



Combined Mean-Field and Semiclassical Limits of Large Fermionic Systems

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Abstract

We study the time dependent Schrödinger equation for large spinless fermions with the semiclassical scale $\hbar = N^{-1/3}$ in three dimensions. By using the Husimi measure defined by coherent states, we rewrite the Schrödinger equation into a BBGKY type of hierarchy for the k particle Husimi measure. Further estimates are derived to obtain the weak compactness of the Husimi measure, and in addition uniform estimates for the remainder terms in the hierarchy are derived in order to show that in the semiclassical regime the weak limit of the Husimi measure is exactly the solution of the Vlasov equation.

Keywords Large fermionic system · Husimi measure · Semiclassical limit · BBGKY · Wasserstein distance · Vlasov equation

1 Introduction

In this paper, we aim to study the combined mean-field and semiclassical limit of N -fermions from time-dependent Schrödinger equation to Vlasov equation.

The following anti-symmetric subspace of $L^2(\mathbb{R}^{3N})$ is considered for fermions,

$$L_a^2(\mathbb{R}^{3N}) := \left\{ \Psi \in L^2(\mathbb{R}^{3N}) : \Psi(q_{\pi(1)}, \dots, q_{\pi(N)}) = \varepsilon(\pi) \Psi(q_1, \dots, q_N) \right\}.$$

It is known that a system of fermions initially confined in a volume of order one have kinetic energy of order $N^{5/3}$ due to the Pauli principle. Therefore, to balance the order, the scale of the interaction term should be of order $N^{-1/3}$, we refer to [6,8] for more details about this

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scaling. After a time rescaling of $N^{1/3}$ the Schrödinger equation for N -fermions is written into

$$N^{1/3}i\partial_t\Psi_{N,t} = \left[-\frac{1}{2} \sum_{j=1}^N \Delta_{q_j} + \frac{1}{2N^{1/3}} \sum_{i \neq j}^N V(q_i - q_j) \right] \Psi_{N,t}.$$

By denoting the semiclassical scale $\hbar = N^{-1/3}$ and multiplying both sides by \hbar^2 , one can recover the N^{-1} , the coupling constant for the mean field interaction. Hence one arrives at the following many body Schrödinger equation

$$\begin{cases} i\hbar\partial_t\Psi_{N,t} = \left[-\frac{\hbar^2}{2} \sum_{j=1}^N \Delta_{q_j} + \frac{1}{2N} \sum_{i \neq j}^N V(q_i - q_j) \right] \Psi_{N,t} =: H_N\Psi_{N,t}, \\ \Psi_{N,0} = \Psi_N, \end{cases} \tag{1.1}$$

where $\Psi_{N,t} \in L_a^2(\mathbb{R}^{3N})$, Ψ_N is the initial data in $L_a^2(\mathbb{R}^{3N})$, and V is the interacting potential.

The limit from many body Schrödinger equation to the Vlasov equation has been studied extensively in the literature. Narnhofer and Sewell [34] and Spohn [46] are the first to prove this limit with the potential V assumed to be analytic and C^2 respectively.

For large N , in the mean field limit regime, the solution of many body fermionic Schrödinger equation can be approximated by the solution of the following nonlinear Hartree–Fock equation,

$$\begin{cases} i\hbar\partial_t\omega_{N,t} = [-\hbar^2\Delta + (V * \varrho_t) - X_t, \omega_{N,t}], \\ \omega_{N,0} = \omega_N, \end{cases}$$

where $\omega_{N,t}$ is the one-particle density matrix, $\varrho_t(q) = N^{-1}\omega_{N,t}(q; q)$ and $X_{N,t}$ is a small term having the kernel $X_t(x, y) = N^{-1}V(x - y)\omega_{N,t}(x; y)$. In [16], for the initial data being a Slater determinant, the approximation has been proved for short time for analytic interaction potential by using BBGKY hierarchy, while [6] proved the approximation with convergence rate for arbitrary time and weakened potential in the framework of second quantization. Similar results have been extended for mixed states in [4] and for relativistic case in [7]. Recently, with the help of Fefferman–de la Llave decomposition [18,26], weaker assumptions on the interaction potential have been considered. Specifically, Coulomb potential has been considered in [38], inverse power law in [41]. Further relevant literature on the fermionic case for the mean-field limit problem of Schrödinger equation can be found in [3,20,35–37].

In parallel, the mean field limit for the bosonic case from many body Schrödinger system to nonlinear Hartree equation was proved in [17] for Coulomb potential. Also for Coulomb potential, the convergence with rate $N^{1/2}$ has been obtained in [40]. Later, it has been optimized to the optimal convergence rate N^{-1} in [11], and furthermore for stronger singular potentials in [10].

The semiclassical limit from Hartree–Fock equation to Vlasov equation has been obtained in the literature by using Wigner–Weyl transformation of the one-particle density matrix $\omega_{N,t}$ defined by

$$W_{N,t}(q, p) = \left(\frac{\hbar}{2\pi}\right)^3 \int dy e^{-ip \cdot y} \omega_{N,t}\left(x + \frac{\hbar}{2}y; x - \frac{\hbar}{2}y\right), \tag{1.2}$$

which has been intensively studied in the semiclassical limit of quantum mechanics by Lions and Paul in [31]. In [5] the authors compared the inverse Wigner transform of the Vlasov solution and the solution of Hartree–Fock and get the convergence rate in the trace norm as well as Hilbert–Schmidt norm with the regular assumptions on the initial data. The works in

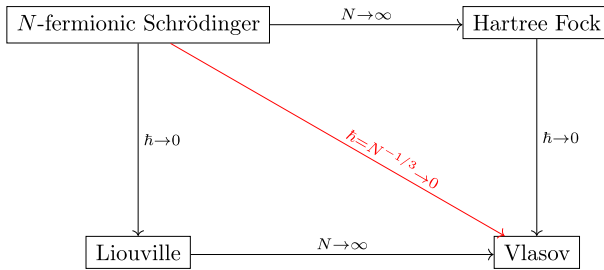


Fig. 1 Relations of N -fermionic Schrödinger systems to other mean-field equations [22,23]

this direction have also been extended for inverse power law potential [43], convergence rate in Schatten norm in [30], and Coulomb potential and mixed states in [42]. The convergence of relativistic Hartree dynamic to relativistic Vlasov equation has also been considered in [14]. Further convergence results from Hartree to Vlasov can be found in [1,2,21,33].

It is known that Wigner transform (1.2) is not a true probability density as it may be negative in certain phase-space. In fact, [27,32,45] concludes that the Wigner measure is non-negative if and only if the pure quantum states are Gaussian, whilst [9] state that the Wigner measure is non-negative if the state is a convex combination of coherent states. Nevertheless, it has been shown that if one convolutes the Wigner measure with a Gaussian function in phase-space, it will yield a non-negative probability measure known as Husimi measure [19,39,48]. In fact, from [19, p.21], the Husimi measure is given by

$$m_{N,t}^{(k)} = \frac{N(N-1) \cdots (N-k+1)}{N^k} W_{N,t}^{(k)} * \mathcal{G}^{\hbar}, \tag{1.3}$$

where $1 \leq k \leq N$, $\mathcal{G}^{\hbar} = (\pi \hbar)^{-3k} \exp(-\hbar^{-1}(\sum_{j=1}^k |q_j|^2 + |p_j|^2))$ and $W_{N,t}^{(k)}$ is the Wigner transform of k -particle density matrix.

In the recent development, the convergence to Vlasov equation in the semiclassical Wasserstein pseudo-distance has been proved in [23–25,28,29]. The semiclassical Wasserstein pseudo-distance is computed between the Husimi measure and Vlasov solution.

One can also show the combined limit by first taking the semiclassical limit and then the mean field limit from many particle Schrödinger to Vlasov via the Liouville equations, and the corresponding BBGKY hierarchy.¹ This has been done in [23]

Our goal, therefore, is to obtain the Vlasov equation from Schrödinger equation directly, as shown in the diagonal line of Figure 1, by taking $N \rightarrow \infty$ and $\hbar \rightarrow 0$ simultaneously. In order to do this, it is convenient for us to introduce the second quantization framework in our study of the quantum many-body systems. In particular, we utilize the notations in [6,8,11] where the fermionic Fock space is defined as

$$\mathcal{F}_a = \bigoplus_{n \geq 0} L_a^2(\mathbb{R}^{3n}, (dx)^{\otimes n}),$$

where we denote $(dx)^{\otimes n} = dx_1 \cdots dx_n$. The creation and annihilation operator in terms of their respective distributive forms,

$$a^*(f) = \int dx a_x^* f(x), \quad a(f) = \int dx a_x \overline{f(x)}. \tag{1.4}$$

¹ See Figure 1.

Due to the canonical anti-commutator relation (CAR) in the fermionic regime, we have that for all $f, g \in H^1(\mathbb{R}^3)$

$$\{a(f), a^*(g)\} = \langle f, g \rangle, \quad \{a^*(f), a^*(g)\} = \{a(f), a(g)\} = 0, \tag{1.5}$$

where $\{A, B\} = AB + BA$ is the anti-commutator. In particular, the CAR for operator kernels hold as follow

$$\{a_x, a_y^*\} = \delta_{x=y}, \quad \{a_x^*, a_y^*\} = \{a_x, a_y\} = 0. \tag{1.6}$$

This CAR in distributive form will be frequently used in our computations.

As in [6], we may write the corresponding Hamiltonian in terms of the operator valued distribution in \mathcal{F}_a by

$$\mathcal{H}_N = \frac{\hbar^2}{2} \int dx \nabla_x a_x^* \nabla_x a_x + \frac{1}{2N} \iint dx dy V(x - y) a_x^* a_y^* a_y a_x. \tag{1.7}$$

Therefore, we rewrite the Schrödinger equation in Fock space as follows,

$$\begin{cases} i\hbar \partial_t \psi_{N,t} = \mathcal{H}_N \psi_{N,t}, \\ \psi_{N,0} = \psi_N, \end{cases} \tag{1.8}$$

for all $\psi_{N,t} \in \mathcal{F}_a^{(N)}$ and $t \in [0, T]$, where $\psi_N \in \mathcal{F}_a^{(N)}$ with $\|\psi_N\| = 1$. The solution to the above Cauchy problem is $\psi_{N,t} = e^{-\frac{i}{\hbar} \mathcal{H}_t} \psi_N$, with a given initial data ψ_N .

Remark 1.1 It should be noted the states $\psi_{N,t}$ in our analysis stays in the N th-sector of \mathcal{F}_a due to the definition of Husimi measure which will be given later. Therefore, denoting $\mathcal{F}_a^{(n)}$ to be the n -th sector in \mathcal{F}_a , we say that $\psi_{N,t} \in \mathcal{F}_a^{(N)}$ for all $t \geq 0$.

Furthermore, we use the definition of the number and kinetic energy operators as follows,

$$\mathcal{N} = \int dx a_x^* a_x \quad \text{and} \quad \mathcal{K} = \hbar^2 \int dx \nabla_x a_x^* \nabla_x a_x, \tag{1.9}$$

respectively. We further explore the properties of the operators in (1.9) in Sect. 2.2.2.

Next, we shall introduce the Husimi measure. In fact, our notation follows closely with the notations in Fournais et al. [19] where it deals with large fermionic particles in stationary case. The main tool in their analysis is the use of coherent state, a subtle tool that proves extremely useful in our work as well.

For any real-valued normalized function f , the coherent state is given by,²

$$f_{q,p}^\hbar(y) := \hbar^{-\frac{3}{4}} f\left(\frac{y - q}{\sqrt{\hbar}}\right) e^{\frac{i}{\hbar} p \cdot y}, \tag{1.10}$$

Similar to [12] and [19], the k -particle Husimi measure is defined as, for any $1 \leq k \leq N$

$$m_N^{(k)}(q_1, p_1, \dots, q_k, p_k) := \left\langle \psi_N, a^*(f_{q_1,p_1}^\hbar) \cdots a^*(f_{q_k,p_k}^\hbar) a(f_{q_k,p_k}^\hbar) \cdots a(f_{q_1,p_1}^\hbar) \psi_N \right\rangle, \tag{1.11}$$

where $\psi_N \in \mathcal{F}_a^{(N)}$ is the N -fermionic states, $a(f_{q,p}^\hbar)$ and $a^*(f_{q,p}^\hbar)$ are the annihilation and creation operators respectively. Husimi measure defined in (1.11) measures how many particles, in particularly fermions, are in the k semiclassical boxes with length scaled of $\sqrt{\hbar}$ centered in its respectively phase-space pair, $(q_1, p_1), \dots, (q_k, p_k)$.

² The function f can be any real-valued function. [19] For this paper, we set f to be compactly supported. See Assumption A1.

In the context of this paper, we use $m_{N,t}^{(k)}$ to be the time dependent Husimi measure defined by the solution of the Schrödinger equation $\psi_{N,t}$. By using operator kernels defined in (1.4), we may rewrite the Husimi measure as follows

$$m_{N,t}^{(k)}(q_1, p_1, \dots, q_k, p_k) := \int \dots \int (dw du)^{\otimes k} \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} \langle \psi_{N,t}, a_{w_1}^* \dots a_{w_k}^* a_{u_k} \dots a_{u_1} \psi_{N,t} \rangle, \tag{1.12}$$

where the tensor products indicate

$$(dw du)^{\otimes k} := dw_1 du_1 \dots dw_k du_k$$

and

$$\left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} := \prod_{j=1}^k f_{q_j,p_j}^{\hbar}(w_j) \overline{f_{q_j,p_j}^{\hbar}(u_j)}.$$

Note that the function f here is a very well localized function in practice [19], therefore we may take the following assumption

Assumption A1 The real-valued function $f \in H^1(\mathbb{R}^3)$ satisfies $\|f\|_2 = 1$, and has compact support.

Additionally, we assume that the interaction potential to satisfy

Assumption A2 V is a real-valued function such that $V(-x) = V(x)$ and $V \in W^{2,\infty}(\mathbb{R}^3)$.

As is well known that in the mean field semiclassical regime, the dynamic of (1.1) can be approximated by a one particle Vlasov equation. Namely, for all $q, p \in \mathbb{R}^3$

$$\partial_t m_t(q, p) + p \cdot \nabla_q m_t(q, p) = \nabla(V * \rho_t)(q) \cdot \nabla_p m_t(q, p), \tag{1.13}$$

with initial data $m_0(q, p)$, where $m_t(q, p)$ is the time dependent one particle probability density function, and $\rho_t(q) = \int m_t(q, p) dp$. Although (1.13) is a non-linear equation, such equation would be more suitable to analyze than the increasingly large systems of Schrödinger equation. The well-posedness of the above Vlasov problem is given by Drobrushin [15] for smooth V .

Now, we are ready to state the our main results.

Theorem 1.1 *Let Assumptions A1 and A2 hold, $\psi_{N,t}$ be the solution of Schrödinger equation (1.8), $m_{N,t}^{(k)}$ be the Husimi measure defined in (1.12). If $m_N^{(1)}$, the 1-particle Husimi measure of the initial data ψ_N , satisfies*

$$\iint dq_1 dp_1 (|p_1|^2 + |q_1|) m_N^{(1)}(q_1, p_1) \leq C. \tag{1.14}$$

Then, for all $t \geq 0$, the k -particle Husimi measure at time t , $m_{N,t}^{(k)}$ has a weakly convergent subsequence which converges to $m_t^{(k)}$ in $L^1(\mathbb{R}^6)$, where $m_t^{(k)}$ is a weak solution of the following infinite hierarchy in the sense of distribution, i.e. it satisfies for all $k \geq 1$ that

$$\begin{aligned} & \partial_t m_t^{(k)}(q_1, p_1, \dots, q_k, p_k) + \mathbf{p}_k \cdot \nabla_{\mathbf{q}_k} m_t^{(k)}(q_1, p_1, \dots, q_k, p_k) \\ &= \frac{1}{(2\pi)^3} \nabla_{\mathbf{p}_k} \cdot \iint dq_{k+1} dp_{k+1} \nabla V(q_j - q_{k+1}) m_t^{(k+1)}(q_1, p_1, \dots, q_{k+1}, p_{k+1}). \end{aligned} \tag{1.15}$$

By using [47, Theorem 7.12], we have the following corollary,

Corollary 1.1 *Suppose assumptions A1 and A2 hold. Assume further that the initial data of (1.15) can be factorized, i.e. for all $k \geq 1$,*

$$\|m_N^{(k)} - m_0^{\otimes k}\|_{L^1} \rightarrow 0, \quad \text{as } N \rightarrow \infty. \tag{1.16}$$

Then, if the infinite hierarchy (1.15) has a unique solution and m_t is the solution to the classical Vlasov equation in (1.13), it holds that

$$W_1\left(m_{N,t}^{(1)}, m_t\right) \rightarrow 0, \quad \text{as } N \rightarrow \infty,$$

for $t \geq 0$.

Remark 1.2 In the pioneering work by Spohn [46], he considered

$$\begin{aligned} & r_n^{(N)}(\xi_1, \eta_1, \dots, \xi_N, \eta_N, t) \\ &= \text{tr} \left[e^{-iH_{Nt}} |\Psi_N\rangle \langle \Psi_N| e^{iH_{Nt}} \prod_{j=1}^N \exp(i(N^{-1/3}\xi_j p_j + \eta_j x_j)) \right] \end{aligned}$$

with $p_j = -i\nabla_j$ and obtained the following Vlasov hierarchy,

$$\begin{aligned} & \frac{\partial}{\partial t} r_n^{(N)}(\xi_1, \eta_1, \dots, \xi_n, \eta_n, t) \\ &= \sum_{j=1}^n \eta_j \frac{\partial}{\partial \xi_j} r_n^{(N)}(\xi_1, \eta_1, \dots, \xi_n, \eta_n, t) \\ &+ \sum_{j=1}^n \int \hat{V}(dk) k \cdot \xi_j r_{n+1}^{(N)}(\xi_1, \eta_1, \dots, \xi_j, \eta_j + k, \dots, \xi_n, \eta_n, 0, -k, t), \end{aligned}$$

which is slightly different from Vlasov hierarchy for Husimi measure given in (1.15), or the version in (2.3) before taking the limit. The benefit of the hierarchy in (2.3) is that one observes directly the mean field and semiclassical structure in the remainder terms. The explicit formulation is helpful in getting estimates for the remainder terms in (2.3). Moreover if one can handle singular potentials (or even the Coulomb potential) for both terms separately, one expects that this new approach can be applied to obtain the limit from many body Schrödinger to Vlasov with singular potentials in the future. Since the mean field limit with singular potential has been studied with convergence rate, for example in [8], then we can utilize similar ideas to handle one of the remainder term which includes the mean field structure. In parallel, we can apply the techniques in semiclassical limit, for example in [43], to get estimates for the other remainder term.

Remark 1.3 Although the results in this article does not yield a convergent rate, the main purpose of this article is to present an alternative approach and framework, namely to rewrite the Schrödinger equation into a BBGKY type of hierarchy, and to derive estimates for the remainder terms that appear in the new hierarchy.

Remark 1.4 In Corollary 1.1, the convergence is stated in terms of 1-Wasserstein distance. For completeness, we give its definition as defined in [47]

$$W_1(\mu, \nu) := \max_{\pi \in \Pi(\mu, \nu)} \int |x - y| d\pi(x, y), \tag{1.17}$$

where μ and ν are probability measures and $\Pi(\mu, \nu)$ the set of all probability measures with marginals μ and ν . The Wasserstein distance, also known as Monge–Kantorovich distance, is a distance on the set of probability measures. In fact, if we interpret the metric in L^p space as the distance that measures two densities “vertically”, the Wasserstein distance measures the distance between two densities “horizontally” [44].

Remark 1.5 The assumptions for initial data (1.14) and (1.16) can be realized by choosing ψ_N to be the Slater-determinant. That is, for all orthonormal basis $\{\varphi_j\}_{j=1}^\infty$, the initial data is given as

$$\psi_N(q_1, \dots, q_N) = \frac{1}{\sqrt{N!}} \det\{\varphi_j(q_i)\}_{1 \leq i, j \leq N}, \tag{1.18}$$

Remark 1.6 Assumptions A1 and A2 are expected to be weakened to the situation that $f \in H^1(\mathbb{R}^3)$, $|x|f(x) \in L^2(\mathbb{R}^3)$, and V to be Coulomb potential. These will be our future projects.

Remark 1.7 In this context, we have applied the BBGKY hierarchy, the intermediate mean field approximation Hartree Fock system has not been benefited. With Hartree Fock approximation, one can do direct factorization in the equation for $m_{N,t}^{(1)}$. In this direction, we expect to derive the rate of convergence in an appropriate distance between the Husimi measure and the solution of the Vlasov equation.

The arrangement of the paper is the following. In Sect. 2, we give the main strategy of the proof. Followed by the reformulation of Schrödinger equation into a hierarchy of the Husimi measure, a sequence of necessary estimates on number operators, the localized number operators, and the kinetic energy operator are given, which will be contributed to do compactness argument for the Husimi measure. We leave the computation of the hierarchy to Sect. 3.1. Furthermore, the uniform estimates for remainder terms in the hierarchy, which is another main contribution of this article, are provided in Sect. 3.2.

2 Proof Strategy Through BBGKY Type Hierarchy for Husimi Measure

We first start from the many particle Schrödinger equation and derive an approximated hierarchy of time dependent Husimi measure by direct computation. Compare to the BBGKY hierarchy of Liouville equation in the classical sense, it has two families of remainder terms, which are determined by the N particle wave function from Schrödinger equation. In order to take a convergent subsequence of the k -particle Husimi measure, we derive the uniform estimates for number operator and the kinetic energy. Together with an additional estimate for localized number operator, we can show that the remainder terms are of order $\hbar^{\frac{1}{2}-\delta}$, for arbitrary small δ . Then the desired result will be obtained by the uniqueness of solution to the infinite hierarchy.

2.1 Reformulation: Hierarchy of Time Dependent Husimi Measure

In this subsection, we begin by examining the dynamics of k -particle Husimi measure by using the N -body fermionic Schrödinger. The proofs of the following propositions are provided in Sect. 3.1.

Proposition 2.1 *Suppose $\psi_{N,t} \in \mathcal{F}_a^{(N)}$ is anti-symmetric N -particle state satisfying the Schrödinger equation in (1.8). Moreover, if $V(-x) = V(x)$ then we have the following equation for $k = 1$,*

$$\begin{aligned} & \partial_t m_{N,t}^{(1)}(q_1, p_1) + p_1 \cdot \nabla_{q_1} m_{N,t}^{(1)}(q_1, p_1) \\ &= \frac{1}{(2\pi)^3} \nabla_{p_1} \cdot \iint dq_2 dp_2 \nabla V(q_1 - q_2) m_{N,t}^{(2)}(q_1, p_1, q_2, p_2) + \nabla_{q_1} \cdot \mathcal{R}_1 + \nabla_{p_1} \cdot \tilde{\mathcal{R}}_1, \end{aligned} \tag{2.1}$$

where the remainder terms \mathcal{R}_1 and $\tilde{\mathcal{R}}_1$, are given by

$$\begin{aligned} \mathcal{R}_1 &:= \hbar \text{Im} \left\langle \nabla_{q_1} a(f_{q_1, p_1}^{\hbar}) \psi_{N,t}, a(f_{q_1, p_1}^{\hbar}) \psi_{N,t} \right\rangle, \\ \tilde{\mathcal{R}}_1 &:= \frac{1}{(2\pi)^3} \cdot \text{Re} \iint dw du \iint dy dv \iint dq_2 dp_2 \int_0^1 ds \\ &\quad \nabla V(su + (1-s)w - y) f_{q_1, p_1}^{\hbar}(w) \overline{f_{q_1, p_1}^{\hbar}(u)} f_{q_2, p_2}^{\hbar}(y) \overline{f_{q_2, p_2}^{\hbar}(v)} \langle a_y a_w \psi_{N,t}, a_v a_u \psi_{N,t} \rangle \\ &\quad - \frac{1}{(2\pi)^3} \iint dq_2 dp_2 \nabla V(q_1 - q_2) m_{N,t}^{(2)}(q_1, p_1, q_2, p_2), \end{aligned} \tag{2.2}$$

Proposition 2.2 *For every $1 \leq i, j \leq k$ and $q_j, p_j \in \mathbb{R}^3$, denote $\mathbf{q}_k = (q_1, \dots, q_k)$ and $\mathbf{p}_k = (p_1, \dots, p_k)$. Under the assumption in Proposition 2.1, then for $1 < k \leq N$, we have the following hierarchy*

$$\begin{aligned} & \partial_t m_{N,t}^{(k)}(q_1, p_1, \dots, q_k, p_k) + \mathbf{p}_k \cdot \nabla_{\mathbf{q}_k} m_{N,t}^{(k)}(q_1, p_1, \dots, q_k, p_k) \\ &= \frac{1}{(2\pi)^3} \nabla_{\mathbf{p}_k} \cdot \iint dq_{k+1} dp_{k+1} \nabla V(q_j - q_{k+1}) m_{N,t}^{(k+1)}(q_1, p_1, \dots, q_{k+1}, p_{k+1}) \\ &\quad + \nabla_{\mathbf{q}_k} \cdot \mathcal{R}_k + \nabla_{\mathbf{p}_k} \cdot \tilde{\mathcal{R}}_k + \hat{\mathcal{R}}_k, \end{aligned} \tag{2.3}$$

where the remainder terms are denoted as

$$\begin{aligned} \mathcal{R}_k &:= \hbar \text{Im} \left\langle \nabla_{\mathbf{q}_k} (a(f_{q_k, p_k}^{\hbar}) \cdots a(f_{q_1, p_1}^{\hbar})) \psi_{N,t}, a(f_{q_k, p_k}^{\hbar}) \cdots a(f_{q_1, p_1}^{\hbar}) \psi_{N,t} \right\rangle, \\ (\tilde{\mathcal{R}}_k)_j &:= \frac{1}{(2\pi)^3} \text{Re} \int \cdots \int (dw du)^{\otimes k} \int dy \left[\int_0^1 ds \nabla V(su_j + (1-s)w_j - y) \right] \left(f_{q, p}^{\hbar}(w) \overline{f_{q, p}^{\hbar}(u)} \right)^{\otimes k} \\ &\quad \iint d\tilde{q} d\tilde{p} \tilde{f}_{\tilde{q}, \tilde{p}}^{\hbar}(y) \int dv \overline{f_{\tilde{q}, \tilde{p}}^{\hbar}(v)} \langle a_{w_k} \cdots a_{w_1} a_y \psi_{N,t}, a_{u_k} \cdots a_{u_1} a_v \psi_{N,t} \rangle \\ &\quad - \frac{1}{(2\pi)^3} \iint dq_{k+1} dp_{k+1} \nabla V(q_j - q_{k+1}) m_{N,t}^{(k+1)}(q_1, p_1, \dots, q_{k+1}, p_{k+1}), \\ \hat{\mathcal{R}}_k &:= \frac{\hbar^2}{2} \text{Im} \int \cdots \int (dw du)^{\otimes k} \sum_{j \neq i}^k \left[V(u_j - u_i) - V(w_j - w_i) \right] \left(f_{q, p}^{\hbar}(w) \overline{f_{q, p}^{\hbar}(u)} \right)^{\otimes k} \\ &\quad \langle a_{w_k} \cdots a_{w_1} \psi_{N,t}, a_{u_k} \cdots a_{u_1} \psi_{N,t} \rangle \end{aligned} \tag{2.4}$$

2.2 A Priori Estimates

In the next steps, we derive estimates in order to have compactness of each k -particle Husimi measure, as well as to prove that the remainder terms converge to zero in the sense of distri-

bution. The estimates are derived directly from the solutions of the N -fermionic Schrödinger equation.

2.2.1 Properties of Coherent States and Husimi Measure

Here we give the properties of coherent states and Husimi measure provided in [19], which will be frequently needed in our computation. Firstly, we observe that the coherent state has a projection property, that is

Lemma 2.1 (Projection of the coherent state, [19]) *For every real-valued function f satisfying $\|f\|_2 = 1$ and the coherent states $f_{q,p}^{\hbar}$ defined as in (1.10), we have that*

$$\frac{1}{(2\pi\hbar)^3} \iint dq dp |f_{q,p}^{\hbar}\rangle \langle f_{q,p}^{\hbar}| = \frac{1}{(2\pi\hbar)^3} \iint dq dp \langle f_{q,p}^{\hbar}, \cdot \rangle f_{q,p}^{\hbar}(y) = \mathbb{1}. \tag{2.5}$$

Secondly, the properties of the k -particle Husimi measure $m_N^{(k)}$ is given as follows

Lemma 2.2 (Properties of k -particle Husimi measure, [19]) *Suppose for $\psi_N \in \mathcal{F}_a^{(N)}$ is normalized. Then, the following properties hold true for $m_N^{(k)}$:*

1. $m_N^{(k)}(q_1, p_1, \dots, q_k, p_k)$ is symmetric,
2. $\frac{1}{(2\pi)^{3k}} \int \dots \int (dq dp)^{\otimes k} m_N^{(k)}(q_1, p_1, \dots, q_k, p_k) = \frac{N(N-1)\dots(N-k+1)}{N^k}$,
3. $\frac{1}{(2\pi\hbar)^3} \iint dq_k dp_k m_N^{(k)}(q_1, p_1, \dots, q_k, p_k) = (N-k+1)m_N^{(k-1)}(q_1, p_1, \dots, q_{k-1}, p_{k-1})$,
and
4. $0 \leq m_N^{(k)}(q_1, p_1, \dots, q_k, p_k) \leq 1$ a.e.,

where $1 \leq k \leq N$.

Remark 2.1 Note that as $\|\psi_N\| = \|\psi_{N,t}\|$, Lemma 2.2 is also valid if we replaced the stationary wave-function ψ_N , to a time-dependent $\psi_{N,t}$, for $t \geq 0$. Moreover, it can be obtained that for any fixed positive integer $1 \leq k \leq N$,

$$0 \leq m_{N,t}^{(k)} \leq 1 \quad \text{a.e. in } \mathbb{R}^{6k}. \tag{2.6}$$

Following [19], we define the \hbar -weighted Fourier transformation as follows,

Definition 2.1 (\hbar -weighted Fourier transform) Let F be any real-valued function in $L^2(\mathbb{R}^3)$. We define the \hbar -weighted Fourier transform of f to be,

$$\mathcal{F}_{\hbar}[f](p) := \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \int_{\mathbb{R}^3} dx f(x) e^{-\frac{i}{\hbar} p \cdot x},$$

and its inverse transform by \mathcal{F}_{\hbar}^{-1} .

From the Definition 2.1, we have the following identity,

$$\int_{\mathbb{R}^3} dy G(y)F(y) = \int_{\mathbb{R}^3} dy G(y) \frac{1}{(2\pi\hbar)^3} \iint_{\mathbb{R}^{3 \cdot 2}} dp_2 dv F(v) e^{\frac{i}{\hbar} p_2 \cdot (y-v)}, \tag{2.7}$$

for any $G, F \in L^2(\mathbb{R}^3)$. In other words, the Dirac-delta distribution is given by

$$\delta_y(v) = \frac{1}{(2\pi\hbar)^3} \int_{\mathbb{R}^3} dp_2 e^{\frac{i}{\hbar} p_2 \cdot (y-v)}. \tag{2.8}$$

2.2.2 Number Operator and Localized Number Operator

In this part, we give the bounds of number operators and its corresponding localized version, both of which are used extensively in estimating the remainder terms in (2.1) and (2.3).

Lemma 2.3 *Let $\psi_{N,t} \in \mathcal{F}_a^{(N)}$ be the solution to Schrödinger equation in (1.1) with initial data $\|\psi_N\| = 1$, the number operator \mathcal{N} defined in (1.9). Then, for finite $1 \leq k \leq N$, we have*

$$\left\langle \psi_{N,t}, \frac{\mathcal{N}^k}{N^k} \psi_{N,t} \right\rangle = 1.$$

Proof Since $\psi_{N,t}$ satisfies the Schrödinger equation, then for $k \geq 1$,

$$i\hbar \frac{d}{dt} \left\langle \psi_{N,t}, \mathcal{N}^k \psi_{N,t} \right\rangle = \left\langle \psi_{N,t}, [\mathcal{N}^k, \mathcal{H}_N] \psi_{N,t} \right\rangle = k \left\langle \psi_{N,t}, \mathcal{N}^{k-1} [\mathcal{N}, \mathcal{H}_N] \psi_{N,t} \right\rangle = 0,$$

where we used the fact that \mathcal{H}_N is self-adjoint and $[\mathcal{H}_N, \mathcal{N}] = 0$. Therefore, integrating the above equation with respect to time, gives us

$$\left\langle \psi_{N,t}, \frac{\mathcal{N}^k}{N^k} \psi_{N,t} \right\rangle = \left\langle \psi_N, \frac{\mathcal{N}^k}{N^k} \psi_N \right\rangle = 1,$$

for any $1 \leq k \leq N$. □

Remark 2.2 The expectation of the number operator is the total mass of Husimi measure. In fact, observe that

$$\langle \psi_{N,t}, \mathcal{N} \psi_{N,t} \rangle = \int dx \langle \psi_{N,t}, a_x^* a_x \psi_{N,t} \rangle = \int dx \langle \psi_{N,t}, a_x^* \mathbb{1}_{a_x} \psi_{N,t} \rangle,$$

Then, by (2.5)

$$\begin{aligned} &= \frac{1}{(2\pi\hbar)^3} \iint dq dp \int dx \left\langle \psi_{N,t}, a_x^* f_{q,p}^\hbar(x) \left(\int dy a_y \overline{f_{q,p}^\hbar(y)} \right) \psi_{N,t} \right\rangle \\ &= \frac{1}{(2\pi\hbar)^3} \iint dq dp \left\langle \psi_{N,t}, a^*(f_{q,p}^\hbar) a(f_{q,p}^\hbar) \psi_{N,t} \right\rangle \\ &= \frac{1}{(2\pi\hbar)^3} \iint dq dp m_{N,t}^{(1)}(q, p) \\ &= N, \end{aligned}$$

where we use Lemma 2.2 in the last equality. Moreover, if we repeat the projection above for k -times, we get

$$\begin{aligned} &\frac{1}{(2\pi)^{3k}} \int \dots \int (dq dp)^{\otimes k} m_{N,t}^{(k)}(q_1, p_1, \dots, q_k, p_k) \\ &\leq \left\langle \psi_{N,t}, \frac{\mathcal{N}^k}{N^k} \psi_{N,t} \right\rangle = 1, \end{aligned} \tag{2.9}$$

where $1 \leq k \leq N$ and $t \geq 0$.

More importantly, we have the following estimates for localized number operators.

Lemma 2.4 (Bound on localized number operator) *Let $\psi_N \in \mathcal{F}_a^{(N)}$ such that $\|\psi_N\| = 1$, and R be the radius of a ball such that the volume is 1. Then, for all $1 \leq k \leq N$, we have*

$$\int \dots \int (dq dx)^{\otimes k} \left\langle \psi_N, \left(\prod_{n=1}^k \chi_{|x_n - q_n| \leq \sqrt{\hbar}R} \right) a_{x_1}^* \dots a_{x_k}^* a_{x_k} \dots a_{x_1} \psi_N \right\rangle \leq \hbar^{-\frac{3}{2}k},$$

where χ is a characteristic function

Proof Consider first the case where $k = 1$. For every $1 \leq j \leq k$, we have

$$\begin{aligned} & \int dx_j \left(\int dq_j \chi_{|x_j - q_j| \leq \sqrt{\hbar}R} \right) \langle \psi_N, a_{x_j}^* a_{x_j} \psi_N \rangle \\ &= \hbar^{\frac{3}{2}} \langle \psi_N, \mathcal{N} \psi_N \rangle = \hbar^{\frac{3}{2}-3} \left\langle \psi_N, \frac{\mathcal{N}}{N} \psi_N \right\rangle \leq \hbar^{-\frac{3}{2}}, \end{aligned}$$

where we used Lemma 2.3. Analogously, for $2 \leq k \leq N$,

$$\begin{aligned} & \int (dx)^{\otimes k} \left(\prod_{n=1}^k \int dq_n \chi_{|x_n - q_n| \leq \sqrt{\hbar}R} \right) \langle \psi_N, a_{x_1}^* \dots a_{x_k}^* a_{x_k} \dots a_{x_1} \psi_N \rangle \\ & \leq \hbar^{\frac{3}{2}k} \langle \psi_N, \mathcal{N}^k \psi_N \rangle = \hbar^{\frac{3}{2}k-3k} \left\langle \psi_N, \frac{\mathcal{N}^k}{N^k} \psi_N \right\rangle \leq \hbar^{\frac{3}{2}k-3k}, \end{aligned}$$

where we applied Lemma 2.3 again. □

Lemma 2.5 (Estimate of oscillation) *For $\varphi(p) \in C_0^\infty(\mathbb{R}^3)$ and*

$$\Omega_{\hbar} := \{x \in \mathbb{R}^3; \max_{1 \leq j \leq 3} |x_j| \leq \hbar^\alpha\}, \tag{2.10}$$

it holds for every $\alpha \in (0, 1)$, $s \in \mathbb{N}$, and $x \in \mathbb{R}^3 \setminus \Omega_{\hbar}$,

$$\left| \int_{\mathbb{R}^3} dp e^{\frac{i}{\hbar} p \cdot x} \varphi(p) \right| \leq C \hbar^{(1-\alpha)s}, \tag{2.11}$$

where C depends on the compact support and the C^s norm of φ .

Proof We will prove the lemma in a single-variable environment. That is, we let the momentum and space to be $p = (p_1, p_2, p_3)$ and $x = (x_1, x_2, x_3)$ such that $x_j, p_j \in \mathbb{R}$ for all $j \in \{1, 2, 3\}$. Then, for arbitrary $x \in \mathbb{R}^3 \setminus \Omega_{\hbar}$, one of the x_j s is bigger than \hbar^α . Without loss of generality, we assume that $|x_1| > \hbar^\alpha$ and $x_2, x_3 \in \mathbb{R}$. Let $\text{supp } \varphi \subset B_r(0) \subset \mathbb{R}^3$, we can rewrite the left hand of (2.11) into the following,

$$\begin{aligned} & \left| \int_{-r}^r dp_1 \int_{-r}^r dp_2 \int_{-r}^r dp_3 e^{\frac{i}{\hbar}(p_1 x_1 + p_2 x_2 + p_3 x_3)} \varphi(p) \right| \\ &= \left| \int_{-r}^r dp_2 e^{\frac{i}{\hbar} p_2 x_2} \int_{-r}^r dp_3 e^{\frac{i}{\hbar} p_3 x_3} \int_{-r}^r dp_1 e^{\frac{i}{\hbar} p_1 x_1} \varphi(p) \right| \end{aligned}$$

Observe that since

$$-i \frac{\hbar}{x_1} \frac{d}{dp_1} e^{\frac{i}{\hbar} p_1 x_1} = e^{\frac{i}{\hbar} p_1 x_1},$$

we have after s times integration by parts in p_1 ,

$$\begin{aligned} & \left| \int_{-r}^r dp_1 \int_{-r}^r dp_2 \int_{-r}^r dp_3 e^{\frac{i}{\hbar}(p_1x_1+p_2x_2+p_3x_3)} \varphi(p) \right| \\ &= \left| \left(-i \frac{\hbar}{x_1}\right)^s \int_{-r}^r dp_2 e^{\frac{i}{\hbar}p_2x_2} \int_{-r}^r dp_3 e^{\frac{i}{\hbar}p_3x_3} \int_{-r}^r dp_1 e^{\frac{i}{\hbar}p_1x_1} \partial_{p_1}^s \varphi(p) \right| \\ &\leq C \frac{\hbar^s}{|x_1|^s} \leq C \hbar^{(1-\alpha)s}, \end{aligned}$$

where s indicates the number of time that integration by parts has been performed. □

2.2.3 Finite Moments of Husimi Measure

To prove that the second moment in p of the Husimi measure is finite, we first show that the kinetic energy is bounded from above. Recall that the definition of the kinetic energy operator \mathcal{K} , i.e.,

$$\mathcal{K} = \frac{\hbar^2}{2} \int dx \nabla_x a_x^* \nabla_x a_x,$$

and the kinetic energy associated with ψ_N is given as $\langle \psi_N, \mathcal{K} \psi_N \rangle$.

Lemma 2.6 *Assume $V \in W^{1,\infty}$, then the kinetic energy is bounded in the following*

$$\left\langle \psi_{N,t}, \frac{\mathcal{K}}{N} \psi_{N,t} \right\rangle \leq 2 \left\langle \psi_N, \frac{\mathcal{K}}{N} \psi_N \right\rangle + Ct^2, \tag{2.12}$$

where C depends on $\|\nabla V\|_\infty$.

Proof From the Schrödinger equation, we get

$$i\hbar \frac{d}{dt} \langle \psi_{N,t}, \mathcal{K} \psi_{N,t} \rangle = \langle \psi_{N,t}, [\mathcal{K}, \mathcal{H}] \psi_{N,t} \rangle. \tag{2.13}$$

Note that since the commutator between kinetic and interaction term is given as

$$\begin{aligned} [\mathcal{K}, \mathcal{H}] &= \frac{\hbar^2}{4} \left[\int dx \nabla_x a_x^* \nabla_x a_x, \iint dydz V(y-z) a_y^* a_z^* a_z a_y \right] \\ &= \frac{\hbar^2}{4} \iint dx dy \nabla_x V(x-y) \left(\nabla_x a_x^* a_y^* a_y a_x - a_x^* a_y^* a_y \nabla_x a_x \right) \\ &= \frac{\hbar^2}{2N} \text{Im} \iint dx dy \nabla_x V(x-y) (\nabla_x a_x^* a_y^* a_y a_x) \end{aligned}$$

Then, from (2.13), we have that

$$\frac{1}{N} \frac{d}{dt} \langle \psi_{N,t}, \mathcal{K} \psi_{N,t} \rangle = \frac{\hbar}{2N^2} \text{Im} \iint dx dy \nabla_x V(x-y) \langle \psi_{N,t}, \nabla_x a_x^* a_y^* a_y a_x \psi_{N,t} \rangle.$$

Now, observe that

$$\begin{aligned} & \left| \frac{\hbar}{2N^2} \iint dx dy \nabla_x V(x-y) \langle \psi_{N,t}, \nabla_x a_x^* a_y^* a_y a_x \psi_{N,t} \rangle \right| \\ &\leq \frac{\hbar}{2N^2} \|\nabla V\|_{L^\infty} \iint dx dy \|a_y \nabla_x a_x \psi_{N,t}\| \|a_y a_x \psi_{N,t}\| \end{aligned}$$

$$\begin{aligned}
 &\leq C \frac{\hbar}{2N^2} \left(\iint dx dy \left\langle \psi_{N,t}, \nabla_x a_x^* a_y^* a_y \nabla_x a_x \psi_{N,t} \right\rangle \right)^{\frac{1}{2}} \left(\iint dx dy \left\langle \psi_{N,t}, a_x^* a_y^* a_y a_x \psi_{N,t} \right\rangle \right)^{\frac{1}{2}} \\
 &= C \left(\frac{\hbar^2}{N} \int dx \left\langle \psi_{N,t}, \nabla_x a_x^* \frac{\mathcal{N}}{N} \nabla_x a_x \psi_{N,t} \right\rangle \right)^{\frac{1}{2}} \left\langle \psi_{N,t}, \frac{\mathcal{N}^2}{N^2} \psi_{N,t} \right\rangle^{\frac{1}{2}} \\
 &\leq C \left(\left\langle \psi_{N,t}, \frac{\mathcal{K}}{N} \psi_{N,t} \right\rangle \right)^{\frac{1}{2}},
 \end{aligned}$$

Thus, we have

$$\frac{d}{dt} \left\langle \psi_{N,t}, \frac{\mathcal{K}}{N} \psi_{N,t} \right\rangle \leq C \left\langle \psi_{N,t}, \frac{\mathcal{K}}{N} \psi_{N,t} \right\rangle^{\frac{1}{2}}.$$

Integrating both sides with respect to time t and we obtain the desired inequality. □

Proposition 2.3 For $t \geq 0$, assume **AI** and let $m_{N,t}^{(k)}$ to be the k -particle Husimi measure. Denoting the phase-space vectors $\mathbf{q}_k = (q_1, \dots, q_k)$ and $\mathbf{p}_k = (p_1, \dots, p_k)$, we have the following finite moments,

$$\int \dots \int (dq dp)^{\otimes k} (|\mathbf{q}_k| + |\mathbf{p}_k|^2) m_{N,t}^{(k)}(q_1, \dots, p_k) \leq C(1 + t^3)$$

where C is a constant dependent on k , $\iint dq_1 dp_1 (|q_1| + |p_1|^2) m_N^{(1)}(q_1, p_1)$, and $\|\nabla V\|_\infty$.

Proof We first consider the case where $k = 1$. Observe that we may rewrite the kinetic energy as follows

$$\begin{aligned}
 \frac{1}{N} \langle \psi_{N,t}, \mathcal{K} \psi_{N,t} \rangle &= \frac{\hbar^2}{N} \int dw \langle \psi_{N,t}, \nabla_w a_w^* \nabla_w a_w \psi_{N,t} \rangle \\
 &= \frac{\hbar^2}{N} (2\pi \hbar)^{-3} \iint dq_1 dp_1 \iint dw du f_{q_1, p_1}^\hbar(w) \overline{f_{q_1, p_1}^\hbar(u)} \langle \psi_{N,t}, \nabla_w a_w^* \nabla_u a_u \psi_{N,t} \rangle \\
 &= \frac{\hbar^2}{(2\pi)^3} \iint dq_1 dp_1 \iint dw du \nabla_w f_{q_1, p_1}^\hbar(w) \overline{\nabla_u f_{q_1, p_1}^\hbar(u)} \langle \psi_{N,t}, a_w^* a_u \psi_{N,t} \rangle \\
 &= \frac{\hbar^2}{(2\pi)^3} \iint dq_1 dp_1 \iint dw du (-\nabla_{q_1} + i\hbar^{-1} p_1) f_{q_1, p_1}^\hbar(w) \\
 &\quad \cdot (-\nabla_{q_1} - i\hbar^{-1} p_1) \overline{f_{q_1, p_1}^\hbar(u)} \langle \psi_{N,t}, a_w^* a_u \psi_{N,t} \rangle,
 \end{aligned}$$

where we used the fact that

$$\nabla_w f \left(\frac{w - q_1}{\sqrt{\hbar}} \right) = -\nabla_{q_1} f \left(\frac{w - q_1}{\sqrt{\hbar}} \right).$$

To continue, we have

$$\begin{aligned}
 \frac{1}{N} \langle \psi_{N,t}, \mathcal{K} \psi_{N,t} \rangle &= \frac{1}{(2\pi)^3} \iint dq_1 dp_1 |p_1|^2 m_{N,t}^{(1)}(q_1, p_1) \\
 &\quad + \frac{\hbar^2}{(2\pi)^3} \iint dq_1 dp_1 \iint dw du \nabla_{q_1} f_{q_1, p_1}^\hbar(w) \cdot \nabla_{q_1} \overline{f_{q_1, p_1}^\hbar(u)} \langle \psi_{N,t}, a_w^* a_u \psi_{N,t} \rangle \\
 &\quad + \hbar \frac{2i}{(2\pi)^3} \text{Im} \iint dq_1 dp_1 \iint dw du p_1 \cdot \nabla_{q_1} f_{q_1, p_1}^\hbar(w) \overline{f_{q_1, p_1}^\hbar(u)} \langle \psi_{N,t}, a_w^* a_u \psi_{N,t} \rangle.
 \end{aligned} \tag{2.14}$$

Since kinetic energy is real-valued, if we take the real part of (2.14), the last term in the right hand side vanishes since it is purely imaginary, yielding

$$\begin{aligned} \frac{1}{N} \langle \psi_{N,t}, \mathcal{K} \psi_{N,t} \rangle &= \frac{1}{(2\pi)^3} \iint dq_1 dp_1 |p_1|^2 m_{N,t}^{(1)}(q_1, p_1) \\ &\quad + \frac{\hbar^2}{(2\pi)^3} \operatorname{Re} \iint dq_1 dp_1 \iint dw du \nabla_{q_1} f_{q_1, p_1}^{\hbar}(w) \cdot \nabla_{q_1} \overline{f_{q_1, p_1}^{\hbar}(u)} \langle \psi_{N,t}, a_w^* a_u \psi_{N,t} \rangle. \end{aligned}$$

Note that by (2.7), we have

$$\begin{aligned} &\frac{\hbar^2}{(2\pi)^3} \iint dq_1 dp_1 \iint dw du \nabla_{q_1} f_{q_1, p_1}^{\hbar}(w) \cdot \nabla_{q_1} \overline{f_{q_1, p_1}^{\hbar}(u)} \langle \psi_{N,t}, a_w^* a_u \psi_{N,t} \rangle \\ &= \hbar^{2+3} \iint dq_1 dw \hbar^{-\frac{3}{2}} \left| \nabla_{q_1} f \left(\frac{w - q_1}{\sqrt{\hbar}} \right) \right|^2 \langle \psi_{N,t}, a_w^* a_w \psi_{N,t} \rangle \\ &= \hbar \int d\tilde{q} |\nabla f(\tilde{q})|^2 \left\langle \psi_{N,t}, \frac{\mathcal{N}}{N} \psi_{N,t} \right\rangle \\ &= \hbar \int d\tilde{q} |\nabla f(\tilde{q})|^2, \end{aligned} \tag{2.15}$$

where we recall that $\hbar^3 = N^{-1}$. Thus, taking the real part of (2.14), we have that

$$\left\langle \psi_{N,t}, \frac{\mathcal{K}}{N} \psi_{N,t} \right\rangle = \frac{1}{(2\pi)^3} \iint dq_1 dp_1 |p_1|^2 m_{N,t}^{(1)}(q_1, p_1) + \hbar \int dq |\nabla f(q)|^2, \tag{2.16}$$

which means,

$$\frac{1}{(2\pi)^3} \iint dq_1 dp_1 |p_1|^2 m_{N,t}^{(1)}(q_1, p_1) \leq \left\langle \psi_{N,t}, \frac{\mathcal{K}}{N} \psi_{N,t} \right\rangle. \tag{2.17}$$

Therefore, (2.17) tells us that the second moment of the 1-particle Husimi measure in momentum space is finite if the kinetic energy is finite.

Now, we turn our focus on the moment with respect to position space. From (2.1), we get

$$\begin{aligned} \partial_t \iint dq_1 dp_1 |q_1| m_{N,t}^{(1)}(q_1, p_1) &= \iint |q_1| \partial_t m_{N,t}^{(1)}(q_1, p_1) \\ &= \iint dq_1 dp_1 |q_1| \left(-p_1 \cdot \nabla_{q_1} m_{N,t}^{(1)}(q_1, p_1) + \frac{1}{(2\pi)^3} \nabla_{p_1} \cdot \iint dw du \iint dx dy \iint dq_2 dp_2 \int_0^1 ds \right. \\ &\quad \left. \nabla V(su + (1-s)w - x) f_{q_1, p_1}^{\hbar}(w) \overline{f_{q_1, p_1}^{\hbar}(u)} f_{q_2, p_2}^{\hbar}(x) \overline{f_{q_2, p_2}^{\hbar}(y)} (a_x a_w \psi_{N,t}, a_y a_u \psi_{N,t}) + \nabla_{q_1} \cdot \mathcal{R}_1 \right). \end{aligned}$$

Then, using intergration by parts with respect to p_1 ,

$$\begin{aligned} &= \iint dq_1 dp_1 \nabla_{q_1} |q_1| \cdot (p_1 m_{N,t}^{(1)}(q_1, p_1) + \mathcal{R}_1) \\ &= \iint dq_1 dp_1 \frac{q_1}{|q_1|} \cdot (p_1 m_{N,t}^{(1)}(q_1, p_1) + \mathcal{R}_1) \\ &\leq \iint dq_1 dp_1 (|p_1| m_{N,t}^{(1)}(q_1, p_1) + |\mathcal{R}_1|), \end{aligned}$$

where R_1 is the remainder term in (2.2).

Note that by Young’s product inequality, we have

$$\begin{aligned} \iint dq_1 dp_1 |p_1| m_{N,t}^{(1)}(q_1, p_1) &\leq \iint dq_1 dp_1 (1 + |p_1|^2) m_{N,t}^{(1)}(q_1, p_1) \\ &\leq (2\pi)^3 \left(1 + 2 \left\langle \psi_N, \frac{\mathcal{K}}{N} \psi_N \right\rangle + Ct^2 \right), \end{aligned}$$

where we used (2.17) and Lemma 2.6 in the last inequality. Next, we want to bound the term associated with \mathcal{R}_1 ,

$$\iint dq_1 dp_1 |\mathcal{R}_1| \leq \hbar \iint dq_1 dp_1 \left| \left\langle \nabla_{q_1} a(f_{q_1,p_1}^\hbar) \psi_{N,t}, a(f_{q_1,p_1}^\hbar) \psi_{N,t} \right\rangle \right|.$$

Observe that we have,

$$\begin{aligned} &\hbar \iint dq_1 dp_1 \left| \left\langle \nabla_{q_1} a(f_{q_1,p_1}^\hbar) \psi_{N,t}, a(f_{q_1,p_1}^\hbar) \psi_{N,t} \right\rangle \right| \\ &\leq \hbar \iint dq_1 dp_1 \left\| \nabla_{q_1} a(f_{q_1,p_1}^\hbar) \psi_{N,t} \right\| \left\| a(f_{q_1,p_1}^\hbar) \psi_{N,t} \right\| \\ &\leq \hbar \left[\iint dq_1 dp_1 \left\langle \nabla_{q_1} a(f_{q_1,p_1}^\hbar) \psi_{N,t}, \nabla_{q_1} a(f_{q_1,p_1}^\hbar) \psi_{N,t} \right\rangle \right]^{\frac{1}{2}} \\ &\quad \left[\iint dq_1 dp_1 \left\langle \psi_{N,t}, a^*(f_{q_1,p_1}^\hbar) a(f_{q_1,p_1}^\hbar) \psi_{N,t} \right\rangle \right]^{\frac{1}{2}} \\ &= \left[\hbar^2 \iint dq_1 dp_1 \iint dw du \nabla_{q_1} f_{q_1,p_1}^\hbar(w) \cdot \nabla_{q_1} \overline{f_{q_1,p_1}^\hbar(u)} \left\langle \psi_{N,t}, a_w^* a_u \psi_{N,t} \right\rangle \right]^{\frac{1}{2}} (2\pi)^{\frac{3}{2}} \\ &\leq (2\pi)^3 \sqrt{\hbar} \left[\int d\tilde{q} |\nabla f(\tilde{q})|^2 \right]^{\frac{1}{2}}, \end{aligned}$$

where we used (2.15), Lemma 2.2. Thus, we have that

$$\partial_t \iint dq_1 dp_1 |q_1| m_{N,t}^{(1)}(q_1, p_1) \leq (2\pi)^3 \left(1 + 2 \left\langle \psi_N, \frac{\mathcal{K}}{N} \psi_N \right\rangle + Ct^2 + C\sqrt{\hbar} \right) \leq C(1+t^2). \tag{2.18}$$

which gives the estimate for first moment after integrating with respect to time t .

We now consider the case of $2 \leq k \leq N$. In this computation, we make use of the properties of k -particle Husimi measure. Namely, that the $m_{N,t}^{(k)}$ is symmetric and satisfies the following equation

$$\begin{aligned} \frac{1}{(2\pi)^3} \iint dq_k dp_k m_{N,t}^{(k)}(q_1, p_1, \dots, q_k, p_k) &= \frac{(N-k+1)}{N} m_{N,t}^{(k-1)}(q_1, p_1, \dots, q_{k-1}, p_{k-1}) \\ &\leq m_{N,t}^{(k-1)}(q_1, p_1, \dots, q_{k-1}, p_{k-1}). \end{aligned} \tag{2.19}$$

Observe that for fixed $1 \leq k \leq N$.

$$\begin{aligned} &\int \dots \int (dq dp)^{\otimes k} \sum_{j=1}^k |p_j|^2 m_{N,t}^{(k)}(q_1, p_1, \dots, q_k, p_k) \\ &= \sum_{j=1}^k \iint dq_j dp_j |p_j|^2 \int \dots \int dq_1 dp_1 \dots \widehat{dq_j dp_j} \dots dq_k dp_k m_{N,t}^{(k)}(q_1, p_1, \dots, q_k, p_k). \end{aligned}$$

Then, by using the symmetricity of $m_{N,t}^{(k)}$ and change of variables, we get

$$\begin{aligned} &= k \iint dq dp |p|^2 \int \dots \int (dq dp)^{\otimes k-1} m_{N,t}^{(k)}(q, p, q_1, p_1, \dots, q_{k-1}, p_{k-1}) \\ &= (2\pi)^{3(k-1)} k \frac{(N-1) \dots (N-k+1)}{N^{k-1}} \iint dq dp |p|^2 m_{N,t}^{(1)}(q, p) \\ &\leq (2\pi)^{3k} k \left(1 + 2 \left\langle \psi_N, \frac{\mathcal{K}}{N} \psi_N \right\rangle + Ct^2 \right) \leq C(1+t^2), \end{aligned}$$

where we denoted $(dq dp)^{\otimes k-1} = dq_1 dp_1 \dots dq_{k-1} dp_{k-1}$.

Similar strategy is used to obtain the first moment with respect to \mathbf{q}_k . That is

$$\begin{aligned} &\int \dots \int (dq dp)^{\otimes k} \sum_{j=1}^k |q_j| m_{N,t}^{(k)}(q_1, p_1, \dots, q_k, p_k) \\ &= (2\pi)^{3(k-1)} k \frac{(N-1) \dots (N-k+1)}{N^{k-1}} \iint dq dp |q| m_{N,t}^{(1)}(q, p) \\ &\leq (2\pi)^{3(k-1)} k \iint dq dp |q| m_{N,t}^{(1)}(q, p) \leq C(1+t^3). \end{aligned}$$

This yields the desired conclusion. □

2.3 Uniform Estimates for the Remainder Terms

In this subsection, we give uniform estimates for the error terms that appear in (2.1) and (2.3). They are all bounded of order $\hbar^{\frac{1}{2}-\delta}$ for arbitrary small $\delta > 0$. The proofs of all the following propositions will be provided in Sect. 3.2.

Proposition 2.4 *Let Assumption A1 holds, then for $1 \leq k \leq N$, we have the following bound for \mathcal{R}_k in (2.1) and (2.3). For arbitrary small $\delta > 0$, the following estimate holds for any test function $\Phi \in C_0^\infty(\mathbb{R}^{6k})$,*

$$\left| \int \dots \int (dq dp)^{\otimes k} \Phi(q_1, p_1, \dots, q_k, p_k) \nabla_{\mathbf{q}_k} \cdot \mathcal{R}_k \right| \leq C \hbar^{\frac{1}{2}-\delta},$$

where C depends on $\|D^{s(\delta)} \Phi\|_\infty$ and k .

Proposition 2.5 *Let Assumption A1 and A2 hold, then we have the following bound for $\tilde{\mathcal{R}}_1$ in (2.2). For arbitrary small $\delta > 0$, the following estimate holds for any test function $\Phi \in C_0^\infty(\mathbb{R}^6)$,*

$$\left| \iint dq_1 dp_1 \Phi(q_1, p_1) \nabla_{p_1} \cdot \tilde{\mathcal{R}}_1 \right| \leq C \hbar^{\frac{1}{2}-\delta}, \tag{2.20}$$

where C depends on $\|D^{s(\delta)} \Phi\|_\infty$.

Proposition 2.6 *Suppose that Assumption A1 and A2 hold. Denote the remainders terms $\tilde{\mathcal{R}}_k$ and $\hat{\mathcal{R}}_k$ as in (2.4). Then for $1 \leq k \leq N$ and arbitrary small $\delta > 0$, the following estimates hold for any test function $\Phi \in C_0^\infty(\mathbb{R}^{6k})$,*

$$\left| \int \dots \int (dq dp)^{\otimes k} \Phi(q_1, p_1, \dots, q_k, p_k) \cdot \hat{\mathcal{R}}_k \right| \leq C \hbar^{3-\delta}, \tag{2.21}$$

and

$$\left| \int \cdots \int (dq dp)^{\otimes k} \Phi(q_1, p_1, \dots, q_k, p_k) \nabla_{\mathbf{p}_k} \cdot \tilde{\mathcal{R}}_k \right| \leq C \hbar^{\frac{1}{2}-\delta}, \tag{2.22}$$

where C depends on $\|D^{s(\delta)}\Phi\|_\infty$ and k .

2.4 Convergence to Infinite Hierarchy

In this subsection, we prove that the k -particle Husimi measure $m_{N,t}^{(k)}$ has subsequence that converges weakly (as $N \rightarrow \infty$) to a limit $m_t^{(k)}$ in L^1 , which is a solution of the infinite hierarchy in the sense of distribution.

The weak compactness of k -particle Husimi measure $m_{N,t}^{(k)}$ can be proved by the use of Dunford–Pettis theorem.³ In particular, we have the following result.

Proposition 2.7 *Let $\{m_{N,t}^{(k)}\}_{N \in \mathbb{N}}$ be the k -particle Husimi measure, then there exists a subsequence $\{m_{N_j,t}^{(k)}\}_{j \in \mathbb{N}}$ that converges weakly in $L^1(\mathbb{R}^{6k})$ to a function $(2\pi)^{3k}m_t^{(k)}$, i.e. for all $\varphi \in L^\infty(\mathbb{R}^{6k})$, it holds*

$$\frac{1}{(2\pi)^{3k}} \int \cdots \int (dq dp)^{\otimes k} m_{N_j,t}^{(k)} \varphi \rightarrow \int \cdots \int (dq dp)^{\otimes k} m_t^{(k)} \varphi,$$

when $j \rightarrow \infty$ for arbitrary fixed $k \geq 1$.

Proof To apply Dunford–Pettis theorem, we need to check that it is uniformly integrable and bounded. From the previous uniform estimates that we have obtained for $m_{N,t}^{(k)}$ from (2.6) and its second finite moment in Proposition 2.3 imply

$$\|m_{N,t}^{(k)}\|_{L^\infty} \leq 1, \quad \|(|\mathbf{q}_k| + |\mathbf{p}_k|)m_{N,t}^{(k)}\|_{L^1} \leq C(t).$$

where $\mathbf{q}_k := (q_1, \dots, q_k)$, $\mathbf{p}_k := (p_1, \dots, p_k)$ and $C(t)$ is a time-dependent constant, we can check the uniform integrability. More precisely, for any $\varepsilon > 0$, by taking $r = \varepsilon^{-1}(2\pi)^{3k}C(t)$ we have that

$$\frac{1}{(2\pi)^{3k}} \int \cdots \int_{|\mathbf{q}_k| + |\mathbf{p}_k| \geq r} (dq dp)^{\otimes k} m_{N,t}^{(k)} \leq \frac{1}{r} \frac{1}{(2\pi)^{3k}} \int \cdots \int (dq dp)^{\otimes k} (|\mathbf{q}_k| + |\mathbf{p}_k|) m_{N,t}^{(k)} \leq \varepsilon. \tag{2.23}$$

Furthermore, for arbitrary $\varepsilon > 0$, by taking $\delta = \varepsilon$, we have that for all $E \subset \mathbb{R}^{6k}$ with $\text{Vol}(E) \leq \delta$, it holds

$$\int \cdots \int_E m_{N,t}^{(k)} \leq \|m_{N,t}^{(k)}\|_\infty \text{Vol}(E) \leq \varepsilon,$$

which means that there is no concentration for the k -particle Husimi measure.

It is shown in (2.9) that the boundedness of k -particle Husimi measure in L^1 , i.e.

$$\|m_{N,t}^{(k)}\|_{L^1} \leq (2\pi)^{3k}.$$

Then applying directly Dunford–Pettis Theorem one obtain that k -particle Husimi measure is weakly compact in L^1 . □

³ See [13] for the treatment of uniform integrability.

Proof (Proof of Theorem 1.1 and Corollary 1.1) Cantor’s diagonal procedure shows that we can take the same convergent subsequence of $m_{N,t}^{(k)}$ for all $k \geq 1$. Then by the error estimates obtained in Propositions 2.4, 2.5, and 2.6, we can obtain that the limit satisfies the infinite hierarchy (1.15) in the sense of distribution, by directly taking the limit in the weak formulation of (2.1) and (2.3).

Observe that the estimates for the remainder terms also show that any convergent subsequence of $m_{N,t}^{(k)}$ converges weakly in L^1 to the solution of the infinite hierarchy. Therefore, if furthermore, the infinite hierarchy has a unique solution, then the sequence $m_{N,t}^{(k)}$ itself converges weakly to the solution of the infinite hierarchy.

As for Corollary 1.1, one only need to combine the facts that the infinite hierarchy has a unique solution and that the tensor products of the solution of the Vlasov equation (1.13), $m_t^{\otimes k}$ is a solution of the infinite hierarchy.

Lastly, by Theorem 7.12 in [47], we would obtain the convergence in 1-Wasserstein metric. □

3 Completion of the Reformulation and Estimates in the Proof

3.1 Proof of the Reformulation in Sect. 2.1

In this subsection we supply the proofs for the reformulation of Schrödinger equation into a hierarchy of k ($1 \leq k \leq N$) particle Husimi measure. The reformulation shares similar structure to the classical BBGKY hierarchy.

Proof (Proof of Proposition 2.1) First, observe that taking the time derivative on the Husimi measure, we have

$$\begin{aligned}
 & 2i\hbar\partial_t m_{N,t}^{(1)}(q_1, p_1) \\
 &= \left(\hbar^2 \iiint dw du dx f_{q_1,p_1}^{\hbar}(w) \overline{f_{q_1,p_1}^{\hbar}(u)} \langle \psi_{N,t}, a_w^* a_u \nabla_x a_x^* \nabla_x a_x \psi_{N,t} \rangle \right. \\
 &\quad \left. - \hbar^2 \iiint dw du dx \overline{f_{q_1,p_1}^{\hbar}(w)} f_{q_1,p_1}^{\hbar}(u) \langle \psi_{N,t}, \nabla_x a_x^* \nabla_x a_x a_w^* a_u \psi_{N,t} \rangle \right) \\
 &\quad + \left(\frac{1}{N} \iint dw du \iint dx dy f_{q_1,p_1}^{\hbar}(w) \overline{f_{q_1,p_1}^{\hbar}(u)} \langle \psi_{N,t}, V(x-y) a_w^* a_u a_x^* a_y a_x \psi_{N,t} \rangle \right) \\
 &\quad - \frac{1}{N} \iint dw du \iint dx dy \overline{f_{q_1,p_1}^{\hbar}(w)} f_{q_1,p_1}^{\hbar}(u) \langle \psi_{N,t}, V(x-y) a_x^* a_y^* a_y a_x a_u^* a_w \psi_{N,t} \rangle \Big) \\
 &=: I_1 + II_1.
 \end{aligned}$$

Now, focus on I_1 , we have

$$\begin{aligned}
 I_1 = & \hbar^2 \iiint dw du dx f_{q_1,p_1}^{\hbar}(w) \overline{f_{q_1,p_1}^{\hbar}(u)} \langle \psi_{N,t}, a_w^* a_u \nabla_x a_x^* \nabla_x a_x \psi_{N,t} \rangle \\
 & - \hbar^2 \iiint dw du dx \overline{f_{q_1,p_1}^{\hbar}(w)} f_{q_1,p_1}^{\hbar}(u) \langle \psi_{N,t}, \nabla_x a_x^* \nabla_x a_x a_w^* a_u \psi_{N,t} \rangle,
 \end{aligned}$$

where the last equality is just change of variable on the complex conjugate term. Then, from CAR, observe we have that

$$\begin{aligned}
 -a_w^* a_u a_x^* \Delta_x a_x &= a_w^* a_x^* a_u \Delta_x a_x - \delta_{u=x} a_w^* \Delta_x a_x \\
 &= a_x^* a_w^* \Delta_x a_x a_u - \delta_{u=x} a_w^* \Delta_x a_x \\
 &= \Delta_x a_x^* a_w^* a_x a_u - \delta_{u=x} a_w^* \Delta_x a_x \\
 &= -\Delta_x a_x^* a_x a_w^* a_u + \delta_{w=x} \Delta_x a_x^* a_u - \delta_{u=x} a_w^* \Delta_x a_x,
 \end{aligned}$$

where integration by parts and CAR of the operator have been used several times. Putting this back, we cancel out the the second term and get

$$\begin{aligned}
 I_1 &= \hbar^2 \iiint dw du dx f_{q_1, p_1}^\hbar(w) \overline{f_{q_1, p_1}^\hbar(u)} \langle \psi_{N,t}, (\delta_{w=x} \Delta_x a_x^* a_u - \delta_{u=x} a_w^* \Delta_x a_x) \psi_{N,t} \rangle \\
 &= \hbar^2 \iint dw du \left(\Delta_w f_{q_1, p_1}^\hbar(w) \right) \overline{f_{q_1, p_1}^\hbar(u)} \langle \psi_{N,t}, a_w^* a_u \psi_{N,t} \rangle \\
 &\quad - \hbar^2 \iint dw du f_{q_1, p_1}^\hbar(w) \left(\Delta_u \overline{f_{q_1, p_1}^\hbar(u)} \right) \langle \psi_{N,t}, a_w^* a_u \psi_{N,t} \rangle.
 \end{aligned} \tag{3.1}$$

Now, observe the following

$$\begin{aligned}
 \nabla_u \overline{f_{q_1, p_1}^\hbar(u)} &= \nabla_u \left(\hbar^{-\frac{3}{4}} f \left(\frac{u - q_1}{\sqrt{\hbar}} \right) e^{-\frac{i}{\hbar} p_1 \cdot u} \right) \\
 &= \hbar^{-\frac{3}{4}} \nabla_u f \left(\frac{u - q_1}{\sqrt{\hbar}} \right) e^{-\frac{i}{\hbar} p_1 \cdot u} + \hbar^{-\frac{3}{4}} f \left(\frac{u - q_1}{\sqrt{\hbar}} \right) \nabla_u e^{-\frac{i}{\hbar} p_1 \cdot u} \\
 &= -\hbar^{-\frac{3}{4}} \nabla_{q_1} f \left(\frac{u - q_1}{\sqrt{\hbar}} \right) e^{-\frac{i}{\hbar} p_1 \cdot u} - i \hbar^{-1} p_1 \cdot \hbar^{-\frac{3}{4}} f \left(\frac{u - q_1}{\sqrt{\hbar}} \right) e^{-\frac{i}{\hbar} p_1 \cdot u} \\
 &= (-\nabla_{q_1} - i \hbar^{-1} p_1) \overline{f_{q_1, p_1}^\hbar(u)},
 \end{aligned}$$

and furthermore,

$$\begin{aligned}
 \Delta_u \overline{f_{q_1, p_1}^\hbar(u)} &= \nabla_u \cdot \nabla_u \overline{f_{q_1, p_1}^\hbar(u)} \\
 &= \nabla_u \cdot (-\nabla_{q_1} - i \hbar^{-1} p_1) \overline{f_{q_1, p_1}^\hbar(u)} \\
 &= (-\nabla_{q_1} - i \hbar^{-1} p_1) \cdot (-\nabla_{q_1} - i \hbar^{-1} p_1) \overline{f_{q_1, p_1}^\hbar(u)} \\
 &= \left(\Delta_{q_1} + 2i \hbar^{-1} p_1 \cdot \nabla_{q_1} - \hbar^{-2} p_1^2 \right) \overline{f_{q_1, p_1}^\hbar(u)}.
 \end{aligned} \tag{3.2}$$

and similarly

$$\Delta_w f_{q_1, p_1}^\hbar(w) = \left(\Delta_{q_1} - 2i \hbar^{-1} p_1 \cdot \nabla_{q_1} - \hbar^{-2} p_1^2 \right) f_{q_1, p_1}^\hbar(w), \tag{3.3}$$

we obtain by putting these back into (3.1),

$$\begin{aligned}
 I_1 = & \hbar^2 \left[\left\langle \Delta_{q_1} \int dw \overline{f_{q_1, p_1}^\hbar(w)} a_w \psi_{N, t}, \int du \overline{f_{q_1, p_1}^\hbar(u)} a_u \psi_{N, t} \right\rangle \right. \\
 & - \left. \left\langle \int dw \overline{f_{q_1, p_1}^\hbar(w)} a_w \psi_{N, t}, \Delta_{q_1} \int du \overline{f_{q_1, p_1}^\hbar(u)} a_u \psi_{N, t} \right\rangle \right] \\
 & - 2i\hbar p_1 \cdot \left[\left\langle \nabla_{q_1} \int dw \overline{f_{q_1, p_1}^\hbar(w)} a_w \psi_{N, t}, \int du \overline{f_{q_1, p_1}^\hbar(u)} a_u \psi_{N, t} \right\rangle \right. \\
 & + \left. \left\langle \int dw \overline{f_{q_1, p_1}^\hbar(w)} a_w \psi_{N, t}, \nabla_{q_1} \int du \overline{f_{q_1, p_1}^\hbar(u)} a_u \psi_{N, t} \right\rangle \right] \\
 = & 2i\hbar^2 \text{Im} \left\langle \Delta_{q_1} a(f_{q_1, p_1}^\hbar) \psi_{N, t}, a(f_{q_1, p_1}^\hbar) \psi_{N, t} \right\rangle - 2i\hbar p_1 \cdot \nabla_{q_1} m_{N, t}^{(1)}(q_1, p_1).
 \end{aligned} \tag{3.4}$$

Since the Husimi measure is actually a real-valued function, we have that

$$\partial_t m_{N, t}^{(1)}(q_1, p_1) + p_1 \cdot \nabla_{q_1} m_{N, t}^{(1)}(q_1, p_1) = \text{Re} \left(\frac{II_1}{2i\hbar} \right) + \hbar \text{Im} \left\langle \Delta_{q_1} a(f_{q_1, p_1}^\hbar) \psi_{N, t}, a(f_{q_1, p_1}^\hbar) \psi_{N, t} \right\rangle. \tag{3.5}$$

Now, we turn our focus on II_1 , i.e.,

$$\begin{aligned}
 II_1 = & \frac{1}{N} \iint dw du \iint dx dy f_{q_1, p_1}^\hbar(w) \overline{f_{q_1, p_1}^\hbar(u)} \\
 & \left\langle \psi_{N, t}, V(x - y) a_w^* a_u a_x^* a_y^* a_y a_x \psi_{N, t} \right\rangle \\
 & - \frac{1}{N} \iint dw du \iint dx dy \overline{f_{q_1, p_1}^\hbar(w)} f_{q_1, p_1}^\hbar(u) \\
 & \left\langle \psi_{N, t}, V(x - y) a_x^* a_y^* a_y a_x a_w^* a_u \psi_{N, t} \right\rangle.
 \end{aligned}$$

Observe that

$$\begin{aligned}
 a_w^* a_u a_x^* a_y^* a_y a_x & = a_x^* a_y^* a_y a_x a_w^* a_u \\
 & + \delta_{w=y} a_x^* a_y^* a_x a_u - \delta_{w=x} a_x^* a_y^* a_y a_u \\
 & + \delta_{u=x} a_w^* a_y^* a_y a_x - \delta_{u=y} a_w^* a_x^* a_y a_x.
 \end{aligned}$$

The first term and the complex conjugate term vanishes under changes of variable, u to w and w to u . Therefore, since from assumption $V(x) = V(-x)$, we have

$$\begin{aligned}
 \Pi_1 &= \frac{1}{N} \iiint dwdudx f_{q_1,p_1}^{\hbar}(w) \overline{f_{q_1,p_1}^{\hbar}(u)} \langle \psi_{N,t}, V(x-w) a_w^* a_x^* a_u \psi_{N,t} \rangle \\
 &\quad - \frac{1}{N} \iiint dwdudx f_{q_1,p_1}^{\hbar}(w) \overline{f_{q_1,p_1}^{\hbar}(u)} \langle \psi_{N,t}, V(x-u) a_w^* a_x^* a_u \psi_{N,t} \rangle \\
 &\quad + \frac{1}{N} \iiint dwdudy f_{q_1,p_1}^{\hbar}(w) \overline{f_{q_1,p_1}^{\hbar}(u)} \langle \psi_{N,t}, V(u-y) a_w^* a_y^* a_u \psi_{N,t} \rangle \\
 &\quad - \frac{1}{N} \iiint dwdudy f_{q_1,p_1}^{\hbar}(w) \overline{f_{q_1,p_1}^{\hbar}(u)} \langle \psi_{N,t}, V(w-y) a_w^* a_y^* a_u \psi_{N,t} \rangle \\
 &= \frac{1}{N} \iiint dwdudx f_{q_1,p_1}^{\hbar}(w) \overline{f_{q_1,p_1}^{\hbar}(u)} \left(V(u-x) - V(w-x) \right) \langle \psi_{N,t}, a_w^* a_x^* a_u \psi_{N,t} \rangle \\
 &\quad + \frac{1}{N} \iiint dwdudy f_{q_1,p_1}^{\hbar}(w) \overline{f_{q_1,p_1}^{\hbar}(u)} \left(V(u-y) - V(w-y) \right) \langle \psi_{N,t}, a_w^* a_y^* a_u \psi_{N,t} \rangle \\
 &= \frac{2}{N} \iiint dwdudy f_{q_1,p_1}^{\hbar}(w) \overline{f_{q_1,p_1}^{\hbar}(u)} \left(V(u-y) - V(w-y) \right) \langle \psi_{N,t}, a_w^* a_y^* a_u \psi_{N,t} \rangle.
 \end{aligned} \tag{3.6}$$

Now, note that mean value theorem gives

$$V(u-y) - V(w-y) = \int_0^1 ds \nabla V(s(u-y) + (1-s)(w-y)) \cdot (u-w), \tag{3.7}$$

and observe that since, $V(s(u-y) + (1-s)(w-y)) = V(su + (1-s)w - y)$, we can have from (3.6) the following

$$\begin{aligned}
 \Pi_1 &= \frac{2}{N} \iiint dwdudy f_{q_1,p_1}^{\hbar}(w) \overline{f_{q_1,p_1}^{\hbar}(u)} \left(\int_0^1 ds \nabla V(su + (1-s)w - y) \right) \\
 &\quad \cdot (u-w) \cdot \langle \psi_{N,t}, a_w^* a_y^* a_u \psi_{N,t} \rangle \\
 &= \frac{2i\hbar}{N} \iiint dwdudy \int_0^1 \nabla V(su + (1-s)w - y) \\
 &\quad \cdot \nabla_{p_1} \left(f_{q_1,p_1}^{\hbar}(w) \overline{f_{q_1,p_1}^{\hbar}(u)} \right) \langle \psi_{N,t}, a_w^* a_y^* a_u \psi_{N,t} \rangle \\
 &= \frac{2i\hbar}{N} \iiint dwdudy \int_0^1 \nabla V(su + (1-s)w - y) \\
 &\quad \cdot \nabla_{p_1} \left(f_{q_1,p_1}^{\hbar}(w) \overline{f_{q_1,p_1}^{\hbar}(u)} \right) \langle a_w a_y \psi_{N,t}, a_u a_y \psi_{N,t} \rangle,
 \end{aligned} \tag{3.8}$$

where we use the fact that

$$\nabla_{p_1} \left(f_{q_1,p_1}^{\hbar}(w) \overline{f_{q_1,p_1}^{\hbar}(u)} \right) = \frac{i}{\hbar} (w-u) \cdot f_{q_1,p_1}^{\hbar}(w) \overline{f_{q_1,p_1}^{\hbar}(u)}. \tag{3.9}$$

Then we get

$$\Pi_1 = \frac{2i\hbar}{N} \iiint dwdudy \int_0^1 \nabla V(su + (1-s)w - y) \cdot \nabla_{p_1} \left(f_{q_1,p_1}^{\hbar}(w) \overline{f_{q_1,p_1}^{\hbar}(u)} \right) \langle a_w a_y \psi_{N,t}, a_u a_y \psi_{N,t} \rangle. \tag{3.10}$$

Applying the following projection

$$\frac{1}{(2\pi\hbar)^3} \iint dq_2 dp_2 | f_{q_2,p_2}^{\hbar} \rangle \langle f_{q_2,p_2}^{\hbar} | = \mathbb{1}, \tag{3.11}$$

onto $a_y \psi_{N,t}$, we get

$$a_y \psi_{N,t} = \frac{1}{(2\pi\hbar)^3} \iint dq_2 dp_2 f_{q_2,p_2}^{\hbar}(y) \int dv \overline{f_{q_2,p_2}^{\hbar}(v)} a_v \psi_{N,t}.$$

Putting this back into (3.10), we get the following

$$\begin{aligned}
 II_1 = & \frac{2i\hbar}{N} \frac{1}{(2\pi\hbar)^3} \iint dwdu \iint dydv \iint dq_2dp_2 \int_0^1 ds \nabla V(su + (1-s)w - y) \\
 & \cdot \nabla_{p_1} \left(f_{q_1,p_1}^\hbar(w) \overline{f_{q_1,p_1}^\hbar(u)} \right) f_{q_2,p_2}^\hbar(y) \overline{f_{q_2,p_2}^\hbar(v)} \langle a_w a_y \psi_{N,t}, a_u a_v \psi_{N,t} \rangle.
 \end{aligned}
 \tag{3.12}$$

Recall that $\hbar^3 = N^{-1}$, we have

$$\begin{aligned}
 II_1 = & \frac{2i\hbar}{(2\pi)^3} \iint dwdu \iint dydv \iint dq_2dp_2 \int_0^1 ds \nabla V(su + (1-s)w - y) \\
 & \cdot \nabla_{p_1} \left(f_{q_1,p_1}^\hbar(w) \overline{f_{q_1,p_1}^\hbar(u)} \right) f_{q_2,p_2}^\hbar(y) \overline{f_{q_2,p_2}^\hbar(v)} \langle a_w a_y \psi_{N,t}, a_u a_v \psi_{N,t} \rangle.
 \end{aligned}
 \tag{3.13}$$

Therefore, we have the last term in (3.5) as

$$\begin{aligned}
 \operatorname{Re} \frac{II_1}{2i\hbar} = & \frac{1}{(2\pi)^3} \operatorname{Re} \iint dwdu \iint dydv \iint dq_2dp_2 \int_0^1 ds \nabla V(su + (1-s)w - y) \\
 & \cdot \nabla_{p_1} \left(f_{q_1,p_1}^\hbar(w) \overline{f_{q_1,p_1}^\hbar(u)} \right) f_{q_2,p_2}^\hbar(y) \overline{f_{q_2,p_2}^\hbar(v)} \langle a_w a_y \psi_{N,t}, a_u a_v \psi_{N,t} \rangle,
 \end{aligned}$$

thus we have derived the equation for $m_{N,t}^{(1)}(q_1, p_1)$. □

We have proved the reformulation from Schrödinger equation into 1-particle Husimi measure. We also observed that it contains a resemblance to the classical Vlasov equation. Next we want to prove the similar result for $2 \leq k \leq N$.

Proof (Proof of Proposition 2.2) Now we focus on the case where $2 \leq k \leq N$. As in the proof for the case of $k = 1$, we first observe that for every $k \in \mathbb{N}$,

$$\begin{aligned}
 & 2i\hbar \partial_t m_{N,t}^{(k)}(q_1, p_1, \dots, q_k, p_k) \\
 = & \left(-\hbar^2 \int \dots \int (dwdu)^{\otimes k} \int dx \left(f_{q,p}^\hbar(w) \overline{f_{q,p}^\hbar(u)} \right)^{\otimes k} \Delta_x \langle \psi_{N,t}, a_{w_1}^* \dots a_{w_k}^* a_{u_k} \dots a_{u_1} a_x^* \psi_{N,t} \rangle \right. \\
 & + \hbar^2 \int \dots \int (dwdu)^{\otimes k} \int dx \left(f_{q,p}^\hbar(w) \overline{f_{q,p}^\hbar(u)} \right)^{\otimes k} \Delta_x \langle \psi_{N,t}, a_x^* a_{w_1}^* \dots a_{w_k}^* a_{u_k} \dots a_{u_1} \psi_{N,t} \rangle \Big) \\
 & + \left(\frac{1}{N} \int \dots \int (dwdu)^{\otimes k} \iint dx dy V(x-y) \left(f_{q,p}^\hbar(w) \overline{f_{q,p}^\hbar(u)} \right)^{\otimes k} \right. \\
 & \left. \langle \psi_{N,t}, a_{w_1}^* \dots a_{w_k}^* a_{u_k} \dots a_{u_1} a_x^* a_y^* \psi_{N,t} \rangle \right) \\
 & - \frac{1}{N} \int \dots \int (dwdu)^{\otimes k} \iint dx dy V(x-y) \left(f_{q,p}^\hbar(w) \overline{f_{q,p}^\hbar(u)} \right)^{\otimes k} \\
 & \left. \langle \psi_{N,t}, a_x^* a_y^* a_{u_k} a_{w_1}^* \dots a_{w_k}^* a_{u_k} \dots a_{u_1} \psi_{N,t} \rangle \right) \\
 =: & I_2 + II_2,
 \end{aligned}
 \tag{3.14}$$

where the tensor product denotes $(dwdu)^{\otimes k} = dw_1 \dots dw_k du_1 \dots du_k$.

We first focus on the I_2 part of (3.14), i.e.,

$$\begin{aligned}
 I_2 = & -\hbar^2 \int \dots \int (dwdu)^{\otimes k} \int dx \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} \Delta_x \left\langle \psi_{N,t}, a_{w_1}^* \dots a_{w_k}^* a_{u_k} \dots a_{u_1} a_x^* a_x \psi_{N,t} \right\rangle \\
 & + \hbar^2 \int \dots \int (dwdu)^{\otimes k} \int dx \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} \Delta_x \left\langle \psi_{N,t}, a_x^* a_x a_{w_1}^* \dots a_{w_k}^* a_{u_k} \dots a_{u_1} \psi_{N,t} \right\rangle.
 \end{aligned}
 \tag{3.15}$$

Observe that we have

$$\begin{aligned}
 a_{w_1}^* \dots a_{w_k}^* a_{u_k} \dots a_{u_1} a_x^* a_x &= (-1)^{4k} a_x^* a_x a_{w_1}^* \dots a_{w_k}^* a_{u_k} \dots a_{u_1} \\
 &+ a_x^* \left(\sum_{j=1}^k (-1)^j \delta_{x=w_j} a_{w_1}^* \dots \widehat{a_{w_j}^*} \dots a_{w_k}^* \right) a_{u_k} \dots a_{u_1} \\
 &- a_{w_1}^* \dots a_{w_k}^* \left(\sum_{j=1}^k (-1)^j \delta_{x=u_j} a_{u_k} \dots \widehat{a_{u_j}} \dots a_{u_1} \right) a_x,
 \end{aligned}
 \tag{3.16}$$

where the *hat* indicates exclusion of that element.

Putting this back into (3.15), we obtain

$$\begin{aligned}
 I_2 = & \hbar^2 \int \dots \int (dwdu)^{\otimes k} \int dx \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} \\
 & \cdot \Delta_x \left\langle \psi_{N,t}, a_{w_1}^* \dots a_{w_k}^* \left(\sum_{j=1}^k (-1)^j \delta_{x=u_j} a_{u_k} \dots \widehat{a_{u_j}} \dots a_{u_1} \right) a_x \psi_{N,t} \right\rangle \\
 & - \hbar^2 \int \dots \int (dwdu)^{\otimes k} \int dx \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} \\
 & \cdot \Delta_x \left\langle \psi_{N,t}, a_x^* \left(\sum_{j=1}^k (-1)^j \delta_{x=w_j} a_{w_1}^* \dots \widehat{a_{w_j}^*} \dots a_{w_k}^* \right) a_{u_k} \dots a_{u_1} \psi_{N,t} \right\rangle \\
 = & \hbar^2 \sum_{j=1}^k (-1)^j \int \dots \int (dwdu)^{\otimes k} \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} \\
 & \cdot \left(\Delta_{u_j} \left\langle \psi_{N,t}, a_{w_1}^* \dots a_{w_k}^* (a_{u_k} \dots \widehat{a_{u_j}} \dots a_{u_1}) a_{u_j} \psi_{N,t} \right\rangle \right. \\
 & \left. - \Delta_{w_j} \left\langle \psi_{N,t}, a_{w_j}^* (a_{w_1}^* \dots \widehat{a_{w_j}^*} \dots a_{w_k}^*) a_{u_k} \dots a_{u_1} \psi_{N,t} \right\rangle \right).
 \end{aligned}
 \tag{3.17}$$

Note that, if we want to move the missing a_{u_j} or $a_{w_j}^*$ back to their original position after applying the delta function, we have for fixed j

$$\begin{aligned}
 (-1)^j a_{w_1}^* \dots a_{w_k}^* [a_{u_k} \dots \widehat{a_{u_j}} \dots a_{u_1}] a_{u_j} &= \frac{(-1)^j}{(-1)^{j-1}} a_{w_1}^* \dots a_{w_k}^* a_{u_k} \dots a_{u_1} \\
 &= (-1)^1 a_{w_1}^* \dots a_{w_k}^* a_{u_k} \dots a_{u_1}, \\
 (-1)^j a_{w_j}^* [a_{w_1}^* \dots \widehat{a_{w_j}^*} \dots a_{w_k}^*] a_{u_k} \dots a_{u_1} &= (-1)^1 a_{w_1}^* \dots a_{w_k}^* a_{u_k} \dots a_{u_1}.
 \end{aligned}$$

Therefore, continuing from (3.17), we have

$$I_2 = -\hbar^2 \sum_{j=1}^k \int \dots \int (dwdu)^{\otimes k} \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} [\Delta_{u_j} - \Delta_{w_j}] \langle \psi_{N,t}, a_{w_1}^* \dots a_{w_k}^* a_{u_k} \dots a_{u_1} \psi_{N,t} \rangle. \tag{3.18}$$

Now, by integration by parts on (3.18) and note that the Laplacian acting on the coherent state would be similar to (3.2) and (3.3), i.e., for fixed j where $1 \leq j \leq k$

$$\begin{aligned} \Delta_{u_j} \left(\overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} &= \left(\Delta_{q_j} + 2i\hbar^{-1} p_j \cdot \nabla_{q_j} - \hbar^{-2} p_j^2 \right) \left(\overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k}, \\ \Delta_{w_j} \left(f_{q,p}^{\hbar}(w) \right)^{\otimes k} &= \left(\Delta_{q_j} - 2i\hbar^{-1} p_j \cdot \nabla_{q_j} - \hbar^{-2} p_j^2 \right) \left(f_{q,p}^{\hbar}(w) \right)^{\otimes k}. \end{aligned}$$

Thus, we have similar for when $k = 1$, the kinetic part as

$$\begin{aligned} I_2 &= -2i\hbar \sum_{j=1}^k p_j \cdot \nabla_{q_j} \int \dots \int (dwdu)^{\otimes k} \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} \langle \psi_{N,t}, a_{w_1}^* \dots a_{w_k}^* a_{u_k} \dots a_{u_1} \psi_{N,t} \rangle \\ &\quad + 2\hbar^2 \text{Im} \sum_{j=1}^k \left\langle \Delta_{q_j} a \left(f_{q_k,p_k}^{\hbar} \right) \dots a \left(f_{q_1,p_1}^{\hbar} \right) \psi_{N,t}, a \left(f_{q_k,p_k}^{\hbar} \right) \dots a \left(f_{q_1,p_1}^{\hbar} \right) \psi_{N,t} \right\rangle \\ &= -2i\hbar \mathbf{p}_k \cdot \nabla_{\mathbf{q}_k} \left\langle a \left(f_{q_k,p_k}^{\hbar} \right) \dots a \left(f_{q_1,p_1}^{\hbar} \right) \psi_{N,t}, a \left(f_{q_k,p_k}^{\hbar} \right) \dots a \left(f_{q_1,p_1}^{\hbar} \right) \psi_{N,t} \right\rangle \\ &\quad + 2i\hbar^2 \text{Im} \sum_{j=1}^k \left\langle \Delta_{q_j} a \left(f_{q_k,p_k}^{\hbar} \right) \dots a \left(f_{q_1,p_1}^{\hbar} \right) \psi_{N,t}, a \left(f_{q_k,p_k}^{\hbar} \right) \dots a \left(f_{q_1,p_1}^{\hbar} \right) \psi_{N,t} \right\rangle. \end{aligned} \tag{3.19}$$

Therefore it follows that

$$\begin{aligned} I_2 &= -2i\hbar \mathbf{p}_k \cdot \nabla_{\mathbf{q}_k} m_{N,t}^{(k)}(q_1, p_1, \dots, q_k, p_k) \\ &\quad + 2i\hbar^2 \text{Im} \sum_{j=1}^k \left\langle \Delta_{q_j} a \left(f_{q_k,p_k}^{\hbar} \right) \dots a \left(f_{q_1,p_1}^{\hbar} \right) \psi_{N,t}, a \left(f_{q_k,p_k}^{\hbar} \right) \dots a \left(f_{q_1,p_1}^{\hbar} \right) \psi_{N,t} \right\rangle. \end{aligned} \tag{3.20}$$

Now, we turn our focus on part I_2 of (3.14),

$$\begin{aligned} I_2 &= \frac{1}{N} \int \dots \int (dwdu)^{\otimes k} \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} \iint dx dy V(x - y) \\ &\quad \left\langle \psi, a_{w_1}^* \dots a_{w_k}^* a_{u_k} \dots a_{u_1} a_x^* a_y^* a_x \psi \right\rangle \\ &\quad - \frac{1}{N} \int \dots \int (dwdu)^{\otimes k} \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} \iint dx dy V(x - y) \\ &\quad \left\langle \psi, a_x^* a_y^* a_x a_{w_1}^* \dots a_{w_k}^* a_{u_k} \dots a_{u_1} \psi \right\rangle. \end{aligned} \tag{3.21}$$

For $1 \leq k \leq N$, observe that from the CAR, we have

$$\begin{aligned}
 & a_{w_1}^* \cdots a_{w_k}^* a_{u_k} \cdots a_{u_1} a_x^* a_y^* a_y a_x - (-1)^{8k} a_x^* a_y^* a_y a_x a_{w_1}^* \cdots a_{w_k}^* a_{u_k} \cdots a_{u_1} \\
 &= -a_{w_1}^* \cdots a_{w_k}^* \left(\sum_{j=1}^k (-1)^j \delta_{x=u_j} a_{u_k} \cdots \widehat{a_{u_j}} \cdots a_{u_1} \right) a_y^* a_y a_x \\
 &\quad - a_x^* a_{w_1}^* \cdots a_{w_k}^* \left(\sum_{j=1}^k (-1)^j \delta_{y=u_j} a_{u_k} \cdots \widehat{a_{u_j}} \cdots a_{u_1} \right) a_y a_x \\
 &\quad + a_x^* a_y^* \left(\sum_{j=1}^k (-1)^j \delta_{y=w_j} a_{w_1}^* \cdots \widehat{a_{w_j}^*} \cdots a_{w_k}^* \right) a_{u_k} \cdots a_{u_1} a_x \\
 &\quad + a_x^* a_y^* a_y \left(\sum_{j=1}^k (-1)^j \delta_{x=w_j} a_{w_1}^* \cdots \widehat{a_{w_j}^*} \cdots a_{w_k}^* \right) a_{u_k} \cdots a_{u_1}.
 \end{aligned} \tag{3.22}$$

From (3.21), we have that

$$\begin{aligned}
 & \iint dx dy V(x - y) \left(a_{w_1}^* \cdots a_{w_k}^* a_{u_k} \cdots a_{u_1} a_x^* a_y^* a_y a_x - a_x^* a_y^* a_y a_x a_{w_1}^* \cdots a_{w_k}^* a_{u_k} \cdots a_{u_1} \right) \\
 &= \iint dx dy V(x - y) \left[-a_{w_1}^* \cdots a_{w_k}^* \left(\sum_{j=1}^k (-1)^j \delta_{x=u_j} a_{u_k} \cdots \widehat{a_{u_j}} \cdots a_{u_1} \right) a_y^* a_y a_x \right. \\
 &\quad - a_x^* a_{w_1}^* \cdots a_{w_k}^* \left(\sum_{j=1}^k (-1)^j \delta_{y=u_j} a_{u_k} \cdots \widehat{a_{u_j}} \cdots a_{u_1} \right) a_y a_x \\
 &\quad + a_x^* a_y^* a_y \left(\sum_{j=1}^k (-1)^j \delta_{x=w_j} a_{w_1}^* \cdots \widehat{a_{w_j}^*} \cdots a_{w_k}^* \right) a_{u_k} \cdots a_{u_1} \\
 &\quad \left. + a_x^* a_y^* a_y \left(\sum_{j=1}^k (-1)^j \delta_{x=w_j} a_{w_1}^* \cdots \widehat{a_{w_j}^*} \cdots a_{w_k}^* \right) a_{u_k} \cdots a_{u_1} \right] \\
 &=: J_1 + J_2 + J_3 + J_4.
 \end{aligned}$$

Note that summing J_1 and J_4 , we have

$$\begin{aligned}
 J_1 + J_4 &= -\sum_{j=1}^k (-1)^j \int dy \left[\left(V(u_j - y) a_{w_1}^* \cdots a_{w_k}^* a_{u_k} \cdots \widehat{a_{u_j}} \cdots a_{u_1} a_y^* a_y a_{u_j} \right) \right. \\
 &\quad \left. - \left(V(w_j - y) a_{w_j}^* a_y^* a_y a_{w_1}^* \cdots \widehat{a_{w_j}^*} \cdots a_{w_k}^* a_{u_k} \cdots a_{u_1} \right) \right] \\
 &= \sum_{j=1}^k \left[\int dy V(u_j - y) a_{w_1}^* \cdots a_{w_k}^* a_{u_k} \cdots a_{u_1} a_y^* a_y - V(0) a_{w_1}^* \cdots a_{w_k}^* a_{u_k} \cdots a_{u_1} \right] \\
 &\quad - \sum_{j=1}^k \left[\int dy V(w_j - y) a_y^* a_y a_{w_1}^* \cdots a_{w_k}^* a_{u_k} \cdots a_{u_1} - V(0) a_{w_1}^* \cdots a_{w_k}^* a_{u_k} \cdots a_{u_1} \right],
 \end{aligned}$$

where the terms with $V(0)$ cancel one another. For the remaining term, we use again CAR to obtain

$$\begin{aligned}
 &= \sum_{j=1}^k \int dy (V(u_j - y) - V(w_j - y)) a_y^* a_{w_1}^* \cdots a_{w_k}^* a_{u_k} \cdots a_{u_1} a_y \\
 &+ \sum_{j=1}^k \sum_{i=1}^k (-1)^i \int dy V(u_j - y) \delta_{u_i=y} a_{w_1}^* \cdots a_{w_k}^* a_{u_k} \cdots \widehat{a_{u_i}} \cdots a_{u_1} a_y \\
 &- \sum_{j=1}^k \sum_{i=1}^k (-1)^i \int dy V(w_j - y) \delta_{w_i=y} a_y^* a_{w_1}^* \cdots \widehat{a_{w_i}^*} \cdots a_{w_k}^* a_{u_k} \cdots a_{u_1} \\
 &= \sum_{j=1}^k \int dy (V(u_j - y) - V(w_j - y)) a_y^* a_{w_1}^* \cdots a_{w_k}^* a_{u_k} \cdots a_{u_1} a_y \\
 &- \sum_{j=1}^k \sum_{i=1}^k (V(u_j - u_i) - V(w_j - w_i)) a_{w_1}^* \cdots a_{w_k}^* a_{u_k} \cdots a_{u_1}.
 \end{aligned}$$

On the other hand, the sum of J_2 and J_2 yield

$$J_2 + J_3 = \sum_{j=1}^k \int dx (V(x - u_j) - V(x - w_j)) a_x^* a_{w_1}^* \cdots a_{w_k}^* a_{u_k} \cdots a_{u_1} a_x.$$

By change of variable and using the fact that $V(-x) = V(x)$, we have from (3.21) that

$$\begin{aligned}
 H_2 &= \frac{2}{N} \int \cdots \int (dwdu)^{\otimes k} \int dy \sum_{j=1}^k [V(y - u_j) - V(w_j - y)] \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} \\
 &\quad \cdot \langle a_{w_k} \cdots a_{w_1} a_y \psi_{N,t}, a_{u_k} \cdots a_{u_1} a_y \psi_{N,t} \rangle \\
 &- \frac{1}{N} \int \cdots \int (dwdu)^{\otimes k} \sum_{j \neq i}^k [V(u_j - u_i) - V(w_j - w_i)] \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} \\
 &\quad \cdot \langle a_{w_k} \cdots a_{w_1} \psi_{N,t}, a_{u_k} \cdots a_{u_1} \psi_{N,t} \rangle
 \end{aligned} \tag{3.23}$$

Applying mean value theorem on the first term on right hand side, we have that

$$\begin{aligned}
 &\frac{2}{N} \sum_{j=1}^k \int \cdots \int (dwdu)^{\otimes k} \int dy (V(y - u_j) - V(w_j - y)) \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} \\
 &\quad \cdot \langle a_{w_k} \cdots a_{w_1} a_y \psi_{N,t}, a_{u_k} \cdots a_{u_1} a_y \psi_{N,t} \rangle \\
 &= \frac{2}{N} \sum_{j=1}^k \int \cdots \int (dwdu)^{\otimes k} \int dy \left[\int_0^1 ds \nabla V(su_j + (1-s)w_j - y) \right] \\
 &\quad \cdot (u_j - w_j) \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} \\
 &\quad \cdot \langle a_{w_k} \cdots a_{w_1} a_y \psi_{N,t}, a_{u_k} \cdots a_{u_1} a_y \psi_{N,t} \rangle \\
 &= \frac{2i\hbar}{N} \sum_{j=1}^k \int \cdots \int (dwdu)^{\otimes k} \int dy \left[\int_0^1 ds \nabla V(su_j + (1-s)w_j - y) \right]
 \end{aligned}$$

$$\begin{aligned} & \cdot \nabla_{p_j} \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} \\ & \cdot \langle a_{w_k} \cdots a_{w_1} a_y \psi_{N,t}, a_{u_k} \cdots a_{u_1} a_y \psi_{N,t} \rangle. \end{aligned} \tag{3.24}$$

As in the case of $k = 1$, we apply the projection (3.11) onto $a_y \psi_{N,t}$ and get further

$$\begin{aligned} & \frac{2i\hbar}{N} \sum_{j=1}^k \int \cdots \int (dwdu)^{\otimes k} \int dy \left[\int_0^1 ds \nabla V(su_j + (1-s)w_j - y) \right] \cdot \nabla_{p_j} \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} \\ & \cdot \langle a_{w_k} \cdots a_{w_1} a_y \psi_{N,t}, a_{u_k} \cdots a_{u_1} \mathbb{1} a_y \psi_{N,t} \rangle \\ & = \frac{2i\hbar}{N} \frac{1}{(2\pi\hbar)^3} \sum_{j=1}^k \int \cdots \int (dwdu)^{\otimes k} \int dy \left[\int_0^1 ds \nabla V(su_j + (1-s)w_j - y) \right] \\ & \cdot \nabla_{p_j} \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} \\ & \cdot \iint d\tilde{q} d\tilde{p} f_{\tilde{q},\tilde{p}}^{\hbar}(y) \int dv \overline{f_{\tilde{q},\tilde{p}}^{\hbar}(v)} \langle a_{w_k} \cdots a_{w_1} a_y \psi_{N,t}, a_{u_k} \cdots a_{u_1} a_v \psi_{N,t} \rangle. \end{aligned} \tag{3.25}$$

Therefore, dividing both equations by $2i\hbar$, we have the following equation

$$\begin{aligned} & \partial_t m_{N,t}^{(k)}(q_1, p_1, \dots, q_k, p_k) + \mathbf{p}_k \cdot \nabla_{\mathbf{q}_k} m_{N,t}^{(k)}(q_1, p_1, \dots, q_k, p_k) \\ & = \hbar \text{Im} \sum_{j=1}^k \langle \Delta_{q_j} a \left(f_{q_k,p_k}^{\hbar} \right) \cdots a \left(f_{q_1,p_1}^{\hbar} \right) \psi_{N,t}, a \left(f_{q_k,p_k}^{\hbar} \right) \cdots a \left(f_{q_1,p_1}^{\hbar} \right) \psi_{N,t} \rangle \\ & + \frac{1}{(2\pi)^3} \sum_{j=1}^k \int \cdots \int (dwdu)^{\otimes k} \int dy \left[\int_0^1 ds \nabla V(su_j + (1-s)w_j - y) \right] \\ & \cdot \nabla_{p_j} \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} \\ & \cdot \iint d\tilde{q} d\tilde{p} f_{\tilde{q},\tilde{p}}^{\hbar}(y) \int dv \overline{f_{\tilde{q},\tilde{p}}^{\hbar}(v)} \langle a_{w_k} \cdots a_{w_1} a_y \psi_{N,t}, a_{u_k} \cdots a_{u_1} a_v \psi_{N,t} \rangle \\ & + \frac{i\hbar^2}{2} \int \cdots \int (dwdu)^{\otimes k} \sum_{j \neq i}^k \left[V(u_j - u_i) - V(w_j - w_i) \right] \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} \\ & \cdot \langle a_{w_k} \cdots a_{w_1} \psi_{N,t}, a_{u_k} \cdots a_{u_1} \psi_{N,t} \rangle. \end{aligned} \tag{3.26}$$

for $1 \leq k \leq N$, $\mathbf{p}_k = (p_1, \dots, p_k)$ and recalling $\hbar^3 = N^{-1}$. At this point we finish the computation of the hierarchy for Husimi measure. □

3.2 Proof of the Uniform Estimates in Section 2.3

This subsection provide the proof of estimates for the error terms that appeared in the equations for $m_{N,t}^{(k)}$. Note that in all the proofs below, we suppose, without loss of generality, that the test function $\Phi \in C_0^\infty(\mathbb{R}^{6k})$ is factorized in phase-space by family of test functions in $C_0^\infty(\mathbb{R}^3)$ space.

3.2.1 Proof of Proposition 2.4

Proof For fixed k , we denote the vector $\mathbf{x}_k = (x_1, \dots, x_k)$ for each $x_j \in \mathbb{R}^3$ with $j = 1, \dots, k$. Then we estimate the integral as follows

$$\begin{aligned}
 & \left| \int \cdots \int (dq dp)^{\otimes k} \nabla_{\mathbf{q}_k} \Phi(q_1, p_1, \dots, q_k, p_k) \cdot \mathcal{R}_k \right| \\
 & \leq \hbar \left| \sum_{j=1}^k \int \cdots \int (dq dp)^{\otimes k} \nabla_{q_j} \Phi(q_1, p_1, \dots, q_k, p_k) \right. \\
 & \quad \cdot \left. \left\langle \nabla_{q_j} (a(f_{q_k, p_k}^\hbar) \cdots a(f_{q_1, p_1}^\hbar)) \psi_{N,t}, a(f_{q_k, p_k}^\hbar) \cdots a(f_{q_1, p_1}^\hbar) \psi_{N,t} \right\rangle \right| \\
 & = \hbar^{1-\frac{3}{2}k} \left| \sum_{j=1}^k \int \cdots \int (dq dp)^{\otimes k} \nabla_{q_j} \Phi(q_1, p_1, \dots, q_k, p_k) \right. \\
 & \quad \cdot \int \cdots \int (dw du)^{\otimes k} \prod_{n=1}^k \left(\chi_{(w_n - u_n) \in \Omega_\hbar^c} + \chi_{(w_n - u_n) \in \Omega_\hbar} \right) \\
 & \quad \cdot \nabla_{q_j} f \left(\frac{w_n - q_n}{\sqrt{\hbar}} \right) f \left(\frac{u_n - q_n}{\sqrt{\hbar}} \right) e^{\frac{i}{\hbar} p_n \cdot (w_n - u_n)} \left\langle a_{w_k} \cdots a_{w_1} \psi_{N,t}, a_{u_k} \cdots a_{u_1} \psi_{N,t} \right\rangle \Big| \\
 & \leq \hbar^{1-\frac{3}{2}k} \sum_{j=1}^k \int \cdots \int (dq dw du)^{\otimes k} \\
 & \quad \left| \int \cdots \int (dp)^{\otimes k} \prod_{n=1}^k \left(\chi_{(w_n - u_n) \in \Omega_\hbar} + \chi_{(w_n - u_n) \in \Omega_\hbar^c} \right) \nabla_{q_j} \Phi \cdot e^{\frac{i}{\hbar} p_n \cdot (w_n - u_n)} \right| \\
 & \quad \cdot \left| \nabla_{q_j} f \left(\frac{w_n - q_n}{\sqrt{\hbar}} \right) \right| \left| f \left(\frac{u_n - q_n}{\sqrt{\hbar}} \right) \right| \| a_{w_k} \cdots a_{w_1} \psi_{N,t} \| \| a_{u_k} \cdots a_{u_1} \psi_{N,t} \| \\
 & = \hbar^{\frac{1}{2}-\frac{3}{2}k} \sum_{j=1}^k \int \cdots \int (dq dw du)^{\otimes k} \\
 & \quad \left| \int \cdots \int (dp)^{\otimes k} \prod_{n=1}^k \left(\chi_{(w_n - u_n) \in \Omega_\hbar} + \chi_{(w_n - u_n) \in \Omega_\hbar^c} \right) \nabla_{q_j} \Phi \cdot e^{\frac{i}{\hbar} p_n \cdot (w_n - u_n)} \right| \\
 & \quad \cdot \prod_{n \neq j}^k \left| f \left(\frac{w_n - q_n}{\sqrt{\hbar}} \right) f \left(\frac{u_n - q_n}{\sqrt{\hbar}} \right) \right| \left| \nabla f \left(\frac{w_j - q_j}{\sqrt{\hbar}} \right) \right| \left| f \left(\frac{u_j - q_j}{\sqrt{\hbar}} \right) \right| \\
 & \quad \cdot \| a_{w_k} \cdots a_{w_1} \psi_{N,t} \| \| a_{u_k} \cdots a_{u_1} \psi_{N,t} \|, \tag{3.27}
 \end{aligned}$$

where Ω_\hbar is defined as in (2.10) and used the fact that

$$\nabla_{q_j} f \left(\frac{w_j - q_j}{\sqrt{\hbar}} \right) = -\frac{1}{\sqrt{\hbar}} \nabla f \left(\frac{w_j - q_j}{\sqrt{\hbar}} \right).$$

Now, the product term $\prod_{n=1}^k \left(\chi_{(w_n - u_n) \in \Omega_\hbar} + \chi_{(w_n - u_n) \in \Omega_\hbar^c} \right)$ in (3.27) includes a summation of $C(k)$ terms of the following type

$$\chi_{(w_1 - u_1) \in \Omega_\hbar} \cdots \chi_{(w_\ell - u_\ell) \in \Omega_\hbar} \chi_{(w_{\ell+1} - u_{\ell+1}) \in \Omega_\hbar^c} \cdots \chi_{(w_k - u_k) \in \Omega_\hbar^c}, \tag{3.28}$$

where $\ell \in \{1, \dots, k\}$. Thus, to continue from (3.27), we have

$$\begin{aligned}
 & \left| \int \cdots \int (dq dp)^{\otimes k} \nabla_{q_k} \Phi(q_1, p_1, \dots, q_k, p_k) \cdot \mathcal{R}_k \right| \\
 & \leq C \hbar^{\frac{1}{2} - \frac{3}{2}k} \sum_{j=1}^k \max_{0 \leq \ell \leq k} \int \cdots \int (dq dw du)^{\otimes k} \prod_{n \neq j}^k \left| f\left(\frac{w_n - q_n}{\sqrt{\hbar}}\right) f\left(\frac{u_n - q_n}{\sqrt{\hbar}}\right) \right| \\
 & \quad \left| \nabla f\left(\frac{w_j - q_j}{\sqrt{\hbar}}\right) \right| \left| f\left(\frac{u_j - q_j}{\sqrt{\hbar}}\right) \right| \\
 & \quad \cdot \left| \int \cdots \int (dp)^{\otimes k} \left(\chi_{(w_1 - u_1) \in \Omega_{\hbar}} \cdots \chi_{(w_{\ell} - u_{\ell}) \in \Omega_{\hbar}} \chi_{(w_{\ell+1} - u_{\ell+1}) \in \Omega_{\hbar}^c} \right. \right. \\
 & \quad \left. \left. \cdots \chi_{(w_k - u_k) \in \Omega_{\hbar}^c} \right) \nabla_{q_j} \Phi \cdot e^{\frac{i}{\hbar} p_k \cdot (w_k - u_k)} \right| \\
 & \quad \cdot \|a_{w_k} \cdots a_{w_1} \psi_{N,t}\| \|a_{u_k} \cdots a_{u_1} \psi_{N,t}\| \\
 & \leq C \hbar^{\frac{1}{2} - \frac{3}{2}k} \sum_{j=1}^k \max_{0 \leq \ell \leq k} \int \cdots \int (dq dw du)^{\otimes k} \prod_{n \neq j}^k \left| f\left(\frac{w_n - q_n}{\sqrt{\hbar}}\right) f\left(\frac{u_n - q_n}{\sqrt{\hbar}}\right) \right| \\
 & \quad \left| \nabla f\left(\frac{w_j - q_j}{\sqrt{\hbar}}\right) \right| \left| f\left(\frac{u_j - q_j}{\sqrt{\hbar}}\right) \right| \\
 & \quad \cdot \left| \int \cdots \int (dp)^{\otimes \ell} \chi_{(w_1 - u_1) \in \Omega_{\hbar}} \right. \\
 & \quad \left. \cdots \chi_{(w_{\ell} - u_{\ell}) \in \Omega_{\hbar}} e^{\frac{i}{\hbar} \sum_{m=1}^{\ell} p_m \cdot (w_m - u_m)} \right. \\
 & \quad \left. \cdot \int \cdots \int (dp)^{\otimes (k-\ell)} \chi_{(w_{\ell+1} - u_{\ell+1}) \in \Omega_{\hbar}^c} \right. \\
 & \quad \left. \cdots \chi_{(w_k - u_k) \in \Omega_{\hbar}^c} e^{\frac{i}{\hbar} \sum_{m=k-\ell}^{\ell} p_m \cdot (w_m - u_m)} \nabla_{q_j} \Phi \right| \\
 & \quad \cdot \|a_{w_k} \cdots a_{w_1} \psi_{N,t}\| \|a_{u_k} \cdots a_{u_1} \psi_{N,t}\|
 \end{aligned}$$

Applying Lemma 2.5 onto the $(k - \ell)$ terms, we have

$$\begin{aligned}
 & \leq C \sum_{j=1}^k \max_{0 \leq \ell \leq k} \hbar^{\frac{1}{2} - \frac{3}{2}k + (1-\alpha)(k-\ell)s} \int \cdots \int (dq dw du)^{\otimes k} \left(\chi_{(w_1 - u_1) \in \Omega_{\hbar}} \cdots \chi_{(w_{\ell} - u_{\ell}) \in \Omega_{\hbar}} \right) \\
 & \quad \cdot \prod_{n \neq j}^k \left| f\left(\frac{w_n - q_n}{\sqrt{\hbar}}\right) f\left(\frac{u_n - q_n}{\sqrt{\hbar}}\right) \right| \cdot \left| \nabla f\left(\frac{w_j - q_j}{\sqrt{\hbar}}\right) \right| \left| f\left(\frac{u_j - q_j}{\sqrt{\hbar}}\right) \right| \\
 & \quad \cdot \|a_{w_k} \cdots a_{w_1} \psi_{N,t}\| \|a_{u_k} \cdots a_{u_1} \psi_{N,t}\|.
 \end{aligned}$$

For a fixed ℓ , observe that since f is compact supported, by using Hölder’s inequality in w and u variables, we have

$$\begin{aligned}
 & \int \cdots \int (dq dw du)^{\otimes k} \left(\chi_{(w_1 - u_1) \in \Omega_{\hbar}} \cdots \chi_{(w_{\ell} - u_{\ell}) \in \Omega_{\hbar}} \right) \prod_{n \neq j}^k \left| f\left(\frac{w_n - q_n}{\sqrt{\hbar}}\right) f\left(\frac{u_n - q_n}{\sqrt{\hbar}}\right) \right| \\
 & \quad \cdot \left| \nabla f\left(\frac{w_j - q_j}{\sqrt{\hbar}}\right) \right| \left| f\left(\frac{u_j - q_j}{\sqrt{\hbar}}\right) \right| \|a_{w_k} \cdots a_{w_1} \psi_{N,t}\| \|a_{u_k} \cdots a_{u_1} \psi_{N,t}\|
 \end{aligned}$$

$$\begin{aligned}
 &= \int \cdots \int (dqdwdu)^{\otimes k} (\chi_{(w_1-u_1) \in \Omega_{\tilde{h}}} \cdots \chi_{(w_\ell-u_\ell) \in \Omega_{\tilde{h}}}) \prod_{n \neq j}^k \left| f\left(\frac{w_n - q_n}{\sqrt{\tilde{h}}}\right) f\left(\frac{u_n - q_n}{\sqrt{\tilde{h}}}\right) \right| \\
 &\quad \cdot \left| \nabla f\left(\frac{w_j - q_j}{\sqrt{\tilde{h}}}\right) \right| \left| f\left(\frac{u_j - q_j}{\sqrt{\tilde{h}}}\right) \right| \prod_{m=1}^k \chi_{|w_m - q_m| \leq \sqrt{\tilde{h}}R} \chi_{|u_m - q_m| \leq \sqrt{\tilde{h}}R} \\
 &\quad \|a_{w_k} \cdots a_{w_1} \psi_{N,t}\| \|a_{u_k} \cdots a_{u_1} \psi_{N,t}\| \\
 &\leq \int \cdots \int (dq)^{\otimes k} \left[\int \cdots \int (dwdu)^{\otimes k} (\chi_{(w_1-u_1) \in \Omega_{\tilde{h}}} \cdots \chi_{(w_\ell-u_\ell) \in \Omega_{\tilde{h}}}) \right. \\
 &\quad \left. \prod_{n \neq j}^k \left| f\left(\frac{w_n - q_n}{\sqrt{\tilde{h}}}\right) f\left(\frac{u_n - q_n}{\sqrt{\tilde{h}}}\right) \right|^2 \right. \\
 &\quad \left. \cdot \left| \nabla f\left(\frac{w_j - q_j}{\sqrt{\tilde{h}}}\right) \right|^2 \left| f\left(\frac{u_j - q_j}{\sqrt{\tilde{h}}}\right) \right|^2 \right]^{\frac{1}{2}} \\
 &\quad \left[\int \cdots \int (dw)^{\otimes k} \prod_{m=1}^k \chi_{|w_m - q_m| \leq \sqrt{\tilde{h}}R} \|a_{w_k} \cdots a_{w_1} \psi_{N,t}\|^2 \right].
 \end{aligned}$$

By change of variables and then applying Lemma 2.4, we have

$$\begin{aligned}
 &= \tilde{h}^{\frac{3}{2}k} \left[\int \cdots \int (d\tilde{w}d\tilde{u})^{\otimes k} \left(\chi_{|\tilde{w}_1 - \tilde{u}_1| \leq \tilde{h}^{\alpha + \frac{1}{2}}} \cdots \chi_{|\tilde{w}_\ell - \tilde{u}_\ell| \leq \tilde{h}^{\alpha + \frac{1}{2}}} \right) \prod_{n \neq j}^k |f(\tilde{w}_n)| |f(\tilde{u}_n)|^2 \right. \\
 &\quad \left. \cdot |\nabla f(\tilde{w}_j)|^2 |f(\tilde{u}_j)|^2 \right]^{\frac{1}{2}} \int \cdots \int (dqdw)^{\otimes k} \prod_{m=1}^k \chi_{|w_m - q_m| \leq \sqrt{\tilde{h}}R} \|a_{w_k} \cdots a_{w_1} \psi_{N,t}\|^2 \\
 &\leq \left[\int \cdots \int (d\tilde{w}d\tilde{u})^{\otimes k} \left(\chi_{|\tilde{w}_1 - \tilde{u}_1| \leq \tilde{h}^{\alpha + \frac{1}{2}}} \cdots \chi_{|\tilde{w}_\ell - \tilde{u}_\ell| \leq \tilde{h}^{\alpha + \frac{1}{2}}} \right) \right. \\
 &\quad \left. \prod_{n \neq j}^k |f(\tilde{w}_n)| |f(\tilde{u}_n)|^2 |\nabla f(\tilde{w}_j)|^2 |f(\tilde{u}_j)|^2 \right]^{\frac{1}{2}}. \tag{3.29}
 \end{aligned}$$

Observe now that by using Hölder inequality with respect to \tilde{u} , we get, for every $1 \leq n \leq k$,

$$\begin{aligned}
 &\int d\tilde{w}_n |f(\tilde{w}_n)|^2 \int d\tilde{u}_n \chi_{|\tilde{w}_n - \tilde{u}_n| \leq \tilde{h}^{\alpha + \frac{1}{2}}} |f(\tilde{u}_n)|^2 \\
 &\leq \int d\tilde{w}_n |f(\tilde{w}_n)|^2 \left(\int d\tilde{u}_n \chi_{|\tilde{w}_n - \tilde{u}_n| \leq \tilde{h}^{\alpha + \frac{1}{2}}} \right)^{\frac{2}{3}} \left(\int d\tilde{u}_n |f(\tilde{u}_n)|^6 \right)^{\frac{1}{3}} \\
 &\leq C \tilde{h}^{2\alpha + 1} \left(\int d\tilde{w}_n |f(\tilde{w}_n)|^2 \right) \left(\int d\tilde{u}_n |f(\tilde{u}_n)|^6 \right)^{\frac{1}{3}} \\
 &\leq C \tilde{h}^{2\alpha + 1},
 \end{aligned} \tag{3.30}$$

where we have used the fact that $f \in H^1$, it is also embedded in the L^6 space. Similarly,

$$\begin{aligned}
 &\iint d\tilde{w}_j d\tilde{u}_j \chi_{|\tilde{w}_j - \tilde{u}_j| \leq \tilde{h}^{\alpha + \frac{1}{2}}} |\nabla f(\tilde{w}_j)|^2 |f(\tilde{u}_j)|^2 \\
 &= \int d\tilde{w}_j |\nabla f(\tilde{w}_j)|^2 \int d\tilde{u}_j \chi_{|\tilde{w}_j - \tilde{u}_j| \leq \tilde{h}^{\alpha + \frac{1}{2}}} |f(\tilde{u}_j)|^2
 \end{aligned}$$

$$\begin{aligned} &\leq \int d\tilde{w}_j |\nabla f(\tilde{w}_j)|^2 \left(\int d\tilde{u}_j \chi_{|\tilde{w}_j - \tilde{u}_j| \leq \hbar^{\alpha + \frac{1}{2}}} \right)^{\frac{2}{3}} \left(\int d\tilde{u}_j |f(\tilde{u}_j)|^6 \right)^{\frac{1}{3}} \\ &\leq C\hbar^{2\alpha + 1}. \end{aligned}$$

Putting this back into (3.29), we have

$$\begin{aligned} &\int \dots \int (dq dw du)^{\otimes k} (\chi_{(w_1 - u_1) \in \Omega_{\hbar}} \dots \chi_{(w_\ell - u_\ell) \in \Omega_{\hbar}}) \prod_{n \neq j}^k \left| f\left(\frac{w_n - q_n}{\sqrt{\hbar}}\right) f\left(\frac{u_n - q_n}{\sqrt{\hbar}}\right) \right| \\ &\cdot \left| \nabla f\left(\frac{w_j - q_j}{\sqrt{\hbar}}\right) \right| \left| f\left(\frac{u_j - q_j}{\sqrt{\hbar}}\right) \right| \|a_{w_k} \dots a_{w_1} \psi_{N,t}\| \|a_{u_k} \dots a_{u_1} \psi_{N,t}\| \\ &\leq C\hbar^{(\alpha + \frac{1}{2})\ell}. \end{aligned}$$

Then, from (3.28), we have

$$\begin{aligned} \left| \int \dots \int (dq dp)^{\otimes k} \nabla_{\mathbf{q}_k} \Phi(q_1, p_1, \dots, q_k, p_k) \cdot \mathcal{R}_k \right| &\leq C \sum_{j=1}^k \max_{0 \leq \ell \leq k} \hbar^{\frac{1}{2} - \frac{3}{2}k + (1-\alpha)(k-\ell)s + (\alpha + \frac{1}{2})\ell} \\ &= Ck \max_{0 \leq \ell \leq k} \hbar^{\frac{1}{2} - \frac{3}{2}k + (1-\alpha)(k-\ell)s + (\alpha + \frac{1}{2})\ell}. \end{aligned} \tag{3.31}$$

Therefore, by picking $s = \left\lceil \frac{1+2\alpha}{2(1-\alpha)} \right\rceil$ we arrive immediately that

$$\left| \int \dots \int (dq dp)^{\otimes k} \nabla_{\mathbf{q}_k} \Phi(q_1, p_1, \dots, q_k, p_k) \cdot \mathcal{R}_k \right| \leq C\hbar^{\frac{1}{2} + (\alpha - 1)k}.$$

Therefore, for all $\delta \ll 1$, we choose $\frac{1}{2} < \alpha < 1$ such that $(\alpha - 1)k \leq -\delta$. □

3.2.2 Proof of Proposition 2.5

Proof Let Φ be an arbitrary test function, then the remainder term $\tilde{\mathcal{R}}_1$ can be written explicitly into

$$\begin{aligned} &\left| \iint dq_1 dp_1 \nabla_{p_1} \Phi(q_1, p_1) \cdot \tilde{\mathcal{R}}_1 \right| \\ &= \left| \iint dq_1 dp_1 \nabla_{p_1} \Phi(q_1, p_1) \cdot \left(\iint dw du \iint dy dv \iint dq_2 dp_2 \right. \right. \\ &\quad \cdot \left[\int_0^1 ds \nabla V(su + (1-s)w - y) - \nabla V(q_1 - q_2) \right] \\ &\quad \cdot \left. f_{q_1, p_1}^{\hbar}(w) \overline{f_{q_1, p_1}^{\hbar}(u)} f_{q_2, p_2}^{\hbar}(y) \overline{f_{q_2, p_2}^{\hbar}(v)} (a_w a_y \psi_{N,t}, a_u a_v \psi_{N,t}) \right) \Big| \\ &= \frac{1}{\hbar^3} \left| \iint dq_1 dp_1 \nabla_{p_1} \Phi(q_1, p_1) \cdot \iint dw du \iint dy dv \iint dq_2 dp_2 \right. \\ &\quad \cdot \left[\int_0^1 ds \nabla V(su + (1-s)w - y) - \nabla V(q_1 - q_2) \right] e^{\frac{i}{\hbar} p_1 \cdot (w-u)} e^{\frac{i}{\hbar} p_2 \cdot (y-v)} \\ &\quad \cdot \left. f\left(\frac{w - q_1}{\sqrt{\hbar}}\right) \overline{f\left(\frac{u - q_1}{\sqrt{\hbar}}\right)} f\left(\frac{y - q_2}{\sqrt{\hbar}}\right) \overline{f\left(\frac{v - q_2}{\sqrt{\hbar}}\right)} (a_w a_y \psi_{N,t}, a_u a_v \psi_{N,t}) \right|. \end{aligned}$$

Then, utilizing (2.7), we may get

$$\begin{aligned}
 & (2\pi)^3 \left| \iint dq_1 dp_1 \nabla_{p_1} \Phi(q_1, p_1) \cdot \iint dw du \iint dy dq_2 \right. \\
 & \quad \cdot \left[\int_0^1 ds \nabla V(su + (1-s)w - y) - \nabla V(q_1 - q_2) \right] \\
 & \quad \cdot f\left(\frac{w - q_1}{\sqrt{\hbar}}\right) \overline{f\left(\frac{u - q_1}{\sqrt{\hbar}}\right)} e^{\frac{i}{\hbar} p_1 \cdot (w-u)} \left| f\left(\frac{y - q_2}{\sqrt{\hbar}}\right) \right|^2 \langle a_w a_y \psi_{N,t}, a_u a_y \psi_{N,t} \rangle \Big| \\
 & = (2\pi)^3 \hbar^{\frac{3}{2}} \left| \iint dq_1 dp_1 \nabla_{p_1} \Phi(q_1, p_1) \cdot \iint dw du \iint dy \tilde{q}_2 \right. \\
 & \quad \cdot \left[\int_0^1 ds \nabla V(su + (1-s)w - y) - \nabla V(q_1 - y + \sqrt{\hbar} \tilde{q}_2) \right] \\
 & \quad \cdot f\left(\frac{w - q_1}{\sqrt{\hbar}}\right) \overline{f\left(\frac{u - q_1}{\sqrt{\hbar}}\right)} e^{\frac{i}{\hbar} p_1 \cdot (w-u)} |f(\tilde{q}_2)|^2 \langle a_w a_y \psi_{N,t}, a_u a_y \psi_{N,t} \rangle \Big|.
 \end{aligned}$$

Then, we insert a term, namely $\nabla V(q_1 - y)$ and use triangle inequality to obtain

$$\begin{aligned}
 & \leq (2\pi)^3 \hbar^{\frac{3}{2}} \left| \iint dq_1 dp_1 \nabla_{p_1} \Phi(q_1, p_1) \cdot \iint dw du \iint dy \tilde{q}_2 \right. \\
 & \quad \cdot \int_0^1 ds \left(\nabla V(su + (1-s)w - y) - \nabla V(q_1 - y) \right) \\
 & \quad \cdot f\left(\frac{w - q_1}{\sqrt{\hbar}}\right) \overline{f\left(\frac{u - q_1}{\sqrt{\hbar}}\right)} e^{\frac{i}{\hbar} p_1 \cdot (w-u)} |f(\tilde{q}_2)|^2 \langle a_w a_y \psi_{N,t}, a_u a_y \psi_{N,t} \rangle \Big| \\
 & \quad + (2\pi)^3 \hbar^{\frac{3}{2}} \left| \iint dq_1 dp_1 \nabla_{p_1} \Phi(q_1, p_1) \right. \\
 & \quad \cdot \iint dw du \iint dy \tilde{q}_2 \left(\nabla V(q_1 - y) - \nabla V(q_1 - y + \sqrt{\hbar} \tilde{q}_2) \right) \\
 & \quad \cdot f\left(\frac{w - q_1}{\sqrt{\hbar}}\right) \overline{f\left(\frac{u - q_1}{\sqrt{\hbar}}\right)} e^{\frac{i}{\hbar} p_1 \cdot (w-u)} |f(\tilde{q}_2)|^2 \langle a_w a_y \psi_{N,t}, a_u a_y \psi_{N,t} \rangle \Big| \\
 & =: I_3 + II_3,
 \end{aligned}$$

where we have used change of variable $\sqrt{\hbar} \tilde{q}_2 = (y - q_2)$ in the second term above.

We first focus on II_3 . We begin by splitting the integral on momentum, by using Lemma 2.5, it follows

$$\begin{aligned}
 II_3 & = (2\pi)^3 \hbar^{\frac{3}{2}} \left| \iint dq_1 dp_1 \nabla_{p_1} \Phi(q_1, p_1) \cdot \iint dw du \iint dy \tilde{q}_2 \left(\chi_{(w-u) \in \Omega_{\hbar}^c} + \chi_{(w-u) \in \Omega_{\hbar}} \right) \right. \\
 & \quad \cdot \left(\nabla V(q_1 - y) - \nabla V(q_1 - y + \sqrt{\hbar} \tilde{q}_2) \right) f\left(\frac{w - q_1}{\sqrt{\hbar}}\right) \overline{f\left(\frac{u - q_1}{\sqrt{\hbar}}\right)} \\
 & \quad \cdot e^{\frac{i}{\hbar} p_1 \cdot (w-u)} |f(\tilde{q}_2)|^2 \langle a_w a_y \psi_{N,t}, a_u a_y \psi_{N,t} \rangle \Big| \\
 & \leq (2\pi)^3 \hbar^{\frac{3}{2} + \frac{1}{2}} \int dq_1 \iint dw du \int dy \left(\left| \int dp_1 e^{\frac{i}{\hbar} p_1 \cdot (w-u)} \chi_{(w-u) \in \Omega_{\hbar}^c} \nabla_{p_1} \Phi(q_1, p_1) \right| \right. \\
 & \quad \left. + \left| \int dp_1 e^{\frac{i}{\hbar} p_1 \cdot (w-u)} \chi_{(w-u) \in \Omega_{\hbar}} \nabla_{p_1} \Phi(q_1, p_1) \right| \right) \cdot \left| f\left(\frac{w - q_1}{\sqrt{\hbar}}\right) \overline{f\left(\frac{u - q_1}{\sqrt{\hbar}}\right)} \right|
 \end{aligned}$$

$$\begin{aligned}
 & \cdot \left(\int d\tilde{q}_2 |\tilde{q}_2| |f(\tilde{q}_2)|^2 \right) | \langle a_w a_y \psi_{N,t}, a_u a_y \psi_{N,t} \rangle | \\
 & \leq C \hbar^{\frac{3}{2} + \frac{1}{2}} \int dq_1 \iint dw du \int dy \left(\left| \int dp_1 e^{\frac{i}{\hbar} p_1 \cdot (w-u)} \chi_{(w-u) \in \Omega_{\hbar}^c} \nabla_{p_1} \Phi(q_1, p_1) \right| \right. \\
 & \quad \left. + \left| \int dp_1 e^{\frac{i}{\hbar} p_1 \cdot (w-u)} \chi_{(w-u) \in \Omega_{\hbar}} \nabla_{p_1} \Phi(q_1, p_1) \right| \right) \cdot \left| f\left(\frac{w-q_1}{\sqrt{\hbar}}\right) \overline{f\left(\frac{u-q_1}{\sqrt{\hbar}}\right)} \right| \\
 & \quad \cdot | \langle a_w a_y \psi_{N,t}, a_u a_y \psi_{N,t} \rangle | \\
 & =: i_{31} + ii_{31}, \tag{3.32}
 \end{aligned}$$

where we used the fact that ∇V is Lipschitz continuous, f has compact support, and the definition of Ω_{\hbar} in (2.10).

The next step is to use Lemmata 2.4 and 2.5 to bound the terms i_{31} and ii_{31} . Then we examine what the appropriate terms α and s should be. By Lemma 2.5, we may bound the term i_{31} , i.e.,

$$\begin{aligned}
 i_{31} & \leq C \hbar^{\frac{3}{2} + \frac{1}{2} + (1-\alpha)s} \int dq_1 \iint dw du \int dy \cdot \left| f\left(\frac{w-q_1}{\sqrt{\hbar}}\right) \overline{f\left(\frac{u-q_1}{\sqrt{\hbar}}\right)} \right| | \langle a_w a_y \psi_{N,t}, a_u a_y \psi_{N,t} \rangle | \\
 & \leq C \hbar^{\frac{3}{2} + \frac{1}{2} + (1-\alpha)s} \int dq_1 \iint dw du \int dy \left| f\left(\frac{w-q_1}{\sqrt{\hbar}}\right) \overline{f\left(\frac{u-q_1}{\sqrt{\hbar}}\right)} \right| \|a_w a_y \psi_{N,t}\| \|a_u a_y \psi_{N,t}\|.
 \end{aligned}$$

Since we assume that f is compactly supported, by Hölder inequality with respect to w and u , we have we have that

$$\begin{aligned}
 i_{31} & \leq C \hbar^{\frac{3}{2} + \frac{1}{2} + (1-\alpha)s} \int dq_1 \left(\iint dw du \left| f\left(\frac{w-q_1}{\sqrt{\hbar}}\right) \right|^2 \left| f\left(\frac{u-q_1}{\sqrt{\hbar}}\right) \right|^2 \right)^{\frac{1}{2}} \\
 & \quad \cdot \left(\iint dw du \chi_{|w-q_1| \leq \sqrt{\hbar}R} \chi_{|u-q_1| \leq \sqrt{\hbar}R} \left(\int dy \|a_w a_y \psi_{N,t}\| \|a_u a_y \psi_{N,t}\| \right)^2 \right)^{\frac{1}{2}} \\
 & = C \hbar^{3 + \frac{1}{2} + (1-\alpha)s} \int dq_1 \left(\int d\tilde{w} |f(\tilde{w})|^2 \right) \\
 & \quad \cdot \left(\iint dw du \chi_{|w-q_1| \leq \sqrt{\hbar}R} \chi_{|u-q_1| \leq \sqrt{\hbar}R} \left(\int dy \|a_w a_y \psi_{N,t}\| \|a_u a_y \psi_{N,t}\| \right)^2 \right)^{\frac{1}{2}},
 \end{aligned}$$

where we used the change of variable $\sqrt{\hbar}\tilde{w} = w - q_1$ in the last inequality. Now, since $\|f\|_2$ is normalized, we continue to have

$$\begin{aligned}
 & \leq C \hbar^{3 + \frac{1}{2} + (1-\alpha)s} \\
 & \quad \int dq_1 \left(\iint dw du \chi_{|w-q_1| \leq \sqrt{\hbar}R} \chi_{|u-q_1| \leq \sqrt{\hbar}R} \left(\int dy \|a_w a_y \psi_{N,t}\| \|a_u a_y \psi_{N,t}\| \right)^2 \right)^{\frac{1}{2}} \\
 & \leq C \hbar^{3 + \frac{1}{2} + (1-\alpha)s} \\
 & \quad \int dq_1 \left(\iint dw du \chi_{|w-q_1| \leq \sqrt{\hbar}R} \chi_{|u-q_1| \leq \sqrt{\hbar}R} \left(\int dy \|a_w a_y \psi_{N,t}\|^2 \right) \left(\int dy \|a_u a_y \psi_{N,t}\|^2 \right) \right)^{\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
 &= C\hbar^{3+\frac{1}{2}+(1-\alpha)s} \int dq_1 \iint dydw \chi_{|w-q_1|\leq\sqrt{\hbar}R} \|a_w a_y \psi_{N,t}\|^2 \\
 &= C\hbar^{3+\frac{1}{2}+(1-\alpha)s} \int dy \iint dq_1 dw \langle a_y \psi_{N,t}, \chi_{|w-q_1|\leq\sqrt{\hbar}R} a_w^* a_w a_y \psi_{N,t} \rangle
 \end{aligned}$$

by Lemma 2.4

$$\begin{aligned}
 i_{31} &\leq C\hbar^{3-\frac{3}{2}+\frac{1}{2}+(1-\alpha)s} \int dy \langle a_y \psi_{N,t}, a_y \psi_{N,t} \rangle \\
 &= C\hbar^{(1-\alpha)s-1} \left\langle \psi_{N,t}, \frac{\mathcal{N}}{N} \psi_{N,t} \right\rangle \leq C\hbar^{(1-\alpha)s-1}.
 \end{aligned} \tag{3.33}$$

On the other hand, from ii_{31} we have

$$\begin{aligned}
 ii_{31} &\leq C\hbar^{\frac{3}{2}+\frac{1}{2}} \int dq_1 \iint dwdu \int dy \left| \int dp_1 e^{\frac{i}{\hbar} p_1 \cdot (w-u)} \chi_{(w-u)\in\Omega_{\hbar}} \nabla_{p_1} \Phi(q_1, p_1) \right| \\
 &\quad \cdot \left| f\left(\frac{w-q_1}{\sqrt{\hbar}}\right) \overline{f\left(\frac{u-q_1}{\sqrt{\hbar}}\right)} \right| |\langle a_w a_y \psi_{N,t}, a_u a_y \psi_{N,t} \rangle| \\
 &\leq C\hbar^{\frac{3}{2}+\frac{1}{2}} \int dq_1 \iint dwdu \int dy \int dp_1 \chi_{(w-u)\in\Omega_{\hbar}} |\nabla_{p_1} \Phi(q_1, p_1)| \\
 &\quad \cdot \left| f\left(\frac{w-q_1}{\sqrt{\hbar}}\right) \overline{f\left(\frac{u-q_1}{\sqrt{\hbar}}\right)} \right| |\langle a_w a_y \psi_{N,t}, a_u a_y \psi_{N,t} \rangle| \\
 &\leq C\hbar^{\frac{3}{2}+\frac{1}{2}} \int dq_1 \iint dwdu \int dy \chi_{(w-u)\in\Omega_{\hbar}} \\
 &\quad \cdot \left| f\left(\frac{w-q_1}{\sqrt{\hbar}}\right) \overline{f\left(\frac{u-q_1}{\sqrt{\hbar}}\right)} \right| |\langle a_w a_y \psi_{N,t}, a_u a_y \psi_{N,t} \rangle|
 \end{aligned}$$

Since f is assumed to be compactly supported, we have

$$\begin{aligned}
 &\leq C\hbar^{\frac{3}{2}+\frac{1}{2}} \int dq_1 \left(\iint dwdu \chi_{(w-u)\in\Omega_{\hbar}} \left| f\left(\frac{w-q_1}{\sqrt{\hbar}}\right) f\left(\frac{u-q_1}{\sqrt{\hbar}}\right) \right|^2 \right)^{\frac{1}{2}} \\
 &\quad \cdot \left(\iint dwdu \chi_{|w-q_1|\leq\sqrt{\hbar}R} \chi_{|u-q_1|\leq\sqrt{\hbar}R} \left(\int dy \|a_w a_y \psi_{N,t}\| \|a_u a_y \psi_{N,t}\| \right)^2 \right)^{\frac{1}{2}},
 \end{aligned}$$

where we use Cauchy–Schwarz inequality and Hölder inequality.

Next, by change of variables as well as Hölder inequality in respect of y , we have

$$\begin{aligned}
 &\leq C\hbar^{3+\frac{1}{2}} \left(\iint d\tilde{w}d\tilde{u} \chi_{|\tilde{w}-\tilde{u}|\leq\hbar^{\alpha+\frac{1}{2}}} \left| f(\tilde{w}) f(\tilde{u}) \right|^2 \right)^{\frac{1}{2}} \\
 &\quad \cdot \int dy \iint dq_1 dw \chi_{|w-q_1|\leq\sqrt{\hbar}R} \langle a_y \psi_{N,t}, a_w^* a_w a_y \psi_{N,t} \rangle \\
 &\leq C\hbar^{-1} \left(\int d\tilde{w} |f(\tilde{w})|^2 \int d\tilde{u} \chi_{|\tilde{w}-\tilde{u}|\leq\hbar^{\alpha+\frac{1}{2}}} |f(\tilde{u})|^2 \right)^{\frac{1}{2}},
 \end{aligned} \tag{3.34}$$

where we applied Lemma 2.4. Observe from (3.30), we get

$$ii_{31} \leq C\hbar^{\alpha-\frac{1}{2}}.$$

Now we compare power of \hbar with the one in (3.33). Namely,

$$\alpha - \frac{1}{2} = (1 - \alpha)s - 1. \tag{3.35}$$

Therefore, we choose $s = \left\lceil \frac{1+2\alpha}{2(1-\alpha)} \right\rceil$ such that I_3 is of order $\hbar^{\alpha-\frac{1}{2}}$. Now, focus on I_3 , we use similar strategy as with I_2 .

$$\begin{aligned} I_3 &\leq C\hbar^{\frac{3}{2}} \int dq_1 \iint dwdu \int dy \int_0^1 ds \left| \nabla V(su + (1-s)w - y) - \nabla V(q_1 - y) \right| \\ &\quad \cdot \left(\left| \int dp_1 e^{\frac{i}{\hbar} p_1 \cdot (w-u)} \chi_{(w-u) \in \Omega_{\hbar}^c} \nabla_{p_1} \Phi(q_1, p_1) \right| \right. \\ &\quad \left. + \left| \int dp_1 e^{\frac{i}{\hbar} p_1 \cdot (w-u)} \chi_{(w-u) \in \Omega_{\hbar}} \nabla_{p_1} \Phi(q_1, p_1) \right| \right) \\ &\quad \cdot \left| f\left(\frac{w-q_1}{\sqrt{\hbar}}\right) \overline{f\left(\frac{u-q_1}{\sqrt{\hbar}}\right)} \right| \left(\int d\tilde{q}_2 |f(\tilde{q}_2)|^2 \right) | \langle a_w a_y \psi_{N,t}, a_u a_y \psi_{N,t} \rangle | \\ &\leq C\hbar^{\frac{3}{2}} \int dq_1 \iint dwdu \int dy \\ &\quad \int_0^1 ds |su + (1-s)w - q_1| \left(\left| \int dp_1 e^{\frac{i}{\hbar} p_1 \cdot (w-u)} \chi_{(w-u) \in \Omega_{\hbar}^c} \nabla_{p_1} \Phi(q_1, p_1) \right| \right. \\ &\quad \left. + \left| \int dp_1 e^{\frac{i}{\hbar} p_1 \cdot (w-u)} \chi_{(w-u) \in \Omega_{\hbar}} \nabla_{p_1} \Phi(q_1, p_1) \right| \right) \cdot \left| f\left(\frac{w-q_1}{\sqrt{\hbar}}\right) \overline{f\left(\frac{u-q_1}{\sqrt{\hbar}}\right)} \right| \\ &\quad \cdot \chi_{|w-q_1| \leq \sqrt{\hbar}R} \chi_{|u-q_1| \leq \sqrt{\hbar}R} \|a_w a_y \psi_{N,t}\| \|a_u a_y \psi_{N,t}\| \\ &=: i_{32} + ii_{32} \end{aligned} \tag{3.36}$$

Again, by Lemma 2.5 and the bounds for number operator and localized number operator, we have for i_{32} that

$$\begin{aligned} i_{32} &\leq C\hbar^{\frac{3}{2}+(1-\alpha)s} \int dq_1 \iint dwdu \int_0^1 ds |su + (1-s)w - q_1| \cdot \left| f\left(\frac{w-q_1}{\sqrt{\hbar}}\right) \overline{f\left(\frac{u-q_1}{\sqrt{\hbar}}\right)} \right| \\ &\quad \cdot \chi_{|w-q_1| \leq \sqrt{\hbar}R} \chi_{|u-q_1| \leq \sqrt{\hbar}R} \int dy \|a_w a_y \psi_{N,t}\| \|a_u a_y \psi_{N,t}\| \\ &\leq C\hbar^{3+\frac{1}{2}+(1-\alpha)s} \int dq_1 \left(\iint d\tilde{w}d\tilde{u} \int_0^1 ds |s\tilde{u} + (1-s)\tilde{w}|^2 \cdot \left| f(\tilde{w}) \overline{f(\tilde{u})} \right|^2 \right)^{\frac{1}{2}} \\ &\quad \cdot \left(\iint dwdu \chi_{|w-q_1| \leq \sqrt{\hbar}R} \chi_{|u-q_1| \leq \sqrt{\hbar}R} \left(\int dy \|a_w a_y \psi_{N,t}\| \|a_u a_y \psi_{N,t}\| \right)^2 \right)^{\frac{1}{2}} \\ &\leq C\hbar^{3+\frac{1}{2}+(1-\alpha)s} \left(\iint d\tilde{w}d\tilde{u} \int_0^1 ds |s\tilde{u} + (1-s)\tilde{w}|^2 \cdot \left| f(\tilde{w}) \overline{f(\tilde{u})} \right|^2 \right)^{\frac{1}{2}} \\ &\quad \cdot \int dy \iint dq_1 dw \chi_{|w-q_1| \leq \sqrt{\hbar}R} \langle a_y \psi_{N,t}, a_w^* a_w a_y \psi_{N,t} \rangle \\ &\leq C\hbar^{(1-\alpha)s-1}, \end{aligned}$$

where we used Lemma 2.4 and the bounds for number operator. Similarly, for ii_{32} , we have

$$\begin{aligned}
 ii_{32} &\leq C\hbar^{\frac{3}{2}} \int dq_1 \iint dwdu \int dy \int_0^1 ds |su + (1-s)w - q_1| \int dp_1 |\chi_{(w-u)\in\Omega_{\hbar}} \nabla_{p_1} \Phi(q_1, p_1)| \\
 &\quad \cdot \left| f\left(\frac{w-q_1}{\sqrt{\hbar}}\right) \overline{f\left(\frac{u-q_1}{\sqrt{\hbar}}\right)} \right| \chi_{|w-q_1|\leq\sqrt{\hbar}R} \chi_{|u-q_1|\leq\sqrt{\hbar}R} \|a_w a_y \psi_{N,t}\| \|a_u a_y \psi_{N,t}\| \\
 &\leq C\hbar^{\frac{3}{2}} \int dq_1 \iint dwdu \int dy \int_0^1 ds |su + (1-s)w - q_1| \chi_{(w-u)\in\Omega_{\hbar}} \left| f\left(\frac{w-q_1}{\sqrt{\hbar}}\right) \overline{f\left(\frac{u-q_1}{\sqrt{\hbar}}\right)} \right| \\
 &\quad \cdot \chi_{|w-q_1|\leq\sqrt{\hbar}R} \chi_{|u-q_1|\leq\sqrt{\hbar}R} \|a_w a_y \psi_{N,t}\| \|a_u a_y \psi_{N,t}\| \\
 &\leq C\hbar^{\frac{3}{2}} \int dq_1 \left(\iint dwdu \int_0^1 ds |su + (1-s)w - q_1|^2 \left| f\left(\frac{w-q_1}{\sqrt{\hbar}}\right) f\left(\frac{u-q_1}{\sqrt{\hbar}}\right) \right|^2 \chi_{(w-u)\in\Omega_{\hbar}} \right)^{\frac{1}{2}} \\
 &\quad \cdot \left(\iint dwdu \chi_{|w-q_1|\leq\sqrt{\hbar}R} \chi_{|u-q_1|\leq\sqrt{\hbar}R} \int dy \|a_w a_y \psi_{N,t}\| \|a_u a_y \psi_{N,t}\| \right)^{\frac{1}{2}} \\
 &\leq C\hbar^{3+\frac{1}{2}} \left(\iint d\tilde{w}d\tilde{u} \int_0^1 ds |s\tilde{u} + (1-s)\tilde{w}|^2 \left| f(\tilde{w}) f(\tilde{u}) \right|^2 \chi_{|\tilde{w}-\tilde{u}|\leq\hbar^{\alpha+\frac{1}{2}}} \right)^{\frac{1}{2}} \\
 &\quad \cdot \int dq_1 \left(\iint dwdu \chi_{|w-q_1|\leq\sqrt{\hbar}R} \chi_{|u-q_1|\leq\sqrt{\hbar}R} \int dy \|a_w a_y \psi_{N,t}\| \|a_u a_y \psi_{N,t}\| \right)^{\frac{1}{2}}.
 \end{aligned}$$

By Lemma 2.4 and the bounds for number operator, we have

$$\leq C\hbar^{-1} \left(\iint d\tilde{w}d\tilde{u} \int_0^1 ds |s\tilde{u} + (1-s)\tilde{w}|^2 \left| f(\tilde{w}) f(\tilde{u}) \right|^2 \chi_{|\tilde{w}-\tilde{u}|\leq\hbar^{\alpha+\frac{1}{2}}} \right)^{\frac{1}{2}}.$$

Then, by using similar computation in (3.30) and the assumption that f is compactly supported, we may get

$$ii_{32} \leq C\hbar^{\alpha-\frac{1}{2}}.$$

Therefore, II_3 and I_3 together, we have the bound of order $\hbar^{\alpha-\frac{1}{2}}$ for $\alpha \in (\frac{1}{2}, 1)$. □

3.2.3 Proof of Proposition 2.6

Proof To calculate the bound in (2.21) for $\widehat{\mathcal{R}}_k$. It has automatically an $1/N$ as a factor, therefore, we expect it has better estimates than the other remainder terms. More precisely, we can split the integrals as before,

$$\begin{aligned}
 &\left| \frac{1}{2N} \int \dots \int (dqdp)^{\otimes k} (dwdu)^{\otimes k} \Phi(q_1, p_1, \dots, q_k, p_k) \sum_{j \neq i}^k \left[V(u_j - u_i) - V(w_j - w_i) \right] \right. \\
 &\quad \cdot \left. \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} \langle a_{w_k} \dots a_{w_1} \psi_{N,t}, a_{u_k} \dots a_{u_1} \psi_{N,t} \rangle \right| \\
 &= \left| \frac{1}{2N} \int \dots \int (dqdp)^{\otimes k} (dwdu)^{\otimes k} \Phi(q_1, p_1, \dots, q_k, p_k) \right. \\
 &\quad \left. \sum_{j \neq i}^k \left[V(u_j - u_i) - V(w_j - w_i) \right] \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} \right|
 \end{aligned}$$

$$\cdot \prod_{n=1}^k \left(\chi_{(w_n - u_n) \in \Omega_{\hbar}^c} + \chi_{(w_n - u_n) \in \Omega_{\hbar}} \right) \left| a_{w_k} \cdots a_{w_1} \psi_{N,t}, a_{u_k} \cdots a_{u_1} \psi_{N,t} \right|,$$

where Ω_{\hbar} is defined as in (2.10). Since $V \in W^{2,\infty}$ and recall $\hbar^3 = N^{-1}$, we have

$$\begin{aligned} &\leq C(k) \|V\|_{\infty} \hbar^{3-\frac{3}{2}k} \int \cdots \int (dqdwdu)^{\otimes k} \prod_{n=1}^k \left| f\left(\frac{w_n - q_n}{\sqrt{\hbar}}\right) f\left(\frac{u_n - q_n}{\sqrt{\hbar}}\right) \right| \\ &\quad \left\| a_{w_k} \cdots a_{w_1} \psi_{N,t} \right\| \left\| a_{u_k} \cdots a_{u_1} \psi_{N,t} \right\| \\ &\quad \cdot \left| \prod_{n=1}^k \left(\chi_{(w_n - u_n) \in \Omega_{\hbar}^c} + \chi_{(w_n - u_n) \in \Omega_{\hbar}} \right) \int \cdots \int (dp)^{\otimes k} e^{\frac{i}{\hbar} \sum_{m=1}^k p_m \cdot (w_m - u_m)} \Phi(q_1, \dots, p_k) \right| \\ &\leq C \hbar^{3-\frac{3}{2}k} \max_{0 \leq \ell \leq k} \int \cdots \int (dqdwdu)^{\otimes k} \prod_{n=1}^k \left| f\left(\frac{w_n - q_n}{\sqrt{\hbar}}\right) f\left(\frac{u_n - q_n}{\sqrt{\hbar}}\right) \right| \\ &\quad \left\| a_{w_k} \cdots a_{w_1} \psi_{N,t} \right\| \left\| a_{u_k} \cdots a_{u_1} \psi_{N,t} \right\| \cdot \left| \int \cdots \int (dp)^{\otimes k} \right. \\ &\quad \left. \left(\chi_{(w_1 - u_1) \in \Omega_{\hbar}} \cdots \chi_{(w_{\ell} - u_{\ell}) \in \Omega_{\hbar}} \chi_{(w_{\ell+1} - u_{\ell+1}) \in \Omega_{\hbar}^c} \cdots \chi_{(w_k - u_k) \in \Omega_{\hbar}^c} \right) \right. \\ &\quad \left. \nabla_{q_j} \Phi \cdot e^{\frac{i}{\hbar} \mathbf{p}_k \cdot (\mathbf{w}_k - \mathbf{u}_k)} \right| \\ &= C \hbar^{3-\frac{3}{2}k} \max_{0 \leq \ell \leq k} \int \cdots \int (dqdwdu)^{\otimes k} \left| \int \cdots \int (dp)^{\otimes \ell} \chi_{(w_1 - u_1) \in \Omega_{\hbar}} \cdots \chi_{(w_{\ell} - u_{\ell}) \in \Omega_{\hbar}} e^{\frac{i}{\hbar} \sum_{m=1}^{\ell} p_m \cdot (w_m - u_m)} \right. \\ &\quad \cdot \left. \int \cdots \int (dp)^{\otimes (k-\ell)} \chi_{(w_{\ell+1} - u_{\ell+1}) \in \Omega_{\hbar}^c} \right. \\ &\quad \left. \cdots \chi_{(w_k - u_k) \in \Omega_{\hbar}^c} e^{\frac{i}{\hbar} \sum_{m=k-\ell}^{\ell} p_m \cdot (w_m - u_m)} \nabla_{q_j} \Phi(q_1, p_1, \dots, q_k, p_k) \right| \\ &\quad \cdot \prod_{n=1}^k \left| f\left(\frac{w_n - q_n}{\sqrt{\hbar}}\right) f\left(\frac{u_n - q_n}{\sqrt{\hbar}}\right) \right| \left\| a_{w_k} \cdots a_{w_1} \psi_{N,t} \right\| \left\| a_{u_k} \cdots a_{u_1} \psi_{N,t} \right\|, \end{aligned}$$

where we apply similar argument in (3.28) in the last inequality. Note here that the constant C above is dependent on k . Applying Lemma 2.5 we have

$$\begin{aligned} &\leq C \max_{0 \leq \ell \leq k} \hbar^{3-\frac{3}{2}k+(1-\alpha)(k-\ell)s} \int \cdots \int (dqdwdu)^{\otimes k} \left(\chi_{(w_1 - u_1) \in \Omega_{\hbar}} \cdots \chi_{(w_{\ell} - u_{\ell}) \in \Omega_{\hbar}} \right) \\ &\quad \cdot \prod_{n=1}^k \left| f\left(\frac{w_n - q_n}{\sqrt{\hbar}}\right) f\left(\frac{u_n - q_n}{\sqrt{\hbar}}\right) \right| \left\| a_{w_k} \cdots a_{w_1} \psi_{N,t} \right\| \left\| a_{u_k} \cdots a_{u_1} \psi_{N,t} \right\| \\ &= C \max_{0 \leq \ell \leq k} \hbar^{3-\frac{3}{2}k+(1-\alpha)(k-\ell)s} \int \cdots \int (dqdwdu)^{\otimes k} \left(\chi_{(w_1 - u_1) \in \Omega_{\hbar}} \cdots \chi_{(w_{\ell} - u_{\ell}) \in \Omega_{\hbar}} \right) \\ &\quad \cdot \prod_{n=1}^k \left| f\left(\frac{w_n - q_n}{\sqrt{\hbar}}\right) f\left(\frac{u_n - q_n}{\sqrt{\hbar}}\right) \right| \chi_{|w_n - q_n| \leq \sqrt{\hbar}R} \chi_{|u_n - q_n| \leq \sqrt{\hbar}R} \\ &\quad \left\| a_{w_k} \cdots a_{w_1} \psi_{N,t} \right\| \left\| a_{u_k} \cdots a_{u_1} \psi_{N,t} \right\| \\ &\leq C \max_{0 \leq \ell \leq k} \hbar^{3-\frac{3}{2}k+(1-\alpha)(k-\ell)s} \int \cdots \int (dq)^{\otimes k} \left[\int \cdots \int (dwdu)^{\otimes k} \left(\chi_{(w_1 - u_1) \in \Omega_{\hbar}} \cdots \chi_{(w_{\ell} - u_{\ell}) \in \Omega_{\hbar}} \right) \right. \\ &\quad \left. \cdot \prod_{n=1}^k \left| f\left(\frac{w_n - q_n}{\sqrt{\hbar}}\right) f\left(\frac{u_n - q_n}{\sqrt{\hbar}}\right) \right|^2 \right]^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
 & \left[\int \cdots \int (dw)^{\otimes k} \prod_{n=1}^k \chi_{|w_n - q_n| \leq \sqrt{\hbar}R} \|a_{w_k} \cdots a_{w_1} \psi_{N,t}\|^2 \right] \\
 = & C \max_{0 \leq \ell \leq k} \hbar^{3+(1-\alpha)(k-\ell)s} \int \cdots \int (dq)^{\otimes k} \\
 & \left[\int \cdots \int (d\tilde{w}d\tilde{u})^{\otimes k} \left(\chi_{|\tilde{w}_1 - \tilde{u}_1| \leq \hbar^{\alpha+\frac{1}{2}}} \cdots \chi_{|\tilde{w}_\ell - \tilde{u}_\ell| \leq \hbar^{\alpha+\frac{1}{2}}} \right) \right. \\
 & \left. \cdot \prod_{n=1}^k |f(\tilde{w}_n) f(\tilde{u}_n)|^2 \right]^{\frac{1}{2}} \\
 & \left[\int \cdots \int (dw)^{\otimes k} \prod_{n=1}^k \chi_{|w_n - q_n| \leq \sqrt{\hbar}R} \|a_{w_k} \cdots a_{w_1} \psi_{N,t}\|^2 \right] \\
 \leq & C \max_{0 \leq \ell \leq k} \hbar^{3-\frac{3}{2}k+(1-\alpha)(k-\ell)s+(\alpha+\frac{1}{2})\ell},
 \end{aligned}$$

where, as in the proof of Proposition 2.4, we applied Lemma 2.4 and (3.30). Therefore, we obtain the desired result by choosing $s = \left\lceil \frac{1+2\alpha}{2(1-\alpha)} \right\rceil$.

Next, we switch to estimate (2.22) for $\tilde{\mathcal{R}}_k$. Repeated the steps in the proof of Proposition 2.5, we have

$$\begin{aligned}
 & \left| \int \cdots \int (dqdp)^{\otimes k} \nabla_{\mathbf{p}_k} \Phi(q_1, p_1, \dots, q_k, p_k) \cdot \tilde{\mathcal{R}}_k \right| \\
 = & \left| \sum_{j=1}^k \int \cdots \int (dqdp)^{\otimes k} (dwdu)^{\otimes k} \nabla_{p_j} \Phi(q_1, p_1, \dots, q_k, p_k) \cdot \iint dydv \iint dq_{k+1} dp_{k+1} \right. \\
 & \cdot \int_0^1 ds [\nabla V(su_j + (1-s)w_j - y) - \nabla V(q_j - y) + \nabla V(q_j - y) - \nabla V(q_j - q_{k+1})] \\
 & \left. \cdot \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} f_{q_{k+1}, p_{k+1}}^{\hbar}(y) \overline{f_{q_{k+1}, p_{k+1}}^{\hbar}(v)} \langle a_{w_k} \cdots a_{w_1} a_y \psi_{N,t}, a_{u_k} \cdots a_{u_1} a_v \psi_{N,t} \rangle \right|.
 \end{aligned}$$

Applying the \hbar -weighted Dirac-delta function as in (2.7), we have

$$\begin{aligned}
 = & (2\pi)^3 \hbar^{3-\frac{3}{2}} \left| \sum_{j=1}^k \int \cdots \int (dqdp)^{\otimes k} (dwdu)^{\otimes k} \nabla_{p_j} \Phi(q_1, p_1, \dots, q_k, p_k) \cdot \iint dydq_{k+1} \right. \\
 & \cdot \int_0^1 ds [\nabla V(su_j + (1-s)w_j - y) - \nabla V(q_j - y) + \nabla V(q_j - y) - \nabla V(q_j - q_{k+1})] \\
 & \left. \cdot \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} \left| f\left(\frac{y - q_{k+1}}{\sqrt{\hbar}}\right) \right|^2 \langle a_{w_k} \cdots a_{w_1} a_y \psi_{N,t}, a_{u_k} \cdots a_{u_1} a_y \psi_{N,t} \rangle \right| \\
 \leq & (2\pi)^3 \hbar^{3-\frac{3}{2}k} \sum_{j=1}^k \int \cdots \int (dqdwdu)^{\otimes k} \prod_{n=1}^k \left| \int \cdots \int (dp)^{\otimes k} \nabla_{p_j} \Phi(q_1, p_1, \dots, q_k, p_k) e^{\frac{i}{\hbar} p_n \cdot (w_n - u_n)} \right| \\
 & \iint dyd\tilde{q}_{k+1} \\
 & \cdot \left(\int_0^1 ds |\nabla V(su_j + (1-s)w_j - y) - \nabla V(q_j - y)| \right. \\
 & \left. + |\nabla V(q_j - y) - \nabla V(q_j - y + \sqrt{\hbar}\tilde{q}_{k+1})| \right)
 \end{aligned}$$

$$\left| f\left(\frac{w_n - q_n}{\sqrt{\hbar}}\right) f\left(\frac{u_n - q_n}{\sqrt{\hbar}}\right) \right| |f(\tilde{q}_{k+1})|^2 |\langle a_{w_k} \cdots a_{w_1} a_y \psi_{N,t}, a_{u_k} \cdots a_{u_1} a_y \psi_{N,t} \rangle|.$$

Using the fact that ∇V is Lipschitz continuous and that f is compactly supported, we have

$$\begin{aligned} &\leq (2\pi)^3 \hbar^{3-\frac{3}{2}k} \sum_{j=1}^k \int \cdots \int (dq dw du)^{\otimes k} \prod_{n=1}^k \left| \int \cdots \int (dp)^{\otimes k} \nabla_{p_j} \Phi(q_1, p_1, \dots, q_k, p_k) e^{\frac{i}{\hbar} p_n \cdot (w_n - u_n)} \right| \\ &\quad \iint dy d\tilde{q}_{k+1} \cdot \left(\int_0^1 ds |su_j + (1-s)w_j - q_j| + |\sqrt{\hbar}\tilde{q}_{k+1}| \right) \\ &\quad \left| f\left(\frac{w_n - q_n}{\sqrt{\hbar}}\right) f\left(\frac{u_n - q_n}{\sqrt{\hbar}}\right) \right| |f(\tilde{q}_{k+1})|^2 \\ &\quad \cdot \chi_{|w_n - q_n| \leq \sqrt{\hbar}R} \chi_{|u_n - q_n| \leq \sqrt{\hbar}R} \|a_{w_k} \cdots a_{w_1} a_y \psi_{N,t}\| \|a_{u_k} \cdots a_{u_1} a_y \psi_{N,t}\| \\ &=: I_4 + II_4 \end{aligned}$$

Focusing on I_4 , we split the integral as follows

$$\begin{aligned} I_4 &= (2\pi)^3 \hbar^{3-\frac{3}{2}k} \sum_{j=1}^k \int \cdots \int (dq dw du)^{\otimes k} \left| \prod_{n=1}^k \left(\chi_{(w_n - u_n) \in \Omega_{\hbar}^c} + \chi_{(w_n - u_n) \in \Omega_{\hbar}} \right) \right. \\ &\quad \left. \int \cdots \int (dp)^{\otimes k} \nabla_{p_j} \Phi(q_1, p_1, \dots, q_k, p_k) \right. \\ &\quad \left. \cdot e^{\frac{i}{\hbar} \sum_{m=1}^k p_m \cdot (w_m - u_m)} \right| \\ &\quad \iint dy d\tilde{q}_{k+1} \int_0^1 ds |su_j + (1-s)w_j - q_j| \left| f\left(\frac{w_n - q_n}{\sqrt{\hbar}}\right) f\left(\frac{u_n - q_n}{\sqrt{\hbar}}\right) \right| |f(\tilde{q}_{k+1})|^2 \\ &\quad \cdot \chi_{|w_n - q_n| \leq \sqrt{\hbar}R} \chi_{|u_n - q_n| \leq \sqrt{\hbar}R} \|a_{w_k} \cdots a_{w_1} a_y \psi_{N,t}\| \|a_{u_k} \cdots a_{u_1} a_y \psi_{N,t}\|. \end{aligned}$$

where Ω_{\hbar} is defined as in (2.10). We do similar computations for II_4 ,

$$\begin{aligned} II_4 &= (2\pi)^3 \hbar^{3-\frac{3}{2}k} \sum_{j=1}^k \int \cdots \int (dq dw du)^{\otimes k} \left| \prod_{n=1}^k \int \cdots \int (dp)^{\otimes k} \left(\chi_{(w_n - u_n) \in \Omega_{\hbar}^c} + \chi_{(w_n - u_n) \in \Omega_{\hbar}} \right) \right. \\ &\quad \left. \nabla_{p_j} \Phi(q_1, p_1, \dots, q_k, p_k) \right. \\ &\quad \left. \cdot e^{\frac{i}{\hbar} p_n \cdot (w_n - u_n)} \right| \iint dy d\tilde{q}_{k+1} |\sqrt{\hbar}\tilde{q}_{k+1}| \left| f\left(\frac{w_n - q_n}{\sqrt{\hbar}}\right) f\left(\frac{u_n - q_n}{\sqrt{\hbar}}\right) \right| |f(\tilde{q}_{k+1})|^2 \\ &\quad \cdot \chi_{|w_n - q_n| \leq \sqrt{\hbar}R} \chi_{|u_n - q_n| \leq \sqrt{\hbar}R} \|a_{w_k} \cdots a_{w_1} a_y \psi_{N,t}\| \|a_{u_k} \cdots a_{u_1} a_y \psi_{N,t}\|. \end{aligned}$$

Repeating the proof of Proposition 2.5, namely in (3.36) and (3.32), as well as the proof for estimate (2.21), we eventually obtain

$$I_4 + II_4 \leq C \max_{0 \leq \ell \leq k} \hbar^{\frac{1}{2} - \frac{3}{2}k + (1-\alpha)(k-\ell)s + (\alpha + \frac{1}{2})\ell},$$

where the constant C depends on k . As before, we choose $s = \left\lceil \frac{1+2\alpha}{2(1-\alpha)} \right\rceil$ and choose $\alpha \in (\frac{1}{2}, 1)$ such that $(\alpha - 1)k \leq -\delta$, and we obtain the desired estimates. \square

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