned}$$

But then (3.17) implies $a \in B$. Thus $B^r \subseteq B$. Equivalence of B being finely closed and containing its regular points is Exercise II(4.9) of Blumenthal and Getoor [5]. $\square$

We have built enough structure around super-finite sets to now prove the final proposition of this section, which establishes that super-finite sets are supportive when the process is issued from them.

Proposition 3.2.6. *For $n \in \mathbb{N}$, $c \in (0,1)$ the super-finite set*

$$M_{n,c} = \Big\{ y \in E : \mathbb{P}_y\Big(\int_0^\infty f(X_s)\,\mathrm{d}s \le n \Big) > c \Big\}$$

is $\mathbb{P}_z$-supportive if and only if $z \in M_{n,c}$. In particular, if for some fixed $z \in E$ it holds that

$$\mathbb{P}_z\Big(\int_0^\infty f(X_s)\,\mathrm{d}s < \infty \Big) > 0,$$

then there exists a super-finite set for (X, f) which is also $\mathbb{P}_z$-supportive.

Proof. For ease of notation let

$$h_n(x) = \mathbb{P}_x\Big(\int_0^\infty f(X_s)\,\mathrm{d}s \le n \Big)$$

and define

$$B_{n,c} = \{x \in E : h_n(x) \le c\} = E \setminus M_{n,c}, \quad n \in \mathbb{N}, c \in (0,1)$$

as in Lemma 3.2.5.

Necessity of $z \in M_{n,c}$ is clear, and we shall prove sufficiency by proving the contrapositive. Let us suppose for the moment that $M_{n,c}$ is not $\mathbb{P}_z$-supportive for some fixed choice of n, c, that is,

$$\mathbb{P}_z(D_{B_{n,c}} < \zeta) = \mathbb{P}_z(D_{E \setminus M_{n,c}} < \zeta) = 1.$$

From this we obtain that

$$\begin{aligned} h_n(z) &= \mathbb{P}_z\Big(\int_0^\infty f(X_s)\,\mathrm{d}s \le n \Big) \\ &= \mathbb{P}_z\Big(\int_0^\infty f(X_s)\,\mathrm{d}s \le n;\, D_{B_{n,c}} < \zeta \Big) \\ &\le \mathbb{P}_z\Big(\int_{D_{B_{n,c}}}^\infty f(X_s)\,\mathrm{d}s \le n;\, D_{B_{n,c}} < \zeta \Big) \\ &= \int_E \mathbb{P}_a\Big(\int_0^\infty f(X_s)\,\mathrm{d}s \le n \Big) \mathbb{P}_z(X_{D_{B_{n,c}}} \in \mathrm{d}a;\, D_{B_{n,c}} < \zeta). \end{aligned} \tag{3.18}$$

Since, due to Lemma 3.2.5, the regular points for $B_{n,c}$ belong to $B_{n,c}$, Lemma 2.1.13 tells us that $\mathbb{P}_y(X_{D_{B_{n,c}}} \in \mathrm{d}a;\, D_{B_{n,c}} < \zeta)$ is concentrated on $B_{n,c}$. We can now return to (3.18) and deduce from the definition of $B_{n,c}$ that

$$h_n(z) \le \int_E \mathbb{P}_a\Big(\int_0^\infty f(X_s)\,\mathrm{d}s \le n \Big) \mathbb{P}_z(X_{D_{B_{n,c}}} \in \mathrm{d}a;\, D_{B_{n,c}} < \zeta) \le c.$$

To recap, we have proven that

$$M_{n,c} \text{ is not } \mathbb{P}_z\text{-supportive} \quad \Rightarrow \quad h_n(z) \le c \quad \Leftrightarrow \quad z \notin M_{n,c},$$

or, equivalently,

$$z \in M_{n,c} \quad \Leftrightarrow \quad h_n(z) > c \quad \Rightarrow \quad M_{n,c} \text{ is } \mathbb{P}_z\text{-supportive}. \tag{3.19}$$

Lastly by continuity of measures $\mathbb{P}_z(\int_0^\infty f(X_s)\,\mathrm{d}s < \infty) > 0$ implies

$$h_{n_0}(z) = \mathbb{P}_z\Big(\int_0^\infty f(X_s)\,\mathrm{d}s \le n_0\Big) > 0$$

for some $n_0 \in \mathbb{N}$. Hence in this case (3.19) implies the existence of some c_0 so that M_{n_0,c_0} is $\mathbb{P}_z$-supportive. □

An Aside

In Theorem II(4.5) of their chapter on the fine topology Blumenthal and Getoor [5] prove that for $q > 0$, the fine topology is the coarsest topology on E which makes all q-excessive functions continuous. Lemma 3.2.5 tells us that super-finite sets are finely open, and it therefore holds that any super-finite set

$$M = \Big\{x \in E : \mathbb{P}_x\Big(\int_0^\infty f(X_s)\,\mathrm{d}s \le n\Big) > c\Big\}.$$

is the pullback of an open set in $[0,\infty)$ by some q-excessive function. By definition of lower-semicontinuity if the map

$$x \mapsto \mathbb{P}_x\Big(\int_0^\infty f(X_s)\,\mathrm{d}s \le n\Big)$$

is lower-semicontinuous then M is open, but it is not clear in general what assumptions on X or f ensure that.

3.3 Proof of Theorem 3.1.1

Perpetual integral proofs typically have a simple and a complicated direction. The simple direction "$\Leftarrow$" uses the potential expression to deduce finiteness of the expectation. In our setting the argument goes as follows. Suppose that M is a $\mathbb{P}_z$-supportive set, and that

$$\mathbb{E}_z\Big[\int_0^\infty \mathbf{1}_M(X_s) f(X_s)\,\mathrm{d}s\Big] = \int_M f(x) U(z,\mathrm{d}x) < \infty.$$

Then $\mathbb{P}_z(\int_0^\infty \mathbf{1}_M(X_s) f(X_s)\,\mathrm{d}s < \infty) = 1$, and thus

$$\mathbb{P}_z\Big(\int_0^\infty f(X_s)\,\mathrm{d}s < \infty\Big) \ge \mathbb{P}_z\Big(\int_0^\infty f(X_s)\,\mathrm{d}s < \infty; X_s \in M\,\forall s < \zeta\Big)$$

$$= \mathbb{P}_z\Big(\int_0^\infty \mathbf{1}_M(X_s)f(X_s)\,\mathrm{d}s < \infty; X_s \in M\,\forall s < \zeta\Big)$$
$$= \mathbb{P}_z(X_s \in M \text{ for all } s < \zeta) > 0.$$

This shows the "$\Leftarrow$" direction of Theorem 3.1.1.

To prove the "$\Rightarrow$" direction we take the super-finite supportive set from Proposition 3.2.6 and obtain the integral test from Proposition 3.2.4. □

3.4 The Almost Sure Case

Without any additional assumptions we can strengthen the integral test in Theorem 3.1.1 in the case that $\int_0^\infty f(X_s)\,\mathrm{d}s$ is almost surely finite, rather than simply finite with positive probability. This is crucially *not* equivalent to the condition that

$$\int_E f(x)\,U(z,\mathrm{d}x) < \infty. \tag{3.20}$$

An example demonstrating this is given below the theorem proof.

Theorem 3.4.1. *Let X be a standard Markov process on state space E and $f \in \overline{\mathcal{E}}_+$. Then the following are equivalent.*

(i) $\mathbb{P}_z\Big(\int_0^\infty f(X_s)\,\mathrm{d}s < \infty\Big) = 1$*;*

(ii) *For every small $\varepsilon > 0$ there exists $M \in \mathcal{E}$ such that $\mathbb{P}_z(D_{E\setminus M} < \infty) \le \varepsilon$ - which implies that M is $\mathbb{P}_z$-supportive - and*

$$\int_M f(x)\,U(z,\mathrm{d}x) < \infty.$$

Proof.

[**(i)**$\Rightarrow$**(ii)**]

Assuming **(i)** it follows that for every $\varepsilon \in (0,1/2)$ there exists an $n \in \mathbb{N}$ such that

$$\mathbb{P}_z\Big(\int_0^\infty f(X_s)\,\mathrm{d}s \le n\Big) > 1-\varepsilon. \tag{3.21}$$

Fix these n, ε and define the super-finite set

$$M = \Big\{y \in E : \mathbb{P}_y\Big(\int_0^\infty f(X_s)\,\mathrm{d}s \le n\Big) > \varepsilon\Big\},$$

which by (3.21) contains z, and thus by Lemma 3.2.6 is $\mathbb{P}_z$-supportive. Moreover,

$$\mathbb{P}_z\Big(\int_0^\infty f(X_s)\,\mathrm{d}s \le n\Big)$$

$$
\begin{aligned}
&= \mathbb{P}_z\Big(\int_0^\infty f(X_s)\,\mathrm{d}s \le n;\ D_{E\setminus M} = \infty \Big) + \mathbb{P}_z\Big(\int_0^\infty f(X_s)\,\mathrm{d}s \le n;\ D_{E\setminus M} < \infty \Big) \\
&\le \mathbb{P}_z(D_{E\setminus M} = \infty) + \mathbb{P}_z\Big(\int_{D_{E\setminus M}}^\infty f(X_s)\,\mathrm{d}s \le n;\ D_{E\setminus M} < \infty \Big).
\end{aligned}
$$

Applying the strong Markov property at $D_{E\setminus M}$ and recalling that $E \setminus M$ contains its regular points, which implies that $X_{D_{E\setminus M}} \in E \setminus M$ almost surely on $\{D_{E\setminus M} < \infty\}$, gives

$$
\le \mathbb{P}_z(D_{E\setminus M} = \infty) + \varepsilon.
$$

In combination with (3.21) this yields

$$
\mathbb{P}_y(D_{E\setminus M} = \infty) > 1 - 2\varepsilon.
$$

Since this holds for arbitrarily small $\varepsilon > 0$, it remains only to note via Proposition 3.2.4 that

$$
\int_M f(x)\,U(z, \mathrm{d}x) < \infty.
$$

[(i)⇐(ii)]

Take $\varepsilon > 0$ small and let M be such that $\mathbb{P}_z(D_{E\setminus M} < \infty) \le \varepsilon$ and

$$
\int_M f(x)\,U(z, \mathrm{d}x) = \mathbb{E}_z\Big[\int_0^\infty f(X_s)\mathbf{1}_M(X_s)\,\mathrm{d}s \Big] < \infty.
$$

Then

$$
\mathbb{P}_z\Big(\int_0^\infty f(X_s)\mathbf{1}_M(X_s)\,\mathrm{d}s < \infty \Big) = 1.
$$

In particular,

$$
\begin{aligned}
\mathbb{P}_z\Big(\int_0^\infty f(X_s)\,\mathrm{d}s < \infty \Big) &\ge \mathbb{P}_z\Big(\int_0^\infty f(X_s)\,\mathrm{d}s < \infty;\ D_{E\setminus M} = \infty \Big) \\
&= \mathbb{P}_z\Big(\int_0^\infty f(X_s)\mathbf{1}_M(X_s)\,\mathrm{d}s < \infty;\ D_{E\setminus M} = \infty \Big) \\
&= \mathbb{P}_z(D_{E\setminus M} = \infty) \\
&> 1 - \varepsilon.
\end{aligned}
$$

This holds for arbitrarily small $\varepsilon > 0$, and thus $\mathbb{P}_z(\int_0^\infty f(X_s)\,\mathrm{d}s < \infty) = 1$. □

We can immedately construct an interesting corollary of Theorem 3.4.1 in the case that f is locally bounded and $U(\cdot, K) < \infty$ for all compact K.

Corollary 3.4.2. *Let X be a standard Markov process on state space E, suppose that E is not compact and that $U(\cdot, K)$ is finite everywhere for all compact K,*[2] *and let $f \in \mathcal{E}_+$ be bounded on compact sets. Then the following are equivalent.*

[2] This is **(iii)** of (2.10), and is satisfied by transient standard Markov processes satisfying (LSC), or as we saw in Definition 2.5.3 by all transient Lévy processes.

(i) $\mathbb{P}_z\Big(\int_0^\infty f(X_s)\,\mathrm{d}s < \infty\Big) = 1;$

(ii) *For every $\varepsilon > 0$ there exists a set $B \in \mathcal{E}$ such that $\mathbb{P}_z(L_B < \zeta) \geq 1 - \varepsilon$ and*

$$\int_{E\setminus B} f(x)\,U(z,\mathrm{d}x) < \infty.$$

Proof. If we assume **(i)** then **(ii)** follows directly from Theorem 3.4.1 and the fact that for any $\mathbb{P}_z$-avoidable set B satisfying $\mathbb{P}_z(D_B < \infty) \leq \varepsilon$ it holds that

$$\mathbb{P}_z(L_B < \zeta) \geq \mathbb{P}_z(L_B = 0) \geq \mathbb{P}_z(D_B = \infty) \geq 1 - \varepsilon.$$

Now suppose **(ii)**. Proposition I(9.3) of Blumenthal and Getoor [5] gives that on the event $\{t < \zeta\}$, the random set $\{X_s : s \in [0,t]\}$, built from the points hit by X up to time t, is almost surely bounded. It follows that for a random time $T \in [0,\infty)$ with $T < \zeta$ almost surely, the path of X up to time T is almost surely contained in a (proper) compact subset of E. Therefore, if (K_n) is an increasing sequence of compact sets with limit E, it holds for fixed $\varepsilon > 0$ that $\mathbb{P}_z(\lim_{n\to\infty} T_{E\setminus K_n} \geq (\zeta - \varepsilon)^+) = 1$. Thus continuity of measure yields

$$\lim_{n\to\infty} T_{E\setminus K_n} \geq \zeta \qquad \mathbb{P}_z\text{-almost surely.}$$

The above argument only works because we have specified that the state space E is unbounded. Fix some $\varepsilon > 0$, and take B the set satisfying $\mathbb{P}_z(L_B < \zeta) \geq 1 - \varepsilon$ and $\int_{E\setminus B} f(x)\,U(z,\mathrm{d}x) < \infty$. Due to continuity of measure,

$$\mathbb{P}_z(L_B \geq T_{E\setminus K_n}) \downarrow \mathbb{P}_z(L_B \geq \zeta) \leq \varepsilon.$$

Then we can find a compact set $K \subseteq E$ such that

$$\mathbb{P}_z(L_B < T_{E\setminus K}) > 1 - 2\varepsilon.$$

Define the set $M = K \cup (E \setminus B)$. This definition implies that on the event $L_B < T_{E\setminus K}$, the process X never leaves M. That is,

$$\mathbb{P}_z(D_{E\setminus M} = \infty) \geq \mathbb{P}_z(L_B < T_{E\setminus K}) > 1 - 2\varepsilon.$$

In addition

$$\int_M f(x)\,U(z,\mathrm{d}x) \leq \int_{E\setminus B} f(x)\,U(z,\mathrm{d}x) + U(z,K)\sup_{x\in K} f(x). \tag{3.22}$$

Taking (3.22) together with the integral test of **(ii)**, boundedness of f on compact sets, and the fact that $U(z,K) < \infty$ implies that

$$\int_M f(x)\,U(z,\mathrm{d}x) < \infty.$$

Our choice of $\varepsilon > 0$ was arbitrary, and so the result follows from Theorem 3.4.1. □

The following example shows that in general condition **(i)** of Theorem 3.4.1 does not imply the integral test (3.20) over the entire state space, although the converse implication (3.20) $\Rightarrow$ **(i)** is clear. This example is instructive for two reasons: first, because it concerns stable processes, and second, because the assumptions it places on f (namely that it has compact support and is not integrable) are not contrived.

Example 3.4.3. *Let X be a symmetric stable process on $\mathbb{R}$ of index $\alpha \in (0,1)$. Fix a point $y \in \mathbb{R}$ not equal zero and let $\varepsilon > 0$ be such that $|y| > \varepsilon$. Now suppose that f has support on $\boldsymbol{B}_\varepsilon(y) = \{x : |x - y| < \varepsilon\}$, in order that the density $u(x) = |x|^{\alpha-1}$ of $U(0, \mathrm{d}x)$ is bounded on the support of f. Then*

$$\int_{\mathbb{R}} f(x)\, U(0, \mathrm{d}x) < \infty \qquad \Leftrightarrow \qquad \int_{\mathbb{R}} f(x)\, \mathrm{d}x < \infty.$$

So if we define such an f which is not integrable, for example $f(x) = (x-y)^{-2}\, \boldsymbol{1}_{\boldsymbol{B}_\varepsilon(y)}(x)$, it holds that $\int_{\mathbb{R}} f(x)\, U(0, \mathrm{d}x) = \infty$. Now recall that X stays a positive distance away from y $\mathbb{P}_0$-almost surely.[3] *It follows that*

$$\xi := \sup_{t \in [0,\infty)} f(X_t) < \infty \qquad \mathbb{P}_0\text{-almost surely.}$$

In addition, $\mathbb{P}_0(L_{\boldsymbol{B}_\varepsilon(y)} < \infty) = 1$. Therefore

$$\int_0^\infty f(X_s)\, \mathrm{d}s \le \xi L_{\boldsymbol{B}_\varepsilon(y)} < \infty \qquad \mathbb{P}_0\text{-almost surely.}$$

3.5 A Zero-One Law

The zero-one law presented below is a nice corollary to Theorem 3.1.1, and unifies the theorems of Döring and Kyprianou [11] and Kolb and Savov [30] which were presented in §3.1.

Let X be a standard Markov process. We say that X has a trivial tail σ-algebra when issued from $x \in E$ if

$$A \in \bigcap_{s \ge 0} \sigma\big(X_t,\, t \ge s\big) \quad \Rightarrow \quad \mathbb{P}_x(A) \in \{0,1\}.$$

As an example, a Lévy process on $\mathbb{R}^d$ has a trivial tail σ-algebra when issued from every $x \in \mathbb{R}^d$.

Theorem 3.5.1. *Let X be a standard Markov process on state space E with trivial tail σ-algebra when issued from $z \in E$, and let $f \in \mathcal{E}_+$ be bounded on compact sets. Suppose in addition that $\mathbb{P}_z(\zeta = \infty) = 1$. Then the following are equivalent.*

(i) $\mathbb{P}_z\Big(\int_0^\infty f(X_s)\, \mathrm{d}s < \infty\Big) > 0$;

[3] This well-known fact can be proven using the density of the point of closest reach of X, see [33].

(ii) $\mathbb{P}_z\Big(\int_0^\infty f(X_s)\,\mathrm{d}s < \infty\Big) = 1$*;*

(iii) *There exists a* $\mathbb{P}_z$*-transient set* B *such that* $\int_{E\setminus B} f(x)U(z,\mathrm{d}x) < \infty$.

It's interesting to recall from Definition 2.4.7 (iv) that for X a recurrent Markov process, the only transient sets are polar sets.

Proof.

[**(iii)⇒(ii)**]

By assumption,

$$\mathbb{E}_z\Big(\int_0^\infty \mathbf{1}_{E\setminus B}(X_s)f(X_s)\,\mathrm{d}s\Big) = \int_{E\setminus B} f(x)U(z,\mathrm{d}x) < \infty,$$

and hence

$$\int_0^\infty \mathbf{1}_{E\setminus B}(X_s)f(X_s)\,\mathrm{d}s < \infty \quad \mathbb{P}_z\text{-almost surely.} \tag{3.23}$$

From the definition of transience of sets, $\mathbb{P}_z(L_B < \infty) = 1$. Proposition I(9.3) of Blumenthal and Getoor [5] therefore yields that the random set

$$\mathcal{S} = \{X_s : s \geq 0,\, X_s \in B\},$$

which contains the points of B reached by X, is $\mathbb{P}_z$-almost surely bounded (that is, contained within a compact set, which implies $\mathrm{Leb}(\mathcal{S}) < \infty$ a.s.). Therefore - because f is bounded on compact sets - for $\mathbb{P}_z$-almost every $\omega \in \Omega$ there exists a compact set K_ω such that $f(X_s(\omega)) \in K_\omega$ for all $s \in \mathcal{S}(\omega)$. In particular,

$$\int_0^\infty \mathbf{1}_B(X_s(\omega))f(X_s(\omega))\,\mathrm{d}s \leq \sup_{x\in K_\omega} f(x)\mathrm{Leb}(S(\omega)) < \infty$$

for $\mathbb{P}_z$-almost every $\omega \in \Omega$. This in combination with (3.23) implies

$$\int_0^\infty f(X_s)\,\mathrm{d}s < \infty \quad \mathbb{P}_z\text{-almost surely.}$$

[**(i)⇒(iii)**]

According to Theorem 3.1.1 there exists a $\mathbb{P}_z$-supportive set M with

$$\int_M f(x)U(z,\,\mathrm{d}x) < \infty.$$

Since M is $\mathbb{P}_z$-supportive, $\mathbb{E} \setminus M$ is avoidable, and

$$\mathbb{P}_z(L_{E\setminus M} < \infty) \geq \mathbb{P}_z(L_{E\setminus M} = 0) \geq \mathbb{P}_z(D_{E\setminus M} = \infty) > 0.$$

For any $B \in \mathcal{E}$ the event $\{L_B < \infty\}$ is in the tail σ-algebra of X, and so by assumption $\mathbb{P}_z(L_B < \infty)$ is a zero-one law. Therefore

$$\mathbb{P}_z(L_{E \setminus M} < \infty) = 1$$

and $E \setminus M$ is transient. $\square$

4 Path Integral Tests

The work of Chapter 3 will be useful to us now as a tool for studying finite-time path integrals. A question of particular interest is under which conditions either of the following hold with positive probability:

$$\text{1} \quad \int_0^t f(X_s)\,\mathrm{d}s < \infty \text{ for every } t < \zeta$$

or

$$\text{2} \quad \int_0^t f(X_s)\,\mathrm{d}s < \infty \text{ for some } t > 0$$

These questions were first answered for the Brownian motion on $\mathbb{R}$ by Engelbert and Schmidt [15] in 1981, in the form of the following zero-one law.

Theorem 4.0.1 (Engelbert and Schmidt [15] Theorem 1)**.** *Let W be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P}_0)$, and let $f \in \overline{\mathcal{B}(\mathbb{R})}_+$. Then the following are equivalent:*

(i) $\mathbb{P}_0\Big(\int_0^t f(W_s)\,\mathrm{d}s < \infty \textit{ for every } t \geq 0\Big) > 0;$

(ii) $\mathbb{P}_0\Big(\int_0^t f(W_s)\,\mathrm{d}s < \infty \textit{ for every } t \geq 0\Big) = 1;$

(iii) *For all compact $K \subseteq \mathbb{R}$,*
$$\int_K f(y)\,\mathrm{d}y < \infty;$$

(iv) *There exists a $t_0 > 0$ such that* $\mathbb{P}_0\Big(\int_0^{t_0} f(W_s)\,\mathrm{d}s < \infty\Big) = 1.$

Engelbert and Schmidt also note that existence of a $t_0 > 0$ such that

$$\mathbb{P}_0\Big(\int_0^{t_0} f(W_s)\,\mathrm{d}s < \infty\Big) > 0$$

is not sufficient to imply **(i)** - **(iv)**. We have already seen that in one dimension the Brownian motion shares many properties with stable processes of index $\alpha \in (1, 2)$, and in 1997 Zanzotto [44] extended Engelbert and Schmidt's result to this class.

Theorem 4.0.2 (Zanzotto [44] Theorem 1.4)**.** *Let X be a stable process of index $\alpha \in (1,2]$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P}_0)$, and let $f \in \overline{\mathcal{B}(\mathbb{R})}_+$. Then the following are equivalent:*

(i) $\mathbb{P}_0\Big(\int_0^t f(X_s)\,\mathrm{d}s < \infty \text{ for every } t \geq 0\Big) > 0;$

(ii) $\mathbb{P}_0\Big(\int_0^t f(X_s)\,\mathrm{d}s < \infty \text{ for every } t \geq 0\Big) = 1;$

(iii) *For all compact $K \subseteq \mathbb{R}$,*

$$\int_K f(y)\,\mathrm{d}y < \infty.$$

Zanzotto's proof is almost identical to the Brownian case, and is rather brief. There is less hope of a similar approach working for $\alpha \in (0,1)$ for two reasons: first, local times play a crucial role in the proof and there is no substitute when they don't exist, and second, understanding supportive sets is far simpler for $\alpha \in (1,2)$ because - up to a difference of a Lebesgue-zero set - there is only one, which is $\mathbb{R}$. That is what allows the neat zero-one laws of Engelbert and Schmidt and Zanzotto, which cannot be achieved in generality without further assumptions on f.

The theorem above also has a related local version. This local version is worth mentioning because it provides a hope of unifying the story of transient and recurrent processes, since those properties don't much affect the small-time behaviour of a process X.

Lemma 4.0.3 (Zanzotto [44] Lemma 1.6)**.** *Let X be a stable process of index $\alpha \in (1,2]$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $f \in \overline{\mathcal{B}(\mathbb{R})}_+$. For a fixed $x \in \mathbb{R}$ suppose there exists a random time τ such that*

$$\mathbb{P}_x(0 < \tau < \infty) = 1 \qquad \textit{and} \qquad \mathbb{P}_x\Big(\int_0^\tau f(X_s)\,\mathrm{d}s < \infty\Big) > 0.$$

Then there exists an $\varepsilon > 0$ such that

$$\int_{-\varepsilon}^{\varepsilon} f(x+y)\,\mathrm{d}y < \infty.$$

The results presented in this chapter are Engelbert-Schmidt-type results for more general classes of Markov processes. First, three theorems for strongly transient Markov processes are proven, which are counterparts to Theorem 4.0.2 and Lemma 4.0.3. They make clear the importance of supportive sets, in particular with respect to the possibility of a zero-one law. These are then used to prove two more precise theorems for Lévy processes, the first of which concerns processes with local times that hit points and which contains the statements of Theorems 4.0.1 and 4.0.2, and the second of which gives a precise result for stable processes on $\mathbb{R}$ with $\alpha \in (0,1)$, achievable via the Wiener Criterion of Chapter 6.

4.1 Transient Markov Processes

The three theorems given below are natural companions to Theorem 4.0.2 and Lemma 4.0.3. Recall from §2.4 that a standard Markov process on E is strongly transient if all compact K have last-exit time $L_K < \zeta$ $\mathbb{P}_x$-almost surely for all $x \in E$. In the case that the lifetime ζ is equal $+\infty$ almost surely, strong transience coincides with transience.

Theorem 4.1.1. *Let X be a strongly transient standard Markov process and $f \in \overline{\mathcal{E}}_+$ a non-negative measurable function. For $z \in E$, the following are equivalent:*

(i) $\mathbb{P}_z\Big(\int_0^t f(X_s)\,\mathrm{d}s < \infty$ *for every* $t < \zeta\Big) > 0$;

(ii) *There is a constant $c > 0$ such that for all compact $K \subseteq E$ there exists a $\mathbb{P}_z$-supportive set M_K satisfying $\mathbb{P}_z(T_{E\setminus M_K} = \zeta) > c$ and*

$$\int_{M_K \cap K} f(y)\,U(z,\mathrm{d}y) < \infty.$$

Theorem 4.1.2. *Let X be a strongly transient standard Markov process and $f \in \overline{\mathcal{E}}_+$. For $z \in E$, the following are equivalent:*

(i) $\mathbb{P}_z\Big(\int_0^t f(X_s)\,\mathrm{d}s < \infty$ *for every* $t < \zeta\Big) = 1$;

(ii) *For every $\varepsilon > 0$ there exists a $\mathbb{P}_z$-supportive set M^ε satisfying $\mathbb{P}_z(T_{E\setminus M^\varepsilon} = \zeta) > 1 - \varepsilon$ and*

$$\int_{M^\varepsilon \cap K} f(y)\,U(z,\mathrm{d}y) < \infty$$

for all compact $K \subseteq E$.

Theorem 4.1.3. *Let X be a strongly transient standard Markov process and $f \in \overline{\mathcal{E}}_+$. For $z \in E$, the following are equivalent:*

(i) *There exists a stopping time τ such that*

$$\mathbb{P}_z(0 < \tau < \zeta) = 1 \qquad \textit{and} \qquad \mathbb{P}_z\Big(\int_0^\tau f(X_s)\,\mathrm{d}s < \infty\Big) = 1$$

(ii) *There exists $\mathbb{P}_z$-thin set B such that* $\displaystyle\int_{E\setminus B} f(y)\,U(z,\mathrm{d}y) < \infty.$

For one argument in the proof of Theorem 4.1.3 it is important that τ is a stopping time. But if X has the property that for all $t > 0$ there $\mathbb{P}_z$-almost surely exists a finely open set G containing z such that $L_G < t$, then τ need only be a random variable. This condition holds for example for transient Lévy processes that don't hit points, because the last exit times of $\mathbf{B}_\varepsilon(z)$ are arbitrarily small.

Proof of Theorem 4.1.1

[(i) ⇒ (ii)]

Let $K_1 \subseteq K_2 \dots$ be an increasing sequence of compact sets with limit E. For $n \in \mathbb{N}$ let L_{K_n} be the last exit time of K_n. Since X is strongly transient, it holds that $\mathbb{P}_z(L_{K_n} < \zeta) = 1$ for all n, and thus

$$\begin{aligned}
& \mathbb{P}_z\Big(\int_0^t f(X_s)\,\mathrm{d}s < \infty \text{ for every } t < \zeta\Big) > 0 \\
\Rightarrow\quad & \mathbb{P}_z\Big(\int_0^{L_{K_n}} f(X_s)\,\mathrm{d}s < \infty \text{ for every } n \in \mathbb{N}\Big) > 0 \\
\Rightarrow\quad & \exists C > 0 \text{ s.t. } \mathbb{P}_z\Big(\int_0^{L_{K_n}} f(X_s)\,\mathrm{d}s < \infty\Big) > C \text{ for every } n \in \mathbb{N} \\
\Rightarrow\quad & \mathbb{P}_z\Big(\int_0^{\infty} f(X_s)\mathbf{1}_{K_n}(X_s)\,\mathrm{d}s < \infty\Big) > C \text{ for every } n \in \mathbb{N}.
\end{aligned} \tag{4.1}$$

Define $f_n = f\mathbf{1}_{K_n}$. For each f_n an application of Theorem 3.1.1 gives a $\mathbb{P}_z$-supportive set M_n such that

$$\int_{M_n \cap K_n} f(x)\,U(z,\mathrm{d}x) = \int_{M_n} f_n(x)\,U(z,\mathrm{d}x) < \infty.$$

In fact we have some control over the form of M_n. For any $n \in \mathbb{N}$ we can by the calculation above choose an $N_n \in \mathbb{N}$ such that

$$\mathbb{P}_z\Big(\int_0^{\infty} f_n(X_s)\,\mathrm{d}s \le N_n\Big) > \frac{C}{2}.$$

Now define the super finite sets

$$M_n := \Big\{y \in E : \mathbb{P}_y\Big(\int_0^{\infty} f_n(X_s)\,\mathrm{d}s \le N_n\Big) > \frac{C}{3}\Big\}, \quad n \in \mathbb{N}.$$

By (3.19), each M_n is $\mathbb{P}_z$-supportive. Moreover

$$\begin{aligned}
\frac{C}{2} &< \mathbb{P}_z\Big(\int_0^{\infty} f_n(X_s)\,\mathrm{d}s \le N_n\Big) \\
&= \mathbb{P}_z\Big(\int_0^{\infty} f_n(X_s)\,\mathrm{d}s \le N_n;\ T_{E\setminus M_n} < \infty\Big) + \mathbb{P}_z\Big(\int_0^{\infty} f_n(X_s)\,\mathrm{d}s \le N_n;\ T_{E\setminus M_n} = \infty\Big) \\
&\le \mathbb{P}_z\Big(\int_{T_{E\setminus M_n}}^{\infty} f_n(X_s)\,\mathrm{d}s \le N_n;\ T_{E\setminus M_n} < \infty\Big) + \mathbb{P}_z(T_{E\setminus M_n} = \infty)
\end{aligned}$$

and because $E \setminus M_n$ contains its regular points

$$\le \frac{C}{3} + \mathbb{P}_0(T_{E\setminus M_n} = \infty),$$

and therefore not only is each M_n supportive, but the probability of remaining in each is uniformly bounded away from 0 because

$$\mathbb{P}_z(T_{E\setminus M_n} = \infty) > \frac{C}{6}.$$

The final property of M_n to recall is from Proposition 3.2.4:

$$\int_{M_n \cap K_n} f(y) U(z, \mathrm{d}y) < \infty.$$

Any compact set $K \subseteq E$ is covered by the interiors of the sequence K_n. By assumption E is locally compact, and therefore K is covered by a finite subcover of these interiors, and in particular K is contained in K_n for some n. Hence **(ii)** follows.

[**(i)** $\Leftarrow$ **(ii)**]

Suppose that for any compact K there exists M such that $\mathbb{P}_z(T_{E\setminus M} = \infty) > c$ and

$$\mathbb{E}_z\Big[\int_0^\infty f(X_s)\mathbf{1}_{M\cap K}(X_s)\,\mathrm{d}s\Big] = \int_{M\cap K} f(y)\,U(z, \mathrm{d}y) < \infty.$$

Let $K_1, K_2, \ldots$ be an increasing sequence of compact sets with limit E, and denote by M_n the supportive sets associated to them by the above relation. Then for all $n \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{E}_z\Big[\int_0^\infty f(X_s)\mathbf{1}_{M_n\cap K_n}(X_s)\,\mathrm{d}s\Big] < \infty \quad &\Rightarrow \quad \mathbb{P}_z\Big(\int_0^\infty f(X_s)\mathbf{1}_{M_n\cap K_n}(X_s)\,\mathrm{d}s < \infty\Big) = 1\\
&\Rightarrow \quad \mathbb{P}_z\Big(\int_0^\infty f(X_s)\mathbf{1}_{K_n}(X_s)\,\mathrm{d}s < \infty\Big) > c\\
&\Rightarrow \quad \mathbb{P}_z\Big(\int_0^{T_{E\setminus K_n}} f(X_s)\,\mathrm{d}s < \infty\Big) > c.
\end{aligned}$$

Thus

$$\begin{aligned}
c &\leq \lim_{n\to\infty} \mathbb{P}_z\Big(\int_0^{T_{E\setminus K_n}} f(X_s)\,\mathrm{d}s < \infty\Big)\\
&= \mathbb{P}_z\Big(\int_0^{T_{E\setminus K_n}} f(X_s)\,\mathrm{d}s < \infty \text{ for all } n \in \mathbb{N}\Big).
\end{aligned} \tag{4.2}$$

Since X is strongly transient, $\lim_{n\to\infty} T_{E\setminus K_n} = \zeta$, and thus

$$\mathbb{P}_z\Big(\int_0^t f(X_s)\,\mathrm{d}s < \infty \text{ for all } t < \zeta\Big) \geq c > 0.$$

□

Proof of Theorem 4.1.2

[(i) ⇒ (ii)]

Let $K_1 \subseteq K_2 \dots$ be an increasing sequence of compact sets with limit E. For $n \in \mathbb{N}$ let L_{K_n} be the last exit time of K_n. Then since X is strongly transient,

$$\begin{aligned}
&\mathbb{P}_z\Big(\int_0^t f(X_s)\,\mathrm{d}s < \infty \text{ for every } t < \zeta\Big) = 1 \\
\Rightarrow\quad &\mathbb{P}_z\Big(\int_0^{L_{K_n}} f(X_s)\,\mathrm{d}s < \infty \text{ for every } n \in \mathbb{N}\Big) = 1 \\
\Rightarrow\quad &\mathbb{P}_z\Big(\int_0^{L_{K_n}} f(X_s)\,\mathrm{d}s < \infty\Big) = 1 \text{ for every } n \in \mathbb{N} \\
\Rightarrow\quad &\mathbb{P}_z\Big(\int_0^{\infty} f(X_s)\mathbf{1}_{K_n}(X_s)\,\mathrm{d}s < \infty\Big) = 1 \text{ for every } n \in \mathbb{N}.
\end{aligned} \tag{4.3}$$

Now fix $\varepsilon > 0$ and $n \in \mathbb{N}$. We can choose a constant N_n such that

$$\mathbb{P}_z\Big(\int_0^{\infty} f(X_s)\mathbf{1}_{K_n}(X_s)\,\mathrm{d}s \le N_n\Big) > 1 - 2^{-n}\varepsilon.$$

Then write $f = f\mathbf{1}_{K_n}$ and define a super-finite set for each (X, f_n),

$$M_n^\varepsilon := \Big\{y \in E : \mathbb{P}_y\Big(\int_0^{\infty} f_n(X_s)\,\mathrm{d}s \le N_n\Big) > 2^{-n}\varepsilon\Big\}, \quad n \in \mathbb{N}.$$

By (3.19), each M_n is $\mathbb{P}_z$-supportive. Moreover

$$\begin{aligned}
1 - 2^{-n}\varepsilon &< \mathbb{P}_z\Big(\int_0^{\infty} f_n(X_s)\,\mathrm{d}s \le N_n\Big) \\
&= \mathbb{P}_z\Big(\int_0^{\infty} f_n(X_s)\,\mathrm{d}s \le N_n;\ T_{E\setminus M_n^\varepsilon} < \infty\Big) \\
&\qquad + \mathbb{P}_z\Big(\int_0^{\infty} f_n(X_s)\,\mathrm{d}s \le N_n;\ T_{E\setminus M_n^\varepsilon} = \infty\Big) \\
&\le \mathbb{P}_z\Big(\int_{T_{E\setminus M_n^\varepsilon}}^{\infty} f_n(X_s)\,\mathrm{d}s \le N_n;\ T_{E\setminus M_n^\varepsilon} < \infty\Big) + \mathbb{P}_z(T_{E\setminus M_n^\varepsilon} = \infty) \\
&\le 2^{-n}\varepsilon + \mathbb{P}_z(T_{E\setminus M_n^\varepsilon} = \infty),
\end{aligned}$$

and therefore not only is each M_n^ε $\mathbb{P}_z$-supportive, but the probability of remaining in each is uniformly bounded away from 0 by

$$\mathbb{P}_z(T_{E\setminus M_n^\varepsilon} = \infty) > 1 - 2^{1-n}\varepsilon.$$

Now let

$$M^\varepsilon := \bigcap_n M_n^\varepsilon.$$

Then by sub-additivity of measure

$$
\begin{aligned}
\mathbb{P}_z(T_{E\setminus M^\varepsilon} = \infty) &\geq 1 - \sum_{n=1}^{\infty} \mathbb{P}_z(T_{E\setminus M_n^\varepsilon} < \infty) \\
&> 1 - \varepsilon \sum_{n=1}^{\infty} 2^{1-n} \\
&= 1 - 2\varepsilon.
\end{aligned}
$$

Moreover for any compact K, the interiors of the sets K_n form a cover of K, and since E is locally compact there is a finite subcover of K, and this implies that there exists an n such that $K \subseteq K_n$. Therefore

$$
\int_{M^\varepsilon \cap K} f(y)\, U(z, \mathrm{d}y) \leq \int_{M_n^\varepsilon \cap K_n} f(y)\, U(z, \mathrm{d}y) < \infty.
$$

[(i) $\Leftarrow$ (ii)]

Suppose that for any $\varepsilon > 0$ there exists a $\mathbb{P}_z$-supportive set M^ε satisfying $\mathbb{P}_z(T_{E\setminus M_\varepsilon} = \zeta) > 1 - \varepsilon$ and

$$
\int_{M^\varepsilon \cap K} f(y)\, U(z, \mathrm{d}y) < \infty
$$

for all compact $K \subseteq E$. Let $K_1 \subseteq K_2, \ldots$ be a nested sequence of compact sets with limit E. Then for all $n \in \mathbb{N}$,

$$
\begin{aligned}
\mathbb{E}_z\Big[\int_0^\infty f(X_s)\mathbf{1}_{M^\varepsilon \cap K_n}(X_s)\,\mathrm{d}s\Big] < \infty \quad &\Rightarrow \quad \mathbb{P}_z\Big(\int_0^\infty f(X_s)\mathbf{1}_{M^\varepsilon \cap K_n}(X_s)\,\mathrm{d}s < \infty\Big) = 1 \\
&\Rightarrow \quad \mathbb{P}_z\Big(\int_0^\infty f(X_s)\mathbf{1}_{K_n}(X_s)\,\mathrm{d}s < \infty\Big) > 1 - \varepsilon \\
&\Rightarrow \quad \mathbb{P}_z\Big(\int_0^{T_{E\setminus K_n}} f(X_s)\,\mathrm{d}s < \infty\Big) > 1 - \varepsilon.
\end{aligned}
$$

Thus, since X is strongly transient, $\lim_{n\to\infty} T_{E\setminus K_n} = \zeta$ and

$$
\begin{aligned}
1 - \varepsilon &< \lim_{n\to\infty} \mathbb{P}_z\Big(\int_0^{T_{E\setminus K_n}} f(X_s)\,\mathrm{d}s < \infty\Big) \\
&= \mathbb{P}_z\Big(\int_0^{T_{E\setminus K_n}} f(X_s)\,\mathrm{d}s < \infty \text{ for all } n \in \mathbb{N}\Big) \qquad (4.4) \\
&= \mathbb{P}_z\Big(\int_0^{t} f(X_s)\,\mathrm{d}s < \infty \text{ for all } t < \zeta\Big).
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary it follows that

$$
\mathbb{P}_z\Big(\int_0^{t} f(X_s)\,\mathrm{d}s < \infty \text{ for all } t < \zeta\Big) = 1.
$$

□

Proof of Theorem 4.1.3

[(i) $\Rightarrow$ (ii)]

First we need to deal with one specific situation, which is when $\mathbb{P}_z(T_{E\setminus\{z\}} > 0) > 0$ (in fact this is a zero-one law, and in this case z is called a *holding point.*). Since $\int_0^\tau f(X_s)\,\mathrm{d}s < \infty$, it must hold that $f(z) < \infty$. In addition in this case $E \setminus \{z\}$ is clearly $\mathbb{P}_z$-thin, and by transience $U(z,\{z\}) < \infty$. So **(i)** holds, because

$$\int_{\{z\}} f(y)\,U(z,\mathrm{d}y) = f(z)U(z,\{z\}) < \infty.$$

Now suppose that $\mathbb{P}_z(T_{E\setminus\{z\}} > 0) = 0$. We shall prove that there exists a $\mathbb{P}_z$-thin set $B \in \mathcal{E}$ such that $z \in E \setminus B$ and $L_{E\setminus B} \leq \tau$ with positive probability. Let μ be the finite measure on $(E,\mathcal{E})$ defined by

$$\mu(A) = \mathbb{P}_z(X_\tau \in A).$$

We can apply Theorem I(10.16) of Blumenthal and Getoor, which tells us that there exists a decreasing sequence of open sets $G_n \in \mathcal{E}$ such that $z \in G_n$ for every n and

$$\mathbb{P}_\mu(T_{G_n} \wedge \zeta \uparrow T_{\{z\}} \wedge \zeta) = 1. \tag{4.5}$$

For $A \in \mathcal{E}_\Delta$ let us introduce the stopping time $T_A^\tau = \inf\{t > \tau : X_t \in A\}$. This is the first hitting time of A after τ, and just like T_A it is a stopping time, but it also satisfies

$$\mathbb{P}_z(T_A^\tau < \zeta) = \mathbb{P}_z(L_A > \tau).$$

It follows from (4.5), and the fact that τ is a stopping time, that

$$\begin{aligned} \mathbb{P}_z(T_{G_n}^\tau \wedge \zeta \uparrow T_{\{z\}}^\tau \wedge \zeta) &= \mathbb{E}_z\big[\mathbb{P}_z(T_{G_n}^\tau \wedge \zeta \uparrow T_{\{z\}}^\tau \wedge \zeta | \mathcal{F}_\tau)\big] \\ &= \mathbb{E}_z\big[\mathbb{P}_{X_\tau}(T_{G_n} \wedge \zeta \uparrow T_{\{z\}} \wedge \zeta)\big] \\ &= \mathbb{P}_\mu(T_{G_n} \wedge \zeta \uparrow T_{\{z\}} \wedge \zeta) = 1. \end{aligned} \tag{4.6}$$

Here is where strong transience comes into play. Clearly

$$T_{\{z\}}^\tau < \zeta \quad \Rightarrow \quad T_{G_n}^\tau < \zeta \quad \text{for every } n.$$

But it does *not* follow from (4.6) alone that

$$T_{G_n}^\tau < \zeta \text{ for every } n \quad \Rightarrow \quad T_{\{z\}}^\tau < \zeta.$$

We need to note that for almost every $\omega \in \Omega$ if $T_{G_n}^\tau < \zeta$ then also $\lim_{n\to\infty} T_{G_n}^\tau \leq L_{G_1}$, and in this case (4.5) yields that $\lim_{n\to\infty} T_{G_n}^\tau = T_{\{z\}}^\tau$. Thus we have that $T_{\{z\}}^\tau \leq L_{G_1}$ almost surely, and from strong transience it follows that $T_{\{z\}}^\tau \leq L_{G_1} < \zeta$ almost surely.

Moreover, transience and the strong Markov property of X tell us that $\mathbb{P}_z(T_{\{z\}}^\tau < \zeta)$ must be less than one, else we would have $\mathbb{P}_z(L_{\{z\}} = \infty) = 1$, which would contradict transience. From our discussion above this implies that there is some choice of n such that $\mathbb{P}_z(T^{\tau}_{G_n} < \zeta) < 1$, and therefore that

$$\mathbb{P}_z(T^{\tau}_{G_n} \geq \zeta) = \mathbb{P}_z(L_{G_n} \leq \tau) > 0.$$

Thus under the assumption of **(i)**,

$$\begin{aligned}\mathbb{P}_z\Big(\int_0^{L_{G_n}} f(X_s)\,\mathrm{d}s < \infty\Big) &\geq \mathbb{P}_z\Big(\int_0^{L_{G_n}} f(X_s)\,\mathrm{d}s < \infty; \tau \geq L_{G_n}\Big)\\ &\geq \mathbb{P}_z\Big(\int_0^{\tau} f(X_s)\,\mathrm{d}s < \infty; \tau \geq L_{G_n}\Big) > 0.\end{aligned}$$

It therefore follows that

$$\mathbb{P}_z\Big(\int_0^{\infty} f(X_s)\mathbf{1}_{G_n}(X_s)\,\mathrm{d}s < \infty\Big) > 0. \tag{4.7}$$

Then Theorem 3.1.1 yields a $\mathbb{P}_z$-supportive set M such that

$$\int_{M\cap G_n} f(x)\,U(z,\mathrm{d}x) < \infty.$$

Since M is $\mathbb{P}_z$-supportive, Blumenthal's zero-one law gives that its complement $E\setminus M$ is $\mathbb{P}_z$-thin. Since G_n is open and contains z, $E\setminus G_n$ is $\mathbb{P}_z$-thin. The union of finitely many $\mathbb{P}_z$-thin sets is again $\mathbb{P}_z$-thin, and thus we define $B = (E\setminus M)\cup(E\setminus G_n) = E\setminus(M\cap G_n)$, which is $\mathbb{P}_z$-thin and satisfies

$$\int_{E\setminus B} f(x)\,U(z,\mathrm{d}x) < \infty.$$

[**(i)** $\Leftarrow$ **(ii)**]

Let B be the $\mathbb{P}_z$-thin set from **(ii)**. Proposition II(4.3) of Blumenthal and Getoor [5] says that there is a compact set $K \subseteq E\setminus B$ such that $z \in K$ and $E\setminus K$ is again $\mathbb{P}_z$-thin. The stopping time we shall define is

$$\tau = T_{E\setminus K}.$$

Since $E\setminus K$ is $\mathbb{P}_z$-thin, τ is $\mathbb{P}_z$-almost surely positive. In addition it follows from strong transience of X that $T_{E\setminus K} \leq L_K < \zeta$ $\mathbb{P}_z$-almost surely, and so $\mathbb{P}_z(0 < \tau < \zeta) = 1$. We have assumed that

$$\int_{E\setminus B} f(x)\,U(z,\mathrm{d}x) = \mathbb{E}_z\Big[\int_0^{\infty} f(X_s)\mathbf{1}_{E\setminus B}(X_s)\,\mathrm{d}s\Big] < \infty.$$

This implies that $\mathbb{P}_z(\int_0^\infty f(X_s)\mathbf{1}_{E\setminus B}(X_s)\,\mathrm{d}s < \infty) = 1$. From $K \subseteq E\setminus B$ it follows that $\tau \leq T_B$. Therefore

$$\begin{aligned}\mathbb{P}_z\Big(\int_0^{\tau} f(X_s)\,\mathrm{d}s < \infty\Big) &\geq \mathbb{P}_z\Big(\int_0^{T_B} f(X_s)\,\mathrm{d}s < \infty\Big)\\ &\geq \mathbb{P}_z\Big(\int_0^{\infty} f(X_s)\mathbf{1}_{E\setminus B}(X_s)\,\mathrm{d}s < \infty\Big) = 1.\end{aligned}$$

□

4.2 Lévy Processes with Local Times

The following lemma and theorem are versions of Zanzotto's in a slightly more general setting. They are also clear counterparts to the results of §4.1. The local time arguments used in the proof of Lemma 4.2.1 are similar those used by Zanzotto and Engelbert and Schmidt. The proof of Theorem 4.2.2 uses the machinery we developed in Theorem 3.4.1 in a novel way to obtain a result for X with local times which are not necessarily jointly continuous.

Lemma 4.2.1. *Let X be a Lévy process on $\mathbb{R}$ which has jointly continuous local times. Let $f \in \overline{\mathcal{B}(\mathbb{R})}_+$. Then the following are equivalent.*

(i) *There exists a random variable τ such that*

$$\mathbb{P}_0(0 < \tau < \infty) = 1 \qquad \textit{and} \qquad \mathbb{P}_0\Big(\int_0^\tau f(X_s)\,\mathrm{d}s < \infty\Big) > 0;$$

(ii) *There exists a random variable τ such that*

$$\mathbb{P}_0(0 < \tau < \infty) = 1 \qquad \textit{and} \qquad \mathbb{P}_0\Big(\int_0^\tau f(X_s)\,\mathrm{d}s < \infty\Big) = 1;$$

(iii) *There exists an $\varepsilon > 0$ such that*

$$\int_{-\varepsilon}^{\varepsilon} f(y)\,\mathrm{d}y < \infty. \tag{4.8}$$

Theorem 4.2.2. *Let X be a Lévy process on $\mathbb{R}$ which has local times[1], and $f \in \overline{\mathcal{B}(\mathbb{R})}_+$. The following are equivalent.*

(i) $\mathbb{P}_0\Big(\int_0^t f(X_s)\,\mathrm{d}s < \infty \textit{ for every } t \geq 0\Big) = 1;$

(ii) *For all compact $K \subseteq \mathbb{R}$,*

$$\int_K f(y)\mathrm{d}y < \infty.$$

*If in addition X is point recurrent and has jointly continuous local times then **(i)** and **(ii)** are also equivalent to*

(iii) $\mathbb{P}_0\Big(\int_0^t f(X_s)\,\mathrm{d}s < \infty \textit{ for every } t \geq 0\Big) > 0.$

[1] Recall that equivalent conditions were given for this in Theorem 2.6.5.

Proof of Lemma 4.2.1

First suppose **(iii)** and let $\tau = \inf\{s > 0 : |X_s| \geq \varepsilon\}$, the first hitting time of $\mathbb{R} \setminus \mathbf{B}_\varepsilon$, which is $\mathbb{P}_0$-almost surely in $(0, \infty)$. Since τ is almost surely finite, $L(\tau, \cdot)$ is almost surely continuous and finite everywhere. This yields that $\delta := \sup_{x \in \mathbf{B}_\varepsilon} L(\tau, x) < \infty$ $\mathbb{P}_0$-a.s. Recall that the occupation measure μ_τ almost surely has support in $\mathbf{B}_\varepsilon$. Then

$$\int_0^\tau f(X_s)\,\mathrm{d}s = \int_{\mathbf{B}_\varepsilon} f(x)L(\tau, x)\,\mathrm{d}x \leq \delta \int_{\mathbf{B}_\varepsilon} f(x)\,\mathrm{d}x < \infty \qquad \mathbb{P}_0\text{-a.s.},$$

and **(ii)** holds.

Now suppose **(i)**. Since $L(t, 0)$ is strictly increasing in t, and τ is almost surely positive, it follows that $L(\tau, 0) > 0$ $\mathbb{P}_0$-almost surely. Since L is continuous in x there exists an almost surely positive random variable γ such that $\inf_{x \in \mathbf{B}_\gamma} L(\tau, x) \geq L(\tau, 0)/2 > 0$. Thus $\mathbb{P}_0$-almost surely

$$\int_0^\tau f(X_s(\omega))\,\mathrm{d}s = \int_{\mathbb{R}^d} f(x)L(x, \tau)\,\mathrm{d}x \geq \frac{L(\tau, 0)}{2} \int_{\mathbf{B}_\gamma} f(x)\,\mathrm{d}x.$$

Then by assumption it follows that $\int_{\mathbf{B}_\gamma} f(x)\,\mathrm{d}x < \infty$ with positive probability. Since γ is almost surely positive, we can choose an $\varepsilon > 0$ such that $\gamma > \varepsilon$ with positive probability on the event that $\int_{\mathbf{B}_\gamma} f(x)\,\mathrm{d}s < \infty$, and therefore (4.8) holds for that choice of ε.

$\square$

Proof of Theorem 4.2.2

This proof makes use of a type of process which we haven't mentioned yet called a *killed Markov process* which for $q > 0$ is denoted X^q and defined by

$$X_t^q = \begin{cases} X_t & \text{if } t \in [0, \tau^q), \\ \Delta & \text{if } t \in [\tau^q, \infty], \end{cases}$$

where X is a standard Markov process on E with cemetary state Δ and τ^q is an independent exponentially distributed random variable with mean $1/q$. In the literature such a process is sometimes called a q-subprocess, see for example Blumenthal and Getoor [5] Example III(3.17). A killed Markov process is again a standard Markov process on E with cemetary state Δ, and has transition semigroup

$$P_t^q(x, A) = \mathbb{P}_x(X_t \in A; t < \tau^q) = \mathrm{e}^{-qt} P_t(x, A) \tag{4.9}$$

for $A \in \mathcal{E}$, and from this it follows that the potential operator of X^q is U^q, the q-potential operator of X. It is worth noting that for T a stopping time,

$$\mathbb{P}_x(T < \tau^q) = \int \mathbb{P}_x(t < \tau^q)\mathbb{P}_x(T \in \mathrm{d}t) = \int \mathrm{e}^{-qt}\mathbb{P}_x(T \in \mathrm{d}t) = \mathbb{E}_x[\mathrm{e}^{-qT}]. \tag{4.10}$$

Since τ^q is almost surely finite, any killed Markov process is transient. But interestingly, if X has lifetime $\zeta = \infty$ almost surely then X^q cannot be strongly transient.

[**(i)** $\Rightarrow$ **(ii)**]

Fix $q > 0$, and let X^q be the killed process as above. Then X^q is a transient standard Markov process on $\mathbb{R}$, and since τ^q has support $(0, \infty)$ our assumption implies that X^q satisfies condition **(i)** of Theorem 3.4.1. Fix some compact $K \subseteq \mathbb{R}$ and take an arbitrary $\varepsilon > 0$. By Theorem 3.4.1, there exists a $\mathbb{P}_0$-supportive - for X^q, that is - set M^ε satisfying $\mathbb{P}_0(X^q_s \in M^\varepsilon \text{ for all } s < \tau^q) > 1 - \varepsilon$ and

$$\int_{M^\varepsilon \cap K} f(y) U^q(0, \mathrm{d}y) \leq \int_{M^\varepsilon} f(y) U^q(0, \mathrm{d}y) < \infty$$

for any compact $K \subseteq \mathbb{R}$, and in particular for our choice of K. Let us use the notation $B^\varepsilon = \mathbb{R} \setminus M^\varepsilon$. Then

$$\begin{aligned}
\mathbb{P}_0(X_s \in B^\varepsilon \cap K \text{ for some } s < \tau^q) &= \mathbb{P}_0(X^q_s \in B^\varepsilon \cap K \text{ for some } s < \tau^q) \\
&\leq \mathbb{P}_0(X^q_s \in B^\varepsilon \text{ for some } s < \tau^q) \\
&< \varepsilon.
\end{aligned}$$

Now suppose $B^\varepsilon \cap K$ is non-empty, so there exists some $x \in B^\varepsilon \cap K$. Then

$$\begin{aligned}
\mathbb{P}_0(X^q_s \in B^\varepsilon \cap K \text{ for some } s \geq 0) &= \mathbb{P}_0(X_s \in B^\varepsilon \cap K \text{ for some } s < \tau^q) \\
&\geq \mathbb{P}_0(X_s = x \text{ for some } s < \tau^q) \\
&= \mathbb{E}_0[\mathrm{e}^{-qT_{\{x\}}}] && \text{by (4.10)}
\end{aligned}$$

where $T_{\{x\}} = \inf\{s > 0 : X_s = x\}$,

$$\begin{aligned}
&= \mathbb{E}_{-x}[\mathrm{e}^{-qT_{\{0\}}}] \\
&= \hat{u}^q(x) C^q. && \text{by (2.15)}
\end{aligned}$$

In the discussion below Theorem 2.6.5 we saw that because X hits points, u and $\hat{u}$ are bounded below on compact sets. In particular there exists some $c > 0$ such that

$$\mathbb{P}_0(X_s \in B^\varepsilon \cap K \text{ for some } s < \tau^q) \geq \hat{u}^q(x) C^q > c \qquad \text{for all } x \in K \text{ (and thus } \in B^\varepsilon \cap K).$$

This constant c is dependent on both q and K, but not upon the choice of ε. Now we have shown that

$$c < \mathbb{P}_0(X_s \in B^\varepsilon \cap K \text{ for some } s < \tau^q) < \varepsilon.$$

But our choice of ε is arbitrary, and we can therefore choose $0 < \varepsilon < c$. The resolution of this apparent contradiction is that in this case there does not exist any point $x \in B^\varepsilon \cap K$. Thus for such ε,

$$\infty > \int_{M^\varepsilon \cap K} f(y) U^q(0, \mathrm{d}y) = \int_K f(y) u^q(y) \, \mathrm{d}y.$$

Again because u^q is bounded below on compacts, there is a $\delta > 0$ such that $u^q(x) > \delta$ for $x \in K$, and thus

$$\delta \int_K f(y)\,\mathrm{d}y \leq \int_K f(y)u^q(y)\,\mathrm{d}y < \infty.$$

We have proven this for one arbitary compact K, and it therefore holds for them all.

[**(i)** $\Leftarrow$ **(ii)**]

Fix $q > 0$. Since X has local times Theorem 2.6.5 gives that u^q is bounded, and thus for arbitrary compact K,

$$\mathbb{E}_0\Big[\int_0^\infty f(X_s^q)\mathbf{1}_K(X_s^q)\,\mathrm{d}s\Big] = \int_K f(y)u^q(y)\,\mathrm{d}y \leq \sup_{x\in\mathbb{R}} u^q(x)\int_K f(y)\,\mathrm{d}y < \infty.$$

Now

$$\mathbb{E}_0\Big[\int_0^\infty f(X_s^q)\mathbf{1}_K(X_s^q)\,\mathrm{d}s\Big] < \infty \quad\Rightarrow\quad \mathbb{P}_0\Big(\int_0^\infty f(X_s^q)\mathbf{1}_K(X_s^q)\,\mathrm{d}s < \infty\Big) = 1.$$

for all compact K. Thus by continuity of measure

$$\mathbb{P}_0\Big(\int_0^\infty f(X_s^q)\,\mathrm{d}s < \infty\Big) = 1.$$

Since τ^q is independent and has support $(0,\infty)$, this implies

$$\mathbb{P}_0\Big(\int_0^t f(X_s)\,\mathrm{d}s < \infty \text{ for every } t < \infty\Big) = 1,$$

and **(ii)** has been proven.

Now we want to prove the zero-one law in the case that X is point recurrent and has jointly continuous local times. The implication **(i)** $\Rightarrow$ **(iii)** is immediate. Suppose **(iii)**, which by Lemma 4.2.1 immediately gives an open set G_0 containing 0 such that

$$\int_{G_0} f(y)\,\mathrm{d}y < \infty. \tag{4.11}$$

Let $T_{\{0\}}$ be the first hitting time of zero, which is almost surely finite from any issuing point under the assumption of point recurrence, and therefore for $x \in \mathbb{R}$,

$$0 < \mathbb{P}_x\Big(\int_0^{T_{\{0\}}} f(X_s)\,\mathrm{d}s < \infty\Big) = \mathbb{P}_0\Big(\int_0^{T_{\{-x\}}} f(X_s + x)\,\mathrm{d}s < \infty\Big).$$

The hitting time $T_{\{-x\}}$ is $\mathbb{P}_0$-almost surely positive for x not equal zero, and therefore Lemma 4.2.1 gives existence of an $\varepsilon > 0$ such that

$$\int_{\mathbf{B}_\varepsilon} f(y + x)\,\mathrm{d}y < \infty.$$

Therefore $G_x \coloneqq \mathbf{B}_\varepsilon + x$ is an open neighbourhood G_x of x such that

$$\int_{G_x} f(y)\,\mathrm{d}y < \infty. \tag{4.12}$$

Since this holds for all $x \in \mathbb{R} \setminus \{0\}$ we combine this with the G_0 already mentioned to get an open cover $\{G_x, x \in \mathbb{R}\}$ of $\mathbb{R}$. Now for any compact $K \subseteq \mathbb{R}$, K is covered by $\{G_x, x \in \mathbb{R}\}$, and thus is covered by a finite sub-cover, and hence via (4.11) and (4.12),

$$\int_K f(y)\,\mathrm{d}y < \infty.$$

So **(ii)** holds. □

4.3 Stable Lévy Processes

The following theorem is a version of Theorem 4.1.3 in the case that X is a transient stable process on $\mathbb{R}$ and f is 'well-behaved' close to the issuing point. Its proof relies on a remarkably precise analytic description of $\mathbb{P}_x$-thin sets, which has its roots in potential theory, and for which we have provided a probabilistic proof in Chapter 6.

Theorem 4.3.1. *Let X be a symmetric stable process on $\mathbb{R}$ with index $\alpha \in (0,1)$, and $f \in \overline{\mathcal{B}(\mathbb{R})}_+$. Suppose that f has an isolated monotone maximum at 0, in the sense that there exists $\delta > 0$ such that f is monotone increasing on $(-\delta, 0)$, monotone decreasing on $(0, \delta)$. Then the following are equivalent.*

- ***(i)*** *There exists a random variable τ such that*

$$\mathbb{P}_0(0 < \tau < \infty) = 1 \qquad \textit{and} \qquad \mathbb{P}_0\Big(\int_0^\tau f(X_s)\,\mathrm{d}s < \infty\Big) = 1;$$

- ***(ii)*** *There exists $\varepsilon \in (0, \delta)$ such that $\displaystyle\int_{-\varepsilon}^{\varepsilon} f(y)|y|^{\alpha-1}\,\mathrm{d}y < \infty$.*

The maximum of f on $\mathbf{B}_\delta$ is allowed to be $+\infty$, so that f has an isolated pole at 0. In fact the result is trivial when this isn't the case.

Theorem 4.3.1 above can be directly extended to 'almost monotone' functions $g \in \overline{\mathcal{B}(\mathbb{R})}_+$, in the sense that there exists a $C < \infty$ such that for all $|x| < \delta$, $|g(x) - f(x)| \le C$ for some $f \in \overline{\mathcal{B}(\mathbb{R})}_+$ which has an isolated monotone pole at 0, by virtue of the fact that in this case

$$\left| \int_{-\varepsilon}^{\varepsilon} g(y)|y|^{\alpha-1}\,\mathrm{d}y - \int_{-\varepsilon}^{\varepsilon} f(y)|y|^{\alpha-1}\,\mathrm{d}y \right| \le \frac{2C\varepsilon^\alpha}{\alpha}$$

and

$$\left| \int_0^\tau g(X_s)\,\mathrm{d}s - \int_0^\tau f(X_s)\,\mathrm{d}y \right| \le C\tau.$$

Proof of Theorem 4.3.1

Recall from (2.17) that $U(0, \mathrm{d}y) = |y|^{\alpha-1}\,\mathrm{d}y$, up to a constant factor which we freely ignore. The implication **(ii)** $\Rightarrow$ **(i)** follows from Theorem 4.1.3 and the fact that $\mathbb{R}\backslash(-\varepsilon, \varepsilon)$ is $\mathbb{P}_0$-thin. Now suppose **(i)**. Again from Theorem 4.1.3 there exists a $\mathbb{P}_0$-thin set B such that

$$\int_{\mathbb{R}\backslash B} f(y)\, U(0, \mathrm{d}y) = \int_{\mathbb{R}\backslash B} f(y)|y|^{\alpha-1}\,\mathrm{d}y < \infty.$$

Take $\varepsilon \in (0, \delta)$ and let the map g be defined by $g(y) = f(y)|y|^{\alpha-1}\mathbf{1}_{(-\varepsilon,\varepsilon)}(y)$. It is immediately seen that g shares the same monotonicity property as f. The intuition to have in mind is that the monotone nature of g will allow its behaviour on B to be determined by its behaviour on $\mathbb{R} \setminus B$.

According to Corollary 6.3.2 (Wiener's Criterion for thin sets), B satisfies

$$\sum_{k=1}^{\infty} 2^{k(1-\alpha)} C(B \cap S_k) < \infty \tag{4.13}$$

where C is capacity and $S_k = \{x \in \mathbb{R} : 2^{-(k+1)} < |x| \leq 2^{-k}\}$ defines a sequence of decreasing shells of Lebesgue measure $2(2^{-k} - 2^{-(k+1)}) = 2^{-k}$. We saw in Lemma 2.6.9 that the isoperimetric inequality states that the α-capacity of $B \cap S_k$ is greater or equal that of the ball of the same volume, which following (2.19) yields

$$C(B \cap S_k) \geq C(\mathbf{B}_{\frac{1}{2}\lambda(B\cap S_k)}) = 2^{\alpha-1}\lambda(B \cap S_k)^{1-\alpha} C_0,$$

where $C_0 = C(\mathbf{B}_1)2^{\alpha-1}$, λ is the Lebesgue measure on $\mathbb{R}$, and $\mathbf{B}_r$ the ball about 0 of radius r. Therefore (4.13) implies

$$\sum_{k=1}^{\infty} 2^{k(1-\alpha)} C_0 \lambda(B \cap S_k)^{1-\alpha} = C_0 \sum_{k=1}^{\infty} (2^k \lambda(B \cap S_k))^{1-\alpha} < \infty.$$

Since $C_0 < \infty$ and $\lambda(S_k) = 2^{-k}$ this implies

$$\sum_{k=1}^{\infty} \Big(\frac{\lambda(B \cap S_k)}{\lambda(S_k)}\Big)^{1-\alpha} < \infty.$$

From this convergent sum it follows that for any fixed $c \in (0,1)$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\lambda(B \cap S_n) \leq c\lambda(S_n) = c2^{-n}$, and therefore that

$$\lambda(B^c \cap S_n) \geq (1-c)\lambda(S_n) = (1-c)2^{-n}. \tag{4.14}$$

We will use this relationship to bound the integral of g over B.

We shall now consider the two halves of S_n separately, using notation $S_n^+ = S_n \cap (0, \infty)$, $S_n^- = S_n \cap (-\infty, 0)$. Taking advantage of the monotonicity of g, with notation $\bar{g}_n = \sup_{S_n^+} g$ and $\underline{g}_n = \inf_{S_n^+} g$, we see for $n \geq N$ that

$$\int_{B\cap S_n^+} g(x)\,\mathrm{d}x \leq \bar{g}_n \lambda(B \cap S_n^+)$$

$$
\begin{aligned}
&\leq \bar{g}_n c\lambda(S_n^+) \\
&= \bar{g}_n 2c\lambda(S_{n+1}^+)
\end{aligned}
$$

Using (4.14) and the fact that $\bar{g}_n \leq \underline{g}_{n+1}$,

$$
\leq \frac{2c}{1-c}\, \underline{g}_{n+1}\lambda(B^c \cap S_{n+1}^+) \leq \frac{2c}{1-c}\int_{B^c\cap S_{n+1}^+} g(x)\,\mathrm{d}x.
$$

Exactly the same procedure works for S_n^-, and adding the two halves gives

$$
\int_{B\cap S_n} g(x)\,\mathrm{d}x \leq \frac{2c}{1-c}\int_{B^c\cap S_{n+1}} g(x)\,\mathrm{d}x.
$$

Summing over $n \geq N$ tells us that

$$
\int_{B\cap \mathbf{B}_{2^{-N}}} g(x)\,\mathrm{d}x \leq \frac{2c}{1-c}\int_{B^c} g(x)\,\mathrm{d}x < \infty.
$$

Let $\tilde{\varepsilon} = \varepsilon \wedge 2^{-N}$. Summing the integrals over B and $\mathbb{R} \setminus B$ then yields

$$
\int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} f(x)|y|^{\alpha-1}\,\mathrm{d}x = \int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} g(x)\,\mathrm{d}x < \infty.
$$

□

5 Stable SDEs

The (driftless) *stable SDE equation* with dispersion $\sigma \in \mathcal{B}(\mathbb{R})_+$ and issuing point $z \in \mathbb{R}$ is defined to be the equation

$$\mathrm{d}Z_t = \sigma(Z_{t-})\,\mathrm{d}X_t, \qquad Z_0 = z. \tag{5.1}$$

This equation represents the following concrete mathematical object. Let X be a stable process on state space $\mathbb{R}$ and probability space $\mathscr{P} = (\Omega, \mathcal{F}, \mathbb{P})$, and Z an $\mathbb{R}$-valued stochastic process on the same probability space satisfying $\mathbb{P}(Z_0 = z) = 1$ for some $z \in \mathbb{R}$. For $B \subseteq \mathbb{R}$ the collection $(X, Z, \mathscr{P})$ is called a *weak solution* to (5.1) on B if

$$Z_t - z = \int_0^t \sigma(Z_{s-})\,\mathrm{d}X_s \qquad \text{for all } t < T_{\mathbb{R}\setminus B}\ \mathbb{P}\text{-a.s.} \tag{5.2}$$

where $T_{\mathbb{R}\setminus B} = \inf\{s > 0 : Z_s \in B\}$. If $B = \mathbb{R}$ then $(X, Z, \mathscr{P})$ is called a *global weak solution* (or simply a weak solution) to (5.1), and if $B \subsetneq \mathbb{R}$ then $(X, Z, \mathscr{P})$ is called a *local weak solution* to (5.1). The process X is called the driving process, and Z is called the solution process.

If $\sigma(z) = 0$ then it is immediately seen that the almost surely constant process $Z_t = z$ for all $t \geq 0$ induces a weak solution $(X, Z, \mathscr{P})$ to (5.1). This solution, if it exists, is called the *trivial solution* to (5.1), and any other weak solution is called non-trivial. This chapter is concerned with deriving conditions on σ regarding existence of non-trivial weak solutions to (5.1).

In Theorem 4.0.1 a zero-one law on finiteness of path integrals of the Brownian motion was given, proven by Engelbert and Schmidt [15]. In the same paper the authors made use of a well known time-change representation of Brownian SDEs - see for example Theorems 3.4.2 and 3.4.6 of Karatzas and Shreve [28] - to prove the following.

Theorem 5.0.1 (Engelbert and Schmidt [15] Theorem 4)**.** *Let $\sigma \in \mathcal{B}(\mathbb{R})_+$. The following are equivalent.*

(i) *For every $z \in \mathbb{R}$ there exists a standard Brownian motion W and a solution process Z, both on state space $\mathbb{R}$ and probability space $\mathscr{P}$, such that $(W, Z, \mathscr{P})$ is a non-trivial weak solution to*

$$\mathrm{d}Z_t = \sigma(Z_{t-})\,\mathrm{d}W_t, \qquad Z_0 = z.$$

(ii) $\displaystyle\int_K \sigma(y)^{-2}\,\mathrm{d}y < \infty$ *for all compact* $K \subseteq \mathbb{R}$.

As before, their result was extended by Zanzotto [44] to stable processes with index $\alpha \in (1,2]$. But unlike before, proving the time-change representation for stable SDEs is significantly harder than for Brownian SDEs, because it is no longer possible to argue using quadratic variation. Despite this difficulty the resultant theorem is almost identical in form to that for the Brownian motion.

Theorem 5.0.2 (Zanzotto [44] Theorem 2.32)**.** *Let* $\sigma \in \mathcal{B}(\mathbb{R})_+$. *The following are equivalent.*

(i) *For every* $z \in \mathbb{R}$ *there exists a stable process of index* $\alpha \in (1,2]$ *and a solution process* Z, *both on state space* $\mathbb{R}$ *and probability space* $\mathscr{P}$, *such that* $(X, Z, \mathscr{P})$ *is a non-trivial weak solution to*

$$\mathrm{d}Z_t = \sigma(Z_{t-})\,\mathrm{d}X_t, \qquad Z_0 = z.$$

(ii) $\displaystyle\int_K \sigma(y)^{-\alpha}\,\mathrm{d}y < \infty$ *for all compact* $K \subseteq \mathbb{R}$.

Zanzotto's success in extending Theorem 5.0.1 might suggest that one could hope for a similar result for stable processes on $\mathbb{R}$ with index $\alpha \in (0,1)$. The following example demonstrates that, on the contrary, any analagous theorem for these processes must necessarily have a different integral test.

Example 5.0.3. *Let* $\sigma(x) = |x|^\beta$ *and fix* $\alpha \in (1,2]$. *We have already seen that because* $\sigma(0) = 0$ *the constant process* $Z \equiv 0$ *induces the trivial weak solution* $(X, Z, \mathscr{P})$ *to*

$$\mathrm{d}Z_t = \sigma(Z_{t-})\,\mathrm{d}X_t, \qquad Z_0 = 0, \tag{5.3}$$

where X *is a stable process on* $\mathbb{R}$ *of index* α. *The function* σ *is locally integrable everywhere away from* 0, *and so it also follows from Theorem 5.0.2 that a non-trivial solution* $(X', Z', \mathscr{P}')$ *to* (5.3) *exists if and only if there exists some* $\varepsilon > 0$ *such that*

$$\int_{-\varepsilon}^{\varepsilon} \sigma(y)^{-\alpha}\,\mathrm{d}y = \int_{-\varepsilon}^{\varepsilon} |y|^{-\alpha\beta}\,\mathrm{d}y < \infty,$$

which holds if and only if $-\alpha\beta > -1$, *that is,* $\beta < 1/\alpha$. *Thus if* $\beta < 1/\alpha$ *two weak solutions to* (5.3) *exist, one trivial and one not.*

But this cannot extend to $\alpha \in (0,1)$. *In that case if we were to take* $1 \leq \beta < 1/\alpha$ *then* σ *would be Lipschitz, and thus there is at most one unique weak solution to* (5.3).[1] *That is, the trivial and non-trivial solutions cannot both exist. The integral condition must therefore be different when* $\alpha \in (0,1)$.

[1] See Theorem 6.2.3 of Applebaum [1] to see that Lipschitz σ implies that solutions are pathwise unique.

The condition that non-trivial weak solutions exist for all issuing points $z \in \mathbb{R}$ is quite strong. There are two approaches to generalising this. The first is to consider the question of existence without distinguishing between trivial and non-trivial, and the second is to consider existence of weak solutions on a smaller set of z. Both cases were considered by Zanzotto [46] in 2002, but here we present only his result for the first. Let us introduce the following notation, for $\sigma \in \mathcal{B}(\mathbb{R})_+$, and X a stable Lévy process on $\mathbb{R}$.

$$\begin{aligned} N(\sigma) &:= \{x \in \mathbb{R} : \sigma(x) = 0\}, \\ \mathcal{O}(\sigma, \alpha) &:= \Big\{ x \in \mathbb{R} : \mathbb{P}_x\Big(\int_0^\tau \sigma(X_s)^{-\alpha} \, \mathrm{d}s = \infty \Big) = 1 \\ &\qquad\qquad \text{for all } \mathbb{P}_x\text{-a.s. positive random times } \tau \Big\}. \end{aligned}$$

It follows immediately from this definition that $\mathcal{O}(\sigma, \alpha)$ contains its regular points, and so is finely closed. For X a stable process of index $\alpha \in (1, 2]$, it follows from Theorem 4.2.1 that

$$\mathcal{O}(\sigma, \alpha) = \Big\{ x \in \mathbb{R} : \int_{x-}^{x+} \sigma(y)^{-\alpha} \, \mathrm{d}y = \infty \Big\},$$

and for X a symmetric stable process of index $\alpha < 1$, it follows from Theorem 4.1.3 and (2.17) that

$$\mathcal{O}(\sigma, \alpha) = \Big\{ x \in \mathbb{R} : \int_{\mathbb{R} \setminus B} \sigma(y)^{-\alpha} |y - x|^{\alpha - 1} \, \mathrm{d}y = \infty \text{ for all } \mathbb{P}_x\text{-thin } B \Big\}.$$

With this notation established we can present the most general result of Zanzotto.

Theorem 5.0.4 (Zanzotto [46] Theorems 2.2 and 2.6)**.** *Let $\sigma \in \mathcal{B}(\mathbb{R})_+$, and fix $\alpha \in (1, 2]$. The following are equivalent:*

- ***(i)*** *For every $z \in \mathbb{R}$ there exists a stable process X of index α and a solution process Z, both on state space $\mathbb{R}$ and probability space $\mathscr{P}$, such that $(X, Z, \mathscr{P})$ is a weak solution to*

$$\mathrm{d}Z_t = \sigma(Z_{t-}) \, \mathrm{d}X_t, \qquad Z_0 = z.$$

- ***(ii)*** *$\mathcal{O}(\sigma, \alpha) \subseteq N(\sigma)$.*

Furthermore, for every $z \in \mathbb{R}$, $(X, Z, \mathscr{P})$ is the unique weak solution if and only if $\mathcal{O}(\sigma, \alpha) = N(\sigma)$.

5.1 Kallenberg-Zanzotto Time-Change Representation

The bridge between the path integral zero-one laws of Engelbert and Schmidt and Zanzotto which were given in Chapter 4 and the SDE existence theorems above is a powerful theorem relating weak solutions of driftless stable SDEs to time-changed versions of the driving stable process. This bridge was explicitly and delicately built by Zanzotto

[44, 45, 46] and in part by Kallenberg [27] by considering the jumps of X as a random Poisson measure on $[0,\infty)\times\mathbb{R}$, and making use of the precise form of the Lévy measure π of a stable process.

Although Zanzotto proved his time-change representation for all stable processes on $\mathbb{R}$, he only proved an Engelbert-Schmidt-type result for $\alpha\in(1,2]$. In this section we shall present a proof of Zanzotto's time-change representation, reformulated and slightly expanded to deal with local solutions, which better suits our needs. Besides the reformulation this is worth doing because Zanzotto's original proof appears in pieces across three papers [44, 45, 46], and builds upon some work which can be hard to access.

This first lemma is a more general version of Zanzotto [46] Lemma 2.3. It is more involved than the original because of the complexity of finely open sets for general Lévy processes, compared to the relatively simple case of recurrent stable processes. The result bears some similarity to those of §10.3 and §10.4 of Helms [23].

Lemma 5.1.1. *Let X be a Lévy process on $\mathbb{R}^d$ satisfying (ACP) and (H), and let $f\in\overline{\mathcal{B}(\mathbb{R})}_+$. Define the path integrals and inverses*

$$I_t=\int_0^t f(X_s)\,\mathrm{d}s,\qquad \varphi_t=\inf\Big\{s>0:\int_0^s f(X_u)\,\mathrm{d}u>t\Big\},\qquad t\in[0,\infty).$$

Then $T_{\mathcal{O}}=\varphi_\infty$ $\mathbb{P}_z$-almost surely for all $z\in\mathbb{R}$, where

$$\mathcal{O}=\Big\{x\in\mathbb{R}:\mathbb{P}_x\Big(\int_0^{\tau} f(X_s)\,\mathrm{d}s=\infty\Big)=1\ \textit{for all } \mathbb{P}_x\textit{-a.s. positive random times } \tau\Big\}.$$

Proof. Fix a $q>0$ and let X^q be the killed process as we saw in (4.9), so that

$$\int_0^\infty f(X^q_s)\,\mathrm{d}s=\int_0^{T^q} f(X_s)\,\mathrm{d}s\qquad\text{almost surely,}$$

where τ^q is an independent exponentially distributed time of mean $1/q$. Define the sets

$$\begin{aligned}M_c&:=\Big\{x\in\mathbb{R}:\mathbb{P}_x\Big(\int_0^\infty f(X^q_s)\,\mathrm{d}s\le\frac{1}{c}\Big)>c\Big\},\qquad c>0,\\ B_c&:=\mathbb{R}\setminus M_c,\qquad c>0.\end{aligned}$$

The sets M_c are super-finite for (X^q,f), and increasing as $c\downarrow 0$. We shall also be interested in the set

$$B=\Big\{x\in\mathbb{R}:\mathbb{P}_x\Big(\int_0^\infty f(X^q_s)\,\mathrm{d}s<\infty\Big)=0\Big\}=\bigcap_{c>0}B_c=\lim_{c\downarrow 0}B_c$$

It is clearly true that $\mathcal{O}\in B$. But we can also note that if $x\in B$ then

$$\int_0^t f(X_s)\,\mathrm{d}s=\infty\qquad\mathbb{P}_x\text{-almost surely for all } t\in(0,\infty),$$

since τ^q is independent and has support $(0,\infty)$. Therefore $x \in \mathcal{O}$, and so it follows that the two sets are equal. We saw in Lemma 3.2.3 that for any choice of c, it holds that for all $y \in \mathbb{R}$

$$\int_0^t f(X_s^q)\mathbf{1}_{M_c}(X_s^q)\,\mathrm{d}s < \infty \qquad \text{for all } t \geq 0 \ \mathbb{P}_y\text{-almost surely.} \tag{5.4}$$

This, in combination with the fact that

$$\int_0^t f(X_s^q)\mathbf{1}_{M_c}(X_s)\,\mathrm{d}s = \int_0^t f(X_s^q)\mathbf{1}_{M_c}(X_s^q)\,\mathrm{d}s$$

almost surely, impies that $\int_0^t f(X_s^q)\,\mathrm{d}s < \infty$ $\mathbb{P}_y$-almost surely for all $t \leq T_{B_c}$, where $T_{B_c} = T_{\mathbb{R}\setminus M_c} = \inf\{s > 0 : X_s \in B_c\}$ is the first exit time of M_c by the unkilled process X. Since the sets B_c are decreasing, the times T_{B_c} are almost surely increasing. In particular, since (5.4) holds for arbitrary $c > 0$, it follows that for any $y \in \mathbb{R}$,

$$\int_0^t f(X_s^q)\,\mathrm{d}s < \infty \qquad \mathbb{P}_y\text{-almost surely for all } t < \lim_{c\downarrow 0} T_{B_c}.$$

Again since τ^q is independent and has support $(0,\infty)$, this implies that

$$\int_0^t f(X_s)\,\mathrm{d}s < \infty \qquad \mathbb{P}_y\text{-almost surely for all } t < \lim_{c\downarrow 0} T_{B_c}.$$

Therefore it follows that $\varphi_\infty \geq \lim_{c\downarrow 0} T_{B_c}$. We shall now show that

$$T \coloneqq \lim_{c\downarrow 0} T_{B_c} \geq T_B \qquad \text{almost surely.} \tag{5.5}$$

Recall that for a Borel set A and any $q > 0$ the function $\Phi_A^q(x) = \mathbb{E}_y[\mathrm{e}^{-qT_A}; T_A < \infty] = \mathbb{E}_y[\mathrm{e}^{-qT_A}]$ is q-excessive. Since our process X satisfies condition (H) and Φ_A^q is bounded, the discussion in Lemma 2.6.8 tells us that Φ_A^q is regular, and thus in particular that

$$\Phi_A^q(X_{T_{A_c}}) \to \Phi_A^q(X_T) \qquad \text{almost surely on } \{T < \infty\} \text{ as } c \downarrow 0.$$

If we fix $A = A_{c_0}$ for some $c_0 > 0$ then $\Phi_A^q(X_{T_{A_c}})$ equals one for all $c \leq c_0$, since the sets A_c are decreasing and contain their regular points by Lemma 3.2.5 so that $X_{T_{A_c}}$ is contained in $A_c \subseteq A_{c_0}$ on the event $\{T < \infty\}$ for any $y \in \mathbb{R}$. Thus the limit $\Phi_A^q(X_T)$ is equal one on $\{T < \infty\}$, that is, $\mathbb{P}_{X_{T(w)}}(T_{A_{c_0}} = 0) = 1$ for $\mathbb{P}_y$-almost every w such that $T(w) < \infty$. Since A_{c_0} contains its regular points, this implies that $X_T \in A_{c_0}$ on $\{T < \infty\}$. Then because our choice of $c_0 > 0$ was arbitrary, it follows that

$$X_T \in A = \bigcap_{c>0} A_c \qquad \text{almost surely on } \{T < \infty\}.$$

This implies (5.5) on $\{T < \infty\}$. On the event $\{T = \infty\}$, (5.5) is trivial, and so it holds almost surely, and thus $\varphi_\infty \geq T_A = T_{\mathcal{O}}$ almost surely.

The inequality $\varphi_\infty \le T_{\mathcal{O}}$ comes from the fact that for any $u > 0$, any $y \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{P}_y(\varphi_\infty \le T_{\mathcal{O}} + u\,; T_{\mathcal{O}} < \infty) &= \mathbb{P}_y\Big(\int_0^{T_{\mathcal{O}}+u} f(X_s)\,\mathrm{d}s = \infty\,; T_{\mathcal{O}} < \infty\Big) \\
&= \mathbb{E}_y\Big[\mathbb{P}_{X_{T_{\mathcal{O}}}}\Big(\int_0^u f(X_s)\,\mathrm{d}s = \infty\Big)\,; T_{\mathcal{O}} < \infty\Big] \\
&= \mathbb{P}_y(T_{\mathcal{O}} < \infty),
\end{aligned}$$

using that $\mathcal{O} = A = \cap_{c>0} A_c$ and that all A_c (and thus the intersection) are finely closed by Lemma 3.2.5 so that $X_{T_{\mathcal{O}}} \in \mathcal{O}$. This implies that $\mathbb{P}_y(\varphi_\infty \le T_{\mathcal{O}} + u) = \mathbb{P}_y(\varphi_\infty \le T_{\mathcal{O}} + u < \infty) + \mathbb{P}_y(T_{\mathcal{O}} = \infty) = 1$. Therefore $\varphi_\infty \le T_{\mathcal{O}} + u$ almost surely for all $u > 0$, and so $\varphi_\infty \le T_{\mathcal{O}}$ almost surely. □

Corollary 5.1.2. *Let X be a symmetric stable Lévy process on $\mathbb{R}$ of index α and*

$$f(x) = \sigma(x)^{-\alpha}$$

for $\sigma \in \mathcal{B}(\mathbb{R})_+$. Then $T_{\mathcal{O}(\sigma,\alpha)} = \varphi_\infty = \bar{\varphi}$ $\mathbb{P}_z$-almost surely for any $z \in \mathbb{R}$.

Proof. Symmetric stable processes satisfy (ACP) and (H), and Lemma 5.1.1 applies. It remains to note that since $\sigma^{-\alpha}$ is non-zero, the path integral over f is almost surely continuous and therefore $\varphi_\infty = \bar{\varphi}$ almost surely. □

Here is the first part of the Kallenberg-Zanzotto time-change. It is inspired by three of Zanzotto's results: [44] Lemma 2.26, [45] Theorem 2, and [46] Theorem 2.2.

Theorem 5.1.3 (Zanzotto's time-change)**.** *Let X be a symmetric stable process on probability space $(\Omega, \mathcal{F}, \mathbb{P}_z)$ of index $\alpha \in (0, 2]$, and let $\sigma \in \mathcal{B}(\mathbb{R})_+$. Define the path integrals*

$$I_t = \int_0^t \sigma(X_s)^{-\alpha}\,\mathrm{d}s, \qquad \varphi_t = \inf\Big\{s > 0 : \int_0^s \sigma(X_u)^{-\alpha}\,\mathrm{d}u > t\Big\}, \qquad t \ge 0,$$

and let X_φ be the time-changed process from (3.4)*. Then there exists an isotropic stable process Y of index α on $\overline{\mathscr{P}} = (\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ an extension of $(\Omega, \mathcal{F}, \mathbb{P}_z)$ such that*

$$X_{\varphi_t} - z = \int_0^t \sigma(X_{\varphi_{s-}})\,\mathrm{d}Y_s \qquad \text{for } t \in [0, \bar{I})\ \overline{\mathbb{P}}\text{-a.s.} \tag{5.6}$$

Corollary 5.1.4. *$(Y, X_\varphi, \overline{\mathscr{P}})$ is a local weak solution to* (5.1) *on $\mathbb{R} \setminus \mathcal{O}(\sigma, \alpha)$. If $z \in \mathbb{R} \setminus \mathcal{O}(\sigma, \alpha)$ then $(Y, X_\varphi, \overline{\mathscr{P}})$ is non-trivial.*

Proof. Since φ is strictly increasing and $\varphi(\bar{I}) = \varphi_\infty$ it follows from Corollary 5.1.2 that $\bar{I} = \inf\{s > 0 : X_{\varphi_s} \in \mathcal{O}(\sigma, \alpha)\}$, and thus from Theorem 5.1.3 that $(Y, X_\varphi, \overline{\mathscr{P}})$ is a local weak solution on $\mathbb{R} \setminus \mathcal{O}(\sigma, \alpha)$ to (5.1).

Suppose $z \in \mathbb{R} \setminus \mathcal{O}(\sigma, \alpha)$. Since $\mathcal{O}(\sigma, \alpha)$ is finely closed, we have that $\bar{I} > 0$ $\mathbb{P}_z$-a.s. It follows that there is some $t > 0$ for which $I_t < \infty$, and thus φ is not constant zero, and therefore that $(Y, X_\varphi, \overline{\mathscr{P}})$ is non-trivial. ☐

Corollary 5.1.5. *If either*

(A1) $\sigma(X_{\varphi_\infty}) = 0$ $\mathbb{P}_z$*-almost surely, or*

(A2) $\bar{I} = \infty$ $\mathbb{P}_z$*-almost surely,*

then (5.6) *reads*

$$X_{\varphi_t} - z = \int_0^t \sigma(X_{\varphi_s -}) \, \mathrm{d}Y_s \qquad \text{for } t \in [0, \infty) \quad \overline{\mathbb{P}}\text{-almost surely,}$$

and $(Y, X_\varphi, \overline{\mathscr{P}})$ *is a global weak solution to* (5.1).

Proof. Follows directly from Theorem 5.1.3. ☐

Corollary 5.1.6. *If*

(A3) $\mathcal{O}(\sigma, \alpha) \subseteq N(\sigma)$

then (A1) holds for all $z \in \mathbb{R}$*, and thus there exists a weak solution* $(Y, X_\varphi, \overline{\mathscr{P}})$ *to* (5.1) *for all* $z \in \mathbb{R}$.

Proof. To see that (A3) implies (A1) for all $z \in \mathbb{R}$, recall that Corollary 5.1.2, alongside the fact that $\mathcal{O}(\sigma, \alpha)$ is finely closed and thus contains its regular points, tells us that $X_{\varphi_\infty} \in \mathcal{O}(\sigma, \alpha)$ $\mathbb{P}_z$-almost surely for all $z \in \mathbb{R}$. Then Corollary 5.1.5 yields the result. ☐

Corollary 5.1.7. *If*

(A4) $\mathcal{O}(\sigma, \alpha) = \emptyset$

then there exists a non-trivial weak solution $(Y, X_\varphi, \overline{\mathscr{P}})$ *to* (5.1) *for all* $z \in \mathbb{R}$.

Proof. Since $\mathbb{R} \setminus \mathcal{O}(\sigma, \alpha) = \mathbb{R}$, Corollary 5.1.4 yields that a non-trivial local weak solution to (5.1) exists for all $z \in \mathbb{R}$, and since $\mathcal{O}(\sigma, \alpha)$ is empty, (A2) holds and the solutions are global. ☐

Now we can present the second half of the Kallenberg-Zanzotto time-change. This theorem was partially drawn from [27] Theorem 4.1.

Theorem 5.1.8 (Kallenberg's time-change)**.** *Let* Z *be a stochastic process on probability space* $\mathscr{P} = (\Omega, \mathcal{F}, \mathbb{P})$*, and suppose there exists an isotropic stable process* X *of index* $\alpha \in (0, 2]$ *on* $\mathscr{P}$ *such that* $(X, Z, \mathscr{P})$ *is a local weak solution on a set* $B \in \mathcal{B}(\mathbb{R})$ *to*

$$\mathrm{d}Z_t = \sigma(Z_{t-}) \, \mathrm{d}X_t, \qquad Z_0 = z.$$

Then there exists an isotropic stable process Y of index α on $\overline{\mathscr{P}} = (\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ an extension of $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$Y_t = Z_{\tilde{\varphi}_t} \qquad \textit{for all } t \in [0, \tilde{I}_{T_{\mathbb{R}\setminus B}})\ \overline{\mathbb{P}}\textit{-almost surely,} \tag{5.7}$$

where

$$\tilde{I}_t = \int_0^t \sigma(Z_{s-})^\alpha \, \mathrm{d}s, \qquad \tilde{\varphi}_t = \inf\{s > 0 : \tilde{I}_s > t\}, \qquad t \in [0, \infty).$$

and $T_{\mathbb{R}\setminus B} = \inf\{t > 0 : Z_t \notin B\}$.

Corollary 5.1.9. *If there exists a non-trivial local weak solution on $\mathbb{R} \setminus A$ to* (5.1) *with issuing point z, where A is $\mathbb{P}_z$-thin, then $z \in \mathbb{R} \setminus \mathcal{O}(\sigma, \alpha)$. This in particular holds when A is the empty set.*

This corollary is closely related to Proposition 2.1 of Zanzotto [46].

Proof. Let $\tilde{\varphi}_t = \inf\{s > 0 : \tilde{I}_s > t\}$, $t \in [0, \infty)$. Since A is $\mathbb{P}_z$-thin, the hitting time of A by Z is $\mathbb{P}_z$-almost surely positive. Since Z is not constant it follows from (5.2) that $\sigma(Z_s) > 0$ for some Lebesgue-positive set of times $s < T_A$, and thus $\tilde{I}$ is also not constant zero, and $\tilde{\varphi}$ does not jump directly to ∞.

Now

$$\begin{aligned}
\tilde{\varphi}_t \geq \int_0^{\tilde{\varphi}_t} \mathbf{1}_{(\sigma(Z_s)>0)} \, \mathrm{d}s &= \int_0^{\tilde{\varphi}_t} \sigma(Z_u)^{-\alpha} \sigma(Z_u)^\alpha \, \mathrm{d}u \\
&= \int_0^t \sigma(Z_{\tilde{\varphi}_s})^{-\alpha} \, \mathrm{d}s \\
&= \int_0^t \sigma(Y_s)^{-\alpha} \, \mathrm{d}s =: I_t.
\end{aligned}$$

As discussed above there is at least some $t \in (0, T_A)$ such that $I_t \leq \tilde{\varphi}_t < \infty$. It therefore holds by definition of $\mathcal{O}(\sigma, \alpha)$ that the issuing point z of Y under $\overline{\mathbb{P}}$ is an element of $\mathbb{R} \setminus \mathcal{O}(\sigma, \alpha)$. □

Corollary 5.1.10. *If there exists a weak solution to* (5.1) *for all $z \in \mathbb{R}$ then*

(A3) $\mathcal{O}(\sigma, \alpha) \subseteq N(\sigma)$.

Proof. Corollary 5.1.9 tells us that if $z \in \mathcal{O}(\sigma, \alpha)$ then there is no non-trivial solution to (5.1) with issuing point z. If we then assume that there exists a weak solution to (5.1) for all issuing points, it follows that the solution for $z \in \mathcal{O}(\sigma, \alpha)$ is trivial, and therefore $z \in N(\sigma)$. □

Corollary 5.1.11. *If there exists a non-trivial weak solution to* (5.1) *for all $z \in \mathbb{R}$ then*

(A4) $\mathcal{O}(\sigma, \alpha) = \emptyset$.

Proof. Once again Corollary 5.1.9 tells us that if $z \in \mathcal{O}(\sigma, \alpha)$ then there is no non-trivial solution to (5.1) with issuing point z. If we then assume that there exists a non-trivial weak solution to (5.1) for all issuing points, it follows that $\mathcal{O}(\sigma, \alpha)$ is empty. $\square$

5.2 Proofs of the Time-Change Representations

Proof of Theorem 5.1.3

The essence of the following proof is to compress (or stretch) X in time and space in a precise way to produce a new process, which due to the scaling property (2.5) has the same law as X, although possibly on a different probability space. The method of compression can result in some of the path of X being 'lost', and the transformed process then needs to be augmented by an independent copy of X, and it is this which leads to the extended probability space in the theorem statement.

The presentation of random measures here is mostly taken from Çinlar [9] and partially from Klenke [29], but the results we use are quite elementary and not specific to that work. In his original proof Zanzotto relied heavily on results from Jacod [25], including the time-change of compensator measures from Theorem 10.27 and the results of §14.4, in particular Theorem 14.56, but we prefer different references here. It's interesting to note that in §14.5 Jacod developed some foundations of the martingale problem, which is an alternative and far more common approach to studying weak solutions of stable SDEs.

Let X be a stable process on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ of index $\alpha \in (0, 2]$, and let $\sigma \in \mathcal{B}(\mathbb{R})_+$. Let $H = [0, \infty] \times \mathbb{R}$ and $\mathcal{H} = \mathcal{B}([0, \infty] \times \mathbb{R})$. The jump measure of X is a Poisson random measure on $(H, \mathcal{H})$, which we denote by μ, and which satisfies

$$\mu(\omega, A) = \sum_{D_\omega} \mathbf{1}_A(t, X_t(\omega) - X_{t-}(\omega)), \qquad D_\omega := \{t \in [0, \infty) : X_t(\omega) - X_{t-}(\omega) > 0\}$$

for $A \in \mathcal{H}$. This is exactly the Poisson random measure from the Lévy-Itô decomposition of Theorem 2.2.4. The first step of this proof is defining a new random measure on $(H, \mathcal{H})$ by shifting each of the points dropped by μ to a random new position, to create a new jump measure - which we will denote $\varphi\mu$ - corresponding to the jumps of X_φ. It is instructive to think of a simple example: if $\varphi_t = 2t$ then the transformation to obtain the jumps of X_φ would map the jump (t, x) of size x at time t to $(t/2, x)$. The first thought might then be to define the transformation from μ to $\varphi\mu$ by

$$(t, x) \mapsto (I_t, x).$$

But as we shall see, this does part of the job, but can cause some problems. Recall that since σ does not take the value infinity, I is strictly increasing, and thus φ is continuous everywhere and $\bar{\varphi} = \varphi_\infty$. The maximum value of I is attained at $\bar{I} = I(\bar{\varphi})$. For any $t > \bar{\varphi}$, $I(t) = \infty$ and any jump (t, x) is sent to the point $(\infty, x) \in H$ by the map above. These jumps are essentially 'forgotten' by the process X_φ. Thus we define the mapping

$$\begin{aligned}\psi_1 &: (H,\mathcal{H}) \to (H,\mathcal{H})\\ &: (t,x) \mapsto \begin{cases}(I_t, x) & \text{if } t < \bar{\varphi}\\ (0,0) & \text{if } t \geq \bar{\varphi}.\end{cases}\end{aligned}$$

The jump measure of X_φ is then given by $\varphi\mu = \mu \circ \psi_1^{-1}$. We already know that $I_s < \infty$ for every $s < \bar{\varphi}$. From the definition of I as a path integral it must follow that $\sigma(X_t) > 0$ for Lebesgue-almost every $t < \bar{\varphi}$. This, combined with the fact that φ is strictly increasing before reaching its maximum, implies that $\sigma(X_{\varphi_t}) > 0$ for Lebesgue-almost all $t \geq 0$.

Any Poisson random measure μ on $(H,\mathcal{H})$ with mean μ^p satisfies

$$\mathbb{P}(\mu(A) = 0) = \mathrm{e}^{-\mu^p(A)}, \qquad A \in \mathcal{H},$$

see for example Çinlar [9] Remark VI.2.4 or Klenke [29] Theorem 24.13. Lévy-Itô (Theorem 2.2.4) tells us that the mean μ^p of the jump measure of a Lévy process has the form $\mu^p = \mathrm{Leb} \times \pi$, and therefore that $B \times \mathbb{R}$ has zero mass almost surely under μ for any Lebesgue-zero Borel subset B of $[0,\infty)$. In particular, there is almost surely no jump (t,x) dropped by μ for which the jump time t satisfies $\sigma(X_{\varphi_t}) = 0$.

We now want to pull a similar trick on $\varphi\mu$ as we just did on μ, by transforming all the points it drops via the mapping

$$(t,x) \mapsto (t, \sigma(X_{\varphi_t})^{-1}x).$$

Taking the reciprocal of $\sigma(X_{\varphi_t})$ isn't problematic due to the discussion directly above. What we therefore define is the random measure ρ on $(H,\mathcal{H})$ with points given by $\varphi\mu \circ \psi_2^{-1}$, where

$$\begin{aligned}\psi_2 &: (H,\mathcal{H}) \to (H,\mathcal{H})\\ &: (t,x) \mapsto (t, \sigma(X_{\varphi_t})^{-1}x).\end{aligned}$$

The relationship between ρ and $\varphi\mu$ will be important later on. But it is easier now to note that the above mapping of the points is equivalent to directly setting $\rho = \mu \circ \psi_3^{-1}$, where

$$\begin{aligned}\psi_3 = \psi_2 \circ \psi_1 &: (H,\mathcal{H}) \to (H,\mathcal{H})\\ &: (t,x) \mapsto \begin{cases}(I_t, \sigma(X_t)^{-1}x) & \text{if } t < \bar{\varphi}, \text{ that is, if } I_t < \bar{I};\\ (0,0) & \text{otherwise,}\end{cases}\end{aligned}$$

where we have used the fact that σ does not take value $+\infty$, which implies that I is strictly increasing everywhere and thus that φ is continuous everywhere, to see that $\sigma(X_{\varphi(I_t)}) = \sigma(X_t)$. The mapping ψ_3 is again random, and for $A \in \mathcal{H}$ has law

$$\mathbb{P}(\psi_3(t,x) \in A) = \int_A \mathbb{P}\big((I_t, \sigma(X_t)^{-1}x) \in (\mathrm{d}s, \mathrm{d}y);\, t < \bar{\varphi}\big) + \mathbf{1}_A(0)\mathbb{P}(t \geq \bar{\varphi}).$$

This defines the random measure $\rho = \mu \circ \psi_3^{-1} = (\mu \circ \psi_1^{-1}) \circ \psi_2^{-1} = \varphi\mu \circ \psi_2^{-1}$ on $(H, \mathcal{H})$. The points it drops have been transformed in time and then afterwards in space, but all in ways determined by the jumps of X, and therefore an application of Theorem VI.3.2 of Çinlar [9] gives that ρ is a Poisson random measure on $(H, \mathcal{H})$ with mean

$$\rho^p(A) = \int_H \mathbb{P}((I_t, \sigma(X_t)^{-1}x) \in A;\, t < \bar{\varphi})\, \mu^p(\mathrm{d}t, \mathrm{d}x) \\ + \mathbf{1}_A(0)\mathbb{P}(t \geq \bar{\varphi})\mu^p(\{(0,0)\}).$$

The mean μ^p of μ is given by Leb$\times\pi$, and π has no mass on $\{0\}$, so the term on the right-hand-side vanishes, leaving

$$= \mathbb{E}\Big[\int_{[0,\bar{\varphi})\times\mathbb{R}} \mathbf{1}_A(I_t, \sigma(X_t)^{-1}x)\, \mathrm{d}t\, \pi(\mathrm{d}x)\Big]$$

Recalling the form of π from (2.6), we substitute $z = \sigma(X_t)^{-1}x$, which yields $\pi(\mathrm{d}x) = \sigma(X_t)^{-\alpha}\pi(\mathrm{d}z)$, to get

$$= \mathbb{E}\Big[\int_{[0,\bar{\varphi})\times\mathbb{R}} \mathbf{1}_A(I_t, z)\sigma(X_t)^{-\alpha}\, \mathrm{d}t\, \pi(\mathrm{d}z)\Big].$$

Substituting $u = I_t$, we get

$$= \mathbb{E}\Big[\int_{[0,I(\bar{\varphi}))\times\mathbb{R}} \mathbf{1}_A(u, z)\, \mathrm{d}u\, \pi(\mathrm{d}z)\Big].$$

Recalling (3.3),

$$= \int_A \mathbb{P}(\bar{I} > u)\, \mathrm{d}u\, \pi(\mathrm{d}z).$$

Although ρ is a Poisson random measure, it is not the jump measure of a Lévy process because it is inhomogeneous in time - recall from the Lévy-Itô decomposition of Theorem 2.2.4 that the jump measure of a Lévy process always has the form Leb$\times\pi$. Although if I were continuous, which would hold for example if σ were strictly positive, then $\bar{I}$ would be almost surely infinite and ρ would be homogeneous.

Now let ν be an independent Poisson random measure on $(H, \mathcal{H})$ and a distinct probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ with the same compensator Leb$\times\pi$ as μ.

We want to define the action of an 'expanded' version $\tilde{\rho}$ of ρ by choosing jumps from either ρ or ν depending on the landscape dictated by $\mathbf{1}\{s < \bar{I}\}$. We won't define $\tilde{\rho}$ via a transformation of points like we have seen before, but rather by cherry-picking points in different parts of H in the following way.

Here is a rough description of the procedure. First $\tilde{\rho}$ will draw from ρ, which drops jumps like Leb$\times\pi$ on parts of the state space, but none on parts where $t \geq \bar{I}$. When $t \geq \bar{I}$, $\tilde{\rho}$ will draw from ν. The action of ν is only relevant insofar as it "rounds out" the compensator of ρ in a convenient way. The jumps dropped by ν have no effect on the time-changed process X_φ, which after all is our ultimate object of interest.

So, formally, for each fixed $\omega \in \Omega$, define the random measure $\tilde{\rho}$ on $(H, \mathcal{H})$, $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}) := (\Omega \times \Omega', \mathcal{F} \times \mathcal{F}', \mathbb{P} \times \mathbb{P}')$ by

$$\tilde{\rho}(A, \omega, \omega') = \rho(A, \omega) + \tilde{\nu}(A, \omega, \omega'), \qquad \tilde{\nu}(A, \omega, \omega') := \int_A \mathbf{1}\{s \geq \bar{I}\}(\omega)\, \nu(\mathrm{d}s, \mathrm{d}x, \omega'). \tag{5.8}$$

Çinlar [9] Theorem VI.3.2 yields that $\tilde{\nu}$ is a Poisson random measure on $(H, \mathcal{H})$, $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ with mean

$$(\tilde{\nu})^p(\mathrm{d}s, \mathrm{d}x) = \overline{\mathbb{P}}(s \geq \bar{I})\, \mathrm{d}s\, \pi(\mathrm{d}x).$$

The two measures ρ and $\tilde{\nu}$ have almost surely disjoint supports on H, and are therefore not independent of one another. But they are conditionally independent given $\bar{I}$, and hence

$$\begin{aligned} \overline{\mathbb{P}}(\tilde{\rho}(A) = 0) &= \overline{\mathbb{E}}[\overline{\mathbb{P}}(\rho(A) = 0;\ \tilde{\nu}(A) = 0 | \bar{I})] \\ &= \overline{\mathbb{E}}[\overline{\mathbb{P}}(\rho(A_1) = 0;\ \tilde{\nu}(A_2) = 0 | \bar{I})] \end{aligned}$$

where A_1 and A_2 are A intersected with $\mathbf{1}\{s < \bar{I}\}$ and $\mathbf{1}\{s \geq \bar{I}\}$ respectively.

$$\begin{aligned} &= \overline{\mathbb{E}}[\mathbb{P}(\rho(A_1) = 0 | \bar{I});\ \tilde{\nu}(A_2) = 0] \\ &= \mathbb{P}(\rho(A) = 0)\mathbb{P}'(\tilde{\nu}(A) = 0) \\ &= \mathrm{e}^{-\rho^p(A)}\mathrm{e}^{-\tilde{\nu}^p(A)}. \end{aligned}$$

It follows from Klenke [29] Theorem 24.13 $\tilde{\rho}$ is a Poisson random measure on $(H, \mathcal{H})$, $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ with compensator

$$\rho^p + \tilde{\nu}^p = \mathrm{Leb} \times \pi.$$

Thus from Lévy-Itô it follows that $\tilde{\rho}$ is the jump measure of a stable process of index α on $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$, which inherits isotropy from X via the form of π. Let us denote this process by Y.

Let us regain our bearings. The process Y has jumps which are points in H dropped by the random measure $\tilde{\rho}$. Some of those jumps are the jumps (t, x) of X_φ which have been multiplied by the value of $\sigma(X_{\varphi_t})$, and the rest are drawn from an independent Poisson random measure ν. We can do some reverse engineering: taking a jump (t, x) dropped by $\tilde{\rho}$ and putting it through the mapping

$$(t, x) \mapsto (t, \sigma(X_{\varphi_t})x)$$

returns us to the original jumps of X_{φ_t}, as long as $\varphi_t < \bar{\varphi}$, that is, $t < \bar{I}$. If it were true that $\sigma(X_{\bar{\varphi}}) = 0$, then jumps at times $t \geq \bar{I}$ (which corresponds to $\varphi_t = \bar{\varphi}$) would be automatically 'forgotten' by this map, because their size would be mapped to zero. But if $\sigma(X_{\bar{\varphi}}) > 0$ these jumps need to be forcibly forgotten, and thus the mapping we use is

$$\psi_4 : (H, \mathcal{H}) \to (H, \mathcal{H})$$
$$: (t, x) \mapsto \begin{cases} (t, \sigma(X_{\varphi_t})x) & \text{if } t < \bar{I}; \\ (0, 0) & \text{if } t \geq \bar{I}. \end{cases}$$

Because $\sigma(X_s)$ is positive for almost every $s < \bar{\varphi}$, it holds that $\psi_2^{-1} \circ \psi_4^{-1}(t, x) = (t, x)$ for Lebesgue-almost every $t < \bar{I}$ and every $x \in \mathbb{R}$, and therefore that

$$\tilde{\rho} \circ \psi_4^{-1} = \varphi\mu \circ \psi_2^{-1} \circ \psi_4^{-1} = \varphi\mu \qquad \overline{\mathbb{P}}\text{-a.s. on } [0, \bar{I}) \times \mathbb{R}.$$

Therefore the pure jump processes that these jump measures describe are also almost surely equal up to time $\bar{I}$, when started from the same issuing point. The pure jump process on $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ started at 0 with jump measure $\tilde{\rho} \circ \psi_4^{-1}$ is

$$\int_0^t \sigma(X_{\varphi_s}) \mathbf{1}_{(s<\bar{I})} \, \mathrm{d}Y_s, \qquad t \geq 0.$$

The process on $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ started at 0 with jump measure $\varphi\mu$ is $X_{\varphi_t} - X_{\varphi_0}$, $t \geq 0$, and these two processes are almost surely equal on $[0, \bar{I})$. Thus (5.6) has been proven. $\square$

Proof of Theorem 5.1.8

As in the proof above let $H = [0, \infty] \times \mathbb{R}$ and $\mathcal{H} = \mathcal{B}([0, \infty] \times \mathbb{R})$, and let μ denote the jump measure of X. As shorthand write $T = T_{\mathbb{R}\setminus B} = \inf\{t > 0 : Z_t \notin B\}$, and let

$$\tilde{I}_t = \int_0^t \sigma(Z_{s-})^\alpha \, \mathrm{d}s, \qquad \tilde{\varphi}_t = \inf\left\{s > 0 : \int_0^s \sigma(Z_{u-})^\alpha \, \mathrm{d}u > t\right\}, \qquad t \geq 0.$$

Since Z satisfies the SDE equation (5.1), the jumps of Z are the jumps of X mapped via

$$\psi_1 : (t, x) \mapsto (t, \sigma(Z_{t-})x).$$

Now take the jumps of Z and map them via

$$: (t, x) \mapsto \begin{cases} (\tilde{I}_t, x) & t < T; \\ (0, 0) & t \geq T. \end{cases}$$

Equivalently, take the points of μ and map them via

$$\psi_2 : (t, x) \mapsto (\tilde{I}_t, \sigma(Z_{t-})x).$$

We give the name ρ to the random measure on state space $(H, \mathcal{H})$ and probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which drops the points of $\mu \circ \psi_2^{-1}$. Theorem 6.3.2 of Çinlar [9] tells us that ρ is a Poisson random measure with mean

$$(\rho)^p(\,\mathrm{d}s, \,\mathrm{d}y) = \int_H \mathbb{P}(\tilde{I}_t \in \,\mathrm{d}s, \, \sigma(Z_{s-})x \in \,\mathrm{d}y) \mathbf{1}_{(t<T)} (\mu)^p(\mathrm{d}t, \mathrm{d}x)$$

$$\begin{aligned}
&= \mathbb{E}\Big[\int_{[0,T)\times\mathbb{R}} \delta\{\tilde{I}_t \in \mathrm{d}s, z \in \mathrm{d}y\}\sigma(Z_{s-})^\alpha \,\mathrm{d}s\,\pi(\mathrm{d}z)\Big] \\
&= \mathbb{E}\Big[\int_{[0,\tilde{I}_T)\times\mathbb{R}} \delta\{u \in \mathrm{d}s, z \in \mathrm{d}y\}\,\mathrm{d}u\,\pi(\mathrm{d}z)\Big] \\
&= \mathbb{P}(s < \tilde{I}_T)\,\mathrm{d}s\,\pi(\mathrm{d}y).
\end{aligned}$$

Let ν be an independent Poisson random measure on $(H,\mathcal{H})$ and a distinct probability space $(\Omega',\mathcal{F}',\mathbb{P}')$ with compensator Leb$\times\pi$. As in the previous proof we want to expand the action of ρ by choosing jumps from either ρ or ν depending on the landscape dictated by $\mathbf{1}\{t < \tilde{I}_T\}$.

For each fixed $\omega \in \Omega$, define a random measure $\tilde{\rho}$ on $(H,\mathcal{H})$, $(\overline{\Omega},\overline{\mathcal{F}},\overline{\mathbb{P}}) := (\Omega\times\Omega',\mathcal{F}\times\mathcal{F}',\mathbb{P}\times\mathbb{P}')$ by

$$\tilde{\rho}(A,\omega,\omega') = \rho(A,\omega) + \tilde{\nu}(A,\omega'), \qquad \tilde{\nu}(A,\omega') := \int_A \mathbf{1}\{t \geq \tilde{I}_t\}\,\nu(\mathrm{d}s,\mathrm{d}x,\omega'). \tag{5.9}$$

The same argument as in the proof of Theorem 5.1.3 yields that $\tilde{\rho}$ is the jump process of an isotropic stable process Y on $(\overline{\Omega},\overline{\mathcal{F}},\overline{\mathbb{P}})$ of index α. Since $\tilde{I}_s < t$ if and only if $s < \tilde{\varphi}_t$,

$$Y_t = Z_{\tilde{\varphi}_t} \qquad \overline{\mathbb{P}}\text{-almost surely, for } t \in [0,\tilde{I}_T).$$

□

5.3 Transient Stable SDEs

With the Zanzotto-Kallenberg time-change representation of stable SDE solutions in place, proving analogues to Theorems 5.0.2 and 5.0.4 is more or less a case of directly translating the Theorems of Chapter 4 to the language of SDEs, with the time-change being the interpretor.

Recall the stable SDE equation (5.1):

$$\mathrm{d}Z_t = \sigma(Z_{t-})\,\mathrm{d}X_t, \qquad Z_0 = z. \tag{5.10}$$

For X a symmetric stable process on $\mathbb{R}$ of index α define as before the sets

$$N(\sigma) = \{x \in \mathbb{R} : \sigma(x) = 0\}$$

and

$$\mathcal{O}(\sigma,\alpha) = \Big\{x \in \mathbb{R} : \mathbb{P}_x\Big(\int_0^\tau \sigma(X_s)^{-\alpha}\,\mathrm{d}s = \infty\Big) = 1 \\ \text{for all } \mathbb{P}_x\text{-a.s. positive random times } \tau\Big\}.$$

In the case that $\alpha \in (1,2]$, Theorem 4.2.1 yields that

$$\mathcal{O}(\sigma,\alpha) = \Big\{x \in \mathbb{R} : \int_{x-\varepsilon}^{x+\varepsilon} \sigma(y)^{-\alpha}\,\mathrm{d}y = \infty \text{ for all } \varepsilon > 0\Big\}.$$

In the case that $\alpha < 1$, according to Theorem 4.1.3, a point $z \in \mathbb{R}$ is an element of $\mathcal{O}(\sigma, \alpha)$ if and only if for all $\mathbb{P}_z$-thin sets B,

$$\int_{\mathbb{R}\setminus B} \sigma(y)^{-\alpha}|y-z|^{\alpha-1}\,\mathrm{d}y = \infty.$$

If in addition σ has an isolated monotone pole at z then Theorem 4.3.1 gives that $z \in \mathcal{O}(\sigma, \alpha)$ if and only if for all $\varepsilon > 0$,

$$\int_{z-\varepsilon}^{z+\varepsilon} \sigma(y)^{-\alpha}|y-z|^{\alpha-1}\,\mathrm{d}y = \infty.$$

Theorem 5.3.1.

(i) *For fixed $z \in \mathbb{R}$ there exists a non-trivial local weak solution $(X, Z, \mathscr{P})$ to* (5.10) *if and only if $z \in \mathbb{R} \setminus \mathcal{O}(\sigma, \alpha)$.*

(ii) *A global weak solution $(X, Z, \mathscr{P})$ to* (5.10) *exists for all $z \in \mathbb{R}$ if and only if $\mathcal{O}(\sigma, \alpha) \subseteq N(\sigma)$.*

(iii) *A non-trivial global weak solution $(X, Z, \mathscr{P})$ to* (5.10) *exists for all $z \in \mathbb{R}$ if and only if $\mathcal{O}(\sigma, \alpha) = \emptyset$.*

Proof.

(i) From Corollary 5.1.4 it follows that if $z \in \mathbb{R}\setminus\mathcal{O}(\sigma, \alpha)$ then there exists a non-trivial local weak solution to (5.10). From Corollary 5.1.9 it follows that if there exists a non-trivial local weak solution to (5.10) then $z \in \mathbb{R} \setminus \mathcal{O}(\sigma, \alpha)$.

(ii) Corollary 5.1.6 tells us that $\mathcal{O}(\sigma, \alpha) \subseteq N(\sigma)$ is a sufficient condition for existence of a global weak solution to (5.10), and Corollary 5.1.10 tells us it is necessary.

(iii) Corollary 5.1.7 yields that $\mathcal{O}(\sigma, \alpha) = \emptyset$ is a sufficient condition for existence of a global weak solution to (5.10), and Corollary 5.1.11 tells us it is necessary.

□

Theorem 5.3.1**(iii)** is well-suited to finishing the story begun in Example 5.0.3. Let X be a stable process on $\mathbb{R}$ of index $\alpha \in (0, 1)$ and let $\sigma(x) = |x|^\beta$, so that a trivial solution issued from 0 exists. Theorem 5.3.1**(iii)** yields that a non-trivial solution issued from 0 exists if $\mathcal{O}(\sigma, \alpha) = \emptyset$, which by Theorem 4.3.1 implies

$$\int_{-\varepsilon}^{\varepsilon} \sigma^{-\alpha}(y)|y|^{\alpha-1}\,\mathrm{d}y = \int_{-\varepsilon}^{\varepsilon} |y|^{-\alpha\beta+\alpha-1}\,\mathrm{d}y < \infty,$$

which holds if and only if $-\alpha\beta + \alpha - 1 > -1$, that is, $\beta < 1$. This ensures that σ is not Lipschitz, and so there is no contradiction like the one in Example 5.0.3. What is notable is the independance of this condition from α.

Now we shall prove a theorem establishing equivalent conditions for existence of *unique* SDE solutions. This turns out to be a reasonable setting to explore questions of explosion and freezing in the next section.

Theorem 5.3.2. *There exists a weak solution to* (5.10) *for all $z \in \mathbb{R}$, each of which is unique in law, if and only if*

(A5) $\mathcal{O}(\sigma, \alpha) = N(\sigma)$.

In that case the solution process Z is given by $Z_t = Y_{\varphi_t}$, $t \in [0, \infty)$ almost surely, where Y is a stable process on $\mathbb{R}$ of index α and

$$\varphi_t = \inf\Big\{ s > 0 : \int_0^s \sigma(Y_u)^{-\alpha} \, \mathrm{d}u > t \Big\}, \qquad t \geq 0.$$

Proof. First suppose that a unique weak solution to (5.10) exists for all $z \in \mathbb{R}$, which by Theorem 5.3.1**(ii)** implies that $\mathcal{O}(\sigma, \alpha) \subseteq N(\sigma)$. Now suppose for contradiction that there is a point $z \in N(\sigma) \cap (\mathbb{R} \setminus \mathcal{O}(\sigma, \alpha))$. Since $z \in N(\sigma)$, the trivial solution is a solution. Since (A3) holds, Corollary 5.1.5 tells us that (A1) holds, and thus there exists a non-trivial weak solution with solution process

$$Z_t = X_{\varphi_t}, \qquad t \in [0, \infty).$$

Then we have two weak solutions issued from z which are not equal in law, and this contradicts uniqueness. So it follows that $N(\sigma) \cap (\mathbb{R} \setminus \mathcal{O}(\sigma, \alpha))$ is empty.

Now suppose that $\mathcal{O}(\sigma, \alpha) = N(\sigma)$. By Theorem 5.3.1**(ii)**, a weak solution to (5.10) exists for all $z \in \mathbb{R}$. We shall now prove the time-change representation for this solution. The argument here is similar to that in the proof of Corollary 5.1.9. We saw in (5.7) that

$$Y_s = Z_{\tilde{\varphi}_s} \qquad \overline{\mathbb{P}}\text{-almost surely, for } s \in [0, \tilde{I}_\infty), \tag{5.11}$$

where Y is a stable process on $\mathbb{R}$ of index α and

$$\tilde{I}_t = \int_0^t \sigma(Z_{s-})^\alpha \, \mathrm{d}s, \qquad \tilde{\varphi}_t = \inf\Big\{ s > 0 : \int_0^s \sigma(Z_{u-})^\alpha \, \mathrm{d}u > t \Big\}, \qquad t \geq 0.$$

Let $T = T_N = T_{\mathcal{O}}$ be the first hitting time of $\mathcal{O}$ by Y, which we saw in Corollary 5.1.2 is almost surely equal φ_∞. Then for all $t \leq T$,

$$I_t = \int_0^t \sigma(Y_s)^{-\alpha} \, \mathrm{d}s = \int_0^t \sigma(Z_{\tilde{\varphi}_s})^{-\alpha} \, \mathrm{d}s = \int_0^{\tilde{\varphi}_t} \sigma(Z_u)^{-\alpha} \sigma(Z_u)^\alpha \, \mathrm{d}u = \tilde{\varphi}_t.$$

For all $t > T$, we have by definition of $\mathcal{O}(\sigma, \alpha)$ that $I_t = \infty$. In addition we have from Corollary 5.1.9 that Z is constant after the first hitting of $\mathcal{O}(\sigma, \alpha)$. Since $\mathcal{O}(\sigma, \alpha) = N$ this implies that $\tilde{I}_t$ is constant after this time, and thus that $\tilde{\varphi}$ jumps to infinity upon hitting $\mathcal{O}(\sigma, \alpha)$, that is, $\tilde{\varphi}_t = \infty$ for $t > T$. Then it holds that

$$I_t = \tilde{\varphi}_t \qquad \text{for all } t \in [0, \infty) \ \overline{\mathbb{P}}\text{-almost surely.}$$

It follows from (5.11) that

$$Z_t = Y_{\varphi_t} \qquad \overline{\mathbb{P}}\text{-almost surely, for } t \in [0, \infty).$$

If $z \in \mathcal{O}(\sigma, \alpha)$ the time-change is identically 0 and the time-change solution and trivial solution coincide, so the solution is unique. If $z \in \mathbb{R} \setminus N(\sigma)$ then the trivial solution does not exist, so the non-trivial time-change solution is unique. Since $\mathcal{O}(\sigma, \alpha) = N(\sigma)$ we have proven uniqueness for all $z \in \mathbb{R}$. □

5.4 Behaviour of the Solution Process

Suppose that X is a symmetric stable process on $\mathbb{R}$ of index α, and suppose further that condition (A5) holds, so that by Theorem 5.3.2 for every issuing point $z \in \mathbb{R}$ the unique (in law) weak solution $(X, Z, \mathscr{P})$ to the SDE equation

$$\mathrm{d}Z_t = \sigma(Z_{t-})\,\mathrm{d}X_t, \qquad Z_0 = z \tag{5.12}$$

is the time change of some stable process, that is, there exists a symmetric stable process Y on $\mathbb{R}$ of index α such that

$$Z_t = Y_{\varphi_t} \text{ for } t \in [0, \infty), \qquad Z_\infty = \Delta, \tag{5.13}$$

where

$$\varphi_t = \inf\left\{s > 0 : \int_0^s \sigma(Y_u)^{-\alpha}\,\mathrm{d}u > t\right\}, \qquad t \geq 0.$$

Since σ takes values in $[0, \infty)$ we see that (3.6) holds, that is, that

$$t \mapsto I_t = \int_0^t \sigma(Y_s)^{-\alpha}\,\mathrm{d}s \tag{5.14}$$

is almost surely continuous on $[0, \infty)$, which from our discussion via Volkonskii [41] in Chapter 3 ensures that Z is a strong Markov process on $\mathbb{R}$. In this setting the solution process Z can itself have some interesting properties.

Explosion

Let ζ denote the lifetime of Z. From (5.13) it follows that $\varphi_\zeta = \infty$ almost surely, and since I is continuous this implies

$$\zeta = \int_0^\infty \sigma(Y_s)^{-\alpha}\,\mathrm{d}s.$$

We say that Z *explodes* if $\zeta < \infty$. If

$$I_t < \infty \text{ for all } t \in [0, \infty) \text{ almost surely,}^2 \tag{5.15}$$

then the event $\{\zeta < \infty\}$ is in the tail-σ-algebra of the Lévy process Y, and therefore has probability in $\{0, 1\}$, and explosion of Z becomes a zero-one law.

We can actually be more explicit: Theorem 3.1.1 gives a necessary and sufficient condition for explosion with positive probability, and Theorem 3.4.1 the same but with probability one. Via (5.15), Theorem 4.1.2 gives a necessary and sufficient condition for explosion to be a zero-one law, in the case that $\alpha < 1$.

Further, if $x \mapsto \sigma(x)^{-\alpha}$ is bounded on compact sets then Theorem 3.5.1 tells us that explosion is a zero-one law - although that is already clear from the fact that in that case (5.15) holds - and also gives a necessary and sufficient condition for explosion. In fact, if we assume the stronger condition that σ is bounded away from zero on compact sets, then according to Theorem 3.5.1, Z explodes if and only if there exists a $\mathbb{P}_z$-transient set B (for Y) such that

$$\int_{\mathbb{R}\setminus B} \sigma(x)^{-\alpha}\, U(z, \mathrm{d}x) < \infty. \tag{5.16}$$

If Y has index $\alpha > 1$ then it is point recurrent and the only choice of B is the empty set. In addition in this case the measure U is infinite on all sets with positive Lebesgue measure and it follows that explosion occurs if and only if $\sigma(x)^{-\alpha}$ is zero almost everywhere. It was an assumption of this Chapter that σ takes values in $[0, \infty)$, and in that case it follows that explosion of Z cannot occur.

The case of Y having index $\alpha = 1$ is similar, because Y is set recurrent, and so B must be a polar set (which in this case means a set with zero Lebesgue measure), and once again U is infinite on all sets with positive Lebesgue measure.

If Y has index $\alpha < 1$ then the integral test in (5.16) can be written

$$\int_{\mathbb{R}\setminus B} \sigma(x)^{-\alpha} |x - z|^{\alpha-1}\, \mathrm{d}x < \infty.$$

In that case it would be nice to remove B from the integral test, but the example of §6.4 shows that in general this is not possible.

Explosion of SDE solutions was also recently studied for positive continuous σ by Döring and Kyprianou [12]. In the same paper those authors also used a theory of time-reversal for Markov processes developed by Nagasawa [38] to consider a related problem, called 'entrance at infinity' of SDE solutions, in which an SDE solution, expressed as the time-reversal of another Markov process, is well-defined and non-trivial under a law which has it issuing from the infinity point. Time-reversal of Markov processes is closely related to duality, and can be quite unwieldy. Getoor and Sharpe [20] demonstrate their commitment to generality by presenting a version of Nagasawa's time reversal for Borel right processes in weak duality.

[2] We have been introduced to this condition before in (3.8), and it played an important role in the proof of Proposition 3.2.4. It is a sufficient condition for (5.14), and it holds for example if σ is continuous and strictly positive.

Freezing

We say that Z is *frozen* if there exists a time $t \in [0, \infty)$ such that $Z_s = Z_t$ for all $s \geq t$, which if Y is transient is equivalent to the event that $\lim_{t\to\infty} Z_t \neq \Delta$. It follows from (5.13) that Z is frozen if and only if $\varphi_\infty < \infty$, which from the definition of φ occurs if and only if there exists a $t \in [0, \infty)$ such that

$$\int_0^t \sigma(Y_s)^{-\alpha} \, \mathrm{d}s = \infty.$$

Thus it is clear that freezing and explosion preclude one another.

If Y is a symmetric stable process of index $\alpha > 1$ then it has local times and is point recurrent, and Theorem 4.2.2 yields that freezing is a zero-one law, and gives a sufficient and necessary condition for freezing to occur. If $\alpha < 1$ then Y is transient, and Theorem 4.1.2 gives a sufficient and necessary condition for freezing with positive probability, while Theorem 4.1.1 gives a sufficient and necessary condition for almost sure freezing. Unfortunately, the case $\alpha = 1$ eludes classification, at least with the methods presented here.

6 Avoidable Sets

At the end of §2.4 we introduced several classes of sets $B \in \mathcal{E}$ defined by path properties of a standard Markov process X on E. Polar sets and thin sets have been studied in detail for both general Markov processes and Lévy processes, and from both a probabilistic and a potential theoretic perspective. A class of sets which are less studied are the avoidable sets, and their complements the supportive sets, which played such a central role in Chapters 3 and 4.

A well-known result on thin sets is the Wiener Criterion, first proved in 1924 by Wiener [42, 43] and generalised by Brelot [8], see Helms [23] Theorem 10.21 for a full statement and proof. The Wiener Criterion gives a sufficient and necessary condition for a set $B \subseteq \mathbb{R}^d$ to be $\mathbb{P}_x$-thin for the Brownian motion on $\mathbb{R}^d$. A more general version of the Wiener Criterion, which covers all stable processes on $\mathbb{R}^d$ with index $\alpha < d$, can be found in Corollary 4.17 of Bliedtner and Hansen [4], and says that a set $B \in \mathcal{B}(\mathbb{R}^d)$ is $\mathbb{P}_x$-thin if and only if

$$\sum_{k=1}^{\infty} \lambda^{k(\alpha-d)} C(B \cap S_k) < \infty$$

where $\lambda \in (0,1)$ is an arbitrary constant and

$$S_k = \{y \in \mathbb{R}^d : \lambda^{k+1} < |y-x| \leq \lambda^k\}, \qquad k \in \mathbb{N}.$$

are the shells of scale λ around x. Bliedtner and Hansen's statement and proof is potential theoretic rather than probabilistic.

More recently Mimica and Vondraček studied unavoidable unions of balls in $\mathbb{R}^d$, for censored stable processes in [36] and for isotropic Lévy processes satisfying a particular scaling condition - which generalises the scaling of stable processes - in [37]. In Proposition A.3 and Corollary A.4 of [37] Mimica and Vondraček give a result analogous to the Wiener Criterion for avoidable sets in $\mathbb{R}^d$. A different perspective on avoidable sets for stable processes and more general Lévy processes can be found in Döring, Kyprianou, and Weißmann [13] and Döring, Watson, and Weißmann [14], in which the laws of stable and Lévy processes are transformed via a form of conditioning in order to make intervals avoidable.

The purpose of this chapter is to prove Wiener criteria for transient isotropic stable processes on $\mathbb{R}^d$ which are not the Brownian motion using purely probabilistic tools.

6.1 Potentials

Equilibrium measures

In order to prove a summation test like that of Bleidtner and Hansen it is necessary that B has an equilibrium measure. We have seen a necessary and sufficient condition for this in Lemma 2.6.2. The following Lemmas will be combined to show that for X a transient isotropic stable process on $\mathbb{R}^d$, any $\mathbb{P}_x$-avoidable set has an equilibrium measure.

Lemma 6.1.1. *Let X be a standard Markov process on $\mathbb{R}^d$, and $B \in \mathcal{B}(\mathbb{R}^d)$. If B is strongly $\mathbb{P}_x$-transient, that is $\mathbb{P}_x(L_B < \zeta) = 1$, then*

$$\mathbb{E}_x[\Phi_B(X_{T_{K^c}})] \downarrow 0 \qquad \text{as } K \uparrow \mathbb{R}^d$$

where $K \subseteq \mathbb{R}^d$ are compact, and $\Phi_B(x) = \mathbb{P}_x(T_B < \infty) = \mathbb{P}_x(L_B > 0)$.

Proof. First note that

$$\begin{aligned}\mathbb{E}_x[\Phi_B(X_{T_{K^c}})] &= \mathbb{E}_x[\mathbb{P}_{X_{T_{K^c}}}(L_B > 0)] \\ &= \mathbb{E}_x[\mathbb{P}_x(L_B > T_{K^c} | \mathcal{F}_{T_{K^c}})] \\ &= \mathbb{P}_x(L_B > T_{K^c}).\end{aligned}$$

Since the T_{K^c} are monotone increasing and have limit in $\{\zeta, \infty\}$ almost surely,[1] continuity of measure yields that $\mathbb{P}_x(L_B > T_{K^c}) \downarrow \mathbb{P}_x(L_B \geq \zeta) = 0$. $\square$

Lemma 6.1.2. *Let X be a Lévy process on $\mathbb{R}^d$ satisfying (ACT) and with support $\mathbb{R}^d$. Let $B \in \mathcal{B}(\mathbb{R}^d)$. If $\mathbb{P}_x(L_B < \infty) = 1$ for some $x \in \mathbb{R}^d$ then $\mathbb{P}_y(L_B < \infty) = 1$ for almost every $y \in \mathbb{R}^d$.*

Proof. If $\mathbb{P}_x(L_B < \infty) = 1$ then $\mathbb{P}_{X_t}(L_B < \infty) = 1$ $\mathbb{P}_x$-almost surely for all $t \geq 0$. Therefore

$$\mathbb{E}_x[\mathbb{P}_{X_t}(L_B = \infty)] = \int \mathbb{P}_y(L_B = \infty) p_t(y - x)\,\mathrm{d}y = 0. \tag{6.1}$$

Sato [39] Exercise 44.1 gives that the support of X is the support of u^q, which here is assumed to be $\mathbb{R}^d$, and from Theorem 2.5.2 and in particular the fact that

$$u^q(x) = \int_0^\infty \mathrm{e}^{-qt} p_t(x)\,\mathrm{d}t$$

it follows that the support of p_t is also $\mathbb{R}^d$. Thus (6.1) yields that $\mathbb{P}_y(L_B = \infty) = 0$ for Lebesgue-almost every $y \in \mathbb{R}^d$. $\square$

Lemma 6.1.3. *Let X be a Lévy process on $\mathbb{R}^d$, $B \in \mathcal{B}(\mathbb{R}^d)$, and $x \in \mathbb{R}^d$. If B is $\mathbb{P}_x$-avoidable, that is $\mathbb{P}_x(D_B < \infty) < 1$, then $\mathbb{P}_x(L_B < \infty) = 1$.*

[1] This relies on $\mathbb{R}^d$ being unbounded, see the proof of Corollary 3.4.2.

Proof. First note that $\mathbb{P}_x(L_B < \infty) \geq \mathbb{P}_x(D_B = \infty) > 0$. Since the tail-$\sigma$-algebra of a Lévy process is trivial, $\mathbb{P}_x(L_B < \infty)$ is a zero-one law, and thus $\mathbb{P}_x(L_B < \infty) = 1$. $\square$

Theorem 6.1.4. *Let X be a transient isotropic stable process on $\mathbb{R}^d$, and $B \in \mathcal{B}(\mathbb{R}^d)$ a $\mathbb{P}_x$-avoidable set. Then there exists a unique measure m_B on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that*

$$Um_B(x) = \mathbb{P}_x(T_B < \infty).$$

Proof. By Lemma 6.1.3 $\mathbb{P}_x(L_B < \infty) = 1$. Since X is an isotropic stable process on $\mathbb{R}^d$, X satisfies (ACT) and has support $\mathbb{R}^d$, and thus by Lemma 6.1.2 $\mathbb{P}_y(L_B < \infty) = 1$ for almost every $y \in \mathbb{R}^d$. Then Lemma 6.1.1 yields that

$$\mathbb{E}_y[\Phi_B(X_{T_{K^c}})] \downarrow 0 \text{ as } K \uparrow \mathbb{R}^d \text{ for almost every } y \in \mathbb{R}^d.$$

Since X is transient and satisfies (ACT), condition (C2) of §2.6 holds, and Lemma 2.6.2 (applied with $q = 0$) gives the result. $\square$

When it exists, the measure m_B in Theorem 6.1.4 is called the capacitary measure or equilibrium measure of B. Justification for that name can be found for example in Definition VI(4.5) of Blumenthal and Getoor [5] and the discussion following it.

Further Lemmas

The following lemmas establish some foundational results for avoidable sets which we shall use in proving the Wiener criteria.

Lemma 6.1.5. *Let X be a transient isotropic stable process on $\mathbb{R}^d$ of index $\alpha \in (0, 2)$, let $\boldsymbol{B} = \overline{\boldsymbol{B}_r}$ be the closed ball of radius r centred at 0. Then*

$$\mathbb{P}_x(T_{\boldsymbol{B}} = \infty) = \mathbb{P}_x(D_{\boldsymbol{B}} = \infty) > 0$$

for all $x \in \mathbb{R}^d \setminus \boldsymbol{B}$. That is, $\boldsymbol{B}$ is $\mathbb{P}_x$-avoidable for $x \notin \boldsymbol{B}$.

This lemma is a direct corollary of Blumenthal, Getoor, and Ray [7] Corollary 2 and an application of the scaling property of X, but we also provide a proof. For a more modern approach to the same problem see Kyprianou, Pardo, and Watson [32].

Proof. Theorem 1.1 of Kyprianou, Rivero, and Satitkanitkul [33] gives the law of the 'point of closest reach' of X, which is the random variable given by $\inf_{t\geq 0}|X_t|$, as

$$\mathbb{P}_x\big(\inf_{t\geq 0}|X_t| \in \mathrm{d}y\big) = \pi^{-d/2}\frac{\Gamma(d/2)^2}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)}\frac{(|x|^2 - |y|^2)^{\alpha/2}}{|x-y|^d|y|^\alpha}\,\mathrm{d}y, \qquad 0 < |y| < |x|.$$

This law has support on the ball $\mathbf{B}_{|x|}$, and in particular if $x \in \mathbb{R}^d \setminus \mathbf{B}$ then it has mass on the shell $\mathbf{B}_{|x|} \setminus \mathbf{B} = \{y : r < |y| < |x|\}$. It follows that for $x \in \mathbb{R}^d \setminus \mathbf{B}$,

$$\mathbb{P}_x(T_{\mathbf{B}} = \infty) = \mathbb{P}_x(D_{\mathbf{B}} = \infty) = \mathbb{P}_x\big(\inf_{t\geq 0}|X_t| > r\big) > 0.$$

$\square$

A far broader perspective on the above lemma can be found in Mimica and Vondraček [37], see in particular Theorem 1.1, and also for the Brownian motion in Gardiner and Ghergu [16], but we only have need for the simpler result as given.

This next lemma establishes a useful corollary of Lemma 6.1.5: a thin set can be 'reduced' to an avoidable set by intersecting it with a compact set.

Lemma 6.1.6. *Let X be an isotropic transient stable process on $\mathbb{R}^d$ of index $\alpha \in (0,2)$, and B be a $\mathbb{P}_x$-thin set. Then there exists $\varepsilon > 0$ such that $\tilde{B} = B \cap \{y : 0 < |y-x| \leq \varepsilon\}$ is $\mathbb{P}_x$-avoidable.*

Proof. Because B is $\mathbb{P}_x$-thin, $\tilde{B}$ is also $\mathbb{P}_x$-thin. Thus by Blumenthal's zero-one law there exists a deterministic time $t > 0$ such that $\mathbb{P}_x(T_{\tilde{B}} > t) > 1/2$. Fix this t. Denote by A_ε the set $\{y : |y-x| > \varepsilon\}$. The probability $\mathbb{P}_x(T_{A_{2\varepsilon}} < t)$ can be made arbitrarily close to 1 by choosing ε small enough. In particular we can choose ε such that $\mathbb{P}_x(T_{A_{2\varepsilon}} < t) > 1/2$, and thus for this choice of ε it holds that

$$\mathbb{P}_x(T_{A_{2\varepsilon}} < T_{\tilde{B}}) \geq \mathbb{P}_x(T_{A_{2\varepsilon}} < t < T_{\tilde{B}}) > 0. \tag{6.2}$$

An application of the strong Markov property at $T_{A_{2\varepsilon}}$ also yields that

$$\begin{aligned}
\mathbb{P}_x(T_{\tilde{B}} = \infty) &= \mathbb{P}_x(T_{\tilde{B}} = \infty;\ T_{A_{2\varepsilon}} < T_{\tilde{B}}) \\
&= \mathbb{E}_x[\mathbb{P}_{X_{T_{A_{2\varepsilon}}}}(T_{\tilde{B}} = \infty);\ T_{A_{2\varepsilon}} < T_{\tilde{B}}] \\
&\geq \mathbb{E}_x[\mathbb{P}_{X_{T_{A_{2\varepsilon}}}}(T_{\mathbb{R}^d \setminus A_\varepsilon} = \infty);\ T_{A_{2\varepsilon}} < T_{\tilde{B}}].
\end{aligned}$$

Because the support of $X_{T_{A_{2\varepsilon}}}$ is contained in $\overline{A_{2\varepsilon}} \subsetneq A_\varepsilon$ we can apply Lemma 6.1.5 to see that the inner probability is almost surely positive. Therefore by (6.2) the expectation on the right-hand-side is positive, and so $\mathbb{P}_x(T_{\tilde{B}} = \infty) > 0$. Since $\tilde{B}$ doesn't contain x, $\mathbb{P}_x(T_{\tilde{B}} = D_{\tilde{B}}) = 1$, and thus $\tilde{B}$ is $\mathbb{P}_x$-avoidable. □

The lemma below establishes that taking the union of an avoidable set and a (bounded) shell yields another avoidable set.

Lemma 6.1.7. *Let X be an isotropic transient stable process on $\mathbb{R}^d$, and let B be a $\mathbb{P}_z$-avoidable set. Then for any constants $\gamma, \lambda \in (0,\infty)$ with $\gamma < \lambda$,*

$$B \cup \{x : |x-z| \in (\gamma, \lambda]\}$$

is also $\mathbb{P}_z$-avoidable.

Proof. Let us write $\mathbf{B} = \overline{\mathbf{B}}_\lambda(z) = \{x : |x-z| \leq \lambda\}$ and $A = \{x : |x-z| \in (\gamma, \lambda]\}$.

It follows from Corollary 2 of Blumenthal, Getoor, and Ray [7] that $\mathbb{P}_y(D_{\mathbf{B}} = \infty) \uparrow 1$. as $|y| \to \infty$. Thus isotropy of X gives that for all $\varepsilon > 0$ there exists some k large enough such that $\mathbb{P}_y(D_{\mathbf{B}} = \infty) > 1 - \varepsilon$ for all $|y| \geq k$.

Now, we have by assumption that $\mathbb{P}_z(D_B = \infty) > 0$. For any $k > 0$ let $T_k = T_{\mathbb{R}^d \setminus \mathbf{B}_k(z)} = \inf\{s > 0 : |X_s - z| \geq k\}$, the first exit time of the ball $\mathbf{B}_k(z)$. Then the strong Markov property at T_k yields

$$0 < \mathbb{P}_z(D_B = \infty) = \mathbb{E}_z[\mathbb{P}_{X_{T_k}}(D_B = \infty);\ D_B \notin [0, T_k)],$$

and therefore that there must exist some Borel set $\tilde{B} \in \mathbb{R}^d \setminus \mathbf{B}_k(z)$ such that

$$\mathbb{P}_z(X_{T_k} \in \tilde{B};\ D_B \notin [0, T_k)) > 0 \tag{6.3}$$

and

$$\mathbb{P}_y(D_B = \infty) \geq \mathbb{P}_z(D_B = \infty) \quad \text{for all } y \in \tilde{B}. \tag{6.4}$$

Since the jump measure of X is isotropic and has support on $\mathbb{R}^d$, (6.3) implies that

$$\mathbb{P}_z(X_{T_{k'}} \in \tilde{B};\ D_B \notin [0, T_{k'})) > 0 \qquad \text{for all } k' \leq k. \tag{6.5}$$

Let us combine what we have seen. Since $\mathbb{P}_z(D_B = \infty) > 0$ (by assumption of $\mathbb{P}_z$-avoidability) we can fix a $k > \lambda$ such that

$$\mathbb{P}_y(D_{\mathbf{B}} = \infty) > 1 - \mathbb{P}_z(D_B = \infty)/2 \qquad \text{for all } |y| \geq k. \tag{6.6}$$

For this same choice of k there exists a set $\tilde{B}$ satisfying (6.3) and (6.4). Combining (6.6) and (6.4) yields that $\mathbb{P}_y(D_B = \infty;\ D_{\mathbf{B}} = \infty) > c$ for all $y \in \tilde{B}$, where $c = \mathbb{P}_z(D_B = \infty)/2$ is a positive constant, and thus that

$$\begin{aligned} \mathbb{P}_y(D_{B\cup A} = \infty) &= \mathbb{P}_y(D_B = \infty;\ D_A = \infty) \\ &\geq \mathbb{P}_y(D_B = \infty;\ D_{\mathbf{B}} = \infty) > c > 0 \qquad \text{for all } y \in \tilde{B}. \end{aligned} \tag{6.7}$$

It lastly remains to note from (6.5) that since $k > \lambda > \gamma$,

$$\mathbb{P}_z(X_{T_\gamma} \in \tilde{B};\ D_{B\cup A} \notin [0, T_\gamma)) = \mathbb{P}_z(X_{T_\gamma} \in \tilde{B};\ D_B \notin [0, T_\gamma)) > 0.$$

This, combined with (6.7), gives

$$\begin{aligned} \mathbb{P}_z(D_{B\cup A} = \infty) &= \mathbb{E}_z[\mathbb{P}_{X_{T_\gamma}}(D_{B\cup A} = \infty);\ D_{B\cup A} \notin [0, T_\gamma)] \\ &\geq \mathbb{E}_z[\mathbb{P}_{X_{T_\gamma}}(D_{B\cup A} = \infty);\ D_{B\cup A} \notin [0, T_\gamma);\ X_{T_\gamma} \in \tilde{B}] \\ &> c\,\mathbb{P}_z(X_{T_\gamma} \in \tilde{B};\ D_{B\cup A} \notin [0, T_\gamma)) \\ &> 0. \end{aligned}$$

That is, $B \cup A$ is $\mathbb{P}_z$-avoidable. $\square$

Corollary 6.1.8. *Let X be an isotropic transient stable process on $\mathbb{R}^d$ of index $\alpha \in (0, 2)$, and B be a bounded $\mathbb{P}_x$-thin set. Then $B \setminus \{x\}$ is $\mathbb{P}_x$-avoidable.*

Proof. According to 6.1.6 there exists $\varepsilon > 0$ such that $\tilde{B} = B \cap \{y : 0 < |y - x| \leq \varepsilon\}$ is $\mathbb{P}_x$-avoidable. Then for any $\delta < \varepsilon$, it clearly holds that $B \cap \{y : 0 < |y - x| \leq \delta\}$ is also $\mathbb{P}_x$-avoidable. For $\delta > \varepsilon$, Lemma 6.1.7 tells us that

$$B \cap \{y : 0 < |y - x| \leq \delta\} \subseteq \tilde{B} \cup \{y : |y - x| \in (\varepsilon, \delta]\}$$

is $\mathbb{P}_x$-avoidable. Since B is bounded there exists some choice of $\delta > 0$ such that $B \cap \{y : 0 < |y - x| \leq \delta\} = B \setminus \{x\}$, and the proof is finished. □

Open Avoidable Sets

This final theorem is the analogue of an existing result for $\mathbb{P}_x$-thin sets,[2] and is useful for addressing a technical point in the proof of Theorem 6.2.1 below.

Theorem 6.1.9. *Let X be a transient Hunt process on $\mathbb{R}^d$, and B be a $\mathbb{P}_x$-avoidable set. Then there exists an open set G such that $B \subseteq G$ and G is $\mathbb{P}_x$-avoidable.*

Proof. For $A \in \mathcal{B}(\mathbb{R}^d)$ and $t \in [0, \infty)$ define the event

$$\begin{aligned} R_t(A) &= \big\{\omega \in \Omega : \text{ there exists } s \in [0, t] \text{ such that } X_s \in A\big\} \\ &= \big\{\omega \in \Omega : D_A(\omega) \leq t\big\}. \end{aligned}$$

The function

$$\begin{aligned} \varphi &: \mathcal{B}(\mathbb{R}^d) \to \mathbb{R} \\ &: A \mapsto \mathbb{P}_x(R_t(B)) \end{aligned}$$

is a Choquet capacity, and this fact implies that for any $A \in \mathcal{B}(\mathbb{R}^d)$,

$$\mathbb{P}_x(R_t(A)) = \inf_{G \supseteq A} \mathbb{P}_x(R_t(G)), \tag{6.8}$$

where the infimum is taken over all open sets G containing A. For details see Definition I(10.5), Theorem I(10.6), and Remark I(10.13) of Blumenthal and Getoor [5]. Now fix the sets

$$m_n = \{x \in \mathbb{R}^d : n - 1 \leq |x| < n\}, \qquad M_n = \bigcup_{k=1}^{n} m_k, \qquad n \in \mathbb{N}.$$

Since X is transient each M_n has an almost surely finite last exit time, and thus for any $\varepsilon > 0$ there exists an increasing sequence of times $t_n \in (0, \infty)$ such that $\mathbb{P}_x(L_{M_n} < t_n) > 1 - \varepsilon$ for every $n \in \mathbb{N}$. In particular, choosing $\delta = (1 - \mathbb{P}_x(D_B < \infty))/2$, which is positive because B is $\mathbb{P}_x$-avoidable, there exists an increasing sequence of times $t_n \in (0, \infty)$ such that

$$\mathbb{P}_x(L_{M_n} < t_n) > 1 - \delta 2^{-n-1} \quad \text{for all } n \in \mathbb{N}. \tag{6.9}$$

[2] See Blumenthal and Getoor [5] Proposition II(4.3) for a probabilistic version, or Bliedtner and Hansen [4] Lemma II.4.1 for a potential theoretic one.

Fix this sequence t_n. Our choice of δ also satisfies $\delta + \mathbb{P}_x(D_B < \infty) < 1$, which will be useful later. Now write $B_n = B \cap m_n$, so that $B = \cup_n B_n$ and the B_n form a partition of B. By (6.8) for every $n \in \mathbb{N}$ we can find an open set $G_n \supseteq B_n$ such that

$$\mathbb{P}_x(D_{G_n} \le t_n) < \mathbb{P}_x(D_{B_n} \le t_n) + \delta 2^{-n-1}, \tag{6.10}$$

with t_n and δ the same as above. Thus it follows from our choice of t_n satisfying (6.9) that

$$\begin{aligned}
\mathbb{P}_x(L_{M_n} < D_{G_n}) &\ge \mathbb{P}_x(L_{M_n} < t_n < D_{G_n}) \\
&\ge 1 - \big(\mathbb{P}_x(L_{M_n} \ge t) + \mathbb{P}_x(D_{G_n} \le t_n)\big) && \text{(subadditivity)} \\
&= \mathbb{P}_x(L_{M_n} < t_n) - \mathbb{P}_x(D_{G_n} \le t_n) \\
&> 1 - \delta 2^{-n-1} - \mathbb{P}_x(D_{G_n} \le t_n) && \text{by (6.9)} \\
&> 1 - \delta 2^{-n-1} - \mathbb{P}_x(D_{B_n} \le t_n) - \delta 2^{-n-1} && \text{by (6.10)} \\
&= \mathbb{P}_x(D_{B_n} > t_n) - \delta 2^{-n}.
\end{aligned}$$

Let $\tilde{G}_n = G_n \cap M_n$, which is an open set containing B_n, and which satisfies

$$\mathbb{P}_x(D_{\tilde{G}_n} < \infty) \le \mathbb{P}_x(D_{G_n} \le L_{M_n}).$$

Therefore

$$\mathbb{P}_x(D_{\tilde{G}_n} < \infty) \le 1 - \mathbb{P}_x(L_{M_n} < D_{G_n}) < \mathbb{P}_x(D_{B_n} \le t_n) + \delta 2^{-n}.$$

From this it follows that

$$\begin{aligned}
\mathbb{P}_x(D_{\tilde{G}_n} < \infty;\ D_{B_n} = \infty) &= \mathbb{P}_x(D_{\tilde{G}_n} < \infty) - \mathbb{P}_x(D_{\tilde{G}_n} < \infty;\ D_{B_n} < \infty) \\
&= \mathbb{P}_x(D_{\tilde{G}_n} < \infty) - \mathbb{P}_x(D_{B_n} < \infty) && \text{(since } B_n \subseteq \tilde{G}_n) \\
&< \mathbb{P}_x(D_{B_n} \le t_n) + \delta 2^{-n} - \mathbb{P}_x(D_{B_n} < \infty) \\
&\le \delta 2^{-n}.
\end{aligned}$$

Now define $\tilde{G} = \cup_n \tilde{G}_n$. Then

$$\begin{aligned}
\mathbb{P}_x(D_{\tilde{G}} < \infty) &= \mathbb{P}_x(\cup_{n=1}^{\infty}\{D_{\tilde{G}_n} < \infty\}) \\
&= \mathbb{P}_x(\cup_{n=1}^{\infty}\{D_{\tilde{G}_n} < \infty;\ D_{B_n} < \infty\} \cup \{D_{\tilde{G}_n} < \infty;\ D_{B_n} = \infty\}) \\
&\le \mathbb{P}_x(\cup_{n=1}^{\infty}\{D_{\tilde{G}_n} < \infty;\ D_{B_n} < \infty\}) + \mathbb{P}_x(\cup_{n=1}^{\infty}\{D_{\tilde{G}_n} < \infty;\ D_{B_n} = \infty\}) \\
&\le \mathbb{P}_x(\cup_{n=1}^{\infty}\{D_{B_n} < \infty\}) + \sum_{n=1}^{\infty} \mathbb{P}_x(D_{\tilde{G}_n} < \infty;\ D_{B_n} = \infty) \\
&\le \mathbb{P}_x(D_B < \infty) + \sum_{n=1}^{\infty} \delta 2^{-n} \\
&< 1.
\end{aligned}$$

The final inequality holds as a result of our choosing $\delta = (1 - \mathbb{P}_x(D_B < \infty))/2$ earlier in the proof. Then $\tilde{G}$ is an open $\mathbb{P}_x$-avoidable set containing B, and the proof is finished.

$\square$

6.2 Summation Tests

The two theorems presented here give necessary and sufficient conditions for avoidability and thinness of $B \in \mathcal{B}(\mathbb{R}^d)$ in terms of the function Φ_B^q from (2.8). In §6.3 we shall give two corollaries which rephrase the conditions in terms of capacity. Let $\lambda \in (0,1)$ be an arbitrary positive constant and denote the shells of scale λ around $z \in \mathbb{R}^d$ by

$$S_k = \{x \in \mathbb{R}^d : \lambda^{k+1} < |x-z| \leq \lambda^k\}, \qquad k \in \mathbb{Z}.$$

Every point of $\mathbb{R}^d \setminus \{z\}$ is in exactly one shell, and the shells decrease as k increases. For any set $B \in \mathcal{B}(\mathbb{R}^d)$ define the sets

$$B_k \coloneqq B \cap S_k, \qquad k \in \mathbb{Z}.$$

When a Lévy process X on $\mathbb{R}^d$ is isotropic its law is rotation-invariant. One result of this is that $u(x,y) = u(|x-y|)$ for all $x, y \in \mathbb{R}^d$, and another is that the dual process $\hat{X} = -X$ has the same law as X, and thus $\hat{u} = u$. This allows us to drop the $\hat{\cdot}$ duality notation in all of what follows.

Theorem 6.2.1. *Let X be a transient isotropic stable process on $\mathbb{R}^d$ of index $\alpha \in (0,2)$, and let $B \in \mathcal{B}(\mathbb{R}^d)$. Then B is $\mathbb{P}_z$-avoidable if and only if*

$$\sum_{k=-\infty}^{\infty} \mathbb{P}_z(T_{B_k} < \infty) < \infty$$

and $z \notin B$.

The proof of Theorem 6.2.1 is given below, but first we present and prove the following corollary, which corresponds to Proposition V.4.15 of Bliedtner and Hansen [4].

Theorem 6.2.2. *Let X be a transient isotropic stable process on $\mathbb{R}^d$ of index $\alpha \in (0,2)$, and let $B \in \mathcal{B}(\mathbb{R}^d)$. Then B is $\mathbb{P}_z$-thin if and only if*

$$\sum_{k=1}^{\infty} \mathbb{P}_z(T_{B_k} < \infty) < \infty. \tag{6.11}$$

Proof. First suppose that (6.11) holds. Fix a $k_0 \in \mathbb{N}$ and define $C = B \cap \{x : |x-z| \leq \lambda^{k_0}\}$. Then

$$\mathbb{P}_z(T_C < \infty) \leq \sum_{k=k_0}^{\infty} \mathbb{P}_z(T_C < \infty;\ T_{C_k} = T_C) \leq \sum_{k=k_0}^{\infty} \mathbb{P}_z(T_{C_k} < \infty).$$

Due to (6.11) there exists a choice of k_0 such that

$$\mathbb{P}_z(T_C < \infty) \leq \sum_{k=k_0}^{\infty} \mathbb{P}_z(T_{C_k} < \infty) < 1.$$

Thus $\mathbb{P}_z(T_C = 0) < 1$, and C is $\mathbb{P}_z$-thin. The union $C \cup \{x : |x-z| > \lambda^{k_0}\}$ of two $\mathbb{P}_z$-thin sets is also $\mathbb{P}_z$-thin, and since $B \subseteq C \cup \{x : |x-z| > \lambda^{k_0}\}$ we have the result.

Suppose now that B is $\mathbb{P}_z$-thin. Lemma 6.1.6 gives an $\varepsilon > 0$ such that $\tilde{B} = B \cap \{x : 0 < |x-z| \leq \varepsilon\}$ is $\mathbb{P}_z$-avoidable. Choosing a $k_0 \in \mathbb{N}$ such that $\lambda^{k_0} < \varepsilon$ it follows from Theorem 6.2.1 that

$$\begin{aligned}\sum_{k=k_0}^{\infty} \mathbb{P}_z(T_{B_k} < \infty) &= \sum_{k=k_0}^{\infty} \mathbb{P}_z(T_{\tilde{B}_k} < \infty) \\ &\leq \sum_{k=-\infty}^{\infty} \mathbb{P}_z(T_{\tilde{B}_k} < \infty) < \infty.\end{aligned}$$

The sum $\sum_{k=1}^{k_0} \mathbb{P}_z(T_{B_k} < \infty)$ is clearly finite, and that finishes the proof. □

Proof of Theorem 6.2.1

First let X be a transient isotropic stable process on $\mathbb{R}^d$ of index $\alpha \in (0,2)$, let $B \in \mathcal{B}(\mathbb{R}^d)$, and suppose

$$\sum_{k=-\infty}^{\infty} \mathbb{P}_z(T_{B_k} < \infty) < \infty.$$

Then there exists a choice of $k_1 \in \mathbb{Z}$ such that

$$\mathbb{P}_z(T_{B^1} < \infty) \leq \sum_{k=k_1}^{\infty} \mathbb{P}_z(T_{B_k} < \infty) < 1/2, \qquad \text{where } B^1 = B \cap \{x : 0 < |x-z| \leq \lambda^{k_1}\},$$

and a choice of $k_2 \in \mathbb{Z}$ such that

$$\mathbb{P}_z(T_{B^2} < \infty) \leq \sum_{k=-\infty}^{k_2} \mathbb{P}_z(T_{B_k} < \infty) < 1/2, \qquad \text{where } B^2 = B \cap \{x : |x-z| > \lambda^{k_2}\}.$$

Then $\mathbb{P}_z(D_{B^1 \cup B^2} < \infty) = \mathbb{P}_z(T_{B^1 \cup B^2} < \infty) \leq \mathbb{P}_z(T_{B^1} < \infty; T_{B^2} < \infty) < 1$, and $B^1 \cup B^2$ is $\mathbb{P}_z$-avoidable. If $k_1 \leq k_2$ we are done as $B \subseteq B^1 \cup B^2$. Otherwise we need to consider the 'forgotten middle' of $B^3 = \{x : \lambda^{k_1} < |x-z| \leq \lambda^{k_2}\}$ and in that case it follows from Lemma 6.1.7 that $B \subseteq B^1 \cup B^2 \cup B^3$ is $\mathbb{P}_z$-avoidable.

Now for the other direction suppose that $B \in \mathcal{B}(\mathbb{R})$ is a $\mathbb{P}_z$-avoidable set. Suppose for now that B is open, and we shall deal with the general case afterwards. Define the set

$$G = \bigcup_{k \in \mathbb{Z}} B_{nk}$$

where $n > 1$ is some particular fixed positive integer in $\mathbb{N}$, chosen large enough so that

$$\mathbb{P}_z(T_G < \infty) < \mathbb{P}_z(T_B < \infty) < (1 - \lambda^{n-1})^{d-\alpha}, \tag{6.12}$$

which is possible because $\mathbb{P}_z(T_B < \infty) < \mathbb{P}_z(D_B < \infty) < 1$. The set G only intersects every nth shell, which is intentional, because the gaps between the shells will be of use to us. We shall first prove the summation test for G, which is also avoidable, and afterwards extend it to B.

For any $n \in \mathbb{N}$ and $x \in B_{nk}$, $y \in B_{nl}$, $l \neq k$, it holds that

$$\begin{aligned} |x-z| &\leq \lambda^{nk} \leq \lambda^{n(l+1)} \leq \lambda^{n-1}|y-z| \qquad &\text{if } k > l, \\ |y-z| &\leq \lambda^{nl} \leq \lambda^{n(k+1)} \leq \lambda^{n-1}|x-z| \qquad &\text{if } k < l. \end{aligned}$$

Therefore if $k > l$,

$$|y-x| = y-x = y-z-|x-z| \geq (1-\lambda^{n-1})|y-z|,$$

and if $k < l$,

$$|y-z| \leq \lambda^{n-1}|x-z| \leq \lambda^{n-1}(|x-y|+|y-z|) \leq |x-y| + \lambda^{n-1}|y-z|$$

and thus in either case

$$|y-x| \geq (1-\lambda^{n-1})|y-z|.$$

Hence because $\alpha < d$,

$$u(x,y) = |y-x|^{\alpha-d} \leq (1-\lambda^{n-1})^{\alpha-d} u(z,y). \tag{6.13}$$

This relationship will be of use to us shortly. See (A.3) of Mimica and Vondraček [37] for a similar but more general version of (6.13). We have from Theorem 6.1.4 that there exists a capacitary measure m_G for G such that

$$Um_G(x) = \mathbb{P}_x(T_G < \infty)$$

for all $x \in \mathbb{R}$. Define the measures corresponding to m_G restricted to S_k and $\mathbb{R}^d \setminus S_k$ respectively by

$$\nu_k := m_G\big|_{S_k} \qquad \text{and} \qquad \mu_k := m_G - \nu_k.$$

For any $x \in B_{nk}$,

$$\begin{aligned} U\mu_{nk}(x) &= \int u(x,y)\,\mu_{nk}(\mathrm{d}y) \\ &\leq (1-\lambda^{n-1})^{\alpha-d} \int u(z,y)\,\mu_{nk}(\mathrm{d}y) \qquad \text{by (6.13)} \\ &\leq (1-\lambda^{n-1})^{\alpha-d} \int u(z,y)\,m_G(\mathrm{d}y) \\ &= (1-\lambda^{n-1})^{\alpha-d}\mathbb{P}_z(T_G < \infty). \end{aligned}$$

This combined with our choice of n satisfying (6.12) means that $U\mu_{nk}(x) = c < 1$. Then for $x \in B_{nk}$

$$\begin{aligned} U\nu_{nk}(x) &= Um_G(x) - U\mu_{nk}(x) \\ &\geq \mathbb{P}_x(T_G < \infty) - c. \end{aligned} \tag{6.14}$$

Since B is open all points of B are regular for B, that is $\mathbb{P}_y(T_B = 0) = 1$ for all $y \in B$.[3] G is not necessarily open, because of the boundary points $|x| = \lambda^{kn}$, $k \in \mathbb{N}$. But at least for every point x of G there exists an 'upper semi-circle' around x in G, that is an $\varepsilon > 0$ such that $\{y : |y - x| < \varepsilon; |y| \geq |x|\} \subseteq G$, and isotropy of X yields that x is regular for this semicircle and therefore for G.[4] Therefore for $x \in B_{nk} \subseteq G$, it holds that $\mathbb{P}_x(T_G < \infty) = 1$, and thus by (6.14),

$$U\nu_{nk}(x) \geq 1 - c.$$

It was mentioned in §2.6 that $U\nu_{nk}$ is excessive, and thus we can define an excessive function f by

$$f(x) = \frac{U\nu_{nk}(x)}{1-c}$$

that satisfies $f(x) \geq 1$ on B_{nk}. It follows from (2.9) that for all $x \in \mathbb{R}^d$,

$$f(x) \geq \mathbb{P}_x(T_{B_{nk}} < \infty)$$

and hence

$$U\nu_{nk}(x) \geq (1-c)\mathbb{P}_x(T_{B_{nk}} < \infty).$$

Therefore for all $x \in \mathbb{R}^d$,

$$\sum_{k\in\mathbb{Z}} \mathbb{P}_x(T_{B_{nk}} < \infty) \leq \frac{1}{1-c} \sum_{k\in\mathbb{Z}} U\nu_{nk}(x) = \frac{1}{1-c} U m_G(x) < \frac{1}{1-c} < \infty.$$

What we have proven for $G = \bigcup_{k\in\mathbb{Z}} B_{nk}$ can be shown in exactly the same way taking $n+1$ instead of n, and likewise for $n+2, \ldots, 2n-1$. Summing the finite number of convergent sums obtained in this way yields

$$\sum_{k\in\mathbb{Z}} \mathbb{P}_x(T_{B_k} < \infty) < \frac{n}{1-c} < \infty. \tag{6.15}$$

This holds for all $x \in \mathbb{R}$, and thus for z.

To finish suppose that $B \in \mathcal{B}(\mathbb{R})$ is a $\mathbb{P}_z$-avoidable set, not necessarily open. According to Theorem 6.1.9, there exists an open set $\tilde{B}$ containing B which is also $\mathbb{P}_z$-avoidable, and which therefore satisfies the summation test in (6.15). From $\tilde{B} \supseteq B$ it follows that $\mathbb{P}_x(T_{B_k} < \infty) \leq \mathbb{P}_x(T_{\tilde{B}_k} < \infty)$ for all $k \in \mathbb{Z}$, $x \in \mathbb{R}^d$, and hence the test also holds for B. $\square$

[3] In fact something stronger holds: B is finely open, and so $\mathbb{R}^d \setminus B$ is thin at all $y \in B$.

[4] Bliedtner and Hansen have a different solution to this problem: they define a larger $G = \bigcup_n B \cap \{x : \lambda^{kn-2} < |x - z| < \lambda^{kn}\}$, which is open. The individual sets are no longer disjoint, but that causes no problems in the proof. Either approach is perfectly valid.

6.3 Wiener Criteria

If a Borel set $B \in \mathcal{B}(\mathbb{R}^d)$ has an equilibrium measure m_B then we saw in §2.6 that the capacity of B is $C(B) = m_B(\mathbb{R}^d)$. In addition m_B satisfies

$$\mathbb{P}_x(T_B < \infty) = \int_{\mathbb{R}^d} u(x,y)\, m_B(\mathrm{d}y) \qquad \text{for all } x \in \mathbb{R}^d.$$

Recall the sets $B_k = B \cap S_k$ from the previous section, where S_k, $k \in \mathbb{Z}$ are the shells centred at z. The sets B_k are bounded, and thus have a capacitary measure. In addition, for $x \in B_k$ and $\alpha < d$, $u(z,x)$ is bounded below by $\lambda^{k(\alpha-d)}$ and above by $\lambda^{(k+1)(\alpha-d)}$. As a result

$$\lambda^{k(\alpha-d)} C(B_k) \le \mathbb{P}_z(T_{B_k} < \infty) \qquad \text{and} \qquad \mathbb{P}_z(T_{B_k} < \infty) \le \lambda^{(k+1)(\alpha-d)} C(B_k).$$

Combining these yields that

$$\sum_{k=-\infty}^{\infty} \mathbb{P}_z(T_{B_k} < \infty) < \infty \qquad \iff \qquad \sum_{k=-\infty}^{\infty} \lambda^{k(\alpha-d)} C(B_k) < \infty$$

and

$$\sum_{k=1}^{\infty} \mathbb{P}_z(T_{B_k} < \infty) < \infty \qquad \iff \qquad \sum_{k=1}^{\infty} \lambda^{k(\alpha-d)} C(B_k) < \infty.$$

Knowing these facts, the following two corollaries follow immediately from Theorems 6.2.1 and 6.2.2. The second corresponds to Corollary V.4.17 (the Wiener Criterion) of Bliedtner and Hansen [4].

Corollary 6.3.1. *Let X be a transient isotropic stable process on $\mathbb{R}^d$ of index $\alpha \in (0,2)$, and let $B \in \mathcal{B}(\mathbb{R}^d)$. Then B is $\mathbb{P}_z$-avoidable if and only if*

$$\sum_{k=-\infty}^{\infty} \lambda^{k(\alpha-d)} C(B_k) < \infty.$$

Corollary 6.3.2. *Let X be a transient isotropic stable process on $\mathbb{R}^d$ of index $\alpha \in (0,2)$, and let $B \in \mathcal{B}(\mathbb{R}^d)$. Then B is $\mathbb{P}_z$-thin if and only if*

$$\sum_{k=1}^{\infty} \lambda^{k(\alpha-d)} C(B_k) < \infty.$$

6.4 An Example

Intuition suggests that avoidability of a set $B \in \mathcal{B}(\mathbb{R}^d)$ is related to potential $U(B)$ of that set, but it is not immediately clear how. Certainly it is possible to find a set which has finite potential but is not avoidable, for example any compact set containing the issuing point. The example below demonstrates that in some cases the converse is also possible, that is, there can exist sets which are avoidable but have infinite potential.

Let X be a stable process on $\mathbb{R}$ of index $\alpha \in (0,1)$, and define the shells $\tilde{S}_k$ via

$$\tilde{S}_k = \{x \in \mathbb{R} : 2^{k-1} < |x - z| \leq 2^k\}, \qquad k \in \mathbb{Z},$$

which grow larger as k increases. This is a reversal of the notation we have seen so far, that is, $\tilde{S}_k = S_{-k}$, and it makes notation in the following example a little easier. Define the sets $A, A_1, \dots \in \mathcal{B}(\mathbb{R})$ by

$$A_n \coloneqq \{x \in \mathbb{R} : 2^n - 2^{(n-1)/3} \leq x < 2^n\}, \qquad A \coloneqq \bigcup_{n=1}^{\infty} A_n.$$

Because $2^n - 2^{(n-1)/3} > 2^n - 2^{(n-1)} = 2^{n-1}$, we have that $A_n \subseteq \tilde{S}_n$ for each $n \in \mathbb{N}$, and in particular that $A \cap \tilde{S}_n = A_n$.

Lemma 6.4.1. *A is $\mathbb{P}_0$-avoidable for all $\alpha \in (0,1)$.*

Proof. Let us use the notation $\mathbf{B}_\varepsilon = \{x \in \mathbb{R} : |x| < \varepsilon\}$ for the ball around 0 of radius ε. Notice that each $A \cap \tilde{S}_n$ is just a translation of the ball of radius $2^{-1}2^{(n-1)/3} = 2^{(n-4)/3}$. Then from Lemma 2.6.4 **(vi)** it follows that

$$C(A \cap \tilde{S}_n) = C(\mathbf{B}_{2^{(n-4)/3}}) = C(2^{(n-4)/3}\mathbf{B}_1).$$

and from (2.18) it follows that

$$C(A \cap \tilde{S}_n) = 2^{(n-4)(1-\alpha)/3} C(\mathbf{B}_1).$$

Therefore the summation test from Corollary 6.3.1 concerns finiteness of

$$\begin{aligned}\sum_{n=-\infty}^{\infty} (2^{-1})^{n(\alpha-1)} C(A \cap S_n) &= \sum_{n=1}^{\infty} 2^{n(\alpha-1)} C(A \cap \tilde{S}_n) \\ &= C(\mathbf{B}_1) \sum_{n=1}^{\infty} 2^{n(\alpha-1)} 2^{(n-4)(1-\alpha)/3} \\ &= 2^{4(\alpha-1)/3} C(\mathbf{B}_1) \sum_{n=1}^{\infty} 2^{(2n/3)(\alpha-1)}.\end{aligned}$$

Since $C(\mathbf{B}_1)$ is a finite constant - see (2.20) - the quantity above is finite if and only if the geometric series $\sum_{n=1}^{\infty} 2^{(2n/3)(\alpha-1)}$ is finite, and this is equivalent to $2^{(2/3)(\alpha-1)} < 1$, that is, $\alpha < 1$. $\square$

Now that we have shown that A is avoidable for all transient stable processes on $\mathbb{R}$, we can show that has infinite potential for a subset of those processes. There is nothing particularly special about the restriction of α here, it is just an artifact of the simple construction of A.

Lemma 6.4.2. *If* $2/3 < \alpha < 1$ *then* $U(0, A) = \infty$.

Proof.

$$\begin{aligned} U(0, A) = \int_A u(x)\,\mathrm{d}x &= \sum_{n=1}^{\infty} \int_{A \cap \tilde{S}_n} u(x)\,\mathrm{d}x \\ &= \sum_{n=1}^{\infty} \int_{2^n - 2^{(n-1)/3}}^{2^n} x^{\alpha-1}\,\mathrm{d}x \\ &\geq \sum_{n=1}^{\infty} 2^{n(\alpha-1)}(2^n - (2^n - 2^{(n-1)/3})) \qquad \text{(monotonicity of } x^{\alpha-1}\text{)} \\ &= 2^{-1/3} \sum_{n=1}^{\infty} 2^{n(\alpha-2/3)}. \end{aligned}$$

This sum is infinite if $\alpha > 2/3$. $\square$

The two lemmas above combine to demonstrate that if X is a stable process on $\mathbb{R}$ of index $2/3 < \alpha < 1$, then A describes a $\mathbb{P}_0$-avoidable set with infinite potential.

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