Three Essays in the Theory of Auctions

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Chapter 1

General Introduction

Auctions have been used since ancient times to quickly and efficiently trade goods and commodities. Roman soldiers often sold their loot at an auction *sub hasta* (under the spear). They drove a spear into the ground and auctioned the booty to the highest bidders. Moreover, auctions were used in raising urgently needed money. "Caligula, for example, auctioned off the furniture and ornaments belonging to his family to help him meet his debts and recoup his losses" (Cassady [1967]). Another Roman emperor, Marcus Aurelius, is said to have auctioned off "royal heirlooms and furniture" to cover a state deficit and raise money for the northern wars (Cassady [1967]). Perhaps the most unusual and most preposterous auction took place after Pertinax was killed and there was no successor who was overall accepted. Flavius Sulpicianus tried to get the praetorian guard to proclaim him emperor, but he was met with little enthusiasm. Didius Julianus began to make promises to the soldiers. "Soon the scene became that of an auction, with Flavius Sulpicianus and Didius Julianus outbidding each other in the size of their donatives to the troops. The Roman empire was for sale to the highest bidder. When Flavius Sulpicianus reached the figure of 20,000 sesterces per soldier, Didius Julianus upped the bid by a whopping 5,000 sesterces [...]" (Online Encyclopedia of Roman Emperors [2002]). Julianus was proclaimed emperor and confirmed by the senate. It should be noted that the people of Rome did not agree with this decision and called for a different emperor. After only 66 days the senate announced a death sentence on Julianus and made Septimus Severus the new emperor.

More current examples are the European auctions of third generation licenses for mobile telecommunication according to the UMTS standard in the year 2000. The auctions were spectacular since licenses for an important new industry[1] were for sale and enormous amounts of money were paid in some countries (e.g. in Germany 50.51 Billion Euros total).[2]

In 1880 Léon Walras described an economic model, where prices are determined endogenously by what became known as the "Walrasian auctioneer".[3] Walras "described the entire price mechanism as an auctioneer, attaching a body to Adam Smith's 'invisible hand'" (The Economist [1999]).

The rigorous analysis of auctions was founded by Vickrey [1961] and became known as auction theory. Here game-theoretic models are used to describe and predict the rational behavior of bidders in auctions. This knowledge allows the seller to anticipate the outcome and therefore optimize his own behavior. Moreover, the designer of the auction is able to choose an auction format that fosters the desired results and eliminates possible loopholes that would jeopardize the successful course of the auction.

This dissertation analyzes three different auction settings. Chapter 2 and 3 introduce discounting into auction theory. In Chapter 2, I focus on sequential auctions where bidders face discounting between the auctions. A bidder's discount factor is private information and thus known only to himself. Chapter 3 considers discounting during a single auction. In Chapter 4, a procurement auction is analyzed, where the buyer has to decide between multiple suppliers willing to sell two different goods.

Chapter 2 originates from collaboration with Thomas Kittsteiner and Professor Eyal Winter and Chapter 3 from collaboration with Thomas Kittsteiner.

[1] In 2000 it was generally accepted that the mobile telephone market would become a major industry and that a UMTS license was the only way to be part of that market.

[2] Numerous papers analyze the UMTS auctions and their outcomes, like Ewerhart and Moldovanu [2001], Jehiel and Moldovanu [2002], and Klemperer [2001].

[3] Note, that the word "auctioneer" is absent in all Walras' writings.

1.1 Private Discounting in Sequential Auctions

In this chapter a model where bidders face a sequence of second-price auctions is presented. In every auction a single object is sold. Auction theory suggests that, as all goods are the same and bidders have unit demand, the expected selling price will be the same in all auctions. The intuition is that bidders can arbitrage away all differences in prices. However, numerous empirical studies show that the same good gets sold at a lower price in later auctions. A possible explanation is that buyers discount the value of the goods in later auctions.

In the presented model, the value of the object in the first auction is identical for all bidders and is commonly known. In each subsequent auction the value for each bidder declines by a discount factor, which is different for every bidder and only privately known. Such a model may give rise to inefficiencies, and symmetric bidding functions leading to an efficient allocation may not exist at all. This turns out to be a result of the countervailing forces that result from impatience. On the one hand, efficiency dictates that those who are highly impatient should be served early. On the other hand, bidders who are extremely impatient should not get the object at all if they fail to get it in the first auction and if there are more bidders than objects. For these bidders the object becomes obsolete after one period of delay. Hence, I focus on the case where all buyers can be served and an efficient allocation is possible. If buyers were completely patient, the expected selling price would be zero in every auction as all buyers could be served. Due to buyers' impatience, the expected selling price in all but the last auction is greater than zero. It is not surprising that impatience causes expected nominal prices to decline between auctions. But a price decline persists even after correcting for the discount factor. Hence, bidders are able to shift some of their loss in valuation to the seller. This reduces the seller's gain, i.e. the price, by more than just the discount factor. Consequently, even if buyers are fairly patient, the price decline may be substantial.

For different impatience settings it is shown that prices are expected to be lower for every auction in the more patient environment. Therefore, the competition effect that pushes prices up is stronger than the value loss effect that drags them down. Consequently, the seller gains from impatience as it increases the competition between buyers. Finally, I show that the expected price dispersion between auctions increases with the degree of impatience.

1.2 Auctions with Impatient Buyers

This chapter analyzes the effects of discounting within a single dynamic auction. Such an auction may last for several days or weeks and the bidders' valuations might change during that period. End-of-Season sales are a prominent example (as they can be understood as a Dutch auction). A buyer purchasing a warm coat at an End-of-Winter sale might be happy to wear it while the sale lasts and it is still cold. Therefore, the good has a higher value early in the auction. Another curiosity of an End-of-Season sale is that people usually wait for the shop doors to open on the first day. This indicates that there is an excess demand at the newly reduced prices, suggesting that such low prices are not optimal from the shop's point of view.

In the model I present both the price and the buyers' valuations decline with time in a linear way (but with possibly different slopes). I show that a low starting price in the auction (such that there is an excess demand at the first instant) can indeed be revenue maximizing. Moreover, it is optimal to either let the auction run very fast (as the decline in the buyers' valuations becomes irrelevant) or very slowly. In a slow auction buyers are discouraged to wait, inducing them to buy at higher prices. Focussing on the welfare generated in the auction, I find that setting a positive reservation price can increase welfare.

Field experiments suggest that, in contrast to the Revenue-Equivalence-Theorem, Dutch auctions generate more revenue than Japanese (English) auctions. A possible explanation is the existence of buyers' declining valuations. In a Dutch auction, a buyer can get the good early by paying a premium. On the contrary, in a Japanese auction the highest buyer has to wait for all other bidders to drop out of the auction. I find that this argument is only partially correct by analyzing a Japanese auction in the described model with declining valuations. In fact, an example is given where the Japanese auction generates more expected revenue than a Dutch auction. A problem with the Dutch format is that buyers with a low initial valuation might be a priori excluded from the auction, since the price always remains above their valuation. This serves as an implicit reservation price, which might be higher than optimal. If the set of participating buyers was the same in both auction formats, the Japanese auction would no longer generate more expected revenue.

1.3 A Simple Procurement Decision for Heterogeneous Goods

Auctions have become a fast, efficient, and popular way for procurement transactions. They boost competition by directly confronting suppliers with their rivals' offers which results in lower procurement prices. But standard auctions face the drawback that the goods have to be specified exactly to ensure that bids are comparable and the desired product is procured. In contrast, a consumer usually shops around with a vague description of the object in mind. He compares offers for different objects fitting his vague description, striking the best deal for him. I combine both approaches and provide an auction design which also selects which good to procure.

The model is formulated as a procurement auction, i.e. a single buyer holds the auction to buy one of two possible goods. The buyer has different valuations for the goods since they are not identical. Each good can be purchased from a different group of sellers. I focus on an auction where the bids for the different goods are made comparable by adding a constant to bids on the less favorable good.[4] The procurement decision is efficient if the constant equals the difference in the buyer's valuations for the two goods. As all suppliers directly compete against each other in this auction, it is superior to the following frequently observed format: The buyer holds two separate auctions – one for each good. The outcome of each auction is only regarded as an offer and the more favorable auction outcome is selected later on.

If the auction where bids are corrected by the additive constant is efficient, it is usually not maximizing the buyer's expected utility. Unfortunately, the optimal auction is usually too complicated to be applicable in real life. Hence, I introduce the concept of *constraint optimal auctions*, where the additive constant for bids on the less favorable good is determined optimally. The optimal constant lies between the difference in valuations for the two goods and the systematic cost difference between the two supplier groups.[5] Consider the following example: The buyer is indifferent between the two goods if good two is 10 Euros less expensive. Moreover, good one is usually about 10 Euros more expensive to produce. Then the optimal additive constant is equal to 10 Euros. The additive constant usually is affected by the number of suppliers. If the number of possible suppliers increases, the additive constant tends to approach its efficient level.

[4] Note that in a procurement context the lowest bid wins.

[5] If no systematic cost difference exists, the constant would be smaller than the efficient choice.

Chapter 2

Private Discounting in Sequential Auctions

2.1 Introduction

Auctions held in sequence are frequently used when numerous items are for sale. They are regularly applied for commodities like timber and fish but also by central banks. The ECB holds liquidity-auctions at regular intervals. In many sequential auctions only a single item/lot is put on the shelf in every period. After the auction is completed an identical or similar item comes up for sale in another auction. Indeed, much of the trade through auctions that takes place over the internet has this property. There is an extensive empirical literature on sequential auctions starting with Ashenfelter [1989]. He analyzed sequential auctions for identical bottles of wine. Surprisingly, he observed that in later auctions the same wine was sold for a lower price. Ashenfelter and Genesove [1992] found declining prices in real-estate auctions. A similar effect has been observed in auctions for art (Beggs and Graddy [1997]), flowers (Van den Berg et al. [2001]), and wool (Jones et al. [1996]). These observations led to the notion of "declining price anomaly" or "afternoon effect", indicating that later auctions for the same goods result in lower prices. The effect is considered an "anomaly" since the expected selling price should be the same in all periods. This was already shown by Milgrom and Weber [2000][6] and Weber [1983] in some of the earliest works on sequential auctions. Intuitively, if the prices were different, bidders could exploit the arbitrage opportunity between auctions, thereby eliminating any differences in prices. However, this argument does not apply if bidders suffer a delay cost or their valuations for the object decline across different auctions. Such an environment introduces interesting strategic considerations that we intend to explore in this chapter.

In order to address the issue of impatience in sequential auctions we follow two main modeling strategies. The first is to assume that bidders share a common discount factor and attribute the private information to the non-discounted value of the object sold. We analyzed this framework in Kittsteiner et al. [2002]. The alternative and somewhat dual approach, which we adopt here, is to attribute all the incomplete information to the impatience. Thus, we assume that the non-discounted value of the object is identical for all agents. However, in many real life environments incomplete information exists on both the discount factor and the non-discounted value of the object. Which of our two models fits better depends very much on the underlying real life environment that is studied. For some environments our present model seems particularly adequate: Business-to-business (B2B) auctions for raw material and equipment often involve objects of common value on which bidders are well informed. The competition between bidders is basically driven by the question how urgent they need the good. Many of the bids by firms for these B2B transactions are linked to specific orders by customers. These orders and their urgency are private information. Another scenario for which our present model seems to be a good approximation is one in which there is a fixed pool of objects that are auctioned sequentially. But the objects are not identical and may vary in quality. Winning an auction means winning the right to choose from the pool of remaining objects. Such auctions are often called right-to-choose auctions. If the bidders have different preferences for quality (which are private information), then the bidders' valuation for the different goods depreciates differently. Bidders for which quality (and

[6] Note that this article was written in 1982 but not published until 2000.

thus winning early) is important face a high cost of waiting. We represent these bidders as having a low discount factor. Auctions of condominiums (including the one described by Ashenfelter and Genesove [1992]), plots, and time sharing vacation apartments often have these features.

In our present model bidders face a sequence of periods in which one object is sold at every period by means of a second-price auction. The value of the object at the first period is identical for all bidders and is commonly known. At each subsequent period the value for bidder i declines by the factor δ_i, which is bidder's i private information. We first point out that such a model may give rise to inefficiencies. Interestingly, for certain specifications of parameters, symmetric bidding functions (not necessarily equilibria) leading to an efficient allocation may not even exist. This turns out to be a result of the countervailing forces that result from impatience. On the one hand, efficiency dictates that those who are highly impatient should be served early. On the other hand, bidders who are extremely impatient should not get the object at all if they fail to get it in the first period and if there are more bidders than objects. For these bidders the object becomes obsolete after one period of delay. We confine our attention to the domain for which the resulting allocation is efficient regardless of the realization of bidders' types. If buyers were completely patient, the expected selling price would be zero in every period as all buyers could be served. Due to buyers' impatience, the expected selling price in all but the last period is greater than zero.

Our main result shows that impatience causes expected nominal prices to decline between periods. A price decline persists even after correcting for the discount factor. Hence, bidders are able to shift some of their loss in valuation to the seller. This reduces the seller's gain, i.e. the price, by more than just the discount factor. We show for a more patient environment (in terms of first order stochastic dominance) that prices are expected to be lower than those prevailing in a less patient one (for every period). This implies that the competition effect that pushes prices up is stronger than the value loss effect that drags them down. This result also has the implication that the seller gains from impatience. It suggests that as much as the seller can control the buyers' impatience he will try to manipulate it, e.g. by appropriately spacing different periods of the auction. This is due to an increase in competition between buyers. Finally, we show that the expected price dispersion between periods increases with the degree of impatience in the market. The price effect may be substantial, even if buyers are very patient.[7]

The existing literature offers several other approaches to sequential auctions. Kittsteiner et al. [2002] consider the case where bidders demand a single unit and all bidders have a common discount factor but different valuations. They find a price decline between periods even after correcting for the discounting. This is shown to be true for different auction formats and information policies of the seller. Pezanis-Christou [1997] points out a similar effect for a less general model. Other papers often use models with two subsequent auctions. So does Jeitschko [1999] for a model with supply uncertainty. McAfee and Vincent [1993] analyze a model with decreasing absolute risk aversion, whereas von der Fehr [1994] uses participation costs. Black and De Meza [1992] use a framework in which bidders may want to buy more than one unit and any winning bidder has the option of buying more units for the winning price. Gale and Hausch [1994] analyze two subsequent auctions where the goods sold are not identical. If the order of sale is determined by the seller, they find that bidders might submit a very low bid for the first object if the less valuable object is sold first ("bottom-fishing"). They also consider right-to-choose auctions briefly for the case of two goods and two bidders. Bernhardt and Scoones [1994] show that prices decline if bidders' valuations are independently drawn at the beginning of each auction. In a similar setting, Engelbrecht-Wiggans [1994] relaxes the assumption of independence among the draws on each stage. Menezes [1993] uses a model with delay costs where a bidder may decide to drop out of the auction. He shows in an example that expected prices are decreasing. If bidders demand more than one unit and economies of scale

[7] Consider the following example: There are three bidders and goods. A bidder's discount factor is $\frac{1}{2}$ with probability p and 1 with probability $1-p$. Even for small p, e.g. $p = 1\%$, the ex ante expected price for the first period is about seven times as large as the expected price for the second period.

exist, Menezes and Monteiro [1999] and Jeitschko and Wolfstetter [2001] show that expected prices are decreasing.

The model is described in Section 2.2 and the inefficiency problem is addressed in Section 2.3. In Section 2.4 we characterize the symmetric bidding equilibrium of the game. We study the price dynamics in Section 2.5. Section 2.6 compares the price dynamics for different impatience environments. We conclude with a discussion in Section 2.7. All proofs are relegated to Appendix A.

2.2 The Model

There are $n \geq 2$ buyers $i = 1, \dots, n$ and one seller who offers $k \leq n$ indivisible and a priori homogeneous goods for sale.[8] The seller has no value for the items and uses a multi-period auction. In each period a second-price auction is conducted with one good for sale. The selling price is announced after each period. The buyers have the same common valuation for getting an object in the first period. This valuation is normalized to 1. Each buyer has a private discount factor (type) $\delta \in D$, where $D \subset [0, 1]$ is an interval. The discount factors are assumed to be stochastically independent and drawn according to a common distribution F with a continuous and strictly positive density f on D. In period l, a buyer with type δ has the valuation δ^{l-1}, where $1 \leq l \leq k$. Each bidder has unit demand. Consequently, the number of interested bidders in period l is $n-l+1$. To simplify notation we write $\delta_{(i)}$ for the ith highest type among all realized discount factors. We restrict our attention to symmetric Bayes-Nash-equilibria in pure strategies that lead to efficient allocations.

2.3 Efficiency

In an efficient allocation the sum of the seller's and bidders' total welfare is maximized (ex post). As prices paid are just transfers between buyers and

[8]Note that the case $k > n$ is also served in the analysis since it is equivalent to the case where there are as many goods as buyers, i.e. $k = n$.

seller, only the buyers' valuations are relevant. Thus, efficient allocations depend only on the bidders' discount factors. Lemma 1 describes properties of efficient allocations depending on the underlying support D of the distribution.

Lemma 1 *(Efficiency)*
An efficient allocation is given as follows: The buyers with the highest discount factors $\delta_{(i)}$, for $i = 1, \dots, k-1$, win all but the first auction. As in the first auction all bidders' valuation is 1, any buyer with a low discount factor $\delta_{(i)}$, for $i = k, \dots, n$, can win. The actual order in which the buyers with highest discount factors $\delta_{(i)}$, for $i = 1, \dots, k-1$, win the latter auctions depends on the support D as follows:

1. *If the support D is a subset of $\left[0, \frac{1}{2}\right]$, then the buyer with discount factor $\delta_{(i)}$ wins the $(i+1)$th auction, for $i = 1, \dots, k-1$. The winners are* ordered by patience*. See also Table 1.*

2. *If the support D is a subset of $\left[\frac{k-2}{k-1}, 1\right]$, then the buyer with discount factor $\delta_{(i)}$ wins the $(k-i+1)$th auction for $i = 1, \dots, k-1$. The winners are* ordered by impatience*. See also Table 2.*

3. *For any other support D the winners of the auctions $i = 2, \dots, k$ are not ordered by patience or impatience.*

Auction	**1**	**2**	$\mathbf{i+1}$	**k**
Winner's Type	$\in \left\{\delta_{(k)} \dots \delta_{(n)}\right\}$	$\delta_{(1)}$	$\delta_{(i)}$	$\delta_{(k-1)}$

Table 1: Ordered by Patience

Auction	**1**	**2**	$\mathbf{i+1}$	**k**
Winner's Type	$\in \left\{\delta_{(k)} \dots \delta_{(n)}\right\}$	$\delta_{(k-1)}$	$\delta_{(k-i)}$	$\delta_{(1)}$

Table 2: Ordered by Impatience

Intuitively, if discount factors are very low, welfare losses from waiting are large. It is efficient to allocate the good to the bidder who values it most and realizes as much welfare as possible early on. In contrast, if discount factors are large, it is efficient to award the object to the most impatient bidder because it is more costly for such a bidder to wait.

Based upon Lemma 1 it is possible to isolate the cases for which efficiency can be obtained.

Corollary 1 *In an efficient allocation: If $k = 1$ any bidder can win the only auction. If $k = 2$ the bidder with the highest discount factor wins the last auction and any other bidder wins the first auction.*

For a support $D \subset \left[0, \frac{1}{2}\right]$ (winners are ordered by patience) inter-period devaluation is very strong: Each bidder's valuation is at least cut in half in each period. Such strong devaluation is rarely observed in real-life situations and we therefore assume from now on $D \subset \left[\frac{1}{2}, 1\right]$.

Theorem 1 *Consider any support $D \subset \left[\frac{1}{2}, 1\right]$ and a setting with more than two auctions, i.e. $k > 2$. If $n > k$ no bidding functions*[9] *exist that guarantee an efficient allocation regardless of the actual realization of types.*

Remarkably, on this domain efficiency is only ensured if all bidders can be served. Note that the necessary condition given in Theorem 1 (as well as the one in Lemma 2 below) is a condition for the existence of an efficient bidding function. It does not imply the existence of an efficient equilibrium. The intuition is straight forward: For an efficient allocation, the good in the first period can be allocated to any of the $n - k + 1$ lowest type bidders. But an efficient allocation requires that the $(k - 1)$th highest bidder wins the second period. If more than $k - 1$ bidders are still present, this bidder is neither the highest nor the lowest bidder still present. It is impossible to construct a symmetric bidding strategy that always guarantees that the $(k - 1)$th highest type bidder of the more than $(k - 1)$ still present bidders submits the highest bid. Therefore, there exists no efficient bidding strategy. As a consequence of this theorem, we from

[9] i.e. strategy profile

now on restrict attention to the case where the number of periods/goods is equal to the number of bidders: $n = k$.[10]

Lemma 2 *Consider a situation with* $n \geq 3$ *and for some* $l \in \{2, \ldots, n-1\}$ *the interval* $\left[\frac{l-1}{l}, \left(\frac{l-1}{n-1}\right)^{\frac{1}{n-l}}\right]$ *is a subset of the support* D*. Then there is no symmetric bidding function leading to an efficient allocation regardless of the actual realization of types.*

Note that $\left[\frac{l-1}{l}, \left(\frac{l-1}{n-1}\right)^{\frac{1}{n-l}}\right] \subset \left[\frac{1}{2}, \sqrt{\frac{n-3}{n-1}}\right]$ for any $l \in \{2, \ldots, n-1\}$. Consequently, it is problematic to achieve an efficient allocation on subsets D of the interval $\left[\frac{1}{2}, \sqrt{\frac{n-3}{n-1}}\right]$. Therefore, we focus on the remaining interval $\left[\sqrt{\frac{n-3}{n-1}}, 1\right]$. It seems to be the most realistic for the applications we have in mind as inter-period devaluation may be fairly small. To simplify our analysis, we from now on restrict our attention further to the slightly smaller interval $\left[\frac{n-2}{n-1}, 1\right] \subset \left[\sqrt{\frac{n-3}{n-1}}, 1\right]$.

Example 1 *For the case of five auctions and bidders* ($n = k = 5$)*, efficiency for bidders' types higher than* $\frac{1}{2}$ *is possible on the intervals* $\left[0.63, \frac{2}{3}\right]$ *and* $[0.71, 1]$.[11] *We focus on* $[0.75; 1] \subset [0.71, 1]$.

Recall that for any support $D \subset \left[\frac{n-2}{n-1}, 1\right]$, winners are ordered by impatience, i.e. every winner is the most impatient bidder still in the auction. Consequently, the lth auction will be won by the bidder with type $\delta_{(n+1-l)}$. Table 3 gives an overview of which type wins which auction in an efficient outcome. Moreover, it states which type sets the price in that period and how many bidders are still present.

[10] It also covers the case $n < k$.

[11] The interval $\left[\frac{l-1}{l}, \left(\frac{l-1}{n-1}\right)^{\frac{1}{n-l}}\right]$ is for $l = 2 : [\frac{1}{2}, 0, 63]$ and for $l = 3 : [\frac{2}{3}, 0.71]$.

Auction	**1**	**l**	**n**
Winning Type	$\delta_{(n)}$	$\delta_{(n+1-l)}$	$\delta_{(1)}$
Price setting Type	$\delta_{(n-1)}$	$\delta_{(n-l)}$	-
Number of Bidders	n	$n+1-l$	1

Table 3: Overview

2.4 The Equilibrium

We analyze the game by using backward induction. In the last period, bidding the own valuation is a dominant strategy.[12] In the period before, the remaining bidders face a trade-off between winning today, when the valuation is still high or winning the last auction. They therefore bid the current valuation for the good reduced by the outside option of winning the good later on. As we show later, the equilibrium can be phrased as "bidding today's valuation minus tomorrow's expected net gain", i.e. tomorrow's valuation minus tomorrow's expected price. Together with the assumption of independence of the discount factors this implies that the types of bidders who already left the auction are not relevant for the remaining bidders. To simplify the following recursive formulas we define $b_i = 0$ for $i > n$ and $\delta_{(0)} = 0$.

Theorem 2 *Consider a support of the discount factors $D \subset \left[\frac{n-2}{n-1}, 1\right]$ and as many goods as bidders $(n = k)$. A symmetric Bayes-Nash equilibrium that leads to an efficient allocation is given by the following bidding functions for every period*[13]

$$\begin{aligned} b_l(\delta) &= \delta^{l-1} - E\left[\delta^l - b_{l+1}\left(\delta_{(n-l-1)}\right) | \delta_{(n-l)} = \delta\right], \\ b_n(\delta) &= \delta^{n-1}, \end{aligned}$$

for $l = 1, \ldots, n-1$.

[12] Actually, as there are as many bidders as stages, the last bidder can submit any kind of bid since he will get the object for the price of zero anyway.

[13] In the last round any positive bidding function is equivalent to the given b_n since there is only one bidder left.

The equilibrium is not in dominant strategies. The bid depends on the outside option which uses an expectation over the opponents' types. If the opponents' types were known, a more profitable bid might be chosen.

The bidding function can be reformulated as

$$\begin{aligned} \delta^{l-1} - b_l(\delta) &= \delta^l - E\left[b_{l+1}\left(\delta_{(n-l-1)}\right)|\delta_{(n-l)} = \delta\right] \\ &= \delta^l - E\left[p_{l+1}|\delta \text{ type bidder wins period } l+1\right], \end{aligned}$$

where p_{l+1} is the price in period $l+1$. This way of writing yields the following intuition: The r.h.s. is the expected utility of winning in period $l+1$. The l.h.s. is the lowest utility a bidder (with type δ) can obtain by winning the lth period (since typically he would pay less than his bid in the second-price regime). Hence, buyers bid such that the maximal price they are willing to pay in period l ensures them the exact same expected utility as winning in period $l+1$.

Corollary 2 *For any period $l < n$ the bidding function b_l is decreasing in δ.*

Note that at any period, the winner is the most impatient buyer still in the auction. In the last auction there is only one bidder left and thus his price will be zero regardless of his bid. The following corollary gives the bidding equilibrium in its direct, i.e. non-recursive form. It becomes apparent that it basically consists of the sum of differences in valuations for all winning types. A buyer bids as if he had the type that would win tomorrow and calculates the future winning types accordingly (in expectation).

Corollary 3 *The non-recursive form of the bidding-function in every period is*

$$\begin{aligned} b_l(\delta) &= \delta^{l-1} - \delta^l + E\left[\sum_{i=l}^{n-2}\left(\delta^i_{(n-1-i)} - \delta^{i+1}_{(n-1-i)}\right)|\delta_{(n-l)} = \delta\right], \\ b_n(\delta) &= \delta^{n-1}, \end{aligned}$$

for $l = 1, \ldots\ n-1$.

2.5 Price Developments

In this section the equilibrium price dynamics are discussed. At first sight it seems plausible that equilibrium prices decline over time since discounting reduces bidders' valuations. This is actually not completely obvious, since more patient buyers win in later auctions. When they win their valuation might still be higher than the one of an early (and impatient) winner. We analyze both the actual sequence of equilibrium prices and the one corrected for the discount factor. As our analysis only considers equilibrium behavior, all prices are prices that emerge in equilibrium.

The following theorem states that the price decline between two periods can be calculated by knowing only the type of the bidder winning the latter auction. Recall, this is the same bidder who sets the price in the previous auction.

Theorem 3 *Consider a support $D \subset \left[\frac{n-2}{n-1}, 1\right]$ and a realized price p_l in period $l = 1, \dots, n-1$. The expected difference in equilibrium prices between periods l and $l+1$ is given by*

$$p_l - E\left[p_{l+1}|p_l\right] = \delta^{l-1} - \delta^l,$$

where $\delta = \delta_{(n-l)}$ is the type of the bidder who sets the price in period l.

The expected difference in prices is therefore equal to the cost of waiting just one more period, i.e. the drop in valuation for the designated winner if prices remain the same. As an immediate consequence we can conclude that unconditional expected prices (in equilibrium) decline as well.

Corollary 4 *The ex ante expected difference in prices between two periods equals the expected difference in valuations for the type winning the latter period, i.e.*

$$E\left[p_l\right] - E\left[p_{l+1}\right] = E\left[\delta_{(n-l)}^{l-1} - \delta_{(n-l)}^{l}\right].$$

Theorem 3 and Corollary 4 demonstrate that expected prices decline across periods. This is true in spite of the fact that patient bidders (with higher valuations) win in later periods. Remarkably, this price decline persists even after correcting the prices for the discounting as the next theorem shows.

Theorem 4 *Consider a support $D \subset \left[\frac{n-2}{n-1}, 1\right]$ and a period l, where $l \in \{1, \ldots, n-1\}$. The expected price in period $l+1$, given the price p_l, is lower than p_l. This is true even after correcting for the discounting any bidder still in the auction will suffer, i.e.*

$$p_l > E\left[\frac{p_{l+1}}{\delta_{(n-j)}} \,|p_l\right],$$

where $\delta_{(n-j)}$ is the type of a bidder $j \in \{l, \ldots, n-1\}$, who is still in the auction.

Note that the theorem implies as a special case that the price decline is stronger than the value depreciation faced by the winner of period $l+1$.

For a bidder with type δ the actual reduction in valuation between periods l and $l+1$ is given by his discount factor δ. A bidder's net utility is his valuation minus the price paid. If the (expected) price in period $l+1$ were $E\left[p_{l+1}\,|p_l\right] = \delta p_l$, then the bidder's net utility in period $l+1$ would be $\delta\left(\delta^{l-1} - p_l\right)$. Hence, it would also be reduced by the factor δ. In that case both the bidder's and the seller's utility would be reduced by the same factor δ. But because of the bidding behavior the bidder's net utility is reduced by less than the factor δ, whereas the seller's expected revenue is reduced by more. This means that the seller is bearing a larger share of the burden imposed by the discounting. However, as we will show in the next section, the seller gains from impatience on the part of the buyers as it boosts competition.

2.6 Comparing Different Impatience Environments

Consider two different impatience environments: In one the distribution for all bidders' types is F, whereas in the other it is $\widetilde{F}$. Suppose the distribution $\widetilde{F}$ puts more mass on high types than F. Formally, $\widetilde{F}$ first order stochastically dominates (FSD) F if $F(x) \geq \widetilde{F}(x)$ for all $x \in [0,1]$ and $F(x) > \widetilde{F}(x)$ for some $x \in (0,1)$. The relevant question is how expected prices and their decline are affected by the two different underlying distributions. To compare both cases we use a "tilde"-notation if $\widetilde{F}$ is the underlying distribution: $\widetilde{\delta}$ is a type drawn according to $\widetilde{F}$, whereas $\widetilde{b}_l$ is the resulting bidding function in period l.

Lemma 3 *For a given type δ, the bid in period $l = 1, \ldots, n-1$ is lower in the more patient environment, i.e.*

$$b_l(\delta) > \widetilde{b}_l(\delta).$$

In other words, if a buyer with a given type knows that his opponents are more patient, he will bid less. This result is driven by the fact that in the more patient environment this buyer expects his opponents to have higher types. As bidding functions are shown to be decreasing, he expects his opponents to bid less. Consequently, the outside option becomes more attractive to this bidder inducing him to reduce his bid.

Next we analyze the price developments in the two environments.

Theorem 5 *Consider a support $D \subset \left[\frac{n-2}{n-1}, 1\right]$ and a period $l = 1, \ldots, n-1$. Then the following statements are true:*

1. *Denote the price realized in period l with p. The expected price in period $l+1$ will be smaller for the first order stochastically dominant distribution, i.e. $E_F[p_{l+1} | p_l = p] > E_{\widetilde{F}}[\widetilde{p}_{l+1} | \widetilde{p}_l = p]$.*

2. *The ex ante expected selling price in period l is lower for the first order stochastically dominant distribution, i.e. $E_F[p_l] > E_{\widetilde{F}}[\widetilde{p}_l]$.*

3. *The ex ante expected price decline between two periods is lower for the first order stochastically dominant distribution, i.e.* $E_F[p_l - p_{l+1}] > E_{\widetilde{F}}[\widetilde{p}_l - \widetilde{p}_{l+1}]$.

The first part of the theorem implies that, given a realized price p_l in period l, the expected price declines more if bidders tend to be more patient. Note that this result does not follow immediately from the fact that bids at each period are lower under the stochastically dominated distribution. When evaluating the price decline at a particular period, the price will be set by different types depending on the distribution. Specifically, under the stochastically dominating distribution, the type which determines the price is lower (see Figure 1).

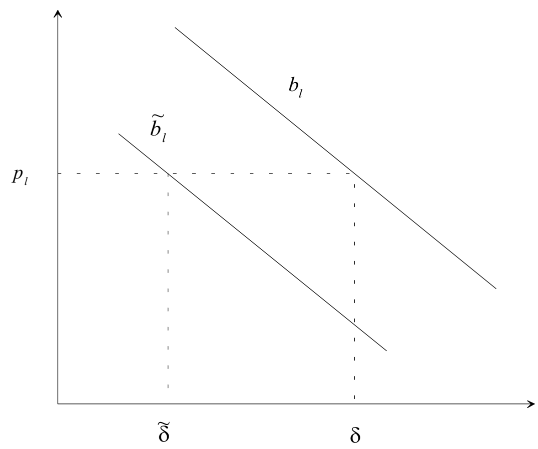

Figure 1: Types Setting Price p_l

To understand the intuition behind the result recall Theorem 3. It states that the expected price dispersion between two periods is equal to the cost of waiting for the person who wins the latter period. Furthermore, the cost of waiting is declining with the discount factor (see Lemma A.1 in Appendix A). Hence, the expected price dispersion between two periods declines with the discount factor. Now, fixing a price p_l for period l, the person who determines this price in the more patient environment has a lower discount factor $\widetilde{\delta} < \delta$. Therefore, the price dispersion is larger in this environment.

The second part of Theorem 5 takes the ex ante perspective and states that the expected selling price for a given period (as long as it is not the last) is lower if bidders are more patient. This result is largely due to the decreasing bidding functions and the fact that bidding is lower if bidders are more patient. Since the expected selling price in the last period is always zero, Theorem 5 implies that the seller always prefers less patient bidders. This is due to the fact that competition is driven be the bidders' impatience. If all bidders were completely patient, the selling price would be zero in every period.

As the third part of Theorem 5 states, not only the expected prices but also their differences are smaller if bidders tend to be more patient.

2.7 Conclusion

We analyzed sequential second-price auctions with private discount factors and unit demand. To ensure an efficient allocation in this setting there need to be as many bidders as goods if discounting is not too strong. The bidding equilibrium leading to an efficient allocation was characterized as: Bidding the present valuation reduced by the outside option of winning tomorrow. In any period the expected decline in equilibrium prices can be interpreted as the cost of waiting for the bidder winning the next auction. The decline is stronger than bidders' value depreciation implies, i.e. the equilibrium price declines even after correcting for the discounting. If bidders tend to be more patient, the equilibrium price for the next period is expected to be lower. Therefore, the seller prefers to have less patient buyers in the auction and thus more devaluation (on average).

Chapter 3

Auctions with Impatient Buyers

3.1 Introduction

Whenever there is an announced sale, a lot of people queue in front of the shop and wait for the doors to open. This indicates an excess demand and suggests that prices could be higher at the beginning of a sale. During the sale prices are usually further reduced to make the unsold products more attractive to buyers with a low willingness to pay. This is due to the fact that shopowners have to clear out their inventories to create space for new products. Therefore, shopowners usually have a very low reservation price. A buyer might want to buy a good early for two reasons: 1. It might be sold to someone else. 2. He can start using the good earlier. For example at an end-of-summer-sale, a buyer might want to enjoy light clothes during the last sunny days. Interviews with sales managers revealed that during a sale the weather plays an important role: Buyers are eager to buy light clothes if they can wear them during the next few days.[14] It is not unusual for prices to be reduced up to six times, down to almost zero.[15] Buyers know

[14]Interviews with sales managers of different fashion and department stores were held by the authors in October 2000 in Mannheim, Germany.

[15]Sales managers told us that they indeed have a very low reservation price. Goods are sold until it is cheaper to dispose them.

that prices drop further during the sale. They face a trade-off between getting the good earlier and paying less. Sales managers are constrained when and how often they set new prices since the new prices have to be posted and observed by potential buyers.

A similar situation occurs for goods, like computers, that become outdated fast. Buyers are impatient to use them and prices are usually decreasing with time. Intel for example publishes so called 'roadmaps' where the company gives out future prices for their CPUs. In 1996, when there was hardly any competition, these roadmaps gave prices for at least three quarters in advance. For example the roadmap published in the second quarter of 1996 gave the prices for the Pentium-133 as \$252 in Q2/96, \$210 in Q3/96, \$173 in Q4/96, and \$131 in Q1/97. The actual price for Pentium-133 CPU in Q1/97 was indeed as predicted one year earlier. For other CPUs the previously announced prices were usually close. The fact that CPUs were nevertheless sold at the higher prices indicates that people find it profitable to acquire the good early. One reason might be the threat of a shortage in supply, as experienced in 1996 for the 200 MHz CPU. Another, maybe even stronger effect, is that people face disadvantages/costs in not being able to use the computer in the meantime. In subsequent years Intel faced an increase in competition. Consequently, prices were lowered faster. This suggests that Intel had previously lowered prices slowly to discourage buyers from waiting.

In each of these examples the buyer has to balance the merits of getting the good early (if at all) against dropping prices. The same is true in numerous instances: Whenever buyers' valuations are declining over time (e.g. for perishable goods like food, show or sport tickets, and news) and when the prices decrease with time. The selling procedure used in the above examples is similar to a Dutch auction: The seller sets a starting price, lowers the price until the good is sold or the price reaches a lowest (reservation) price and the good goes unsold. In practice the seller will be able to choose a starting price freely, but for technical reasons cannot decrease the price arbitrarily fast. The seller might also be unable to commit himself to a reservation price, especially if it is common knowledge that his own valuation is lower than the announced reservation price.

At www.reverseauction.com private Dutch auctions à la Ebay were run.[16] The seller was free to choose a starting price, a reservation price and the total duration of the auction in a range from three to thirty days. The management of www.reverseauction.com highlighted the buyers' impatience while waiting for the auction to end. Further, they pointed out that in a Dutch auction the buyer can determine the time of getting the good. Reverseauction.com's slogan was "Get what you want, when you want it". The winning bidder can influence the end of the auction directly by accepting the current price – unlike in an English or Japanese auction.

In the present chapter, a single unit Dutch auction is analyzed where buyers have preferences that change over time. The price and the buyers' valuations decline in a linear fashion while the auction is running. The seller has two tools to influence the outcome of the auction: The starting price and the speed of the auction. For low starting prices, bidders with high valuations (high types) find it optimal to stop the auction immediately. When analyzing the seller's expected revenue, we show that a low starting price might be optimal. This is true even if multiple bidders stop the auction immediately. In contrast, in a standard Dutch auction (without time preferences) it is never optimal to set a starting price below the bid of the highest possible buyer's type. In our model, a high starting price entails that buyers with low types do not participate in the auction. This might exclude more buyers than optimal. The speed of the auction also influences the seller's expected revenue. We present a setting where it is optimal for the seller to run the auction either very slowly or very fast. This illustrates that it might be beneficial for the seller to run the auction slowly to discourage buyers from waiting for lower prices.

If the seller's objective is to maximize the welfare generated in the auction, we find that a reservation price can raise welfare. But since time costs reduce welfare, it is not possible to achieve a welfare maximizing allocation as long as time cannot run infinitely fast. In an example we demonstrate that a slow auction generates less welfare.

[16] Unfortunately, the auction site went out of service during the ".com bust" of 2001.

The Revenue-Equivalence-Theorem no longer holds in dynamic auctions with time preferences. We analyze whether a dynamic first-price auction, i.e. Dutch auction, or a dynamic second-price auction, i.e. Japanese auction, generates more expected revenue. Even though intuition suggests that buyers might be willing to pay a premium for getting the object early and therefore increase the seller's expected revenue in a Dutch auction, we give an example where this is not true.

This research was sparked by Lucking-Reiley [1999] who finds in field experiments that "Dutch auctions earn statistically significantly more revenues than first-price auctions". He gives the possible explanation that bidders may be impatient and are willing to pay a premium for getting the good earlier.

Impatience/discounting is an important aspect in the bargaining literature such as in Rubinstein [1982]. It influences the market power, therefore the bargaining behavior and thus the resulting outcome. To our knowledge discounting has not been considered in the context of auctions. Kittsteiner et al. [2002] analyze models with sequential auctions where discounting occurs between the auction stages.

In Section 3.2 we describe our model of a Dutch auction with time preferences on the buyers' side. The symmetric Bayes-Nash-Equilibrium is derived in Section 3.3, whereas we focus on the revenue maximizing behavior of the seller in Section 3.4. In Section 3.5 we analyze the generated welfare. In Section 3.6 we allow the seller to set a positive reservation price. The Japanese auction is described in Section 3.7 and the expected revenue is compared to the Dutch auction. We conclude with a discussion in Section 3.8. All proofs are relegated to Appendix B.

3.2 The Model

A seller offers one indivisible good for sale in a Dutch auction. There are n buyers $i = 1 \ldots n$ with privately known types θ_i. The types θ_i are independently drawn from the interval $\left[\underline{\theta}, \overline{\theta}\right]$ with $0 \leq \underline{\theta} < \overline{\theta}$, according to a distribution F. The distribution is continuous and has a strictly positive density f. Buyer i's valuation at time $t \geq 0$ is $v_i(t) = \theta_i - ct$, where $c > 0$ is the rate of value depreciation which is the same for every buyer. The seller's valuation for the good is assumed to be zero at all times. In a Dutch auction the seller sets a starting price $\overline{p}$ and possibly a reservation price r. The price is decreased continuously with time, i.e. the selling price at time t is given by $p(t) = \overline{p} - et$, where $e \in (0, e_{\max}]$ is the rate of price depreciation and $e_{\max}$ is an upper bound on price reduction. The bidder who first stops the auction at a time t gets the object and pays the current price $p(t)$. The lowest possible price r is assumed to be zero, which is reached at time $\overline{t} = \frac{\overline{p}}{e}$ if no bidder stops the auction before. Then the seller keeps the object and no money is paid. Bidder i's strategy $t_i(\theta_i)$ is a stopping time depending on his type θ_i. Note that the buyer with the lowest stopping time wins. In the case that more than one buyer ends the auction at the same time, each is given the object with equal probability. We restrict attention to symmetric Bayes-Nash-Equilibria in pure strategies.

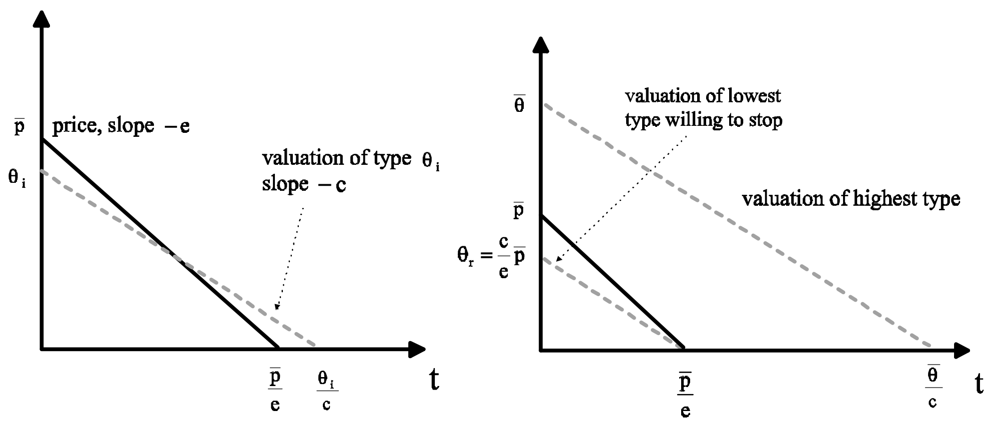

Figure 2: Development of the Price and Buyer's Valuation

The left graph in Figure 2 indicates how the valuation of a buyer with type θ_i declines over time with the given rate c. The starting price $\overline{p}$ depreciates with the rate e. The right graph illustrates how the valuation of the buyer with the highest possible type develops. Moreover, it shows the buyer whose valuation is always below the price except when both are zero.

3.3 The Symmetric Equilibrium

In a standard Dutch auction without time preferences, the agents' bidding behavior is determined by the following trade off: If an agent decides to stop the auction early, his winning probability is high but his net profit is low. In our model buyers bid aggressively to get the object early, since waiting is costly for them. It might be optimal for some buyers' types to stop the auction immediately. Consider the case that the buyers' valuations of the object depreciate at least as fast as its price ($c \geq e$). Then a bidder i stops the auction at time 0 if his valuation θ_i exceeds the starting price $\overline{p}$. But if his valuation at time 0 is below $\overline{p}$ he never stops.[17] We assume that a buyer who is indifferent between stopping times always stops as early as possible. Therefore, we can restrict our attention to the case $c \leq e$ since the case $c > e$ is already covered by the analysis for the case $c = e$.

Denote by θ_r the highest type whose valuation is always below the price while it is positive. For $r = 0$, the right graph in Figure 2 shows that this type is $\theta_r = \frac{c}{e}\overline{p}$. This buyer stops the auction at time $\frac{\overline{p}}{e}$, when the price is no longer above his valuation – both are zero at that time. To avoid redundancies in Section 3.6 we state the equilibrium for a general reservation price $r \geq 0$.

Lemma 4 *The lowest type willing to stop the auction at some time is given by*

$$\theta_r = \left(1 - \frac{c}{e}\right) r + \frac{c}{e}\overline{p}.$$

For a reservation price $r = 0$, the right graph in Figure 2 shows for a type $\theta < \theta_r$ that the auction price is always above his valuation. Hence, it is not profitable for such a bidder to stop the auction. Consequently, a necessary condition for trade to take place is a starting price $\overline{p} \leq \frac{e}{c}\overline{\theta}$. Otherwise, the price will always remain above all possible buyers' valuations and the seller will always keep the good.

[17] If $c > e$ a type $\overline{p}$ bidder stops the auction immediately or never, whereas in the case $c = e$ a type $\overline{p}$ bidder is always indifferent between different stopping times.

If the starting price is low, bidders with high types might want to stop the auction immediately. The winner is then selected by a randomization device. The feature that bids are bounded from above can also be found in Gavious et al. [2002] where endogenous bid caps are analyzed. Even though independently derived, their symmetric equilibrium has in some aspects similar properties.

We define a starting price to be *high enough* if

$$\overline{p} \geq \overline{\theta} - \int_{\theta_r}^{\overline{\theta}} F^{n-1}(x)\, dx.$$

Otherwise the starting price is said to be *not high enough.*

If the starting price is not high enough, some bidders might want to stop the auction in the first instant. Denote by $\widetilde{\theta}$ the bidders' type who is indifferent between stopping immediately and not. The type $\widetilde{\theta}$ is given as the solution of

$$\left(\widetilde{\theta} - \overline{p}\right) \frac{F^n\left(\widetilde{\theta}\right) - 1}{n\left(F\left(\widetilde{\theta}\right) - 1\right)} = \int_{\theta_r}^{\widetilde{\theta}} F^{n-1}(x)\, dx.$$

The left hand side of the equation gives the expected utility of a bidder with type $\widetilde{\theta} \in \left[\overline{p}, \overline{\theta}\right]$ if he is the lowest type stopping the auction immediately. Recall that if more than one buyer stops in the first instant, each buyer gets the good with equal probability. The right hand side is the expected utility if the type $\widetilde{\theta}$ buyer has the highest type and bids according to the bidding equilibrium derived in Theorem 6. Consequently, the type $\widetilde{\theta}$ bidder is indifferent between stopping immediately and bidding according to the equilibrium. If the starting price is high enough, define $\widetilde{\theta} = \overline{\theta}$.

Lemma 5 *The above equation has always a unique solution.*

Theorem 6 *A symmetric bidding strategy for $\theta \in [\theta_r, \overline{\theta}]$ is given by*[18]

$$t(\theta) = \begin{cases} \frac{1}{e-c}\left(\overline{p} - \theta + \int_{\theta_r}^{\theta} \frac{F^{n-1}(x)}{F^{n-1}(\theta)} dx\right) & \textit{if } \theta \in \left[\theta_r, \widetilde{\theta}\right), \\ 0 & \textit{if } \quad \theta \geq \widetilde{\theta}. \end{cases}$$

For a high enough starting price, the equilibrium bidding function t is strictly decreasing and continuous. This is not true for a not high enough starting price since there exists an interval of high types who prefer to stop the auction at time zero. The bidding function is discontinuous at $\widetilde{\theta}$ but strictly decreasing and continuous for types below $\widetilde{\theta}$. Note that a lower starting price $\overline{p}$ results in more aggressive bidding for types that do not stop immediately, i.e. $\frac{dt(\theta)}{d\overline{p}} > 0$ for $\theta \in [\theta_r, \widetilde{\theta}]$. Recall that the considered bidding function is a stopping time. It can also be translated into the price at the stopping time

$$p(t(\theta)) = \overline{p} - et(\theta) = \frac{e}{e-c}\left(\theta - \frac{c}{e}\overline{p} - \int_{\frac{c}{e}\overline{p}}^{\theta} \frac{F^{n-1}(x)}{F^{n-1}(\theta)} dx\right).$$

Example 2 *Assume that there are two bidders and $f(\theta) = 1_{[0,1]}(\theta)$. All types $\theta_i \in [0, \theta_r)$ never find it profitable to win and thus never stop the auction. The starting price is high enough if $\overline{p} \geq \frac{e}{c^2}\left(e - \sqrt{(e^2 - c^2)}\right)$. If the starting price is not high enough, the boundary type is $\widetilde{\theta} = \frac{\overline{p}}{1-\overline{p}}\left(1 - \frac{c^2\overline{p}}{e^2}\right)$. The symmetric bidding strategy for a type $\theta \geq \theta_r$ is given by*

$$t(\theta) = \begin{cases} \frac{1}{e-c}\left(\overline{p} - \frac{\theta}{2} - \frac{1}{2\theta}\frac{c^2\overline{p}^2}{e^2}\right) & \textit{if } \theta \in [\theta_r, \widetilde{\theta}], \\ 0 & \textit{if } \theta \in (\widetilde{\theta}, 1]. \end{cases}$$

In a model without time preferences on the buyers' side, the seller will never use a starting price below the highest possible price a bidder might be willing to pay. The auction simply continues until the first buyer stops without any loss of valuation due to waiting for the buyers. Hence, setting

[18] The case of $e = c$ is also covered in the following sense: For $e \to c$ we get $t(\theta_i) \to 0$ for all types that might stop the auction.

a high starting price does not have any negative consequences. But setting a low starting price is always bad for efficiency and revenue maximization, since the seller cannot discriminate between high types. Therefore, in standard models the starting price has always been assumed to be high enough to avoid the immediate stopping of the auction.

In our model, where the valuation of the bidders decreases with time, the starting price influences the auction in numerous ways. It determines whether high types immediately stop the auction. Moreover, it influences the set of buyers $[\underline{\theta}, \theta_r)$ that never stop the auction. Increasing the starting price entails that

- more types never stop the auction,
- buyers not stopping instantaneously bid less aggressively,
- fewer buyers' types immediately stop the auction,
- buyers who immediately stop pay a higher price.

With a higher starting price, the buyers who do not stop the auction instantaneously (and did not do so before) have to wait longer for the good to become affordable. As their valuation for the good decreases with time, they stop the auction at a lower price. Because of the different effects pulling in different directions, it is difficult to determine the overall effect of a change in the starting price.

Similarly, a change in the speed of the price drop has numerous effects. If the seller raises the speed of the price drop, e.g. increases e, more types stop the auction immediately and fewer types never stop the auction. But as the bidding behavior also changes, the overall effect is not easily identified.

3.4 Revenue Maximization

We assume that the seller is interested in maximizing his expected revenue. One of the questions we address is whether the seller should set a high starting price or allow immediate stopping. We assume the seller does not set a positive reservation price. Hence, the price could decline until zero. This might be a realistic assumption in environments where the seller's commitment power is low. Therefore, the seller has two tools that influence his expected revenue: The starting price and the speed of the auction. In most end-of-season sales there is an excess demand at the starting price. Even though this might seem counterintuitive at first, we demonstrate in Example 3 that setting such a low starting price can indeed be optimal. The seller's expected revenue is given by the starting price multiplied with the probability that at least one buyer is willing to pay this price. If no buyer is willing to pay the starting price, the seller receives the price of the auction at the time the first buyer stops the auction

$$U_S = \overline{p}\left(1 - F^n\left(\widetilde{\theta}\right)\right) + \int_{\theta_r}^{\widetilde{\theta}} \left(\overline{p} - et\left(\theta\right)\right) f_{1:n}\left(\theta\right) d\theta,$$

where $f_{1:n}$ denotes the density of the first order statistic with n buyers.

Lemma 6 *The seller's expected revenue can be written as*

$$U_S = \overline{p}\left(1 - F^n\left(\theta_r\right)\right) - e\int_{\theta_r}^{\widetilde{\theta}} t\left(\theta\right) f_{1:n}\left(\theta\right) d\theta.$$

This allows the following interpretation: The seller receives $\overline{p}$ with the probability that at least one buyer participates in the auction, reduced by the costs of time. If no buyer stops the auction immediately, the highest bidder's expected type determines the costs of time.

Lemma 7 *The seller will always select a starting price*

$$\overline{p} \leq \overline{\theta} - \int_{\theta_r}^{\overline{\theta}} F^{n-1}\left(x\right) dx.$$

The seller will therefore choose as a starting price either the boundary $\overline{p} = \overline{\theta} - \int_{\theta_r}^{\overline{\theta}} F^{n-1}\left(x\right) dx$ or an even lower starting price and thus allow for multiple buyers to stop the auction at the first instant.

The general problem of finding the seller's optimal values for $e \leq e_{\max}$ and $\overline{p}$ is complex.

Example 3 *Consider the specification given in Example 2. We can restrict our attention to the case where* $\overline{p} \leq \overline{\theta} - \int_{\theta_r}^{\overline{\theta}} F^{n-1}(x)\,dx$. *The lowest bidders' type that stops the auction immediately is* $\widetilde{\theta} = \frac{\overline{p}}{1-\overline{p}}(1-\theta_r)$. *The seller's expected revenue is given by*

$$U_S = \frac{1}{e-c}\left(\frac{e}{3}\widetilde{\theta}^3 - e\overline{p}\widetilde{\theta}^2 + \frac{c^2}{e}\overline{p}^2\widetilde{\theta} - \frac{1}{3}\frac{c^3}{e^2}\overline{p}^3 + (e-c)\,\overline{p}\right).$$

The seller's problem is therefore

$$\max_{e,\overline{p}} U_S(\overline{p}, e).$$

Standard optimization reveals that there exist no interior maxima. Hence, we concentrate on finding boundary solutions. Let $\overline{p}^*$ *denote the optimal starting price depending on* e. *The seller's expected revenue as a function of the speed of the auction* $U_S(\overline{p}^*(e), e)$ *is convex (as can be seen for a value* $c = 0.2$ *in Figure 3). Consequently, the seller's utility is highest in the two extreme cases* $e = c$ *and* $e = e_{\max} \to \infty$.

- $e = c$: *The seller's expected revenue is* $U_S = \overline{p} - \overline{p}^3$. *It is maximized for a starting price* $\overline{p} = \frac{1}{\sqrt{3}}$. *The resulting expected revenue for the seller is* $\frac{2}{3\sqrt{3}} = 0.38$.
- $e = e_{\max} \to \infty$: *The expected revenue becomes*

$$\lim_{e=e_{\max}\to\infty} U_S = -\frac{\overline{p}}{3}\frac{7\overline{p}^2 + 3 - 9\overline{p}}{(\overline{p}-1)^3}.$$

  It is maximized for $\overline{p} = \frac{1}{2}$ *with a resulting expected utility of* $\frac{1}{3}$.

Therefore, the global optimum for the seller is to set the time-depreciation rate just as high as the rate of value-depreciation $(e = c)$ *and a starting price of* $\overline{p} = \frac{1}{\sqrt{3}}$. *Consequently, a winning buyer's utility remains unchanged during the auction. Thus, all trade takes place immediately at the beginning of the auction (if at all). All stopping buyers receive the good with equal probability ("distortion at the top").*

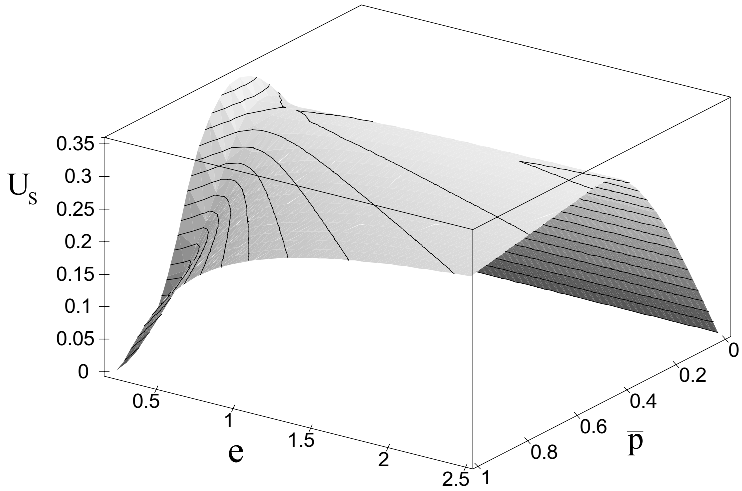

Figure 3: Seller's Expected Revenue for $c = 0.2$

Note that this auction has a higher seller's expected revenue than a first-price-sealed-bid auction without time preferences and without reservation price (where the expected revenue is $\frac{1}{3}$). But the expected revenue is less than in an optimal mechanism without time preferences. An interesting observation is that for any e, it is optimal for the seller to set the starting price in such a way that some types stop the auction in the first instant.[19] *This is in contrast to an environment without time costs where discriminating between high types is costless and therefore always optimal ("no distortion at the top").*

This example demonstrates how Intel's pricing strategy can be optimal in the sense that lowering the price very slowly (and announcing that strategy) can maximize the expected revenue. Intel's potential buyers know that prices will fall slowly. If they face high opportunity costs, they will be reluctant to wait and buy CPUs for the high price instead.

The next example uses a different distribution and derives a different result.

[19]The derivative of the seller's utility at the lowest starting price where high types stop the auction immediately is negative: $\frac{d}{dp}U_S(\overline{p}, e)\Big|_{\overline{p}=\frac{1}{1+\frac{1}{e}\sqrt{e^2-c^2}}} = -2\frac{c}{e+\sqrt{(e^2-c^2)}} < 0$, i.e. it increases revenue to decrease $\overline{p}$.

Example 4 *There are two buyers and agents' types are distributed according to* $F(x) = x^3$ *on the unit-interval. As before, there exist no interior maxima and the partially optimized function* $U_S(\overline{p}^*(e), e)$ *is again convex. In the case of a slow auction the seller's expected revenue (for optimal* $\overline{p} = 7^{-\frac{1}{6}}$*) is* $\lim_{e \to c} U_S = \frac{6}{49} 7^{\frac{5}{6}} = 0.61973$*. In case of an infinitely fast auction the expected revenue (for optimal* $\overline{p} = \frac{3}{4}$*) becomes* $\lim_{e=e_{\max} \to \infty} U_S = \frac{9}{14} = 0.643$*. In contrast to Example 3, the global optimum for the seller is to set the time-depreciation rate as high as possible (if* $e_{\max}$ *is sufficiently high).*

All other examples we calculated also exhibit a convex seller's partially optimized utility function $U_S(\overline{p}^*(e), e)$. The global maximum in all cases was either a slow auction (value depreciation equals price depreciation) or a very fast auction.

3.4.1 General Analysis of two Extreme Cases

Since a general optimization result is very difficult to obtain, we focus on two extreme cases.

- $e = c$: The price-depreciation in the auction is just as fast as the buyers' value depreciation. Consequently, a winning buyer's utility remains unchanged during the auction. Therefore, he stops the auction at the first instant – if at all. As a result, the auction coincides with a simultaneous take-it-or-leave-it offer (fixed-price-offer) to the n buyers. If more than one buyer accepts the offer, each of these buyers gets the object with the same probability at the price $\overline{p}$. The seller's expected revenue is $U_S = \overline{p}(1 - F^n(\overline{p}))$, which constitutes a local maximum.

- $e = e_{\max} \to \infty$: The price falls very fast and the auction resembles a first-price-sealed-bid-auction without time preferences. The seller is able to discriminate perfectly between all buyers (if $\overline{p}$ is high enough). The auction is over (almost) immediately and time costs converge to zero. All buyers are willing to stop the auction. The seller's expected revenue is maximized for $\overline{p} = \overline{\theta} - \int_{\theta_r}^{\overline{\theta}} F^{n-1}(\theta)\, d\theta$. This results in a boundary type $\widetilde{\theta} = \overline{\theta}$ and a seller's expected revenue of

$$U_S = \int_{\theta_r}^{\overline{\theta}} \left(1 - nF^{n-1}(\theta) + (n-1) F^n(\theta)\right) d\theta.$$

As seen in the previous examples, each of the two cases can constitute the global maximum – depending on the underlying distribution.

3.5 Welfare Considerations

The welfare generated by the auction is the sum of buyers' and seller's expected utility.[20] Since the seller's valuation for the good is normalized to be zero and payments are just transfers between buyers and the seller, welfare is equal to the sum of the expected buyers' valuations conditional on winning. The efficient allocation is the one maximizing the welfare generated by the auction. We can describe the resulting allocation of a Dutch auction by the vector $k = (k_1, \ldots, k_n)$, where $k_i(\theta)$ denotes the probability that agent i gets the good if the buyers' types are given by the vector $\theta = (\theta_1, \ldots, \theta_n)$. Therefore, welfare is given by

$$W = \sum_{i=1}^{n} E_\theta \left[(\theta_i - ct(\theta_i))\, k_i(\theta_i, \theta_{-i}) \right].$$

Let $m(\theta)$ denote the number of bidders that stop the auction in the first instant, e.g. $m(\theta) = \#\{j \mid \theta_j \geq \widetilde{\theta}\}$. The equilibrium derived in Section 3.3 results in the following allocation[21]

$$k_i(\theta) = \begin{cases} 1 & \text{if } \widetilde{\theta} \geq \theta_i > \theta_j \text{ and } \theta_i \geq \theta_r \text{ for all } j \neq i, \\ \frac{1}{m(\theta)} & \text{if } \theta_i \geq \widetilde{\theta}, \\ 0 & \text{if } \theta_i < \min\left\{\widetilde{\theta}, \theta_j\right\} \text{ or } \theta_i < \theta_r \text{ for some } j \neq i. \end{cases}$$

[20] Note that unless otherwise stated we take an ex ante point of view.

[21] If $\widetilde{\theta} \geq \theta_i \geq \theta_j$ and $\theta_i \geq \theta_r$ $\forall j \neq i$ and if $\theta_i = \theta_l$ for some l, any tie-breaking rule can be applied since this almost never happens.

Putting all this together, the welfare generated by the Dutch auction is given by

$$W(e, c, \overline{p}) = \left(1 - F^n\left(\widetilde{\theta}\right)\right) \int_{\widetilde{\theta}}^{\overline{\theta}} x \frac{f(x)}{1 - F\left(\widetilde{\theta}\right)} dx + \int_{\theta_r}^{\widetilde{\theta}} (x - ct(x))\, dF^n(x).$$

The first part gives the generated welfare if at least one buyer has a type higher than $\widetilde{\theta}$. In that case the welfare is just the expected type of such a bidder. The second part describes the welfare if all bidders are below $\widetilde{\theta}$. Then the generated welfare is the valuation of the highest bidder at his stopping time (if he stops the auction at all). Recall that bidders with types below θ_r never stop the auction.

We call a mechanism *(ex post) efficient* if it maximizes welfare. A Dutch auction is (ex post) efficient if it allocates the good at $t = 0$ to the bidder with the highest valuation. An immediate allocation of the good only occurs if $e = c$ (see Section 3.4.1). In this case the allocation is not efficient since the good is allocated randomly among the bidders with $\theta_i \geq \overline{p}$ and never given to a buyer with type $\theta_i < \overline{p}$. We get arbitrarily close to an efficient outcome if $e = e_{\max}$ becomes large and the starting price is high enough.

Lemma 8 *It is never efficient to start with a price* $\overline{p} > \overline{\theta} - \int_0^{\overline{\theta}} F^{n-1}(x)\, dx$.

The following example shows an auction that realizes less welfare the slower it runs.

Example 5 *Consider the specification given in Example 2. The welfare generated by the Dutch auction is given by*

$$W(e, \overline{p}) =$$
$$\begin{cases} \frac{1-\widetilde{\theta}^2}{1-\widetilde{\theta}} \int_{\widetilde{\theta}}^1 x dx + 2\int_{\theta_r}^{\widetilde{\theta}} (x - ct(x))\, x dx & \text{if } \overline{p} < \frac{e}{c^2}\left(e - \sqrt{(e^2 - c^2)}\right), \\ 2\int_{\theta_r}^1 (x - ct(x))\, x dx & \text{if } \overline{p} \geq \frac{e}{c^2}\left(e - \sqrt{(e^2 - c^2)}\right). \end{cases}$$

For large e it turns out that, as long as the starting price is relatively high, we are close to the highest possible welfare of $\frac{2}{3}$. *For fast auctions time costs do not matter much and almost every type bids a stopping time greater than zero. The starting price allows to discriminate perfectly in-between buyers. In this example it means setting a starting price close or equal to* 0.5. *For a slow auction, i.e. if* $e \to c$, *the welfare of the Dutch auction decreases. Winning with a stopping time above zero becomes very costly and consequently agents either stop the auction immediately or never. Therefore, it is optimal to choose a starting price close to* $\overline{p} = \frac{1}{3}$. *The realized welfare is close to* 0.6. *The graphs in Figure 4 show the welfare* $W(e, \overline{p})$ *for* $c = 0.2$. *The left graph is a contour plot that shows iso-welfare curves, whereas the right plot shows the full three dimensional picture. A lighter color indicates a higher welfare level.*

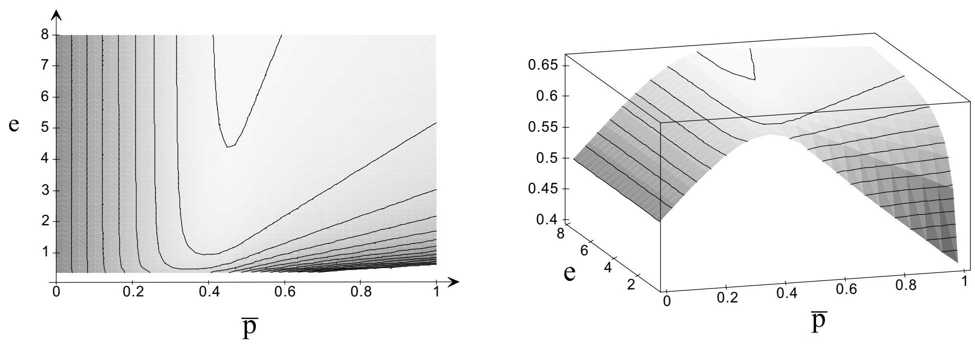

Figure 4: Welfare

3.6 Dutch Auction with Reservation Price

If the seller can commit himself to stop the auction after it reached a fixed reservation price $r > 0$ (which is common knowledge to all participants), he can realize more revenue. If $e = e_{\max} \to \infty$, i.e. the price falls arbitrarily fast, the seller can obtain the highest possible revenue by setting an optimal reservation price.

Lemma 9 *An increase in the reservation price r leads to a lower boundary type $\widetilde{\theta}$ who stops the auction at the first instant.*

An increase in the reservation price prevents more low types from stopping the auction at some time. Moreover, it induces more high types to stop the auction in the first instant. Therefore, both effects lead to fewer types stopping at a time greater than zero.

In models without time preferences an increase of the reservation price always reduces efficiency. This is not generally true in our model. We find that an increase in the reservation price can increase the welfare. This is due to the fact that with a higher reservation price buyers tend to stop the auction earlier.

Example 6 *Consider the setting given in Example 2. The price decline during the auction is given by $e = 0.5$. The bidders' decline in valuation is $c = 0.2$. As there are only two bidders, an optimal starting price will be quite low (both for revenue and welfare). For a high starting price the probability of selling the good is low. Note that both the seller's expected revenue and welfare are continuous in the reservation price as long as it is not above the starting price. If the reservation price is equal to the starting price, the auction is a take-it-or-leave-it offer. In that case both expected revenue and welfare are strictly greater than zero if the asked price is such that an interval of bidders' types is willing to accept the offer, i.e. $r = \overline{p} < 1$.*

Seller's Expected Revenue
The graphs in Figure 5 show the seller's expected revenue for $\overline{p} \in [0, 1]$ and a reservation price r.[22] As $r \leq \overline{p}$ the domains in the graphs are triangles. The left graph shows a contour plot with iso-revenue-curves. The right graph shows the corresponding 3-dimensional picture. A lighter color indicates a higher expected revenue.For a given starting price $\overline{p}$, the seller can improve his expected revenue by setting a positive reservation price. The optimal combination of starting and reservation price is given by $\overline{p} = 0.59$ and

[22]The irregular border is due to the discontinuity that arises between $r = \overline{p}$ and $r > \overline{p}$.

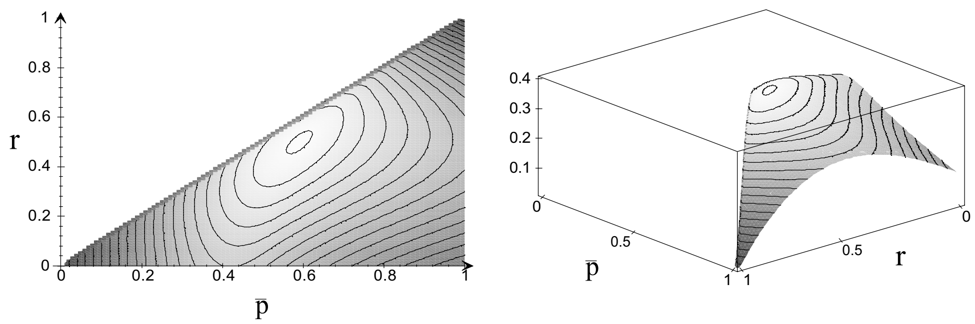

Figure 5: Expected Revenue

$r = 0.50$. *It seems surprising that the optimal reservation price is quite high compared to the optimal starting price. Basically, the optimal auction is like a take-it-or-leave-it offer to high types* ($\theta > \overline{p}$) *and an auction for intermediate types. Due to the reservation price, the intermediate types can only very little shade the price they bid.*

The maximal expected revenue the seller gets in this example is 0.4019. *In an optimal auction without time preferences it would be only slightly higher, i.e.* $\frac{5}{12} = 0.4167$. *Despite the decline in valuation the seller's expected revenue is only marginally reduced.*

Welfare

The graphs in Figure 6 show how welfare develops for $\overline{p} \in [0, 1]$ *and for a reservation price* $r \leq \overline{p}$. *It can be seen that, given a sufficiently high starting price* $\overline{p}$, *the auction becomes more efficient if the reservation price is raised from* $r = 0$ *(for example for a starting price slightly higher than* $\overline{p} = 0.6$*). Choosing a reservation price too close to* $\overline{p}$ *reduces efficiency. As in models without time preferences, a reservation price reduces efficiency since low types never receive the good. A reservation price also induces bidders to bid an earlier stopping time, which – in contrast to models without time preferences – increases the realized welfare.*

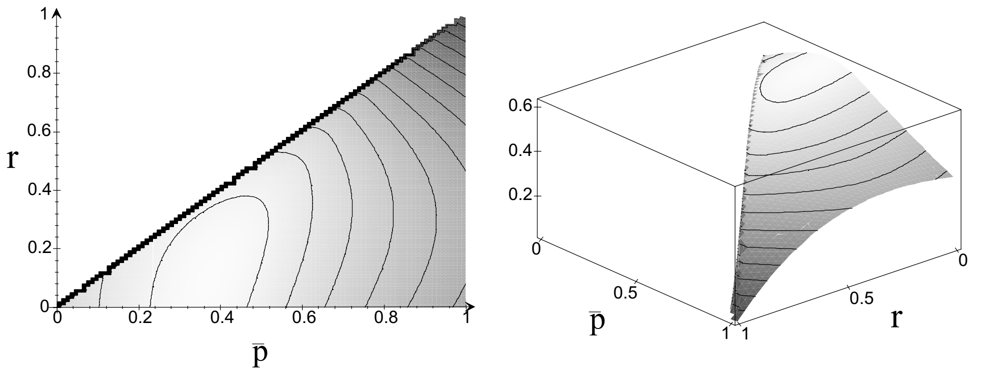

Figure 6: Welfare

3.7 Japanese Auction with Time Preferences

Consider a Japanese auction in the setting described in Section 3.2. In a Japanese auction the seller sets a minimum price $\underline{p}$. The price rises continuously with speed $e > 0$. During the auction all present bidders keep a button pressed as long as they are willing to pay the posted price. A bidder who no longer presses the button drops out of the auction. The last bidder still pressing the button gets the object for the price at which the last of the other bidders left the auction. The selling price at time t is given by $p(t) = \underline{p} + et$. A bidder's valuation at time t is, as before, $v_i(t) = \theta_i - ct$. A bidder's strategy is again a stopping or exit time at which he releases the button. Note that the bidder with the highest exit time wins the auction.

Figure 7 shows how the price and the valuations of two bidders with type θ_i and type $\underline{p}$ develop. The buyer with type $\underline{p}$ is the lowest buyer who takes part in the auction. In fact this buyer releases the button after the first instant as the price becomes higher than his valuation.

Theorem 7 *The unique symmetric equilibrium is in dominant strategies. It is given by*

$$t(\theta) = \begin{cases} \frac{\theta - \underline{p}}{e+c} & \text{if } \theta \geq \underline{p}, \\ 0 & \text{if } \theta < \underline{p}. \end{cases}$$

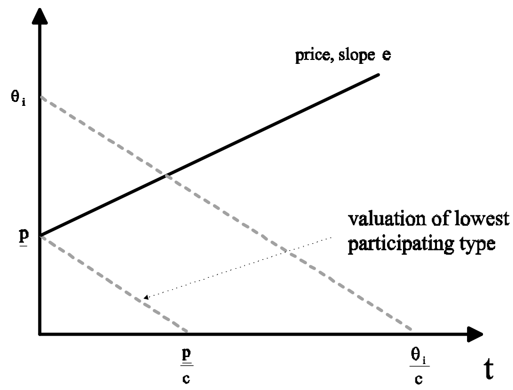

Figure 7: Price and Valuation Development

The bidding strategy can be easily translated into price-bidding: It is optimal to stay in the auction as long as the current price is below the own current valuation, i.e. $\underline{p} + et(\theta) \leq \theta - ct(\theta)$.

As the bidding function is strictly monotone for bidders taking part in the auction (i.e. with types not below the starting price), bidders' types can easily be identified by their bid. In contrast to the Dutch auction with time preferences, the Japanese auction allocates the good efficiently – if it is allocated at all. As the resulting allocation is different to the one resulting from the Dutch auction, the seller's expected revenue will also be different in general.

Example 7 *Consider the setting given in Example 2. The seller's expected revenue in the Japanese auction is given by*

$$U_S = 2\underline{p}^2\left(1-\underline{p}\right) + \frac{2}{e+c}\int_{\underline{p}}^{1}\left(e\theta\left(1-\theta\right) + c\underline{p}\left(1-\theta\right)\right)d\theta.$$

Note that the expected revenue is strictly increasing in e. This is intuitive since a fast Japanese auction minimizes the bidders' loss in valuation – without an opposing force as in the Dutch auction. For a given speed e, the optimal starting price $\underline{p}$ is $\frac{e+\sqrt{e^2+3c^2+4epc}}{3c+4e}$. In the limit $e = e_{\max} \to \infty$ the optimal starting price becomes $\underline{p} = \frac{1}{2}$. This results in a seller's expected revenue of $\frac{5}{12}$. This corresponds to the expected revenue of an optimal auction without time preferences.

For a given rate of price change e, we compare the seller's expected revenues in a Dutch and a Japanese auction. For the Japanese auction we use the optimal starting price. For the Dutch auction we take the optimal starting price and the optimal reservation price. Recall that both auctions yield the same expected revenue if the price decrease is arbitrarily fast, e.g. $e = e_{\max} \to \infty$. *For any* $e \geq c$ *the Japanese auction results in a higher expected revenue than the Dutch auction (as can be seen for* $c = 0.2$ *in Figure 8). This is somewhat surprising: In a Dutch auction buyers can get*

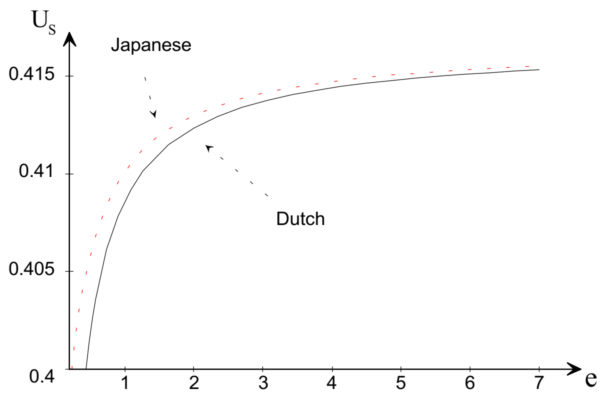

Figure 8: Seller's Expected Revenue

the object earlier if they pay a premium. This results in a higher net gain for the buyer and the seller. Therefore, it seems intuitive that the expected revenue is greater in case of the Dutch auction. This is not true in the given example. An explanation is that the set of participating buyers is not the same.

The biggest difference in revenue between the two auction formats occurs for the case $e = c$, *which we now analyze in more detail: Buyers with types* $\theta \in \left[\frac{1}{\sqrt{3}}, 1\right] = [0.58, 1]$ *participate in the Dutch auction. In the Japanese auction buyers with types* $\theta \in \left[\frac{1+2\sqrt{2}}{7}, 1\right] = [0.55, 1]$ *participate. Hence, more buyers are taking part actively in the Japanese auction. This also results in a higher probability of the good being sold.*

Next, we analyze the expected payments by participating bidders. Recall that in the considered case $e = c$ *the Dutch auction corresponds to a take-it-*

or-leave-it offer. Hence, all participating bidders would pay the same price $\overline{p} = \frac{1+\sqrt{3}}{6}$*. The seller's expected revenue is just this price if the object is sold. In a Japanese auction the expected payment of a participating bidder depends on his type as the bidding function is monotonically increasing. Figure 9 shows that for low types the expected payment is lower in the Japanese case, whereas for high types it is higher. If the same set of buyers' types took part in the Japanese auction as in the Dutch auction, then the Dutch auction would yield more expected revenue. On average, a bidder with type in the interval* $\left[\frac{1}{\sqrt{3}}, 1\right]$ *pays more in the Dutch auction, which accords to our intuition that agents are willing to pay a premium for getting the good earlier. But since fewer buyers participate in the Dutch than in the Japanese auction, the overall expected revenue for the seller is lower.*

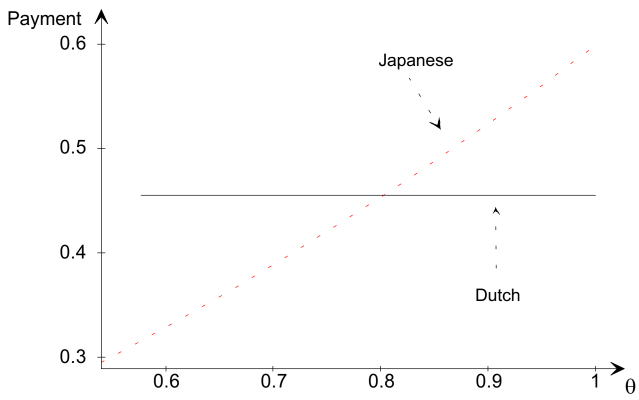

Figure 9: A Bidder's Expected Payment

3.8 Conclusion

We studied a Dutch auction model where bidders' valuations decrease during the auction. If the price depreciation during the auction is as fast as the bidders' decrease in valuation, the auction is a take-it-or-leave-it offer to all bidders. If the price falls very fast the auction resembles a first-price-sealed-bid-auction without time preferences. Both cases might be optimal for the seller. We gave an example where setting a positive reservation price increased the welfare realized in the auction. Finally, we compared the seller's expected revenue resulting from a Japanese and a Dutch auction with time preferences. Surprisingly, we found that the expected revenue can be higher in a Japanese auction.

Chapter 4

A Simple Procurement Decision for Heterogeneous Goods

4.1 Introduction

With the rise of electronic procurement, business-to-business auctions have become a fast, efficient, and popular way for business transactions. The main advantage of auctions is that they provide an environment boosting competition by directly confronting suppliers with their rivals' offers. As a result procurement prices are driven down. The prime drawback of standard auctions is the fact that the goods have to be specified in minute detail to ensure that bids are directly comparable and the right product is procured. On the contrary, a consumer usually does not have the option of holding an auction. Often the consumer shops around with a vague description of the object in mind. He typically compares offers for different objects fitting his vague description, aiming to strike the best deal for him. We want to combine both approaches and provide an auction design which also selects which good to procure.

Consider the following example: A firm plans to procure numerous identical desktop computers.[23] For an auction the firm needs to give the exact specifications of the computer, e.g. the CPU speed, the size of RAM, and hard-disk. Bidders in the auction compete for virtually identical computers. A consumer usually goes to a number of different stores, views the given offers and then decides which computer-price combination is best suited for him. He is willing to accept different specifications if the price is right. Therefore, if a store has a cost advantage for a good still acceptable for the consumer, he can profit from the advantage.

Generally, firms pursue two different approaches in combining auctions and allowing for a certain variety in the good. Either they do not specify the good in full detail, but give certain ranges for the specification. For a computer it might be minimal requirements, e.g. the CPU speed must be at least 2 GHz, the hard-disk must be at least 60 GB. A drawback of this approach is that suppliers will always bid for the cheapest specification just fulfilling the requirement, e.g. the CPU speed will be 2 GHz, the hard-disk will have exactly 60 GB. More often, firms hold not one but numerous auctions for different (exact) specifications. After observing the results, the firm decides on the auction that determines the outcome and the specification. For example one specification might be a 2 GHz CPU, 60 GB hard-disk computer, another 2.4 GHz, 100 GB. After both auctions the firm decides which offer it will accept. The second specification is more valuable to the buyer since the computer is more powerful. But in general it will be more expensive. The firm can select the better good-price combination. A disadvantage of this approach is that bidders might become discontent due to the non-transparent decision process, since they do not know how they could have won the contract.

We introduce a modified second-price auction. It combines the strengths of procurement auctions and allows for different goods, picking the best deal overall. The main idea is that instead of holding different auctions for different specifications, these auctions are combined into one. In this auction bids on the different specifications are made comparable. This is

[23] An example for such a procurement auction is given in the press release DaimlerChrysler [2001] where 10,000 computers were procured online.

achieved by handicapping bids on less favorable specifications. The auction therefore determines not only the price but also which good is to be procured.

We consider a simple environment with two different goods and quasi-linear utility for the buyer. First, we analyze a modified second-price auction which always implements the efficient procurement decision. Then we compare our result to holding two separate second-price auctions. Since the buyer is usually more interested in his utility than in efficiency, we analyze the optimal auction. It suffers from the drawback that the necessary comparison between bids is hard to implement in reality and difficult to understand for participating bidders. Hence, we introduce *constrained optimal auctions*, which only allow bids to be handicapped via an additive constant. This constant is then determined optimally.

The aim of this chapter is to develop concepts and solutions easy and simple enough for real world applications. The basic idea of holding a single auction for different goods was introduced by the Consulting Group Prof. Moldovanu for DaimlerChrysler Research [2002], albeit without a scientific analysis. Close connections exist to the optimal auctions literature introduced by Myerson [1981]. Bulow and Roberts [1989] describe the optimal auction problem as price discrimination with nice intuitions. In a procurement setting McAfee and McMillan [1989] analyze the optimal auction with variable quantity for a homogeneous good, allowing for domestic and foreign firms. In Branco [1994] the auctioneer (government) is interested in maximizing domestic welfare and thus discriminating between domestic and foreign firms. In a broad sense there is a relation to multidimensional auctions, where the good is characterized via different attributes like price and quality.[24] The differences in the specification are assigned values which are summarized via a scoring rule. In this language we focus on a sub-class of scoring rules. A fundamental paper in this area is Che [1993], who analyzes different scoring rules combining bids on price and quality in a quasi-linear environment. Branco [1997] generalizes the

[24]In information technology research multidiminsional auctions are viewed from a different perspective, see for example Teich et al. [1999].

model of Che [1993] and allows for cost correlations.

The model is described in Section 4.2. In Section 4.3 the efficient auction is characterized, whereas in Section 4.4 optimality is focused on. Section 4.5 concludes and all proofs are relegated to Appendix C.

4.2 The Model

There is a single buyer who plans to procure one of two different objects. His valuation for object i is θ_i for $i = 1, 2$. The buyer's utility is quasi-linear, i.e. his utility for getting object i at a price p is $\theta_i - p$. For object $i = 1, 2$ there is a group of n_i potential suppliers with privately known production costs c_{ij} for $j = 1, \ldots, n_i$ drawn according to a distribution F_i. The distribution function F_i has the density f_i which is strictly positive on $\left[\underline{c_i}, \overline{c_i}\right]$ and zero elsewhere. We are interested in symmetric equilibria, i.e. bidders in group i bid according to a bidding function b_i for $i = 1, 2$. Denote by $b_{i,(l)}$ the lth highest bid for object i and by $c_{i,(l)}$ the lth highest cost in group i, where $l \in \{1, \ldots, n_i\}$, $i = 1, 2$. Assume without loss of generality $\theta_1 > \theta_2$ and define the difference as $d = \theta_1 - \theta_2$.

The buyer's problem is to decide which of the two objects he should procure and for which price. Consider the following modified version of a second-price auction with a handicap to bidders from group 2: Bidders submit bids for the object they are willing to supply. The bids for the different objects are not compared directly: bids for object 2 are corrected by an additive handicap e to reflect the buyer's preferences. Moreover, a winning bidder receives as payment for the good the minimum of the second lowest bid submitted from his own group and a modified lowest bid from the other group. More exactly, if, on the one hand, the lowest bid from group 1 is below the (handicapped) lowest bid from group 2, i.e. $b_{1,(n_1)} < b_{2,(n_2)} + e$, the bidder submitting $b_{1,(n_1)}$ wins the auction and gets paid $\min\left\{b_{1,(n_1-1)}, b_{2,(n_2)} + e\right\}$. If, on the other hand, the lowest bid from group 1 is higher, i.e. $b_{1,(n_1)} > b_{2,(n_2)} + e$, then the bidder submitting $b_{2,(n_2)}$ wins the auction and gets paid $\min\left\{b_{1,(n_1)} - e, b_{2,(n_2-1)}\right\}$. In case of a tie any randomization rule can be applied. To simplify the analysis we assume trade always to be beneficial, i.e. $\theta_i > \overline{c}_i$, for $i = 1, 2$. Otherwise, a reservation price can ensure that trade only occurs if it is advantageous for the buyer.

By construction of the auction, bidding behavior is analogous to a regular second-price auction.

Theorem 8 *For bidders in both groups it is a dominant strategy to bid the true costs, i.e.* $b_i(c) = c$ *for* $c \in [\underline{c_i}, \overline{c_i}]$ *and* $i = 1, 2$.

Since bidders will report their costs truthfully in equilibrium the procuring firm's utility can be easily calculated. It is given by

$$\begin{cases} \theta_1 - \min\left\{c_{1,(n_1-1)}, c_{2,(n_2)} + e\right\}, & \text{if } c_{1,(n_1)} < c_{2,(n_2)} + e, \\ \theta_2 - \min\left\{c_{1,(n_1)} - e, c_{2,(n_2-1)}\right\}, & \text{if } c_{1,(n_1)} \geq c_{2,(n_2)} + e, \end{cases} \tag{U}$$

where the first line corresponds to the case where the lowest bidder from group 1 wins and supplies good 1. The second line gives the utility when the lowest bidder from the second group wins.

4.3 Efficient Auction

As an efficient procurement decision we understand the welfare maximizing one, i.e. the decision that maximizes the value of the object to the procuring firm reduced by the incurred supplier's cost.

The parameter e can be chosen such that the resulting procurement decision is always efficient. This is achieved by comparing bids between the two groups in the same way as the goods differ from the procuring firm's point of view.

Theorem 9 *The auction is efficient if* $e = d$*, i.e. bids are corrected according to the procuring firm's preferences.*

The procuring firm's utility (U) becomes

$$\theta_1 - \begin{cases} \min\left\{c_{1,(n_1-1)}, c_{2,(n_2)} + d\right\}, & \text{if } c_{1,(n_1)} < c_{2,(n_2)} + d, \\ \min\left\{c_{1,(n_1)}, c_{2,(n_2-1)} + d\right\}, & \text{if } c_{1,(n_1)} \geq c_{2,(n_2)} + d. \end{cases}$$

Note that the model with the handicap e set to the procuring firms' difference in valuations d is equivalent to a regular second-price auction where the following conditions are met: The procuring firm has the same valuation for both goods and the support of the distribution function for the second group is shifted from $[\underline{c}_2, \overline{c}_2]$ to $[\underline{c}_2 + d, \overline{c}_2 + d]$ with a distribution function $F(c) = F_2(c - d)$.

4.3.1 Two Separate Second-Price Auctions

Keeping the environment unchanged, we now want to compare the efficient auction to two separate second-price auctions — one for each good. The result of each auction is just an offer to the buyer. After observing the outcomes of the auctions, the buyer picks a good and the corresponding outcome of the auction that maximizes his utility and only buys that good. The other auction becomes irrelevant. Again it is each bidder's dominant strategy in both auctions to bid his true cost. The outcome of each auction is therefore a price of $c_{i,(n_i-1)}$ for $i = 1, 2$. Selecting the more favorable outcome, the procuring firm therefore picks the good which maximizes its utility, i.e.

$$i = \arg\max \left\{\theta_i - c_{i,(n_i-1)}\right\}.$$

Equivalently, the two auctions can be run in sequence, where the result of the first auction determines a reservation price for the second auction. The reservation price for the second auction that guarantees at least the utility for the buyer that can be obtained from the first auction is $c_{1,(n_1-1)} - d$.[25] Since in both separate second-price auction formats the "better" second-lowest bidder sets the price and determines which good is procured, the auctions are equivalent in all respects.

Theorem 10 *The procuring firm's utility of the modified second-price auction is never below its utility after picking one of the two separate second-*

[25]The statement is formulated for the case where good number one is auctioned in auction one and good number two is auctioned in auction two.

price auctions. In some cases (which occur with positive probability[26]) it is strictly greater.

Hence, it is never beneficial for the procuring firm to hold two separate second-price auctions. The key feature driving the result is that bidders in the two separate auctions do not directly compete against each other. If the second-lowest and therefore price determining bid is high, the price is high. Thus, the auction is not attractive for the buyer and he will probably select the other auction. The bidder who submitted the lowest bid is unable to take the indirect competition induced by the other auction into account, regardless of his own cost realization. Because of the second-price nature of the auctions, he is unable to increase the chance of his auction being selected by making the price more attractive to the buyer, even though it might be in his interest to do so. In contrast, in the modified second-price auction all bidders directly compete against each other. Hence, all potentials for price reductions are realized since the overall lowest bid (after correction) always wins.

4.4 Optimality

From an interim point of view, the above efficiently modified second-price auction is, in general, not utility maximizing for the procuring firm. We now analyze the procuring firm's optimization problem. First, an optimal auction is described which not only allows to compare bids from the different bidder groups, but also includes a reservation price. The optimal auction is difficult to implement in practice due to complex calculations needed for the bid comparisons. Next, we introduce the concept of constrained optimal auctions, where bids are compared in a very simple way.

[26] Excluding the degenerate case where the supports do not overlap.

4.4.1 Optimal Auction

To simplify the analysis we assume from now on that the distribution function for the second group is a 'translation' of the first group's distribution, i.e. the cost parameters for group one are drawn according to the distribution function $F_1 = F$ on $[\underline{c}_1, \overline{c}_1] = [\underline{c}, \overline{c}]$, whereas for group two it is according to $F_2(c) = F(c+a)$ on $[\underline{c}_2, \overline{c}_2] = [\underline{c} - a, \overline{c} - a]$ for some $a \in \mathbb{R}$. The parameter a takes care of the fact that production costs might be systematically different for the two objects. Note that a can be both positive or negative, hence the more favored good is not necessarily more expensive to produce. Assume that $\frac{F(c)}{f(c)}$ is increasing in c, which is the analog to the monotone hazard rate assumption in non-procurement auctions and corresponds to assuming that F is log-concave.[27] We define a function $M(c) = \left(c + \frac{F(c)}{f(c)}\right)$ that will play the role of what Bulow and Roberts [1989] call *Marginal Revenue*. Consequently, $M(c)$ is strictly increasing in c, which corresponds to a decreasing marginal revenue assumption in non-procurement settings.

To simplify notation we use c_i^{low} to denote the lowest cost type in group i, i.e. $c_i^{low} = c_{i,(n_i)}$ for $i = 1, 2$.

Theorem 11 *The optimal procurement rule for the different goods is given by:*

$$\begin{cases} \textit{no object is bought, if } \theta_1 < \min\left\{M\left(c_1^{low}\right), M\left(c_2^{low} + a\right) + d - a\right\}, \\ \textit{object 1 is bought from bidder } c_1^{low}, \textit{ if } M\left(c_1^{low}\right) < M\left(c_2^{low} + a\right) + d - a, \\ \textit{object 2 is bought from bidder } c_2^{low}, \textit{ if } M\left(c_1^{low}\right) > M\left(c_2^{low} + a\right) + d - a. \end{cases}$$

It is easily seen that this procurement rule is not efficient. Consider the case where no object is bought: Trade might nevertheless be beneficial for the buyer, e.g. if the price for object one is strictly below the buyer's valuation.

Bidders willing to supply the less favorable object face a handicap, since their bids are modified. Consider the case that all bidders' distributions

[27] For a detailed analysis of log-concave functions and numerous examples of distribution functions that are log-concave, see Bagnoli and Bergstrom [1989].

are the same, i.e. $a = 0$. When comparing $M\left(c_1^{low}\right)$ and $M\left(c_2^{low}\right) + d$, the handicap for bidders in the second group is just an additive d, which is the difference in buyer's valuation. Since the function M is applied to a bid, this does not correspond to adding the handicap d to bids from the second group. Now consider the case that buyers' distributions are not the same but the two goods have the same value for the buyer ($\theta_1 = \theta_2$). Then the translation a of the distribution function becomes the sole source of unequal treatment of the two groups. Again the handicap is not just an additive constant to the bids submitted from a bidder in the second group.

As seen in the two exemplary cases, not only the realized cost types but also the underlying distribution function (via the function M) determine the winning bidder in the optimal auction. Bids are transformed in a (for the buyer) non-transparent way to a "priority level", which determines the winner and bidders might find it hard to understand why a worse bid is preferred to their own.[28] Therefore, optimal auctions are hard to apply in real life situations.

4.4.2 Constrained Optimal Auction

In this section we introduce constrained optimal auctions, where "constrained" means that bids are modified for comparison by the additional handicap e. The question arises how to optimally select the handicap e in order to maximize the procuring firm's expected utility.

Denote by $F_{(l:n)}$ the distribution function of the lth order statistic for n bidders and by $f_{(l:n)}$ the corresponding density.

The procuring firm's expected utility is given by weighting the different cases in (U) with the corresponding probabilities. Applying the Revenue-Equivalence-Theorem allows us to write the expected utility as stated in the following lemma.

[28] The term "priority level" is used in a non-procurement auction by Wolfstetter [1994] and stands for the same as marginal revenue or virtual valuation.

Lemma 10 *The procuring firm's expected utility depending on the handicap e is*

$$U(e) = \theta_1 - \int_{\underline{c}}^{\overline{c}} M(c)\left(1 - F(c + a - e)\right)^{n_2} f_{(n_1:n_1)}(c)\, dc \\ - \int_{\underline{c}}^{\overline{c}} \left(M(c) - a + d\right)\left(1 - F(c - a + e)\right)^{n_1} f_{(n_2:n_2)}(c)\, dc.$$

Note that the densities are given by the distribution of the lowest of n_1 and n_2 bidders. They are $f_{(n_1:n_1)}(c) = \frac{d}{dc}\left(1 - (1 - F(c))^{n_1}\right)$ and $f_{(n_2:n_2)}(c) = \frac{d}{dc}\left(1 - (1 - F(c))^{n_2}\right)$. We introduce $\widetilde{M}(c, e) = M(c + a - e) - M(c) - a + d$ to simplify the notation.

Optimizing the selection of the handicap e results in the following first order condition.

Corollary 5 *The first order condition with respect to e is given by*

$$\frac{\partial U(e^*)}{\partial e} = \int_{\underline{c}}^{\overline{c}} \widetilde{M}(c, e^*)\, f_{(n_1:n_1)}(c)\, f_{(n_2:n_2)}(c + a - e^*)\, dc \\ = 0.$$

Only for c in the interval $[\max\{\underline{c}, \underline{c} - a + e\},\ \min\{\overline{c}, \overline{c} - a + e\}]$ the integrand is non-zero. Outside at least one of the two densities is zero.

The first order condition is basically the integral over a function that looks like the one in Figure 10 multiplied by two (positive) densities.[29] The handicap parameter e has to be adjusted in such a way that the "area under the function" (with densities) is zero. In the described case the lowest possible c is $\underline{c} - a + e$ with a function value of $d - e - \frac{F(\underline{c}-a+e)}{f(\underline{c}-a+e)}$. Moreover, the function is monotonically decreasing. For the first order condition to hold, $d - e$ must be greater than zero and the function value for $\overline{c}$ must be below zero. For a simple class of distribution functions the following example gives the explicit solution of the optimization problem.

[29] Note that Figure 10 depicts the case where $a < d$ and F fulfills the assumption that $\frac{F(c)}{f(c)}$ is convex, which will be introduced later.

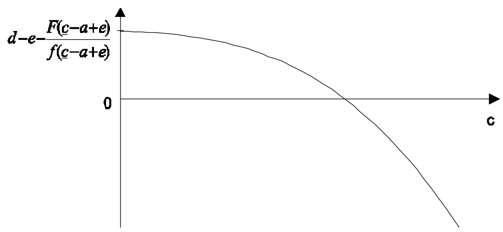

Figure 10: $\widetilde{M}(c,e)$

Example 8 *If the distribution function is of the form $F(c) = c^m$ on the unit interval for some $m \in \mathbb{R}^+$, then the optimal e^* is given by the following convex combination between d and a (independent of n_1 and n_2): $e^* = \frac{m}{m+1}d + \frac{1}{m+1}a$. In this case the constrained optimal auction is also the optimal auction.*[30]

Corollary 6 *Consider two distribution functions of the form $\widehat{F}(c) = c^{\widehat{m}}$ and $\widetilde{F}(c) = c^{\widetilde{m}}$ for some $\widehat{m} > \widetilde{m}$ (i.e. $\widehat{F}$ first order stochastically dominates $\widetilde{F}$, i.e. costs tend to be higher for the distribution function $\widehat{F}$). Then the optimal e^* is closer to the efficient handicap d for the distribution function $\widehat{F}$ than for $\widetilde{F}$.*

Since it is impossible to calculate the optimal handicap e^* for a general distribution function F, we elaborate on its properties for different cases that relate the difference in valuations d with the systematic difference in costs a. The following theorem states that the optimal handicap e^* is usually strictly between d and a. Only if a and d coincide this is also the optimal handicap.[31]

Theorem 12 *1. If the procuring firm's valuation difference d is equal to the supplying firms' cost parameter a, then the optimal handicap e^* is equal to d (Figure 11).*

---

[30] It is the optimal auction if trade trade only occurs in the case that θ_1 is below $\min\left\{\frac{m+1}{m}c_1^{low}, \frac{m+1}{m}\left(c_2^{low}+a\right)+d-a\right\}$.

[31] Note that for optimal auctions it is true that the lowest bidder from group 1 wins if $c_1^{low} < c_2^{low} + a$ and the lowest bidder from group 2 wins if $c_1^{low} > c_2^{low} + d$.

2. *If the procuring firm's valuation difference d is* not *equal to the supplying firms' cost parameter a, then the optimal handicap e^* lies between a and d, i.e. $e^* \in \left]\min\{a,d\}, \max\{a,d\}\right[$ (Figure 12).*

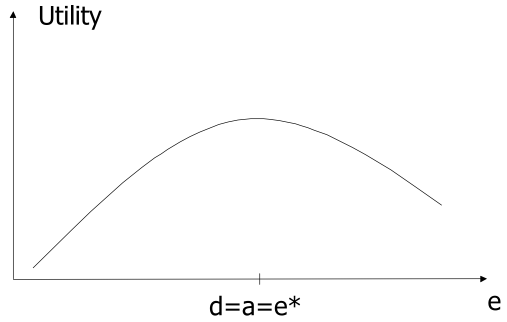

Figure 11: $d = a = e^*$

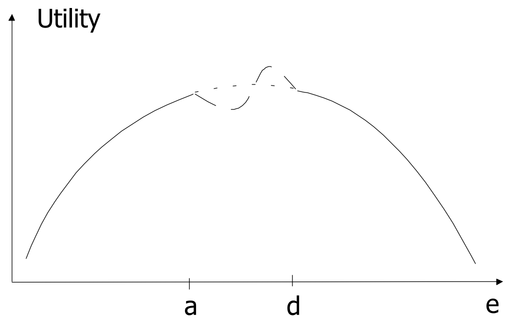

Figure 12: $a < d$

To further characterize the optimal handicap, we introduce the additional assumption that $\frac{F(c)}{f(c)}$ is not only increasing in c, but also convex.

Example 9 *Numerous distributions fulfill the assumption that $\frac{F(c)}{f(c)}$ is increasing and convex like the truncated logistic distribution, the truncated exponential distribution and the truncated Weibull distribution. The trun-*

cated versions of the distribution functions on the finite support $[\underline{c}, \overline{c}]$ *are given by*

$$F^{trunc}(c) = \begin{cases} 0 & \textit{if } c \le \underline{c}, \\ \frac{F(c)-F(\underline{c})}{F(\overline{c})-F(\underline{c})} & \textit{if } \underline{c} < c < \overline{c}, \\ 1 & \textit{if } c > \overline{c}. \end{cases}$$

The following theorem gives a sufficient condition for the uniqueness of the optimal handicap.

Theorem 13 *If* $a < d$ *then the optimal handicap* e^* *is unique if*

$$\int_{\underline{c}-a+e^*}^{\overline{c}} \Bigg(\underbrace{\widetilde{M}(c,e^*)\, f_{(n_1:n_1)}(c)\, f_{(n_2:n_2)}(c+a-e^*)}_{(I)} \underbrace{\left[\frac{(n_2-1)\, f(c+a-e^*)}{1-F(c+a-e^*)} - \frac{f'(c+a-e^*)}{f(c+a-e^*)} - 1\right]}_{(II)} \Bigg) dc \le 0.$$

It means that (II) *is small for small* c *and big for large* c.

If the term (II) was equal to one, the expression would be the first order condition evaluated at the optimum and thus zero. The term (II) modifies the weights in the first order condition. Recall that (I) looks like in Figure 11 (without the two positive densities $f_{(n_1:n_1)}$ and $f_{(n_2:n_2)}$). For the second order condition to hold, (II) has to put only little weight on small c (where (I) is greater than zero) and more weight on large c (where (I) is below zero). A change in n_2 has numerous effects, like a change in the optimal handicap e^*, a modification of the densities and a change in (II). The overall effect is thus hard to determine.

Next, we analyze the effects of more bidders on the optimal handicap e^*. In Example 8 changes in n_1 or n_2 do not change the optimal handicap. This is not true in general. As the next theorem states, the optimal handicap e^* tends to increase if the number of bidders is increased in any of the two bidder groups.

Theorem 14 *If $a < d$ and if f' is bounded by some real number on the interval $[\underline{c}, \overline{c} + a - e^*]$, where e^* is the unique solution of the first order condition, then there exists an N such that e^* is increasing in n_1 and n_2 for all $n_1, n_2 > N$.*

The minimal number of bidders N, required for the comparative static results, depends on the density f. As a rule of thumb it is true that the higher the bound on f' the higher N must be. If f' is below zero everywhere then N is equal to one.

Example 10 *For the truncated exponential distribution the minimal number is $N = 1$. This follows from Theorem 14 since $f'(c) = -\lambda^2 e^{-\lambda x} < 0$.*

4.5 Conclusion

We analyzed a procurement setting in which a single buyer plans to buy one of two goods. We showed that the widely applied practice of holding two separate auctions and afterwards selecting the relevant auction is inferior to an efficient modified auction. In such a modified auction one group of bidders is handicapped and the result of the auction is not only the price but also the decision which variety to buy. The result of the modified auction can be further improved by looking at the optimal auction for this setting. Unfortunately, it is difficult to implement in real life. As a remedy, constrained optimal auctions, where only an additive handicap to bids is allowed, were introduced. The optimal handicap was characterized and a condition for uniqueness was given. Moreover, it was shown that an increase in the number of bidders tends to increase the optimal handicap, although it always remained below the efficient level.

Appendix A

Appendix Chapter 2: Private Discounting in Seq. Auctions

The following technical lemmata are of great importance throughout the discussion. The first deals with the monotonicity of the function that describes the difference of a bidder's valuation between two periods. Depending on the domain of bidders' types this function is increasing, decreasing or neither.

Lemma A.1 *Let i and j be integers with $2 \leq i < j \leq k$. We analyze the behavior of $x^{i-1} - x^{j-1}$ as a function of x.*

1. *On the domain $\left[0, \frac{1}{2}\right]$, the function $x^{i-1} - x^{j-1}$ is increasing in x.*
2. *On the domain $\left[\frac{k-2}{k-1}, 1\right]$, the function $x^{i-1} - x^{j-1}$ is decreasing in x.*
3. *On the domain $\left[\frac{1}{2}, \frac{k-2}{k-1}\right]$, the function $x^{i-1} - x^{j-1}$ can be either increasing or decreasing in x, depending on i and j.*

Proof of Lemma A.1:
The function $x^{i-1} - x^{j-1}$ has a unique extremum in $[0, 1]$. It is a maximum and the maximizer is $x^{\max} = \left(\frac{i-1}{j-1}\right)^{\frac{1}{j-i}}$. Note that $x^{\max}$ is increasing both

in i and in j, since $\ln(y) + \frac{1}{y} > 1$ for all $y > 0$ and $y \neq 1$. The lowest value $x^{\max}$ can take is given by $x^{\max} = \frac{1}{2}$ for $i = 2$ and $j = 3$. The highest value is given by $x^{\max} = \frac{k-2}{k-1}$ for $j = k$ and $i = k - 1$. ■

The next lemma shows how properties of the function $x^{i-1} - x^{j-1}$ translate into conditions about which bidder's type is to win which auction in a welfare maximizing environment.

Lemma A.2 *Consider the ith and the jth auction with $2 \leq i < j \leq k$. On the interval $\left[0, \left(\frac{i-1}{j-1}\right)^{\frac{1}{j-i}}\right]$, efficiency requires that the bidder's type winning auction i must be higher than the one winning auction j (note $\delta^{i-1} - \delta^{j-1}$, as a function of δ, is increasing on this interval). On the interval $\left[\left(\frac{i-1}{j-1}\right)^{\frac{1}{j-i}}, 1\right]$, the bidder's type winning auction i must be smaller than the one winning auction j (note $\delta^{i-1} - \delta^{j-1}$, as a function of δ, is decreasing on this interval).*

Proof of Lemma A.2:
We show: If $\delta^{i-1} - \delta^{j-1}$ is increasing (decreasing) then the type winning the ith auction must be higher (lower) than the type winning the jth auction (in an efficient allocation). We give the proof for the case that $\delta^{i-1} - \delta^{j-1}$ is increasing. The other case is a direct analog.
Consider the welfare generated in two auctions i and j if the winning types are $\widetilde{\delta}$ and $\widehat{\delta}$ with $\widehat{\delta} > \widetilde{\delta}$: On the one hand, if $\widetilde{\delta}$ wins the ith and $\widehat{\delta}$ wins the jth auction, welfare is $\left(\widetilde{\delta}\right)^{i-1} + \left(\widehat{\delta}\right)^{j-1}$. On the other hand, if $\widetilde{\delta}$ wins the jth and $\widehat{\delta}$ wins the ith auction welfare is $\left(\widehat{\delta}\right)^{i-1} + \left(\widetilde{\delta}\right)^{j-1}$. Since $\delta^{i-1} - \delta^{j-1}$ is increasing, we have

$$\left(\widehat{\delta}\right)^{i-1} - \left(\widehat{\delta}\right)^{j-1} > \left(\widetilde{\delta}\right)^{i-1} - \left(\widetilde{\delta}\right)^{j-1}$$

and equivalently

$$\left(\widehat{\delta}\right)^{i-1} + \left(\widetilde{\delta}\right)^{j-1} > \left(\widetilde{\delta}\right)^{i-1} + \left(\widehat{\delta}\right)^{j-1}.$$

Hence, welfare is larger if the higher type wins the ith and the lower type the jth auction. ■

Proof of Lemma 1:
Welfare is given by $1+\delta_2^*+(\delta_3^*)^2+(\delta_4^*)^3+\ldots+(\delta_k^*)^{k-1}$, where δ_i^* denotes the discount factor of the bidder who wins the ith auction, $i=2\ldots k$. For welfare maximization the buyers with the highest discount factors have to win the auctions $i=2\ldots k$, i.e. $\delta_i^*\in\left\{\delta_{(1)},\ldots,\delta_{(k-1)}\right\}$. The optimal sequence depends on the discount factors' support. Consider two periods l and j with $2\leq l<j\leq k$:

1. On the domain $\left[0,\frac{1}{2}\right]$, $\delta^{l-1}-\delta^{j-1}$ (as a function of δ) is increasing. Lemma A.2 implies that the type winning auction l is larger than the type winning auction j, i.e. $\delta_l^*>\delta_j^*$, for all $l<j$.
2. On the domain $\left[\frac{k-2}{k-1},1\right]$, $\delta^{l-1}-\delta^{j-1}$ (as a function of δ) is decreasing. By Lemma A.2 the type winning auction l is smaller than the type winning auction j, i.e. $\delta_l^*<\delta_j^*$, for all $l<j$.
3. On the domain $\left[\frac{1}{2},\frac{k-2}{k-1}\right]$, $\delta^{l-1}-\delta^{j-1}$ (as a function of δ) can be both increasing or decreasing, depending on l and j (and is neither increasing nor decreasing for all l and j). Hence, it is neither true that $\delta_l^*<\delta_j^*$ nor $\delta_l^*>\delta_j^*$ for all $l<j$.

■

Proof of Theorem 1:
Since $\delta-\delta^2$ (as a function of δ) is decreasing on $\left[\frac{1}{2},1\right]$, it is efficient that the winning type in the second auction is lower than the winning type in the third auction, i.e. $\delta_2^*<\delta_3^*$. Hence, the bidding function in the second auction cannot be increasing. As Lemma 1 states it cannot be decreasing since $n>k$ and only the buyers with discount factors $\delta_{(i)}$ for $i=1,\ldots,k-1$ win the auctions in periods $2,\ldots,n$. Therefore, the bidding function in the second period must have a maximum in the interval $\left]\frac{1}{2},1\right[$. We denote the corresponding maximizer by $\widehat{\delta}$. Suppose the realization of types is such that all types are above $\widehat{\delta}$, i.e. $\widehat{\delta}<\delta_{(k)}<\delta_{(k-1)}$. For $\delta_{(k)}$ close to $\widehat{\delta}$ the buyer with

type $\delta_{(k)}$ will submit the highest bid in the second period and consequently win – which is not efficient. Hence, there is no bidding function that ensures efficiency regardless of the realization of types. ■

Proof of Lemma 2:
Consider three auctions with indices i, j, $o \in \mathbb{N}$ with $2 \leq i < j < o \leq k$. For the moment, we focus on the interval $\left[\left(\frac{i-1}{j-1}\right)^{\frac{1}{j-i}}, \left(\frac{i-1}{o-1}\right)^{\frac{1}{o-i}}\right]$. The function $\delta^{i-1} - \delta^{j-1}$ is decreasing in δ on this domain. By Lemma A.2 efficiency requires that the type winning auction i must be smaller than the one winning auction j, i.e. $\delta_i^* < \delta_j^*$. Moreover, the function $\delta^{i-1} - \delta^{o-1}$ is increasing on this domain. Thus, the type winning auction i must be bigger than the one winning auction o, i.e. $\delta_i^* > \delta_o^*$. Together, it must be true on this interval that the type winning the first of the three auctions (index i) must be between the type winning auctions o and j, i.e. $\delta_o^* < \delta_i^* < \delta_j^*$. That cannot be achieved by a symmetric bidding function for all possible realizations: No symmetric bidding function always results in a winning 'middle type' bidder - regardless of the actual realization of types (see also the proof of Theorem 1). Therefore, efficiency is impossible to ensure on any interval of the form $\left[\left(\frac{i-1}{j-1}\right)^{\frac{1}{j-i}}, \left(\frac{i-1}{o-1}\right)^{\frac{1}{o-i}}\right]$. Given i, this interval's length is maximized by selecting j as low and o as high as possible (since Lemma 1 states that the maximizer $x^{\max}$ is increasing in both parameters), resulting in $\left[\left(\frac{i-1}{i}\right)^{\frac{1}{j-i}}, \left(\frac{i-1}{k-1}\right)^{\frac{1}{n-1-i}}\right]$. Defining $l = i$ and using $n = k$ results in the stated formula. ■

Proof of Theorem 2:
Let $v_l\left(\delta; x_1, \dots, x_{n-l}\right)$ denote the expected utility of a buyer with type δ who finds himself in period l of the auction, given his remaining opponents have types $x_1 \dots x_{n-l}$ and announce their types truthfully. If $x_i > \delta$ for all $i = 1 \dots n-l$, the δ type bidder wins the lth auction and we have $v_l\left(\delta; x_1 \dots x_{n-l}\right)) = \delta^{l-1} - b_l\left(\min\left\{x_1, \dots x_{n-l}\right\}\right)$. To simplify notation we use $f\left(x_i\right) dx_i\big|_{i=n-l\dots2}$, which stands for $f\left(x_{n-l}\right) dx_{n-l} \dots f\left(x_2\right) dx_2$. Note that the order of integration is given by the order the index runs.

Assume that all bidders other than i are bidding according to the bidding function b_l in all periods $l = 1, \ldots, n$. We show by backward induction that it is optimal for i to bid according to the bidding function $b_l(\delta_i)$ as well. In the last period the auction is a standard second-price auction with the symmetric equilibrium that the remaining bidder bids his own valuation, which is for a type δ in the last period $b_n(\delta) = \delta^{n-1}$. The optimality for previous periods is shown for two different cases.

Case 1 *In period $l-1$ bidder i submitted a bid of $b_{l-1}\left(\widetilde{\delta}\right)$, which was the second highest bid in that period and determined the price paid in that period. Therefore, bidder i knows that all bidders remaining in period l have a type higher than $\widetilde{\delta}$. This case also covers period $l = 1$ where no previous prices exist (which is covered if $\widetilde{\delta} = 0$).*

Case 2 *In period $l-1$ a different bidder than bidder i submitted the second highest bid. Since the price was announced, bidder i knows the lowest of the other bidders' types who are still in the auction.*

Case 1:

Note that the expected utility of bidder i in period l (if he is still in the auction) does only depend on his type δ. His bid in period l is given by $b_l(\delta)$, whereas his bid in period $l-1$ is given by $b_{l-1}\left(\widetilde{\delta}\right)$.

Bidder i's expected utility in period l if he

- has a discount factor δ,
- bids (in period l) as if it were $\widehat{\delta}$,
- submitted $b_{l-1}\left(\widetilde{\delta}\right)$ in period $l-1$,

is given by

$$
\begin{aligned}
&U_l\left(\delta,\widehat{\delta},\widetilde{\delta}\right) = \\
&\quad \frac{n-l}{\left(1-F\left(\widetilde{\delta}\right)\right)^{n-l}}\left[\int_{\widehat{\delta}}^{1}\left[\delta^{l-1}-b_l\left(x_1\right)\right]\left(1-F\left(x_1\right)\right)^{n-l-1}f\left(x_1\right)dx_1\right. \\
&\quad +\int_{\widetilde{\delta}}^{\widehat{\delta}}\int_{\max\{x_1,\delta\}}^{1}\cdots\int_{\max\{x_1,\delta\}}^{1}v_{l+1}\left(\delta;x_2,\ldots,x_{n-l}\right)f\left(x_i\right)dx_i\Big|_{i=n-l,\ldots,1} \\
&+\left(n-l-1\right)\int_{\widetilde{\delta}}^{\widehat{\delta}}\int_{x_1}^{\max\{x_1,\delta\}}\int_{x_2}^{1}\cdots\int_{x_2}^{1}v_{l+1}\left(\delta;x_2,\ldots,x_{n-l}\right)f\left(x_i\right)dx_i\Big|_{i=n-l,\ldots,1}\left.\vphantom{\int}\right].
\end{aligned}
$$

The first additive term describes the case where bidder i wins period l, when there are $(n-l)$ other bidders still present. The second additive term stands for the case where bidder i does not win period l but wins period $l+1$. The last additive term represents the case where bidder i does neither win period l nor period $l+1$.

The first order condition of the optimization problem is given by

$$
\left.\frac{\partial U_l\left(\delta,\widehat{\delta},\widetilde{\delta}\right)}{\partial\widehat{\delta}}\right|_{\widehat{\delta}=\delta}=0,
$$

i.e.

$$
\begin{aligned}
&-\frac{n-l}{\left(1-F\left(\widetilde{\delta}\right)\right)^{n-l}}f\left(\delta\right)\left[\delta^{l-1}-b_l\left(\delta\right)\right]\left(1-F\left(\delta\right)\right)^{n-l-1} \\
&+\frac{n-l}{\left(1-F\left(\widetilde{\delta}\right)\right)^{n-l}}f\left(\delta\right)\int_{\delta}^{1}\cdots\int_{\delta}^{1}v_{l+1}\left(\delta;x_2,\ldots,x_{n-l}\right)f\left(x_i\right)dx_i\Big|_{i=n-l,\ldots,2} \\
=\;&0.
\end{aligned}
$$

Solving for $b_l(\delta)$ yields

$$\begin{aligned}
& b_l(\delta) && (1)\\
= \; & \delta^{l-1} - \frac{1}{(1-F(\delta))^{n-l-1}} \int_\delta^1 \cdots \int_\delta^1 v_{l+1}(\delta; x_2, \ldots, x_{n-l})\, f(x_i)\, dx_i|_{i=n-l,\ldots,2} \\
= \; & \delta^{l-1} - E\left[\delta^l - b_{l+1}\left(\delta_{(n-l-1)}\right) \big| \delta_{(n-l)} = \delta\right].
\end{aligned}$$

To show that this solution constitutes a global maximum we show

$$\frac{\partial}{\partial \widehat{\delta}} U_l\left(\delta, \widehat{\delta}, \widetilde{\delta}\right) \begin{cases} > 0 & \text{for} \quad \widehat{\delta} < \delta, \\ < 0 & \text{for} \quad \widehat{\delta} > \delta. \end{cases}$$

We use the following form of the derivative

$$\begin{aligned}
& \frac{\left(1 - F\left(\widetilde{\delta}\right)\right)^{n-l}}{f\left(\widehat{\delta}\right)(n-l)} \frac{\partial}{\partial \widehat{\delta}} U_l\left(\delta, \widehat{\delta}, \widetilde{\delta}\right) && (2)\\
= \; & -\left[\delta^{l-1} - b_l\left(\widehat{\delta}\right)\right]\left(1 - F\left(\widehat{\delta}\right)\right)^{n-l-1} \\
& + \int_{\max\left\{\widehat{\delta}, \delta\right\}}^1 \cdots \int_{\max\left\{\widehat{\delta}, \delta\right\}}^1 v_{l+1}(\delta; x_2, \ldots, x_{n-l})\, f(x_i)\, dx_i|_{i=n-l,\ldots,2} \\
& + (n-l-1) \int_{\widehat{\delta}}^{\max\left\{\widehat{\delta}, \delta\right\}} \int_{x_2}^1 \cdots \int_{x_2}^1 v_{l+1}(\delta; x_2, \ldots, x_{n-l})\, f(x_i)\, dx_i|_{i=n-l,\ldots,2}.
\end{aligned}$$

Since $\frac{\left(1-F\left(\widetilde{\delta}\right)\right)^{n-l}}{f\left(\widehat{\delta}\right)(n-l)} > 0$, it remains to determine the sign of (2). Using equation (1) for b_l gives

$$\begin{aligned}
&\frac{\left(1-F\left(\widetilde{\delta}\right)\right)^{n-l}}{f\left(\widehat{\delta}\right)(n-l)}\frac{\partial}{\partial\widehat{\delta}}U_l\left(\delta,\widehat{\delta},\widetilde{\delta}\right) \qquad (3)\\
= & -\int_{\widehat{\delta}}^{1}\cdots\int_{\widehat{\delta}}^{1}\left[\delta^{l-1}-\widehat{\delta}^{l-1}+v_{l+1}\left(\widehat{\delta};x_2,\ldots,x_{n-l}\right)\right]f\left(x_i\right)dx_i|_{i=n-l,\ldots,2}\\
& +\int_{\max\left\{\widehat{\delta},\delta\right\}}^{1}\cdots\int_{\max\left\{\widehat{\delta},\delta\right\}}^{1}v_{l+1}\left(\delta;x_2,\ldots,x_{n-l}\right)f\left(x_i\right)dx_i|_{i=n-l,\ldots,2}\\
& +(n-l-1)\int_{\widehat{\delta}}^{\max\left\{\widehat{\delta},\delta\right\}}\int_{x_2}^{1}\cdots\int_{x_2}^{1}v_{l+1}\left(\delta;x_2,\ldots,x_{n-l}\right)f\left(x_i\right)dx_i|_{i=n-l,\ldots,2}.
\end{aligned}$$

Consider the **sub-case** $\widehat{\delta} > \delta$ where we have to show that $\frac{\partial}{\partial\widehat{\delta}}U_l\left(\delta,\widehat{\delta},\widetilde{\delta}\right) < 0$.

By using (3) this is equivalent to

$$\begin{aligned}
-\int_{\widehat{\delta}}^{1}\cdots\int_{\widehat{\delta}}^{1}&\left[\delta^{l-1}-\widehat{\delta}^{l-1}+v_{l+1}\left(\widehat{\delta};x_2,\ldots,x_{n-l}\right)\right.\\
&\left.-v_{l+1}\left(\delta;x_2,\ldots,x_{n-l}\right)\right]f\left(x_i\right)dx_i|_{i=n-l,\ldots,2}<0. \qquad (4)
\end{aligned}$$

Note that all other bidders have a discount factor bigger than $\widehat{\delta}$. Hence, types δ and $\widehat{\delta}$ would win period l. The expected utilities are

$$v_{l+1}\left(\widehat{\delta};x_2,\ldots,x_{n-l}\right)=\widehat{\delta}^{l}-b_{l+1}\left(\min\left\{x_1,\ldots x_{n-l-1}\right\}\right)$$

and

$$v_{l+1}\left(\delta;x_2,\ldots,x_{n-l}\right)=\delta^{l}-b_{l+1}\left(\min\left\{x_1,\ldots x_{n-l-1}\right\}\right).$$

Equation (4) therefore becomes

$$\int_{\widehat{\delta}}^{1}\cdots\int_{\widehat{\delta}}^{1}\left[\left(\widehat{\delta}^{l-1}-\widehat{\delta}^{l}\right)-\left(\delta^{l-1}-\delta^{l}\right)\right]f\left(x_i\right)dx_i|_{i=n-l,\ldots,2}<0. \qquad (5)$$

Since the function $\delta^{l-1}-\delta^{l}$ is decreasing in the relevant interval (see Lemma A.1), this implies $\left(\widehat{\delta}^{l-1}-\widehat{\delta}^{l}\right)<\left(\widehat{\delta}^{l-1}-\widehat{\delta}^{l}\right)$. Hence, (5) is true.

The **sub-case** $\widehat{\delta}<\delta$ is shown by induction over the periods. For $l=n-1$ equation (3) becomes $\widehat{\delta}^{n-2}-\widehat{\delta}^{n-1}-\left(\delta^{n-2}-\delta^{n.-1}\right)$, which is positive.

For any period $l<n-1$, we have to show that equation (3) is positive, i.e.

$$\begin{aligned}
&\int_{\delta}^{1}\cdots\int_{\delta}^{1}\left[\widehat{\delta}^{l-1}-\delta^{l-1}-v_{l+1}\left(\widehat{\delta};x_2,\ldots,x_{n-l}\right)\right.\\
&\qquad\qquad\left.+v_{l+1}\left(\delta;x_2,\ldots,x_{n-l}\right)\right]f\left(x_i\right)dx_i|_{i=n-l,\ldots,2}\\
&\quad+\left(n-l-1\right)\int_{\widehat{\delta}}^{\delta}\int_{x_2}^{1}\cdots\int_{x_2}^{1}\left[\widehat{\delta}^{l-1}-\delta^{l-1}-v_{l+1}\left(\widehat{\delta};x_2,\ldots,x_{n-l}\right)\right.\\
&\qquad\qquad\qquad\left.+v_{l+1}\left(\delta;x_2,\ldots,x_{n-l}\right)\right]f\left(x_i\right)dx_i|_{i=n-l,\ldots,2}>0. \quad (6)
\end{aligned}$$

In the first integral of (6) the x_i, $i=2,\ldots,n-l$, are always bigger than δ and $\widehat{\delta}$. Therefore, we have $v_{l+1}\left(\widehat{\delta};x_2,\ldots,x_{n-l}\right)=\widehat{\delta}^{l}-b_{l+1}\left(\min\left\{x_1,\ldots x_{n-l-1}\right\}\right)$ and $v_{l+1}\left(\delta;x_2,\ldots,x_{n-l}\right)=\delta^{l}-b_{l+1}\left(\min\left\{x_1,\ldots x_{n-l-1}\right\}\right)$. Hence, (6) becomes

$$\begin{aligned}
&\int_{\delta}^{1}\cdots\int_{\delta}^{1}\left[\widehat{\delta}^{l-1}-\widehat{\delta}^{l}-\left(\delta^{l-1}-\delta^{l}\right)\right]f\left(x_i\right)dx_i|_{i=n-l,\ldots,2}\\
&\quad+\left(n-l-1\right)\int_{\widehat{\delta}}^{\delta}\int_{x_2}^{1}\cdots\int_{x_2}^{1}\left[\widehat{\delta}^{l-1}-\delta^{l-1}-v_{l+1}\left(\widehat{\delta};x_2,\ldots,x_{n-l}\right)\right.\\
&\qquad\qquad\qquad\left.+v_{l+1}\left(\delta;x_2,\ldots,x_{n-l}\right)\right]f\left(x_i\right)dx_i|_{i=n-l,\ldots,2}>0.
\end{aligned}$$

Since the function $\delta^{l-1}-\delta^{l}$ is decreasing on the relevant interval, the first integral is positive. Thus, it remains to show that the second term is also positive. Note that $\widehat{\delta}<x_i$ for $i=2,\ldots,n-l$, and x_2 denotes the lowest

of the other bidders' types (remaining in the auction). Hence, we have

$$\begin{aligned}
& v_{l+1}\left(\widehat{\delta}; x_2, \ldots, x_{n-l}\right) \\
= {} & \widehat{\delta}^{l} - b_{l+1}\left(x_2\right) \\
= {} & \widehat{\delta}^{l} - x_2^l \\
& + \frac{1}{\left(1-F\left(x_2\right)\right)^{n-l-2}} \int_{x_2}^{1} \cdots \int_{x_2}^{1} v_{l+1}\left(x_{2;}\widehat{x}_3, \ldots, \widehat{x}_{n-l}\right) f\left(\widehat{x}_i\right) d\widehat{x}_i\big|_{i=n-l,\ldots,3}.
\end{aligned}$$

By construction of the integral we have $\delta > x_2$ and therefore the x_2 bidder will win period $l+1$, which implies $v_{l+1}\left(\delta; x_2, \ldots, x_{n-l}\right) = v_{l+2}\left(\delta; x_3, \ldots, x_{n-l}\right)$. Together, it remains to show

$$\begin{aligned}
& (n-l-1) \int_{\widehat{\delta}}^{\delta} \int_{x_2}^{1} \cdots \int_{x_2}^{1} \left[\widehat{\delta}^{l-1} - \widehat{\delta}^{l} - \delta^{l-1} + x_2^l\right. \\
& \left. - v_{l+1}\left(x_{2;}x_3, \ldots, x_{n-l}\right) + v_{l+2}\left(\delta; x_3, \ldots, x_{n-l}\right)\right] f\left(x_i\right) dx_i\big|_{i=n-l,\ldots,2} > 0. \quad (7)
\end{aligned}$$

Since the function $\delta^{l-1} - \delta^l$ is decreasing, we have $\widehat{\delta}^{l-1} - \widehat{\delta}^{l} - \delta^{l-1} > -\delta^l$, which results in

$$\begin{aligned}
& (7) \\
> {} & (n-l-1) \int_{\widehat{\delta}}^{\delta} \int_{x_2}^{1} \cdots \int_{x_2}^{1} \left[x_2^l - \delta^l\right. \\
& \left. - v_{l+1}\left(x_{2;}x_3, \ldots, x_{n-l}\right) + v_{l+2}\left(\delta; x_3, \ldots, x_{n-l}\right)\right] f\left(x_i\right) dx_i\big|_{i=n-l,\ldots,2}
\end{aligned}$$

$$\begin{aligned}
= (n-l-1) \int_{\widehat{\delta}}^{\delta} & \left[\int_{\delta}^{1} \cdots \int_{\delta}^{1} \left[x_2^l - \delta^l - v_{l+1}\left(x_{2;}x_3, \ldots, x_{n-l}\right)\right.\right. \\
& \left. + v_{l+2}\left(\delta; x_3, \ldots, x_{n-l}\right)\right] f\left(x_i\right) dx_i\big|_{i=n-l,\ldots,3} \\
& + (n-l-2) \int_{x_2}^{\delta} \int_{x_3}^{1} \cdots \int_{x_3}^{1} \left[x_2^l - \delta^l - v_{l+1}\left(x_{2;}x_3, \ldots, x_{n-l}\right)\right. \\
& \left.\left. + v_{l+2}\left(\delta; x_3, \ldots, x_{n-l}\right)\right] f\left(x_i\right) dx_i\big|_{i=n-l,\ldots,3}\right] f\left(x_2\right) dx_2.
\end{aligned}$$

The integrand of the outer integral is greater than zero by induction (it is equation (6) for one period later) since $x_2 < \delta$. This ends the sub-case $\widehat{\delta} < \delta$.

Case 2:
Assume y to be the lowest of the rivals' types. This is known to bidder i, since it can be inferred from the announced price of the previous period $l-1$. We show that for $\delta < y$ it is optimal for bidder i to win period l. This is done by bidding according to b_l. If $\delta > y$ bidder i finds it optimal not to win period l, which is achieved by bidding according to b_l as well. If bidder i wins period l, his profit is given by

$$\delta^{l-1} - b_l\left(y\right).$$

If he does not win this period, his profit can be written as (by splitting up the integral)

$$\frac{1}{\left(1-F\left(y\right)\right)^{n-l-1}} \int\limits_{\max\{y,\delta\}}^{1} \cdots \int v_{l+1}\left(\delta; x_2, \ldots, x_{n-l}\right) f\left(x_i\right) dx_i|_{i=n-l,\ldots,2}$$
$$+\frac{(n-l-1)}{\left(1-F\left(y\right)\right)^{n-l-1}} \int_y^{\max\{y,\delta\}} \int\limits_{x_2}^{1} \cdots \int v_{l+1}\left(\delta; x_2, \ldots, x_{n-l}\right) f\left(x_i\right) dx_i|_{i=n-l,\ldots,2}.$$

Hence, the difference in utility is

$$\begin{aligned}
&\delta^{l-1} - b_l\left(y\right) &\quad (8)\\
&-\frac{1}{\left(1-F\left(y\right)\right)^{n-l-1}} \int\limits_{\max\{y,\delta\}}^{1} \cdots \int v_{l+1}\left(\delta; x_2, \ldots, x_{n-l}\right) f\left(x_i\right) dx_i|_{i=n-l,\ldots,2}\\
&-\frac{(n-l-1)}{\left(1-F\left(y\right)\right)^{n-l-1}} \int_y^{\max\{y,\delta\}} \int\limits_{x_2}^{1} \cdots \int v_{l+1}\left(\delta; x_2, \ldots, x_{n-l}\right) f\left(x_i\right) dx_i|_{i=n-l,\ldots,2}.
\end{aligned}$$

The sign is just the opposite of (2) if we set $\widehat{\delta} = y$. Therefore, we have already proved (in case 1) that (8) is negative for $y < \delta$ and positive for $y > \delta$. This shows that bidder i prefers to lose in the lth period if his type is greater than y, which he achieves by bidding $b_l\left(\delta\right)$. If $y > \delta$, he prefers to win period l. This again is achieved by bidding according to b_l. ■

Proof of Corollary 2:
For any $j > 1$ denote the distribution function for the $(j-1)$th highest order-statistic, given the realization $\delta_{(j)} = \delta$, by $F_{(j-1)|\delta}$. Due to independence of types this is equal to the distribution function of the lowest order statistic with $(j-1)$ bidders on $[\delta, 1]$

$$F_{(j-1)|\delta}(x) = 1 - \left(\frac{1-F(x)}{1-F(\delta)}\right)^{j-1}. \tag{9}$$

For $\delta < 1$, $F_{(j-1)|\delta}(x)$ is decreasing in δ for every $x \in [\delta, 1)$, i.e. for $\widehat{\delta} < \widetilde{\delta} < 1$ the distribution function $F_{(j-1)|\widetilde{\delta}}$ dominates (according to first order stochastic dominance) the distribution function $F_{(j-1)|\widehat{\delta}}$.
Note that for any decreasing and positive function m we have (due to partial integration)

$$\int_{\widehat{\delta}}^{1} m(x)\, dF_{(j-1)|\widehat{\delta}}(x) > \int_{\widetilde{\delta}}^{1} m(x)\, dF_{(j-1)|\widetilde{\delta}}(x).$$

The statement is shown by induction. The bidding function $b_{n-1}(\delta) = \delta^{n-2} - \delta^{n-1}$ is decreasing (by Lemma A.1). Hence, $E\left[b_{l+1}\left(\delta_{(n-l-1)}\right) | \delta_{(n-l)} = \delta\right]$ is decreasing in δ. Consequently, $b_l(\delta) = \delta^{l-1} - \delta^l + E\left[b_{l+1}\left(\delta_{(n-l-1)}\right) | \delta_{(n-l)} = \delta\right]$ is decreasing in δ. ■

Proof of Corollary 3:
The proof is done by induction. In period $n-1$ the bidding function is $b_{n-1}(\delta) = \delta^{n-2} - \delta^{n-1}$. The induction hypothesis is

$$b_{l+1}(\delta) = \delta^l - \delta^{l+1} + E\left[\sum_{i=l+1}^{n-2} \left(\delta^i_{(n-1-i)} - \delta^{i+1}_{(n-1-i)}\right) | \delta_{(n-l-1)} = \delta\right].$$

In period l the bidding function is

$$\begin{aligned}
b_l(\delta) &= \delta^{l-1} - E\left[\delta^l - b_{l+1}\left(\delta_{(n-l-1)}\right)|\delta_{(n-l)} = \delta\right] \\
&= \delta^{l-1} - \delta^l + E\left[\delta^l_{(n-l-1)} - \delta^{l+1}_{(n-l-1)}|\delta_{(n-l)} = \delta\right] \\
&\quad +E\left[E\left[\sum_{i=l+1}^{n-2}\left(\widetilde{\delta}^i_{(n-1-i)} - \widetilde{\delta}^{i+1}_{(n-1-i)}\right)\middle|\widetilde{\delta}_{(n-l-1)} = \delta_{(n-l-1)}\right]|\delta_{(n-l)} = \delta\right] \\
&= \delta^{l-1} - \delta^l + E\left[\sum_{i=l}^{n-2}\left(\delta^i_{(n-1-i)} - \delta^{i+1}_{(n-1-i)}\right)|\delta_{(n-l)} = \delta\right].
\end{aligned}$$

■

Proof of Theorem 3:
Let the price in period $l = 1, \ldots, n-1$ be determined by the bidder with type δ, whose type is common knowledge in period $l+1$, i.e.

$$\begin{aligned}
p_l &= b_l(\delta) \\
&= \delta^{l-1} - \delta^l + E\left[b_{l+1}\left(\widetilde{\delta}_{(n-l-1)}\right)\middle|\widetilde{\delta}_{(n-l)} = \delta\right].
\end{aligned}$$

Then

$$E\left[p_{l+1}|p_l\right] = E\left[b_{l+1}\left(\widetilde{\delta}_{(n-l-1)}\right)\middle|\widetilde{\delta}_{(n-l)} = \delta\right]$$

is the expected price in $l+1$ given p_l. ■

Proof of Corollary 4:
For any period $l = 1, \ldots, n-1$ the expected price is given by

$$\begin{aligned}
E\left[p_l\right] &= E\left[b_l\left(\delta_{(n-l)}\right)\right] \\
&= E\left[\delta^{l-1}_{(n-l)} - \delta^l_{(n-l)} + E\left[b_{l+1}\left(\widetilde{\delta}_{(n-l-1)}\right)\middle|\widetilde{\delta}_{(n-l)} = \delta_{(n-l)}\right]\right] \\
&= E\left[b_{l+1}\left(\delta_{(n-l-1)}\right)\right] + E\left[\delta^{l-1}_{(n-l)} - \delta^l_{(n-l)}\right] \\
&= E\left[p_{l+1}\right] + E\left[\delta^{l-1}_{(n-l)} - \delta^l_{(n-l)}\right].
\end{aligned}$$

■

Proof of Theorem 4:
It is sufficient to show the statement for the bidder with the lowest δ still in the auction $\left(\delta = \delta_{(n-l)}\right)$, i.e. the bidder who sets the price in period l and will win period $l+1$. Using Theorem 3, the discounted price in period l can be written as

$$\begin{aligned}\delta p_l &= \delta\left(\delta^{l-1} - \delta^l + E\left[p_{l+1}|p_l\right]\right)\\ &= \delta^l\left(1-\delta\right) + \delta E\left[b_{l+1}\left(\delta_{(n-l-1)}\right)\left|\delta_{(n-l)} = \delta\right.\right].\end{aligned}$$

Hence, we have to show that

$$\delta^l > E\left[b_{l+1}\left(\delta_{(n-l-1)}\right)\left|\delta_{(n-l)} = \delta\right.\right].$$

Using the non-recursive form of b_{l+1}, we get

$$\begin{aligned}&E\left[b_{l+1}\left(\delta_{(n-l-1)}\right)\left|\delta_{(n-l)} = \delta\right.\right]\\ =\ &E\left[\sum_{i=l}^{n-2}\left(\delta^i_{(n-1-i)} - \delta^{i+1}_{(n-1-i)}\right)\left|\delta_{(n-l)} = \delta\right.\right].\end{aligned}$$

Note that $\delta_{(n-i-1)} \geq \delta_{(n-l)} = \delta$ for $i = l, \dots, n-2$. As shown in Lemma A.1 the function $\delta^{i-1} - \delta^i$ is decreasing for all i if $\delta \in \left]\frac{n-2}{n-1}, 1\right[$. Consequently, it results that $\delta^{i-1} - \delta^i \geq \delta^{i-1}_{(n-i)} - \delta^i_{(n-i)}$ for all i. Therefore, we get

$$\begin{aligned}&E\left[\sum_{i=l}^{n-2}\left(\delta^i_{(n-1-i)} - \delta^{i+1}_{(n-1-i)}\right)\left|\delta_{(n-l)} = \delta\right.\right]\\ \leq\ &\sum_{i=l}^{n-2}\left(\delta^i - \delta^{i+1}\right) = \delta^l - \delta^{n-1} < \delta^l.\end{aligned}$$

■

Proof of Lemma 3:
For the stochastically dominant distribution $\widetilde{F}$ the order statistics are expected to be higher than in the F case. Recall, that the function $\delta^i - \delta^{i+1}$ is decreasing for $\delta \in \left[\frac{n-2}{n-1}, 1\right]$ for all relevant i. Consequently, each term in

the following sum is smaller in the $\widetilde{F}$ case. Thus, the sum is also smaller, *i.e.*

$$\begin{aligned} b_l(\delta) &= \delta^{l-1} - \delta^l + E_F\left[\sum_{i=l}^{n-2}\left(\delta^i_{(n-1-i)} - \delta^{i+1}_{(n-1-i)}\right)\Big|\delta_{(n-l)} = \delta\right] \\ &> \delta^{l-1} - \delta^l + E_{\widetilde{F}}\left[\sum_{i=l}^{n-2}\left(\widetilde{\delta}^i_{(n-1-i)} - \widetilde{\delta}^{i+1}_{(n-1-i)}\right)\Big|\widetilde{\delta}_{(n-l)} = \delta\right] = \widetilde{b}_l(\delta). \end{aligned}$$

■

Proof of Theorem 5:
Part 1:
As noted in Theorem 3, the expected price in period $l+1$, given the price in period l, can be written as

$$E\left[p_{l+1}|p_l = p\right] = p - \left(\delta^{l-1} - \delta^l\right),$$

for $\delta = b_l^{-1}(p)$ being the type that set the price in period l. If the underlying distribution is $\widetilde{F}$, we have $\widetilde{\delta} = \widetilde{b}_l^{-1}(p)$ with $E_{\widetilde{F}}\left[\widetilde{p}_{l+1}|\widetilde{p}_l = p\right] = p - \left(\widetilde{\delta}^{l-1} - \widetilde{\delta}^l\right)$. Since $b_l > \widetilde{b}_l$ and both bidding functions are decreasing, it follows that $\delta > \widetilde{\delta}$ (see Figure 1). Recall, that the function $\delta^i - \delta^{i+1}$ is decreasing for all i. Hence $p - \left(\delta^{l-1} - \delta^l\right) > p - \left(\widetilde{\delta}^{l-1} - \widetilde{\delta}^l\right)$.

Part 2:
Given the stochastic dominance of $\widetilde{F}$ over F, it follows that the distribution for the random variable $\delta_{(n-l)}$ is first order stochastically dominated by the distribution for $\widetilde{\delta}_{(n-l)}$. Since the bidding function b_l is decreasing (if $l < n$) this implies $E\left[b_l\left(\delta_{(n-l)}\right)\right] > E\left[b_l\left(\widetilde{\delta}_{(n-l)}\right)\right]$. Using Lemma 3 gives $E\left[b_l\left(\widetilde{\delta}_{(n-l)}\right)\right] > E\left[\widetilde{b}_l\left(\widetilde{\delta}_{(n-l)}\right)\right]$. Together we obtain

$$E\left[p_l\right] = E\left[b_l\left(\delta_{(n-l)}\right)\right] > E\left[\widetilde{b}_l\left(\widetilde{\delta}_{(n-l)}\right)\right] = E\left[\widetilde{p}_l\right].$$

Part 3:
The proof is similar to the one of part 2: As Corollary 4 states

$$E_F\left[p_l - p_{l+1}\right] = E_F\left[\delta_{(n-l)}^{l-1} - \delta_{(n-l)}^{l}\right].$$

Since the distribution for $\delta_{(n-l)}$ is first order stochastically dominated by the distribution for $\widetilde{\delta}_{(n-l)}$ and since the function $\delta^{l-1} - \delta^{l}$ is decreasing we have

$$\begin{aligned} E_F\left[\delta_{(n-l)}^{l-1} - \delta_{(n-l)}^{l}\right] &> E_{\widetilde{F}}\left[\widetilde{\delta}_{(n-l)}^{l-1} - \widetilde{\delta}_{(n-l)}^{l}\right] \\ &= E_{\widetilde{F}}\left[\widetilde{p}_l - \widetilde{p}_{l+1}\right]. \end{aligned}$$

■

Appendix B

Appendix Chapter 3: Auctions with Impatient Buyers

Proof of Lemma 4:
At time $t = \frac{\overline{p}-r}{e}$ the price of the auction reaches the reservation price r if it is not stopped by any bidder. The bidder's type who is willing to pay r at this time is given by $\left(1-\frac{c}{e}\right) r+\frac{c}{e}\overline{p}$. This bidder is indifferent between stopping at time $t = \frac{\overline{p}-r}{e}$ and not stopping at all. For any bidder with a lower type the price is always above his valuation. ■

Proof of Lemma 5:
To simplify our analysis we give the proof for a reservation price $r = 0$. The proof for the general case works in an analogue way.
We show that the equation

$$\left(\widetilde{\theta}-\overline{p}\right)\frac{F^{n}\left(\widetilde{\theta}\right)-1}{n\left(F\left(\widetilde{\theta}\right)-1\right)}=\int_{\theta_r}^{\widetilde{\theta}} F^{n-1}\left(x\right)dx.$$

has always a unique solution.
Define

$$U_{NP}\left(\theta\right)=\int_{\theta_r}^{\theta} F^{n-1}\left(x\right)dx$$

and

$$U_P(\theta) = (\theta - \overline{p}) \frac{F^n(\theta) - 1}{n(F(\theta) - 1)}.$$

We get $\widetilde{\theta}$ as the solution of

$$U_{NP}\left(\widetilde{\theta}\right) = U_P\left(\widetilde{\theta}\right).$$

Uniqueness and existence of this solution follow because 'low' types $\frac{c}{e}\overline{p} \leq \theta < \overline{p}$ strictly prefer not to stop at time zero. More precisely, we have $U_P(\theta) < 0$ and $U_{NP}(\theta) > 0$. For 'high' types $\theta \in \left[\overline{p}, \overline{\theta}\right)$ we find that the derivative of the utility with respect to the type is greater if a bidder stops at time zero (note that both are positive), i.e.

$$\frac{d}{d\theta}U_P(\theta) \geq \frac{F^n(\theta) - 1}{n(F(\theta) - 1)} = \frac{1}{n}\sum_{j=0}^{n-1} F^j(\theta) > F^{n-1}(\theta) = \frac{d}{d\theta}U_{NP}(\theta).$$

As we are in the case where the starting price is low, i.e. $\overline{p} < \overline{\theta} - \int_{\frac{c}{e}\overline{p}}^{\overline{\theta}} F^{n-1}(x)\,dx$, we find that

$$U_P\left(\overline{\theta}\right) = \overline{\theta} - \overline{p} > \int_{\theta_r}^{\overline{\theta}} F^{n-1}(x)\,dx = U_{NP}\left(\overline{\theta}\right).$$

Hence, intuitively U_P starts lower, is steeper and ends up higher than U_{NP}. Therefore, U_P and U_{NP} always intersect exactly for one type $\widetilde{\theta}$. The solution of

$$(\theta - \overline{p}) \frac{F^n(\theta) - 1}{n(F(\theta) - 1)} = \int_{\theta_r}^{\theta} F^{n-1}(x)\,dx$$

gives this intersection point. ■

Proof of Theorem 6:
To simplify our analysis we give the proof for a reservation price $r = 0$. The proof for the general case works in an analogue way. Assume that all

buyers except i bid according to the stopping time bidding strategy $t(\theta_j)$, $j \neq i$ with $t(\theta_j) \geq 0$. Buyer $i's$ maximization problem is given by

$$\max_{\tau \geq 0} (\theta_i - c\tau - \overline{p} + e\tau) \prod_{j \neq i} \Pr\{\tau < t(\theta_j)\}$$

or equivalently by

$$\max_{\tau \geq 0} (\theta_i - c\tau - \overline{p} + e\tau) F^{n-1}\left(t^{-1}(\tau)\right).$$

The first-order condition is given by

$$\begin{aligned} 0 &= (\theta_i - \overline{p} + (e-c)\tau)(n-1) F^{n-2}\left(t^{-1}(\tau)\right) \frac{dt^{-1}(\tau)}{d\tau} f\left(t^{-1}(\tau)\right) \\ &\quad + (e-c) F^{n-1}\left(t^{-1}(\tau)\right). \end{aligned}$$

In addition, we have the initial condition $t(\theta_r) = \frac{\overline{p}}{e}$ since the type $\theta_i = \theta_r$ stops the auction when the price for the object reaches zero (see Figure 2). A solution of this initial value problem is given by

$$t(\theta_i) = \frac{1}{e-c}\left(\overline{p} - \theta_i + \int_{\theta_r}^{\theta_i} \frac{F^{n-1}(x)}{F^{n-1}(\theta_i)} dx\right).$$

On the one hand, for a high enough starting price, i.e. $\overline{p} \geq \overline{\theta} - \int_{\frac{c}{e}\overline{p}}^{\overline{\theta}} F^{n-1}(x)\,dx$, the stopping time is non-negative for all types $\theta_i \in [\theta_r, \overline{\theta}]$. On the other hand, if the starting price is not high enough, some bidders' types prefer to stop the auction even before it starts, i.e. $t(\theta_i) < 0$. Of course this is not possible and the best they can do is to stop the auction at time zero. Assuming for the moment that there exists a type $\widetilde{\theta} \in [\underline{\theta}, \overline{\theta}]$ such that all types higher or equal to $\widetilde{\theta}$ stop the auction at time zero. Then it must be true that type $\widetilde{\theta}$ is indifferent between stopping the auction at once and bidding according to $t\left(\widetilde{\theta}\right)$. For a bidder with type θ the utility of bidding according to $t(\theta)$ is

$$\begin{aligned} U_{NP}(\theta) &= \left(\theta - \overline{p} + (e-c)\, t(\theta)\right) F^{n-1}(\theta) \\ &= \int_{\frac{c}{e}\overline{p}}^{\theta} F^{n-1}(x)\, dx \\ &= \int_{\theta_r}^{\theta} F^{n-1}(x)\, dx. \end{aligned}$$

On the other hand, stopping the auction at time zero results in a gamble and all immediate stoppers receive the good with equal probability. Hence, the utility is

$$\begin{aligned} U_P(\theta) &= (\theta - \overline{p}) \sum_{j=0}^{n-1} \frac{1}{j+1} \binom{n-1}{j} (1 - F(\theta))^j F(\theta)^{n-1-j} \\ &= (\theta - \overline{p}) \frac{F^n(\theta) - 1}{n\,(F(\theta) - 1)}. \end{aligned}$$

Next, we show that the solution $\widetilde{\theta}$ makes sense as we have $t\left(\widetilde{\theta}\right) \geq 0$. By definition we have

$$0 = U_{NP}\left(\widetilde{\theta}\right) - U_P\left(\widetilde{\theta}\right) = \int_{\frac{c}{e}\overline{p}}^{\widetilde{\theta}} F^{n-1}(x)\, dx - \left(\widetilde{\theta} - \overline{p}\right) \frac{F^n\left(\widetilde{\theta}\right) - 1}{n\left(F\left(\widetilde{\theta}\right) - 1\right)}.$$

Moreover, note that for any $\theta \in \left[\overline{p}, \overline{\theta}\right]$ the following is true

$$F^{n-1}(\theta) \leq \frac{1}{n} \sum_{j=0}^{n-1} F^j(\theta) = \frac{F^n(\theta) - 1}{n\,(F(\theta) - 1)}.$$

As $e \geq \theta_B$ the sign of $t\left(\widetilde{\theta}\right)$ is the same as in

$$
\begin{aligned}
& (e-c) F^{n-1}\left(\widetilde{\theta}\right) t\left(\widetilde{\theta}\right) \\
= & \int_{\frac{c}{e}\overline{p}}^{\widetilde{\theta}} F^{n-1}(x)\, dx-\left(\widetilde{\theta}-\overline{p}\right) F^{n-1}\left(\widetilde{\theta}\right) \\
\geq & \int_{\frac{c}{e}\overline{p}}^{\widetilde{\theta}} F^{n-1}(x)\, dx-\left(\widetilde{\theta}-\overline{p}\right) \frac{F^n(\theta)-1}{n(F(\theta)-1)} \\
= & \ 0 .
\end{aligned}
$$

Further note that if e is increased, $\widetilde{\theta}$ becomes smaller, since U_{NP} increases for every type θ and thus intersects for a lower type with U_P. ■

Proof of Lemma 6:

Rearranging the seller's expected revenue gives

$$
\overline{p}\left(1-F^n\left(\widetilde{\theta}\right)\right)+\overline{p}\left(F^n\left(\widetilde{\theta}\right)-F^n\left(\theta_r\right)\right)-e \int_{\theta_r}^{\widetilde{\theta}} t(\theta)\, n F^{n-1}(\theta)\, f(\theta)\, d\theta
$$

results immediately in

$$
\overline{p}\left(1-F^n\left(\frac{c}{e} \overline{p}\right)\right)-e \int_{\frac{c}{e}\overline{p}}^{\widetilde{\theta}} t(\theta)\, f_{1: n}(\theta)\, d\theta .
$$

■

Proof of Lemma 7:
The seller's expected revenue is

$$U_S = \overline{p}\left(1 - F^n\left(\widetilde{\theta}\right)\right) + \int_{\theta_r}^{\widetilde{\theta}} \left(\overline{p} - et\left(\theta\right)\right) nF^{n-1}\left(\theta\right) f\left(\theta\right) d\theta.$$

Inserting the bidding function and simplifying leads to

$$\begin{aligned} U_S &= \overline{p}\left(1 - F^n\left(\widetilde{\theta}\right)\right) - \frac{e}{e-c}\theta_r\left(F^n\left(\widetilde{\theta}\right) - F^n\left(\theta_r\right)\right) \\ &\quad + \frac{e}{e-c}n\int_{\theta_r}^{\widetilde{\theta}} \theta F^{n-1}\left(\theta\right) f\left(\theta\right) d\theta - \frac{e}{e-c}n\int_{\theta_r}^{\widetilde{\theta}}\int_{\theta_r}^{\theta} F^{n-1}\left(x\right) f\left(\theta\right) dx d\theta. \end{aligned}$$

Using

$$n\int_{\theta_r}^{\widetilde{\theta}} \theta F^{n-1}\left(\theta\right) f\left(\theta\right) d\theta = F^n\left(\widetilde{\theta}\right)\widetilde{\theta} - F\left(\theta_r\right)\theta_r - \int_{\theta_r}^{\widetilde{\theta}} F^n\left(\theta\right) d\theta$$

and

$$\int_{\theta_r}^{\widetilde{\theta}}\int_{\theta_r}^{\theta} F^{n-1}\left(x\right) f\left(\theta\right) dx d\theta = \int_{\theta_r}^{\widetilde{\theta}} F^{n-1}\left(\theta\right)\left(F\left(\widetilde{\theta}\right) - F\left(\theta\right)\right) d\theta$$

gives

$$\begin{aligned} U_S &= \overline{p}\left(1 - F^n\left(\widetilde{\theta}\right)\right) \\ &\quad + \frac{e}{e-c}\int_{\frac{c}{e}\overline{p}}^{\widetilde{\theta}} \left(F^n\left(\widetilde{\theta}\right) - nF\left(\widetilde{\theta}\right)F^{n-1}\left(\theta\right) + \left(n-1\right)F^n\left(\theta\right)\right) d\theta. \end{aligned}$$

If the starting price is high enough (no stopping at the first instant), i.e. $\overline{p} \geq \overline{\theta} - \int_{\theta_r}^{\overline{\theta}} F^{n-1}\left(x\right) dx$, the seller's expected revenue is

$$U_S = \frac{e}{e-c}\int_{\theta_r}^{\overline{\theta}} \left(1 - nF^{n-1}\left(\theta\right) + \left(n-1\right)F^n\left(\theta\right)\right) d\theta.$$

Note that the seller's expected revenue is decreasing in $\overline{p}$.[32] Hence, the seller will never choose a starting price above the boundary $\overline{p} = \overline{\theta} - \int_{\theta_r}^{\overline{\theta}} F^{n-1}\left(x\right) dx$. ■

[32] Note that $1 - nF^{n-1}\left(\theta\right) + \left(n-1\right)F^n\left(\theta\right) \geq 0$.

Proof of Lemma 8:
Since the starting price is high enough, i.e. $\overline{p} \geq \overline{\theta} - \int_0^{\overline{\theta}} F^{n-1}(x)\,dx$, the bidding equilibrium is given by

$$t(\theta) = \frac{1}{e-c}\left(\overline{p} - \theta + \int_{\theta_r}^{\theta} \frac{F^{n-1}(x)}{F^{n-1}(\theta)}dx\right),$$

for bidders with types $\theta > \theta_r$ (recall $\widetilde{\theta} = \overline{\theta}$ in that case).
The welfare can therefore be written as

$$\begin{aligned}
W &= \int_{\frac{c}{e}\overline{p}}^{\overline{\theta}} \left[\theta - \frac{c}{e-c}\left(\overline{p} - \theta + \int_{\frac{c}{e}\overline{p}}^{\theta} \frac{F^{n-1}(x)}{F^{n-1}(\theta)}dx\right)\right] nF^{n-1}(\theta)\, f(\theta)\, d\theta \\
&= n\frac{c}{e-c}\int_{\frac{c}{e}\overline{p}}^{\overline{\theta}} \left[\frac{e}{c}\theta - \overline{p} - \int_{\frac{c}{e}\overline{p}}^{\theta} \frac{F^{n-1}(x)}{F^{n-1}(\theta)}dx\right] F^{n-1}(\theta)\, f(\theta)\, d\theta \\
&= n\frac{c}{e-c}\int_{\frac{c}{e}\overline{p}}^{\overline{\theta}} \left[\frac{e}{c}\int_{\frac{c}{e}\overline{p}}^{\theta} dx - \int_{\frac{c}{e}\overline{p}}^{\theta} \frac{F^{n-1}(x)}{F^{n-1}(\theta)}dx\right] F^{n-1}(\theta)\, f(\theta)\, d\theta \\
&= n\frac{e}{e-c}\int_{\frac{c}{e}\overline{p}}^{\overline{\theta}} \int_{\frac{c}{e}\overline{p}}^{\theta} \left[F^{n-1}(\theta) - \frac{c}{e}F^{n-1}(x)\right] dx f(\theta)\, d\theta.
\end{aligned}$$

As the upper bound of the inner integral is θ, we have that $x \leq \theta$ (by construction). Since $c \leq e$ (by assumption), we find that the integrand $F^{n-1}(\theta) - \frac{c}{e}F^{n-1}(x) \geq 0$ for all relevant θ and x. If therefore the starting price $\overline{p}$ is decreased, the lower bounds $\frac{c}{d}\overline{p}$ both decrease. Thus, the range of the integrals increases and therefore welfare increases. Consequently, it can never be efficient to start with a price $\overline{p} > \overline{\theta} - \int_0^{\overline{\theta}} F^{n-1}(x)\,dx$. ■

Proof of Lemma 9:
As already shown in the proof of Theorem 6, the derivative of U_p is greater than the one of U_{NP} for the relevant types $\theta \in \in \left[\overline{p}, \overline{\theta}\right)$, where $U_{NP}(\theta) = \int_{\theta_r}^{\theta} F^{n-1}(x)\,dx$. Consider a type $\widehat{\theta}$ such that $U_{NP}\left(\widehat{\theta}\right) = U_P\left(\widehat{\theta}\right)$. An increase in the reservation price reduces U_{NP}. Thus the new intersection point $\widetilde{\theta}$ must be below $\widehat{\theta}$. ■

Proof of Theorem 7:
Since the equilibrium is in dominant strategies, the standard argument works. Intuitively, a buyer remains in the auction as long as the current price is below or equal to his own current valuation. ■

Appendix C

Appendix Chapter 4: A Simple Procurement Decision...

Proof of Theorem 8

The standard argument for second-price auctions is also valid in this setting. It will be briefly shown for a bidder in group 1. Suppose all other bids from his group and all bids from group 2 corrected by e are above his cost, then he finds it profitable to win — which is ensured if he bids his true cost. If any other bidder from his group or from group 2 corrected by e is below his cost, he finds it unprofitable to win and he does not win if he bids his cost. ■

Proof of Theorem 9

Within each group the supplier with the lowest cost is the most efficient. Suppose c_i is the lowest cost in group $i = 1, 2$. It is efficient to give the supplier from group 1 the contract iff $\theta_1 - c_1 > \theta_2 - c_2$. Since $\theta_1 = \theta_2 + d$ this is equivalent to $c_2 + d > c_1$. ■

Proof of Theorem 10

With separate auctions, the buyer will select auction 1 iff $\theta_1 - c_{1,(n_1-1)} >$

$\theta_2 - c_{2,(n_2-1)}$ which is equivalent to $c_{2,(n_2-1)} + d > c_{1,(n_1-1)}$. This is the same condition as in Theorem 9 (first line) resulting in the utility of

$$\theta_1 - \min\left\{c_{1,(n_1-1)}, c_{2,(n_2)} + d\right\},$$

which is never lower than $\theta_1 - c_{1,(n_1-1)}$. If, moreover, $c_{2,(n_2)} + d < c_{1,(n_1-1)}$ then the utility from the combined auction is strictly greater. The same reasoning applies for picking auction 2 in case of the two separate auctions. ■

Proof of Theorem 11

The proof is based on Bulow and Roberts [1989] (whose notation is used) and McAfee and McMillan [1989]. Define $n = n_1 + n_2$. We index the bidders by $l = 1, \ldots, n$, where $l = 1, \ldots, n_1$ refers to the bidders in the first group and $l = n_1 + 1, \ldots, n$ to the bidders in the second group. We write $p_l(c_l, c_{-l})$ for the probability that bidder l wins the auction, given the realization of all costs. The interim expected probability that bidder l wins is $\overline{p}_l(c) = E_{c_{-l}}[p_l(c_l, c_{-l})]$. The expected surplus of the lth supplier is denoted by $S_l(c)$.

We are interested in the procuring firm's expected utility $U(\theta_1, \theta_2)$. By definition this is the expected social value (ESV) reduced by the suppliers' expected surplus (SES).

The expected social value ESV is the value for the procuring company minus the production cost, i.e. in expectation

$$\begin{aligned} & ESV \\ = {} & \sum_{l=1}^{n_1} \int_{\underline{c}}^{\overline{c}} (\theta_1 - c) f(c) \overline{p}_l(c) \, dc + \sum_{l=n_1+1}^{n} \int_{\underline{c}_2}^{\overline{c}_2} (\theta_2 - c) f_2(c) \overline{p}_l(c) \, dc \\ = {} & \theta_1 - \sum_{l=1}^{n_1} \int_{\underline{c}}^{\overline{c}} c f(c) \overline{p}_l(c) \, dc - \sum_{l=n_1+1}^{n} \int_{\underline{c}_2}^{\overline{c}_2} (c+d) f_2(c) \overline{p}_l(c) \, dc. \quad (10) \end{aligned}$$

As Myerson [1981] notes an equilibrium induced by any set of action rules requires that $\frac{\partial S_l(c)}{\partial c} = \overline{p}_l(c)$. Integrating from c to $\overline{c}_1$ or $\overline{c}_2$ and using the fact that $S_l(\overline{c}_l) = 0$ gives $S_l(c) = \int_c^{\overline{c}} \overline{p}_l(t)\,dt$ for $l = 1,\ldots,n_1$ and $S_l(c) = \int_c^{\overline{c}_2} \overline{p}_l(t)\,dt$ for $l > n_1$. The suppliers' *expected* surplus is therefore

$$\begin{aligned} SES &= \sum_{l=1}^{n_1} \int_{\underline{c}}^{\overline{c}} \int_c^{\overline{c}} \overline{p}_l(t)\,dt f(c)\,dc - \sum_{l=n_1+1}^{n} \int_{\underline{c}_2}^{\overline{c}_2} \int_c^{\overline{c}_2} \overline{p}_l(t)\,dt f_2(c)\,dc \\ &= \sum_{l=1}^{n_1} \int_{\underline{c}}^{\overline{c}} \overline{p}_l(c)\,F(c)\,dc - \sum_{l=n_1+1}^{n} \int_{\underline{c}_2}^{\overline{c}_2} \overline{p}_l(c)\,F_2(c)\,dc. \end{aligned} \tag{11}$$

Thus, the procuring firm's expected utility is ESV minus SES

$$\begin{aligned} U(\theta_1,\theta_2) = \theta_1 - \sum_{l=1}^{n_1} \int_{\underline{c}}^{\overline{c}} \left(c + \frac{F(c)}{f(c)}\right) f(c)\,\overline{p}_l(c)\,dc \\ - \sum_{l=n_1+1}^{n} \int_{\underline{c}_2}^{\overline{c}_2} \left(c + d + \frac{F_2(c)}{f_2(c)}\right) f_2(c)\,\overline{p}_l(c)\,dc. \end{aligned} \tag{12}$$

Using the definition of $\overline{p}_l$ gives

$$\begin{aligned} U(\theta_1,\theta_2) = \theta_1 - \sum_{l=1}^{n} \int_{\underline{c}_2}^{\overline{c}_2} \ldots \int_{\underline{c}_2}^{\overline{c}_2} \int_{\underline{c}}^{\overline{c}} \ldots \int_{\underline{c}}^{\overline{c}} \widehat{M}_l(c_l) \\ f(c_1)\ldots f(c_{n_1})\,f_2(c_{n_1+1})\ldots f_2(c_n)\,p_l(c_l,c_{-l})\,dc_1\ldots dc_n, \end{aligned}$$

where

$$\widehat{M}_l(c) = \begin{cases} M(c), & \text{for } \ l \le n_1 \\ c + d + \frac{F_2(c)}{f_2(c)}, & \text{for } \ n_1 < l \le n. \end{cases}$$

Given that $F_2(c) = F(c+a)$, we get

$$\widehat{M}_l(c) = \begin{cases} M(c), & \text{for } \ l \le n_1 \\ M(c+a) + d - a, & \text{for } \ n_1 < l \le n. \end{cases}$$

Note that $\widehat{M}_l$ is decreasing in c. For each vector $(c_1,\ldots,c_n)$, the expected utility $U(\theta_1,\theta_2)$ is maximized by procuring the good with probability one from the firm with the lowest $\widehat{M}_l(c_l)$, as long as $\theta_1 > \widehat{M}_l(c_l)$. ■

Proof of Lemma 10

Recall that $F_{(l:n)}$ denotes the distribution function of the lth order statistic for n bidders and $f_{(l:n)}$ the corresponding density. If there is no doubt about the underlying bidder group (and the number of bidders, i.e. n_1 or n_2) we also simply use $F_{(l)}$, where $l \in \{1, \ldots, n_i\}$, $i = 1, 2$.

The procuring firm's expected utility is directly calculated via weighing the different cases in (U). This gives

$$\begin{aligned} & n_1 \int_{\underline{c}}^{\overline{c}} (\theta_1 - c_3) F(c_3) \left(1 - F_{2,(n_2)}(c_3 - e)\right) f_{1,(n_1-1)}(c_3)\, dc_3 \\ & + n_1 \int_{\underline{c}_2}^{\overline{c}_2 - e} (\theta_1 - (c_2 + e)) \left(1 - F_{(n_1-1)}(c_2 + e)\right) F(c_2 + e) f_{2,(n_2)}(c_2)\, dc_2 \\ & \quad + n_2 \int_{\underline{c}_2}^{\overline{c}_2 - e} (\theta_2 - c_4) \left(1 - F_{(n_1)}(c_4 + e)\right) F(c_4) f_{2,(n_2-1)}(c_4)\, dc_4 \\ & + n_2 \int_{\underline{c}+e}^{\overline{c}} (\theta_2 - (c_1 - e)) \left(1 - F_{2,(n_2-1)}(c_1 - e)\right) F(c_1 - e) f_{1,(n_1)}(c_1)\, dc_1. \end{aligned} \tag{13}$$

Using Theorem 11, the expected utility can also be written differently: We only focus on winning rules $p_l(c_l, c_{-l})$ that involve a direct comparison between costs (possible modified by an additional handicap e). The interim expected probability that bidder l wins is therefore

$$\overline{p}_l(c) = \begin{cases} (1 - F(c))^{n_1-1} (1 - F_2(c - e))^{n_2}, & \text{if } \; i \leq n_1 \\ (1 - F(c + e))^{n_1} (1 - F_2(c))^{n_2-1}, & \text{if } \; n_1 < i \leq n. \end{cases}$$

The expected utility (12) becomes

$$\begin{aligned} & U(\theta_1, \theta_2) \\ = \; & \theta_1 - n_1 \int_{\underline{c}}^{\overline{c}} \left(c + \frac{F(c)}{f(c)}\right) f(c) (1 - F(c))^{n_1-1} (1 - F_2(c - e))^{n_2}\, dc \\ & - n_2 \int_{\underline{c}_2}^{\overline{c}_2} \left(c + d + \frac{F_2(c)}{F_2(c)}\right) f_2(c) (1 - F(c + e))^{n_1} (1 - F_2(c))^{n_2-1}\, dc. \end{aligned}$$

Substituting the definition of F_2 gives

$$\begin{aligned}
& U(\theta_1,\theta_2) \\
= \; & \theta_1 - \int_{\underline{c}}^{\overline{c}} M(c)\,(1-F(c+a-e))^{n_2}\,\frac{d}{dc}\left(1-(1-F(c))^{n_1}\right)dc \\
& - \int_{\underline{c}}^{\overline{c}} (M(c)-a+d)\,(1-F(c-a+e))^{n_1}\,\frac{d}{dc}\left(1-(1-F(c))^{n_2}\right)dc.
\end{aligned}$$

Using the notation $f_{(n_1:n_1)}(c) = \frac{d}{dc}\left(1-(1-F(c))^{n_1}\right)$ as the density for the lowest type among n_1 bidders results in the stated formula.
Since the requirements for the Revenue-Equivalence-Theorem are satisfied the procuring firm's expected utility is the same as in (13). ∎

Proof of Corollary 5.
Differentiating the expected utility with respect to e and transforming the second integral gives

$$\begin{aligned}
& \frac{\partial U(e)}{\partial e} = \\
& -n_1 n_2 \int_{\underline{c}}^{\overline{c}} M(c)\,f(c)\,(1-F(c))^{n_1-1}\,f_2(c-e)\,(1-F_2(c-e))^{n_2-1}\,dc \\
& \qquad + n_1 n_2 \int_{\underline{c}_2+e}^{\overline{c}_2+e} \left(c-e+d+\frac{F_2(c-e)}{F_2(c-e)}\right) \\
& \qquad\qquad f(c)\,(1-F(c))^{n_1-1}\,f_2(c-e)\,(1-F_2(c-e))^{n_2-1}\,dc.
\end{aligned}$$

Combining both integrals and using the fact that f is zero outside $[\underline{c},\overline{c}]$ and f_2 is zero outside $[\underline{c}_2,\overline{c}_2]$ we get

$$\begin{aligned}
\frac{\partial U(e)}{\partial e} = n_1 n_2 & \int_{\max\{\underline{c},\underline{c}-a+e\}}^{\min\{\overline{c},\overline{c}-a+e\}} \left(c-e+d+\frac{F_2(c-e)}{F_2(c-e)}-\left(c+\frac{F(c)}{f(c)}\right)\right) \\
& f(c)\,(1-F(c))^{n_1-1}\,f_2(c-e)\,(1-F_2(c-e))^{n_2-1}\,dc.
\end{aligned}$$

Using the definition $F_2(c) = F(c+a)$ on $[\underline{c}_2, \overline{c}_2] = [\underline{c} - a, \overline{c} - a]$ we have

$$\frac{\partial U(e)}{\partial e} =$$
$$n_1 n_2 \int_{\max\{\underline{c}, \underline{c}-a+e\}}^{\min\{\overline{c}, \overline{c}-a+e\}} \left(c + a - e + \frac{F(c+a-e)}{f(c+a-e)} - \left(c + \frac{F(c)}{f(c)} \right) + d - a \right)$$
$$f(c)(1 - F(c))^{n_1 - 1} f(c+a-e)(1 - F(c+a-e))^{n_2 - 1} dc$$

$$= \int_{\max\{\underline{c}, \underline{c}-a+e\}}^{\min\{\overline{c}, \overline{c}-a+e\}} \left(d - e + \frac{F(c+a-e)}{f(c+a-e)} - \frac{F(c)}{f(c)} \right)$$
$$\frac{d}{dc}(1 - (1 - F(c))^{n_1}) \frac{d}{dc}(1 - (1 - F(c+a-e))^{n_2}) dc.$$

Using the first order condition $\frac{\partial U(e)}{\partial e} = 0$ gives the result. ■

Proof of Example 8
Since $\frac{F(c)}{f(c)} = \frac{c}{m}$ for all c we find that for $e^* = \frac{m}{m+1} d + \frac{1}{m+1} a$ the first part under the integral $\left(d - e + \frac{F(c+a-e)}{f(c+a-e)} - \frac{F(c)}{f(c)} \right)$ is equal to zero for all c. Therefore, the first order condition is satisfied. Note that this is also an optimal auction (except for the reservation value). ■

Proof of Corollary 6
Since $\widehat{m} > \widetilde{m}$ we have that $c^{\widehat{m}} < c^{\widetilde{m}}$. Hence, $\widehat{F}$ first order stochastically dominates $\widetilde{F}$. Therefore, the optimal $\widehat{e}^*$ for the distribution $\widehat{F}$ is closer to d than the optimal $\widetilde{e}^*$ for the distribution $\widetilde{F}$, i.e. $|\widehat{e}^* - d| < |\widetilde{e}^* - d|$. ■

Proof of Theorem 12

1. Case $a = d$: Since $\frac{F}{f}$ is assumed to be increasing we have for $e < d$, that $\left(d - e + \frac{F(c+d-e)}{f(c+d-e)} - \frac{F(c)}{f(c)} \right) > 0$. Hence, the derivative is greater than zero. For $e > d$, we find that the first part under the integral and therefore the entire derivative is negative. For $e = d$ we have
$$\left(d - e + \frac{F(c+d-e)}{f(c+d-e)} - \frac{F(c)}{f(c)} \right) = 0$$
and thus a unique maximum.

2. Case $a \neq d$: For $e \leq \min\{a,d\}$ we have (since $\frac{F}{f}$ is assumed to be increasing)
$$\left(d - e + \frac{F(c+a-e)}{f(c+a-e)} - \frac{F(c)}{f(c)}\right) > 0.$$
Hence, the entire derivative is greater than zero.

   For $e \geq \max\{a,d\}$, we have
$$\left(d - e + \frac{F(c+a-e)}{f(c+a-e)} - \frac{F(c)}{f(c)}\right) < 0.$$
Hence, the entire derivative is negative.

■

Proof of Theorem 13
We consider the case where $a < d$ and differentiate $\frac{\partial U(e)}{\partial e}$ with respect to e (recall $\widetilde{M}(c,e) = M(c+a-e) - M(c) - a + d$)

$$\frac{\partial^2 U(e)}{\partial e^2} \tag{14}$$

$$= \frac{\partial}{\partial e}\left[\int_{\underline{c}-a+e}^{\overline{c}} \widetilde{M}(c,e)\, f_{(n_1:n_1)}(c)\, f_{(n_2:n_2)}(c+a-e)\, dc\right] \tag{15}$$

$$= \underbrace{-\left(d - e - \frac{F(\underline{c}-a+e)}{f(\underline{c}-a+e)}\right)}_{<0} \underbrace{f_{(n_1:n_1)}(\underline{c}-a+e)\, n_2 f(\underline{c})}_{>0} \tag{16}$$

$$+ \int_{\underline{c}-a+e}^{\overline{c}} \underbrace{f_{(n_1:n_1)}(c)}_{\geq 0} \left[\underbrace{\left(\frac{-1 + f'F - f^2}{f^2}\right)\Bigg|_{(c+a-e)}}_{<0} \underbrace{f_{(n_2:n_2)}(c+a-e)}_{\geq 0} \right. \tag{17}$$

$$\left. + \underbrace{\left(d - e - \left(\frac{F(c)}{f(c)} - \frac{F(c+a-e)}{f(c+a-e)}\right)\right)}_{>0 \text{ for small c, } <0 \text{ for large } c} \underbrace{\frac{d}{de} f_{(n_2:n_2)}(c+a-e)}_{\geq 0} \right] dc. \tag{18}$$

Note that (16) is smaller than zero. The parts in line (17) (without the integral) result in a negative sign. In line (18) the first part will be greater or smaller than zero, depending on c, but the second part is equal to

$$n_2 \left(1 - F\left(c + a - e\right)\right)^{n_2 - 1} f\left(c + a - e\right) \left[\frac{\left(n_2 - 1\right) f\left(c + a - e\right)}{1 - F\left(c + a - e\right)} - \frac{f'\left(c + a - e\right)}{f\left(c + a - e\right)}\right]$$

and not below zero. To summarize, if the first part in (18) is below zero we have that, starting from the first order condition, a further increase in e will decrease the expected payoff, hence solution of the first order condition constitutes a maximum.
We split up the integral to focus on (18)

$$\int_{\underline{c}-a+e}^{\overline{c}} f_{(n_1:n_1)}\left(c\right) \left(d - e - \left(\frac{F\left(c\right)}{f\left(c\right)} - \frac{F\left(c + a - e\right)}{f\left(c + a - e\right)}\right)\right) \\ n_2 \left(1 - F\right)^{n_2 - 1} f \left[\frac{\left(n_2 - 1\right) f}{1 - F} - \frac{f'}{f}\right]\Bigg|_{c+a-e} dc. \quad (19)$$

The first order condition on the other hand is given for $e = e^*$ by

$$0 = \int_{\underline{c}-a+e}^{\overline{c}} f_{(n_1:n_1)}\left(c\right) \left(d - e - \left(\frac{F\left(c\right)}{f\left(c\right)} - \frac{F\left(c + a - e\right)}{f\left(c + a - e\right)}\right)\right) \\ n_2 \left(1 - F\right)^{n_2 - 1} f\Big|_{c+a-e} dc. \quad (20)$$

If (19) $<$ (20) $= 0$, we can conclude that (14)is below zero. Therefore, every solution to the first order condition constitutes a maximum, therefore there can only be one maximum. ■

Proof of Theorem 14
In the case $a < d$, it results from Theorem 12 that $a < e^* < d$. The derivative can be written as

$$\int_{\underline{c}-a+e}^{\overline{c}} \left(\underbrace{d - e}_{>0} - \underbrace{\left(\frac{F\left(c\right)}{f\left(c\right)} - \frac{F\left(c + a - e\right)}{f\left(c + a - e\right)}\right)}_{>0}\right) f_{(n_1:n_1)}\left(c\right) f_{(n_{21}:n_2)}\left(c + a - e\right) dc \quad (21)$$

which is zero for $e = e^*$.
Note that $d - e - \left(\frac{F(c)}{f(c)} - \frac{F(c+a-e)}{f(c+a-e)}\right)$ is decreasing in c. Furthermore we have

$$\frac{d}{dc}\left(1 - (1 - F(c))^{n_1}\right) = n_1 (1 - F(c))^{n_1 - 1} f(c).$$

This derivative is decreasing in c if

$$f'(c)(1 - F(c)) < (n_1 - 1) f^2(c) \tag{22}$$

for all $c \in [\underline{c} - a + e^*, \overline{c}]$, since

$$\begin{aligned} & \frac{d^2}{dc^2}\left(1 - (1 - F(c))^{n_1}\right) \\ = \; & n_1 (1 - F(c))^{n_1 - 2}\left[f'(c)(1 - F(c)) - (n_1 - 1) f^2(c)\right]. \end{aligned}$$

Moreover

$$\begin{aligned} & \frac{d^2}{dc^2}\left(1 - (1 - F(c + a - e))^{n_2}\right) \\ = \; & n_2 (1 - F(c + a - e))^{n_2 - 2} \\ & \left[f'(c + a - e)(1 - F(c + a - e)) - (n_2 - 1) f^2(c + a - e)\right] \end{aligned}$$

is decreasing in c if

$$f'(c)(1 - F(c)) < (n_2 - 1) f^2(c) \tag{23}$$

for all $c \in [\underline{c} - a + e^*, \overline{c}]$. Since f' is bounded by assumption, (22) and (23) are true if n_1 and n_2 are big enough. Define N such that (22) and (23) are true for all $n_1 > N$ and $n_2 > N$.
Note that (21) is an integral over a decreasing function multiplied with a density, hence we can think of it as an expected value.
Without loss of generality we consider an increase in n_1 and use as density $f_{(n_1:n_1)}(c) = \frac{d}{dc}\left(1 - (1 - F(c))^{n_1}\right)$. It is the density of the distribution of the lowest of n_1 bidders. For any $m > n_1$ we have that $1 - (1 - F(c))^m > 1 - (1 - F(c))^{n_1}$. Hence, $1 - (1 - F(c))^{n_1}$ dominates (in the sense of first order stochastic dominance) $1 - (1 - F(c))^m$. Intuitively, it means that

for higher n_1 the distribution puts more mass on low types. Increasing n_1 in (21) more mass is put on higher function values and therefore (21) is increased.
If e^* is the unique solution of the first order condition and constitutes a maximum, (21) is positive for $e < e^*$ and negative for $e > e^*$. Since increasing n_1 implies increasing (21), it remains positive for $e < e^*$ but lifts (21) above zero for some $e \in [e^*, \widehat{e}]$. Therefore, the new solution e^{**} of the first order condition must be bigger than e^*.
Note that for an increased n_1 it is no longer guaranteed that there is only one solution of the first order condition. But all solutions are bigger than e^*. ■

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