Rigorous Derivation of the Degenerate Parabolic-Elliptic Keller-Segel System from a Moderately Interacting Stochastic Particle System

Inauguraldissertation
zur Erlangung des akademischen Grades
eines Doktors der Naturwissenschaften
der Universität Mannheim

vorgelegt von

Veniamin Gvozdik
aus Sotschi, Russland

Mannheim, 2022

Dekan: Dr. Bernd Lübcke, Universität Mannheim
Referentin: Professorin boshi Li Chen, Universität Mannheim
Korreferent: Professor Doktor Peter Pickl, Universität Tübingen

Tag der mündlichen Prüfung: 28. November 2022

Abstract

The main goal of this thesis is a rigorous derivation of the degenerate parabolic-elliptic Keller-Segel system of porous medium type on the whole space $\mathbb{R}^d$ from a moderately interacting stochastic particle system. After we review some existing results on this topic and introduce the setting of the problem as well as the main results of this thesis, we establish the classical solution theory of the degenerate parabolic-elliptic Keller-Segel system and its non-local version. This classical solution theory is used later to obtain required estimates on the particle level. Because of the non-linearity in diffusion and the singularity in aggregation we perform an approximation of the stochastic moderately interacting particle system using the cut-offed potential. The stochastic effect is introduced as a parabolic regularization of the system. Then we compare this new system with another cut-offed system of mean-field type. We present the propagation of chaos result with two different types of cut-off scaling, namely logarithmic and algebraic scaling. For the logarithmic scaling we prove the convergence of trajectories in expectation. For the algebraic scaling we obtain it in the sense of probability. Consequently, the propagation of chaos follows directly from these convergence results and the vanishing viscosity of the system.

Key words: Moderately interacting particle systems; Stochastic particle systems; Mean-field limit; Chemotaxis; Keller–Segel model; Degenerate parabolic-elliptic system; Propagation of chaos

Zusammenfassung

Das Hauptziel dieser Arbeit ist eine rigorose Herleitung des entarteten parabolisch-elliptischen Keller-Segel Systems vom Poröse-Medien-Typ auf dem ganzen Raum $\mathbb{R}^d$ aus einem mäßig wechselwirkenden stochastischen Teilchensystem. Als erstes erläutern wir manche bereits existierende Resultate aus diesem Gebiet und präsentieren die Problemstellung und die Hauptresultate dieser Arbeit. Danach leiten wir die klassische Lösungstheorie für das entartete parabolisch-elliptische Keller-Segel System und seine nicht-lokale Version her. Diese klassische Lösungstheorie wird später benutzt, um die notwendigen Abschätzungen auf dem Teilchenniveau zu erhalten. Wegen der Nichtlinearität der Diffusion und der Singularität des Interaktionspotentials muss man das mäßig wechselwirkende stochastische Teilchensystem mit Hilfe eines abgeschnittenen Potentials approximieren. Der stochastische Effekt dient als eine parabolische Regularisierung des Systems. Dann vergleicht man dieses abgeschnittene System mit einem weiteren abgeschnittenen Teilchensystem vom Mean-Field-Typ. Wir präsentieren zwei Resultate der Ausbreitung des Chaos für unterschiedliche Skalierungsresultate. Mit der logarithmischen Skalierung erhalten wir Konvergenz der Trajektorien im Erwartungswert und mit der algebraischen Skalierung bekommen wir die Konvergenz in Wahrscheinlichkeit. Ausbreitung des Chaos folgt direkt aus den beiden Konvergenzresulten und der verschwindenden Viskosität des Systems.

Schlüsselwörter: Mäßig wechselwirkende Teilchensysteme; Stochastische Teilchensysteme; Mean-field Limit; Chemotaxis; Keller–Segel Model; Entartetes parabolisch-elliptisches System; Ausbreitung des Chaos

Acknowledgments

First of all, I would like to express my deepest gratitude to my supervisor, Prof. boshi. Li Chen, for her mentoring and support. Next, I want to extend my special thanks to Yue Li for fruitful discussions and her valuable insights. I am also grateful to Alexandra Holzinger for her important comments on this thesis. I would like to acknowledge Prof. Dr. Peter Pickl, who has kindly agreed to be a referee for my thesis. Finally, I would like to thank my family members and friends who supported me on this journey.

Contents

Chapter 1

Introduction

Interacting particle systems of mean-field type belong to the well-known and fast growing fields in mathematics. They are used to construct microscopic models of different physical and biological phenomena, such as dynamics of ions in plasma, distribution of bacteria in fluids or flocking behavior of animals. One of the easiest ways to describe this kind of behavior mathematically is to consider a system of ordinary (or stochastic) differential equations with a suitable two-particle interaction effect to describe the dynamics of a large group of identical particles. However, since the number of particles could be very high, one would like to find a way to approximate these microscopic models and so reduce their complexity. In order to achieve this goal, one can derive the scaling limit of the interacting particle systems. This macroscopic model is often given by a partial differential equation.

1.1 Background and State of the Art

The idea of the mean-field theory comes initially from statistical physics. First kinds of such equations were introduced by Jeans in [Jeans, 1915] as he studied stellar dynamics. A very successful implementation of the mean-field theory was done by Vlasov in [Vlasov, 1938] and [Vlasov, 1968], as he studied evolution of plasma. He obtained that interaction of ions can be approximated by equations of Vlasov-type. For the general setting and background of Vlasov equations and the mean-field theory we refer to [Jabin, 2014], [Mayer, 2014] and [Golse, 2016].

Although Vlasov's first paper in this field was published in 1940s, the rigorous derivation of the Vlasov equation as a mean-field limit for Coulomb interaction potentials remains a hot topic in plasma physics. Let us recall some classical results from this field. Braun and Hepp in [Braun and Hepp, 1977] established a connection between Vlasov dynamics and particle systems with regular interaction potentials. Then Dobrushin obtained in [Dobrushin, 1979] uniqueness of the solution to the Vlasov equation from [Braun and Hepp, 1977] with the help of the Kantorovich-Rubinstein metric. An overview about Vlasov equations and some classical results in this field can be found in [Spohn, 2004].

Parallel to the mean-field theory, Kac studied in [Kac, 1956] interacting particle systems and introduced a mathematical definition of chaos. He proposed an idea that the independence of two particles in the system is preserved in time as the number of particles tends to infinity. This effect is now called propagation of chaos. The definition of Kac's chaos and some extensions can be found in [Jabin and Wang, 2017] or [Sznitman, 1991].

A significant enhancement of the theory was done by McKean in [McKean, 1967] as he studied new diffusion models which also satisfy propagation of chaos. McKean developed a so-called synchronous coupling method which was later generalized by Sznitman in [Sznitman, 1991]. It allows to approximate a system of stochastic interacting particle systems by another symmetric system of stochastic differential equations. Thanks to this method, one can reduce this symmetric system to only one equation. Then using Ito's formula one obtains a mean-field equation from this single stochastic differential equation. The proof of McKean as well as other details about this method can be found in [Chaintron and Diez, 2021] and [Sznitman, 1991].

Many important interaction potentials are not Lipschitz continuous therefore one needs to use other techniques to handle them. Now we would like to mention some significant contributions for the Coulomb potential. Lazarovici and Pickl in [Lazarovici and Pickl, 2017] proposed a new method which considers a particle system with a cut-offed potential and random initial data. They derived a mean-field limit in the sense of probability for the Coulomb potential in three dimensions. Later Leblé and Serfaty proved in [Leblé and Serfaty, 2018] a version of the central limit theorem for 2-d gases with the Coulomb potential. Moreover, Serfaty established in [Serfaty, 2020] the mean-field limit for the particle system with the Coulomb-type potential in arbitrary dimensions. For further recent results about particle systems with Coulomb interactions we refer to [Serfaty, 2018].

The classical method of the mean-field theory, especially for smooth interactions, is based on the direct trajectory estimates. For deterministic particle systems we refer to [Golse, 2016] and for stochastic systems we refer to [Sznitman, 1991]. Many particle models contain singularity and therefore one cannot apply classical methods. One needs to introduce additional assumptions or apply regularization techniques in order to derive mean-field limits for this kind of particle systems. This kind of results were studied in [García and Pickl, 2017], [Boers and Pickl, 2016] , [Lazarovici and Pickl, 2017], [Godinho and Quininao, 2015] , [Bolley et al., 2011], [Chen et al., 2017], [Chen et al., 2020] with different cut-off techniques. For more information about interacting particle systems with singular potentials as well as some generalizations see [Jabin and Wang, 2017] and references therein.

Now we would like to draw attention to the relative entropy method which gives the strong convergence in the propagation of chaos. Moreover, one can directly handle weaker interaction potentials without mollification. In this case, instead of considering trajectories of the particles, one studies directly a high-dimensional partial differential equation of the joint law of all particles. Then one computes the relative entropy of the joint law and the factorized law from the limiting equation. The relative entropy was used by H-T Yao in [Yau, 1991] to study hydrodynamics of Ginzburg–Landau and was later applied to other hydrodynamics limits which can be found in [Kipnis and Landim, 1998]. Then Serfaty introduced in [Serfaty, 2017] the modulated energy approach which was successfully used to derive mean-field limits for quantum vortices. This method was later used by Duerinckx in [Duerinckx, 2016] for the gradient flows with potentials of Riesz-type and by Serfaty in [Serfaty, 2020] for the flows of Coulomb-type. In [Jabin and Wang, 2016] Jabin and Wang considered the second order interacting particle systems with the interaction potential K which belongs to L^∞. Jabin and Wang used the relative entropy in order to control the distance between the N-particle distribution and the expected limiting equation which is nothing but the solution to the corresponding Vlasov system. In [Jabin and Wang, 2018] Jabin and Wang used the same method and derived mean-field limits for the interaction kernel K such that K and $\nabla \cdot K$ belong to $\dot{W}^{-1,\infty}$. Then Bresch, Jabin and Wang combined in [Bresch et al., 2019] the tools from [Jabin and Wang, 2018] together with the modulated potential energy method from [Serfaty, 2020] and obtained mean-field filmits for two dimensional Keller-Segel systems or logarithm potentials in higher dimensions. More precisely, they derived similar estimates as those in [Jabin and Wang, 2018] for less singular parts of the potential and also used tools from [Serfaty, 2020] to control logarithm singularities of the potential. The application of the relative entropy method for the derivation of the degenerate parabolic-elliptic Keller-Segel system seems to be challenging and will be covered in our future projects.

1.2 Main Concepts of the Mean-field Theory

In this section we discuss the key ideas of the mean-field theory for both deterministic and stochastic interacting particle systems.

1.2.1 Deterministic Models with Regular Potentials

We use the toy model from [Golse, 2016] in order to illustrate the deterministic theory. Denote by $\mathbb{R}^d$ the d-dimensional Euclidean space and by $\mathbb{N}$ the set of natural numbers. Consider a system of $N \in \mathbb{N}$ identical particles

$$x^i(t) \in \mathbb{R}^d, \quad i \in \{1, \dots, N\},$$

where $x^i(t)$ describes the position of the i-th particle at time point t. It is important to mention that particles should be identical in order to describe the dynamics of the whole system by only one individual particle. In this deterministic model we assume that only two-body interaction takes place. The interaction force between the i-th and the j-th particle is denoted by $K(x^i, x^j)$, where

$$\begin{aligned} K : \mathbb{R}^d \times \mathbb{R}^d &\to \mathbb{R}^d, \\ (x^i, x^j) &\mapsto K(x^i, x^j). \end{aligned}$$

In this example we consider Newton dynamics. So, we need to take Newton's third law of mechanics into account and therefore assume that

$$K(x,x') = -K(x',x)$$

holds for all $x, x' \in \mathbb{R}^d$. Moreover, for simplicity we assume that $K \in C^1(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d)$ and there exists $L > 0$ such that

$$\sup_{x' \in \mathbb{R}^d} |\nabla_x K(x,x')| \leq L \quad \text{and} \quad \sup_{x \in \mathbb{R}^d} |\nabla_{x'} K(x,x')| \leq L.$$

The dynamics of the particle system is determined by the system of ordinary differential equations which is given by

$$\frac{dx^i(t)}{dt} = \sum_{\substack{j=1 \\ j\neq i}}^{N} K(x^i(t), x^j(t)), \qquad i \in \{1, \ldots, N\}.$$

Now we need to rescale the time variable of this system in order to ensure that after this procedure $\dfrac{dx^i(t)}{dt}$ is bounded for every $i \in \{1, \ldots, N\}$ as N tends to infinity. Since $\sum\limits_{\substack{j=1 \\ j\neq i}}^{N} K(x^i(t), x^j(t))$ contains $N-1$ terms, the desired property could be achieved by the new time scaling $\dfrac{t}{N}$.

For the example above one obtains a classical result, which is stated in Theorem 1.3.1 and Theorem 1.4.4 from [Golse, 2016]:

1. For each $N \in \mathbb{N}$ and each N-tuple $X_0^N := (x_0^1, \cdots, x_0^N)$ the Cauchy problem for the system of ordinary differential equations

$$\begin{cases} \frac{dx^i(t)}{dt} = \frac{1}{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} K(x^i(t), x^j(t)), \\ x^i(0) = x_0^i, \qquad i \in \{1, \ldots, N\}, \end{cases} \tag{1.1}$$

   has a unique C^1 solution on $\mathbb{R}$ which is denoted by

$$t \mapsto X^N(t) := (x^1(t), \cdots, x^N(t)) =: T_t X_0^N.$$

2. The empirical measure $\mu_{T_t X_0^N} := \frac{1}{N} \sum_{j=1}^{N} \delta_{x^j(t)}$ is a weak solution of the Cauchy problem for the mean-field partial differential equation which is given by

$$\partial_t f + \nabla \cdot (f \mathcal{K} f) = 0, \tag{1.2}$$
$$f|_{t=0} = \mu_{X_0^N},$$

where

$$\begin{aligned}\mathcal{K} : \mathcal{P}_1(\mathbb{R}^d) &\to \mathbb{R}^d,\\ \mu &\mapsto \mathcal{K}\mu(\cdot) := \int_{\mathbb{R}^d} K(\cdot, x')\mu(dx'),\end{aligned}$$

and $\mathcal{P}_1(\mathbb{R}^d)$ denotes the set of Borel probability measures on $\mathbb{R}^d$ with the finite first moment.

3. If f_0 is a probability density such that $\int_{\mathbb{R}^d} |x| f_0(x)\, dx < \infty$, then (1.2) has a unique weak solution $f \in C(\mathbb{R}^d; L^1(\mathbb{R}^d))$. Let $\mathrm{dist}_{MK,1}$ be the Monge-Kantarovich distance. If for f_0 it holds that $\mathrm{dist}_{MK,1}(\mu_0, \mu_{X_0^N}) \to 0$ as $N \to \infty$ where $d\mu_0 = f_0 dx$, then $\mu_{T_t X_0^N}$ converges to $\mu(t, \cdot)$ in the weak topology of probability measures where $d\mu(t.\cdot) = f(t, \cdot)dx$.

1.2.2 Stochastic Models with Regular Potentials

In real world applications it is natural to consider that particle dynamics is under the influence of uncertainties. One of the possibilities to perturb the system is to add an independent noise to each single particle. This kind of particle systems is typically described by a system of stochastic differential equations.

At the same time we can consider a general type of stochastic interacting particle systems where coefficients of stochastic differential equations depend on a probability measure. For the motivation let us recall (1.1) and have a closer look at $\frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^N K(x^i(t), x^j(t)),\ i \in \{1, \dots, N\}$. One can see that it is possible to redefine this sum as the convolution of K with $\mu_{T_t X_0^N} = \frac{1}{N}\sum_{j=1}^N \delta_{x^j(t)}$. So, using this idea we consider the following type of stochastic interacting particle systems from sections 2.2.2 and 5.1 in [Chaintron and Diez, 2021] . Let $b : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^{d\times d}$ where $\mathcal{P}(\mathbb{R}^d)$ is the set of probability measures on $\mathbb{R}^d$. The stochastic interacting particle system is given by

$$\begin{cases} dX_t^i = b(X_t^i, \mu_t^N)dt + \sigma(X_t^i, \mu_t^N)dB_t^i, \\ X_0^i \text{ are i.i.d. with the density } u_0, \quad i \in \{1, \dots, N\}, \end{cases} \tag{1.3}$$

where B_t^i , $i \in \{1, \dots, N\}$ are N independent d-dimensional Brownian motions and $\mu_t^N := \dfrac{1}{N}\displaystyle\sum_{i=1}^N \delta_{X_t^i}$ is the empirical measure of the above system. Moreover, we need to assume that B_t^i , $i \in \{1, \dots, N\}$ are independent from the initial data.

If we compare (1.1) and (1.3), we recognize some significant differences. First, μ_t^N is now a measure-valued random variable, namely a random probability measure, and not a deterministic function as it was before. Moreover, one cannot expect that μ_t^N solves the mean-field partial differential equation for every $N \in \mathbb{N}$ as it was the case in Theorem 1.3.1 from [Golse, 2016]. However, μ_t^N still gives us information about the N-particle distribution. We notice that $X_0^i,\ i \in \{1, \dots, N\}$ are independent and identically distributed random variables and each of it has the same density function u_0. Therefore, the density of the joint probability distribution is given by $u_0^{\otimes N}$. Moreover, since Brownian motions are mutually independent, under some additional conditions on the coefficients $b(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$ we are able to reduce (1.3) to the following symmetric system of stochastic differential equations

$$\begin{cases} d\bar{X}_t^i = b(\bar{X}_t^i, u(t,\bar{X}_t^i))dt + \sigma(\bar{X}_t^i, u(t,\bar{X}_t^i))dB_t^i, \\ X_0^i \text{ are i.i.d. with the density } u_0, \qquad i \in \{1,\dots,N\}, \\ u(t,\cdot) \text{ is the law of } \bar{X}_t^i. \end{cases} \tag{1.4}$$

Since every equation in (1.4) is independent of each other, we can reduce it to one stochastic differential equation

$$\begin{cases} d\bar{X}_t = b(\bar{X}_t, u(t,\bar{X}_t))dt + \sigma(\bar{X}_t, u(t,\bar{X}_t))dB_t, \\ X_0 \text{ has a density } u_0, \\ u(t,\cdot) \text{ is the law of } \bar{X}_t. \end{cases}$$

By Itô's formula we deduce that for $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ where f is C^1 in the time variable t and C^2 in the space variable x it holds that

$$\begin{aligned} d(f(t,\bar{X}_t)) = {} & \frac{\partial f}{\partial t}(t,\bar{X}_t)dt + \sum_{i=1}^d \frac{\partial f}{\partial x_i}(t,\bar{X}_t) b_i(\bar{X}_t, u(t,\bar{X}_t))dt \\ & + \frac{1}{2}\sum_{i,j=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j}(t,\bar{X}_t) \sum_{l=1}^d \sigma_{il}(\bar{X}_t, u(t,\bar{X}_t))\sigma_{jl}(\bar{X}_t, u(t,\bar{X}_t))dt \\ & + \sum_{i=1}^d \Big(\frac{\partial f}{\partial x_i}(\bar{X}_t, u(t,\bar{X}_t)) \sum_{j=1}^d \sigma_{ij}(\bar{X}_t, u(t,\bar{X}_t)) \Big) dB_t^j. \end{aligned} \tag{1.5}$$

Then we write equation (1.5) in the integral form and take the expected value in order to obtain that for $0 \le s \le r \le T$ it holds that

$$\begin{aligned} \mathbb{E}\left[f(r,\bar{X}_r) - f(s,\bar{X}_s)\right] = {} & \mathbb{E}\left[\int_s^r \frac{\partial f}{\partial t}(t,\bar{X}_t)dt\right] + \mathbb{E}\left[\int_s^r \sum_{i=1}^d \frac{\partial f}{\partial x_i}(t,\bar{X}_t) b_i(\bar{X}_t, u(t,\bar{X}_t))dt\right] \\ & + \mathbb{E}\left[\int_s^r \frac{1}{2}\sum_{i,j=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j}(t,\bar{X}_t) \sum_{l=1}^d \sigma_{il}(\bar{X}_t, u(t,\bar{X}_t))\sigma_{jl}(\bar{X}_t, u(t,\bar{X}_t))dt\right] \end{aligned}$$

At this point we need to assume that u has a density function which is denoted by u as well. So, the density function of the law of $\bar{X}_t$ is a weak solution to the mean-field partial differential equation which is given by

$$\partial_t u = \sum_{i=1}^d \partial_{x_i} \left(b_i(\cdot, u)u\right) + \frac{1}{2}\sum_{i,j=1}^d \partial_{x_i}\partial_{x_j}\left(\sum_{l=1}^d \sigma_{il}(\cdot, u)\sigma_{jl}(\cdot, u)u\right),$$
$$u(0,\cdot) = u_0.$$

In this short example we only present a formal idea of the proof. In order to derive the mean-field partial differential equation rigorously, one needs to ensure existence of density functions, to derive their connection to partial differential equations and to prove that (1.3) can be in fact approximated by (1.4). A common strategy to compare these two particle systems is to prove with the help of Grönwall's lemma that

$$\sup_{t\geq 0}\sup_{i=1,\dots,N} \mathbb{E}\left[\left|X_t^i - \bar{X}_t^i\right|^2\right] \to 0 \text{ as } N \to \infty.$$

So, one proves that the empirical measure μ_t^N converges weakly to $u(t,\cdot)$ as $N \to \infty$. As a byproduct one obtains propagation of chaos, namely that the joint distribution of k particles from the system (1.3) where $k \in \{1,\dots,N\}$ converges weakly to the the joint distribution of k particles from the system (1.4) as $N \to \infty$.

1.2.3 Particle Models with Singular Potentials

In the past two subsections we considered only regular interaction potentials. Many particle models contain singularities and therefore one cannot apply classical methods we discussed above. Now we present some common strategies to overcome this difficulty.

The authors of [Bolley et al., 2011] extended the classical McKean's result to the locally Lipschitz interactions with the polynomial growth by requiring additional moment estimates for the particle system and for the limiting equation as well.

In [Fournier et al., 2017] the authors derived a the so-called Keller-Segel model, which we explain in the next section, from the following system of two dimensional stochastic differential equations:

$$X_t^{i,N} = X_0^i + \sqrt{2}B_t^i + \frac{\chi}{N}\sum_{j=1}^N \int_0^t K(X_s^{i,N} - X_s^{j,N})ds, \quad i = 1,\dots,N. \tag{1.6}$$

In (1.6) χ is a positive number, $N \geq 2$ and $K(x) := \dfrac{-x}{2\pi|x|^2}$ with the convention $K(0) = 0$ and B_t^i, $i \in \{1,\dots,N\}$ are N independent 2-dimensional Brownian motions. Since K is singular, the authors of [Fournier et al., 2017] proved in Proposition 4 that for any $N \geq 2$ and $T > 0$:

$$\mathbb{P}(\exists\, t \in [0,T] : \exists 1 \leq i < j \leq N : X_t^{i,N} = X_t^{j,N}) > 0.$$

This means that the singularity of K is visited and consequently the model (1.6) is not well-defined.

So, the authors considered a regularized system where K is approximated by $K_\varepsilon(x) := \frac{-x}{2\pi(|x|^2+\varepsilon^2)}$. It allows to obtain a globally Lipschitz continuous kernel K_ε and therefore a global strong solution of the interacting particle system. Then letting $\varepsilon \to 0$ and $N \to \infty$ the expected mean-field limit is given by

$$\begin{aligned}&\partial_t f_t(x) + \chi \nabla_x \cdot ((K * f_t(x)) f_t(x)) = \Delta_x f_t(x)\\ &f_0 = \mu_0.\end{aligned}$$

For more information about interacting particle systems with singular potentials as well as some generalizations see [Jabin and Wang, 2017] and references therein.

In this work we study so-called moderate interactions which we present in the next subsection.

1.2.4 Moderately Interacting Particle Systems

In the models described above only weak interaction of particles, namely the interaction of order $O\left(\frac{1}{N}\right)$, has been considered. In [Oelschläger, 1985] Oelschläger introduced and studied moderately interacting particle systems which are used to obtain local non-linear partial differential equations. The idea is to take a radially symmetric function V and then consider the interaction potential $V^\varepsilon(x) := \frac{1}{\varepsilon^d} V(\frac{x}{\varepsilon})$ for $\varepsilon > 0$, which converges to the Dirac delta in the sense of distributions. The resulted interaction of particles is of order $\varepsilon^{-d} N^{-1}$. With an appropriate chosen scaling $\varepsilon(N)$ one can obtain a non-linear diffusion equation by taking the limit $\varepsilon \to 0$ and $N \to \infty$.

Oelschläger studied moderately interacting particle systems in [Oelschläger, 1990] in order to derive equations of porous medium type. The deterministic Oelschläger's model was later extended by Philipowski in [Philipowski, 2007] where he considered a system of stochastic interacting particles in $\mathbb{R}^d$ which is given by

$$\begin{cases} dX_t^{N,i,\varepsilon,\delta} = -\frac{1}{N}\sum_{j=1}^N \nabla V^\varepsilon(X_t^{N,i,\varepsilon,\delta} - X_t^{N,j,\varepsilon,\delta})dt + \delta dB_t^i, \quad i = 1,\dots,N\\ X_0^{N,i,\varepsilon,\delta} = \zeta_i, \end{cases}$$

where $(B^i)_{i\in\mathbb{N}}$ is a sequence of independent Brownian motions and $(\zeta_i)_{i\in\mathbb{N}}$ is a sequence of i.i.d. random variables which are independent of the Brownian motions. The distribution of $(\zeta_i)_{i\in\mathbb{N}}$ is given by a smooth density function u_0. Philipowski proved that for each $t \geq 0$ the empirical measure $\mu_t^{N,\varepsilon,\delta} := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{N,i,\varepsilon,\delta}}$ converges weakly to a deterministic measure P_t on $\mathbb{R}^d$ as $N \to \infty, \varepsilon \to 0, \delta \to 0$ in a way that $N >> \frac{1}{\varepsilon}$ and $\varepsilon << \delta$. The measure P_t has a density $u(t,\cdot)$ which solves the porous medium equation

$$\begin{cases} \partial_t u = \frac{1}{2}\Delta(u^2) \;\text{ in } (0,\infty)\times\mathbb{R}^d\\ u(0,\cdot) = u_0. \end{cases}$$

He obtained the above result and the propagation of chaos for the logarithmic scaling. The Oelschläger's approach was generalized by [Chen et al., 2019] to derive a cross-diffusion system, in [Chen et al., 2021a] to obtain the SKT system where the moderate interaction appeared in the diffusion coefficient and in [Chen et al., 2021b] to derive a non-local porous medium equation.

1.2.5 Choice of the Scaling

Let us come back to the question how we compare particle systems (1.3) and (1.4). In the subsections before before we mentioned that we can derive estimates of $\mathbb{E}\left[\left|X_t^i - \bar{X}_t^i\right|^2\right]$ by using Grönwall's argument and obtain the convergence result. However, for moderately interacting particle systems this procedure works only for the logarithmic scaling which yields a relatively slow convergence rate. There exists another way how to establish that (1.3) can be indeed approximated by (1.4). However, due to the singularity of the potential one cannot expect that the trajectories of the regularized particle system and the particle system of the mean-field type remain close to each other on any finite time interval. Nevertheless, one can prove that the probability of the event that the maximal distance between these two trajectories is bigger than N^{-a}, where a is some real number between 0 and $\frac{1}{2}$, could be bounded by $N^{-\gamma}$ for an arbitrary $\gamma > 0$. This kind of results were studied in [García and Pickl, 2017], [Boers and Pickl, 2016] and [Lazarovici and Pickl, 2017]. The main idea is to define an auxiliary function to control the probability of the events described above and then precisely choose parameters of this function in order to bound it by $N^{-\gamma}$.

1.3 Keller-Segel Model

In this section we present a heuristic mathematical model from biology which describes chemotaxis, namely the collective motion of cells which are attracted or repelled by a chemical substance and are able to emit it. From biology there are two types of chemotaxis. If organisms move from a lower concentration towards a higher concentration of the chemical substance, it is called a positive chemotaxis. The opposite phenomenon is called a negative chemotaxis. Substances that induce chemotaxis are called chemoattractants or chemorepelents. One of the easiest ways to model chemotaxis is to assume that the chemotactic flux F is given by

$$F = \chi u \nabla v,$$

where u is the organism's density, v is the chemical concentration and χ is a so-called chemotactic sensitivity (see e.g. [Arumugam and Tyagi, 2021]). So, if we choose $\chi > 0$ then we obtain positive chemotaxis and vice versa. Since the chemical substance diffuses as well, one typically considers v as a solution to some elliptic or parabolic equations which are coupled with u.

The so-called Keller-Segel model was introduced by [Keller and Segel, 1970], [Keller and Segel, 1971] and [Patlak, 1953], which nowadays has many different modifications. Let us consider a general

Keller-Segel model from [Arumugam and Tyagi, 2021]

$$\begin{cases} \partial_t u = \nabla \cdot (\phi(u,v)\nabla u - \psi(u,v)\nabla v) + f(u,v), \\ \tau \partial_t v = d\Delta v + g(u,v) - h(u,v)v \end{cases} \tag{1.7}$$

In this system u denotes the density of organisms and v represents the concentration of some chemicals substance on a domain $\Omega \subset \mathbb{R}^n$. In the system (1.7) τ is a binary variable, i.e. if $\tau = 0$ then we call system (1.7) parabolic-elliptic and if $\tau = 1$ then parabolic-parabolic. Therefore, it is clear that for $\tau = 1$ it is necessary to have information about initial conditions for both unknown functions, namely $u(0,\cdot) = u_0$ and $v(0,\cdot) = v_0$, and for the case $\tau = 0$ only $u(0,\cdot) = u_0$ is required.

Now we explain the remaining terms appeared in (1.7). $\phi(u,v)$ describes organisms' diffusion, $\psi(u,v)$ their chemotactic sensitivity and $f(u,v)$ the growth rate. Therefore, we obtain the main feature of this model, namely the competition between the aggregation term $-\psi(u,v)\nabla v$ and the diffusion term $\phi(u,v)\nabla u$. So, the behavior of u depends on the choice of functions and parameters in (1.7). Especially it decides whether the aggregation will be balanced by the diffusion term, which ensures the global existence of u, or u will blow up. Moreover, $g(u,v)$ and $h(u,v)$ represent production and degradation of the chemical substances. For concrete examples we refer to Section 1.3 of [Arumugam and Tyagi, 2021].

For higher dimensions the aggregation effect of the Keller-Segel system is stronger than the diffusion effect, so the porous media type nonlinear diffusion has been introduced to balance the aggregation. Let us consider the following classical degenerate Keller-Segel system with a non-linear diffusion of porous medium type

$$\begin{cases} \partial_t u = -\nabla \cdot (u\nabla c) + \nabla \cdot (\nabla u^m), \\ -\Delta c = u(t,x), \\ u(0,x) = u_0(x) \qquad x \in \mathbb{R}^d, t > 0. \end{cases} \tag{1.8}$$

The main result of this work is to rigorously derive (1.8) from a stochastic interacting particle system for the sub-critical m.

Now let us discuss one of the main properties of the Keller-Segel model, which is the conservation of mass, namely

$$M_0 := \int_{\mathbb{R}^d} u_0(x)dx = \int_{\mathbb{R}^d} u(t,x)dx \text{ for } t \geq 0.$$

It is well-known that solution's behavior depends on the quantity M_0. Choosing $d = 2$ and $m = 1$ in (1.8) one obtains a system of partial differential equations which corresponds to the original work of [Keller and Segel, 1970]. The authors of [Blanchet et al., 2006] proved that for the sub-critical case, namely $0 < M_0 < 8\pi$, the solution exists globally and for the super-critical case, that is $M_0 > 8\pi$, there is a blow up in finite time. The main tool of [Blanchet et al., 2006] is a so-called free energy which is defined as

$$F[u] = \int_{\mathbb{R}^2} u \ln(u)\, dx - \frac{1}{2}\int_{\mathbb{R}^2} uc\, dx.$$

With the help this functional one obtains a priori estimates and then the asymptotic behavior of u. Later using the free energy method the authors of [Blanchet et al., 2008] considered the critical case, namely $M_0 = 8\pi$, and proved that the solution of (1.8) for $d = 2$ and $m = 1$ exists globally and blows up at the center of mass as time goes to infinity.

Let us now review some more general results. The behavior of the solution to (1.8) for $d \geq 3$ and $m = \frac{2(d-1)}{d} \in (1, 2)$ was studied in [Blanchet et al., 2009]. The main tool in this article is the following free energy functional

$$t \mapsto \mathcal{F}[u(t)] := \int_{\mathbb{R}^d} \frac{u^m(t,x)}{m-1} dx - \frac{C_d}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-2}} u(t,x)u(t,y)\, dx\, dy$$

where $C_d := \frac{\Gamma(d/2+1)}{d(d-2)\pi^{d/2}}$ as well as the entropy/entropy-dissipation relation. This means that $\mathcal{F}$ is connected to its time derivative which is the so-called entropy production

$$\frac{d}{dt}\mathcal{F}[u(t)] = -\int_{\mathbb{R}^d} u(t,x)\left|\nabla\left(\frac{m}{m-1}u^{m-1}(t,x) - c(t,x)\right)\right|^2 dx$$

where u is a positive fast-decaying solution to (1.8). Then using the Hardy-Littlewood-Sobolev inequality (see Theorem 4.3 [Lieb and Loss, 2001]) authors of [Blanchet et al., 2009] obtained a critical mass M_c and studied behavior of the solution to (1.8) for different M_0. The scaling $m = \frac{2(d-1)}{d}$ is invariant for the conservation of mass.

Authors of [Luckhaus and Sugiyama, 2006], [Sugiyama, 2006], [Sugiyama, 2007] and [Sugiyama and Kunii, 2006] proved that if $m > 2 - \frac{2}{d}$ then the uniqueness and existence of the solution to (1.8) holds for any initial data and if $1 < m \leq 2 - \frac{2}{d}$ then one observers either global existence or blow up for different initial data. Since the methods from [Sugiyama, 2007] play an important role in this thesis, we discuss them briefly here. The first step is to regularize degeneracy of the studied system of partial differential equations which is caused by the porous medium term. Using a fixed point argument one obtains a local solution to the regularized system. In order to obtain global existence, one needs to establish uniform L^m a priory estimates and then perform Moser's iteration which yields an L^∞ bound. Moreover, it is crucial to obtain estimates of the L^∞ norm of the gradient of the solution to the regularized problem. This is done by the so-called Bernstein's method which is explained in [Oleinik and Kruzhkov, 1961]. Then using standard estimates one obtains a weak solution of the degenerate system as a limit case of the regularized system of partial differential equations.

It was first proposed by Chen, Liu and Wang in [Chen et al., 2012] that the nonlinear diffusion term with $m_c = \frac{2d}{d+2}$ is also important for Patlak-Keller-Segel system from many points of view. In this case, the system has a family of stationary solutions which are quite similar to the two dimensional case from [Keller and Segel, 1971], which is also the sharp profile in Hardy-Littlewood-Sobolev inequality. Moreover, the entropy is conformal invariant. For $\frac{2d}{d+2} < m < \frac{2(d+1)}{d}$ a clear criterion for the classification of initial data is given in [Wang et al., 2016]. We refer to [Wang et al., 2016] for detailed information about these two critical exponents , namely, the exponent $\frac{2(d-1)}{d}$ which comes from the scaling invariance of the mass, and the exponent $\frac{2d}{d+2}$ which comes from the conformal invariance of the entropy.

1.4 Problem Setting

In this section we introduce main assumptions and the problem setting of this thesis.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_{t\geq 0}), \mathbb{P})$ be a complete filtered probability space. Let $N \in \mathbb{N}$ and $d \geq 2$. We consider d-dimensional $\mathcal{F}_t$-Brownian motions $\{(B_t^i)_{t\geq 0}\}_{i=1}^N$ which are assumed to be independent of each other. Now we introduce u_0.

Assumption 1 (Assumptions for u_0). $0 \leq u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$,

For this given u_0 there exists a sequence of initial data $\{u_0^\sigma\}_\sigma$, which satisfies the following conditions:

Initial data u_0^σ

$$\begin{aligned} &\{u_0^\sigma\}_\sigma \subset C_0^\infty(\mathbb{R}^d), \\ &u_0^\sigma \to u_0 \text{ in } L^s(\mathbb{R}^d) \text{ for } s \in [1,\infty) \text{ as } \sigma \to 0, \\ &\|u_0^\sigma\|_{L^q(\mathbb{R}^d)} \leq \|u_0\|_{L^q(\mathbb{R}^d)} \text{ for } q \in [1,\infty]. \end{aligned} \tag{1.9}$$

Now we assume that $\{\zeta_i\}_{i=1}^N$ are i.i.d. random variables, independent of $\{(B_t^i)_{t\geq 0}\}_{i=1}^N$ and have a common density function u_0^σ.

Recall the fundamental solution of Laplace equation which is denoted by Φ, namely

$$\Phi(x) = \begin{cases} \frac{C_d}{|x|^{d-2}}, & \text{if } d \geq 3, \\ -\frac{1}{2\pi}\ln|x|, & \text{if } d = 2, \end{cases}$$

where $C_d := \frac{\Gamma(d/2+1)}{d(d-2)\pi^{d/2}}$.

Let $W(|x|) \in C_0^\infty(\mathbb{R}^d)$ such that $W \geq 0$, $\int_{\mathbb{R}^d} W(x)dx = 1$ and define $V := W * W$. Since $V \in C_0^\infty(\mathbb{R}^d)$, it follows directly that $\forall n \in \mathbb{N}$: $\int_{\mathbb{R}^d} |x|^n V(x)dx < \infty$ which means that all moments of V are bounded. Furthermore, we denote

$$\begin{aligned} W^\varepsilon(x) &:= \frac{1}{\varepsilon^d} W(x/\varepsilon), \\ V^\varepsilon(x) &:= \frac{1}{\varepsilon^d} V(x/\varepsilon), \\ \Phi^\varepsilon &:= \Phi * V^\varepsilon. \end{aligned}$$

Moreover, we obtain that V is symmetric and ∇V is bounded and Lipschitz continuous with some constant L. We also know that ∇V^ε is Lipschitz continuous with constant $\frac{L}{\varepsilon^{d+2}}$ and bounded by $\frac{\|\nabla V\|_{L^\infty(\mathbb{R}^d)}}{\varepsilon^{d+1}}$. By the same argument we obtain that D^2V^ε and D^3V^ε are bounded by $\frac{\|D^2 V\|_{L^\infty(\mathbb{R}^d)}}{\varepsilon^{d+2}}$ and $\frac{\|D^3 V\|_{L^\infty(\mathbb{R}^d)}}{\varepsilon^{d+3}}$ respectively.

In this thesis we consider different assumptions for p and m.

Assumption 2 (Weak assumptions for p and m)**.**

$$p(u) = \frac{m}{m-1}u^{m-1} \textit{ for } m = 2 \textit{ or } m \geq 3.$$

Assumption 3 (Strong assumptions for p and m)**.**

$$p(u) = \frac{m}{m-1}u^{m-1} \textit{ for } m \in \Big(\mathbb{N} \cap (1, \frac{d}{2}+3]\Big) \cup (\frac{d}{2}+3, \infty).$$

Now let p_λ be the approximation of p such that $p_\lambda \in C^3(\mathbb{R}_+)$ and

$$p_\lambda(r) = \begin{cases} p(2/\lambda), & \text{if } r > 2/\lambda, \\ p(r), & \text{if } 2\lambda < r < 1/\lambda, \\ p(\lambda), & \text{if } 0 < r < \lambda. \end{cases} \tag{1.10}$$

So, we obtain that

$$\begin{aligned} \|p_\lambda'\|_{L^\infty(\mathbb{R}_+)} &\leq p'(1/\lambda) \leq C(m)\Big(\frac{1}{\lambda}\Big)^{m-2}, \\ \|p_\lambda''\|_{L^\infty(\mathbb{R}_+)} &\leq \max\{p''(2\lambda), p''(1/\lambda)\} \leq C(m)\Big(\frac{1}{\lambda}\Big)^{|m-3|}, \\ \|p_\lambda'''\|_{L^\infty(\mathbb{R}_+)} &\leq \max\{p'''(2\lambda), p'''(1/\lambda)\} \leq C(m)\Big(\frac{1}{\lambda}\Big)^{|m-4|}. \end{aligned}$$

Since our model contains an aggregation term and a diffusion term, we introduce $\varepsilon_k > 0$ and $\varepsilon_p > 0$. So, the regularized interacting particle system is described by the following system of stochastic differential equations, namely for $i \in \{1, \dots, N\}$ the equation for the i-th particle is given by

Regularized Particle Model

$$\begin{aligned} &dX_t^{N,i,\varepsilon,\sigma} = \frac{1}{N}\sum_{j=1}^N \nabla\Phi^{\varepsilon_k}(X_t^{N,i,\varepsilon,\sigma} - X_t^{N,j,\varepsilon,\sigma})dt - \nabla p_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_t^{N,i,\varepsilon,\sigma} - X_t^{N,j,\varepsilon,\sigma})\Big)dt + \sqrt{2\sigma}\, dB_t^i, \\ &X_0^{N,i,\varepsilon,\sigma} = \zeta^i, \end{aligned} \tag{1.11}$$

where $\sigma > 0$ is a positive real number which corresponds to the individual noise of each particle.

Then we estimate the difference between (1.11) and the following particle system of mean-field type.

Intermediate Particle Model

$$\begin{cases} d\bar{X}_t^{i,\varepsilon,\sigma} = \nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(t,\bar{X}_t^{i,\varepsilon,\sigma})dt - \nabla p_\lambda\Big(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(t,\bar{X}_t^{i,\varepsilon,\sigma})\Big)dt + \sqrt{2\sigma}dB_t^i, \\ \bar{X}_0^{i,\varepsilon,\sigma} = \zeta^i, \end{cases} \tag{1.12}$$

where $u^{\varepsilon,\sigma}(t,x)$ is the probability density function of $\bar{X}_t^{i,\varepsilon,\sigma}$ and solves the following system of partial differential equations

$$\begin{cases} \partial_t u^{\varepsilon,\sigma} = \sigma\Delta u^{\varepsilon,\sigma} - \nabla(u^{\varepsilon,\sigma}\nabla c^{\varepsilon,\sigma}) + \nabla\cdot(\nabla p_\lambda(V^{\varepsilon_p} * u^{\varepsilon,\sigma})u^{\varepsilon,\sigma}), \\ -\Delta c^{\varepsilon,\sigma} = u^{\varepsilon,\sigma} * V^{\varepsilon_k}(t,x), \\ u^{\varepsilon,\sigma}(0,x) = u_0^\sigma(x), \qquad x\in\mathbb{R}^d, t>0. \end{cases} \tag{1.13}$$

Then we estimate the difference between (1.12) and the particle system (1.14) for small $\varepsilon_k, \varepsilon_p$ and λ.

Final Particle Model

$$\begin{cases} d\hat{X}_t^{i,\sigma} = \nabla\Phi * u^\sigma(t,\hat{X}_t^{i,\sigma})dt - \nabla p(u^\sigma(t,\hat{X}_t^{i,\sigma}))dt + \sqrt{2\sigma}dB_t^i, \\ \hat{X}_0^{i,\sigma} = \zeta^i, \end{cases} \tag{1.14}$$

where $u^\sigma(t,x)$ is the probability density function of $\hat{X}_t^{i,\sigma}$ and solves the following system of partial differential equations

$$\begin{cases} \partial_t u^\sigma = \sigma\Delta u^\sigma - \nabla\cdot(u^\sigma\nabla c^\sigma) + \nabla\cdot(u^\sigma\nabla p(u^\sigma)), \\ -\Delta c^\sigma = u^\sigma(t,x), \\ u^\sigma(0,x) = u_0^\sigma(x), \qquad x\in\mathbb{R}^d, t>0. \end{cases} \tag{1.15}$$

Then we consider (1.15) and take $\sigma\to 0$ on the PDE level and so achieve that for each $t\geq 0$ the empirical measure $\mu_t^{N,\varepsilon,\sigma} := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{N,i,\varepsilon,\sigma}}$ of the particle system (1.11) converges to $u(t,x)$, as $N\to\infty$, $\varepsilon_k,\varepsilon_p\to 0$, $\lambda\to 0$ and $\sigma\to 0$ in the weak sense and $u(t,x)$ solves the following system of partial differential equations

$$\begin{cases} \partial_t u = -\nabla\cdot(u\nabla c) + \nabla\cdot(u\nabla p(u)), \\ -\Delta c = u(t,x), \\ u(0,x) = u_0(x) \qquad x\in\mathbb{R}^d, t>0. \end{cases} \tag{1.16}$$

As a corollary we prove propagation of chaos result.

1.5 Organization of the Thesis and Main Results

In Chapter 2 we study existence, uniqueness and regularity of u^σ and $u^{\varepsilon,\sigma}$. First, we prove the following theorem

Theorem 1.1. *Let $T>0$, u_0^σ be given by (1.9) and Assumption 2 holds, then (1.15) possesses a unique global solution u^σ such that*

- $u^\sigma \in L^\infty(0,T;W^{1,\infty}(\mathbb{R}^d)) \cap L^\infty(0,T;L^1(\mathbb{R}^d))$,
- $u^\sigma \in W_q^{3,1}((0,T)\times\mathbb{R}^d)$ *for any* $1<q<\infty$.

In order to prove this theorem we consider

$$\begin{cases} \partial_t u^\sigma = \nabla\cdot\left(\sigma\nabla u^\sigma - u^\sigma\nabla c^\sigma + \nabla(u^\sigma)^m\right), \\ -\Delta c^\sigma = u^\sigma(t,x), \\ u^\sigma(0,x) = u_0^\sigma(x), \qquad x\in\mathbb{R}^d, t>0. \end{cases}$$

and approximate it by

$$\begin{cases} \partial_t u_\eta^\sigma = \nabla\cdot\Big(\sigma\nabla u_\eta^\sigma - u_\eta^\sigma\nabla\Phi * u_\eta^\sigma + \nabla(u_\eta^\sigma+\eta)^m\Big), \\ u_\eta^\sigma(0,x) = u_0^\sigma(x), \end{cases}$$

in order to handle degeneracy from the porous medium term. Since this partial differential equation contains aggregation and diffusion terms, which are both non-linear, we need to perform a two-step fixed point argument. We introduce two spaces X_T and Y_T and consider a linear parabolic partial differential equation

$$\begin{cases} \partial_t w = \nabla\cdot\left(\sigma\nabla w - w\nabla\Phi*\xi + m(\rho+\eta)^{m-1}\nabla w\right), \\ w(0,x) = u_0^\sigma(x), \end{cases}$$

for $x\in\mathbb{R}^d, t>0$ and fixed $\xi\in X_T$, $\rho\in Y_T$, $\eta>0$. Then we ensure that it has a non-negative weak solution. Next, with the help of Schauder fixed-point theorem we handle the non-linearity caused by the porous medium term. In order to apply this technique, we need to derive H^s estimate for w, which allows us to construct a map for Schauder fixed-point theorem. Using Aubin-Lion's argument, we prove that this map is a compact operator and so obtain existence of the local solution in X_T to

$$\begin{cases} \partial_t v = \nabla\cdot\left(\sigma\nabla v - v\nabla\Phi*\xi + \nabla(v+\eta)^m\right), \\ v(0,x) = u_0^\sigma(x), \qquad x\in\mathbb{R}^d, t>0. \end{cases}$$

Then we ensure that the resulted fixed point is unique.

After we handle the porous medium term, we study the above equation for v in order to deal with the non-linearity caused by the aggregation term. Here we construct another map and using

Banach fixed-point theorem prove that it has a unique fixed point. In order to achieve it, we ensure that this map is well-defined and choose $t > 0$ small enough such that it is a contraction. So, we obtain that the partial differential equation for u_η^σ possesses a unique local solution in X_T.

The next step is to derive L^q estimate for u_η^σ using classical tools of the PDE analysis. Then we perform Moser's iteration technique in order to prove that the L^∞ norm of u_η^σ is bounded. Next, we show that the L^∞ norm of ∇u_η^σ is bounded. This procedure is quite technical. First, we assume that u_η^σ can be written as $u_\eta^\sigma = \psi(\bar{\mathrm{u}})$ where ψ is a function which will be determined later. Then using some basic computations we determine a corresponding partial differential equation for $|\nabla\bar{\mathrm{u}}|^2$. Next, we decompose $\mathbb{R}^d$ into finitely many sets and define a small number $\tilde{\omega}$ as well as a family of sets

$$\tilde{\Omega}_k(t) := \{x \in \mathbb{R}^d \mid (k-1)\tilde{\omega} \le u_\eta^\sigma(t,x) < k\tilde{\omega}\} \text{ for } k \in \mathbb{N}.$$

Later we show that there exists $k_0 \in \mathbb{N}$ such that

$$\tilde{\Omega}_k(t) \cap \tilde{\Omega}_j(t) = \emptyset \ \forall j,k \in \{1,\dots,k_0\} \text{ and } \mathbb{R}^d = \bigcup_{k=1}^{k_0} \tilde{\Omega}_k(t).$$

Then for any fixed $t \in [0,T]$, we define the operator $\psi_k(\bar{\mathrm{u}})$ as

$$u_\eta^\sigma(t,x) = \psi_k(\bar{\mathrm{u}}(t,x)) = (k-3)\tilde{\omega} + 4e\tilde{\omega}\int_0^{\bar{\mathrm{u}}(t,x)} e^{-s^2} ds \text{ in } x \in \bigcup_{i=k-1}^{k+1} \tilde{\Omega}_i(t).$$

and introduce $\phi_k(x)$ as a smooth function in $C^2(\mathbb{R}^d)$ such that $0 \le \phi_k(x) \le 1$ and

$$\phi_k(x) := \begin{cases} 1, & \text{for } x \in \tilde{\Omega}_k(t), \\ 0, & \text{for } x \in \mathbb{R}^d \setminus \bigcup_{j=k-1}^{k+1} \tilde{\Omega}_j(t). \end{cases}$$

After proving some basic estimates, we rigorously derive a Bochner type inequality for $|\nabla\bar{\mathrm{u}}(t,x)|^2\phi_k$ on $\tilde{\Omega} = \bigcup_{j=k-1}^{k+1} \tilde{\Omega}_j(t)$ in order to ensure that

$$\frac{d}{dt}\int_{\tilde{\Omega}} ||\nabla\bar{\mathrm{u}}(t,x)|^2\phi_k|^r dx \le -\frac{\sigma(r-1)}{r}\int_{\tilde{\Omega}} |\nabla(|\nabla\bar{\mathrm{u}}(t,x)|^2\phi_k)^{\frac{r}{2}}|^2 dx + r^3\mathcal{M}\int_{\tilde{\Omega}} ||\nabla\bar{\mathrm{u}}(t,x)|^2\phi_k|^r dx + r^3\mathcal{M}$$

where $\mathcal{M}$ is a positive number. Then using Moser's iteration technique we prove that $\||\nabla\bar{\mathrm{u}}(t,x)|^2\phi_k\|_{L^\infty(0,T;L^\infty(\tilde{\Omega}))}$ is bounded. This property implies that $\sup_{0<t<T}\|\nabla u_\eta^\sigma(t,\cdot)\|_{L^\infty(\tilde{\Omega}_k)}$ is bounded and by repeating this argument in $\tilde{\Omega}_k$ for $k \in \{1,\dots,k_0\}$ we obtain the bound for $\sup_{0<t<T}\|\nabla u_\eta^\sigma(t,\cdot)\|_{L^\infty(\mathbb{R}^d)}$.

Combining all above arguments we define for $s > \frac{d}{2} + 3$

$$T_{\max} := \sup\{T' \in (0,T) \mid (2.1) \text{ possesses solution } u \in L^\infty(0,T';H^s(\mathbb{R}^d))\}.$$

and prove that

$$\|u_\eta^\sigma\|_{L^\infty(0,T';H^s(\mathbb{R}^d))} \le C, \ \forall T' < T_{\max}$$

and so obtain global existence of u_η^σ.

Later we take limit of u_η^σ as $\eta \to 0$ and so obtain a solution u^σ to (2.1). Next, we derive L^∞ estimates of u^σ independent of σ and establish its regularity results.

Next we study the non-local equation (1.13). Since $u^{\varepsilon,\sigma}$ solves a non-local partial differential equation, we use the perturbation method and study $u^\sigma - u^{\varepsilon,\sigma}$. Applying this technique we use regularity of u^σ and the advantage of the initial data being 0. Furthermore, in order to get uniform estimates, we consider only sufficiently small ε_k, ε_p and λ. In Section 2.2 we prove the following theorem.

Theorem 1.2. *Let $T>0$, $s>\frac{d}{2}+2$, u_0^σ be given by (1.9) and Assumption 3 holds. For ε_k, ε_p and λ small enough there exists a unique global solution $u^{\varepsilon,\sigma}\in L^\infty(0,T;H^s(\mathbb{R}^d))$ of (1.13) and a constant $C>0$ independent of $\varepsilon_k,\varepsilon_p$ and λ such that*

$$\|u^\sigma - u^{\varepsilon,\sigma}\|_{L^\infty(0,T;H^s(\mathbb{R}^d))} \le C(\varepsilon_k+\varepsilon_p).$$

The proof of this theorem is based on the fixed point argument. First, we study the following linear partial differential equation

$$\begin{aligned}
\partial_t \mathbf{u} - \sigma\Delta\mathbf{u} = \nabla\cdot\Big(&(\mathbf{u}+u^\sigma)\nabla(p_\lambda - p)(V^{\varepsilon_p} * (\mathbf{v}+u^\sigma)) + \mathbf{u}\nabla p(V^{\varepsilon_p} * (\mathbf{v}+u^\sigma))\\
&+ u^\sigma\nabla\int_0^1 p'(V^{\varepsilon_p} * u^\sigma + sV^{\varepsilon_p} * \mathbf{v})V^{\varepsilon_p} * \mathbf{u}\,ds + u^\sigma\Big(\nabla p(V^{\varepsilon_p} * u^\sigma) - \nabla p(u^\sigma)\Big)\Big)\\
&- \nabla\cdot\Big(\mathbf{u}\nabla\Phi * V^{\varepsilon_k} * \mathbf{v} + \mathbf{u}\nabla\Phi * V^{\varepsilon_k} * u^\sigma + u^\sigma\nabla\Phi * V^{\varepsilon_k} * (\mathbf{u}+u^\sigma) - u^\sigma\nabla\Phi * u^\sigma\Big),\\
\mathbf{u}(0,x) = 0,&
\end{aligned}$$

where $\mathbf{v}$ is a function from some space Y which we define later. Then we apply Banach fixed point theorem and so obtain a unique local solution of the equation

$$\begin{aligned}
\partial_t \mathbf{u} - \sigma\Delta\mathbf{u} = \nabla\cdot\Big(&(\mathbf{u}+u^\sigma)\nabla(p_\lambda - p)(V^{\varepsilon_p} * (\mathbf{u}+u^\sigma)) + \mathbf{u}\nabla p(V^{\varepsilon_p} * (\mathbf{u}+u^\sigma))\\
&+ u^\sigma\nabla\int_0^1 p'(V^{\varepsilon_p} * u^\sigma + sV^{\varepsilon_p} * \mathbf{u})V^{\varepsilon_p} * \mathbf{u}\,ds + u^\sigma\Big(\nabla p(V^{\varepsilon_p} * u^\sigma) - \nabla p(u^\sigma)\Big)\Big)\\
&- \nabla\cdot\Big(\mathbf{u}\nabla\Phi * V^{\varepsilon_k} * (\mathbf{u}+u^\sigma) + u^\sigma\nabla\Phi * V^{\varepsilon_k} * \mathbf{u} + u^\sigma(\nabla\Phi * V^{\varepsilon_k} * u^\sigma - \nabla\Phi * u^\sigma)\Big),\\
\tilde{\mathbf{u}}(0,x) = 0.&
\end{aligned}$$

Then we derive H^s estimates for $\mathbf{u}$ and so prove that

$$\begin{aligned}\partial_t \mathbf{u} - \sigma\Delta\mathbf{u} &= \nabla\cdot\Big((\mathbf{u}+u^\sigma)\nabla p_\lambda(V^{\varepsilon_p} * (\mathbf{u}+u^\sigma)) - u^\sigma\nabla p(u^\sigma)\Big)\\ &\quad - \nabla\cdot\Big((\mathbf{u}+u^\sigma)\mathbf{u}\nabla\Phi * V^{\varepsilon_k} * (\mathbf{u}+u^\sigma) - u^\sigma\nabla(\Phi * u^\sigma)\Big),\\ \mathbf{u}(0,x) &= 0.\end{aligned}$$

possesses a unique global solution in the strong sense. Then by Sobolev's embedding theorem we get the desired result.

In Chapter 3 we explain the connection between partial differential equations from Chapter 2 and our interacting particle systems. With the help of the theory of stochastic differential equations and duality analysis we prove that density functions of (1.12) and (1.14) solve (1.13) and (1.15) respectively. Then using results from Chapter 2 we derive estimates of $\sup\limits_{t\in[0,T]}\mathbb{E}\left[\sup\limits_{i=1,\dots,N}\left|\bar{X}_t^{i,\varepsilon,\sigma} - \hat{X}_t^{i,\sigma}\right|^2\right]$ and prove the following theorem

Theorem 1.3. *Let $T>0$ and assumptions from Theorem 1.1 and Theorem 1.2 be satisfied, then there exists a constant $C>0$ independent of N,λ,ε_k and ε_p such that*

$$\sup_{t\in[0,T]}\mathbb{E}\left[\sup_{i=1,\dots,N}\left|\bar{X}_t^{i,\varepsilon,\sigma} - \hat{X}_t^{i,\sigma}\right|^2\right] \le C(\varepsilon_k+\varepsilon_p)^2\ \exp\Big(CT\Big).$$

where $\bar{X}_t^{i,\varepsilon,\sigma}$ and $\hat{X}_t^{i,\sigma}$ solve (1.12) and (1.14) in the classical sense with the probability density functions (1.13) and (1.15) respectively.

In Chapter 4 we prove two important results. First, using standard techniques from analysis and independence of stochastic differential equations from (1.12) we derive estimates of

$$\sup_{t\in[0,T]}\sup_{i=1,\dots,N}\mathbb{E}\left[\left|X_t^{N,i,\varepsilon,\sigma} - \bar{X}_t^{i,\varepsilon,\sigma}\right|^2\right]$$

in order to obtain the logarithmic rate of convergence, namely

Theorem 1.4. *Let $T>0$, then for $\beta\in(0,1)$ and N large enough there exits a constant $C>0$ independent of N such that*

$$\sup_{t\in[0,T]}\sup_{i=1,\dots,N}\mathbb{E}\left[\left|X_t^{N,i,\varepsilon,\sigma} - \bar{X}_t^{i,\varepsilon,\sigma}\right|^2\right] \le CN^{-\beta}$$

where $X_t^{N,i,\varepsilon,\sigma}$ and $\bar{X}_t^{i,\varepsilon,\sigma}$ solve (1.11) and (1.12) respectively such that

$$\varepsilon_k = \Big(\frac{1}{\ln(N^{\alpha_k})}\Big)^{\frac{1}{d}}, \varepsilon_p = \Big(\frac{1}{\ln(N^{\alpha_p})}\Big)^{\frac{1}{dm-d+2}},\ \ \lambda = \frac{\varepsilon_p^d}{2},\ \ \textit{for } 0<\alpha_k+\alpha_p<\frac{1-\delta\cdot\frac{2dm-2d+2}{dm-d+2}-\beta}{\tilde{C}(T)}.$$

Here $\delta>0$ is an arbitrary small number and $\tilde{C}(T)$ is a positive constant which depends on T.

Secondly, using fine estimates we obtain convergence in probability result with the algebraic scaling, namely

Theorem 1.5. *Let $T > 0$ and β_k, β_p and a be positive numbers such that the following condition is satisfied*

$$\max\{(d+1)\beta_k, (d|m-4| + 3(d+1))\beta_p\} < a < \min\{\frac{1}{2} - (d-1)\beta_k, \frac{1}{2} - (2d+1+d|m-3|)\beta_p\}.$$

Then for an arbitrary $\gamma > 0$ there exists a constant $C(\gamma) > 0$, which is independent of N, such that

$$\sup_{0 \le t \le T} \mathbb{P}\left(\max_{i \in \{1,\dots,N\}} \left| X_t^{N,i,\varepsilon,\sigma} - \bar{X}_t^{i,\varepsilon,\sigma} \right| > N^{-a} \right) \le C(\gamma) N^{-\gamma},$$

where $X_t^{N,i,\varepsilon,\sigma}$ and $\bar{X}_t^{i,\varepsilon,\sigma}$ solve (1.11) and (1.12) respectively with $\varepsilon_k = N^{-\beta_k}$, $\varepsilon_p = N^{-\beta_p}$ and $\lambda = \frac{\varepsilon_p^d}{2}$.

Remark 1.6. From Theorem (1.5) we see that the algebraic cut-off rate can be chosen such that the following conditions are satisfied

$$0 < \beta_k < \frac{1}{4d}, \qquad 0 < \beta_p < \frac{1}{2(d|m-4| + d|m-3| + 5d + 4)}.$$

One of the main contributions of this thesis is the proof of Theorem 1.5. It is motivated by [García and Pickl, 2017], [Boers and Pickl, 2016] and [Lazarovici and Pickl, 2017]. Since the interaction potential is singular, one cannot expect convergence in expectation on the particle level. However, using an appropriate algebraic cut-off, one can obtain convergence in the sense of probability. Using the algebraic cut-off we construct a new framework to get the convergence rate in the sense of probability. We build up a suitable stopped process and then apply Markov's inequality, so that the problem is reduced to the estimation of the expectation of the stopped process. Then we study the time evolution of this expectation. Even for the dynamics of the stopped process we do not have Lipschitz continuity, hence we apply Taylor's expansion at the point which corresponds to (1.12). Moreover, we use the regularity of the solutions from the PDE analysis to absorb the singularity caused by the interaction potential. In the whole process we used several times a generalized version of the law of large numbers which is presented in Lemma 4.2.

As a corollary we get that

Corollary 1.7. *Let $T > 0$ and assumptions from Theorem 1.3 be satisfied.*

(a) If the assumptions of Theorem 1.4 are satisfied then for N large enough there exists a constant $C > 0$ independent of N such that

$$\sup_{t \in [0,T]} \sup_{i=1,\dots,N} \mathbb{E}\left[\left| X_t^{N,i,\varepsilon,\sigma} - \hat{X}_t^{i,\sigma} \right|^2 \right] \le C(\varepsilon_k + \varepsilon_p)^2$$

(b) If the assumptions of Theorem 1.5 are satisfied then there exists a constant $C > 0$ independent of N such that

$$\sup_{0\le t\le T} \mathbb{P}\left(\max_{i\in\{1,\dots,N\}} \left|X_t^{N,i,\varepsilon,\sigma} - \hat{X}_t^{i,\sigma}\right| > N^{-\frac{\min\{\beta_k,\beta_p\}}{2}}\right) \le C(\varepsilon_k + \varepsilon_p).$$

Finally, in Chapter 5 we let $\sigma \to 0$ and present the propagation of chaos result

Theorem 1.8. *Let $T > 0$ and Assumptions 1 and 2 be satisfied. Then there exists a subsequence of (u^σ) which converges weakly in $L^q(0,T;L^q(\mathbb{R}^d))$, $q \in [1,\infty)$ to u as $\sigma \to 0$ where u is a weak solution to (1.16) such that $u \in L^\infty(0,T;L^2(\mathbb{R}^d))$ and $u^m \in L^2(0,T;H^1(\mathbb{R}^d))$.*

Theorem 1.9. *Let $k \in \mathbb{N}$ and consider a k-tuple $(X_t^{N,1,\varepsilon,\sigma},\dots,X_t^{N,k,\varepsilon,\sigma})$. We denote by $P^k_{N,\varepsilon,\sigma}(t)$ the joint distribution of $(X_t^{N,1,\varepsilon,\sigma},\dots,X_t^{N,k,\varepsilon,\sigma})$. Then it holds that*

$$P^k_{N,\varepsilon,\sigma}(t) \text{ convergres weakly to } P^{\otimes k}(t)$$

as $N \to \infty$, $\varepsilon_k, \varepsilon_p, \lambda \to 0$ in the sense of Corollary 1.7 (a) or 1.7 (b), and $\sigma \to 0$ where $P(t)$ is a measure which is absolutely continuous with respect to the Lebesgue measure and has a probability density function $u(t,x)$ which solves (1.16) in the weak sense.

The results of this thesis are presented in the following two scientific articles: [Chen, Gvozdik and Li, 2022] and [Chen, Gvozdik, Holzinger and Li, 2022].

Chapter 2

Estimates on the PDE Level

In this chapter we prove Theorem 1.1 and Theorem 1.2.

Remark 2.1.

1. Throughout this work we denote by C a generic positive constant which can vary from line to line. If we want to specify parameters which influence C then we list them in the brackets $C(\cdot)$.

2. For simplicity we focus only on the case $d \geq 3$. Using the same methods, we can easily obtain the desired results for the case $d = 2$.

Here is some notation which is used in this thesis.

Definition 2.2 (Notation)**.**

- $|\cdot|$ is the Euclidean norm,
- $\nabla u = (\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_d})$, $D^2 u = (\frac{\partial^2 u}{\partial x_1 \partial x_1}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \dots, \frac{\partial^2 u}{\partial x_d \partial x_{d-1}}, \frac{\partial^2 u}{\partial x_d \partial x_d})$,
- $D^\alpha u := \Big\{ \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_d} x_d} \;\Big|\; \text{for } |\alpha| := \alpha_1 + \dots + \alpha_d \text{ and } \alpha_1, \dots, \alpha_d \in \mathbb{N} \Big\}$,
- $Q_T := \mathbb{R}^d \times (0, T)$,
- $H^s(\mathbb{R}^d) = W^{s,2}(\mathbb{R}^d)$,
- For $p \geq 1$ we define $W_p^{1,2}(Q_T)$ as

$$W_p^{1,2}(Q_T) := \Big\{ u \in L^p(0, T; W^{2,p}(\mathbb{R}^d)) \cap W^{1,p}(0, T; L^p(\mathbb{R}^d)) \;\Big|\; \|u\|_{W_p^{1,2}(Q_T)} < \infty \Big\}$$

where $\|u\|_{W_p^{1,2}(Q_T)} := \|u\|_{L^p(Q_T)} + \|\nabla u\|_{L^p(Q_T)} + \|D^2 u\|_{L^p(Q_T)} + \|\partial_t u\|_{L^p(Q_T)}$.

2.1 Partial Differential Equation from the Final Particle Model

The main goal of this section is to prove Theorem 1.1.

2.1.1 Existence of the Local Solution to (2.1). $H^s(\mathbb{R}^d)$ Theory

First, we consider the approximation of (1.15), namely

$$\begin{cases} \partial_t u_\eta^\sigma = \nabla \cdot \left(\sigma \nabla u_\eta^\sigma - u_\eta^\sigma \nabla \Phi * u_\eta^\sigma + \nabla (u_\eta^\sigma + \eta)^m\right), \\ u_\eta^\sigma(0,x) = u_0^\sigma(x), \end{cases} \tag{2.1}$$

for $x \in \mathbb{R}^d, t > 0$ and $\eta > 0$.

Moreover, we take $s \in (\frac{d}{2}+3, \infty) \cap \mathbb{N}$ and denote $M := 2\|u_0^\sigma\|_{H^s(\mathbb{R}^d)} + 1$. We assume that $\eta \leq M$ and consider the spaces

$$\begin{aligned} X_T :=& \Big\{ u \in L^\infty(0,T;H^s(\mathbb{R}^d)) \;\Big|\; \|u\|_{L^\infty(0,T;H^s(\mathbb{R}^d))} \leq M; u \geq 0 \text{ in } Q_T \Big\}, \\ Y_T :=& \Big\{ u \in L^4(0,T;H^s(\mathbb{R}^d)) \cap L^\infty(0,T;H^{s-1}(\mathbb{R}^d)) \;\Big|\; \|u\|_{L^4(0,T;H^s(\mathbb{R}^d)) \cap L^\infty(0,T;H^{s-1}(\mathbb{R}^d))} \leq M; \\ & u \geq 0 \text{ in } Q_T \Big\}. \end{aligned} \tag{2.2}$$

For fixed $\xi \in X_T$, $\rho \in Y_T$ and $\eta > 0$ consider the following linear equation

$$\begin{cases} \partial_t w = \nabla \cdot \left(\sigma \nabla w - w \nabla \Phi * \xi + m(\rho+\eta)^{m-1} \nabla w\right), \\ w(0,x) = u_0^\sigma(x), \end{cases} \tag{2.3}$$

for $x \in \mathbb{R}^d, t > 0$. By the theory of parabolic differential equations (see for example [Wu et al., 2006]) it follows that there exists a weak solution w to (2.3).

Proposition 2.3. *The weak solution w to (2.3) is non-negative.*

Proof. The solution w to (2.3) is non-negative by the following argument. Let w be a solution of (2.3) and $w^- := \min\{0, w\}$. If we multiply (2.3) by w^- and integrate it over $\mathbb{R}^d$, we obtain that

$$\begin{aligned} \int_{\mathbb{R}^d} w^- \partial_t w dx =& \int_{\mathbb{R}^d} w^- \nabla \cdot \left(\sigma \nabla w - w \nabla \Phi * \xi + m(\rho+\eta)^{m-1} \nabla w\right) dx \\ =& -\int_{\mathbb{R}^d} \sigma \nabla w^- \cdot \nabla w + \int_{\mathbb{R}^d} w \nabla w^- \cdot \nabla \Phi * \xi dx - m \int_{\mathbb{R}^d} (\rho+\eta)^{m-1} \nabla w^- \cdot \nabla w dx \\ =& -\int_{\mathbb{R}^d} \left(\sigma + m(\rho+\eta)^{m-1}\right) \nabla w^- \cdot \nabla w dx + \int_{\mathbb{R}^d} w \nabla w^- \cdot \nabla \Phi * \xi dx \end{aligned}$$

So, computations above together with Young's inequality imply that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w^-\|^2_{L^2(\mathbb{R}^d)} &= -\int_{\mathbb{R}^d}\left(\sigma + m(\rho+\eta)^{m-1}\right)|\nabla w^-|^2 dx + \int_{\mathbb{R}^d} w^- \nabla w^- \cdot \nabla\Phi * \xi dx \\
&= -\int_{\mathbb{R}^d}\left(\sigma + m(\rho+\eta)^{m-1}\right)|\nabla w^-|^2 dx \\
&+ \int_{\mathbb{R}^d}\left(\sigma + m(\rho+\eta)^{m-1}\right)^{\frac{1}{2}} \frac{1}{(\sigma + m(\rho+\eta)^{m-1})^{\frac{1}{2}}} w^- \nabla w^- \cdot \nabla\Phi * \xi dx \\
&\le -\int_{\mathbb{R}^d}\left(\sigma + m(\rho+\eta)^{m-1}\right)|\nabla w^-|^2 dx + \frac{1}{2}\int_{\mathbb{R}^d}\left(\sigma + m(\rho+\eta)^{m-1}\right)|\nabla w^-|^2 dx \\
&+ \frac{1}{2}\int_{\mathbb{R}^d}\frac{1}{\sigma + m(\rho+\eta)^{m-1}}|w^-|^2|\nabla\Phi * \xi|^2 dx \\
&\le -\frac{1}{2}\int_{\mathbb{R}^d}\left(\sigma + m(\rho+\eta)^{m-1}\right)|\nabla w^-|^2 dx + \frac{1}{2\sigma}\|\nabla\Phi * \xi\|^2_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\int_{\mathbb{R}^d}|w^-|^2 dx \\
&\le \frac{1}{2\sigma}\|\nabla\Phi * \xi\|^2_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\int_{\mathbb{R}^d}|w^-|^2 dx.
\end{aligned}
$$

Since $\xi \in X_T$ and $-\Delta(\Phi * \xi) = \xi$, there exists a constant $C > 0$ such that

$$\|\nabla\Phi * \xi\|_{L^\infty(0,T;H^s(\mathbb{R}^d))} \le C\|\xi\|_{L^\infty(0,T;H^s(\mathbb{R}^d))} \le CM.$$

Since $s > \frac{d}{2}$, Sobolev's inequality implies that

$$\|\nabla\Phi * \xi\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \le C\|\nabla\Phi * \xi\|_{L^\infty(0,T;H^s(\mathbb{R}^d))}.$$

So, by Grönwall's inequality we deduce that

$$\sup_{t\in(0,T)}\|w^-(t,\cdot)\|_{L^2(\mathbb{R}^d)} \le \exp\left(\frac{1}{\sigma}\|\nabla\Phi * \xi\|^2_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}T\right)\|w^-(0,\cdot)\|_{L^2(\mathbb{R}^d)} = 0$$

since $w^-(0,\cdot) = u^-(0,\cdot) = 0$.

This implies that $w \ge 0$ for all $t \in [0,T)$.

□

Proposition 2.4. *There exists $T_1 > 0$ such that for a fixed $\xi \in X_T$ the following equation*

$$\begin{cases} \partial_t v = \nabla\cdot\left(\sigma\nabla v - v\nabla\Phi * \xi + \nabla(v+\eta)^m\right), \\ v(0,x) = u_0^\sigma(x), \qquad x \in \mathbb{R}^d, t > 0. \end{cases}$$

posses a unique local solution in $L^\infty(0,T_1;H^s(\mathbb{R}^d)) \cap L^2(0,T_1;H^{s+1}(\mathbb{R}^d))$.

Proof. First, we prove the existence of a local solution using Schauder fixed-point theorem (see e.g. [Novotny and Straskraba, 2004] p. 72). Then we establish uniqueness of this local solution.

Existence

In order to apply Schauder fixed-point theorem we need to show that $\|w(t)\|_{H^s(\mathbb{R}^d)} \le M$ for $t > 0$ small enough.

Since, $\|D^0(\rho+\eta)^{m-1}\|_{L^2(\mathbb{R}^d)}$ and $\|D^{1-1}(\rho+\eta)\|_{L^2(\mathbb{R}^d)}$ are not bounded, we need to handle the cases $\alpha = 0$ and $\alpha = 1$ separately.

Multiplying (2.3) by w and integrating it over $\mathbb{R}^d$ we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} w^2 dx + \sigma\int_{\mathbb{R}^d} |\nabla w|^2 dx + m\int_{\mathbb{R}^d} (\rho+\eta)^{m-1}|\nabla w|^2 dx \\
&= \int_{\mathbb{R}^d} w\nabla\Phi * \xi \cdot \nabla w dx \\
&\leq \frac{\sigma}{2}\int_{\mathbb{R}^d} |\nabla w|^2 dx + C(\sigma,m)\int_{\mathbb{R}^d} |w\nabla\Phi * \xi|^2 dx \\
&\leq \frac{\sigma}{2}\int_{\mathbb{R}^d} |\nabla w|^2 dx + C(\sigma,m)M^2\int_{\mathbb{R}^d} w^2 dx
\end{aligned} \tag{2.4}$$

Then we see that

$$\partial_t D^1 w = \nabla\cdot\Big(\sigma\nabla D^1 w - D^1(w\nabla\Phi * \xi) + mD^1((\rho+\eta)^{m-1}\nabla w)\Big).$$

If we multiply the equation above by $D^1 w$ and integrate it over $\mathbb{R}^d$, we obtain by product rule and Young's inequality that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |D^1 w|^2 dx + \sigma\int_{\mathbb{R}^d} |\nabla D^1 w|^2 dx + m\int_{\mathbb{R}^d} (\rho+\eta)^{m-1}|\nabla D^1 w|^2\, dx \\
&\leq m(m-1)\int_{\mathbb{R}^d} (\rho+\eta)^{m-2}|\nabla\rho|\cdot|\nabla w|\cdot|\nabla D^1 w|\, dx \\
&+ \int_{\mathbb{R}^d} |D^1(w\nabla\Phi * \xi)|\cdot|\nabla D^1 w|\, dx \\
&\leq \frac{\sigma}{2}\int_{\mathbb{R}^d} |\nabla D^1 w|^2 dx + C(\sigma,m)\|\rho+\eta\|^{2(m-2)}_{L^\infty(\mathbb{R}^d)}\|\nabla\rho\|^2_{L^\infty(\mathbb{R}^d)}\int_{\mathbb{R}^d} |\nabla w|^2 dx \\
&+ C(\sigma,m)\int_{\mathbb{R}^d} |D^1(w\nabla\Phi * \xi)|^2 dx \\
&\leq \frac{\sigma}{2}\int_{\mathbb{R}^d} |\nabla D^1 w|^2 dx + C(\sigma,m)M^{2(m-1)}\int_{\mathbb{R}^d} |\nabla w|^2 dx + C(\sigma,m)M^2\int_{\mathbb{R}^d} |D^1 w|^2 dx
\end{aligned} \tag{2.5}$$

In this step we denote by C a positive constant which depends on σ, η, m, d and s. Let α be a multi-index such that $2 \leq |\alpha| \leq s$. Now take the D^α derivative of (2.3) and obtain

$$\partial_t D^\alpha w = \nabla\cdot\Big(\sigma\nabla D^\alpha w - D^\alpha(w\nabla\Phi * \xi) + mD^\alpha((\rho+\eta)^{m-1}\nabla w)\Big).$$

Then multiply the above equation by $D^\alpha w$ and integrate it over $\mathbb{R}^d$ in order to obtain that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |D^\alpha w|^2 dx + \sigma\int_{\mathbb{R}^d} |\nabla D^\alpha w|^2 dx = &-\int_{\mathbb{R}^d} \nabla D^\alpha(w\nabla\Phi * \xi)\cdot D^\alpha w\, dx \\
&- m\int_{\mathbb{R}^d} D^\alpha((\rho+\eta)^{m-1}\nabla w)\cdot\nabla D^\alpha w\, dx.
\end{aligned}$$

Now we are going to estimate the terms on the right-hand side. By Young's inequality and

Proposition 2.1 on page 43 from [Majda, 1984] there exists a constant $C > 0$ such that

$$
\begin{aligned}
& -\int_{\mathbb{R}^d} \nabla D^\alpha (w \nabla \Phi * \xi) \cdot D^\alpha w \, dx \\
& = \int_{\mathbb{R}^d} D^\alpha (w \nabla \Phi * \xi) \cdot \nabla D^\alpha w \, dx \\
& \leq \frac{\sigma}{4} \int_{\mathbb{R}^d} |\nabla D^\alpha w|^2 \, dx + \frac{1}{\sigma} \int_{\mathbb{R}^d} |D^\alpha (w \nabla \Phi * \xi)|^2 dx \\
& \leq \frac{\sigma}{4} \int_{\mathbb{R}^d} |\nabla D^\alpha w|^2 \, dx + C \left(\|w\|_{L^\infty(\mathbb{R}^d)} \|D^\alpha (\nabla \Phi * \xi)\|_{L^2(\mathbb{R}^d)} + \|\nabla \Phi * \xi\|_{L^\infty(\mathbb{R}^d)} \|D^\alpha w\|_{L^2(\mathbb{R}^d)} \right)^2,
\end{aligned}
$$

Using $\|\nabla \Phi * \xi\|_{L^\infty(0,T;H^{s+1}(\mathbb{R}^d))} \leq C \|\xi\|_{L^\infty(0,T;H^s(\mathbb{R}^d))} \leq CM$ and the estimate above we obtain that

$$
-\int_{\mathbb{R}^d} \nabla D^\alpha (w \nabla \Phi * \xi) \cdot D^\alpha w \, dx \leq \frac{\sigma}{4} \int_{\mathbb{R}^d} |\nabla D^\alpha w|^2 \, dx + CM^2 \|w\|^2_{H^s(\mathbb{R}^d)}.
$$

Now let us come to the next term

$$
\begin{aligned}
& -m \int_{\mathbb{R}^d} D^\alpha ((\rho + \eta)^{m-1} \nabla w) \cdot \nabla D^\alpha w \, dx \\
& = -m \int_{\mathbb{R}^d} (\rho + \eta)^{m-1} D^\alpha \nabla w \cdot \nabla D^\alpha w \, dx \\
& - m \int_{\mathbb{R}^d} \left(D^\alpha ((\rho + \eta)^{m-1} \nabla w) - (\rho + \eta)^{m-1} D^\alpha \nabla w \right) \cdot \nabla D^\alpha w \, dx
\end{aligned}
$$

Using Young's inequality and Proposition 2.1 on page 43 from [Majda, 1984] , we have

$$
\begin{aligned}
& -m \int_{\mathbb{R}^d} D^\alpha ((\rho + \eta)^{m-1} \nabla w) \cdot \nabla D^\alpha w \, dx \\
& \leq -m \int_{\mathbb{R}^d} (\rho + \eta)^{m-1} |\nabla D^\alpha w|^2 dx + \frac{\sigma}{4} \int_{\mathbb{R}^d} |\nabla D^\alpha w|^2 dx \\
& + C \int_{\mathbb{R}^d} |D^\alpha ((\rho + \eta)^{m-1} \nabla w) - (\rho + \eta)^{m-1} D^\alpha \nabla w|^2 dx \\
& \leq -m \int_{\mathbb{R}^d} (\rho + \eta)^{m-1} |\nabla D^\alpha w|^2 dx + \frac{\sigma}{4} \int_{\mathbb{R}^d} |\nabla D^\alpha w|^2 dx \\
& + C \left(\|\nabla (\rho + \eta)^{m-1}\|_{L^\infty(\mathbb{R}^d)} \|D^\alpha w\|_{L^2(\mathbb{R}^d)} + \|\nabla w\|_{L^\infty(\mathbb{R}^d)} \|D^\alpha (\rho + \eta)^{m-1}\|_{L^2(\mathbb{R}^d)} \right)^2.
\end{aligned}
$$

So, combining the above estimate with (2.4) and (2.5) we deduce that

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|w\|^2_{H^s(\mathbb{R}^d)} + \frac{\sigma}{2} \sum_{|\alpha|=2}^{s} \int_{\mathbb{R}^d} |\nabla D^\alpha w|^2 dx + m \sum_{|\alpha|=2}^{s} \int_{\mathbb{R}^d} (\rho + \eta)^{m-1} |\nabla D^\alpha w|^2 dx \\
& \leq CM^{2(m-1)} \|w\|^2_{H^s(\mathbb{R}^d)} + C \|\nabla (\rho + \eta)^{m-1}\|^2_{L^\infty(\mathbb{R}^d)} \|w\|^2_{H^s(\mathbb{R}^d)} + C \sum_{|\alpha|=2}^{s} \|D^\alpha (\rho + \eta)^{m-1}\|^2_{L^2(\mathbb{R}^d)} \|w\|^2_{H^s(\mathbb{R}^d)}
\end{aligned}
\tag{2.6}
$$

With the help of Grönwall's lemma we obtain that

$$
\|w(t)\|^2_{H^s(\mathbb{R}^d)} \leq \|u_0^\sigma\|^2_{H^s(\mathbb{R}^d)} \exp \left(C \int_0^t \left(M^{2(m-1)} + \|\nabla (\rho + \eta)^{m-1}\|^2_{L^\infty(\mathbb{R}^d)} + \sum_{|\alpha|=2}^{s} \|D^\alpha (\rho + \eta)^{m-1}\|^2_{L^2(\mathbb{R}^d)} \right) d\tau \right)
$$

Since $\rho \in Y_T$, using Hölder's inequality we obtain that

$$
\begin{aligned}
\int_0^t \|\nabla(\rho+\eta)^{m-1}\|^2_{L^\infty(\mathbb{R}^d)} d\tau &\leq C \int_0^t \|(\rho+\eta)^{m-2}\|^2_{L^\infty(\mathbb{R}^d)} \|\nabla \rho\|^2_{L^\infty(\mathbb{R}^d)} d\tau \\
&\leq C \operatorname*{ess\,sup}_{\tau \in (0,t)} \|(\rho+\eta)^{m-2}\|^2_{L^\infty(\mathbb{R}^d)} \int_0^t \|\rho\|^2_{H^s(\mathbb{R}^d)} d\tau \\
&\leq C \operatorname*{ess\,sup}_{\tau \in (0,t)} \|(\rho+\eta)^{m-2}\|^2_{L^\infty(\mathbb{R}^d)} \left(\int_0^t \|\rho\|^2_{H^s(\mathbb{R}^d)} d\tau \right)^{\frac{1}{2}} \left(\int_0^t 1 d\tau \right)^{\frac{1}{2}} \\
&= C \operatorname*{ess\,sup}_{\tau \in (0,t)} \|(\rho+\eta)^{m-2}\|^2_{L^\infty(\mathbb{R}^d)} t^{\frac{1}{2}} \|\rho\|^2_{L^4(0,T;H^s(\mathbb{R}^d))} \\
&\leq C \operatorname*{ess\,sup}_{\tau \in (0,t)} \|(\rho+\eta)^{m-2}\|^2_{L^\infty(\mathbb{R}^d)} t^{\frac{1}{2}} M^2 \\
&\leq C M^{2(m-1)} t^{\frac{1}{2}}.
\end{aligned}
$$

Now let us consider the term $\int_0^t \|D^\alpha(\rho+\eta)^{m-1}\|^2_{L^2(\mathbb{R}^d)} d\tau$

$$
\begin{aligned}
&\int_0^t \|D^\alpha(\rho+\eta)^{m-1}\|^2_{L^2(\mathbb{R}^d)} d\tau \\
&\leq C \int_0^t \left(\|(\rho+\eta)^{m-2} D^\alpha(\rho+\eta)\|^2_{L^2(\mathbb{R}^d)} + \|D^\alpha((\rho+\eta)^{m-2}(\rho+\eta)) - (\rho+\eta)^{m-2} D^\alpha(\rho+\eta)\|^2_{L^2(\mathbb{R}^d)} \right) d\tau \\
&\leq C \operatorname*{ess\,sup}_{\tau \in (0,t)} \|(\rho+\eta)^{m-2}\|^2_{L^\infty(\mathbb{R}^d)} \int_0^t \|D^\alpha(\rho+\eta)\|^2_{L^2(\mathbb{R}^d)} d\tau \\
&+ C \int_0^t \left(\|\nabla(\rho+\eta)^{m-2}\|_{L^\infty(\mathbb{R}^d)} \|D^{\alpha-1}(\rho+\eta)\|_{L^2(\mathbb{R}^d)} + \|(\rho+\eta)\|_{L^\infty(\mathbb{R}^d)} \|D^\alpha(\rho+\eta)^{m-2}\|_{L^2(\mathbb{R}^d)} \right)^2 d\tau.
\end{aligned}
$$

Now using Hölder's inequality we deduce that

$$
\begin{aligned}
\int_0^t \|D^\alpha(\rho+\eta)^{m-1}\|^2_{L^2(\mathbb{R}^d)} d\tau &\leq C M^{2(m-1)} t^{\frac{1}{2}} + C \operatorname*{ess\,sup}_{\tau \in (0,t)} \|D^{\alpha-1}(\rho+\eta)\|^2_{L^2(\mathbb{R}^d)} \int_0^t \|\nabla(\rho+\eta)^{m-2}\|^2_{L^\infty(\mathbb{R}^d)} d\tau \\
&+ C \operatorname*{ess\,sup}_{\tau \in (0,t)} \|(\rho+\eta)\|^2_{L^\infty(\mathbb{R}^d)} \int_0^t \|D^\alpha(\rho+\eta)^{m-2}\|^2_{L^2(\mathbb{R}^d)} d\tau \\
&\leq C M^{2(m-1)} t^{\frac{1}{2}} + C(M+\eta)^2 \int_0^t \|D^\alpha(\rho+\eta)^{m-2}\|^2_{L^2(\mathbb{R}^d)} d\tau,
\end{aligned}
$$

since $\operatorname*{ess\,sup}_{\tau \in (0,t)} \|D^{\alpha-1}(\rho+\eta)\|^2_{L^2(\mathbb{R}^d)} \leq \operatorname*{ess\,sup}_{\tau \in (0,t)} \|D^{\alpha-1}(\rho)\|^2_{L^2(\mathbb{R}^d)} \leq M^2$.

By iteration we obtain that

$$
\begin{aligned}
\int_0^t \|D^\alpha(\rho+\eta)^{m-1}\|^2_{L^2(\mathbb{R}^d)} d\tau &\leq C M^{2(m-1)} t^{\frac{1}{2}} + C M^{2\lfloor m \rfloor} \int_0^t \|D^\alpha(\rho+\eta)^{m-\lfloor m \rfloor}\|^2_{L^2(\mathbb{R}^d)} d\tau \\
&\leq C(M^{2(m-1)} + M^{2(\lfloor m \rfloor + s)}) t^{\frac{1}{2}}.
\end{aligned}
$$

So, we deduce that

$$
\|w(t)\|_{H^s(\mathbb{R}^d)} \leq \|u_0^\sigma\|_{H^s(\mathbb{R}^d)} \exp\left(C M^{2(m-1)} t + C M^{2(\lfloor m \rfloor + s)} t^{\frac{1}{2}} \right). \tag{2.7}
$$

Therefore,

$$\|w(t)\|_{L^\infty(0,T_1;H^s(\mathbb{R}^d))} \leq \|u_0^\sigma\|_{H^s(\mathbb{R}^d)} \exp\left(CM^{2(m-1)}T_1 + CM^{2(\lfloor m\rfloor+s)}T_1^{\frac{1}{2}}\right),$$

which is less or equal than M if we choose $T_1 > 0$ small enough.

Now we are able to construct a map

$$\begin{aligned} \mathcal{T} : Y_{T_1} &\to X_{T_1} \subset Y_{T_1}, \\ \rho &\mapsto w. \end{aligned} \tag{2.8}$$

Our goal is to apply Schauder fixed-point theorem, so we need to ensure that $\mathcal{T}$ is a compact operator.

Let $(\rho_i)_{i\in\mathbb{N}}$ be a uniformly bounded sequence in Y_{T_1}. Consider $w_i := \mathcal{T}(\rho_i)$ for $i \in \mathbb{N}$. Since we cannot apply Aubin-Lions Lemma (see for example [Simon, 1986]) directly, we use the following argument.

In order to prove that $H^s(\mathbb{R}^d)$ is compactly embedded in $H^{s-1}(\mathbb{R}^d)$ it is enough to prove that $H^1(\mathbb{R}^d)$ is compactly embedded in $L^2(\mathbb{R}^d)$. From Lemma A.1 in Appendix A we should prove that $\int_{\mathbb{R}^d} w(\cdot,x)|x|dx < \infty$. So, we multiply $\partial_t w = \nabla\cdot(\sigma\nabla w - w\nabla\Phi * \xi + m(\rho+\eta)^{m-1}\nabla w)$ by $|x|$ and integrate it over $\mathbb{R}^d$ in order to deduce that

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^d} w|x|dx &= -\sigma\int_{\mathbb{R}^d}\nabla w\cdot\nabla|x|dx + \int_{\mathbb{R}^d}(\nabla\Phi * \xi w)\cdot\nabla|x|dx - m\int_{\mathbb{R}^d}(\rho+\eta)^{m-1}\nabla w\cdot\nabla|x|dx \\ &\leq \sigma\int_{\mathbb{R}^d}|\nabla w|dx + \int_{\mathbb{R}^d}|\nabla\Phi * \xi w|dx + m\int_{\mathbb{R}^d}(\rho+\eta)^{m-1}|\nabla w|dx. \end{aligned}$$

The fact $\rho \in L^\infty(0,T_1;H^{s-1}(\mathbb{R}^d))$ implies that $\Delta\rho \in L^\infty(0,T_1;H^{s-3}(\mathbb{R}^d))$. Since $s > \frac{d}{2}+3$ we obtain that $L^\infty(0,T_1;H^{s-3}(\mathbb{R}^d)) \hookrightarrow L^\infty(0,T_1;L^\infty(\mathbb{R}^d))$. So,

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^d} w|x|dx &\leq \sigma\int_{\mathbb{R}^d}|\nabla w|dx + \int_{\mathbb{R}^d}|\nabla\Phi * \xi w|dx + m\int_{\mathbb{R}^d}(\rho+\eta)^{m-1}|\nabla w|dx \\ &\leq \sigma\|w\|_{H^s(\mathbb{R}^d)} + \left(\int_{\mathbb{R}^d}|\nabla\Phi * \xi|^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^d}|w|^2dx\right)^{\frac{1}{2}} + m\|\rho+\eta\|_{L^\infty(\mathbb{R}^d)}^{m-1}\int_{\mathbb{R}^d}|\nabla w|dx \\ &\leq \sigma M + M^2 + C(M+\eta)^{m-1}M \\ &\leq \sigma M + M^2 + C(2M)^m \\ &\leq CM^m, \end{aligned}$$

where we have used that

$$\|\nabla w\|_{L^1(\mathbb{R}^d)} \leq C\|\nabla w\|_{W^{1,d}(\mathbb{R}^d)} \leq C\|w\|_{W^{2,d}(\mathbb{R}^d)} \leq C\|w\|_{W^{1+\frac{d}{2},2}(\mathbb{R}^d)},$$

which holds by Sobolev's embedding theorem.

Therefore,

$$\int_{\mathbb{R}^d} w|x|dx \leq \left(CM^m + \int_{\mathbb{R}^d} u_0^\sigma |x|dx\right),$$

which is bounded since $\int_{\mathbb{R}^d} u_0^\sigma |x|dx < \infty$.

Let us come back to the sequence $(w_i)_{i\in\mathbb{N}}$. Since $(\rho_i)_{i\in\mathbb{N}}$ is a uniformly bounded sequence in Y_{T_1} we obtain by (2.6) that $\|w_i\|_{L^\infty(0,T_1;H^s(\mathbb{R}^d))} + \|w_i\|_{L^2(0,T_1;H^{s+1}(\mathbb{R}^d))} \leq C$ and $\|\partial_t w_i\|_{L^2(0,T_1;H^{s-1}(\mathbb{R}^d))} \leq C$ for all $i \in \mathbb{N}$. By the argument which was discussed above we can use Aubin-Lions Lemma and obtain that there exists $\bar{w} \in Y_{T_1}$ such that

$$w_i \to \bar{w} \text{ in } L^\infty(0,T_1;H^{s-1}(\mathbb{R}^d))\ ,\ w_i \to \bar{w} \text{ in } L^2(0,T_1;H^s(\mathbb{R}^d)) \text{ and } w_i \overset{*}{\rightharpoonup} \bar{w} \text{ in } L^\infty(0,T_1;H^s(\mathbb{R}^d)).$$

This implies that

$$\|\bar{w}\|_{L^\infty(0,T_1;H^s(\mathbb{R}^d))} \leq \liminf_{i\to\infty} \|w_i\|_{L^\infty(0,T_1;H^s(\mathbb{R}^d))} \leq C$$

By interpolation (see for example Chapter 1 from [Novotny and Straskraba, 2004]) we obtain that $\forall \gamma \in (0,1)$:

$$\begin{aligned}\|w_i - \bar{w}\|_{L^\infty(0,T_1;H^{s-\gamma}(\mathbb{R}^d))} &\leq \|w_i - \bar{w}\|^{1-\theta}_{L^\infty(0,T_1;H^{s-1}(\mathbb{R}^d))} \|w_i - \bar{w}\|^{\theta}_{L^\infty(0,T_1;H^s(\mathbb{R}^d))} \\ &\leq C\|w_i - \bar{w}\|^{1-\theta}_{L^\infty(0,T_1;H^{s-1}(\mathbb{R}^d))} \to 0 \text{ as } i \to \infty,\end{aligned}$$

where θ satisfies $s - \gamma = (1-\theta)(s-1) + \theta s$. This implies that

$$w_i \to \bar{w} \text{ in } L^\infty(0,T_1;H^{s-\frac{1}{2}}(\mathbb{R}^d)) \cap L^2(0,T_1;H^{s+\frac{1}{2}}(\mathbb{R}^d)) \quad \text{as } i \to \infty$$

and so by interpolation it follows that

$$w_i \to \bar{w} \text{ in } L^4(0,T_1;H^s(\mathbb{R}^d)) \text{ as } i \to \infty.$$

This proves that $\mathcal{T} : Y_{T_1} \to Y_{T_1}$ is a compact operator.

Therefore, by Schauder's fixed point theorem we obtain that there exists a local solution of (2.3) with ρ replaced by w which lies in X_{T_1} by (2.7).

Uniqueness

It remains to prove that the fixed point of $\mathcal{T}$ is unique. Let w_1 and w_2 be two fixed points of $\mathcal{T}$. So,

$$\begin{cases}\partial_t(w_1 - w_2) = \nabla\cdot(\sigma\nabla(w_1 - w_2) - (w_1 - w_2)\nabla\Phi * \xi + \nabla((w_1+\eta)^m - (w_2+\eta)^m)), \\ w_1(0,x) - w_2(0,x) = 0,\end{cases}$$

for $x \in \mathbb{R}^d$ and $t > 0$.

If we multiply the equation above by $w_1 - w_2$ and integrate it over $\mathbb{R}^d$ we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}|w_1-w_2|^2dx + \sigma\int_{\mathbb{R}^d}|\nabla(w_1-w_2)|^2dx\\
&= \int_{\mathbb{R}^d}(w_1-w_2)\nabla\Phi * \xi\cdot\nabla(w_1-w_2)dx\\
&- \int_{\mathbb{R}^d}\nabla\left((w_1+\eta)^m-(w_2+\eta)^m\right)\cdot\nabla(w_1-w_2)dx.
\end{aligned}$$

Since

$$\begin{aligned}
(w_1+\eta)^m(t,x)-(w_2+\eta)^m(t,x) &= \int_0^1 m(z(w_1+\eta)+(1-z)(w_2+\eta))^{m-1}dz((w_1+\eta)-(w_2+\eta))\\
&= \int_0^1 m(zw_1+(1-z)w_2+\eta)^{m-1}dz(w_1-w_2),
\end{aligned}$$

we can rewrite the term $-\int_{\mathbb{R}^d}\nabla\left((w_1+\eta)^m-(w_2+\eta)^m\right)\cdot\nabla(w_1-w_2)dx$ as

$$-\int_{\mathbb{R}^d}\nabla\left(\int_0^1 m(zw_1+(1-z)w_2+\eta)^{m-1}dz(w_1-w_2)\right)\cdot\nabla(w_1-w_2)dx.$$

Therefore, since $\nabla w_1, \nabla w_2 \in L^\infty(0,T;H^{s-1}(\mathbb{R}^d)) \hookrightarrow L^\infty(0,T;L^\infty(\mathbb{R}^d))$

$$\begin{aligned}
&-\int_{\mathbb{R}^d}\nabla\left(\int_0^1 m(zw_1+(1-z)w_2+\eta)^{m-1}dz(w_1-w_2)\right)\cdot\nabla(w_1-w_2)dx\\
&= -\int_{\mathbb{R}^d}\int_0^1 m(zw_1+(1-z)w_2+\eta)^{m-1}dz|\nabla(w_1-w_2)|^2dx\\
&-\int_{\mathbb{R}^d}\int_0^1 m(m-1)(zw_1+(1-z)w_2+\eta)^{m-2}(z\nabla w_1+(1-z)\nabla w_2)dz(w_1-w_2)\cdot\nabla(w_1-w_2))dx\\
&\le -\int_{\mathbb{R}^d}\int_0^1 m(zw_1+(1-z)w_2+\eta)^{m-1}dz|\nabla(w_1-w_2)|^2dx\\
&+\left\|\int_0^1 m(m-1)(zw_1+(1-z)w_2+\eta)^{m-2}(z\nabla w_1+(1-z)\nabla w_2)dz\right\|_{L^\infty(\mathbb{R}^d)}\\
&\quad\cdot\int_{\mathbb{R}^d}|w_1-w_2|\;|\nabla(w_1-w_2)|dx\\
&\le -\int_{\mathbb{R}^d}\int_0^1 m(zw_1+(1-z)w_2+\eta)^{m-1}dz|\nabla(w_1-w_2)|^2dx + CM^{m-1}\int_{\mathbb{R}^d}|w_1-w_2|\;|\nabla(w_1-w_2)|dx\\
&\le \frac{\sigma}{4}\int_{\mathbb{R}^d}|\nabla(w_1-w_2)|^2dx + CM^{m-1}\int_{\mathbb{R}^d}|w_1-w_2|^2dx.
\end{aligned}$$

Let us now analyze the term $\int_{\mathbb{R}^d}(w_1-w_2)\nabla\Phi*\xi\cdot\nabla(w_1-w_2)dx$. Since $\|\nabla\Phi*\xi\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \le CM$, we obtain by Young's inequality that

$$\int_{\mathbb{R}^d}(w_1-w_2)\nabla\Phi*\xi\cdot\nabla(w_1-w_2)dx \le \frac{\sigma}{4}\int_{\mathbb{R}^d}|\nabla(w_1-w_2)|^2dx + CM^2\int_{\mathbb{R}^d}|w_1-w_2|^2dx.$$

This implies that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}|w_1-w_2|^2dx+\frac{\sigma}{2}\int_{\mathbb{R}^d}|\nabla(w_1-w_2)|^2dx\\&\le C(M^{m-1}+M^2)\int_{\mathbb{R}^d}|w_1-w_2|^2dx.\end{aligned}$$

So,

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^d}|w_1(t,\cdot)-w_2(t,\cdot)|^2dx+\frac{\sigma}{2}\int_0^t\int_{\mathbb{R}^d}|\nabla(w_1-w_2)|^2dxds\\&\le C(M^{m-1}+M^2)\int_0^t\int_{\mathbb{R}^d}|w_1-w_2|^2dxds.\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^d}|w_1(t,\cdot)-w_2(t,\cdot)|^2dx\le 0.$$

This proves that the fixed point of $\mathcal{T}$ is unique. It implies that (2.3) with ρ replaced by w possesses a unique local solution in X_{T_1}.

□

Proposition 2.5. *There exists $T_2\in(0,T_1]$ such that (2.1) possesses a unique solution in X_{T_2}.*

Proof. We have proved so far that for a fixed $\xi\in X_{T_1}$ the following equation

$$\begin{cases}\partial_t v=\nabla\cdot\left(\sigma\nabla v-v\nabla\Phi*\xi+\nabla(v+\eta)^m\right),\\ v(0,x)=u_0^\sigma(x),\qquad x\in\mathbb{R}^d,t>0.\end{cases}\tag{2.9}$$

possesses a unique local solution in $L^\infty(0,T_1;H^s(\mathbb{R}^d))\cap L^2(0,T_1;H^{s+1}(\mathbb{R}^d))$.

Using the same technique as for the H^s estimates of (2.3) which we obtained in (2.7), we deduce that for $t>0$ small enough, the following operator is well-defined:

$$\begin{aligned}\mathcal{T}':X_{T_1}&\to X_{T_1},\\ \xi&\mapsto v.\end{aligned}\tag{2.10}$$

It remains to prove that $\mathcal{T}'$ is a contraction. Let $v_1,v_2\in X_{T_1}$ are solutions of (2.9) for $\xi_1,\xi_2\in X_{T_1}$ respectively. We deduce that

$$\partial_t(v_1-v_2)=\sigma\Delta(v_1-v_2)-\nabla\cdot(v_1\nabla\Phi*\xi_1-v_2\nabla\Phi*\xi_2)+\Delta((v_1+\eta)^m-(v_2+\eta)^m)$$

If we multiply the equation above by $v_1 - v_2$ and integrate it over $\mathbb{R}^d$ we obtain that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}|v_1-v_2|^2dx+\sigma\int_{\mathbb{R}^d}|\nabla(v_1-v_2)|^2dx\\&=\int_{\mathbb{R}^d}(v_1\nabla\Phi*\xi_1-v_2\nabla\Phi*\xi_2)\cdot\nabla(v_1-v_2)dx-\int_{\mathbb{R}^d}\nabla((v_1+\eta)^m-(v_2+\eta)^m)\cdot\nabla(v_1-v_2)dx.\end{aligned}$$

Since

$$\begin{aligned}(v_1+\eta)^m(t,x)-(v_2+\eta)^m(t,x)&=\int_0^1 m(z(v_1+\eta)+(1-z)(v_2+\eta))^{m-1}dz((v_1+\eta)-(v_2+\eta))\\&=\int_0^1 m(zv_1+(1-z)v_2+\eta)^{m-1}dz(v_1-v_2),\end{aligned}$$

we rewrite the term $-\int_{\mathbb{R}^d}\nabla((v_1+\eta)^m-(v_2+\eta)^m)\cdot\nabla(v_1-v_2)dx$ as

$$-\int_{\mathbb{R}^d}\nabla\left(\int_0^1 m(zv_1+(1-z)v_2+\eta)^{m-1}dz(v_1-v_2)\right)\cdot\nabla(v_1-v_2)dx.$$

Therefore, together with the embedding $\nabla v_1, \nabla v_2 \subset L^\infty(0,T_1;H^{s-1}(\mathbb{R}^d)) \hookrightarrow L^\infty(0,T_1;L^\infty(\mathbb{R}^d))$ it follows that

$$\begin{aligned}&-\int_{\mathbb{R}^d}\nabla\left(\int_0^1 m(zv_1+(1-z)v_2+\eta)^{m-1}dz(v_1-v_2)\right)\cdot\nabla(v_1-v_2)dx\\&=-\int_{\mathbb{R}^d}\int_0^1 m(zv_1+(1-z)v_2+\eta)^{m-1}dz|\nabla(v_1-v_2)|^2dx\\&-\int_{\mathbb{R}^d}\int_0^1 m(m-1)(zv_1+(1-z)v_2+\eta)^{m-2}(z\nabla v_1+(1-z)\nabla v_2)dz(v_1-v_2)\cdot\nabla(v_1-v_2))dx\\&\leq-\int_{\mathbb{R}^d}\int_0^1 m(zv_1+(1-z)v_2+\eta)^{m-1}dz|\nabla(v_1-v_2)|^2dx\\&+\left\|\int_0^1 m(m-1)(zv_1+(1-z)v_2+\eta)^{m-2}(z\nabla v_1+(1-z)\nabla v_2)dz\right\|_{L^\infty(\mathbb{R}^d)}\int_{\mathbb{R}^d}|v_1-v_2|\,|\nabla(v_1-v_2)|dx\\&\leq-\int_{\mathbb{R}^d}\int_0^1 m(zv_1+(1-z)v_2+\eta)^{m-1}dz|\nabla(v_1-v_2)|^2dx+CM^{m-1}\int_{\mathbb{R}^d}|v_1-v_2|\,|\nabla(v_1-v_2)|dx\\&\leq\frac{\sigma}{4}\int_{\mathbb{R}^d}|\nabla(v_1-v_2)|^2dx+CM^{m-1}\int_{\mathbb{R}^d}|v_1-v_2|^2dx,\end{aligned}$$

where C appeared in this step denotes a positive constant which depends on σ, d and m.

Let us now analyze the term $\int_{\mathbb{R}^d}(v_1\nabla\Phi*\xi_1-v_2\nabla\Phi*\xi_2)\cdot\nabla(v_1-v_2)dx$. Since $-\Delta\Phi*(\xi_1-\xi_2)=\xi_1-\xi_2$, we obtain that

$$\begin{aligned}&\int_{\mathbb{R}^d}(v_1\nabla\Phi*\xi_1-v_2\nabla\Phi*\xi_2)\cdot\nabla(v_1-v_2)dx\\&=\int_{\mathbb{R}^d}(v_1-v_2)\nabla\Phi*\xi_1\cdot\nabla(v_1-v_2)dx+\int_{\mathbb{R}^d}v_2(\nabla\Phi*\xi_1-\nabla\Phi*\xi_2)\cdot\nabla(v_1-v_2)dx\\&\leq CM\int_{\mathbb{R}^d}|v_1-v_2|\,|\nabla(v_1-v_2)|dx+CM\int_{\mathbb{R}^d}|\nabla\Phi*(\xi_1-\xi_2)||\nabla(v_1-v_2)|dx\end{aligned}$$

Since

$$\begin{aligned}
&\|\Delta\Phi * (\xi_1 - \xi_2)\|_{L^2(\mathbb{R}^d)} \le C\|\xi_1 - \xi_2\|_{L^2(\mathbb{R}^d)},\\
&\|\nabla\Phi * \xi_1\|_{L^\infty(\mathbb{R}^d)} \le C\|\nabla\Phi * \xi_1\|_{H^s(\mathbb{R}^d)} \le C\|\xi_1\|_{H^s(\mathbb{R}^d)},
\end{aligned}$$

we obtain that

$$\begin{aligned}
&CM\int_{\mathbb{R}^d} |v_1 - v_2|\,|\nabla(v_1 - v_2)|dx + CM\int_{\mathbb{R}^d} |\nabla\Phi * (\xi_1 - \xi_2)||\nabla(v_1 - v_2)|dx\\
&\le CM^2\int_{\mathbb{R}^d} |v_1 - v_2|^2 dx + \frac{\sigma}{4}\int_{\mathbb{R}^d} |\nabla(v_1 - v_2)|^2 dx\\
&+ CM\int_{\mathbb{R}^d} |\xi_1 - \xi_2|^2 dx + \frac{\sigma}{4}\int_{\mathbb{R}^d} |\nabla(v_1 - v_2)|^2 dx.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |v_1 - v_2|^2 dx + \frac{\sigma}{4}\int_{\mathbb{R}^d} |\nabla(v_1 - v_2)|^2 dx\\
&\le C(M^{m-1} + M^2)\int_{\mathbb{R}^d} |v_1 - v_2|^2 dx + CM^2\int_{\mathbb{R}^d} |\xi_1 - \xi_2|^2 dx.
\end{aligned}$$

So,

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^d} |v_1(t,\cdot) - v_2(t,\cdot)|^2 dx + \frac{\sigma}{4}\int_0^t\int_{\mathbb{R}^d} |\nabla(v_1 - v_2)|^2 dxds\\
&\le C(M^{m-1} + M^2)\int_0^t\int_{\mathbb{R}^d} |v_1 - v_2|^2 dxds + CM^2\int_0^t\int_{\mathbb{R}^d} |\xi_1 - \xi_2|^2 dxds\\
&\le C(M^{m-1} + M^2)\int_0^t\int_{\mathbb{R}^d} |v_1 - v_2|^2 dxds + CM^2 t \operatorname*{ess\,sup}_{\tau\in(0,t)}\int_{\mathbb{R}^d} |\xi_1 - \xi_2|^2 dx
\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^d} |v_1(t,\cdot) - v_2(t,\cdot)|^2 dx \le CM^2 \exp(Ct(M^{m-1} + M^2))t\|\xi_1 - \xi_2\|^2_{L^\infty(0,T_1;L^2(\mathbb{R}^d))}.$$

Hence,

$$\sup_{t\in(0,T_2)}\int_{\mathbb{R}^d} |v_1(t,\cdot) - v_2(t,\cdot)|^2 dx \le CM^2 \exp(CT_2(M^{m-1} + M^2))T_2\|\xi_1 - \xi_2\|^2_{L^\infty(0,T_1;L^2(\mathbb{R}^d))}$$

If we choose $T_2 > 0$ small enough we obtain that $\mathcal{T}'$ is a contraction. So, (2.1) possesses a unique local solution in X_T. $\square$

2.1.2 L^q Estimates of (2.1)

Proposition 2.6. *Let u_η^σ be the weak solution to* (2.1)*. Then it holds that*

$$\sup_{t\in(0,T_2)} \|u_\eta^\sigma\|_{L^q(\mathbb{R}^d)} \le C(d,m,q,\|u_0\|_{L^1(\mathbb{R}^d)},\|u_0\|_{L^q(\mathbb{R}^d)}).$$

Proof. If we multiply $\partial_t u_\eta^\sigma = \nabla\cdot\Big(\sigma\nabla u_\eta^\sigma - u_\eta^\sigma\nabla\Phi * u_\eta^\sigma + \nabla(u_\eta^\sigma+\eta)^m\Big)$ by $(u_\eta^\sigma)^{q-1}$ and integrate it over $\mathbb{R}^d$ we obtain that

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\int_{\mathbb{R}^d}(u_\eta^\sigma)^q dx = -\sigma(q-1)\int_{\mathbb{R}^d}(u_\eta^\sigma)^{q-2}|\nabla u_\eta^\sigma|^2dx + \int_{\mathbb{R}^d} u_\eta^\sigma\nabla\Phi * u_\eta^\sigma\cdot\nabla(u_\eta^\sigma)^{q-1}dx\\ - \int_{\mathbb{R}^d}\nabla(u_\eta^\sigma+\eta)^m\cdot\nabla(u_\eta^\sigma)^{q-1}dx\end{aligned}$$

Since u_η^σ is non-negative and

$$\Big|\nabla(u_\eta^\sigma)^{\frac{m+q-1}{2}}\Big|^2 = \Big|\frac{m+q-1}{2}(u_\eta^\sigma)^{\frac{m+q-3}{2}}\nabla u_\eta^\sigma\Big|^2 = \frac{(m+q-1)^2}{4}(u_\eta^\sigma)^{m+q-3}|\nabla u_\eta^\sigma|^2.$$

we deduce that

$$\begin{aligned}-\int_{\mathbb{R}^d}\nabla(u_\eta^\sigma+\eta)^m\cdot\nabla(u_\eta^\sigma)^{q-1}dx &= -m(q-1)\int_{\mathbb{R}^d}(u_\eta^\sigma)^{q-2}(u_\eta^\sigma+\eta)^{m-1}|\nabla u_\eta^\sigma|^2dx\\ &\le -m(q-1)\int_{\mathbb{R}^d}(u_\eta^\sigma)^{m+q-3}|\nabla u_\eta^\sigma|^2dx\\ &= -\frac{4m(q-1)}{(m+q-1)^2}\int_{\mathbb{R}^d}|\nabla(u_\eta^\sigma)^{\frac{m+q-1}{2}}|^2dx.\end{aligned}$$

Moreover,

$$\begin{aligned}\int_{\mathbb{R}^d}u_\eta^\sigma\nabla\Phi * u_\eta^\sigma\cdot\nabla(u_\eta^\sigma)^{q-1}dx &= (q-1)\int_{\mathbb{R}^d}\nabla\Phi * u_\eta^\sigma(u_\eta^\sigma)^{q-1}\cdot\nabla u_\eta^\sigma dx\\ &= \frac{q-1}{q}\int_{\mathbb{R}^d}\nabla\Phi * u_\eta^\sigma\cdot\nabla(u_\eta^\sigma)^q dx\\ &= -\frac{q-1}{q}\int_{\mathbb{R}^d}\Delta(\Phi * u_\eta^\sigma)(u_\eta^\sigma)^q dx\\ &= \frac{q-1}{q}\int_{\mathbb{R}^d}(u_\eta^\sigma)^{q+1}dx.\end{aligned}$$

Therefore, using interpolation (see [Novotny and Straskraba, 2004] Theorem 1.49 page 47) and Gagliardo-Nirenberg-Sobolev inequality (see [Evans, 2010] Theorem 1 page 263) we deduce that

$$\begin{aligned}
&\frac{d}{dt}\|u_\eta^\sigma\|_{L^q(\mathbb{R}^d)}^q + \sigma q(q-1)\int_{\mathbb{R}^d}(u_\eta^\sigma)^{q-2}|\nabla u_\eta^\sigma|^2 dx + \frac{4mq(q-1)}{(m+q-1)^2}\|\nabla (u_\eta^\sigma)^{\frac{m+q-1}{2}}\|_{L^2(\mathbb{R}^d)}^2 \\
&\le (q-1)\|u_\eta^\sigma\|_{L^{q+1}(\mathbb{R}^d)}^{q+1} \\
&\le (q-1)\left(\|u_\eta^\sigma\|_{L^1(\mathbb{R}^d)}^{1-\theta_1}\|u_\eta^\sigma\|_{L^{\frac{d(q+m-1)}{d-2}}(\mathbb{R}^d)}^{\theta_1}\right)^{q+1} \\
&= (q-1)\left(\|u_\eta^\sigma\|_{L^1(\mathbb{R}^d)}^{1-\theta_1}\|(u_\eta^\sigma)^{\frac{q+m-1}{2}}\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}^{\frac{2\theta_1}{q+m-1}}\right)^{q+1} \\
&\le C(d)(q-1)\left(\|u_\eta^\sigma\|_{L^1(\mathbb{R}^d)}^{1-\theta_1}\|\nabla(u_\eta^\sigma)^{\frac{q+m-1}{2}}\|_{L^2(\mathbb{R}^d)}^{\frac{2\theta_1}{p+m-1}}\right)^{q+1}
\end{aligned}$$

where $\frac{1}{q+1} = (1-\theta_1) + \frac{\theta_1}{\frac{d(q+m-1)}{d-2}}$

Since $m = 2$ or $m \ge 3$, it follows that $\frac{2\theta_1(q+1)}{q+m-1} < 2$, so we can ensure that $\frac{q+m-1}{\theta_1(q+1)} > 1$ and apply Young's inequality which implies that

$$\begin{aligned}
&C(d)(q-1)\left(\|u_\eta^\sigma\|_{L^1(\mathbb{R}^d)}^{1-\theta_1}\|\nabla(u_\eta^\sigma)^{\frac{q+m-1}{2}}\|_{L^2(\mathbb{R}^d)}^{\frac{2\theta_1}{q+m-1}}\right)^{q+1} \\
&\le \frac{2mq(q-1)}{(m+q-1)^2}\|\nabla(u_\eta^\sigma)^{\frac{q+m-1}{2}}\|_{L^2(\mathbb{R}^d)}^2 + C(d,q,m,\|u_0\|_{L^1(\mathbb{R}^d)}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{d}{dt}\|u_\eta^\sigma\|_{L^q(\mathbb{R}^d)}^q + \sigma q(q-1)\int_{\mathbb{R}^d}(u_\eta^\sigma)^{q-2}|\nabla u_\eta^\sigma|^2 dx + \frac{2mq(q-1)}{(m+q-1)^2}\|\nabla (u_\eta^\sigma)^{\frac{m+q-1}{2}}\|_{L^2(\mathbb{R}^d)}^2 \\
&\le C(d,q,m,\|u_0\|_{L^1(\mathbb{R}^d)})
\end{aligned}$$

By interpolation, Gagliardo-Nirenberg-Sobolev inequality and Young's inequality it follows that

$$\begin{aligned}
\|u_\eta^\sigma\|_{L^q(\mathbb{R}^d)}^q &\le C\|u_\eta^\sigma\|_{L^1(\mathbb{R}^d)}^{(1-\theta_2)q}\ \|\nabla(u_\eta^\sigma)^{\frac{m+q-1}{2}}\|_{L^2(\mathbb{R}^d)}^{\frac{2\theta_2}{m+q-1}q} \\
&\le \frac{2mq(q-1)}{(m+q-1)^2}\int_{\mathbb{R}^d}|\nabla(u_\eta^\sigma)^{\frac{m+q-1}{2}}|^2 dx + C(m,q,\|u_0\|_{L^1(\mathbb{R}^d)})
\end{aligned}$$

where $\frac{1}{q} = (1-\theta_2) + \frac{\theta_2}{\frac{d(q+m-1)}{d-2}}$.

Since

$$\sigma q(q-1)\int_{\mathbb{R}^d}(u_\eta^\sigma)^{q-2}|\nabla u_\eta^\sigma|^2 dx \ge 0,$$

we deduce that

$$\frac{d}{dt}\|u_\eta^\sigma\|_{L^q(\mathbb{R}^d)}^q \le -\|u_\eta^\sigma\|_{L^q(\mathbb{R}^d)}^q + C(d,m,q,\|u_0\|_{L^1(\mathbb{R}^d)}).$$

If we multiply both sides by e^t, it follows that

$$\frac{d}{dt}\left(e^t \|u_\eta^\sigma\|^q_{L^q(\mathbb{R}^d)}\right) \le Ce^t.$$

After applying integral on both side, we obtain that

$$e^t \|u_\eta^\sigma\|^q_{L^q(\mathbb{R}^d)} \le \|u_0\|^q_{L^q(\mathbb{R}^d)} + Ce^t.$$

After multiplying both sides by e^{-t}, it follows that

$$\sup_{t\in(0,T_2)} \|u_\eta^\sigma\|_{L^q(\mathbb{R}^d)} \le C(d, m, q, \|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^q(\mathbb{R}^d)}).$$

□

2.1.3 L^∞ estimates of (2.1)

Proposition 2.7. *Let u_η^σ be the weak solution to (2.1). Then it holds that*

$$\|u_\eta^\sigma(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \le C$$

for $t \in (0,T_2)$ where C is a positive constant which depends on $d, \sigma, m, \|u_0\|_{L^1(\mathbb{R}^d)}$ and $\|u_0\|_{L^\infty(\mathbb{R}^d)}$ but is independent of η.

Proof. In order to obtain the desired L^∞ regularity in time and $W^{2,\infty}$ regularity in space we will use Moser's iteration technique. Let $(q_k)_{k\in\mathbb{N}}$ such that $q_k \in (1,\infty)$ for all $k \in \mathbb{N}$ and $q_k \to \infty$ as $k \to \infty$.

First of all, we multiply the first equation from (2.1) by $q_k(u_\eta^\sigma)^{q_k-1}$ and obtain

$$\partial_t u_\eta^\sigma q_k(u_\eta^\sigma)^{q_k-1} = \sigma\Delta u_\eta^\sigma q_k(u_\eta^\sigma)^{q_k-1} - \nabla\cdot(u_\eta^\sigma\nabla\Phi * u_\eta^\sigma)q_k(u_\eta^\sigma)^{q_k-1} + \nabla\cdot(\nabla(u_\eta^\sigma+\eta)^m)q_k(u_\eta^\sigma)^{q_k-1}.$$

Then we integrate it with respect to $x \in \mathbb{R}^d$ and get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k}dx =& -\sigma q_k(q_k-1)\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k-2}|\nabla u_\eta^\sigma(t,x)|^2dx \\ &+ q_k(q_k-1)\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k-1}\nabla\Phi * u_\eta^\sigma(t,x)\cdot\nabla u_\eta^\sigma(t,x)dx \\ &- q_k(q_k-1)\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k-2}\nabla(u_\eta^\sigma(t,x)+\eta)^m\cdot\nabla u_\eta^\sigma(t,x)dx \\ =&: I_1+I_2+I_3\end{aligned} \tag{2.11}$$

We estimate (2.11) in three steps.

<u>**Step 1:** Estimates of I_2 and I_1.</u>

Let us start with the estimates of the term I_2. Taking $\|\nabla\Phi * u_\eta^\sigma(t,\cdot)\|_{L^\infty(\mathbb{R}^d)}$ out and using Hölder's inequality we deduce that

$$\begin{aligned} I_2 &= q_k(q_k-1)\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k-1}\nabla\Phi * u_\eta^\sigma(t,x)\cdot\nabla u_\eta^\sigma(t,x)dx \\ &\le q_k(q_k-1)\|\nabla\Phi * u_\eta^\sigma(t,\cdot)\|_{L^\infty(\mathbb{R}^d)}\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k-1}|\nabla u_\eta^\sigma(t,x)|dx \\ &\le q_k(q_k-1)\|\nabla\Phi * u_\eta^\sigma(t,\cdot)\|_{L^\infty(\mathbb{R}^d)}\left(\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k}dx\right)^{\frac12}\left(\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k-2}|\nabla u_\eta^\sigma(t,x)|^2dx\right)^{\frac12}\end{aligned}$$

Using Young's inequality for $\sigma_1 > 0$ we deduce that

$$\begin{aligned} &q_k(q_k-1)\|\nabla\Phi * u_\eta^\sigma(t,\cdot)\|_{L^\infty(\mathbb{R}^d)}\left(\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k}dx\right)^{\frac12}\left(\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k-2}|\nabla u_\eta^\sigma(t,x)|^2dx\right)^{\frac12} \\ &\le \sigma_1\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k-2}|\nabla u_\eta^\sigma(t,x)|^2dx + \frac{q_k^2(q_k-1)^2\|\nabla\Phi * u_\eta^\sigma(t,\cdot)\|^2_{L^\infty(\mathbb{R}^d)}}{4\sigma_1}\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k}dx.\end{aligned}$$

Choosing $\sigma_1 = \frac{\sigma q_k(q_k-1)}{4}$ we obtain that

$$\begin{aligned}
&q_k(q_k-1)\|\nabla\Phi * u_\eta^\sigma(t,\cdot)\|_{L^\infty(\mathbb{R}^d)}\left(\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k}dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k-2}|\nabla u_\eta^\sigma(t,x)|^2dx\right)^{\frac{1}{2}}\\
&\le \frac{\sigma q_k(q_k-1)}{4}\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k-2}|\nabla u_\eta^\sigma(t,x)|^2dx + \frac{q_k(q_k-1)}{\sigma}\|\nabla\Phi * u_\eta^\sigma(t,\cdot)\|^2_{L^\infty(\mathbb{R}^d)}\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k}dx.
\end{aligned}$$

Furthermore, we notice that $\frac{\sigma q_k(q_k-1)}{4}\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k-2}|\nabla u_\eta^\sigma(t,x)|^2dx$ can be written as

$$\frac{\sigma q_k(q_k-1)}{4}\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k-2}|\nabla u_\eta^\sigma(t,x)|^2dx = \frac{\sigma q_k(q_k-1)}{4}\frac{4}{q_k^2}\int_{\mathbb{R}^d}|\nabla(u_\eta^\sigma(t,x))^{\frac{q_k}{2}}|^2dx.$$

Since I_1 is nothing but

$$-\sigma q_k(q_k-1)\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k-2}|\nabla u_\eta^\sigma(t,x)|^2dx = -\sigma q_k(q_k-1)\frac{4}{q_k^2}\int_{\mathbb{R}^d}|\nabla(u_\eta^\sigma(t,x))^{\frac{q_k}{2}}|^2dx$$

and

$$\frac{\sigma q_k(q_k-1)}{4}\frac{4}{q_k^2} - \sigma q_k(q_k-1)\frac{4}{q_k^2} = -3\sigma\frac{(q_k-1)}{q_k},$$

it follows that

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k}dx \le &-3\sigma\frac{(q_k-1)}{q_k}\int_{\mathbb{R}^d}|\nabla(u_\eta^\sigma(t,x))^{\frac{q_k}{2}}|^2dx\\
&+\frac{q_k(q_k-1)}{\sigma}\|\nabla\Phi * u_\eta^\sigma(t,\cdot)\|^2_{L^\infty(\mathbb{R}^d)}\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k}dx\\
&-q_k(q_k-1)\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k-2}\nabla(u_\eta^\sigma(t,x)+\eta)^m\cdot\nabla u_\eta^\sigma(t,x)dx
\end{aligned}$$

Step 2: Estimates of $\|\nabla\Phi * u_\eta^\sigma(t,\cdot)\|_{L^\infty(\mathbb{R}^d)}$ and $\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k}dx$.

Choosing $\tilde{p} > d$ we obtain by Morrey's inequality (Theorem 9.12 p. 282 [Brezis and Brézis, 2011]), Calderon-Zygmund inequality (Theorem 9.9 p. 230 [Gilbarg and Trudinger, 2015]) and L^q estimates from the previous subsection that

$$\|\nabla\Phi * u_\eta^\sigma(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \le C\|\nabla\Phi * u_\eta^\sigma(t,\cdot)\|_{W^{1,\tilde{p}}(\mathbb{R}^d)} \le C\|u_\eta^\sigma(t,\cdot)\|_{L^{\tilde{p}}(\mathbb{R}^d)} \le C, \tag{2.12}$$

where C appeared in this subsection denotes a positive constant which depends on $d, m, \tilde{p}$, $\|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^{\tilde{p}}(\mathbb{R}^d)}$ and σ but independent of q_k. It is important to mention that we can use Theorem 9.9 p. 230 [Gilbarg and Trudinger, 2015] on $\mathbb{R}^d$ since the constant there does not dependent on the domain.

Moreover, for

$$\theta_1 = \frac{\frac{1}{q_{k-1}} - \frac{1}{q_k}}{\frac{1}{q_{k-1}} - \frac{d-2}{dq_k}}, \tag{2.13}$$

we obtain by interpolation that

$$
\begin{aligned}
\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k}dx &= \|u_\eta^\sigma(t,\cdot)\|^{q_k}_{L^{q_k}(\mathbb{R}^d)} \\
&\le \left(\|u_\eta^\sigma(t,\cdot)\|^{1-\theta_1}_{L^{q_k-1}(\mathbb{R}^d)}\|u_\eta^\sigma(t,\cdot)\|^{\theta_1}_{L^{\frac{dq_k}{d-2}}(\mathbb{R}^d)}\right)^{q_k} \\
&= \left(\|u_\eta^\sigma(t,\cdot)\|^{1-\theta_1}_{L^{q_k-1}(\mathbb{R}^d)}\|(u_\eta^\sigma(t,\cdot))^{\frac{q_k}{2}}\|^{\theta_1\frac{2}{q_k}}_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}\right)^{q_k}.
\end{aligned}
$$

By Gagliardo–Nirenberg-Sobolev inequality we deduce that

$$
\|(u_\eta^\sigma(t,\cdot))^{\frac{q_k}{2}}\|^{\theta_1\frac{2}{q_k}}_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \le S_d^{-\frac{\theta_1}{q_k}}\|\nabla(u_\eta^\sigma(t,\cdot))^{\frac{q_k}{2}}\|^{\theta_1\frac{2}{q_k}}_{L^2(\mathbb{R}^d)},
$$

where $S_d := \frac{d(d-2)}{4}2^{\frac{2}{d}}\pi^{1+\frac{1}{d}}\Gamma(\frac{d+1}{2})^{-\frac{2}{d}}$ is the best constant of the Sobolev inequality (see e.g. [Lieb and Loss, 2001]).

Therefore,

$$
\begin{aligned}
\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k}dx &\le \left(S_d^{-\frac{\theta_1}{q_k}}\|u_\eta^\sigma(t,\cdot)\|^{1-\theta_1}_{L^{q_k-1}(\mathbb{R}^d)}\|\nabla(u_\eta^\sigma(t,\cdot))^{\frac{q_k}{2}}\|^{\theta_1\frac{2}{q_k}}_{L^2(\mathbb{R}^d)}\right)^{q_k} \\
&= S_d^{-\theta_1}\|u_\eta^\sigma(t,\cdot)\|^{(1-\theta_1)q_k}_{L^{q_k-1}(\mathbb{R}^d)}\|\nabla(u_\eta^\sigma(t,\cdot))^{\frac{q_k}{2}}\|^{2\theta_1}_{L^2(\mathbb{R}^d)}
\end{aligned} \tag{2.14}
$$

So, (2.12) and (2.14) imply that

$$
\begin{aligned}
&\frac{q_k(q_k-1)}{\sigma}\|\nabla\Phi * u_\eta^\sigma(t,\cdot)\|^2_{L^\infty(\mathbb{R}^d)}\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k}dx \\
&\le \frac{q_k(q_k-1)}{\sigma}C\Big(S_d^{-\theta_1}\|u_\eta^\sigma(t,\cdot)\|^{(1-\theta_1)q_k}_{L^{q_k-1}(\mathbb{R}^d)}\|\nabla(u_\eta^\sigma(t,\cdot))^{\frac{q_k}{2}}\|^{2\theta_1}_{L^2(\mathbb{R}^d)}\Big)
\end{aligned}
$$

Using the following version of the Young's inequality from [Evans, 2010] page 622.

$$
ab \le c_1 a^{l_1} + (c_l l_1)^{-\frac{l_2}{l_1}}\ l_2^{-1}b^{l_2} \text{ where } \frac{1}{l_1}+\frac{1}{l_2}=1
$$

with

$$
\begin{aligned}
&a = \|\nabla(u_\eta^\sigma(t,\cdot))^{\frac{q_k}{2}}\|^{2\theta_1}_{L^2(\mathbb{R}^d)}, \qquad c_1 = \frac{\sigma(q_k-1)}{q_k}, \qquad l_1 = \frac{1}{\theta_1}, \qquad l_2 = \frac{1}{1-\theta_1}, \\
&b = \frac{q_k(q_k-1)}{\sigma}CS_d^{-\theta_1}\|u_\eta^\sigma(t,\cdot)\|^{(1-\theta_1)q_k}_{L^{q_k-1}(\mathbb{R}^d)},
\end{aligned}
$$

we obtain that

$$
\begin{aligned}
&\frac{q_k(q_k-1)}{\sigma} C\Big(S_d^{-\theta_1}\|u_\eta^\sigma(t,\cdot)\|_{L^{q_{k-1}}(\mathbb{R}^d)}^{(1-\theta_1)q_k}\|\nabla(u_\eta^\sigma(t,\cdot))^{\frac{q_k}{2}}\|_{L^2(\mathbb{R}^d)}^{2\theta_1}\Big)\\
&\leq \frac{\sigma(q_k-1)}{q_k}\|\nabla(u_\eta^\sigma(t,\cdot))^{\frac{q_k}{2}}\|_{L^2(\mathbb{R}^d)}^2\\
&+\left(\frac{\sigma(q_k-1)}{q_k}\frac{1}{\theta_1}\right)^{-\frac{\theta_1}{1-\theta_1}}(1-\theta_1)\left(\frac{q_k(q_k-1)}{\sigma}C\right)^{\frac{1}{1-\theta_1}} S_d^{-\frac{\theta_1}{1-\theta_1}}\|u_\eta^\sigma(t,\cdot)\|_{L^{q_{k-1}}(\mathbb{R}^d)}^{q_k}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k}dx\\
&\leq -2\sigma\frac{(q_k-1)}{q_k}\int_{\mathbb{R}^d}|\nabla(u_\eta^\sigma(t,x))^{\frac{q_k}{2}}|^2dx\\
&+\left(\frac{\sigma(q_k-1)}{q_k}\frac{1}{\theta_1}\right)^{-\frac{\theta_1}{1-\theta_1}}(1-\theta_1)\left(\frac{q_k(q_k-1)}{\sigma}C\right)^{\frac{1}{1-\theta_1}} S_d^{-\frac{\theta_1}{1-\theta_1}}\|u_\eta^\sigma(t,\cdot)\|_{L^{q_{k-1}}(\mathbb{R}^d)}^{q_k}\\
&-q_k(q_k-1)\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k-2}\nabla(u_\eta^\sigma+\eta)^m(t,x))\cdot\nabla u_\eta^\sigma(t,x)dx.
\end{aligned}
$$

Step 3 : <u>Estimates of the porous medium term I_3 and $-2\sigma\frac{(q_k-1)}{q_k}\int_{\mathbb{R}^d}|\nabla(u_\eta^\sigma(t,x))^{\frac{q_k}{2}}|^2dx$.</u>
Since

$$
\begin{aligned}
I_3 &= -q_k(q_k-1)\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k-2}\nabla(u_\eta^\sigma+\eta)^m(t,x))\cdot\nabla u_\eta^\sigma(t,x)dx\\
&= -mq_k(q_k-1)\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k-2}(u_\eta^\sigma(t,x)+\eta)^{m-1}|\nabla u_\eta^\sigma(t,x)|^2dx\\
&\leq -mq_k(q_k-1)\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{m+q_k-3}|\nabla u_\eta^\sigma(t,x)|^2dx\\
&= -\frac{4mq_k(q_k-1)}{(m+q_k-1)^2}\int_{\mathbb{R}^d}|\nabla(u_\eta^\sigma(t,x))^{\frac{m+q_k-1}{2}}|^2dx,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k}dx\\
&\leq -2\sigma\frac{(q_k-1)}{q_k}\int_{\mathbb{R}^d}|\nabla(u_\eta^\sigma(t,x))^{\frac{q_k}{2}}|^2dx\\
&+\left(\frac{\sigma(q_k-1)}{q_k}\frac{1}{\theta_1}\right)^{-\frac{\theta_1}{1-\theta_1}}(1-\theta_1)\left(\frac{q_k(q_k-1)}{\sigma}C\right)^{\frac{1}{1-\theta_1}} S_d^{-\frac{\theta_1}{1-\theta_1}}\|u_\eta^\sigma(t,\cdot)\|_{L^{q_{k-1}}(\mathbb{R}^d)}^{q_k}\\
&-\frac{4mq_k(q_k-1)}{(m+q_k-1)^2}\int_{\mathbb{R}^d}|\nabla(u_\eta^\sigma(t,x))^{\frac{m+q_k-1}{2}}|^2dx.
\end{aligned}
$$

Now we handle the term $-2\sigma\frac{(q_k-1)}{q_k}\int_{\mathbb{R}^d}|\nabla(u_\eta^\sigma(t,x))^{\frac{q_k}{2}}|^2dx$. Let us come back to the inequality

(2.14), namely

$$\int_{\mathbb{R}^d} (u_\eta^\sigma(t,x))^{q_k} dx \leq S_d^{-\theta_1} \|u_\eta^\sigma(t,\cdot)\|_{L^{q_k-1}(\mathbb{R}^d)}^{(1-\theta_1)q_k} \|\nabla (u_\eta^\sigma(t,\cdot))^{\frac{q_k}{2}}\|_{L^2(\mathbb{R}^d)}^{2\theta_1}.$$

Using Young's inequality

$$ab \leq c_1 a^{l_1} + (c_l l_1)^{-\frac{l_2}{l_1}} \, l_2^{-1} b^{l_2} \text{ where } \frac{1}{l_1} + \frac{1}{l_2} = 1$$

with

$$a = \|\nabla (u_\eta^\sigma(t,\cdot))^{\frac{q_k}{2}}\|_{L^2(\mathbb{R}^d)}^{2\theta_1}, \qquad c_1 = \frac{\sigma(q_k-1)}{q_k}, \qquad l_1 = \frac{1}{\theta_1}, \qquad l_2 = \frac{1}{1-\theta_1},$$
$$b = S_d^{-\theta_1} \|u_\eta^\sigma(t,\cdot)\|_{L^{q_k-1}(\mathbb{R}^d)}^{(1-\theta_1)q_k},$$

we obtain that

$$\int_{\mathbb{R}^d} (u_\eta^\sigma(t,x))^{q_k} dx \leq \frac{\sigma(q_k-1)}{q_k} \|\nabla (u_\eta^\sigma(t,\cdot))^{\frac{q_k}{2}}\|_{L^2(\mathbb{R}^d)}^2 + \left(\frac{\sigma(q_k-1)}{q_k} \frac{1}{\theta_1}\right)^{-\frac{\theta_1}{1-\theta_1}} (1-\theta_1) S_d^{-\frac{\theta_1}{1-\theta_1}} \|u_\eta^\sigma(t,\cdot)\|_{L^{q_k-1}(\mathbb{R}^d)}^{q_k}.$$

Therefore,

$$-\frac{\sigma(q_k-1)}{q_k} \|\nabla (u_\eta^\sigma(t,\cdot))^{\frac{q_k}{2}}\|_{L^2(\mathbb{R}^d)}^2 \leq -\int_{\mathbb{R}^d} (u_\eta^\sigma(t,x))^{q_k} dx + \left(\frac{\sigma(q_k-1)}{q_k} \frac{1}{\theta_1}\right)^{-\frac{\theta_1}{1-\theta_1}} (1-\theta_1) S_d^{-\frac{\theta_1}{1-\theta_1}} \|u_\eta^\sigma(t,\cdot)\|_{L^{q_k-1}(\mathbb{R}^d)}^{q_k}$$

Since

$$-2\sigma \frac{(q_k-1)}{q_k} \int_{\mathbb{R}^d} |\nabla (u_\eta^\sigma(t,x))^{\frac{q_k}{2}}|^2 dx \leq -\sigma \frac{(q_k-1)}{q_k} \int_{\mathbb{R}^d} |\nabla (u_\eta^\sigma(t,x))^{\frac{q_k}{2}}|^2 dx$$

and

$$-\frac{4mq_k(q_k-1)}{(m+q_k-1)^2} \int_{\mathbb{R}^d} |\nabla (u_\eta^\sigma(t,x))^{\frac{m+q_k-1}{2}}|^2 dx \leq 0,$$

we get that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k}dx\\
&\leq -\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k}dx\\
&+\left(\frac{\sigma(q_k-1)}{q_k}\frac{1}{\theta_1}\right)^{-\frac{\theta_1}{1-\theta_1}}(1-\theta_1)\left(\frac{q_k(q_k-1)}{\sigma}\right)^{\frac{1}{1-\theta_1}}S_d^{-\frac{\theta_1}{1-\theta_1}}\|u_\eta^\sigma(t,\cdot)\|_{L^{q_{k-1}}(\mathbb{R}^d)}^{q_k}\left(1+C^{\frac{1}{1-\theta_1}}\right).
\end{aligned}$$

Step 4 : Conditions for $(q_k)_{k\in\mathbb{N}}$.

Recall from (2.13) that $\theta_1=\frac{\frac{1}{q_{k-1}}-\frac{1}{q_k}}{\frac{1}{q_{k-1}}-\frac{d-2}{dq_k}}$. Since $1-\theta_1=\frac{\frac{1}{q_k}-\frac{d-2}{dq_k}}{\frac{1}{q_{k-1}}-\frac{d-2}{dq_k}}=\frac{\frac{2}{d}}{\frac{q_k}{q_{k-1}}-\frac{d-2}{d}}$, we need to ensure that condition $\frac{1}{1-\theta_1}\leq d$ holds. Therefore,

$$\begin{aligned}
&\frac{\frac{q_k}{q_{k-1}}-\frac{d-2}{d}}{\frac{2}{d}}\leq d\\
\Leftrightarrow\ &\frac{q_k}{q_{k-1}}-\frac{d-2}{d}\leq 2\\
\Leftrightarrow\ &\frac{q_k}{q_{k-1}}\leq 2+1-\frac{2}{d}=3-\frac{2}{d}.
\end{aligned}$$

So, we obtain that $(q_k)_{k\in\mathbb{N}}$ should satisfy

$$q_k\leq\left(3-\frac{2}{d}\right)q_{k-1},\quad\text{for } k\geq 2. \tag{2.15}$$

Let us choose $(q_k)_{k\in\mathbb{N}}$ as $q_k:=2^k+4d+4$ for $k\in\mathbb{N}$. Since $d\geq 3$, we obtain that $3-\frac{2}{d}\geq 2$. So $\left(3-\frac{2}{d}\right)q_{k-1}\geq 2q_{k-1}\geq q_k$. Therefore, $(q_k)_{k\in\mathbb{N}}$ with $q_k:=2^k+4d+4$ for $k\in\mathbb{N}$ satisfies (2.15).

Since

$$\theta_1=\frac{d(q_k-q_{k-1})}{d(q_k-q_{k-1})+2q_{k-1}}<1,$$

and $q_k(q_k-1)\leq q_k^2$, then

$$\frac{d}{dt}\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k}dx\leq-\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_k}dx+Cq_k^{2d}\left(\int_{\mathbb{R}^d}(u_\eta^\sigma(t,x))^{q_{k-1}}dx\right)^{\frac{q_k}{q_{k-1}}}.$$

Step 5 : Moser's iteration.

Let $y_k(t) := \|u_\eta^\sigma(t,\cdot)\|_{L^{q_k}(\mathbb{R}^d)}^{q_k}$. Since $\frac{q_k}{q_{k-1}} \leq 2$ we obtain that

$$\begin{aligned}
y_k'(t) &= \frac{d}{dt}\int_{\mathbb{R}^d} (u_\eta^\sigma(t,x))^{q_k}dx \\
&\leq -\int_{\mathbb{R}^d} (u_\eta^\sigma(t,x))^{q_k}dx + Cq_k^{2d}\left(\int_{\mathbb{R}^d} (u_\eta^\sigma(t,x))^{q_{k-1}}dx\right)^{\frac{q_k}{q_{k-1}}} \\
&= -y_k(t) + Cq_k^{2d} y_{k-1}^{\frac{q_k}{q_{k-1}}}(t) \\
&\leq -y_k(t) + Cq_k^{2d}\max\{1, y_{k-1}^2(t)\}.
\end{aligned}$$

So, mathematical induction implies that

$$\begin{aligned}
(e^t y_k(t))' &= e^t(y_k'(t) + y_k(t)) \\
&\leq e^t C q_k^{2d}\max\{1, \sup_{t\geq 0} y_{k-1}^2(t)\} \\
&\leq e^t C(4d)^{2d}4^{dk}\max\{1, \sup_{t\geq 0} y_{k-1}^2(t)\}.
\end{aligned}$$

Then we integrate both sides and deduce that

$$\begin{aligned}
e^t y_k(t) &\leq y_k(0) + C(4d)^{2d}4^{dk}\max\{1, \sup_{t\geq 0} y_{k-1}^2(t)\}\int_0^t e^s ds \\
&= y_k(0) + C(4d)^{2d}4^{dk}\max\{1, \sup_{t\geq 0} y_{k-1}^2(t)\}(e^t - 1).
\end{aligned}$$

We know that by interpolation it holds that

$$\begin{aligned}
y_k(0) &= \|u_0\|_{L^{q_k}(\mathbb{R}^d)}^{q_k} \\
&\leq (\max\{\|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)}\})^{q_k} \\
&\leq K_0^{q_k} = (K_0^{\frac{q_k}{2^k}})^{2^k} := K^{2^k},
\end{aligned}$$

where $K_0 := \max\{1, \|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)}\}$.

So, now we define $a_k := C(4d)^{2d}4^{dk}$. Since $e^{-t} \leq 1$ and $1 - e^{-t} \leq 1$ for $t \geq 0$, we deduce that

$$\begin{aligned}
y_k(t) &\leq e^{-t}y_k(0) + a_k\max\{1, \sup_{t\geq 0} y_{k-1}^2(t)\}e^{-t}(e^t - 1) \\
&= e^{-t}y_k(0) + a_k\max\{1, \sup_{t\geq 0} y_{k-1}^2(t)\}(1 - e^{-t}) \\
&\leq y_k(0) + a_k\max\{1, \sup_{t\geq 0} y_{k-1}^2(t)\} \\
&\leq a_k(K^{2^k} + \sup_{t\geq 0}(y_{k-1}^2(t)) \\
&\leq 2a_k\max\{K^{2^k}, \sup_{t\geq 0}(y_{k-1}^2(t))\}.
\end{aligned}$$

Therefore, by iteration it follows that

$$\begin{aligned}
y_k(t) &\le 2a_k \max\{K^{2^k}, \sup_{t\ge 0}(y_{k-1}^2(t))\} \\
&\le \prod_{j=0}^{k-1}(2a_{k-j})^{2^j} \max\{K^{2^k}, \sup_{t\ge 0}(y_0^{2^k}(t))\} \\
&= \prod_{j=0}^{k-1}\left(2C(4d)^{2d}4^{d(k-j)}\right)^{2^j} \max\{K^{2^k}, \sup_{t\ge 0}(y_0^{2^k}(t))\} \\
&= \prod_{j=0}^{k-1}\left(2C(4d)^{2d}4^{dk}\right)^{2^j} \prod_{j=0}^{k-1}\left(4^{-dj}\right)^{2^j} \max\{K^{2^k}, \sup_{t\ge 0}(y_0^{2^k}(t))\}.
\end{aligned}$$

Since

$$\prod_{j=0}^{k-1}\left(2C(4d)^{2d}4^{dk}\right)^{2^j} = \left(2C(4d)^{2d}4^{dk}\right)^{\sum_{j=0}^{k-1}2^j} = \left(2C(4d)^{2d}4^{dk}\right)^{\frac{1-2^k}{1-2}} = \left(2C(4d)^{2d}4^{dk}\right)^{2^k-1}$$

and

$$\prod_{j=0}^{k-1}\left(4^{-dj}\right)^{2^j} = \prod_{j=0}^{k-1}2^{-2dj2^j} = 2^{\sum_{j=0}^{k-1}-2dj2^j} = 2^{-2d\sum_{j=0}^{k-1}j2^j} = 2^{-2d\sum_{j=1}^{k-1}j2^j} = 2^{-2d((k-2)2^k+2)},$$

by $\sum_{j=1}^{k} j2^j = (k-1)2^{k+1}+2$, we deduce that

$$\begin{aligned}
y_k(t) &\le \left(2C(4d)^{2d}4^{dk}\right)^{2^k-1}\left(2^{-2d((k-2)2^k+2)}\right) \max\{K^{2^k}, \sup_{t\ge 0}(y_0^{2^k}(t))\} \\
&= \left(2C(4d)^{2d}\right)^{2^k-1}\left(4^d\right)^{k(2^k-1)}\left(4^d\right)^{-(k-2)2^k-2} \max\{K^{2^k}, \sup_{t\ge 0}(y_0^{2^k}(t))\} \\
&= \left(2C(4d)^{2d}\right)^{2^k-1}\left(4^d\right)^{2^{k+1}-k-2} \max\{K^{2^k}, \sup_{t\ge 0}(y_0^{2^k}(t))\}.
\end{aligned}$$

So,

$$\|u_\eta^\sigma(t,\cdot)\|_{L^{q_k}(\mathbb{R}^d)} \le \left(2C(4d)^{2d}\right)^{\frac{2^k-1}{q_k}}\left(4^d\right)^{\frac{2^{k+1}-k-2}{q_k}} \max\{K^{\frac{2^k}{q_k}}, \sup_{t\ge 0}(y_0^{\frac{2^k}{q_k}}(t))\}.$$

Since $q_k = 2^k + 4d + 4$ for $k \in \mathbb{N}$, it follows that $\frac{2^k-1}{q_k} \le 1$, $\frac{2^{k+1}-k-2}{q_k} \le 2$, $\frac{2^k}{q_k} \le 1$ therefore

$$\|u_\eta^\sigma(t,\cdot)\|_{L^{q_k}(\mathbb{R}^d)} \le 2C(4d)^{2d}4^{2d} \max\{K, \sup_{t\ge 0}(y_0(t))\}.$$

$\square$

2.1.4 L^∞ Estimates of the Gradient of (2.1)

Proposition 2.8. *Let u_η^σ be the weak solution to (2.1). Then it holds that*

$$\sup_{0<t<T_2} \|\nabla u_\eta^\sigma(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \leq C$$

where C is a positive constant which depends on $d, \sigma, m, \|u_0\|_{L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)}$ and $\|\nabla u_0^\sigma\|_{L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)}$ but is independent of η.

Proof. In this subsection we adapt the proof of Lemma 13 from [Sugiyama et al., 2007]. The main difference is that we use $\sigma\Delta u$ and not $\Delta(u+\eta)^m$ to control some terms as we derive the required estimates. Therefore, the resulted constant does not depend on η.

Step 1 : Some basic computations.

Let $u_\eta^\sigma = \psi(\bar{\mathrm{u}})$. So, we obtain that

$$\partial_t u_\eta^\sigma = \psi'(\bar{\mathrm{u}})\partial_t \bar{\mathrm{u}},$$

$$\begin{aligned}
\sigma\nabla\cdot\nabla u_\eta^\sigma &= \sigma\nabla\cdot(\psi'(\bar{\mathrm{u}})\nabla\bar{\mathrm{u}}) \\
&= \sigma\psi''(\bar{\mathrm{u}})\nabla\bar{\mathrm{u}}\cdot\nabla\bar{\mathrm{u}} + \sigma\psi'(\bar{\mathrm{u}})\nabla\cdot\nabla\bar{\mathrm{u}} \\
&= \sigma\left(\psi''(\bar{\mathrm{u}})|\nabla\bar{\mathrm{u}}|^2 + \psi'(\bar{\mathrm{u}})\Delta\bar{\mathrm{u}}\right),
\end{aligned}$$

$$\begin{aligned}
-\nabla\cdot(u_\eta^\sigma\nabla\Phi * u_\eta^\sigma) &= -\nabla u_\eta^\sigma\cdot\nabla\Phi * u_\eta^\sigma - u_\eta^\sigma\Delta\Phi * u_\eta^\sigma \\
&= -\left(\psi'(\bar{\mathrm{u}})\nabla\bar{\mathrm{u}}\cdot\nabla\Phi * u_\eta^\sigma + u_\eta^\sigma\Delta\Phi * u_\eta^\sigma\right),
\end{aligned}$$

$$\begin{aligned}
\nabla\cdot\nabla(u_\eta^\sigma+\eta)^m &= \nabla\cdot(m(u_\eta^\sigma+\eta)^{m-1}\nabla u_\eta^\sigma) \\
&= m(m-1)(u_\eta^\sigma+\eta)^{m-2}|\nabla u_\eta^\sigma|^2 + m(u_\eta^\sigma+\eta)^{m-1}\nabla\cdot(\psi'(\bar{\mathrm{u}})\nabla\bar{\mathrm{u}}) \\
&= m(m-1)(u_\eta^\sigma+\eta)^{m-2}|\nabla u_\eta^\sigma|^2 + m(u_\eta^\sigma+\eta)^{m-1}\left(\psi''(\bar{\mathrm{u}})|\nabla\bar{\mathrm{u}}|^2 + \psi'(\bar{\mathrm{u}})\Delta\bar{\mathrm{u}}\right).
\end{aligned}$$

This implies that

$$\begin{aligned}
\partial_t\bar{\mathrm{u}} = &\left(\sigma + m(u_\eta^\sigma+\eta)^{m-1}\right)\Delta\bar{\mathrm{u}} + \left(\sigma + m(u_\eta^\sigma+\eta)^{m-1}\right)\frac{\psi''(\bar{\mathrm{u}})}{\psi'(\bar{\mathrm{u}})}|\nabla\bar{\mathrm{u}}|^2 \\
&+ \frac{m(m-1)(u_\eta^\sigma+\eta)^{m-2}|\nabla u_\eta^\sigma|^2}{\psi'(\bar{\mathrm{u}})} - \frac{\nabla\cdot(u_\eta^\sigma\nabla\Phi * u_\eta^\sigma)}{\psi'(\bar{\mathrm{u}})}.
\end{aligned} \tag{2.16}$$

Now we take the derivative of (2.16) with respect to x_k for $k\in\{1,\ldots,d\}$.

Since

$$\partial_{x_k}\left(\frac{\psi''(\bar{u})}{\psi'(\bar{u})}\right)=\left(\frac{\psi'''(\bar{u})\psi'(\bar{u})-(\psi''(\bar{u}))^2}{(\psi'(\bar{u}))^2}\right)\partial_{x_k}\bar{u},$$
$$\partial_{x_k}|\nabla\bar{u}|^2=2\nabla\bar{u}\cdot\nabla\partial_{x_k}\bar{u},$$

we obtain that

$$\begin{aligned}\partial_{x_k}\left(\left(\sigma+m(u_\eta^\sigma+\eta)^{m-1}\right)\Delta\bar{u}\right)=&\left(m(m-1)(u_\eta^\sigma+\eta)^{m-2}\psi'(\bar{u})\partial_{x_k}\bar{u}\right)\Delta\bar{u}\\&+\left(\sigma+m(u_\eta^\sigma+\eta)^{m-1}\right)\partial_{x_k}\Delta\bar{u},\end{aligned}$$

$$\begin{aligned}\partial_{x_k}\left(\left(\sigma+m(u_\eta^\sigma+\eta)^{m-1}\right)\frac{\psi''(\bar{u})}{\psi'(\bar{u})}|\nabla\bar{u}|^2\right)=&\,m(m-1)(u_\eta^\sigma+\eta)^{m-2}\psi''(\bar{u})|\nabla\bar{u}|^2\partial_{x_k}\bar{u}\\&+\left(\sigma+m(u_\eta^\sigma+\eta)^{m-1}\right)|\nabla\bar{u}|^2\left(\frac{\psi'''(\bar{u})\psi'(\bar{u})-(\psi''(\bar{u}))^2}{(\psi'(\bar{u}))^2}\right)\partial_{x_k}\bar{u}\\&+2\left(\sigma+m(u_\eta^\sigma+\eta)^{m-1}\right)\frac{\psi''(\bar{u})}{\psi'(\bar{u})}\nabla\bar{u}\cdot\nabla\partial_{x_k}\bar{u},\end{aligned}$$

$$\begin{aligned}\partial_{x_k}\left(\frac{m(m-1)(u_\eta^\sigma+\eta)^{m-2}|\nabla u_\eta^\sigma|^2}{\psi'(\bar{u})}\right)=&\,m(m-1)(m-2)(u_\eta^\sigma+\eta)^{m-3}(\psi'(\bar{u}))^2|\nabla\bar{u}|^2\partial_{x_k}\bar{u}\\&+2m(m-1)(u_\eta^\sigma+\eta)^{m-2}\nabla\bar{u}\cdot\nabla\partial_{x_k}u_\eta^\sigma\\&-m(m-1)(u_\eta^\sigma+\eta)^{m-2}\psi''(\bar{u})|\nabla\bar{u}|^2\partial_{x_k}\bar{u}.\end{aligned}$$

Since the term $(m-1)(u_\eta^\sigma+\eta)^{m-2}\psi''(\bar{u})|\nabla\bar{u}|^2\partial_{x_k}\bar{u}$ cancels out, we deduce that

$$\begin{aligned}\partial_{x_k}\partial_t\bar{u}=&\left(m(m-1)(u_\eta^\sigma+\eta)^{m-2}\psi'(\bar{u})\partial_{x_k}\bar{u}\right)\Delta\bar{u}+\left(\sigma+m(u_\eta^\sigma+\eta)^{m-1}\right)\partial_{x_k}\Delta\bar{u}\\&+\left(\sigma+m(u_\eta^\sigma+\eta)^{m-1}\right)|\nabla\bar{u}|^2\left(\frac{\psi'''(\bar{u})\psi'(\bar{u})-(\psi''(\bar{u}))^2}{(\psi'(\bar{u}))^2}\right)\partial_{x_k}\bar{u}\\&+2\left(\sigma+m(u_\eta^\sigma+\eta)^{m-1}\right)\frac{\psi''(\bar{u})}{\psi'(\bar{u})}\nabla\bar{u}\cdot\nabla\partial_{x_k}\bar{u}\\&+m(m-1)(m-2)(u_\eta^\sigma+\eta)^{m-3}(\psi'(\bar{u}))^2|\nabla\bar{u}|^2\partial_{x_k}\bar{u}\\&+2m(m-1)(u_\eta^\sigma+\eta)^{m-2}\nabla\bar{u}\cdot\nabla\partial_{x_k}u_\eta^\sigma\\&-\frac{1}{\psi'(\bar{u})}\left(\nabla\partial_{x_k}u_\eta^\sigma\cdot\nabla\Phi*u_\eta^\sigma+\nabla u_\eta^\sigma\cdot\nabla\partial_{x_k}\Phi*u_\eta^\sigma+\partial_{x_k}u_\eta^\sigma\Delta\Phi*u_\eta^\sigma+u_\eta^\sigma\partial_{x_k}\Delta\Phi*u_\eta^\sigma\right)\\&+\frac{\psi''(\bar{u})}{(\psi'(\bar{u}))^2}\partial_{x_k}\bar{u}\left(\nabla u_\eta^\sigma\cdot\nabla\Phi*u_\eta^\sigma+u_\eta^\sigma\Delta\Phi*u_\eta^\sigma\right).\end{aligned}\tag{2.17}$$

Now we multiply (2.17) by $\partial_{x_k}\bar{u}$ and denote $|\partial_{x_k}\bar{u}|^2$ by $\bar{\mathfrak{U}}_k$, then we obtain that

$$
\begin{aligned}
\frac{1}{2}\partial_t\bar{\mathfrak{U}}_k =& \left(m(m-1)(u_\eta^\sigma+\eta)^{m-2}\psi'(\bar{\mathrm{u}})\right)\Delta\bar{\mathrm{u}}\ \bar{\mathfrak{U}}_k + \left(\sigma+m(u_\eta^\sigma+\eta)^{m-1}\right)\partial_{x_k}\Delta\bar{\mathrm{u}}\partial_{x_k}\bar{\mathrm{u}} \\
&+ \left(\sigma+m(u_\eta^\sigma+\eta)^{m-1}\right)\left(\frac{\psi'''(\bar{\mathrm{u}})\psi'(\bar{\mathrm{u}})-(\psi''(\bar{\mathrm{u}}))^2}{(\psi'(\bar{\mathrm{u}}))^2}\right)|\nabla\bar{\mathrm{u}}|^2\bar{\mathfrak{U}}_k \\
&+ 2\left(\sigma+m(u_\eta^\sigma+\eta)^{m-1}\right)\frac{\psi''(\bar{\mathrm{u}})}{\psi'(\bar{\mathrm{u}})}\nabla\bar{\mathrm{u}}\cdot\nabla\partial_{x_k}\bar{\mathrm{u}}\partial_{x_k}\bar{\mathrm{u}} \\
&+ m(m-1)(m-2)(u_\eta^\sigma+\eta)^{m-3}(\psi'(\bar{\mathrm{u}}))^2|\nabla\bar{\mathrm{u}}|^2\bar{\mathfrak{U}}_k \\
&+ 2m(m-1)(u_\eta^\sigma+\eta)^{m-2}\nabla\bar{\mathrm{u}}\ \cdot\nabla\partial_{x_k}u_\eta^\sigma\ \partial_{x_k}\bar{\mathrm{u}} \\
&- \frac{\partial_{x_k}\bar{\mathrm{u}}}{\psi'(\bar{\mathrm{u}})}\left(\nabla\partial_{x_k}u_\eta^\sigma\cdot\nabla\Phi*u_\eta^\sigma+\nabla u_\eta^\sigma\cdot\nabla\partial_{x_k}\Phi*u_\eta^\sigma+\partial_{x_k}u_\eta^\sigma\Delta\Phi*u_\eta^\sigma+u_\eta^\sigma\partial_{x_k}\Delta\Phi*u_\eta^\sigma\right) \\
&+ \frac{\psi''(\bar{\mathrm{u}})}{(\psi'(\bar{\mathrm{u}}))^2}\left(\nabla u_\eta^\sigma\cdot\nabla\Phi*u_\eta^\sigma+u_\eta^\sigma\Delta\Phi*u_\eta^\sigma\right)\bar{\mathfrak{U}}_k.
\end{aligned}
\tag{2.18}
$$

One can see that

$$
\begin{aligned}
&\Delta\bar{\mathfrak{U}}_k = \Delta(\partial_{x_k}\bar{\mathrm{u}})^2 = 2\nabla\cdot(\partial_{x_k}\bar{\mathrm{u}}\nabla\partial_{x_k}\bar{\mathrm{u}}) = 2\partial_{x_k}\Delta\bar{\mathrm{u}}\partial_{x_k}\bar{\mathrm{u}} + 2|\partial_{x_k}\nabla\bar{\mathrm{u}}|^2, \\
&\partial_{x_k}\Delta\bar{\mathrm{u}}\partial_{x_k}\bar{\mathrm{u}} = \frac{1}{2}\Delta\bar{\mathfrak{U}}_k - |\partial_{x_k}\nabla\bar{\mathrm{u}}|^2, \\
&\partial_{x_k}\nabla\bar{\mathrm{u}}\partial_{x_k}\bar{\mathrm{u}} = \frac{1}{2}\nabla(\partial_{x_k}\bar{\mathrm{u}})^2 = \frac{1}{2}\nabla\bar{\mathfrak{U}}_k
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\partial_{x_k}\nabla u_\eta^\sigma\partial_{x_k}\bar{\mathrm{u}} &= \partial_{x_k}\left(\psi'(\bar{\mathrm{u}})\nabla\bar{\mathrm{u}}\right)\partial_{x_k}\bar{\mathrm{u}} \\
&= \left(\psi''(\bar{\mathrm{u}})\partial_{x_k}\bar{\mathrm{u}}\nabla\bar{\mathrm{u}} + \psi'(\bar{\mathrm{u}})\nabla\partial_{x_k}\bar{\mathrm{u}}\right)\partial_{x_k}\bar{\mathrm{u}} \\
&= \psi''(\bar{\mathrm{u}})\nabla\bar{\mathrm{u}}\ \bar{\mathfrak{U}}_k + \frac{1}{2}\psi'(\bar{\mathrm{u}})\nabla\bar{\mathfrak{U}}_k.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
-\frac{\nabla\partial_{x_k}u_\eta^\sigma\cdot\nabla\Phi*u_\eta^\sigma}{\psi'(\bar{\mathrm{u}})}\partial_{x_k}\bar{\mathrm{u}} &= -\frac{1}{\psi'(\bar{\mathrm{u}})}\left(\psi'(\bar{\mathrm{u}})\nabla\partial_{x_k}\bar{\mathrm{u}} + \psi''(\bar{\mathrm{u}})\partial_{x_k}\bar{\mathrm{u}}\nabla\bar{\mathrm{u}}\right)\cdot\nabla\Phi*u_\eta^\sigma\partial_{x_k}\bar{\mathrm{u}} \\
&= -\partial_{x_k}\bar{\mathrm{u}}\nabla\partial_{x_k}\bar{\mathrm{u}}\cdot\nabla\Phi*u_\eta^\sigma - \frac{\psi''(\bar{\mathrm{u}})}{\psi'(\bar{\mathrm{u}})}|\partial_{x_k}\bar{\mathrm{u}}|^2\nabla\bar{\mathrm{u}}\cdot\nabla\Phi*u_\eta^\sigma \\
&= -\frac{1}{2}\nabla\bar{\mathfrak{U}}_k\cdot\nabla\Phi*u_\eta^\sigma - \frac{\psi''(\bar{\mathrm{u}})}{\psi'(\bar{\mathrm{u}})}\bar{\mathfrak{U}}_k\nabla\bar{\mathrm{u}}\cdot\nabla\Phi*u_\eta^\sigma.
\end{aligned}
$$

So, we obtain that

$$\begin{aligned}
\frac{1}{2}\partial_t\bar{\mathfrak{U}}_k &= \Big(m(m-1)(u^\sigma_\eta+\eta)^{m-2}\psi'(\bar{\mathfrak{u}})\Big)\Delta\bar{\mathfrak{u}}\;\bar{\mathfrak{U}}_k + \Big(\sigma+m(u^\sigma_\eta+\eta)^{m-1}\Big)\left(\frac{1}{2}\Delta\bar{\mathfrak{U}}_k - |\partial_{x_k}\nabla\bar{\mathfrak{u}}|^2\right) \\
&+ \Big(\sigma+m(u^\sigma_\eta+\eta)^{m-1}\Big)\left(\frac{\psi'''(\bar{\mathfrak{u}})\psi'(\bar{\mathfrak{u}})-(\psi''(\bar{\mathfrak{u}}))^2}{(\psi'(\bar{\mathfrak{u}}))^2}\right)|\nabla\bar{\mathfrak{u}}|^2\bar{\mathfrak{U}}_k \\
&+ 2m(m-1)(u^\sigma_\eta+\eta)^{m-2}\psi''(\bar{\mathfrak{u}})|\nabla\bar{\mathfrak{u}}|^2\bar{\mathfrak{U}}_k \\
&+ m(m-1)(m-2)(u^\sigma_\eta+\eta)^{m-3}(\psi'(\bar{\mathfrak{u}}))^2|\nabla\bar{\mathfrak{u}}|^2\bar{\mathfrak{U}}_k \\
&+ \Big(\sigma+m(u^\sigma_\eta+\eta)^{m-1}\Big)\frac{\psi''(\bar{\mathfrak{u}})}{\psi'(\bar{\mathfrak{u}})}\nabla\bar{\mathfrak{u}}\cdot\nabla\bar{\mathfrak{U}}_k \\
&+ m(m-1)(u^\sigma_\eta+\eta)^{m-2}\psi'(\bar{\mathfrak{u}})\nabla\bar{\mathfrak{u}}\cdot\nabla\bar{\mathfrak{U}}_k \\
&- \frac{1}{2}\nabla\bar{\mathfrak{U}}_k\cdot\nabla\Phi * u^\sigma_\eta - \nabla\bar{\mathfrak{u}}\cdot\nabla\partial_{x_k}\Phi * u^\sigma_\eta\partial_{x_k}\bar{\mathfrak{u}} - \frac{u^\sigma_\eta\partial_{x_k}\Delta\Phi * u^\sigma_\eta\partial_{x_k}\bar{\mathfrak{u}}}{\psi'(\bar{\mathfrak{u}})} \\
&+ \left(\frac{\psi''(\bar{\mathfrak{u}})}{(\psi'(\bar{\mathfrak{u}}))^2}u^\sigma_\eta - 1\right)\Delta\Phi * u^\sigma_\eta\bar{\mathfrak{U}}_k.
\end{aligned} \tag{2.19}$$

We denote $\sum_{k=1}^d \bar{\mathfrak{U}}_k = |\nabla\bar{\mathfrak{u}}|^2$ by $\mathfrak{D}$. Since $\sum_{k=1}^d \bar{\mathfrak{U}}_k^2 \le \mathfrak{D}^2$, we obtain that

$$\begin{aligned}
\frac{1}{2}\partial_t\mathfrak{D} &\le \Big(m(m-1)(u^\sigma_\eta+\eta)^{m-2}\psi'(\bar{\mathfrak{u}})\Big)\Delta\bar{\mathfrak{u}}\;\mathfrak{D} + \Big(\sigma+m(u^\sigma_\eta+\eta)^{m-1}\Big)\left(\frac{1}{2}\Delta\mathfrak{D} - \sum_{k=1}^d|\partial_{x_k}\nabla\bar{\mathfrak{u}}|^2\right) \\
&+ \Big(\sigma+m(u^\sigma_\eta+\eta)^{m-1}\Big)\left(\frac{\psi'''(\bar{\mathfrak{u}})\psi'(\bar{\mathfrak{u}})-(\psi''(\bar{\mathfrak{u}}))^2}{(\psi'(\bar{\mathfrak{u}}))^2}\right)\mathfrak{D}^2 \\
&+ m(m-1)(m-2)(u^\sigma_\eta+\eta)^{m-3}(\psi'(\bar{\mathfrak{u}}))^2\mathfrak{D}^2 + 2m(m-1)(u^\sigma_\eta+\eta)^{m-2}\psi''(\bar{\mathfrak{u}})\mathfrak{D}^2 \\
&+ \Big(\sigma+m(u^\sigma_\eta+\eta)^{m-1}\Big)\frac{\psi''(\bar{\mathfrak{u}})}{\psi'(\bar{\mathfrak{u}})}\nabla\bar{\mathfrak{u}}\cdot\nabla\mathfrak{D} + m(m-1)(u^\sigma_\eta+\eta)^{m-2}\psi'(\bar{\mathfrak{u}})\nabla\bar{\mathfrak{u}}\cdot\nabla\mathfrak{D} \\
&- \frac{1}{2}\nabla\mathfrak{D}\cdot\nabla\Phi * u^\sigma_\eta - \sum_{k=1}^d\nabla\bar{\mathfrak{u}}\cdot\nabla\partial_{x_k}\Phi * u^\sigma_\eta\partial_{x_k}\bar{\mathfrak{u}} - \sum_{k=1}^d\frac{u^\sigma_\eta\partial_{x_k}\Delta\Phi * u^\sigma_\eta\partial_{x_k}\bar{\mathfrak{u}}}{\psi'(\bar{\mathfrak{u}})} \\
&+ \left(\frac{\psi''(\bar{\mathfrak{u}})}{(\psi'(\bar{\mathfrak{u}}))^2}u^\sigma_\eta - 1\right)\Delta\Phi * u^\sigma_\eta\mathfrak{D}.
\end{aligned} \tag{2.20}$$

Step 2 : Decomposition of $\mathbb{R}^d$

Let $t \in [0, T_2]$ be fixed. Define $\tilde{\omega} > 0$ such that

$$\tilde{\omega} := \begin{cases} \min\left\{ \frac{\sqrt{\sigma}}{16e\sqrt{m(m-1)(m-2)}(2\|u_\eta^\sigma\|_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))})^{\frac{m-3}{2}}}, \frac{\sigma}{16em(m-1)(2\|u_\eta^\sigma\|_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))})^{m-2}} \right\}, & \text{for } m \geq 3, \\ \frac{\sigma}{16em(m-1)}, & \text{for } m = 2. \end{cases} \tag{2.21}$$

Now we define

$$\tilde{\Omega}_k(t) := \{x \in \mathbb{R}^d \mid (k-1)\tilde{\omega} \leq u_\eta^\sigma(t,x) < k\tilde{\omega}\} \text{ for } k \in \mathbb{N}. \tag{2.22}$$

Since $\|u_\eta^\sigma\|_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))} < \infty$ there exists $k_0 \in \mathbb{N}$ such that

$$\tilde{\Omega}_k(t) \cap \tilde{\Omega}_j(t) = \emptyset \ \forall j,k \in \{1,\dots,k_0\} \text{ and } \mathbb{R}^d = \bigcup_{k=1}^{k_0} \tilde{\Omega}_k(t)$$

For any fixed $t \in [0, T_2]$, we define operator $\psi_k(\bar{\mathrm{u}})$ as

$$u_\eta^\sigma(t,x) = \psi_k(\bar{\mathrm{u}}(t,x)) = (k-3)\tilde{\omega} + 4e\tilde{\omega}\int_0^{\bar{\mathrm{u}}(t,x)} e^{-s^2} ds \text{ in } x \in \bigcup_{i=k-1}^{k+1} \tilde{\Omega}_i(t).$$

Moreover, let $\phi_k(x)$ be a smooth function in $C^2(\mathbb{R}^d)$ such that $0 \leq \phi_k(x) \leq 1$ and

$$\phi_k(x) := \begin{cases} 1, & \text{for } x \in \tilde{\Omega}_k(t), \\ 0, & \text{for } x \in \mathbb{R}^d \setminus \bigcup_{j=k-1}^{k+1} \tilde{\Omega}_j(t). \end{cases}$$

Then there exists a constant $C_{\phi_k}\left(\mathrm{dist}(\tilde{\Omega}_k, \partial\tilde{\Omega}_{k-1}), \mathrm{dist}(\tilde{\Omega}_k, \partial\tilde{\Omega}_{k+1})\right)$ such that

$$|\nabla\phi_k(x)| \leq C_{\phi_k}(\phi_k)^{\frac{3}{4}}, \qquad |\Delta\phi_k(x)| \leq C_{\phi_k}$$

For details about the existence of this cut-off function we refer to [Sugiyama et al., 2007] and [Ishida et al., 2013] .

Step 3 : Lemma for the coefficient before $\mathfrak{D}^2$

Now we are going to prove that the coefficient of $\mathfrak{D}^2$ in (2.20) is negative.

Lemma 2.9. *For $x \in \bigcup_{i=k-1}^{k+1} \tilde{\Omega}_i(t)$ it holds that*

$$
\begin{aligned}
&\left(\sigma + m(u_\eta^\sigma + \eta)^{m-1}\right)\left(\frac{\psi_k'''(\bar{\mathfrak{u}})\psi_k'(\bar{\mathfrak{u}}) - (\psi_k''(\bar{\mathfrak{u}}))^2}{(\psi_k'(\bar{\mathfrak{u}}))^2}\right)\mathfrak{D}^2 \\
&+ m(m-1)(m-2)(u_\eta^\sigma + \eta)^{m-3}(\psi_k'(\bar{\mathfrak{u}}))^2\mathfrak{D}^2 + 2m(m-1)(u_\eta^\sigma + \eta)^{m-2}\psi_k''(\bar{\mathfrak{u}})\mathfrak{D}^2 \\
&\leq -2\sigma\mathfrak{D}^2 - 2m(u_\eta^\sigma + \eta)^{m-1}\mathfrak{D}^2 + \frac{1}{8}\sigma\mathfrak{D}^2 \\
&< 0.
\end{aligned}
$$

Proof. Let $x \in \bigcup_{i=k-1}^{k+1} \tilde{\Omega}_i(t)$ and recall that $\tilde{\Omega}_k(t) := \{x \in \mathbb{R}^d \mid (k-1)\tilde{\omega} \leq u_\eta^\sigma(t,x) < k\tilde{\omega}\}$ for $k \in \mathbb{N}$. We know that for any fixed $t \in [0, T_2]$, it holds that

$$
u_\eta^\sigma(t,x) = \psi_k(\bar{\mathfrak{u}}(t,x)) = (k-3)\tilde{\omega} + 4e\tilde{\omega}\int_0^{\bar{\mathfrak{u}}(t,x)} e^{-s^2}ds \text{ in } x \in \bigcup_{i=k-1}^{k+1} \tilde{\Omega}_i(t).
$$

First, we prove that ψ_k is injective. Take without loss of generality $\bar{\mathfrak{u}}_1(t,x)$ and $\bar{\mathfrak{u}}_2(t,x)$ such that $\bar{\mathfrak{u}}_2(t,x) \leq \bar{\mathfrak{u}}_1(t,x)$ and $\psi_k(\bar{\mathfrak{u}}_1(t,x)) = \psi_k(\bar{\mathfrak{u}}_2(t,x))$. So it implies that

$$
(k-3)\tilde{\omega} + 4e\tilde{\omega}\int_0^{\bar{\mathfrak{u}}_1(t,x)} e^{-s^2}ds = (k-3)\tilde{\omega} + 4e\tilde{\omega}\int_0^{\bar{\mathfrak{u}}_2(t,x)} e^{-s^2}ds \text{ in } x \in \bigcup_{i=k-1}^{k+1} \tilde{\Omega}_i(t).
$$

Hence, it should hold that $\int_{\bar{\mathfrak{u}}_2(t,x)}^{\bar{\mathfrak{u}}_1(t,x)} e^{-s^2}ds = 0$. So, $e^{-s^2} \geq 0$ implies that $\bar{\mathfrak{u}}_2(t,x) = \bar{\mathfrak{u}}_1(t,x)$.

Moreover, since

$$
0 \leq \frac{u_\eta^\sigma(t,x) - (k-3)\tilde{\omega}}{4e\tilde{\omega}} \leq \frac{1}{e} \leq \frac{\sqrt{\pi}}{2}
$$

we obtain that ψ_k is surjective. This implies that ψ_k is a bijection.

Since $e^{-s^2} \geq 0$ for all s between 0 and $\bar{\mathfrak{u}}(t,x)$ we obtain that ψ_k is non-decreasing in $\bar{\mathfrak{u}}$. Therefore, if $u_\eta^\sigma(t,x) = \psi_k(\bar{\mathfrak{u}}(t,x)) = (k-2)\tilde{\omega}$, then $\bar{\mathfrak{u}}(t,x)$ is the minimum in this case. Moreover, since $(k-2)\tilde{\omega} = (k-3)\tilde{\omega} + 4e\tilde{\omega}\int_0^{\psi_k^{-1}((k-2)\tilde{\omega})} e^{-s^2}ds$, it follows that $\int_0^{\psi_k^{-1}((k-2)\tilde{\omega})} e^{-s^2}ds = \frac{(k-2)\tilde{\omega}-(k-3)\tilde{\omega}}{4e\tilde{\omega}} = \frac{1}{4e}$. All together it yields

$$
\int_0^{\min_{(k-2)\tilde{\omega} \leq u_\eta^\sigma \leq (k+1)\tilde{\omega}} \bar{\mathfrak{u}}} e^{-s^2}ds = \int_0^{\psi_k^{-1}((k-2)\tilde{\omega})} e^{-s^2}ds = \frac{1}{4e} = \int_0^{\frac{1}{4e}} 1ds \geq \int_0^{\frac{1}{4e}} e^{-s^2}ds.
$$

Let us now analyze the upper bound. Since $s^2 \leq 1$, so $\frac{1}{e} \leq \frac{1}{e^{s^2}}$. Moreover, since $(k+1)\tilde{\omega} = (k-2)\tilde{\omega} + 4e\tilde{\omega}\int_0^{\psi_k^{-1}((k+1)\tilde{\omega})} e^{-s^2}ds$, it follows that $\int_0^{\psi_k^{-1}((k+1)\tilde{\omega})} e^{-s^2}ds = \frac{(k+1)\tilde{\omega}-(k-2)\tilde{\omega}}{4e\tilde{\omega}} = \frac{3}{4e}$. Therefore, we deduce that

$$
\int_0^{\max_{(k-2)\tilde{\omega} \leq u_\eta^\sigma \leq (k+1)\tilde{\omega}} \bar{\mathfrak{u}}} e^{-s^2}ds \leq \int_0^{\psi_k^{-1}((k+1)\tilde{\omega})} e^{-s^2}ds = \frac{3}{4e} = \int_0^{\frac{3}{4e}} 1ds \leq \int_0^{\frac{3}{4}} e^{-s^2}ds \leq \int_0^1 e^{-s^2}ds.
$$

Therefore, we obtain that

$$\frac{1}{4e} \le \bar{\mathfrak{u}}(t,x) = \psi_k^{-1}(u_\eta^\sigma) \le 1 \text{ in } x \in \bigcup_{i=k-1}^{k+1} \tilde{\Omega}_i(t).$$

This implies that

$$\begin{aligned}
0 < &\psi_k'(\bar{\mathfrak{u}}) = 4e\tilde{\omega}e^{-\bar{\mathfrak{u}}^2} \le 4e\tilde{\omega}, \\
&\psi_k''(\bar{\mathfrak{u}}) = -8e\tilde{\omega}\bar{\mathfrak{u}}e^{-\bar{\mathfrak{u}}^2} \le -8e\tilde{\omega}\frac{1}{4e}\cdot\frac{1}{e} = -\frac{2}{e}\tilde{\omega} < 0, \\
&\psi_k'''(\bar{\mathfrak{u}}) = 8e\tilde{\omega}e^{-\bar{\mathfrak{u}}^2}(2\bar{\mathfrak{u}}^2 - 1).
\end{aligned}$$

Therefore,

$$\frac{\psi_k'''(\bar{\mathfrak{u}})\psi_k'(\bar{\mathfrak{u}}) - (\psi_k''(\bar{\mathfrak{u}}))^2}{(\psi_k'(\bar{\mathfrak{u}}))^2} = \frac{8e\tilde{\omega}e^{-\bar{\mathfrak{u}}^2}(2\bar{\mathfrak{u}}^2-1)4e\tilde{\omega}e^{-\bar{\mathfrak{u}}^2} - (-8e\tilde{\omega}\bar{\mathfrak{u}}e^{-\bar{\mathfrak{u}}^2})^2}{(4e\tilde{\omega}e^{-\bar{\mathfrak{u}}^2})^2} = -2.$$

Since

$$\begin{aligned}
m(m-1)(m-2)(u_\eta^\sigma+\eta)^{m-3}(\psi_k'(\bar{\mathfrak{u}}))^2 &\le 16e^2\tilde{\omega}^2 e^{-2\bar{\mathfrak{u}}^2} m(m-1)(m-2)(u_\eta^\sigma+\eta)^{m-3} \\
&\le 16e^2\tilde{\omega}^2 m(m-1)(m-2)(2\|u_\eta^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))})^{m-3},
\end{aligned}$$

we need to ensure that $16e^2\tilde{\omega}^2 m(m-1)(m-2)(2\|u_\eta^\sigma\|_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))})^{m-3} \le \frac{1}{8}\sigma$, so we obtain that $\tilde{\omega}$ should satisfy the inequality

$$\tilde{\omega} \le \frac{\sqrt{\sigma}}{8\sqrt{2}e\sqrt{m(m-1)(m-2)}(2\|u_\eta^\sigma\|_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))})^{\frac{m-3}{2}}}.$$

Finally, we obtain that

$$\begin{aligned}
&\left(\sigma + m(u_\eta^\sigma+\eta)^{m-1}\right)\left(\frac{\psi_k'''(\bar{\mathfrak{u}})\psi_k'(\bar{\mathfrak{u}}) - (\psi_k''(\bar{\mathfrak{u}}))^2}{(\psi_k'(\bar{\mathfrak{u}}))^2}\right)\mathfrak{D}^2 \\
&+ m(m-1)(m-2)(u_\eta^\sigma+\eta)^{m-3}(\psi_k'(\bar{\mathfrak{u}}))^2\mathfrak{D}^2 + 2m(m-1)(u_\eta^\sigma+\eta)^{m-2}\psi_k''(\bar{\mathfrak{u}})\mathfrak{D}^2 \\
&\le -2\left(\sigma + m(u_\eta^\sigma+\eta)^{m-1}\right)\mathfrak{D}^2 + m(m-1)(m-2)(u_\eta^\sigma+\eta)^{m-3}(\psi_k'(\bar{\mathfrak{u}}))^2\mathfrak{D}^2 \\
&\le -2\sigma\mathfrak{D}^2 - 2m(u_\eta^\sigma+\eta)^{m-1}\mathfrak{D}^2 + \frac{1}{8}\sigma\mathfrak{D}^2 \\
&< 0 \text{ in } \bigcup_{i=k-1}^{k+1}\tilde{\Omega}_i(t).
\end{aligned}$$

□

Step 4 : Bochner type inequality for $V_\phi := |\nabla\bar{\mathfrak{u}}(t,x)|^2\phi$

In order to simplify the notation in this step we use ϕ and ψ instead of ϕ_k and ψ_k respectively.

Multiplying (2.20) by ϕ and using Lemma 2.9 together with

$$\psi'(\bar{\mathfrak{u}}) \leq 4e\tilde{\omega}, \qquad \psi''(\bar{\mathfrak{u}}) < 0, \qquad \psi'''(\bar{\mathfrak{u}}) = 8e\tilde{\omega}e^{-\bar{\mathfrak{u}}^2}(2\bar{\mathfrak{u}}^2 - 1),$$
$$\frac{\psi'''(\bar{\mathfrak{u}})\psi'(\bar{\mathfrak{u}}) - (\psi''(\bar{\mathfrak{u}}))^2}{(\psi'(\bar{\mathfrak{u}}))^2} = -2, \qquad -2 \leq \frac{\psi''(\bar{\mathfrak{u}})}{\psi'(\bar{\mathfrak{u}})} = -2\bar{\mathfrak{u}} \leq -\frac{1}{2e},$$

we obtain that

$$\frac{1}{2}\partial_t V_\phi \leq \left(m(m-1)(u_\eta^\sigma + \eta)^{m-2}4e\tilde{\omega}\right)\sum_{i=1}^d |\partial_{ii}\bar{\mathfrak{u}}|\; V_\phi + \left(\sigma + m(u_\eta^\sigma + \eta)^{m-1}\right)\left(\frac{1}{2}\Delta\mathfrak{D}\phi - \sum_{k=1}^d |\partial_{x_k}\nabla\bar{\mathfrak{u}}|^2\phi\right) \tag{2.23}$$
$$- 2\left(\sigma + m(u_\eta^\sigma + \eta)^{m-1}\right)\mathfrak{D}^2\phi + \sigma\frac{\psi''(\bar{\mathfrak{u}})}{\psi'(\bar{\mathfrak{u}})}\nabla\bar{\mathfrak{u}}\cdot\nabla\mathfrak{D}\;\phi + \frac{1}{8}\sigma\mathfrak{D}^2\phi$$
$$+ m\left((u_\eta^\sigma + \eta)\frac{\psi''(\bar{\mathfrak{u}})}{\psi'(\bar{\mathfrak{u}})} + (m-1)\psi'(\bar{\mathfrak{u}})\right)(u_\eta^\sigma + \eta)^{m-2}\nabla\bar{\mathfrak{u}}\cdot\nabla\mathfrak{D}\;\phi$$
$$- \frac{1}{2}\nabla\mathfrak{D}\cdot\nabla\Phi * u_\eta^\sigma\phi - \sum_{k=1}^d \nabla\bar{\mathfrak{u}}\cdot\nabla\partial_{x_k}\Phi * u_\eta^\sigma\partial_{x_k}\bar{\mathfrak{u}}\phi - \sum_{k=1}^d \frac{u_\eta^\sigma\partial_{x_k}\Delta\Phi * u_\eta^\sigma\partial_{x_k}\bar{\mathfrak{u}}}{\psi'(\bar{\mathfrak{u}})}\phi$$
$$+ \left(\frac{\psi''(\bar{\mathfrak{u}})}{(\psi'(\bar{\mathfrak{u}}))^2}u_\eta^\sigma - 1\right)\Delta\Phi * u_\eta^\sigma\mathfrak{D}\phi.$$

Now we multiply (2.23) by V_ϕ^{r-1} and integrate it over $\tilde{\Omega} := \bigcup_{j=k-1}^{k+1}\tilde{\Omega}_j(t)$. Since

$$\begin{aligned}
m\int_{\tilde{\Omega}}(u_\eta^\sigma + \eta)^{m-1}\Delta\mathfrak{D}\phi(V_\phi)^{r-1}dx &= m\int_{\tilde{\Omega}}(u_\eta^\sigma + \eta)^{m-1}\Delta V_\phi(V_\phi)^{r-1}dx \\
&\quad - 2m\int_{\tilde{\Omega}}(u_\eta^\sigma + \eta)^{m-1}\nabla\mathfrak{D}\cdot\nabla\phi(V_\phi)^{r-1}dx \\
&\quad - m\int_{\tilde{\Omega}}(u_\eta^\sigma + \eta)^{m-1}\mathfrak{D}\Delta\phi(V_\phi)^{r-1}dx \\
&= -m(m-1)\int_{\tilde{\Omega}}(u_\eta^\sigma + \eta)^{m-2}\nabla u_\eta^\sigma\cdot\nabla V_\phi(V_\phi)^{r-1}dx \\
&\quad - m(r-1)\int_{\tilde{\Omega}}(u_\eta^\sigma + \eta)^{m-1}|\nabla V_\phi|^2(V_\phi)^{r-2}dx \\
&\quad - 2m\int_{\tilde{\Omega}}(u_\eta^\sigma + \eta)^{m-1}\nabla\mathfrak{D}\cdot\nabla\phi(V_\phi)^{r-1}dx \\
&\quad - m\int_{\tilde{\Omega}}(u_\eta^\sigma + \eta)^{m-1}\mathfrak{D}\Delta\phi(V_\phi)^{r-1}dx,
\end{aligned}$$

$$\sigma\int_{\tilde{\Omega}}\Delta\mathfrak{D}\phi(V_\phi)^{r-1}dx = -\sigma(r-1)\int_{\tilde{\Omega}}|\nabla V_\phi|^2(V_\phi)^{r-2}dx - 2\sigma\int_{\tilde{\Omega}}\nabla\mathfrak{D}\cdot\nabla\phi(V_\phi)^{r-1}dx - \sigma\int_{\tilde{\Omega}}\mathfrak{D}\Delta\phi(V_\phi)^{r-1}dx$$

and

$$\nabla V_\phi = \nabla\mathfrak{D}\phi + \nabla\phi\mathfrak{D}$$

we deduce that

$$
\begin{aligned}
\frac{1}{r}\frac{d}{dt}\|V_\phi\|^r_{L^r(\tilde\Omega)} \le& -m(r-1)\int_{\tilde\Omega}(u^\sigma_\eta+\eta)^{m-1}|\nabla V_\phi|^2(V_\phi)^{r-2}dx \\
&-\sigma(r-1)\int_{\tilde\Omega}|\nabla V_\phi|^2(V_\phi)^{r-2}dx \\
&-2m\int_{\tilde\Omega}(u^\sigma_\eta+\eta)^{m-1}\sum_{k=1}^d|\partial_{x_k}\nabla\bar{\mathfrak{u}}|^2\phi V_\phi^{r-1}dx-2\sigma\int_{\tilde\Omega}\sum_{k=1}^d|\partial_{x_k}\nabla\bar{\mathfrak{u}}|^2\phi V_\phi^{r-1}dx \\
&-4m\int_{\tilde\Omega}(u^\sigma_\eta+\eta)^{m-1}\mathfrak{D}V_\phi^r dx-4\sigma\int_{\tilde\Omega}\mathfrak{D}V_\phi^r dx+\frac{1}{4}\sigma\int_{\tilde\Omega}\mathfrak{D}V_\phi^r dx \\
&-m(m-1)\int_{\tilde\Omega}(u^\sigma_\eta+\eta)^{m-2}\nabla u^\sigma_\eta\cdot\nabla V_\phi(V_\phi)^{r-1}dx \\
&-2m\int_{\tilde\Omega}(u^\sigma_\eta+\eta)^{m-1}\nabla\mathfrak{D}\cdot\nabla\phi(V_\phi)^{r-1}dx-m\int_{\tilde\Omega}(u^\sigma_\eta+\eta)^{m-1}\mathfrak{D}\Delta\phi(V_\phi)^{r-1}dx \\
&-2\sigma\int_{\tilde\Omega}\nabla\mathfrak{D}\cdot\nabla\phi(V_\phi)^{r-1}dx-\sigma\int_{\tilde\Omega}\mathfrak{D}\Delta\phi(V_\phi)^{r-1}dx \\
&+(8e\tilde\omega m(m-1))\int_{\tilde\Omega}(u^\sigma_\eta+\eta)^{m-2}\sum_{i=1}^d|\partial_{ii}\bar{\mathfrak{u}}|\ V_\phi^r dx \\
&+2m\int_{\tilde\Omega}\left((u^\sigma_\eta+\eta)\frac{\psi''(\bar{\mathfrak{u}})}{\psi'(\bar{\mathfrak{u}})}+(m-1)\psi'(\bar{\mathfrak{u}})\right)(u^\sigma_\eta+\eta)^{m-2}\nabla\bar{\mathfrak{u}}\cdot\nabla V_\phi(V_\phi)^{r-1}dx \\
&-2m\int_{\tilde\Omega}\left((u^\sigma_\eta+\eta)\frac{\psi''(\bar{\mathfrak{u}})}{\psi'(\bar{\mathfrak{u}})}+(m-1)\psi'(\bar{\mathfrak{u}})\right)(u^\sigma_\eta+\eta)^{m-2}\nabla\bar{\mathfrak{u}}\cdot\nabla\phi\mathfrak{D}(V_\phi)^{r-1}dx \\
&+2\sigma\int_{\tilde\Omega}\frac{\psi''(\bar{\mathfrak{u}})}{\psi'(\bar{\mathfrak{u}})}\nabla\bar{\mathfrak{u}}\cdot\nabla V_\phi(V_\phi)^{r-1}dx-2\sigma\int_{\tilde\Omega}\frac{\psi''(\bar{\mathfrak{u}})}{\psi'(\bar{\mathfrak{u}})}\nabla\bar{\mathfrak{u}}\cdot\nabla\phi\mathfrak{D}(V_\phi)^{r-1}dx \\
&-\int_{\tilde\Omega}\nabla V_\phi\cdot\nabla\Phi*u^\sigma_\eta(V_\phi)^{r-1}dx+\int_{\tilde\Omega}\mathfrak{D}\nabla\phi\cdot\nabla\Phi*u^\sigma_\eta(V_\phi)^{r-1}dx \\
&-2\sum_{k=1}^d\int_{\tilde\Omega}\nabla\bar{\mathfrak{u}}\cdot\nabla\partial_{x_k}\Phi*u^\sigma_\eta\partial_{x_k}\bar{\mathfrak{u}}\phi(V_\phi)^{r-1}dx \\
&+2\int_{\tilde\Omega}\left(\frac{\psi''(\bar{\mathfrak{u}})}{(\psi'(\bar{\mathfrak{u}}))^2}u^\sigma_\eta-1\right)\Delta\Phi*u^\sigma_\eta(V_\phi)^r dx \\
&-2\sum_{k=1}^d\int_{\tilde\Omega}\frac{u^\sigma_\eta\partial_{x_k}\Delta\Phi*u^\sigma_\eta\partial_{x_k}\bar{\mathfrak{u}}}{\psi'(\bar{\mathfrak{u}})}\phi(V_\phi)^{r-1}dx
\end{aligned} \tag{2.24}
$$

Since

$$
\begin{aligned}
-2m\int_{\tilde\Omega}(u^\sigma_\eta+\eta)^{m-1}\nabla\mathfrak{D}\cdot\nabla\phi(V_\phi)^{r-1}dx =& 2m(m-1)\int_{\tilde\Omega}(u^\sigma_\eta+\eta)^{m-2}\mathfrak{D}\nabla u^\sigma_\eta\cdot\nabla\phi(V_\phi)^{r-1}dx \\
&+2m\int_{\tilde\Omega}(u^\sigma_\eta+\eta)^{m-1}\mathfrak{D}\Delta\phi(V_\phi)^{r-1}dx \\
&+2m(r-1)\int_{\tilde\Omega}(u^\sigma_\eta+\eta)^{m-1}\mathfrak{D}(V_\phi)^{r-2}\nabla\phi\cdot\nabla V_\phi dx,
\end{aligned}
$$

and

$$
-2\sigma\int_{\tilde\Omega}\nabla\mathfrak{D}\cdot\nabla\phi(V_\phi)^{r-1}dx=2\sigma\int_{\tilde\Omega}\mathfrak{D}\Delta\phi(V_\phi)^{r-1}dx+2\sigma(r-1)\int_{\tilde\Omega}\mathfrak{D}(V_\phi)^{r-2}\nabla\phi\cdot\nabla V_\phi dx,
$$

we deduce that

$$
\begin{aligned}
\frac{1}{r}\frac{d}{dt}\|V_\phi\|^r_{L^r(\tilde\Omega)} \le\ & -m(r-1)\int_{\tilde\Omega}(u^\sigma_\eta+\eta)^{m-1}|\nabla V_\phi|^2(V_\phi)^{r-2}dx \\
& -\sigma(r-1)\int_{\tilde\Omega}|\nabla V_\phi|^2(V_\phi)^{r-2}dx \\
& -2m\int_{\tilde\Omega}(u^\sigma_\eta+\eta)^{m-1}\sum_{k=1}^d|\partial_{x_k}\nabla\bar{\mathfrak{u}}|^2\phi V_\phi^{r-1}dx-2\sigma\int_{\tilde\Omega}\sum_{k=1}^d|\partial_{x_k}\nabla\bar{\mathfrak{u}}|^2\phi V_\phi^{r-1}dx \\
& -4m\int_{\tilde\Omega}(u^\sigma_\eta+\eta)^{m-1}\mathfrak{D}V_\phi^r dx-4\sigma\int_{\tilde\Omega}\mathfrak{D}V_\phi^r dx+\frac{1}{4}\sigma\int_{\tilde\Omega}\mathfrak{D}V_\phi^r dx \\
& +m(m-1)\int_{\tilde\Omega}(u^\sigma_\eta+\eta)^{m-2}\nabla u^\sigma_\eta\cdot\nabla V_\phi(V_\phi)^{r-1}dx \\
& +2m(r-1)\int_{\tilde\Omega}(u^\sigma_\eta+\eta)^{m-1}\mathfrak{D}(V_\phi)^{r-2}\nabla\phi\cdot\nabla V_\phi dx \\
& +2\sigma(r-1)\int_{\tilde\Omega}\mathfrak{D}(V_\phi)^{r-2}\nabla\phi\cdot\nabla V_\phi dx \\
& +m\int_{\tilde\Omega}(u^\sigma_\eta+\eta)^{m-1}\mathfrak{D}\Delta\phi(V_\phi)^{r-1}dx \\
& +\sigma\int_{\tilde\Omega}\mathfrak{D}\Delta\phi(V_\phi)^{r-1}dx \\
& +(8e\tilde\omega m(m-1))\int_{\tilde\Omega}(u^\sigma_\eta+\eta)^{m-2}\sum_{i=1}^d|\partial_{ii}\bar{\mathfrak{u}}|\ V_\phi^r dx \\
& +2m\int_{\tilde\Omega}(u^\sigma_\eta+\eta)^{m-1}\left|\frac{\psi''(\bar{\mathfrak{u}})}{\psi'(\bar{\mathfrak{u}})}\right||\nabla\bar{\mathfrak{u}}|\ |\nabla V_\phi|\ |V_\phi|^{r-1}dx \\
& +2\sigma\int_{\tilde\Omega}\left|\frac{\psi''(\bar{\mathfrak{u}})}{\psi'(\bar{\mathfrak{u}})}\right||\nabla\bar{\mathfrak{u}}|\ |\nabla V_\phi||V_\phi|^{r-1}dx \\
& +2m\int_{\tilde\Omega}(u^\sigma_\eta+\eta)^{m-1}\left|\frac{\psi''(\bar{\mathfrak{u}})}{\psi'(\bar{\mathfrak{u}})}\right||\nabla\bar{\mathfrak{u}}|\ |\nabla\phi|\mathfrak{D}(V_\phi)^{r-1}dx \\
& +2\sigma\int_{\tilde\Omega}\left|\frac{\psi''(\bar{\mathfrak{u}})}{\psi'(\bar{\mathfrak{u}})}\right||\nabla\bar{\mathfrak{u}}|\ |\nabla\phi|\mathfrak{D}(V_\phi)^{r-1}dx \\
& -\int_{\tilde\Omega}\nabla V_\phi\cdot\nabla\Phi*u^\sigma_\eta(V_\phi)^{r-1}dx+\int_{\tilde\Omega}\mathfrak{D}\nabla\phi\cdot\nabla\Phi*u^\sigma_\eta(V_\phi)^{r-1}dx \\
& -2\sum_{k=1}^d\int_{\tilde\Omega}\nabla\bar{\mathfrak{u}}\cdot\nabla\partial_{x_k}\Phi*u^\sigma_\eta\partial_{x_k}\bar{\mathfrak{u}}\phi(V_\phi)^{r-1}dx \\
& +2\int_{\tilde\Omega}\left(\frac{\psi''(\bar{\mathfrak{u}})}{(\psi'(\bar{\mathfrak{u}}))^2}u^\sigma_\eta-1\right)\Delta\Phi*u^\sigma_\eta(V_\phi)^r dx \\
& -2\sum_{k=1}^d\int_{\tilde\Omega}\frac{u^\sigma_\eta\partial_{x_k}\Delta\Phi*u^\sigma_\eta\partial_{x_k}\bar{\mathfrak{u}}}{\psi'(\bar{\mathfrak{u}})}\phi(V_\phi)^{r-1}dx \\
& =:-I_1-I_1'-I_2-I_2'-I_3-I_3'+\frac{1}{16}I_3'+II_1+II_2+II_3+II_4+II_5+III \\
& +IV_1+IV_1'+IV_2+IV_2'+J_1^1+J_1^2+J_2+J_3+J_4. \qquad (2.25)
\end{aligned}
$$

Now we are going to estimate 15 terms.

By Young's inequality we obtain that

$$
\begin{aligned}
II_1 &= m(m-1)\int_{\tilde{\Omega}} (u_\eta^\sigma+\eta)^{m-2}\nabla u_\eta^\sigma \cdot \nabla V_\phi (V_\phi)^{r-1}dx \\
&= m(m-1)\int_{\tilde{\Omega}} (u_\eta^\sigma+\eta)^{m-2}\psi'(\bar{\mathfrak{u}})\nabla \bar{\mathfrak{u}} \cdot \nabla V_\phi (V_\phi)^{r-1}dx \\
&\le \int_{\tilde{\Omega}} \left(\frac{\sqrt{\sigma(r-1)}}{2}(V_\phi)^{\frac{r-2}{2}}|\nabla V_\phi|\right)\left(\frac{2m(m-1)}{\sqrt{\sigma(r-1)}}(u_\eta^\sigma+\eta)^{m-2}\psi'(\bar{\mathfrak{u}})|\nabla\bar{\mathfrak{u}}||V_\phi|^{\frac{r}{2}}\right)dx \\
&\le \frac{\sigma(r-1)}{8}\int_{\tilde{\Omega}} |V_\phi|^{r-2}|\nabla V_\phi|^2 dx \\
&+ \frac{2m^2(m-1)^2}{\sigma(r-1)}\left(2\|u_\eta^\sigma\|_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))}\right)^{2(m-2)}\int_{\tilde{\Omega}} (\psi'(\bar{\mathfrak{u}}))^2|\nabla\bar{\mathfrak{u}}|^2|V_\phi|^r dx \\
&\le \frac{\sigma(r-1)}{8}\int_{\tilde{\Omega}} |V_\phi|^{r-2}|\nabla V_\phi|^2 dx \\
&+ \frac{2m^2(m-1)^2}{\sigma(r-1)}\left(2\|u_\eta^\sigma\|_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))}\right)^{2(m-2)}\int_{\tilde{\Omega}} 16e^2\tilde{\omega}^2 e^{-2\bar{\mathfrak{u}}^2}\mathfrak{D}|V_\phi|^r dx.
\end{aligned}
$$

Since $e^{-2\bar{\mathfrak{u}}^2} \le 1$, at this step we need to ensure that

$$
\frac{32m^2(m-1)^2e^2\tilde{\omega}^2(2\|u_\eta^\sigma\|_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))})^{2(m-2)}}{\sigma(r-1)} \le \frac{1}{8}4\sigma = \frac{\sigma}{2},
$$

which is equivalent to

$$
\begin{aligned}
\tilde{\omega} &\le \left(\frac{\sigma}{2}\cdot\frac{\sigma(r-1)}{32m^2(m-1)^2e^2(2\|u_\eta^\sigma\|_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))})^{2(m-2)}}\right)^{\frac{1}{2}} \\
&= \frac{\sigma\sqrt{r-1}}{8m(m-1)e(2\|u_\eta^\sigma\|_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))})^{m-2}}.
\end{aligned}
$$

So, the above condition is satisfied if

$$
\tilde{\omega} \le \frac{\sigma}{8m(m-1)e(2\|u_\eta^\sigma\|_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))})^{m-2}}.
$$

Therefore,

$$
II_1 \le \frac{1}{8}I_1' + \frac{1}{8}I_3'. \tag{2.26}
$$

Next,

$$
\begin{aligned}
II_2 &= 2m(r-1)\int_{\tilde{\Omega}}(u_\eta^\sigma+\eta)^{m-1}\mathfrak{D}(V_\phi)^{r-2}\nabla\phi\cdot\nabla V_\phi dx\\
&\leq \int_{\tilde{\Omega}}\left(\frac{\sqrt{m(r-1)}}{2}(u_\eta^\sigma+\eta)^{\frac{m-1}{2}}(V_\phi)^{\frac{r-2}{2}}|\nabla V_\phi|\right)\left(4\sqrt{m(r-1)}(u_\eta^\sigma+\eta)^{\frac{m-1}{2}}\mathfrak{D}(V_\phi)^{\frac{r-2}{2}}|\nabla\phi|\right)dx\\
&\leq \frac{1}{8}I_1+8m(r-1)\int_{\tilde{\Omega}}(u_\eta^\sigma+\eta)^{m-1}\mathfrak{D}^2(V_\phi)^{r-2}|\nabla\phi|^2dx\\
&= \frac{1}{8}I_1+\int_{\tilde{\Omega}}\left(\sqrt{m}(u_\eta^\sigma+\eta)^{\frac{m-1}{2}}\mathfrak{D}^{\frac{1}{2}}(V_\phi)^{\frac{r}{2}}\right)\left(8\sqrt{m}(r-1)(u_\eta^\sigma+\eta)^{\frac{m-1}{2}}\mathfrak{D}^{\frac{3}{2}}(V_\phi)^{\frac{r}{2}-2}|\nabla\phi|^2\right)dx\\
&\leq \frac{1}{8}I_1+\frac{1}{2}m\int_{\tilde{\Omega}}(u_\eta^\sigma+\eta)^{m-1}\mathfrak{D}(V_\phi)^r dx+32m(r-1)^2\int_{\tilde{\Omega}}(u_\eta^\sigma+\eta)^{m-1}\mathfrak{D}^3(V_\phi)^{r-4}|\nabla\phi|^4dx.
\end{aligned}
$$

Since $|\nabla\phi|^4\leq C_\phi^4\phi^3$, we obtain by Hölder's inequality that

$$
II_2\leq\frac{1}{8}I_1+\frac{1}{8}I_3+32m(r-1)^2C_\phi^4(2\|u_\eta^\sigma\|_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))})^{m-1}|\tilde{\Omega}|^{\frac{1}{r}}\|V_\phi\|_{L^r(\tilde{\Omega})}^{r-1}, \tag{2.27}
$$

For II_3 we obtain similarly that

$$
\begin{aligned}
II_3 &= 2\sigma(r-1)\int_{\tilde{\Omega}}\mathfrak{D}(V_\phi)^{r-2}\nabla\phi\cdot\nabla V_\phi dx\\
&\leq \int_{\tilde{\Omega}}\left(\frac{\sqrt{\sigma(r-1)}}{2}(V_\phi)^{\frac{r-2}{2}}|\nabla V_\phi|\right)\left(4\sqrt{\sigma(r-1)}\mathfrak{D}(V_\phi)^{\frac{r-2}{2}}|\nabla\phi|\right)dx\\
&\leq \frac{1}{8}I_1'+8\sigma(r-1)\int_{\tilde{\Omega}}\mathfrak{D}^2(V_\phi)^{r-2}|\nabla\phi|^2dx\\
&\leq \frac{1}{8}I_1'+\int_{\tilde{\Omega}}\left(\sqrt{\sigma}\mathfrak{D}^{\frac{1}{2}}(V_\phi)^{\frac{r}{2}}\right)\left(8\sqrt{\sigma}(r-1)\mathfrak{D}^{\frac{3}{2}}(V_\phi)^{\frac{r}{2}-2}|\nabla\phi|^2\right)dx\\
&\leq \frac{1}{8}I_1'+\frac{\sigma}{2}\int_{\tilde{\Omega}}\mathfrak{D}(V_\phi)^r dx+32\sigma(r-1)^2\int_{\tilde{\Omega}}\mathfrak{D}^3(V_\phi)^{r-4}|\nabla\phi|^4dx.
\end{aligned}
$$

So, by Hölder's inequality it follows that

$$
II_3\leq\frac{1}{8}I_1'+\frac{1}{8}I_3'+32\sigma(r-1)^2C_\phi^4|\tilde{\Omega}|^{\frac{1}{r}}\|V_\phi\|_{L^r(\tilde{\Omega})}^{r-1}. \tag{2.28}
$$

Next, we come to II_4. First, we see that

$$
\begin{aligned}
II_4 &= m\int_{\tilde{\Omega}}(u_\eta^\sigma+\eta)^{m-1}\mathfrak{D}\Delta\phi(V_\phi)^{r-1}dx\\
&= -m(m-1)\int_{\tilde{\Omega}}(u_\eta^\sigma+\eta)^{m-2}\psi'(\bar{\mathfrak{u}})\nabla\bar{\mathfrak{u}}\cdot\mathfrak{D}\nabla\phi(V_\phi)^{r-1}dx\\
&\quad -m\int_{\tilde{\Omega}}(u_\eta^\sigma+\eta)^{m-1}\nabla\mathfrak{D}\cdot\nabla\phi(V_\phi)^{r-1}dx\\
&\quad -m(r-1)\int_{\tilde{\Omega}}(u_\eta^\sigma+\eta)^{m-1}\mathfrak{D}\nabla\phi\cdot(V_\phi)^{r-2}\nabla V_\phi dx.
\end{aligned}
$$

Let us now estimate these three terms of II_4. Since $\psi'(\bar{\mathfrak{u}})\leq 4e\tilde{\omega}e^{-\bar{\mathfrak{u}}^2}$, $|\nabla\bar{\mathfrak{u}}|\leq\mathfrak{D}^{\frac{1}{2}}$ and $|\nabla\phi|\leq C_\phi\phi^{\frac{3}{4}}$

we obtain that

$$
\begin{aligned}
&- m(m-1)\int_{\tilde{\Omega}}(u_\eta^\sigma+\eta)^{m-2}\psi'(\bar{\mathfrak{u}})\nabla\bar{\mathfrak{u}}\cdot\mathfrak{D}\nabla\phi(V_\phi)^{r-1}dx\\
&\le m(m-1)\int_{\tilde{\Omega}}4e\tilde{\omega}(u_\eta^\sigma+\eta)^{m-2}\mathfrak{D}^{\frac{3}{2}}C_\phi\phi^{\frac{3}{4}}(V_\phi)^{r-1}dx\\
&\le \int_{\tilde{\Omega}}\left(\frac{\sqrt{\sigma}}{2}\mathfrak{D}^{\frac{1}{2}}(V_\phi)^{\frac{r}{2}}\right)\left(8e\tilde{\omega}m(m-1)\frac{1}{\sqrt{\sigma}}(u_\eta^\sigma+\eta)^{m-2}\mathfrak{D}C_\phi\phi^{\frac{3}{4}}(V_\phi)^{\frac{r}{2}-1}\right)dx\\
&\le \frac{1}{32}\cdot 4\sigma\int_{\tilde{\Omega}}\mathfrak{D}(V_\phi)^r dx+32e^2\tilde{\omega}^2m^2(m-1)^2\frac{1}{\sigma}(2\|u_\eta^\sigma\|_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))})^{2(m-2)}C_\phi^2\int_{\tilde{\Omega}}\mathfrak{D}^2\phi^{\frac{3}{2}}(V_\phi)^{r-2}dx\\
&\le \frac{1}{32}I_3'+\int_{\tilde{\Omega}}\left(\frac{\sqrt{\sigma}}{2}\mathfrak{D}^{\frac{1}{2}}(V_\phi)^{\frac{r}{2}}\right)\left(64e^2\tilde{\omega}^2m^2(m-1)^2\frac{1}{\sigma^{\frac{3}{2}}}(2\|u_\eta^\sigma\|_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))})^{2(m-2)}C_\phi^2\mathfrak{D}^{\frac{3}{2}}\phi^{\frac{3}{2}}(V_\phi)^{\frac{r}{2}-2}\right)dx\\
&\le \frac{1}{16}I_3'+64^2e^4\tilde{\omega}^4m^4(m-1)^4\frac{1}{\sigma^3}(2\|u_\eta^\sigma\|_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))})^{4(m-2)}C_\phi^4\int_{\tilde{\Omega}}(V_\phi)^{r-1}dx\\
&\le \frac{1}{16}I_3'+64^2e^4m^4(m-1)^4\frac{1}{\sigma^3}(2\|u_\eta^\sigma\|_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))})^{4(m-2)}C_\phi^4\|V_\phi\|_{L^r(\tilde{\Omega})}^{r-1}|\tilde{\Omega}|^{\frac{1}{r}}
\end{aligned}
$$

Since $\nabla\mathfrak{D}=\nabla|\nabla\bar{\mathfrak{u}}|^2=2(\nabla\bar{\mathfrak{u}})(D^2\bar{\mathfrak{u}})$ we obtain that

$$
\begin{aligned}
&- m\int_{\tilde{\Omega}}(u_\eta^\sigma+\eta)^{m-1}\nabla\mathfrak{D}\cdot\nabla\phi(V_\phi)^{r-1}dx\\
&\le m\int_{\tilde{\Omega}}(u_\eta^\sigma+\eta)^{m-1}|\nabla\mathfrak{D}|C_\phi\phi^{\frac{3}{4}}(V_\phi)^{r-1}dx\\
&\le 2mC_\phi\int_{\tilde{\Omega}}(u_\eta^\sigma+\eta)^{m-1}|\nabla\bar{\mathfrak{u}}|\ |D^2\bar{\mathfrak{u}}|\phi^{\frac{3}{4}}|V_\phi|^{r-1}dx\\
&\le \int_{\tilde{\Omega}}\left(\frac{\sqrt{m}}{\sqrt{2}}(u_\eta^\sigma+\eta)^{\frac{m-1}{2}}|D^2\bar{\mathfrak{u}}|\phi^{\frac{1}{2}}|V_\phi|^{\frac{r-1}{2}}\right)\left(2\sqrt{2}\sqrt{m}C_\phi(u_\eta^\sigma+\eta)^{\frac{m-1}{2}}|\nabla\bar{\mathfrak{u}}|\phi^{\frac{1}{4}}|V_\phi|^{\frac{r-1}{2}}\right)dx\\
&\le \frac{1}{8}\cdot 2m\int_{\tilde{\Omega}}(u_\eta^\sigma+\eta)^{m-1}\sum_{k=1}^{d}|\nabla\partial_{x_k}\bar{\mathfrak{u}}|\phi|V_\phi|^{r-1}dx+4mC_\phi^2\int_{\tilde{\Omega}}(u_\eta^\sigma+\eta)^{m-1}\mathfrak{D}\phi^{\frac{1}{2}}|V_\phi|^{r-1}dx\\
&\le \frac{1}{8}I_2+\int_{\tilde{\Omega}}\left(\sqrt{m}(u_\eta^\sigma+\eta)^{\frac{m-1}{2}}\mathfrak{D}^{\frac{1}{2}}(V_\phi)^{\frac{r}{2}}\right)\left(4\sqrt{m}C_\phi^2(u_\eta^\sigma+\eta)^{\frac{m-1}{2}}\mathfrak{D}^{\frac{1}{2}}\phi^{\frac{1}{2}}|V_\phi|^{\frac{r}{2}-1}\right)dx\\
&\le \frac{1}{8}I_2+\frac{1}{8}I_3+8mC_\phi^4(2\|u_\eta^\sigma\|_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))})^{m-1}\int_{\tilde{\Omega}}|V_\phi|^{r-1}dx\\
&\le \frac{1}{8}I_2+\frac{1}{8}I_3+8mC_\phi^4(2\|u_\eta^\sigma\|_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))})^{m-1}\|V_\phi\|_{L^r(\tilde{\Omega})}^{r-1}|\tilde{\Omega}|^{\frac{1}{r}},
\end{aligned}
$$

For the third term we deduce that

$$
\begin{aligned}
&- m(r-1)\int_{\tilde\Omega}(u_\eta^\sigma+\eta)^{m-1}\mathfrak{D}\nabla\phi\cdot(V_\phi)^{r-2}\nabla V_\phi dx\\
&\le \int_{\tilde\Omega}\left(\frac{\sqrt{m(r-1)}}{2}(u_\eta^\sigma+\eta)^{\frac{m-1}{2}}(V_\phi)^{\frac{r-2}{2}}|\nabla V_\phi|\right)\left(2\sqrt{m(r-1)}C_\phi(u_\eta^\sigma+\eta)^{\frac{m-1}{2}}\mathfrak{D}\phi^{\frac34}(V_\phi)^{\frac{r-2}{2}}\right)dx\\
&\le \frac18 m(r-1)\int_{\tilde\Omega}(u_\eta^\sigma+\eta)^{m-1}|V_\phi|^{r-2}|\nabla V_\phi|^2dx\\
&+2m(r-1)C_\phi^2\int_{\tilde\Omega}(u_\eta^\sigma+\eta)^{m-1}\mathfrak{D}^2\phi^{\frac32}(V_\phi)^{r-2}dx\\
&\le \frac18 I_1+\int_{\tilde\Omega}\left(\sqrt{m}(u_\eta^\sigma+\eta)^{\frac{m-1}{2}}\mathfrak{D}^{\frac12}(V_\phi)^{\frac r2}\right)\left(2\sqrt{m}(r-1)C_\phi^2(u_\eta^\sigma+\eta)^{\frac{m-1}{2}}\mathfrak{D}^{\frac32}\phi^{\frac32}(V_\phi)^{\frac r2-2}\right)dx\\
&\le \frac18 I_1+\frac18 I_3+2m(r-1)^2C_\phi^4(2\|u_\eta^\sigma\|_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))})^{m-1}\|V_\phi\|_{L^r(\tilde\Omega)}^{r-1}|\tilde\Omega|^{\frac1r}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
II_4 \le{}& \frac18 I_1+\frac18 I_2+\frac14 I_3+\frac1{16}I_3' \\
&+64^2e^4m^4(m-1)^4\frac{1}{\sigma^3}(2\|u_\eta^\sigma\|_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))})^{4(m-2)}C_\phi^4\|V_\phi\|_{L^r(\tilde\Omega)}^{r-1}|\tilde\Omega|^{\frac1r}\\
&+8mC_\phi^4(2\|u_\eta^\sigma\|_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))})^{m-1}\|V_\phi\|_{L^r(\tilde\Omega)}^{r-1}|\tilde\Omega|^{\frac1r}\\
&+2m(r-1)^2C_\phi^4(2\|u_\eta^\sigma\|_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))})^{m-1}\|V_\phi\|_{L^r(\tilde\Omega)}^{r-1}|\tilde\Omega|^{\frac1r}.
\end{aligned}
\tag{2.29}
$$

Similarly,

$$
II_5=\sigma\int_{\tilde\Omega}\mathfrak{D}\Delta\phi(V_\phi)^{r-1}dx\le\frac18 I_1'+\frac18 I_2'+\frac1{16}I_3'+\left(32\sigma C_\phi^4+32\sigma(r-1)^2C_\phi^4\right)\|V_\phi\|_{L^r(\tilde\Omega)}^{r-1}|\tilde\Omega|^{\frac1r}. \tag{2.30}
$$

Now consider III.

$$
\begin{aligned}
III &= (8e\tilde\omega m(m-1))\int_{\tilde\Omega}(u_\eta^\sigma+\eta)^{m-2}\sum_{i=1}^d|\partial_{ii}\bar{\mathfrak{u}}|\ V_\phi^r dx\\
&\le \int_{\tilde\Omega}\left(\frac{\sqrt\sigma}{\sqrt2}\sum_{i=1}^d|\partial_{ii}\bar{\mathfrak{u}}|\phi^{\frac12}|V_\phi|^{\frac{r-1}{2}}\right)\left(\frac{1}{\sqrt\sigma}8\sqrt2 e\tilde\omega m(m-1)(u_\eta^\sigma+\eta)^{m-2}\mathfrak{D}^{\frac12}(V_\phi)^{\frac r2}\right)dx\\
&\le \frac18 2\sigma\int_{\tilde\Omega}\sum_{i=1}^d|\partial_{ii}\bar{\mathfrak{u}}|^2\phi|V_\phi|^{r-1}dx+\frac{64}{\sigma}\tilde\omega^2e^2m^2(m-1)^2(2\|u_\eta^\sigma\|_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))})^{2(m-2)}\int_{\tilde\Omega}\mathfrak{D}(V_\phi)^rdx.
\end{aligned}
$$

In order to ensure that

$$
\frac{64}{\sigma}\tilde\omega^2e^2m^2(m-1)^2(2\|u_\eta^\sigma\,|_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))})^{2(m-2)}\le\frac{\sigma}{2},
$$

$\tilde{\omega}$ should satisfy

$$\begin{aligned}\tilde{\omega} &\leq \left(\frac{\sigma^2}{2\cdot 64e^2m^2(m-1)^2(2\|u_\eta^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))})^{2(m-2)}}\right)^{\frac{1}{2}}\\ &\leq \frac{\sigma}{8\sqrt{2}em(m-1)(2\|u_\eta^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))})^{m-2}}.\end{aligned}$$

So, we obtain that

$$III \leq \frac{1}{8}I_2' + \frac{1}{8}I_3'. \tag{2.31}$$

Now we focus on IV_1.

$$\begin{aligned}IV_1 &= 2m\int_{\tilde{\Omega}}(u_\eta^\sigma+\eta)^{m-1}\left|\frac{\psi''(\bar{\mathfrak{u}})}{\psi'(\bar{\mathfrak{u}})}\right||\nabla\bar{\mathfrak{u}}|\ |\nabla V_\phi|\ |V_\phi|^{r-1}dx\\ &\leq 4m\int_{\tilde{\Omega}}(u_\eta^\sigma+\eta)^{m-1}|\nabla\bar{\mathfrak{u}}|\ |\nabla V_\phi|\ |V_\phi|^{r-1}dx\\ &\leq \int_{\tilde{\Omega}}\left(\frac{\sqrt{m(r-1)}}{2}(u_\eta^\sigma+\eta)^{\frac{m-1}{2}}|V_\phi|^{\frac{r-2}{2}}|\nabla V_\phi|\right)\left(\frac{8\sqrt{m}}{\sqrt{r-1}}(u_\eta^\sigma+\eta)^{\frac{m-1}{2}}|\nabla\bar{\mathfrak{u}}|\ |V_\phi|^{\frac{r}{2}}\right)dx\\ &\leq \frac{m(r-1)}{8}\int_{\tilde{\Omega}}(u_\eta^\sigma+\eta)^{m-1}|V_\phi|^{r-2}|\nabla V_\phi|^2dx + \frac{32m}{r-1}\int_{\tilde{\Omega}}(u_\eta^\sigma+\eta)^{m-1}\mathfrak{D}|V_\phi|^r dx.\end{aligned}$$

Choosing $r \geq 65$ we obtain that $\frac{32m}{r-1} \leq \frac{1}{2}m = \frac{1}{8}\cdot 4m$. Therefore,

$$IV_1 \leq \frac{1}{8}I_1 + \frac{1}{8}I_3. \tag{2.32}$$

Similarly,

$$\begin{aligned}IV_1' &= 2\sigma\int_{\tilde{\Omega}}\left|\frac{\psi''(\bar{\mathfrak{u}})}{\psi'(\bar{\mathfrak{u}})}\right||\nabla\bar{\mathfrak{u}}|\ |\nabla V_\phi||V_\phi|^{r-1}dx\\ &\leq 4\sigma\int_{\tilde{\Omega}}|\nabla\bar{\mathfrak{u}}|\ |\nabla V_\phi||V_\phi|^{r-1}dx\\ &\leq \int_{\tilde{\Omega}}\left(\frac{\sqrt{\sigma(r-1)}}{2}|V_\phi|^{\frac{r-2}{2}}|\nabla V_\phi|\right)\left(\frac{8\sqrt{\sigma}}{\sqrt{r-1}}|\nabla\bar{\mathfrak{u}}|\ |V_\phi|^{\frac{r}{2}}\right)dx\\ &\leq \frac{\sigma(r-1)}{8}\int_{\tilde{\Omega}}|V_\phi|^{r-2}|\nabla V_\phi|^2dx + \frac{32\sigma}{r-1}\int_{\tilde{\Omega}}\mathfrak{D}|V_\phi|^r dx.\end{aligned}$$

Therefore,

$$IV_1' \leq \frac{1}{8}I_1' + \frac{1}{8}I_3'. \tag{2.33}$$

Since $\left|\frac{\psi''(\bar{\mathfrak{u}})}{\psi'(\bar{\mathfrak{u}})}\right| \leq 2$, $|\nabla \bar{\mathfrak{u}}| \leq \mathfrak{D}^{\frac{1}{2}}$ and $|\nabla \phi| \leq C_\phi \phi^{\frac{3}{4}}$, we deduce that

$$\begin{aligned}
IV_2 &= 2m \int_{\tilde{\Omega}} (u^\sigma_\eta + \eta)^{m-1} \left|\frac{\psi''(\bar{\mathfrak{u}})}{\psi'(\bar{\mathfrak{u}})}\right| |\nabla \bar{\mathfrak{u}}| \; |\nabla \phi| \mathfrak{D} (V_\phi)^{r-1} dx \\
&\leq 4mC_\phi \int_{\tilde{\Omega}} (u^\sigma_\eta + \eta)^{m-1} \mathfrak{D}^{\frac{1}{2}} \mathfrak{D} \phi^{\frac{3}{4}} |V_\phi|^{r-1} dx \\
&\leq \int_{\tilde{\Omega}} \left(\frac{\sqrt{m}}{2} (u^\sigma_\eta + \eta)^{\frac{m-1}{2}} \mathfrak{D}^{\frac{1}{2}} |V_\phi|^{\frac{r}{2}} \right) \left(8\sqrt{m} C_\phi (u^\sigma_\eta + \eta)^{\frac{m-1}{2}} \mathfrak{D} \phi^{\frac{3}{4}} |V_\phi|^{\frac{r}{2}-1} \right) dx \\
&\leq \frac{1}{32} I_3 + 32mC_\phi^2 \int_{\tilde{\Omega}} (u^\sigma_\eta + \eta)^{m-1} \mathfrak{D}^2 \phi^{\frac{3}{2}} |V_\phi|^{r-2} dx \\
&\leq \frac{1}{32} I_3 + \int_{\tilde{\Omega}} \left(\frac{\sqrt{m}}{2} (u^\sigma_\eta + \eta)^{\frac{m-1}{2}} \mathfrak{D}^{\frac{1}{2}} |V_\phi|^{\frac{r}{2}} \right) \left(64\sqrt{m} C_\phi^2 (u^\sigma_\eta + \eta)^{\frac{m-1}{2}} \mathfrak{D}^{\frac{3}{2}} \phi^{\frac{3}{2}} |V_\phi|^{\frac{r}{2}-2} \right) dx \\
&\leq \frac{1}{16} I_3 + 64^2 m C_\phi^4 \int_{\tilde{\Omega}} (u^\sigma_\eta + \eta)^{m-1} \mathfrak{D}^3 \phi^3 |V_\phi|^{r-4} dx.
\end{aligned}$$

So,

$$IV_2 \leq \frac{1}{16} I_3 + 64^2 m C_\phi^4 (2\|u^\sigma_\eta\|_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))})^{m-1} \|V_\phi\|^{r-1}_{L^r(\tilde{\Omega})} |\tilde{\Omega}|^{\frac{1}{r}}. \tag{2.34}$$

Similarly,

$$IV_2' = 2\sigma \int_{\tilde{\Omega}} \left|\frac{\psi''(\bar{\mathfrak{u}})}{\psi'(\bar{\mathfrak{u}})}\right| |\nabla \bar{\mathfrak{u}}| \; |\nabla \phi| \mathfrak{D} (V_\phi)^{r-1} dx \leq \frac{1}{16} I_3' + 64^2 \sigma C_\phi^4 \|V_\phi\|^{r-1}_{L^r(\tilde{\Omega})} |\tilde{\Omega}|^{\frac{1}{r}}. \tag{2.35}$$

Recall that there exists a constant $C > 0$ such that for $q > d$ we obtain that

$$\|\nabla \Phi * u^\sigma_\eta\|_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))} \leq C \|\nabla \Phi * u^\sigma_\eta\|_{L^\infty(0,T_2;W^{1,q}(\mathbb{R}^d))} \leq C \|u^\sigma_\eta\|_{L^\infty(0,T_2;L^q(\mathbb{R}^d))} \leq C.$$

So we obtain that

$$\begin{aligned}
J_1^1 &= -\int_{\tilde{\Omega}} \nabla V_\phi \cdot \nabla \Phi * u^\sigma_\eta (V_\phi)^{r-1} dx \\
&\leq \int_{\tilde{\Omega}} \left(\frac{\sqrt{\sigma(r-1)}}{2} |V_\phi|^{\frac{r-2}{2}} |\nabla V_\phi| \right) \left(\frac{2}{\sqrt{\sigma(r-1)}} |\nabla \Phi * u^\sigma_\eta| \; |V_\phi|^{\frac{r}{2}} \right) dx \\
&\leq \frac{1}{8} I_1' + \frac{2\|\nabla \Phi * u^\sigma_\eta\|^2_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))}}{\sigma(r-1)} \int_{\tilde{\Omega}} |V_\phi|^r dx.
\end{aligned}$$

Hence,

$$J_1^1 \leq \frac{1}{8} I_1' + \frac{\|\nabla \Phi * u^\sigma_\eta\|^2_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))}}{32\sigma} \|V_\phi\|^r_{L^r(\tilde{\Omega})}. \tag{2.36}$$

By the same argument it follows that

$$\begin{aligned}
J_1^2 &= \int_{\tilde{\Omega}} \mathfrak{D}\nabla\phi \cdot \nabla\Phi * u_\eta^\sigma (V_\phi)^{r-1} dx \\
&\leq \int_{\tilde{\Omega}} \left(\sqrt{\sigma}\mathfrak{D}^{\frac{1}{2}}|V_\phi|^{\frac{r}{2}}\right)\left(\frac{1}{\sqrt{\sigma}}\mathfrak{D}^{\frac{1}{2}}C_\phi \phi^{\frac{3}{4}}|\nabla\Phi * u_\eta^\sigma|\ |V_\phi|^{\frac{r}{2}-1}\right) dx \\
&\leq \frac{1}{8}I_3' + \frac{1}{2\sigma}C_\phi^2 \int_{\tilde{\Omega}} \mathfrak{D}\phi^{\frac{3}{2}}|\nabla\Phi * u_\eta^\sigma|^2\ |V_\phi|^{r-2} dx \\
&\leq \frac{1}{8}I_3' + \frac{1}{2\sigma}C_\phi^2 \|\nabla\Phi * u_\eta^\sigma\|^2_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))} \int_{\tilde{\Omega}} |V_\phi|^{r-1} dx.
\end{aligned}$$

So,

$$J_1^2 \leq \frac{1}{8}I_3' + \frac{1}{2\sigma}C_\phi^2 \|\nabla\Phi * u_\eta^\sigma\|^2_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))} \|V_\phi\|^{r-1}_{L^r(\tilde{\Omega})} |\tilde{\Omega}|^{\frac{1}{r}}. \tag{2.37}$$

Next, we analyze J_2.

$$\begin{aligned}
J_2 &= -2\sum_{k=1}^d \int_{\tilde{\Omega}} \nabla\bar{\mathfrak{u}} \cdot \nabla\partial_{x_k}\Phi * u_\eta^\sigma \partial_{x_k}\bar{\mathfrak{u}}\phi(V_\phi)^{r-1} dx \\
&= 2\sum_{k=1}^d \int_{\tilde{\Omega}} \nabla\partial_{x_k}\bar{\mathfrak{u}} \cdot \nabla\Phi * u_\eta^\sigma \partial_{x_k}\bar{\mathfrak{u}}\phi(V_\phi)^{r-1} dx \\
&\quad + 2\sum_{k=1}^d \int_{\tilde{\Omega}} \nabla\bar{\mathfrak{u}} \cdot \nabla\Phi * u_\eta^\sigma \partial_{x_k}\partial_{x_k}\bar{\mathfrak{u}}\phi(V_\phi)^{r-1} dx \\
&\quad + 2\sum_{k=1}^d \int_{\tilde{\Omega}} \nabla\bar{\mathfrak{u}} \cdot \nabla\Phi * u_\eta^\sigma \partial_{x_k}\bar{\mathfrak{u}}\partial_{x_k}\phi(V_\phi)^{r-1} dx \\
&\quad + 2(r-1)\sum_{k=1}^d \int_{\tilde{\Omega}} \nabla\bar{\mathfrak{u}} \cdot \nabla\Phi * u_\eta^\sigma \partial_{x_k}\bar{\mathfrak{u}}\phi(V_\phi)^{r-2}\partial_{x_k}V_\phi dx.
\end{aligned}$$

Now we are going to estimate these four terms

$$\begin{aligned}
&2\sum_{k=1}^d \int_{\tilde{\Omega}} \nabla\partial_{x_k}\bar{\mathfrak{u}} \cdot \nabla\Phi * u_\eta^\sigma \partial_{x_k}\bar{\mathfrak{u}}\phi(V_\phi)^{r-1} dx \\
&\leq \sum_{k=1}^d \int_{\tilde{\Omega}} \left(\frac{\sqrt{\sigma}}{\sqrt{2}}|\nabla\partial_{x_k}\bar{\mathfrak{u}}|\phi^{\frac{1}{2}}|V_\phi|^{\frac{r-1}{2}}\right)\left(\frac{2\sqrt{2}}{\sqrt{\sigma}}|\nabla\Phi * u_\eta^\sigma|\ |\nabla\bar{\mathfrak{u}}|\phi^{\frac{1}{2}}|V_\phi|^{\frac{r-1}{2}}\right) dx \\
&\leq \frac{1}{8}\cdot 2\sigma \int_{\tilde{\Omega}} \sum_{k=1}^d |\nabla\partial_{x_k}\bar{\mathfrak{u}}|^2\phi|V_\phi|^{r-1} dx + \frac{4}{\sigma}\int_{\tilde{\Omega}} |\nabla\Phi * u_\eta^\sigma|^2\mathfrak{D}\phi|V_\phi|^{r-1} dx \\
&\leq \frac{1}{8}I_2' + \frac{4\|\nabla\Phi * u_\eta^\sigma\|^2_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))}}{\sigma}\|V_\phi\|^r_{L^r(\tilde{\Omega})},
\end{aligned}$$

$$
\begin{aligned}
&2\sum_{k=1}^{d}\int_{\tilde{\Omega}} \nabla\bar{\mathbf{u}}\cdot\nabla\Phi * u_\eta^\sigma \partial_{x_k}\partial_{x_k}\bar{\mathbf{u}}\phi(V_\phi)^{r-1}dx\\
&\le \sum_{k=1}^{d}\int_{\tilde{\Omega}}\left(\frac{\sqrt{\sigma}}{\sqrt{2}}|\nabla\partial_{x_k}\bar{\mathbf{u}}|\phi^{\frac{1}{2}}|V_\phi|^{\frac{r-1}{2}}\right)\left(\frac{2\sqrt{2}}{\sqrt{\sigma}}|\nabla\Phi * u_\eta^\sigma|\ |\nabla\bar{\mathbf{u}}|\phi^{\frac{1}{2}}|V_\phi|^{\frac{r-1}{2}}\right)dx\\
&\le \frac{1}{8}\cdot 2\sigma\int_{\tilde{\Omega}}\sum_{k=1}^{d}|\nabla\partial_{x_k}\bar{\mathbf{u}}|^2\phi|V_\phi|^{r-1}dx+\frac{4}{\sigma}\int_{\tilde{\Omega}}|\nabla\Phi * u_\eta^\sigma|^2\mathfrak{D}\phi|V_\phi|^{r-1}dx\\
&\le \frac{1}{8}I_2'+\frac{4\|\nabla\Phi * u_\eta^\sigma\|^2_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))}}{\sigma}\|V_\phi\|^r_{L^r(\tilde{\Omega})},
\end{aligned}
$$

$$
\begin{aligned}
&2\sum_{k=1}^{d}\int_{\tilde{\Omega}} \nabla\bar{\mathbf{u}}\cdot\nabla\Phi * u_\eta^\sigma \partial_{x_k}\bar{\mathbf{u}}\partial_{x_k}\phi(V_\phi)^{r-1}dx\\
&\le 2C_\phi\int_{\tilde{\Omega}}\mathfrak{D}|\nabla\Phi * u_\eta^\sigma|\phi^{\frac{3}{4}}|V_\phi|^{r-1}dx\\
&\le \int_{\tilde{\Omega}}\left(\frac{\sqrt{\sigma}}{\sqrt{2}}\mathfrak{D}^{\frac{1}{2}}|V_\phi|^{\frac{r}{2}}\right)\left(\frac{2\sqrt{2}}{\sqrt{\sigma}}C_\phi\mathfrak{D}^{\frac{1}{2}}|\nabla\Phi * u_\eta^\sigma|\phi^{\frac{3}{4}}|V_\phi|^{\frac{r}{2}-1}\right)dx\\
&\le \frac{1}{16}\cdot 4\sigma\int_{\tilde{\Omega}}\mathfrak{D}|V_\phi|^r dx+\frac{4C_\phi^2}{\sigma}\int_{\tilde{\Omega}}\mathfrak{D}|\nabla\Phi * u_\eta^\sigma|^2\phi^{\frac{3}{2}}|V_\phi|^{r-2}dx\\
&\le \frac{1}{16}I_3'+\frac{4C_\phi^2\|\nabla\Phi * u_\eta^\sigma\|^2_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))}}{\sigma}|\tilde{\Omega}|^{\frac{1}{r}}\|V_\phi\|^r_{L^r(\tilde{\Omega})},
\end{aligned}
$$

$$
\begin{aligned}
&2(r-1)\sum_{k=1}^{d}\int_{\tilde{\Omega}} \nabla\bar{\mathbf{u}}\cdot\nabla\Phi * u_\eta^\sigma \partial_{x_k}\bar{\mathbf{u}}\phi(V_\phi)^{r-2}\partial_{x_k}V_\phi dx\\
&\le \int_{\tilde{\Omega}}\left(\frac{\sqrt{\sigma(r-1)}}{2\sqrt{2}}|V_\phi|^{\frac{r-2}{2}}|\nabla V_\phi|\right)\left(\frac{4\sqrt{2}\cdot\phi\sqrt{r-1}}{\sqrt{\sigma}}|\nabla\bar{\mathbf{u}}|^2|\nabla\Phi * u_\eta^\sigma|\ |V_\phi|^{\frac{r-2}{2}}\right)dx\\
&\le \frac{1}{16}\sigma(r-1)\int_{\tilde{\Omega}}|V_\phi|^{r-2}|\nabla V_\phi|^2dx+\frac{16(r-1)}{\sigma}\int_{\tilde{\Omega}}\phi^2|\nabla\bar{\mathbf{u}}|^4|\nabla\Phi * u_\eta^\sigma|^2|V_\phi|^{r-2}dx\\
&\le \frac{1}{16}I_1'+\frac{16r\|\nabla\Phi * u_\eta^\sigma\|^2_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))}}{\sigma}\int_{\tilde{\Omega}}|V_\phi|^r dx.
\end{aligned}
$$

Therefore, we obtain that

$$
\begin{aligned}
J_2 \le\ &\frac{1}{16}I_1'+\frac{1}{4}I_2'+\frac{1}{16}I_3'\\
&+\frac{16r\|\nabla\Phi * u_\eta^\sigma\|^2_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))}}{\sigma}\|V_\phi\|^r_{L^r(\tilde{\Omega})}\\
&+\frac{4C_\phi^2\|\nabla\Phi * u_\eta^\sigma\|^2_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))}}{\sigma}|\tilde{\Omega}|^{\frac{1}{r}}\|V_\phi\|^{r-1}_{L^r(\tilde{\Omega})}
\end{aligned}
\tag{2.38}
$$

We handle the last two terms of J_3 and J_4 (2.25) together. First, we rewrite J_4 and obtain that

$$\begin{aligned}J_4 &= -2\sum_{k=1}^{d}\int_{\tilde\Omega}\frac{u_\eta^\sigma\partial_{x_k}\Delta\Phi * u_\eta^\sigma\partial_{x_k}\bar{\mathrm{u}}}{\psi'(\bar{\mathrm{u}})}\phi(V_\phi)^{r-1}dx\\ &= 2\sum_{k=1}^{d}\int_{\tilde\Omega}\partial_{x_k}\left(\frac{u_\eta^\sigma\partial_{x_k}\bar{\mathrm{u}}}{\psi'(\bar{\mathrm{u}})}\phi(V_\phi)^{r-1}\right)\Delta\Phi * u_\eta^\sigma dx.\end{aligned}$$

Since

$$\begin{aligned}\sum_{k=1}^{d}\partial_{x_k}\left(\frac{u_\eta^\sigma\partial_{x_k}\bar{\mathrm{u}}}{\psi'(\bar{\mathrm{u}})}\phi(V_\phi)^{r-1}\right) &= \sum_{k=1}^{d}\left(\frac{(\psi'(\bar{\mathrm{u}}))^2\partial_{x_k}\bar{\mathrm{u}} - u_\eta^\sigma\psi''(\bar{\mathrm{u}})\partial_{x_k}\bar{\mathrm{u}}}{(\psi'(\bar{\mathrm{u}}))^2}\right)\partial_{x_k}\bar{\mathrm{u}}\phi(V_\phi)^{r-1}\\ &+\frac{u_\eta^\sigma\Delta\bar{\mathrm{u}}}{\psi'(\bar{\mathrm{u}})}\phi(V_\phi)^{r-1} + \frac{u_\eta^\sigma\nabla\bar{\mathrm{u}}}{\psi'(\bar{\mathrm{u}})}\cdot\nabla\phi(V_\phi)^{r-1} + (r-1)\frac{u_\eta^\sigma\nabla\bar{\mathrm{u}}}{\psi'(\bar{\mathrm{u}})}\cdot\phi(V_\phi)^{r-2}\nabla V_\phi\end{aligned}$$

and

$$\sum_{k=1}^{d}\left(\frac{(\psi'(\bar{\mathrm{u}}))^2\partial_{x_k}\bar{\mathrm{u}} - u_\eta^\sigma\psi''(\bar{\mathrm{u}})\partial_{x_k}\bar{\mathrm{u}}}{(\psi'(\bar{\mathrm{u}}))^2}\right)\partial_{x_k}\bar{\mathrm{u}} = \frac{(\psi'(\bar{\mathrm{u}}))^2|\nabla\bar{\mathrm{u}}|^2 - u_\eta^\sigma\psi''(\bar{\mathrm{u}})|\nabla\bar{\mathrm{u}}|^2}{(\psi'(\bar{\mathrm{u}}))^2} = \left(1-\frac{\psi''(\bar{\mathrm{u}})}{(\psi'(\bar{\mathrm{u}}))^2}u_\eta^\sigma\right)|\nabla\bar{\mathrm{u}}|^2,$$

we obtain that

$$\begin{aligned}J_4 &= 2\int_{\tilde\Omega}\left(1-\frac{\psi''(\bar{\mathrm{u}})}{(\psi'(\bar{\mathrm{u}}))^2}u_\eta^\sigma\right)(V_\phi)^r\Delta\Phi * u_\eta^\sigma dx + 2\int_{\tilde\Omega}\frac{u_\eta^\sigma\Delta\bar{\mathrm{u}}}{\psi'(\bar{\mathrm{u}})}\phi(V_\phi)^{r-1}\Delta\Phi * u_\eta^\sigma dx\\ &+2\int_{\tilde\Omega}\frac{u_\eta^\sigma\nabla\bar{\mathrm{u}}}{\psi'(\bar{\mathrm{u}})}\cdot\nabla\phi(V_\phi)^{r-1}\Delta\Phi * u_\eta^\sigma dx + 2(r-1)\int_{\tilde\Omega}\frac{u_\eta^\sigma\nabla\bar{\mathrm{u}}}{\psi'(\bar{\mathrm{u}})}\cdot\phi(V_\phi)^{r-2}\nabla V_\phi\Delta\Phi * u_\eta^\sigma dx\\ &= -2\int_{\tilde\Omega}\left(\frac{\psi''(\bar{\mathrm{u}})}{(\psi'(\bar{\mathrm{u}}))^2}u_\eta^\sigma - 1\right)(V_\phi)^r\Delta\Phi * u_\eta^\sigma dx + J_4^1 + J_4^2 + J_4^3\end{aligned}$$

where

$$\begin{aligned}J_4^1 &:= 2\int_{\tilde\Omega}\frac{u_\eta^\sigma\Delta\bar{\mathrm{u}}}{\psi'(\bar{\mathrm{u}})}\phi(V_\phi)^{r-1}\Delta\Phi * u_\eta^\sigma dx,\\ J_4^2 &:= 2\int_{\tilde\Omega}\frac{u_\eta^\sigma\nabla\bar{\mathrm{u}}}{\psi'(\bar{\mathrm{u}})}\cdot\nabla\phi(V_\phi)^{r-1}\Delta\Phi * u_\eta^\sigma dx,\\ J_4^3 &:= 2(r-1)\int_{\tilde\Omega}\frac{u_\eta^\sigma\nabla\bar{\mathrm{u}}}{\psi'(\bar{\mathrm{u}})}\cdot\phi(V_\phi)^{r-2}\nabla V_\phi\Delta\Phi * u_\eta^\sigma dx.\end{aligned}$$

From the definition of J_3 we deduce that

$$\begin{aligned}J_3 + J_4 &= 2\int_{\tilde\Omega}\left(\frac{\psi''(\bar{\mathrm{u}})}{(\psi'(\bar{\mathrm{u}}))^2}u_\eta^\sigma - 1\right)\Delta\Phi * u_\eta^\sigma(V_\phi)^r dx\\ &- 2\int_{\tilde\Omega}\left(\frac{\psi''(\bar{\mathrm{u}})}{(\psi'(\bar{\mathrm{u}}))^2}u_\eta^\sigma - 1\right)(V_\phi)^r\Delta\Phi * u_\eta^\sigma dx + J_4^1 + J_4^2 + J_4^3\\ &= J_4^1 + J_4^2 + J_4^3.\end{aligned}$$

Let us now estimate these three terms.

$$
\begin{aligned}
J_4^1 &= 2\int_{\tilde{\Omega}} \frac{u_\eta^\sigma \Delta\bar{\mathrm{u}}}{\psi'(\bar{\mathrm{u}})}\phi(V_\phi)^{r-1}\Delta\Phi * u_\eta^\sigma dx \\
&= \int_{\tilde{\Omega}} \left(\frac{\sqrt{\sigma}}{\sqrt{2}}|\Delta\bar{\mathrm{u}}|\phi^{\frac{1}{2}}|V_\phi|^{\frac{r-1}{2}}\right)\left(\frac{u_\eta^\sigma}{\sqrt{2\sigma}e\tilde{\omega}e^{-\bar{\mathrm{u}}^2}}\phi^{\frac{1}{2}}|V_\phi|^{\frac{r-1}{2}}|\Delta\Phi * u_\eta^\sigma|\right)dx \\
&\le \frac{1}{8}\cdot 2\sigma\int_{\tilde{\Omega}} |\Delta\bar{\mathrm{u}}|^2\phi|V_\phi|^{r-1}dx + \frac{e^{2(\bar{\mathrm{u}}^2-1)}}{4\sigma\tilde{\omega}^2}\int_{\tilde{\Omega}}(u_\eta^\sigma)^2\phi|V_\phi|^{r-1}|\Delta\Phi * u_\eta^\sigma|^2dx \\
&\le \frac{1}{8}\cdot 2\sigma\int_{\tilde{\Omega}} |\Delta\bar{\mathrm{u}}|^2\phi|V_\phi|^{r-1}dx + \frac{1}{4\sigma\tilde{\omega}^2}\int_{\tilde{\Omega}}(u_\eta^\sigma)^2\phi|V_\phi|^{r-1}|\Delta\Phi * u_\eta^\sigma|^2dx \\
&\le \frac{1}{8}I_2' + \frac{1}{4\sigma\tilde{\omega}^2}\left(\int_{\tilde{\Omega}}(u_\eta^\sigma)^{2r}|\Delta\Phi * u_\eta^\sigma|^{2r}\phi^r dx\right)^{\frac{1}{r}}\left(\int_{\tilde{\Omega}}|V_\phi|^r dx\right)^{\frac{r-1}{r}} \\
&= \frac{1}{8}I_2' + \frac{1}{4\sigma\tilde{\omega}^2}\|u_\eta^\sigma\Delta\Phi * u_\eta^\sigma\phi^{\frac{1}{2}}\|^2_{L^{2r}(\tilde{\Omega})}\|V_\phi\|^{r-1}_{L^r(\tilde{\Omega})},
\end{aligned}
$$

$$
\begin{aligned}
J_4^2 &= 2\int_{\tilde{\Omega}} \frac{u_\eta^\sigma\nabla\bar{\mathrm{u}}}{\psi'(\bar{\mathrm{u}})}\cdot\nabla\phi(V_\phi)^{r-1}\Delta\Phi * u_\eta^\sigma dx \\
&= 2\int_{\tilde{\Omega}} \frac{u_\eta^\sigma\nabla\bar{\mathrm{u}}}{4e\tilde{\omega}e^{-\bar{\mathrm{u}}^2}}\cdot\nabla\phi(V_\phi)^{\frac{r}{2}-1}\Delta\Phi * u_\eta^\sigma(V_\phi)^{\frac{r}{2}}dx \\
&\le \int_{\tilde{\Omega}}|V_\phi|^r dx + \int_{\tilde{\Omega}}\frac{(u_\eta^\sigma)^2|\nabla\bar{\mathrm{u}}|^2}{16\tilde{\omega}^2e^{2(1-\bar{\mathrm{u}}^2)}}|\nabla\phi|^2|V_\phi|^{r-2}|\Delta\Phi * u_\eta^\sigma|^2dx \\
&\le \int_{\tilde{\Omega}}|V_\phi|^r dx + \int_{\tilde{\Omega}}\frac{(u_\eta^\sigma)^2\mathfrak{D}}{16\tilde{\omega}^2e^{2(1-\bar{\mathrm{u}}^2)}}C_\phi^2\phi^{\frac{3}{2}}|V_\phi|^{r-2}|\Delta\Phi * u_\eta^\sigma|^2dx \\
&\le \int_{\tilde{\Omega}}|V_\phi|^r dx + \frac{C_\phi^2}{16\tilde{\omega}^2}\int_{\tilde{\Omega}}(u_\eta^\sigma)^2\phi^{\frac{1}{2}}|V_\phi|^{r-1}|\Delta\Phi * u_\eta^\sigma|^2dx \\
&\le \int_{\tilde{\Omega}}|V_\phi|^r dx + \frac{C_\phi^2}{16\tilde{\omega}^2}\left(\int_{\tilde{\Omega}}(u_\eta^\sigma)^{2r}|\Delta\Phi * u_\eta^\sigma|^{2r}\phi^{\frac{1}{4}\cdot 2r}dx\right)^{\frac{1}{r}}\left(\int_{\tilde{\Omega}}|V_\phi|^r dx\right)^{\frac{r-1}{r}} \\
&= \|V_\phi\|^r_{L^r(\tilde{\Omega})} + \frac{C_\phi^2}{16\tilde{\omega}^2}\|u_\eta^\sigma\Delta\Phi * u_\eta^\sigma\phi^{\frac{1}{4}}\|^2_{L^{2r}(\tilde{\Omega})}\|V_\phi\|^{r-1}_{L^r(\tilde{\Omega})},
\end{aligned}
$$

$$
\begin{aligned}
J_4^3 &= 2(r-1)\int_{\tilde{\Omega}} \frac{u_\eta^\sigma\nabla\bar{\mathrm{u}}}{\psi'(\bar{\mathrm{u}})}\cdot\phi(V_\phi)^{r-2}\nabla V_\phi\Delta\Phi * u_\eta^\sigma dx \\
&= \int_{\tilde{\Omega}}\left(\frac{\sqrt{\sigma(r-1)}}{2\sqrt{2}}(V_\phi)^{\frac{r-2}{2}}|\nabla V_\phi|\right)\left(\frac{\sqrt{2(r-1)}u_\eta^\sigma|\nabla\bar{\mathrm{u}}|}{\sqrt{\sigma}e\tilde{\omega}e^{-\bar{\mathrm{u}}^2}}\phi(V_\phi)^{\frac{r-2}{2}}|\Delta\Phi * u_\eta^\sigma|\right)dx \\
&\le \frac{1}{16}\sigma(r-1)\int_{\tilde{\Omega}}|V_\phi|^{r-2}|\nabla V_\phi|^2dx + \frac{r-1}{\sigma\tilde{\omega}^2e^{2(1-\bar{\mathrm{u}})}}\int_{\tilde{\Omega}}(u_\eta^\sigma)^2|\nabla\bar{\mathrm{u}}|^2\phi^2|V_\phi|^{r-2}|\Delta\Phi * u_\eta^\sigma|^2dx \\
&= \frac{1}{16}\sigma(r-1)\int_{\tilde{\Omega}}|V_\phi|^{r-2}|\nabla V_\phi|^2dx + \frac{r-1}{\sigma\tilde{\omega}^2e^{2(1-\bar{\mathrm{u}})}}\int_{\tilde{\Omega}}(u_\eta^\sigma)^2\phi|V_\phi|^{r-1}|\Delta\Phi * u_\eta^\sigma|^2dx \\
&\le \frac{1}{16}I_1' + \frac{r-1}{\sigma\tilde{\omega}^2}\left(\int_{\tilde{\Omega}}(u_\eta^\sigma)^{2r}|\Delta\Phi * u_\eta^\sigma|^{2r}\phi^r dx\right)^{\frac{1}{r}}\left(\int_{\tilde{\Omega}}|V_\phi|^r dx\right)^{\frac{r-1}{r}} \\
&= \frac{1}{16}I_1' + \frac{r-1}{\sigma\tilde{\omega}^2}\|u_\eta^\sigma\Delta\Phi * u_\eta^\sigma\phi^{\frac{1}{2}}\|^2_{L^{2r}(\tilde{\Omega})}\|V_\phi\|^{r-1}_{L^r(\tilde{\Omega})}.
\end{aligned}
$$

Therefore, we obtain that

$$J_3 + J_4 \le \frac{1}{16} I_1' + \frac{1}{8} I_2' + rC \| u_\eta^\sigma \Delta\Phi * u_\eta^\sigma \phi^{\frac{1}{4}} \|^2_{L^{2r}(\tilde{\Omega})} \| V_\phi \|^{r-1}_{L^r(\tilde{\Omega})} \tag{2.39}$$

where C is a positive number depending on $\sigma, C_\phi, m, \|u_\eta^\sigma\|_{L^\infty(0,T_2;L^\infty(\tilde{\Omega}))}$. Moreover, it holds by interpolation that

$$\begin{aligned}
\| u_\eta^\sigma \Delta\Phi * u_\eta^\sigma \phi^{\frac{1}{4}} \|_{L^{2r}(\tilde{\Omega})} &\le \| \Delta\Phi * u_\eta^\sigma \|_{L^{4r}(\tilde{\Omega})} \| u_\eta^\sigma \phi^{\frac{1}{4}} \|_{L^{4r}(\tilde{\Omega})} \\
&\le \| u_\eta^\sigma \|_{L^{4r}(\tilde{\Omega})} \| u_\eta^\sigma \|_{L^{4r}(\tilde{\Omega})} \| \phi^{\frac{1}{4}} \|_{L^\infty(\tilde{\Omega})} \\
&\le C \| u_\eta^\sigma \|^2_{L^{4r}(\tilde{\Omega})} \\
&\le C(\| u_\eta^\sigma \|_{L^1(0,T_2;L^1(\mathbb{R}^d))}, \| u_\eta^\sigma \|_{L^\infty(0,T_2;L^\infty(\mathbb{R}^d))}).
\end{aligned}$$

Finally, combining (2.26) - (2.39) and

$$-\frac{\sigma(r-1)}{4} \int_{\tilde{\Omega}} |V_\phi|^{r-2} |\nabla V_\phi|^2 dx = -\frac{\sigma(r-1)}{4} \cdot \frac{4}{r^2} \int_{\tilde{\Omega}} |\nabla (V_\phi)^{\frac{r}{2}}|^2 dx,$$

we get that

$$\frac{1}{r}\frac{d}{dt} \| V_\phi \|^r_{L^r(\tilde{\Omega})} \le -\frac{\sigma(r-1)}{r^2} \int_{\tilde{\Omega}} |\nabla (V_\phi)^{\frac{r}{2}}(t,x)|^2 dx + r^2 \mathcal{M} \| V_\phi \|^r_{L^r(\tilde{\Omega})} + r^2 \mathcal{M}$$

where $\mathcal{M}$ is a positive number which depends only on $d, \sigma, C_\phi, m, \|u_\eta^\sigma\|_{L^1(0,T_2;L^1(\tilde{\Omega}))}, \|u_\eta^\sigma\|_{L^\infty(0,T_2;L^\infty(\tilde{\Omega}))}, |\tilde{\Omega}|$. This implies that

$$\frac{d}{dt} \int_{\tilde{\Omega}} |V_\phi|^r dx \le -\frac{\sigma(r-1)}{r} \int_{\tilde{\Omega}} |\nabla (V_\phi)^{\frac{r}{2}}(t,x)|^2 dx + r^3 \mathcal{M} \int_{\tilde{\Omega}} |V_\phi|^r dx + r^3 \mathcal{M} \tag{2.40}$$

It remains to prove that $\|V_\phi\|_{L^\infty(0,T;L^\infty(\tilde{\Omega}))}$ is bounded. We are going to apply the same technique as we used proving that the L^∞ norm in t and x of u_η^σ is bounded.

Let $q_k := 2^k + 4d + 4$ for $k \in \mathbb{N}$ and $k \ge 6$. We obtain that

$$\frac{d}{dt} \int_{\tilde{\Omega}} |V_\phi|^{q_k} dx \le -\frac{\sigma(q_k - 1)}{q_k} \int_{\tilde{\Omega}} |\nabla (V_\phi)^{\frac{q_k}{2}}(t,x)|^2 dx + q_k^3 \mathcal{M} \int_{\tilde{\Omega}} |V_\phi|^{q_k} dx + q_k^3 \mathcal{M}$$

Recall that $\theta_1 = \frac{\frac{1}{q_{k-1}} - \frac{1}{q_k}}{\frac{1}{q_{k-1}} - \frac{d-2}{dq_k}}$. Using the following inequality

$$ab \le C_1 a^{l_1} + (C_1 l_1)^{-\frac{l_2}{l_1}} l_2^{-1} b^{l_2},$$

where

$$\frac{1}{l_1}+\frac{1}{l_2}=1, \qquad C_1=\frac{\sigma(q_k-1)}{2q_k}, \qquad l_1=\frac{1}{\theta_1},$$
$$a=\|\nabla(V_\phi)^{\frac{q_k}{2}}\|_{L^2(\tilde\Omega)}^{2\theta_1}, \qquad b=S_d^{-\theta_1}q_k^3\mathcal{M}\|V_\phi\|_{L^{q_{k-1}}(\tilde\Omega)}^{(1-\theta_1)q_k},$$

we notice that

$$\begin{aligned}
q_k^3\mathcal{M}\int_{\tilde\Omega}|V_\phi|^{q_k}dx &\le S_d^{-\theta_1}q_k^3\mathcal{M}\|V_\phi\|_{L^{q_{k-1}}(\tilde\Omega)}^{(1-\theta_1)q_k}\|\nabla(V_\phi)^{\frac{q_k}{2}}\|_{L^2(\tilde\Omega)}^{2\theta_1}\\
&\le \frac{\sigma(q_k-1)}{2q_k}\int_{\tilde\Omega}|\nabla(V_\phi)^{\frac{q_k}{2}}|^2dx\\
&\quad+\left(\frac{\sigma(q_k-1)}{2q_k}\cdot\frac{1}{\theta_1}\right)^{-\frac{\theta_1}{1-\theta_1}}(1-\theta_1)\left(S_d^{-\theta_1}q_k^3\mathcal{M}\right)^{\frac{1}{1-\theta_1}}\|V_\phi\|_{L^{q_{k-1}}(\tilde\Omega)}^{q_k}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\frac{d}{dt}\int_{\tilde\Omega}|V_\phi|^{q_k}dx &\le -\frac{\sigma(q_k-1)}{2q_k}\int_{\tilde\Omega}|\nabla(V_\phi)^{\frac{q_k}{2}}|^2dx+q_k^3\mathcal{M}\\
&\quad+\left(\frac{\sigma(q_k-1)}{2q_k}\cdot\frac{1}{\theta_1}\right)^{-\frac{\theta_1}{1-\theta_1}}(1-\theta_1)\left(S_d^{-\theta_1}q_k^3\mathcal{M}\right)^{\frac{1}{1-\theta_1}}\|V_\phi\|_{L^{q_{k-1}}(\tilde\Omega)}^{q_k}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\int_{\tilde\Omega}|V_\phi|^{q_k}dx &\le S_d^{-\theta_1}\|V_\phi\|_{L^{q_{k-1}}(\tilde\Omega)}^{(1-\theta_1)q_k}\|\nabla(V_\phi)^{\frac{q_k}{2}}\|_{L^2(\tilde\Omega)}^{2\theta_1}\\
&\le \frac{\sigma(q_k-1)}{2q_k}\int_{\tilde\Omega}|\nabla(V_\phi)^{\frac{q_k}{2}}|^2dx+(1-\theta_1)\left(\frac{\sigma(q_k-1)}{2q_k}\cdot\frac{1}{\theta_1}\right)^{-\frac{\theta_1}{1-\theta_1}}S_d^{\frac{-\theta_1}{1-\theta_1}}\|V_\phi\|_{L^{q_{k-1}}(\tilde\Omega)}^{q_k}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\frac{d}{dt}\int_{\tilde\Omega}|V_\phi|^{q_k}dx &\le -\int_{\tilde\Omega}|V_\phi|^{q_k}dx+q_k^3\mathcal{M}\\
&\quad+(1-\theta_1)\left(\frac{\sigma(q_k-1)}{2q_k}\cdot\frac{1}{\theta_1}\right)^{-\frac{\theta_1}{1-\theta_1}}S_d^{\frac{-\theta_1}{1-\theta_1}}\left(1+(q_k^3\mathcal{M})^{\frac{1}{1-\theta_1}}\right)\|V_\phi\|_{L^{q_{k-1}}(\tilde\Omega)}^{q_k}\\
&\le -\int_{\tilde\Omega}|V_\phi|^{q_k}dx+q_k^3\mathcal{M}+Cq_k^{\frac{3}{1-\theta_1}}\|V_\phi\|_{L^{q_{k-1}}(\tilde\Omega)}^{q_k}
\end{aligned}$$

where C appeared in this step denotes a positive constant independent of q_k.

Recall that $\frac{1}{1-\theta_1}\le d$.

Let $y_k(t):=\|V_\phi(t,\cdot)\|_{L^{q_k}(\tilde\Omega)}^{q_k}$. Since $\frac{q_k}{q_{k-1}}\le 2$ we obtain that

$$
\begin{aligned}
y_k'(t) &= \frac{d}{dt}\int_{\tilde{\Omega}} |V_\phi(t,x)|^{q_k}dx \\
&\leq -y_k(t) + Cq_k^{3d}(y_{k-1}(t))^{\frac{q_k}{q_{k-1}}} + Cq_k^{3d} \\
&\leq -y_k(t) + Cq_k^{3d}\max\{1, y_{k-1}^2(t)\}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
(e^t y_k(t))' &\leq e^t Cq_k^{3d}\max\{1, \sup_{t\geq 0} y_{k-1}^2(t)\} \\
&\leq e^t C(4d)^{3d}16^{dk}\max\{1, \sup_{t\geq 0} y_{k-1}^2(t)\}.
\end{aligned}
$$

So,

$$
\begin{aligned}
e^t y_k(t) &\leq y_k(0) + C(4d)^{3d}16^{dk}\max\{1, \sup_{t\geq 0} y_{k-1}^2(t)\}\int_0^t e^s ds \\
&= y_k(0) + C(4d)^{3d}16^{dk}\max\{1, \sup_{t\geq 0} y_{k-1}^2(t)\}(e^t - 1).
\end{aligned}
$$

We know that

$$
\begin{aligned}
y_k(0) &= \|\mathfrak{D}(0)\phi\|_{L^{q_k}(\tilde{\Omega})}^{q_k} \\
&\leq K_0^{q_k} = (K_0^{\frac{q_k}{2^k}})^{2^k} := K^{2^k},
\end{aligned}
$$

where $K_0 := \max\{1, \|\mathfrak{D}(0)\phi\|_{L^1(\tilde{\Omega})}, \|\mathfrak{D}(0)\phi\|_{L^\infty(\tilde{\Omega})}\}$.

Now we define $a_k := C(4d)^{3d}16^{dk}$. Since $e^{-t} \leq 1$ and $1 - e^{-t} \leq 1$ for $t \geq 0$ we deduce that

$$
\begin{aligned}
y_k(t) &\leq e^{-t}y_k(0) + a_k\max\{1, \sup_{t\geq 0} y_{k-1}^2(t)\}e^{-t}(e^t - 1) \\
&= e^{-t}y_k(0) + a_k\max\{1, \sup_{t\geq 0} y_{k-1}^2(t)\}(1 - e^{-t}) \\
&\leq y_k(0) + a_k\max\{1, \sup_{t\geq 0} y_{k-1}^2(t)\} \\
&\leq a_k(K^{2^k} + \sup_{t\geq 0}(y_{k-1}^2(t)) \\
&\leq 2a_k\max\{K^{2^k}, \sup_{t\geq 0}(y_{k-1}^2(t))\}.
\end{aligned}
$$

Therefore, we obtain that

$$\begin{aligned}
y_k(t) &\le 2a_k \max\{K^{2^k}, \sup_{t\ge 0}(y_{k-1}^2(t))\}\\
&\le \prod_{j=0}^{k-1}(2a_{k-j})^{2^j} \max\{K^{2^k}, \sup_{t\ge 0}(y_0^{2^k}(t))\}\\
&= \prod_{j=0}^{k-1}\Big(2C(4d)^{3d}16^{d(k-j)}\Big)^{2^j} \max\{K^{2^k}, \sup_{t\ge 0}(y_0^{2^k}(t))\}\\
&= \prod_{j=0}^{k-1}\Big(2C(4d)^{3d}16^{dk}\Big)^{2^j}\prod_{j=0}^{k-1}\Big(16^{-dj}\Big)^{2^j} \max\{K^{2^k}, \sup_{t\ge 0}(y_0^{2^k}(t))\}.
\end{aligned}$$

Since

$$\begin{aligned}
\prod_{j=0}^{k-1}\Big(2C(4d)^{3d}16^{dk}\Big)^{2^j} &= \Big(2C(4d)^{3d}16^{dk}\Big)^{\sum_{j=0}^{k-1}2^j}\\
&= \Big(2C(4d)^{3d}16^{dk}\Big)^{\frac{1-2^k}{1-2}}\\
&= \Big(2C(4d)^{3d}16^{dk}\Big)^{2^k-1}
\end{aligned}$$

and

$$\prod_{j=0}^{k-1}\Big(16^{-dj}\Big)^{2^j} = \prod_{j=0}^{k-1}4^{-2dj2^j} = 4^{\sum_{j=0}^{k-1}-2dj2^j} = 4^{-2d\sum_{j=0}^{k-1}j2^j} = 4^{-2d\sum_{j=1}^{k-1}j2^j} = (16^d)^{2^{k+1}-k2^k-2},$$

by $\sum_{j=1}^{k} j2^j = (k-1)2^{k+1}+2$, we deduce that

$$\begin{aligned}
y_k(t) &\le \Big(2C(4d)^{3d}16^{dk}\Big)^{2^k-1}(16^d)^{2^{k+1}-k2^k-2}\max\{K^{2^k}, \sup_{t\ge 0}(y_0^{2^k}(t))\}\\
&= \Big(2C(4d)^{3d}\Big)^{2^k-1}\Big(16^d\Big)^{2^{k+1}-k-2}\max\{K^{2^k}, \sup_{t\ge 0}(y_0^{2^k}(t))\}.
\end{aligned}$$

So,

$$\|V_\phi(t,\cdot)\|_{L^{q_k}(\tilde\Omega)} \le \Big(2C(4d)^{3d}\Big)^{\frac{2^k-1}{q_k}}\Big(16^d\Big)^{\frac{2^{k+1}-k-2}{q_k}}\max\{K^{\frac{2^k}{q_k}}, \sup_{t\ge 0}(y_0^{\frac{2^k}{q_k}}(t))\}.$$

Since $q_k = 2^k+4d+4$ for $k\in\mathbb{N}$ and $k\ge 6$, it follows that $\frac{2^k-1}{q_k}\le 1$, $\frac{2^{k+1}-k-2}{q_k}\le 2$, $\frac{2^k}{q_k}\le 1$ therefore

$$\|V_\phi(t,\cdot)\|_{L^{q_k}(\tilde\Omega)} \le 2C(4d)^{3d}16^{2d}\max\{K, \sup_{t\ge 0}(y_0(t))\}.$$

Therefore, we get that

$$\|V_\phi(t,\cdot)\|_{L^\infty(0,T;L^\infty(\tilde{\Omega}))} \leq 2C(4d)^{3d}16^{2d}\max\{K, \sup_{t\geq 0}(y_0(t))\} =: C_\infty \tag{2.41}$$

where C_∞ is independent of η.

Recall that $u^\sigma_\eta = \psi(\bar{\mathfrak{u}})$ and $\mathfrak{D} = |\nabla\bar{\mathfrak{u}}|^2$. So, we proved that

$$\begin{aligned}\sup_{0<t<T}\|\nabla u^\sigma_\eta(t,\cdot)\|_{L^\infty(\tilde{\Omega}_k)} &= \sup_{0<t<T}\|\psi'(\bar{\mathfrak{u}})\nabla\bar{\mathfrak{u}}\|_{L^\infty(\tilde{\Omega}_k)} \\ &\leq \sup_{0<t<T}\|\psi'(\bar{\mathfrak{u}})\|_{L^\infty(\tilde{\Omega}_k)}\left(\sup_{0<t<T}\|\mathfrak{D}(t,\cdot)\|_{L^\infty(\tilde{\Omega}_k)}\right)^{\frac{1}{2}} \\ &\leq 4e\tilde{\omega}(C_\infty)^{\frac{1}{2}}.\end{aligned} \tag{2.42}$$

By repeating a similar argument in $\tilde{\Omega}_k$ for $k \in \{1,\ldots,k_0\}$ we obtain the bound for $\sup_{0<t<T}\|\nabla u^\sigma_\eta(t,\cdot)\|_{L^\infty(\mathbb{R}^d)}$.

□

2.1.5 Global Existence of the Solution to (2.1)

Proposition 2.10. *For any $T > 0$ (2.1) possesses a weak solution u_η^σ on $[0, T]$.*

Define

$$T_{\max} := \sup\{T_2 \in (0, T) \mid (2.1) \text{ possesses solution } u \in L^\infty(0, T_2; H^s(\mathbb{R}^d))\}.$$

In order to obtain $T_{\max} = T$ we only need to prove that

$$\|u_\eta^\sigma\|_{L^\infty(0,T_2;H^s(\mathbb{R}^d))} \le C, \ \forall T_2 < T_{\max},$$

where C appeared in this subsection is a positive constant which depends on σ, d, m, s, $\|u_0\|_{L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)}$, $\|\nabla u_0^\sigma\|_{L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)}$ and η.

We take the D^α derivative of $\partial_t u_\eta^\sigma = \nabla \cdot \big(\sigma \nabla u_\eta^\sigma - u_\eta^\sigma \nabla \Phi * u_\eta^\sigma + \nabla (u_\eta^\sigma + \eta)^m\big)$, multiply the equation by $D^\alpha u_\eta^\sigma$ and integrate it over $\mathbb{R}^d$ in order to obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |D^\alpha u_\eta^\sigma|^2 dx + \sigma \int_{\mathbb{R}^d} |\nabla D^\alpha u_\eta^\sigma|^2 dx = \int_{\mathbb{R}^d} D^\alpha (u_\eta^\sigma \nabla \Phi * u_\eta^\sigma) \cdot \nabla D^\alpha u_\eta^\sigma \, dx \\ - \int_{\mathbb{R}^d} D^\alpha \nabla ((u_\eta^\sigma + \eta)^m) \cdot \nabla D^\alpha u_\eta^\sigma \, dx.\end{aligned}$$

Let us start with the term $\int_{\mathbb{R}^d} D^\alpha (u_\eta^\sigma \nabla \Phi * u_\eta^\sigma) \cdot \nabla D^\alpha u_\eta^\sigma \, dx$ which can be estimated as follows

$$\begin{aligned}&\int_{\mathbb{R}^d} D^\alpha (u_\eta^\sigma \nabla \Phi * u_\eta^\sigma) \cdot \nabla D^\alpha u_\eta^\sigma \, dx \\ &\le \|\nabla D^\alpha u_\eta^\sigma\|_{L^2(\mathbb{R}^d)} \|D^\alpha (u_\eta^\sigma \nabla \Phi * u_\eta^\sigma)\|_{L^2(\mathbb{R}^d)} \\ &\le \|\nabla D^\alpha u_\eta^\sigma\|_{L^2(\mathbb{R}^d)} \Big(\|u_\eta^\sigma\|_{L^\infty(\mathbb{R}^d)} \|D^\alpha (\nabla \Phi * u_\eta^\sigma)\|_{L^2(\mathbb{R}^d)} + \|D^\alpha u_\eta^\sigma\|_{L^2(\mathbb{R}^d)} \|\nabla \Phi * u_\eta^\sigma\|_{L^\infty(\mathbb{R}^d)}\Big)\end{aligned}$$

Since we know that $\|u_\eta^\sigma\|_{L^q(\mathbb{R}^d)}$ is bounded for any $q > d$ and so

$$\|\nabla \Phi * u_\eta^\sigma\|_{L^\infty(\mathbb{R}^d)} \le C \|\nabla \Phi * u_\eta^\sigma\|_{W^{1,q}(\mathbb{R}^d)} \le C \|u_\eta^\sigma\|_{L^q(\mathbb{R}^d)} \le C,$$

we obtain that

$$\begin{aligned}&\int_{\mathbb{R}^d} D^\alpha (u_\eta^\sigma \nabla \Phi * u_\eta^\sigma) \cdot \nabla D^\alpha u_\eta^\sigma \, dx \\ &\le C \|\nabla D^\alpha u_\eta^\sigma\|_{L^2(\mathbb{R}^d)} \|D^\alpha (\nabla \Phi * u_\eta^\sigma)\|_{L^2(\mathbb{R}^d)} + C \|\nabla D^\alpha u_\eta^\sigma\|_{L^2(\mathbb{R}^d)} \|D^\alpha u_\eta^\sigma\|_{L^2(\mathbb{R}^d)} \\ &\le \frac{\sigma}{4} \int_{\mathbb{R}^d} |\nabla D^\alpha u_\eta^\sigma|^2 dx + C \int_{\mathbb{R}^d} |D^\alpha (\nabla \Phi * u_\eta^\sigma)|^2 dx + C \int_{\mathbb{R}^d} |D^\alpha u_\eta^\sigma|^2 dx \\ &\le \frac{\sigma}{4} \int_{\mathbb{R}^d} |\nabla D^\alpha u_\eta^\sigma|^2 dx + C \int_{\mathbb{R}^d} |D^\alpha u_\eta^\sigma|^2 dx\end{aligned}$$

Now we consider the term $-\int_{\mathbb{R}^d} D^\alpha \nabla ((u_\eta^\sigma + \eta)^m) \cdot \nabla D^\alpha u_\eta^\sigma \, dx$.

$$
\begin{aligned}
&- \int_{\mathbb{R}^d} D^\alpha \nabla((u_\eta^\sigma+\eta)^m)\cdot \nabla D^\alpha u_\eta^\sigma \, dx \\
&= -m \int_{\mathbb{R}^d} D^\alpha \left((u_\eta^\sigma+\eta)^{m-1}\nabla u_\eta^\sigma\right)\cdot \nabla D^\alpha u_\eta^\sigma \, dx \\
&= -m \int_{\mathbb{R}^d} (u_\eta^\sigma+\eta)^{m-1} |\nabla D^\alpha u_\eta^\sigma|^2 dx \\
&\quad - m \int_{\mathbb{R}^d} \left(D^\alpha \left((u_\eta^\sigma+\eta)^{m-1}\nabla u_\eta^\sigma\right) - (u_\eta^\sigma+\eta)^{m-1}\nabla D^\alpha u_\eta^\sigma\right)\cdot \nabla D^\alpha u_\eta^\sigma \, dx \\
&\le -m \int_{\mathbb{R}^d} (u_\eta^\sigma+\eta)^{m-1} |\nabla D^\alpha u_\eta^\sigma|^2 dx \\
&\quad + \frac{\sigma}{4}\int_{\mathbb{R}^d} |\nabla D^\alpha u_\eta^\sigma|^2 dx + C\int_{\mathbb{R}^d} \left|D^\alpha \left((u_\eta^\sigma+\eta)^{m-1}\nabla u_\eta^\sigma\right) - (u_\eta^\sigma+\eta)^{m-1}\nabla D^\alpha u_\eta^\sigma\right|^2 dx \\
&\le -m \int_{\mathbb{R}^d} (u_\eta^\sigma+\eta)^{m-1} |\nabla D^\alpha u_\eta^\sigma|^2 dx + \frac{\sigma}{4}\int_{\mathbb{R}^d} |\nabla D^\alpha u_\eta^\sigma|^2 dx \\
&\quad + C\|\nabla(u_\eta^\sigma+\eta)^{m-1}\|_{L^\infty(\mathbb{R}^d)}^2 \|D^\alpha u_\eta^\sigma\|_{L^2(\mathbb{R}^d)}^2 + C(\sigma,m)\|\nabla u_\eta^\sigma\|_{L^\infty(\mathbb{R}^d)}^2 \|D^\alpha (u_\eta^\sigma+\eta)^{m-1}\|_{L^2(\mathbb{R}^d)}^2 .
\end{aligned}
$$

So,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |D^\alpha u_\eta^\sigma|^2 dx + \frac{\sigma}{2}\int_{\mathbb{R}^d} |\nabla D^\alpha u_\eta^\sigma|^2 dx + m\int_{\mathbb{R}^d} (u_\eta^\sigma+\eta)^{m-1}|\nabla D^\alpha u_\eta^\sigma|^2 dx \\
&\le C\int_{\mathbb{R}^d} |D^\alpha u_\eta^\sigma|^2 dx \\
&+ C\|\nabla(u_\eta^\sigma+\eta)^{m-1}\|_{L^\infty(\mathbb{R}^d)}^2 \|D^\alpha u_\eta^\sigma\|_{L^2(\mathbb{R}^d)}^2 + C\|\nabla u_\eta^\sigma\|_{L^\infty(\mathbb{R}^d)}^2 \|D^\alpha (u_\eta^\sigma+\eta)^{m-1}\|_{L^2(\mathbb{R}^d)}^2 .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_\eta^\sigma\|_{H^s(\mathbb{R}^d)}^2 + \frac{\sigma}{2}\sum_{|\alpha|=0}^{s}\int_{\mathbb{R}^d} |\nabla D^\alpha u_\eta^\sigma|^2 dx + m\sum_{|\alpha|=0}^{s}\int_{\mathbb{R}^d} (u_\eta^\sigma+\eta)^{m-1}|\nabla D^\alpha u_\eta^\sigma|^2 dx \\
&\le C\|u_\eta^\sigma\|_{H^s(\mathbb{R}^d)}^2 + C\|u_\eta^\sigma+\eta\|_{L^\infty(\mathbb{R}^d)}^{2(m-2)} \|\nabla u_\eta^\sigma\|_{L^\infty(\mathbb{R}^d)}^2 \|u_\eta^\sigma\|_{H^s(\mathbb{R}^d)}^2 \\
&+ C\sum_{|\alpha|=1}^{s} \|D^\alpha (u_\eta^\sigma+\eta)^{m-1}\|_{L^2(\mathbb{R}^d)}^2 .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\|D^\alpha (u_\eta^\sigma+\eta)^{m-1}\|_{L^2(\mathbb{R}^d)}^2 \\
&\le \|(u_\eta^\sigma+\eta)^{m-2} D^\alpha (u_\eta^\sigma+\eta)\|_{L^2(\mathbb{R}^d)}^2 \\
&\quad + \|D^\alpha \left((u_\eta^\sigma+\eta)^{m-2}(u_\eta^\sigma+\eta)\right) - (u_\eta^\sigma+\eta)^{m-2} D^\alpha (u_\eta^\sigma+\eta)\|_{L^2(\mathbb{R}^d)}^2 \\
&\le C\|D^\alpha (u_\eta^\sigma+\eta)\|_{L^2(\mathbb{R}^d)}^2 \\
&\quad + C\left(\|\nabla(u_\eta^\sigma+\eta)^{m-2}\|_{L^\infty(\mathbb{R}^d)} \|D^{\alpha-1}(u_\eta^\sigma+\eta)\|_{L^2(\mathbb{R}^d)} + \|u_\eta^\sigma+\eta\|_{L^\infty(\mathbb{R}^d)} \|D^\alpha (u_\eta^\sigma+\eta)^{m-2}\|_{L^2(\mathbb{R}^d)}\right)^2 .
\end{aligned}
$$

Since

$$\|\nabla(u_\eta^\sigma+\eta)^{m-2}\|_{L^\infty(\mathbb{R}^d)} \le (m-2)\|(u_\eta^\sigma+\eta)^{m-3}\|_{L^\infty(\mathbb{R}^d)}\|\nabla u_\eta^\sigma\|_{L^\infty(\mathbb{R}^d)} \le C,$$

we obtain that

$$\begin{aligned}&\|D^\alpha(u_\eta^\sigma+\eta)^{m-1}\|^2_{L^2(\mathbb{R}^d)}\\ &\le C\|D^\alpha(u_\eta^\sigma+\eta)\|^2_{L^2(\mathbb{R}^d)} + C\|D^{\alpha-1}(u_\eta^\sigma+\eta)\|^2_{L^2(\mathbb{R}^d)} + C\|D^\alpha(u_\eta^\sigma+\eta)^{m-2}\|^2_{L^2(\mathbb{R}^d)}\end{aligned}$$

This implies that

$$\begin{aligned}&\frac{d}{dt}\|u_\eta^\sigma\|^2_{H^s(\mathbb{R}^d)} + \sigma\sum_{|\alpha|=0}^{s}\int_{\mathbb{R}^d}|\nabla D^\alpha u_\eta^\sigma|^2dx + 2m\sum_{|\alpha|=0}^{s}\int_{\mathbb{R}^d}(u_\eta^\sigma+\eta)^{m-1}|\nabla D^\alpha u_\eta^\sigma|^2dx\\ &\le C\|u_\eta^\sigma\|^2_{H^s(\mathbb{R}^d)} + C\|D^\alpha(u_\eta^\sigma+\eta)^{m-[m]}\|^2_{L^2(\mathbb{R}^d)}\end{aligned}$$

Therefore, by Grönwall's inequality we deduce that

$$\sup_{t\in(0,T_2)}\|u_\eta^\sigma\|^2_{H^s(\mathbb{R}^d)} \le \left(C+\|u_0\|^2_{H^s(\mathbb{R}^d)}\right)e^{CT_{\max}} \text{ for any } T_2<T_{\max}.$$

So, we obtain that

$$\|u_\eta^\sigma\|_{L^\infty(0,T_2;H^s(\mathbb{R}^d))} \le C,\ \forall T_2<T_{\max}.$$

2.1.6 Limit of (2.1) as $\eta \to 0$

Now we consider the weak formulation of (2.1) and let $\eta \to 0$. Let us start with the following lemma.

Lemma 2.11. *Let u_η^σ be the solution of (2.1). Then the following estimates hold*

(i) $\|u_\eta^\sigma\|_{L^\infty(0,T;W^{1,\infty}(\mathbb{R}^d))} + \|u_\eta^\sigma\|_{L^\infty(0,T;L^q(\mathbb{R}^d))} \le C \ \forall q \in [1,\infty]$,

(ii) $\|\Phi * u_\eta^\sigma\|_{L^\infty(0,T;W^{2,q}(\mathbb{R}^d))} \le C \ \forall q \in (1,\infty)$,

(iii) $\|u_\eta^\sigma\|_{L^2(0,T;H^1(\mathbb{R}^d))} \le C$,

where C is a positive constant independent of η.

Proof. (i) This is a direct consequence of subsection 2.1.4

(ii) Using basic results from the theory of elliptic partial differential and (i), we obtain (ii) directly as well.

(iii) Multiplying equation (2.1) by u_η^σ and integrating over $\mathbb{R}^d$, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |u_\eta^\sigma|^2 dx + \sigma \int_{\mathbb{R}^d} |\nabla u_\eta^\sigma|^2 dx + m\int_{\mathbb{R}^d} (u_\eta^\sigma + \eta)^{m-1} |\nabla u_\eta^\sigma|^2 dx \\
&= \int_{\mathbb{R}^d} u_\eta^\sigma \nabla\Phi * u_\eta^\sigma \cdot \nabla u_\eta^\sigma dx \\
&\le \frac{\sigma}{2}\int_{\mathbb{R}^d} |\nabla u_\eta^\sigma|^2 dx + C\|u_\eta^\sigma\|^2_{L^\infty(\mathbb{R}^d)} \|\nabla\Phi * u_\eta^\sigma\|^2_{L^2(\mathbb{R}^d)}
\end{aligned}$$

Using (i) and (ii) we obtain (iii).

□

Now we are ready to prove next result.

Lemma 2.12. *There exists a subsequence of (u_η^σ) and a function u^σ such that*

(i) $u_\eta^\sigma \overset{*}{\rightharpoonup} u^\sigma$ *in* $L^q((0,T)\times\mathbb{R}^d)$ *for* $q \in (1,\infty]$,

(ii) $u_\eta^\sigma \to u^\sigma$ *in* $L^2(0,T;L^2(\mathbb{R}^d))$,

(iii) $\nabla(u_\eta^\sigma + \eta)^m \overset{*}{\rightharpoonup} \nabla(u^\sigma)^m$ *in* $L^2(0,T;L^2(\mathbb{R}^d))$,

(iv) $\Phi * u_\eta^\sigma \overset{*}{\rightharpoonup} \Phi * u^\sigma$ *in* $L^\infty(0,T;W^{2,q}(\mathbb{R}^d)) \ \forall q \in (1,\infty)$.

Proof. (i) and (iv) are direct consequences of Lemma 2.11 (i) and Lemma 2.11 (iii) respectively. Since

$$\partial_t u_\eta^\sigma = \nabla\cdot\left(\sigma\nabla u_\eta^\sigma - u_\eta^\sigma\nabla\Phi * u_\eta^\sigma + m(u_\eta^\sigma+\eta)^{m-1}\nabla u_\eta^\sigma\right)$$

and using Lemma 2.11 we obtain that

$$\|\partial_t u_\eta^\sigma\|_{L^2(0,T;H^{-1}(\mathbb{R}^d))} \leq C. \tag{2.43}$$

Estimates from Lemma 2.11 (ii) and (2.43) allow us to use Aubin-Lions Lemma and so obtain (ii). Since it is not possible to apply Aubin-Lions Lemma directly one needs to consider 2.11 not on the whole space but on the sequence of growing d-dimensional balls and then use diagonal argument.

From Lemma 2.11 (i) and Lemma 2.11 (ii) we deduce that

$$\|\nabla(u_\eta^\sigma+\eta)^m\|_{L^2(0,T;L^2(\mathbb{R}^d))} \leq m\|(u_\eta^\sigma+\eta)^{m-1}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\|\nabla u_\eta^\sigma\|_{L^2(0,T;L^2(\mathbb{R}^d))} \leq C.$$

This estimate together with (ii) we obtain that for any $\psi \in C_0^\infty((0,T)\times\mathbb{R}^d)$

$$\int_0^T\int_{\mathbb{R}^d}\nabla(u_\eta^\sigma+\eta)^m\psi\,dx\,dt = -m\int_0^T\int_{\mathbb{R}^d}(u_\eta^\sigma+\eta)^m\nabla\psi\,dx\,dt.$$

converges to

$$-m\int_0^T\int_{\mathbb{R}^d}(u^\sigma)^m\nabla\psi\,dx\,dt = m\int_0^T\int_{\mathbb{R}^d}\nabla(u^\sigma)^m\psi\,dx\,dt,$$

as $\eta\to 0$, which implies (iii). $\square$

So we proved that we can let $\eta\to 0$ in the weak formulation of (2.1). More precisely, for any $\varphi \in C_0^\infty((0,T)\times\mathbb{R}^d)$ it holds that

$$\int_0^T\int_{\mathbb{R}^d}\Big(u_\eta^\sigma\partial_t\varphi + u_\eta^\sigma\nabla\Phi * u_\eta^\sigma\cdot\nabla\varphi - \sigma\nabla u_\eta^\sigma\cdot\nabla\varphi - \nabla(u_\eta^\sigma+\eta)^m\cdot\nabla\varphi\Big)\,dx\,dt + \int_{\mathbb{R}^d}u_0(x)\varphi(0,x)dx = 0,$$
$$\int_0^T\int_{\mathbb{R}^d}\Big(\nabla\Phi * u_\eta^\sigma\cdot\nabla\varphi - u_\eta^\sigma\varphi\Big)\,dx\,dt = 0.$$

2.1.7 L^∞ Estimates of (1.15) Independent of σ

Proposition 2.13. *Let u^σ be the weak solution to (1.15). Then there exists a positive constant C which depends on d, m and $\|u_0\|_{L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)}$ such that*

$$\|u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \leq C.$$

First of all we multiply the first equation from (1.15) by $q_k(u^\sigma)^{q_k-1}$ and obtain

$$\partial_t u^\sigma q_k(u^\sigma)^{q_k-1} = \sigma\Delta u^\sigma q_k(u^\sigma)^{q_k-1} - \nabla\cdot(u^\sigma\nabla\Phi * u^\sigma)q_k(u^\sigma)^{q_k-1} + \nabla\cdot(\nabla(u^\sigma)^m)q_k(u^\sigma)^{q_k-1}$$

where $q_k = 2^k + 4d + 4$ for $k \in \mathbb{N}$. Then we integrate it with respect to $x \in \mathbb{R}^d$ and get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}(u^\sigma(t,x))^{q_k}dx = &-\sigma q_k(q_k-1)\int_{\mathbb{R}^d}(u^\sigma(t,x))^{q_k-2}|\nabla u^\sigma(t,x)|^2dx\\ &+ q_k(q_k-1)\int_{\mathbb{R}^d}(u^\sigma(t,x))^{q_k-1}\nabla\Phi * u^\sigma(t,x)\cdot\nabla u^\sigma(t,x)dx\\ &- q_k(q_k-1)\int_{\mathbb{R}^d}(u^\sigma(t,x))^{q_k-2}\nabla(u^\sigma(t,x))^m\cdot\nabla u^\sigma(t,x)dx\\ =:&\ I_1 + I_2 + I_3\end{aligned}$$

Let us now analyze the term I_2.

$$\begin{aligned}&q_k(q_k-1)\int_{\mathbb{R}^d}(u^\sigma(t,x))^{q_k-1}\nabla\Phi * u^\sigma(t,x)\cdot\nabla u^\sigma(t,x)dx\\ &\leq q_k(q_k-1)\|\nabla\Phi * u^\sigma(t,\cdot)\|_{L^\infty(\mathbb{R}^d)}\int_{\mathbb{R}^d}(u^\sigma(t,x))^{\frac{m+q_k-3}{2}}|\nabla u^\sigma(t,x)|(u^\sigma(t,x))^{\frac{q_k-m+1}{2}}dx\\ &\leq \frac{mq_k(q_k-1)}{2}\int_{\mathbb{R}^d}(u^\sigma(t,x))^{m+q_k-3}|\nabla u^\sigma(t,x)|^2dx\\ &\quad + C\frac{q_k^2(q_k-1)^2}{2mq_k(q_k-1)}\int_{\mathbb{R}^d}(u^\sigma(t,x))^{q_k-m+1}dx\\ &= \frac{mq_k(q_k-1)}{2}\int_{\mathbb{R}^d}(u^\sigma(t,x))^{m+q_k-3}|\nabla u^\sigma(t,x)|^2dx + C\frac{q_k(q_k-1)}{m}\int_{\mathbb{R}^d}(u^\sigma(t,x))^{q_k-m+1}dx\end{aligned}$$

where C appeared in this subsection is independent of q_k.

Since for $\theta_2 = \frac{\frac{1}{q_{k-1}} - \frac{1}{q_k+1-m}}{\frac{1}{q_{k-1}} - \frac{d-2}{d(m+q_k-1)}}$ it holds that

$$\begin{aligned}&C\frac{q_k(q_k-1)}{m}\int_{\mathbb{R}^d}(u^\sigma(t,x))^{q_k-m+1}dx\\ &\leq C\frac{q_k(q_k-1)}{m}S_d^{-\frac{\theta_2(q_k-m+1)}{m+q_k-1}}\|u^\sigma(t,\cdot)\|_{L^{q_{k-1}}(\mathbb{R}^d)}^{(1-\theta_2)(q_k-m+1)}\|\nabla(u^\sigma(t,\cdot))^{\frac{m+q_k-1}{2}}\|_{L^2(\mathbb{R}^d)}^{\frac{2\theta_2(q_k-m+1)}{m+q_k-1}},\end{aligned}$$

using the following version of the Young's inequality

$$ab \leq C_1a^{l_3} + (C_1l_3)^{-\frac{l_4}{l_3}}l_4^{-1}b^{l_4}$$

where

$$a = \|\nabla(u^\sigma(t,\cdot))^{\frac{m+q_k-1}{2}}\|_{L^2(\mathbb{R}^d)}^{\frac{2\theta_2(q_k-m+1)}{m+q_k-1}}, \quad b = C\frac{q_k(q_k-1)}{m}S_d^{-\frac{\theta_2(q_k-m+1)}{m+q_k-1}}\|u^\sigma(t,\cdot)\|_{L^{q_k-1}(\mathbb{R}^d)}^{(1-\theta_2)(q_k-m+1)},$$
$$C_1 = \frac{mq_k(q_k-1)}{(m+q_k-1)^2}, \quad l_3 = \frac{m+q_k-1}{\theta_2(q_k-m+1)}, \quad \frac{1}{l_4} = 1-\frac{1}{l_3} = 1-\frac{\theta_2(q_k-m+1)}{m+q_k-1}$$

we obtain that

$$\begin{aligned}
&C\frac{q_k(q_k-1)}{m}\int_{\mathbb{R}^d}(u^\sigma(t,x))^{q_k-m+1}dx\\
&\leq \frac{mq_k(q_k-1)}{(m+q_k-1)^2}\int_{\mathbb{R}^d}\left|\nabla(u^\sigma(t,x))^{\frac{m+q_k-1}{2}}\right|^2dx\\
&+\left(\frac{mq_k(q_k-1)}{\theta_2(q_k-m+1)(m+q_k-1)}\right)^{-\frac{l_4}{l_3}}l_4^{-1}\left(C\frac{q_k(q_k-1)}{m}S_d^{-\frac{\theta_2(q_k-m+1)}{m+q_k-1}}\right)^{l_4}\|u^\sigma(t,\cdot)\|_{L^{q_k-1}(\mathbb{R}^d)}^{q_k-1\left(\frac{(1-\theta_2)(q_k-m+1)l_4}{q_k-1}\right)}.
\end{aligned}$$

Since $I_1 \leq 0$ and

$$\begin{aligned}
I_3 &= -q_k(q_k-1)\int_{\mathbb{R}^d}(u^\sigma(t,x))^{q_k-2}\nabla(u^\sigma(t,x))^m\cdot\nabla u^\sigma(t,x)dx\\
&= -q_k(q_k-1)m\int_{\mathbb{R}^d}(u^\sigma(t,x))^{m+q_k-3}|\nabla u^\sigma(t,x)|^2dx
\end{aligned}$$

we deduce that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}(u^\sigma(t,x))^{q_k}dx\\
&\leq -\frac{mq_k(q_k-1)}{2}\int_{\mathbb{R}^d}(u^\sigma(t,x))^{m+q_k-3}|\nabla u^\sigma(t,x)|^2dx\\
&\quad+\frac{mq_k(q_k-1)}{(m+q_k-1)^2}\int_{\mathbb{R}^d}\left|\nabla(u^\sigma(t,x))^{\frac{m+q_k-1}{2}}\right|^2dx\\
&\quad+\left(\frac{mq_k(q_k-1)}{\theta_2(q_k-m+1)(m+q_k-1)}\right)^{-\frac{l_4}{l_3}}l_4^{-1}\left(C\frac{q_k(q_k-1)}{m}S_d^{-\frac{\theta_2(q_k-m+1)}{m+q_k-1}}\right)^{l_4}\|u^\sigma(t,\cdot)\|_{L^{q_k-1}(\mathbb{R}^d)}^{q_k-1\left(\frac{(1-\theta_2)(q_k-m+1)l_4}{q_k-1}\right)}.
\end{aligned}$$

Since $|\nabla(u^\sigma(t,x))^{\frac{m+q_k-1}{2}}|^2 = |\frac{m+q_k-1}{2}\nabla u^\sigma(t,x)(u^\sigma(t,x))^{\frac{m+q_k-3}{2}}|^2$ it implies that

$$\begin{aligned}
&-\frac{mq_k(q_k-1)}{2}\int_{\mathbb{R}^d}(u^\sigma(t,x))^{m+q_k-3}|\nabla u^\sigma(t,x)|^2dx\\
&= -\frac{mq_k(q_k-1)}{2}\frac{4}{(m+q_k-1)^2}\int_{\mathbb{R}^d}|\nabla(u^\sigma(t,x))^{\frac{m+q_k-1}{2}}|^2dx\\
&= -\frac{2mq_k(q_k-1)}{(m+q_k-1)^2}\int_{\mathbb{R}^d}|\nabla(u^\sigma(t,x))^{\frac{m+q_k-1}{2}}|^2dx
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}(u^\sigma(t,x))^{q_k}dx\\
&\le -\frac{mq_k(q_k-1)}{(m+q_k-1)^2}\int_{\mathbb{R}^d}|\nabla(u^\sigma(t,x))^{\frac{m+q_k-1}{2}}|^2dx\\
&+\left(\frac{mq_k(q_k-1)}{\theta_2(q_k-m+1)(m+q_k-1)}\right)^{-\frac{l_4}{l_3}} l_4^{-1}\left(C\frac{q_k(q_k-1)}{m}S_d^{-\frac{\theta_2(q_k-m+1)}{m+q_k-1}}\right)^{l_4}\|u^\sigma(t,\cdot)\|_{L^{q_{k-1}}(\mathbb{R}^d)}^{q_{k-1}\left(\frac{(1-\theta_2)(q_k-m+1)l_4}{q_{k-1}}\right)}
\end{aligned}
$$

Performing the same calculation as in Section 3 of [Wang et al., 2019] we know that

$$
\begin{aligned}
&l_4\le d,\\
&\frac{(1-\theta_2)(q_k-m+1)l_4}{q_{k-1}}\le 2.
\end{aligned}
$$

On the other hand we know that

$$
\int_{\mathbb{R}^d}(u^\sigma(t,x))^{q_k}dx\le S_d^{-\frac{\theta_1 q_k}{m+q_k-1}}\|u^\sigma(t,\cdot)\|_{L^{q_{k-1}}(\mathbb{R}^d)}^{(1-\theta_1)q_k}\|\nabla(u^\sigma(t,\cdot))^{\frac{m+q_k-1}{2}}\|_{L^2(\mathbb{R}^d)}^{\frac{2\theta_1 q_k}{m+q_k-1}}
$$

where $\theta_1=\frac{\frac{1}{q_{k-1}}-\frac{1}{q_k}}{\frac{1}{q_{k-1}}-\frac{d-2}{d(m+q_k-1)}}$.

Now using the following version of the Young's inequality

$$
\tilde a\tilde b\le C_1\tilde a^{l_1}+(C_1l_1)^{-\frac{l_2}{l_1}}l_2^{-1}\tilde b^{l_2}
$$

where

$$
\begin{aligned}
&\tilde a=\|\nabla(u^\sigma(t,\cdot))^{\frac{m+q_k-1}{2}}\|_{L^2(\mathbb{R}^d)}^{\frac{2\theta_1 q_k}{m+q_k-1}},\quad \tilde b=S_d^{-\frac{\theta_1 q_k}{m+q_k-1}}\|u^\sigma(t,\cdot)\|_{L^{q_{k-1}}(\mathbb{R}^d)}^{(1-\theta_1)q_k},\\
&C_1=\frac{mq_k(q_k-1)}{(m+q_k-1)^2},\quad l_1=\frac{m+q_k-1}{\theta_1 q_k},\quad l_2=\frac{m+q_k-1}{m+(1-\theta_1)q_k-1}
\end{aligned}
$$

we obtain that

$$
\begin{aligned}
&\int_{\mathbb{R}^d}(u^\sigma(t,x))^{q_k}dx\\
&\le\frac{mq_k(q_k-1)}{(m+q_k-1)^2}\int_{\mathbb{R}^d}|\nabla(u^\sigma(t,x))^{\frac{m+q_k-1}{2}}|^2dx\\
&\quad+\left(\frac{m(q_k-1)}{\theta_1(m+q_k-1)}\right)^{-\frac{l_2}{l_1}}\frac{m+(1-\theta_1)q_k-1}{m+q_k-1}\left(S_d^{-\frac{\theta_1 q_k}{m+q_k-1}}\|u^\sigma(t,\cdot)\|_{L^{q_{k-1}}(\mathbb{R}^d)}^{(1-\theta_1)q_k}\right)^{\frac{m+q_k-1}{m+(1-\theta_1)q_k-1}}.
\end{aligned}
$$

So,

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}(u^\sigma(t,x))^{q_k}dx \\
&\leq -\int_{\mathbb{R}^d}(u^\sigma(t,x))^{q_k}dx \\
&+\left(\frac{m(q_k-1)}{\theta_1(m+q_k-1)}\right)^{-\frac{l_2}{l_1}}\frac{m+(1-\theta_1)q_k-1}{m+q_k-1}S_d^{-\frac{\theta_1 q_k}{m+(1-\theta_1)q_k-1}}\|u^\sigma(t,\cdot)\|_{L^{q_{k-1}}(\mathbb{R}^d)}^{q_{k-1}\cdot\frac{(1-\theta_1)q_k(m+q_k-1)}{q_{k-1}(m+(1-\theta_1)q_k-1)}} \\
&+\left(\frac{mq_k(q_k-1)}{\theta_2(q_k-m+1)(m+q_k-1)}\right)^{-\frac{l_4}{l_3}}l_4^{-1}\left(C\frac{q_k(q_k-1)}{m}S_d^{-\frac{\theta_2(q_k-m+1)}{m+q_k-1}}\right)^{l_4}\|u^\sigma(t,\cdot)\|_{L^{q_{k-1}}(\mathbb{R}^d)}^{q_{k-1}\left(\frac{(1-\theta_2)(q_k-m+1)l_4}{q_{k-1}}\right)}.
\end{aligned}
$$

Now we need to prove that $\frac{(1-\theta_1)q_k}{q_{k-1}}\frac{m+q_k-1}{m+(1-\theta_1)q_k-1}\leq 2$. Since

$$
\begin{aligned}
&\frac{(1-\theta_1)q_k}{q_{k-1}}\frac{m+q_k-1}{m+(1-\theta_1)q_k-1}\leq 2 \\
&\Leftrightarrow (1-\theta_1)q_k(m+q_k-1)\leq 2q_{k-1}(m+(1-\theta_1)q_k-1) \\
&\Leftarrow (1-\theta_1)q_k(m+q_k-1)\leq q_k(m+(1-\theta_1)q_k-1) \\
&\Leftrightarrow (1-\theta_1)(m+q_k-1)\leq m+(1-\theta_1)q_k-1 \\
&\Leftrightarrow (1-\theta_1)(m-1)\leq m-1
\end{aligned}
$$

we obtain that $\frac{(1-\theta_1)q_k}{q_{k-1}}\frac{m+q_k-1}{m+(1-\theta_1)q_k-1}\leq 2$ holds. Therefore, it implies that there exists a constant C independent of q_k such that

$$
\frac{d}{dt}\int_{\mathbb{R}^d}(u^\sigma(t,x))^{q_k}dx\leq -\int_{\mathbb{R}^d}(u^\sigma(t,x))^{q_k}dx+Cq_k^{2d}\left(\int_{\mathbb{R}^d}(u^\sigma(t,x))^{q_{k-1}}dx\right)^2
$$

Now we are ready to do Moser's iteration as we already did before.

Let $y_k(t):=\|u^\sigma(t,\cdot)\|_{L^{q_k}(\mathbb{R}^d)}^{q_k}$. Choosing $\frac{q_k}{q_{k-1}}\leq 2$, we obtain that

$$
\begin{aligned}
y_k'(t)&=\frac{d}{dt}\int_{\mathbb{R}^d}(u^\sigma(t,x))^{q_k}dx \\
&\leq -\int_{\mathbb{R}^d}(u^\sigma(t,x))^{q_k}dx+Cq_k^{2d}\left(\int_{\mathbb{R}^d}(u^\sigma(t,x))^{q_{k-1}}dx\right)^{\frac{q_k}{q_{k-1}}} \\
&=-y_k(t)+Cq_k^{2d}y_{k-1}^{\frac{q_k}{q_{k-1}}}(t) \\
&\leq -y_k(t)+Cq_k^{2d}\max\{1,y_{k-1}^2(t)\}.
\end{aligned}
$$

This implies that

$$\begin{aligned}(e^t y_k(t))' &= e^t(y_k'(t) + y_k(t)) \\ &\le e^t C q_k^{2d} \max\{1, \sup_{t\ge 0} y_{k-1}^2(t)\} \\ &\le e^t C(4d)^{2d} 4^{dk} \max\{1, \sup_{t\ge 0} y_{k-1}^2(t)\}.\end{aligned}$$

So,

$$\begin{aligned}e^t y_k(t) &\le y_k(0) + \mathcal{C}(4d)^{2d} 4^{dk} \max\{1, \sup_{t\ge 0} y_{k-1}^2(t)\} \int_0^t e^s ds \\ &= y_k(0) + \mathcal{C}(4d)^{2d} 4^{dk} \max\{1, \sup_{t\ge 0} y_{k-1}^2(t)\}(e^t - 1).\end{aligned}$$

We know that

$$\begin{aligned}y_k(0) &= \|u_0\|_{L^{q_k}(\mathbb{R}^d)}^{q_k} \\ &\le (\max\{\|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)}\})^{q_k} \\ &\le K_0^{q_k} = (K_0^{\frac{q_k}{2^k}})^{2^k} := K^{2^k},\end{aligned}$$

where $K_0 := \max\{1, \|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)}\}$.

Now we define $a_k := C(4d)^{2d}4^{dk}$. Since $e^{-t} \le 1$ and $1 - e^{-t} \le 1$ for $t \ge 0$ we deduce that

$$\begin{aligned}y_k(t) &\le e^{-t} y_k(0) + a_k \max\{1, \sup_{t\ge 0} y_{k-1}^2(t)\} e^{-t}(e^t - 1) \\ &= e^{-t} y_k(0) + a_k \max\{1, \sup_{t\ge 0} y_{k-1}^2(t)\}(1 - e^{-t}) \\ &\le y_k(0) + a_k \max\{1, \sup_{t\ge 0} y_{k-1}^2(t)\} \\ &\le a_k (K^{2^k} + \sup_{t\ge 0}(y_{k-1}^2(t)) \\ &\le 2a_k \max\{K^{2^k}, \sup_{t\ge 0}(y_{k-1}^2(t))\}.\end{aligned}$$

Therefore, we obtain that

$$
\begin{aligned}
y_k(t) &\le 2a_k \max\{K^{2^k}, \sup_{t\ge 0}(y_{k-1}^2(t))\} \\
&\le \prod_{j=0}^{k-1} (2a_{k-j})^{2^j} \max\{K^{2^k}, \sup_{t\ge 0}(y_0^{2^k}(t))\} \\
&= \prod_{j=0}^{k-1} \left(2C(4d)^{2d}4^{d(k-j)}\right)^{2^j} \max\{K^{2^k}, \sup_{t\ge 0}(y_0^{2^k}(t))\} \\
&= \prod_{j=0}^{k-1} \left(2C(4d)^{2d}4^{dk}\right)^{2^j} \prod_{j=0}^{k-1} \left(4^{-dj}\right)^{2^j} \max\{K^{2^k}, \sup_{t\ge 0}(y_0^{2^k}(t))\}.
\end{aligned}
$$

Since

$$
\prod_{j=0}^{k-1} \left(2C(4d)^{2d}4^{dk}\right)^{2^j} = \left(2C(4d)^{2d}4^{dk}\right)^{\sum_{j=0}^{k-1} 2^j} = \left(2C(4d)^{2d}4^{dk}\right)^{\frac{1-2^k}{1-2}} = \left(2C(4d)^{2d}4^{dk}\right)^{2^k-1}
$$

and

$$
\prod_{j=0}^{k-1} \left(4^{-dj}\right)^{2^j} = \prod_{j=0}^{k-1} 2^{-2dj2^j} = 2^{\sum_{j=0}^{k-1} -2dj2^j} = 2^{-2d\sum_{j=0}^{k-1} j2^j} = 2^{-2d\sum_{j=1}^{k-1} j2^j} = 2^{-2d((k-2)2^k+2)},
$$

by $\sum_{j=1}^{k} j2^j = (k-1)2^{k+1}+2$, we deduce that

$$
\begin{aligned}
y_k(t) &\le \left(2C(4d)^{2d}4^{dk}\right)^{2^k-1} \left(2^{-2d((k-2)2^k+2)}\right) \max\{K^{2^k}, \sup_{t\ge 0}(y_0^{2^k}(t))\} \\
&= \left(2C(4d)^{2d}\right)^{2^k-1} \left(4^d\right)^{k(2^k-1)} \left(4^d\right)^{-(k-2)2^k-2} \max\{K^{2^k}, \sup_{t\ge 0}(y_0^{2^k}(t))\} \\
&= \left(2C(4d)^{2d}\right)^{2^k-1} \left(4^d\right)^{2^{k+1}-k-2} \max\{K^{2^k}, \sup_{t\ge 0}(y_0^{2^k}(t))\}.
\end{aligned}
$$

So,

$$
\|u^\sigma(t,\cdot)\|_{L^{q_k}(\mathbb{R}^d)} \le \left(2C(4d)^{2d}\right)^{\frac{2^k-1}{q_k}} \left(4^d\right)^{\frac{2^{k+1}-k-2}{q_k}} \max\{K^{\frac{2^k}{q_k}}, \sup_{t\ge 0}(y_0^{\frac{2^k}{q_k}}(t))\}.
$$

Since $q_k = 2^k + 4d + 4$ for $k \in \mathbb{N}$, it follows that $\frac{2^k-1}{q_k} \le 1$, $\frac{2^{k+1}-k-2}{q_k} \le 2$, $\frac{2^k}{q_k} \le 1$ therefore

$$
\|u^\sigma(t,\cdot)\|_{L^{q_k}(\mathbb{R}^d)} \le 2C(4d)^{2d}4^{2d} \max\{K, \sup_{t\ge 0}(y_0(t))\}.
$$

2.1.8 Regularity of the Solution to (1.15)

Proposition 2.14. *If u^σ is a weak solution to (1.15), then $u^\sigma \in W_q^{3,1}((0,T)\times\mathbb{R}^d)$ for any $1 < q < \infty$.*

Proof. So far we have proved that $u^\sigma \in L^\infty(0,T;W^{1,\infty}(\mathbb{R}^d)) \cap L^\infty(0,T;L^1(\mathbb{R}^d))$.

Using the same arguments as in subsection 2.1.5 we get that

$$u^\sigma \in L^\infty(0,T;H^1(\mathbb{R}^d)) \cap L^2(0,T;H^2(\mathbb{R}^d))$$

and applying theory of the elliptic partial differential equation we deduce that

$$\Phi * u^\sigma \in L^\infty(0,T;W^{2,q}(\mathbb{R}^d)) \text{ for all } q \in (1,\infty).$$

Recall that

$$\partial_t u^\sigma = \nabla \cdot \left(\sigma \nabla u^\sigma - u^\sigma \nabla \Phi * u^\sigma + \nabla (u^\sigma)^m\right).$$

So it implies that

$$\partial_t u^\sigma \in L^2(0,T;L^2(\mathbb{R}^d)),$$

and therefore

$$u^\sigma \in W^{1,2}(0,T;L^2(\mathbb{R}^d)).$$

Noticing that

$$\partial_t u^\sigma - (\sigma + m(u^\sigma)^{m-1})\Delta u^\sigma - m(m-1)(u^\sigma)^{m-2}\nabla u^\sigma \cdot \nabla u^\sigma + \nabla\Phi * u^\sigma \cdot \nabla u^\sigma - u^\sigma u^\sigma = 0 \tag{2.44}$$

we combine it with $u^\sigma \in L^\infty(0,T;W^{1,\infty}(\mathbb{R}^d))$ and Theorem 9.2.2 from [Wu et al., 2006] in order to obtain that

$$u^\sigma \in W_q^{2,1}((0,T)\times\mathbb{R}^d) \text{ for any } q \in (1,\infty).$$

Let $v^\sigma := \partial_{x_j} u^\sigma$, then it holds that

$$\begin{aligned}
&\partial_t v^\sigma - (\sigma + m(u^\sigma)^{m-1})\Delta v^\sigma - m(m-1)(u^\sigma)^{m-2}\Delta u^\sigma \partial_{x_j} u^\sigma \\
&- m(m-1)(m-2)(u^\sigma)^{m-3}|\nabla u^\sigma|^2 \partial_{x_j} u^\sigma - 2m(m-1)(u^\sigma)^{m-2}\nabla u^\sigma \cdot \nabla \partial_{x_j} u^\sigma \\
&+ \nabla\Phi * u^\sigma \cdot \nabla \partial_{x_j} u^\sigma + \nabla u^\sigma \cdot \nabla\Phi * \partial_{x_j} u^\sigma - 2u^\sigma \partial_{x_j} u^\sigma = 0.
\end{aligned}$$

By means of Theorem 9.2.2 from [Wu et al., 2006] we deduce that

$$\partial_{x_j} u^\sigma \in W_q^{2,1}((0,T) \times \mathbb{R}^d) \text{ for any } q \in (1, \infty).$$

Therefore,

$$\nabla u^\sigma \in W_q^{2,1}((0,T) \times \mathbb{R}^d) \text{ for any } q \in (1, \infty). \tag{2.45}$$

Hence,

$$u^\sigma \in W_q^{3,1}((0,T) \times \mathbb{R}^d) \text{ for any } 1 < q < \infty.$$

□

2.2 Partial Differential Equation from the Intermediate Particle Model

In this section we present the proof of Theorem 1.2. Under the assumptions of Theorem 1.2, we know that $u^\sigma \in L^\infty(0,T;H^{s+2}(\mathbb{R}^d))$.

Take $s > \frac{d}{2} + 2$ and define

$$Y := \{\mathbf{v} \in L^\infty(0,T;H^s(\mathbb{R}^d)) \cap L^2(0,T;H^{s+1}(\mathbb{R}^d)) \Big| \\ \|\mathbf{v}\|^2_{L^\infty(0,T;H^s(\mathbb{R}^d))} + \|\nabla \mathbf{v}\|^2_{L^2(0,T;H^s(\mathbb{R}^d))} \le \varepsilon_k + \varepsilon_p \text{ and } \mathbf{v}(0,x) = 0\}$$

For any $\mathbf{v} \in Y$ we consider the following linear partial differential equation

$$\begin{aligned} \partial_t \mathbf{u} - \sigma \Delta \mathbf{u} = \nabla \cdot \Big(&(\mathbf{u} + u^\sigma)\nabla(p_\lambda - p)(V^{\varepsilon_p} * (\mathbf{v} + u^\sigma)) + \mathbf{u}\nabla p(V^{\varepsilon_p} * (\mathbf{v} + u^\sigma)) \\ &+ u^\sigma \nabla \int_0^1 p'(V^{\varepsilon_p} * u^\sigma + sV^{\varepsilon_p} * \mathbf{v})V^{\varepsilon_p} * \mathbf{u}\, ds + u^\sigma\Big(\nabla p(V^{\varepsilon_p} * u^\sigma) - \nabla p(u^\sigma)\Big)\Big) \\ &- \nabla \cdot \Big(\mathbf{u}\nabla\Phi * V^{\varepsilon_k} * \mathbf{v} + \mathbf{u}\nabla\Phi * V^{\varepsilon_k} * u^\sigma + u^\sigma\nabla\Phi * V^{\varepsilon_k} * (\mathbf{u} + u^\sigma) - u^\sigma \nabla\Phi * u^\sigma\Big), \\ \mathbf{u}(0,x) = 0.& \end{aligned} \tag{2.46}$$

The linear problem (2.46) possesses a unique global solution $\mathbf{u} \in L^\infty(0,T;H^s(\mathbb{R}^d)) \cap L^2(0,T;H^{s+1}(\mathbb{R}^d))$.

Let $\alpha \in \mathbb{N}^d$ be a multi-index of order $|\alpha| \le s$. We apply D^α to (2.46), multiply the resulting equation bu $D^\alpha \mathbf{u}$ and integrate over $\mathbb{R}^d$ and so obtain that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |D^\alpha \mathbf{u}|^2 dx + \sigma \int_{\mathbb{R}^d} |\nabla D^\alpha \mathbf{u}|^2 dx \\ &= -\int_{\mathbb{R}^d} D^\alpha\Big((\mathbf{u} + u^\sigma)\nabla(p_\lambda - p)(V^{\varepsilon_p} * (\mathbf{v} + u^\sigma))\Big) \cdot \nabla D^\alpha \mathbf{u}\, dx \\ &- \int_{\mathbb{R}^d} D^\alpha\Big(\mathbf{u}\nabla p(V^{\varepsilon_p} * (\mathbf{v} + u^\sigma))\Big) \cdot \nabla D^\alpha \mathbf{u}\, dx \\ &- \int_{\mathbb{R}^d} D^\alpha\Big(u^\sigma \nabla \int_0^1 p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v})V^{\varepsilon_p} * \mathbf{u}\, dz\Big) \cdot \nabla D^\alpha \mathbf{u}\, dx \\ &- \int_{\mathbb{R}^d} D^\alpha\Big(u^\sigma(\nabla p(V^{\varepsilon_p} * u^\sigma) - \nabla p(u^\sigma))\Big) \cdot \nabla D^\alpha \mathbf{u}\, dx \\ &+ \int_{\mathbb{R}^d} D^\alpha\Big(\mathbf{u}\nabla\Phi * V^{\varepsilon_k} * \mathbf{v}\Big) \cdot \nabla D^\alpha \mathbf{u}\, dx \\ &+ \int_{\mathbb{R}^d} D^\alpha\Big(\mathbf{u}\nabla\Phi * V^{\varepsilon_k} * u^\sigma\Big) \cdot \nabla D^\alpha \mathbf{u}\, dx \\ &+ \int_{\mathbb{R}^d} D^\alpha\Big(u^\sigma\nabla\Phi * V^{\varepsilon_k} * (\mathbf{u} + u^\sigma) - u^\sigma\nabla\Phi * u^\sigma)\Big) \cdot \nabla D^\alpha \mathbf{u}\, dx \\ &=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7. \end{aligned}$$

Since $\|\mathbf{v}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}$ and $\|u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}$ are bounded, by definition of p_λ we can take λ

small enough such that

$$\begin{aligned}
\|\mathbf{v}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} &\leq \frac{1}{2\lambda},\\
\|u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} &\leq \frac{1}{2\lambda}.
\end{aligned}$$

Therefore, it follows that $I_1 = 0$.

Next, we estimate I_2.

$$\begin{aligned}
I_2 &\leq \frac{\sigma}{12}\|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)} + C\|D^\alpha\Big(\mathbf{u}\nabla p(V^{\varepsilon_p} * (\mathbf{v}+u^\sigma))\Big)\|^2_{L^2(\mathbb{R}^d)}\\
&\leq \frac{\sigma}{12}\|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)} + C\|\mathbf{u}\|^2_{L^\infty(\mathbb{R}^d)}\|D^\alpha \nabla p(V^{\varepsilon_p} * (\mathbf{v}+u^\sigma))\|^2_{L^2(\mathbb{R}^d)}\\
&+ C\|D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)}\|\nabla p(V^{\varepsilon_p} * (\mathbf{v}+u^\sigma))\|^2_{L^\infty(\mathbb{R}^d)}\\
&\leq \frac{\sigma}{12}\|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)} + C\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)}\|D^\alpha \nabla p(V^{\varepsilon_p} * (\mathbf{v}+u^\sigma))\|^2_{L^2(\mathbb{R}^d)}\\
&+ C\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)}\|p(V^{\varepsilon_p} * (\mathbf{v}+u^\sigma))\|^2_{H^s(\mathbb{R}^d)}\\
&\leq \frac{\sigma}{12}\|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)} + C\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)} + C\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)}\|\nabla \mathbf{v}\|^2_{H^s(\mathbb{R}^d)}
\end{aligned}$$

where C in this section denotes a positive constant which depends on σ and s.

$$\begin{aligned}
I_3 &= -\int_{\mathbb{R}^d} D^\alpha\Big(u^\sigma \nabla \int_0^1 p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v})V^{\varepsilon_p} * \mathbf{u}\,dz\Big)\cdot \nabla D^\alpha \mathbf{u}\,dx\\
&= -\int_{\mathbb{R}^d}\int_0^1 D^\alpha\Big(u^\sigma \nabla\left(p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v})V^{\varepsilon_p} * \mathbf{u}\right)\Big)\cdot \nabla D^\alpha \mathbf{u}\,dz\,dx\\
&= -\int_{\mathbb{R}^d}\int_0^1 u^\sigma \nabla D^\alpha\left(p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v})V^{\varepsilon_p} * \mathbf{u}\right)\cdot \nabla D^\alpha \mathbf{u}\,dz\,dx\\
&- \int_{\mathbb{R}^d}\int_0^1 \Big(D^\alpha\Big(u^\sigma \nabla(p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v})V^{\varepsilon_p} * \mathbf{u})\Big)\\
&- u^\sigma \nabla D^\alpha\Big(p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v})V^{\varepsilon_p} * \mathbf{u}\Big)\Big)\cdot \nabla D^\alpha \mathbf{u}\,dz\,dx\\
&= A + B
\end{aligned}$$

Therefore,

$$\begin{aligned}
B &\leq \frac{\sigma}{36}\|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)}\\
&+ C\sup_{0<z<1}\|D^\alpha\Big(u^\sigma \nabla(p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v})V^{\varepsilon_p} * \mathbf{u})\Big) - u^\sigma D^\alpha \nabla\Big(p'(V^{\varepsilon_p} * u^\sigma + V^{\varepsilon_p} * \mathbf{v})\Big)\|^2_{L^2(\mathbb{R}^d)}\\
&\leq \frac{\sigma}{36}\|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)} + C\|\nabla u^\sigma\|^2_{L^\infty(\mathbb{R}^d)}\sup_{0<z<1}\|D^{\alpha-1}\nabla\Big(p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v})V^{\varepsilon_p} * \mathbf{u}\Big)\|^2_{L^2(\mathbb{R}^d)}\\
&+ C\|D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)}\sup_{0<z<1}\|\nabla\Big(p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v})V^{\varepsilon_p} * \mathbf{u}\Big)\|^2_{L^\infty(\mathbb{R}^d)}\\
&\leq \frac{\sigma}{36}\|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)} + C\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)}
\end{aligned}$$

$$
\begin{aligned}
A = &-\int_{\mathbb{R}^d}\int_0^1 u^\sigma p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v})V^{\varepsilon_p} * (\nabla D^\alpha \mathbf{u}) \cdot \nabla D^\alpha \mathbf{u}\, dz\, dx \\
&- \int_{\mathbb{R}^d}\int_0^1 u^\sigma \Big(\nabla D^\alpha \Big(p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v})V^{\varepsilon_p} * \mathbf{u}\Big) \\
&- p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v})\nabla D^\alpha V^{\varepsilon_p} * \mathbf{u}\Big) \cdot \nabla D^\alpha \mathbf{u}\, dz\, dx \\
&=: G + D
\end{aligned}
$$

So, it follows that

$$
\begin{aligned}
D \le& \frac{\sigma}{36}\|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)} \\
&+ C\sup_{0<z<1}\|u^\sigma\Big(\nabla D^\alpha\Big(p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v})V^{\varepsilon_p} * \mathbf{u}\Big) - p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v})\nabla D^\alpha V^{\varepsilon_p} * \mathbf{u}\Big)\|^2_{L^2(\mathbb{R}^d)} \\
\le& \frac{\sigma}{36}\|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)} \\
&+ C\|\nabla\Big(p'(V^{\varepsilon_p} * u^\sigma + V^{\varepsilon_p} * \mathbf{v})\Big)\|^2_{L^\infty(\mathbb{R}^d)}\|D^\alpha V^{\varepsilon_p} * \mathbf{u}\|^2_{L^2(\mathbb{R}^d)} \\
&+ C\|\nabla D^\alpha\Big(p'(V^{\varepsilon_p} * u^\sigma + V^{\varepsilon_p} * \mathbf{v})\Big)\|^2_{L^2(\mathbb{R}^d)}\|V^{\varepsilon_p} * \mathbf{u}\|^2_{L^\infty(\mathbb{R}^d)} \\
\le& \frac{\sigma}{36}\|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)} + C\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)} + C\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)}\|\nabla \mathbf{v}\|^2_{H^s(\mathbb{R}^d)}
\end{aligned}
$$

and

$$
\begin{aligned}
G =& -\int_{\mathbb{R}^d}\int_0^1 W^{\varepsilon_p} * \Big(u^\sigma p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v})\nabla D^\alpha \mathbf{u}\Big) \cdot W^{\varepsilon_p} * \nabla D^\alpha \mathbf{u}\, dz\, dx \\
=& -\int_{\mathbb{R}^d}\int_0^1 u^\sigma p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v})|W^{\varepsilon_p} * \nabla D^\alpha \mathbf{u}|^2\, dz\, dx \\
&- \int_{\mathbb{R}^d}\int_0^1 \Big(W^{\varepsilon_p} * \Big(u^\sigma p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v})\nabla D^\alpha \mathbf{u}\Big) \\
&\qquad - u^\sigma W^{\varepsilon_p} * \Big(p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v})\nabla D^\alpha \mathbf{u}\Big)\Big) \cdot \nabla D^\alpha \mathbf{u}\, dz\, dx \\
&- \int_{\mathbb{R}^d}\int_0^1 u^\sigma\Big(W^{\varepsilon_p} * \Big(p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v})\nabla D^\alpha \mathbf{u}\Big) \\
&\qquad - p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v})W^{\varepsilon_p} * \nabla D^\alpha \mathbf{u}\Big) \cdot \nabla D^\alpha \mathbf{u}\, dz\, dx \\
\le& \, 0 + E + F
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E \le& \sup_{0<z<1}\Big\|W^{\varepsilon_p} * \Big(u^\sigma p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v})\nabla D^\alpha \mathbf{u}\Big) \\
&\qquad - u^\sigma W^{\varepsilon_p} * \Big(p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v})\nabla D^\alpha \mathbf{u}\Big)\Big\|_{L^2(\mathbb{R}^d)}\Big\|\nabla D^\alpha \mathbf{u}\Big\|_{L^2(\mathbb{R}^d)} \\
\le& \, C\varepsilon_p\|\nabla u^\sigma\|_{L^\infty(\mathbb{R}^d)}\|\nabla D^\alpha \mathbf{u}\|_{L^2(\mathbb{R}^d)}\sup_{0<z<1}\|p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v})\nabla D^\alpha \mathbf{u}\|_{L^2(\mathbb{R}^d)} \\
\le& \, C\varepsilon_p\sup_{0<z<1}\|p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v})\|_{L^\infty(\mathbb{R}^d)}\|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)} \\
\le& \, C\varepsilon_p\|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}
$$

and

$$
\begin{aligned}
F \le \sup_{0<z<1} \Big\| & W^{\varepsilon_p} * \Big(p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v})\nabla D^\alpha \mathbf{u}\Big) \\
& - p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v}) W^{\varepsilon_p} * \nabla D^\alpha \mathbf{u}\Big\|_{L^2(\mathbb{R}^d)} \Big\| u^\sigma \Big\|_{L^\infty(\mathbb{R}^d)} \Big\| \nabla D^\alpha \mathbf{u}\Big\|_{L^2(\mathbb{R}^d)} \\
\le C\varepsilon_p & \sup_{0<z<1} \|p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v})\|_{L^\infty(\mathbb{R}^d)} \|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)} \\
\le C\varepsilon_p & \|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}
$$

where we used the following lemma.

Lemma 2.15. *Let $\varepsilon > 0$, $q \in [1, \infty)$, $f \in L^q(\mathbb{R}^d)$ and $g \in W^{1,\infty}(\mathbb{R}^d)$. Then the following inequality holds*

$$
\|V^\varepsilon * (fg) - (V^\varepsilon * f)g\|^q_{L^q(\mathbb{R}^d)} \le C\varepsilon^q \|\nabla g\|^q_{L^\infty(\mathbb{R}^d)} \|f\|^q_{L^q(\mathbb{R}^d)}
$$

Proof.

$$
\begin{aligned}
& \|V^\varepsilon * (fg) - (V^\varepsilon * f)g\|^q_{L^q(\mathbb{R}^d)} \\
& = \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} V^\varepsilon(x-y)\Big(f(y)g(y) - f(y)g(x)\Big)\,dy\Big|^q\,dx \\
& = \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} (V^\varepsilon(x-y))^{\frac{q-1}{q}} (V^\varepsilon(x-y))^{\frac{1}{q}} \Big(g(y) - g(x)\Big) f(y)\,dy\Big|^q\,dx \\
& \le \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} V^\varepsilon(x-y)\,dy\Big)^{q-1} \Big(\int_{\mathbb{R}^d} V^\varepsilon(x-y)|g(y)-g(x)|^q|f(y)|^q\,dy\Big)\,dx \\
& \le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} V^\varepsilon(x-y)|f(y)|^q|g(x)-g(y)|^q\,dy\,dx \\
& \overset{x-y=z}{=} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} V^\varepsilon(z)|f(y)|^q|g(y+z)-g(y)|^q\,dy\,dx \\
& = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} V^\varepsilon(z)|z|^q|f(y)|^q\frac{|g(y+z)-g(y)|^q}{|z|^q}\,dy\,dx \\
& \le \|\nabla g\|^q_{L^\infty(\mathbb{R}^d)}\Big(\int_{\mathbb{R}^d}|f(y)|^q\,dy\Big)\Big(\int_{\mathbb{R}^d}V^\varepsilon(z)|z|^q\,dz\Big) \\
& = \|\nabla g\|^q_{L^\infty(\mathbb{R}^d)}\|f\|^q_{L^q(\mathbb{R}^d)}\frac{1}{\varepsilon^d}\int_{\mathbb{R}^d}V(\frac{z}{\varepsilon})|z|^q\,dz \\
& \overset{\frac{z}{\varepsilon}=z'}{=} \|\nabla g\|^q_{L^\infty(\mathbb{R}^d)}\|f\|^q_{L^q(\mathbb{R}^d)}\int_{\mathbb{R}^d}V(\frac{z'}{\varepsilon})|\varepsilon z'|^q\,dz' \\
& \le C\varepsilon^q\|\nabla g\|^q_{L^\infty(\mathbb{R}^d)}\|f\|^q_{L^q(\mathbb{R}^d)}
\end{aligned}
$$

□

So, we obtain that

$$
I_3 \le \frac{\sigma}{18}\|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)} + C\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)} + C\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)}\|\nabla \mathbf{v}\|^2_{H^s(\mathbb{R}^d)} + C\varepsilon_p\|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)}
$$

Now take ε_p small enough such that $C\varepsilon_p \le \frac{\sigma}{36}$ and deduce that

$$I_3 \le \frac{\sigma}{12}\|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)} + C\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)} + C\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)}\|\nabla \mathbf{v}\|^2_{H^s(\mathbb{R}^d)}.$$

Next, we handle I_4 using Lemma 2.15.

$$\begin{aligned}
I_4 &= -\int_{\mathbb{R}^d} D^\alpha\Big(u^\sigma(\nabla p(V^{\varepsilon_p} * u^\sigma) - \nabla p(u^\sigma))\Big)\cdot \nabla D^\alpha \mathbf{u}\,dx\\
&\le \frac{\sigma}{12}\|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)} + C\|D^\alpha\Big(u^\sigma(\nabla p(V^{\varepsilon_p} * u^\sigma) - \nabla p(u^\sigma))\Big)\|^2_{L^2(\mathbb{R}^d)}\\
&\le \frac{\sigma}{12}\|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)} + C\|D^\alpha u^\sigma\|^2_{L^2(\mathbb{R}^d)}\|\nabla p(V^{\varepsilon_p} * u^\sigma) - \nabla p(u^\sigma)\|^2_{L^\infty(\mathbb{R}^d)}\\
&+ C\|u^\sigma\|^2_{L^\infty(\mathbb{R}^d)}\|D^\alpha\nabla\Big(p(V^{\varepsilon_p} * u^\sigma) - p(u^\sigma)\Big)\|^2_{L^2(\mathbb{R}^d)}\\
&\le \frac{\sigma}{12}\|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)} + C\Big\|\nabla\int_0^1 p'(zV^{\varepsilon_p} * u^\sigma + (1-z)u^\sigma)(V^{\varepsilon_p} * u^\sigma - u^\sigma)\,dz\Big\|^2_{H^s(\mathbb{R}^d)}\\
&\le \frac{\sigma}{12}\|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)} + C\Big\|\int_0^1 p'(zV^{\varepsilon_p} * u^\sigma + (1-z)u^\sigma)(V^{\varepsilon_p} * u^\sigma - u^\sigma)\,dz\Big\|^2_{H^s(\mathbb{R}^d)}\\
&+ C\Big\|\int_0^1 p''(zV^{\varepsilon_p} * u^\sigma + (1-z)u^\sigma)\nabla\Big(zV^{\varepsilon_p} * u^\sigma + (1-z)u^\sigma\Big)(V^{\varepsilon_p} * u^\sigma - u^\sigma)\,dz\Big\|^2_{H^s(\mathbb{R}^d)}\\
&\le \frac{\sigma}{12}\|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)} + C\varepsilon_p^2.
\end{aligned}$$

For I_5 we get the following estimate

$$\begin{aligned}
I_5 &= \int_{\mathbb{R}^d} D^\alpha\Big(\mathbf{u}\nabla\Phi * V^{\varepsilon_k} * \mathbf{v}\Big)\cdot\nabla D^\alpha \mathbf{u}\,dx\\
&\le \frac{\sigma}{12}\|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)} + \|D^\alpha\Big(\mathbf{u}\nabla\Phi * V^{\varepsilon_k} * \mathbf{v}\Big)\|^2_{L^2(\mathbb{R}^d)}\\
&\le \frac{\sigma}{12}\|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)} + C\|D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)}\|\nabla\Phi * V^{\varepsilon_k} * \mathbf{v}\|^2_{L^\infty(\mathbb{R}^d)} + \|\mathbf{u}\|^2_{L^\infty(\mathbb{R}^d)}\|D^\alpha\Big(\nabla\Phi * V^{\varepsilon_k} * \mathbf{v}\Big)\|^2_{L^2(\mathbb{R}^d)}\\
&\le \frac{\sigma}{12}\|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)} + C\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)}.
\end{aligned}$$

Similarly, we obtain that

$$I_6 = \int_{\mathbb{R}^d} D^\alpha\Big(\mathbf{u}\nabla\Phi * V^{\varepsilon_k} * u^\sigma\Big)\cdot\nabla D^\alpha \mathbf{u}\,dx \le \frac{\sigma}{12}\|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)} + C\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)}.$$

Next, we estimate I_7

$$
\begin{aligned}
I_7 &= \int_{\mathbb{R}^d} D^\alpha\Big(u^\sigma\nabla\Phi * V^{\varepsilon_k} * (\mathbf{u}+u^\sigma) - u^\sigma\nabla\Phi * u^\sigma)\Big)\cdot\nabla D^\alpha\mathbf{u}\,dx \\
&\le \frac{\sigma}{12}\|\nabla D^\alpha\mathbf{u}\|^2_{L^2(\mathbb{R}^d)} + C\|D^\alpha\Big(u^\sigma\nabla\Phi * V^{\varepsilon_k} * (\mathbf{u}+u^\sigma) - u^\sigma\nabla\Phi * u^\sigma)\Big)\|^2_{L^2(\mathbb{R}^d)} \\
&\le \frac{\sigma}{12}\|\nabla D^\alpha\mathbf{u}\|^2_{L^2(\mathbb{R}^d)} + C\|D^\alpha\Big(u^\sigma(\nabla\Phi * V^{\varepsilon_k} * (\mathbf{u}+u^\sigma) - \nabla\Phi * V^{\varepsilon_k} * u^\sigma)\Big)\|^2_{L^2(\mathbb{R}^d)} \\
&+ \|D^\alpha\Big(u^\sigma(\nabla\Phi * V^{\varepsilon_k} * u^\sigma - \nabla\Phi * u^\sigma)\Big)\|^2_{L^2(\mathbb{R}^d)} \\
&\le \frac{\sigma}{12}\|\nabla D^\alpha\mathbf{u}\|^2_{L^2(\mathbb{R}^d)} + C\|D^\alpha u^\sigma\|^2_{L^2(\mathbb{R}^d)}\|\nabla\Phi * V^{\varepsilon_k} * \mathbf{u}\|^2_{L^\infty(\mathbb{R}^d)} \\
&+ C\|u^\sigma\|^2_{L^\infty(\mathbb{R}^d)}\|D^\alpha(\nabla\Phi * V^{\varepsilon_k} * \mathbf{u})\|^2_{L^2(\mathbb{R}^d)} + C\|D^\alpha u^\sigma\|^2_{L^2(\mathbb{R}^d)}\|V^{\varepsilon_k} * \nabla\Phi * u^\sigma - \nabla\Phi * u^\sigma)\|^2_{L^\infty(\mathbb{R}^d)} \\
&+ C\|u^\sigma\|^2_{L^\infty(\mathbb{R}^d)}\|D^\alpha\Big(V^{\varepsilon_k} * \nabla\Phi * u^\sigma - \nabla\Phi * u^\sigma\Big)\|^2_{L^2(\mathbb{R}^d)} \\
&\le \frac{\sigma}{12}\|\nabla D^\alpha\mathbf{u}\|^2_{L^2(\mathbb{R}^d)} + C\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)} + C\varepsilon_k^2
\end{aligned}
$$

Therefore, we deduce that

$$
\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)} + \frac{\sigma}{2}\|\nabla\mathbf{u}\|^2_{H^s(\mathbb{R}^d)} \le \|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)} + C\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)}\|\nabla\mathbf{v}\|^2_{H^s(\mathbb{R}^d)} + C(\varepsilon_k+\varepsilon_p)^2.
$$

Integrating the above inequality from 0 to t, it follows that

$$
\|\mathbf{u}(t)\|^2_{H^s(\mathbb{R}^d)} + \frac{\sigma}{2}\int_0^t \|\nabla\mathbf{u}\|^2\,d\tilde{s} \le C\int_0^t \|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)}(1+\|\nabla\mathbf{v}\|^2_{H^s(\mathbb{R}^d)})\,d\tilde{s} + C(\varepsilon_k+\varepsilon_p)^2.
$$

By Grönwall's inequality, we obtain that

$$
\|\mathbf{u}(t)\|^2_{H^s(\mathbb{R}^d)} \le C(\varepsilon_k+\varepsilon_p)^2 e^{C\int_0^T(1+\|\nabla\mathbf{v}\|^2_{H^s(\mathbb{R}^d)})\,d\tilde{s}} \le C(\varepsilon_k+\varepsilon_p)^2, \forall t\in[0,T].
$$

Therefore,

$$
\int_0^T \|\nabla\mathbf{u}\|^2_{H^s(\mathbb{R}^d)}\,d\tilde{s} \le C(\varepsilon_k+\varepsilon_p)^2\int_0^T(1+\|\nabla\mathbf{v}\|_{H^s(\mathbb{R}^d)})\,d\tilde{s} + C(\varepsilon_k+\varepsilon_p)^2 \le C(\varepsilon_k+\varepsilon_p)^2.
$$

We can take ε_k and ε_p small enough such that

$$
C(\varepsilon_k+\varepsilon_p)^2 \le \varepsilon_k+\varepsilon_p.
$$

This implies that $u\in Y$. Hence, we can define an operator

$$
\begin{aligned}
\mathcal{T}: Y &\to Y \\
\mathbf{v} &\mapsto \mathbf{u}
\end{aligned}
$$

where $\mathbf{u}$ solves (2.46). In order to apply Banach fixed-point theorem, it remains to prove that $\mathcal{T}$ is a contraction.

Let $\mathbf{u}_1, \mathbf{u}_2 \in Y$ be solutions of (2.46) for $\mathbf{v}_1, \mathbf{v}_2 \in Y$ respectively. Then $\mathbf{u}_1 - \mathbf{u}_2$ solves the following equation

$$
\begin{aligned}
&\partial_t(\mathbf{u}_1 - \mathbf{u}_2) - \sigma\Delta(\mathbf{u}_1 - \mathbf{u}_2) \qquad (2.47)\\
&= \nabla\cdot\Big((\mathbf{u}_1 + u^\sigma)\nabla(p_\lambda - p)(V^{\varepsilon_p} * (\mathbf{v}_1 + u^\sigma)) - (\mathbf{u}_2 + u^\sigma)\nabla(p_\lambda - p)(V^{\varepsilon_p} * (\mathbf{v}_2 + u^\sigma))\\
&+ \mathbf{u}_1\nabla p(V^{\varepsilon_p} * (\mathbf{v}_1 + u^\sigma)) - \mathbf{u}_2\nabla p(V^{\varepsilon_p} * (\mathbf{v}_2 + u^\sigma))\\
&+ u^\sigma\nabla\int_0^1 p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v}_1)V^{\varepsilon_p} * \mathbf{u}_1\,dz - u^\sigma\nabla\int_0^1 p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v}_2)V^{\varepsilon_p} * \mathbf{u}_2\,dz\Big)\\
&- \nabla\cdot\Big(\mathbf{u}_1\nabla\Phi * V^{\varepsilon_k} * \mathbf{v}_1 - \mathbf{u}_2\nabla\Phi * V^{\varepsilon_k} * \mathbf{v}_2 + (\mathbf{u}_1 - \mathbf{u}_2)\nabla\Phi * V^{\varepsilon_k} * u^\sigma + u^\sigma\nabla\Phi * V^{\varepsilon_k} * (\mathbf{u}_1 - \mathbf{u}_2)\Big),\\
&(\mathbf{u}_1 - \mathbf{u}_2)(0, x) = 0.
\end{aligned}
$$

Multiplying (2.47) by $\mathbf{u}_1 - \mathbf{u}_2$ and integrating it over $\mathbb{R}^d$, we deduce that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}|\mathbf{u}_1 - \mathbf{u}_2|^2\,dx + \sigma\int_{\mathbb{R}^d}|\nabla(\mathbf{u}_1 - \mathbf{u}_2)|^2\,dx\\
&= \int_{\mathbb{R}^d} -(\mathbf{u}_1 + u^\sigma)\nabla(p_\lambda - p)(V^{\varepsilon_p} * (\mathbf{v}_1 + u^\sigma))\cdot\nabla(\mathbf{u}_1 - \mathbf{u}_2)\\
&\qquad + (\mathbf{u}_2 + u^\sigma)\nabla(p_\lambda - p)(V^{\varepsilon_p} * (\mathbf{v}_2 + u^\sigma))\cdot\nabla(\mathbf{u}_1 - \mathbf{u}_2)\,dx\\
&- \int_{\mathbb{R}^d}(\mathbf{u}_1 - \mathbf{u}_2)\nabla p(V^{\varepsilon_p} * (\mathbf{v}_1 + u^\sigma))\cdot\nabla(\mathbf{u}_1 - \mathbf{u}_2)\,dx\\
&- \int_{\mathbb{R}^d}\mathbf{u}_2\Big(\nabla p(V^{\varepsilon_p} * (\mathbf{v}_1 + u^\sigma)) - \nabla p(V^{\varepsilon_p} * (\mathbf{v}_2 + u^\sigma))\Big)\cdot\nabla(\mathbf{u}_1 - \mathbf{u}_2)\,dx\\
&- \int_{\mathbb{R}^d}u^\sigma\nabla\int_0^1 p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v}_1)V^{\varepsilon_p} * (\mathbf{u}_1 - \mathbf{u}_2)\,dz\cdot\nabla(\mathbf{u}_1 - \mathbf{u}_2)\,dx\\
&- \int_{\mathbb{R}^d}u^\sigma\nabla\int_0^1\Big(p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v}_1) - p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v}_2)\Big)V^{\varepsilon_p} * \mathbf{u}_2\,dz\cdot\nabla(\mathbf{u}_1 - \mathbf{u}_2)\,dx\\
&+ \int_{\mathbb{R}^d}(\mathbf{u}_1 - \mathbf{u}_2)\nabla\Phi * V^{\varepsilon_k} * \mathbf{v}_1\nabla(\mathbf{u}_1 - \mathbf{u}_2)\,dx\\
&+ \int_{\mathbb{R}^d}\mathbf{u}_2\nabla\Phi * V^{\varepsilon_k} * (\mathbf{v}_1 - \mathbf{v}_2)\cdot\nabla(\mathbf{u}_1 - \mathbf{u}_2)\,dx\\
&+ \int_{\mathbb{R}^d}(\mathbf{u}_1 - \mathbf{u}_2)\nabla\Phi * V^{\varepsilon_k} * u^\sigma\cdot\nabla(\mathbf{u}_1 - \mathbf{u}_2)\,dx\\
&+ \int_{\mathbb{R}^d}u^\sigma\nabla\Phi * V^{\varepsilon_k} * (\mathbf{u}_1 - \mathbf{u}_2)\cdot\nabla(\mathbf{u}_1 - \mathbf{u}_2)\,dx\\
&=: J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8 + J_9.
\end{aligned}
$$

First, we can take λ small enough such that

$$
\begin{aligned}
J_1 = \int_{\mathbb{R}^d} &-(\mathbf{u}_1 + u^\sigma)\nabla(p_\lambda - p)(V^{\varepsilon_p} * (\mathbf{v}_1 + u^\sigma))\cdot\nabla(\mathbf{u}_1 - \mathbf{u}_2)\\
&+ (\mathbf{u}_2 + u^\sigma)\nabla(p_\lambda - p)(V^{\varepsilon_p} * (\mathbf{v}_2 + u^\sigma))\cdot\nabla(\mathbf{u}_1 - \mathbf{u}_2)\,dx = 0
\end{aligned}
$$

Next, we handle J_2.

$$\begin{aligned}J_2 &= -\int_{\mathbb{R}^d} (\mathbf{u}_1-\mathbf{u}_2)\nabla p(V^{\varepsilon_p} * (\mathbf{v}_1+u^\sigma))\cdot\nabla(\mathbf{u}_1-\mathbf{u}_2)\,dx\\ &\le \frac{\sigma}{16}\int_{\mathbb{R}^d}|\nabla(\mathbf{u}_1-\mathbf{u}_2)|^2\,dx+\hat{C}\int_{\mathbb{R}^d}|\mathbf{u}_1-\mathbf{u}_2|^2\,dx.\end{aligned}$$

where $\hat{C}$ in this subsection denotes a positive constant which depends on σ, s and ε_p. Similarly we obtain that J_6, J_7, J_8 and J_9 are bounded by

$$\frac{\sigma}{16}\int_{\mathbb{R}^d}|\nabla(\mathbf{u}_1-\mathbf{u}_2)|^2\,dx+\hat{C}\int_{\mathbb{R}^d}|\mathbf{u}_1-\mathbf{u}_2|^2\,dx.$$

Next, we estimate J_3.

$$\begin{aligned}J_3 &= -\int_{\mathbb{R}^d}\mathbf{u}_2\Big(\nabla p(V^{\varepsilon_p} * (\mathbf{v}_1+u^\sigma))-\nabla p(V^{\varepsilon_p} * (\mathbf{v}_2+u^\sigma))\Big)\cdot\nabla(\mathbf{u}_1-\mathbf{u}_2)\,dx\\ &\le \frac{\sigma}{16}\int_{\mathbb{R}^d}|\nabla(\mathbf{u}_1-\mathbf{u}_2)|^2\,dx+\hat{C}\|\mathbf{u}_2\|^2_{L^\infty(\mathbb{R}^d)}\|\nabla p(V^{\varepsilon_p} * (\mathbf{v}_2+u^\sigma))-\nabla p(V^{\varepsilon_p} * (\mathbf{v}_1+u^\sigma))\|^2_{L^2(\mathbb{R}^d)}\\ &\le \frac{\sigma}{16}\int_{\mathbb{R}^d}|\nabla(\mathbf{u}_1-\mathbf{u}_2)|^2\,dx\\ &+\hat{C}\int_{\mathbb{R}^d}\Big|\nabla\int_0^1 p'\Big(zV^{\varepsilon_p} * (\mathbf{v}_2+u^\sigma)+(1-z)V^{\varepsilon_p} * (\mathbf{v}_1+u^\sigma)\Big)V^{\varepsilon_p} * \Big((\mathbf{v}_2+u^\sigma)-(\mathbf{v}_1+u^\sigma)\Big)\,dz\Big|^2\,dx\\ &\le \frac{\sigma}{16}\int_{\mathbb{R}^d}|\nabla(\mathbf{u}_1-\mathbf{u}_2)|^2\,dx\\ &+\hat{C}\int_{\mathbb{R}^d}\Big|\int_0^1 p''\Big(zV^{\varepsilon_p} * (\mathbf{v}_2+u^\sigma)+(1-z)V^{\varepsilon_p} * (\mathbf{v}_1+u^\sigma)\Big)\\ &\qquad\cdot\nabla V^{\varepsilon_p} * \Big(z\mathbf{v}_2+(1-z)\mathbf{v}_1+u^\sigma\Big)V^{\varepsilon_p} * (\mathbf{v}_2-\mathbf{v}_1)\Big|^2\,dz\,dx\\ &+\hat{C}\int_{\mathbb{R}^d}\int_0^1\Big|p'\Big(zV^{\varepsilon_p} * (\mathbf{v}_2+u^\sigma)+(1-z)V^{\varepsilon_p} * (\mathbf{v}_1+u^\sigma)\Big)\nabla V^{\varepsilon_p} * (\mathbf{v}_2-\mathbf{v}_1)\Big|^2\,dz\,dx\\ &= \frac{\sigma}{16}\int_{\mathbb{R}^d}|\nabla(\mathbf{u}_1-\mathbf{u}_2)|^2\,dx+J_{31}+J_{32}.\end{aligned}$$

Then by Young's convolution inequality we obtain that

$$\begin{aligned}J_{31} &\le \hat{C}\sup_{0<z<1}\|\nabla V^{\varepsilon_p} * \Big(z\mathbf{v}_2+(1-z)\mathbf{v}_1+u^\sigma\Big)V^{\varepsilon_p} * (\mathbf{v}_2-\mathbf{v}_1)\|^2_{L^2(\mathbb{R}^d)}\\ &\le \hat{C}\sup_{0<z<1}\|\nabla V^{\varepsilon_p} * \Big(z\mathbf{v}_2+(1-z)\mathbf{v}_1+u^\sigma\Big)\|^2_{L^\infty(\mathbb{R}^d)}\|\mathbf{v}_2-\mathbf{v}_1\|^2_{L^2(\mathbb{R}^d)}\\ &\le \hat{C}\|\nabla V^{\varepsilon_p}\|^2_{L^1(\mathbb{R}^d)}\|\mathbf{v}_2+\mathbf{v}_1+u^\sigma\|^2_{L^\infty(\mathbb{R}^d)}\|\mathbf{v}_2-\mathbf{v}_1\|^2_{L^2(\mathbb{R}^d)}\\ &\le \hat{C}\|\mathbf{v}_2-\mathbf{v}_1\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

For J_{32} we deduce that

$$J_{32}\le \hat{C}\|\nabla V^{\varepsilon_p} * (\mathbf{v}_1-\mathbf{v}_2)\|^2_{L^2(\mathbb{R}^d)}\le \hat{C}\|\nabla V^{\varepsilon_p}\|^2_{L^1(\mathbb{R}^d)}\|\mathbf{v}_2-\mathbf{v}_1\|^2_{L^2(\mathbb{R}^d)}\le \hat{C}\|\mathbf{v}_2-\mathbf{v}_1\|^2_{L^2(\mathbb{R}^d)}.$$

This implies that

$$J_3 \leq \frac{\sigma}{16} \int_{\mathbb{R}^d} |\nabla(\mathbf{u}_1 - \mathbf{u}_2)|^2 \, dx + \hat{C} \|\mathbf{v}_2 - \mathbf{v}_1\|^2_{L^2(\mathbb{R}^d)}.$$

Next, we estimate J_4.

$$\begin{aligned}
J_4 &= -\int_{\mathbb{R}^d} u^\sigma \nabla \int_0^1 p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v}_1) V^{\varepsilon_p} * (\mathbf{u}_1 - \mathbf{u}_2) \, dz \cdot \nabla(\mathbf{u}_1 - \mathbf{u}_2) \, dx \\
&\leq \frac{\sigma}{16} \int_{\mathbb{R}^d} |\nabla(\mathbf{u}_1 - \mathbf{u}_2)|^2 \, dx \\
&+ \hat{C} \|u^\sigma\|^2_{L^\infty(\mathbb{R}^d)} \| \int_0^1 \nabla\Big(p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v}_1) V^{\varepsilon_p} * (\mathbf{u}_1 - \mathbf{u}_2)\Big) \, dz \|^2_{L^2(\mathbb{R}^d)} \\
&\leq \frac{\sigma}{16} \int_{\mathbb{R}^d} |\nabla(\mathbf{u}_1 - \mathbf{u}_2)|^2 \, dx \\
&+ \hat{C} \| \int_0^1 p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v}_1) \nabla V^{\varepsilon_p} * (\mathbf{u}_1 - \mathbf{u}_2) \, dz \|^2_{L^2(\mathbb{R}^d)} \\
&+ \hat{C} \| \int_0^1 p''(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v}_1) V^{\varepsilon_p} * \nabla(u^\sigma + z\mathbf{v}_1) V^{\varepsilon_p} * (\mathbf{u}_1 - \mathbf{u}_2) \, dz \|^2_{L^2(\mathbb{R}^d)} \\
&\leq \frac{\sigma}{16} \int_{\mathbb{R}^d} |\nabla(\mathbf{u}_1 - \mathbf{u}_2)|^2 \, dx + \hat{C} \|\nabla V^{\varepsilon_p} * (\mathbf{u}_1 - \mathbf{u}_2)\|^2_{L^2(\mathbb{R}^d)} + \hat{C} \|V^{\varepsilon_p} * (\mathbf{u}_1 - \mathbf{u}_2)\|^2_{L^2(\mathbb{R}^d)} \\
&\leq \frac{\sigma}{16} \int_{\mathbb{R}^d} |\nabla(\mathbf{u}_1 - \mathbf{u}_2)|^2 \, dx + \hat{C} \|\nabla V^{\varepsilon_p}\|^2_{L^1(\mathbb{R}^d)} \|\mathbf{u}_1 - \mathbf{u}_2\|^2_{L^2(\mathbb{R}^d)} + \hat{C} \|\mathbf{u}_1 - \mathbf{u}_2\|^2_{L^2(\mathbb{R}^d)} \\
&\leq \frac{\sigma}{16} \int_{\mathbb{R}^d} |\nabla(\mathbf{u}_1 - \mathbf{u}_2)|^2 \, dx + \hat{C} \|\mathbf{u}_1 - \mathbf{u}_2\|^2_{L^2(\mathbb{R}^d)}
\end{aligned}$$

For J_5 we obtain that

$$\begin{aligned}
J_5 &= -\int_{\mathbb{R}^d} u^\sigma \nabla \int_0^1 \Big(p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v}_1) - p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v}_2)\Big) V^{\varepsilon_p} * \mathbf{u}_2 \, dz \cdot \nabla(\mathbf{u}_1 - \mathbf{u}_2)\, dx \\
&\leq \frac{\sigma}{16} \int_{\mathbb{R}^d} |\nabla(\mathbf{u}_1 - \mathbf{u}_2)|^2 \, dx \\
&+ \hat{C} \|u^\sigma\|^2_{L^\infty(\mathbb{R}^d)} \| \int_0^1 \nabla \Big(\Big(p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v}_1) - p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{v}_2)\Big) V^{\varepsilon_p} * \mathbf{u}_2 \Big) \, dz \|^2_{L^2(\mathbb{R}^d)} \\
&\leq \frac{\sigma}{16} \int_{\mathbb{R}^d} |\nabla(\mathbf{u}_1 - \mathbf{u}_2)|^2 \, dx \\
&+ \hat{C} \| \int_0^1 \nabla \int_0^1 p''\Big(V^{\varepsilon_p} * u^\sigma + z\tau V^{\varepsilon_p} * \mathbf{v}_1 + z(1-\tau) V^{\varepsilon_p} * \mathbf{v}_2\Big) z V^{\varepsilon_p} * (\mathbf{v}_1 - \mathbf{v}_2) \, d\tau \, dz \|^2_{L^2(\mathbb{R}^d)} \\
&\leq \frac{\sigma}{16} \int_{\mathbb{R}^d} |\nabla(\mathbf{u}_1 - \mathbf{u}_2)|^2 \, dx \\
&+ \hat{C} \| \int_0^1 \int_0^1 p'''\Big(V^{\varepsilon_p} * u^\sigma + z\tau V^{\varepsilon_p} * \mathbf{v}_1 + z(1-\tau) V^{\varepsilon_p} * \mathbf{v}_2\Big) \\
&\qquad \cdot \nabla V^{\varepsilon_p} * \Big(u^\sigma + z\tau \mathbf{v}_1 + z(1-\tau)\mathbf{v}_2\Big) z V^{\varepsilon_p} * (\mathbf{v}_1 - \mathbf{v}_2) \, d\tau \, dz \|^2_{L^2(\mathbb{R}^d)} \\
&+ \hat{C} \| \int_0^1 \int_0^1 p''\Big(V^{\varepsilon_p} * u^\sigma + z\tau V^{\varepsilon_p} * \mathbf{v}_1 + z(1-\tau) V^{\varepsilon_p} * \mathbf{v}_2\Big) z \nabla V^{\varepsilon_p} * (\mathbf{v}_1 - \mathbf{v}_2) \, d\tau \, dz \|^2_{L^2(\mathbb{R}^d)} \\
&\leq \frac{\sigma}{16} \int_{\mathbb{R}^d} |\nabla(\mathbf{u}_1 - \mathbf{u}_2)|^2 \, dx \\
&+ \hat{C} \sup_{0<z,\tau<1} \|V^{\varepsilon_p} * \nabla\Big(u^\sigma + z\tau \mathbf{v}_1 + z(1-\tau)\mathbf{v}_2\Big)\|^2_{L^\infty(\mathbb{R}^d)} \|V^{\varepsilon_p} * (\mathbf{v}_1 - \mathbf{v}_2)\|^2_{L^2(\mathbb{R}^d)} \\
&+ \hat{C} \|\mathbf{v}_1 - \mathbf{v}_2\|^2_{L^2(\mathbb{R}^d)} \\
&\leq \frac{\sigma}{16} \int_{\mathbb{R}^d} |\nabla(\mathbf{u}_1 - \mathbf{u}_2)|^2 \, dx + \hat{C} \|\mathbf{v}_1 - \mathbf{v}_2\|^2_{L^2(\mathbb{R}^d)}
\end{aligned}$$

Therefore, it follows that

$$\frac{d}{dt} \|\mathbf{u}_1 - \mathbf{u}_2\|^2_{L^2(\mathbb{R}^d)} + \frac{\sigma}{2} \|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|^2_{L^2(\mathbb{R}^d)} \leq \hat{C} \|\mathbf{u}_1 - \mathbf{u}_2\|^2_{L^2(\mathbb{R}^d)} + \hat{C} \|\mathbf{v}_1 - \mathbf{v}_2\|^2_{L^2(\mathbb{R}^d)}.$$

So,

$$\|(\mathbf{u}_1 - \mathbf{u}_2)(t)\|^2_{L^2(\mathbb{R}^d)} + \frac{\sigma}{2} \int_0^t \|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|^2_{L^2(\mathbb{R}^d)} \, ds \leq \hat{C} \int_0^t \|\mathbf{u}_1 - \mathbf{u}_2\|^2_{L^2(\mathbb{R}^d)} \, ds + \hat{C} \int_0^t \|\mathbf{v}_1 - \mathbf{v}_2\|^2_{L^2(\mathbb{R}^d)} \, ds$$

By Grönwall's inequality

$$\|(\mathbf{u}_1 - \mathbf{u}_2)(t)\|^2_{L^2(\mathbb{R}^d)} \leq e^{\int_0^t \hat{C} \, ds} \hat{C} \int_0^t \|\mathbf{v}_1 - \mathbf{v}_2\|^2_{L^2(\mathbb{R}^d)} \, ds \leq e^{\hat{C} T_*} \hat{C} T_* \|\mathbf{v}_1 - \mathbf{v}_2\|^2_{L^\infty(0,T_*;L^2(\mathbb{R}^d))}, \ \forall t \in [0, T_*]$$

We can take $T_*(\varepsilon_p)$ small enough such that $e^{\hat{C} T_*} \hat{C} T_* < 1$.

Therefore, the operator $\mathcal{T}$ possesses a unique fixed point $\mathbf{u}$ such that $\mathcal{T}(\mathbf{u}) = \mathbf{u}$. So, we find a unique solution $\mathbf{u} \in Y$ on $[0, T_*]$ to the following system.

$$\begin{aligned}\partial_t \mathbf{u} - \sigma\Delta\mathbf{u} &= \nabla\cdot\Big((\mathbf{u}+u^\sigma)\nabla(p_\lambda - p)(V^{\varepsilon_p} * (\mathbf{u}+u^\sigma)) + \mathbf{u}\nabla p(V^{\varepsilon_p} * (\mathbf{u}+u^\sigma)) \\ &\quad + u^\sigma\nabla\int_0^1 p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{u})V^{\varepsilon_p} * \mathbf{u}\,dz + u^\sigma\Big(\nabla p(V^{\varepsilon_p} * u^\sigma) - \nabla p(u^\sigma)\Big)\Big) \\ &\quad - \nabla\cdot\Big(\mathbf{u}\nabla\Phi * V^{\varepsilon_k} * (\mathbf{u}+u^\sigma) + u^\sigma\nabla\Phi * V^{\varepsilon_k} * \mathbf{u} + u^\sigma(\nabla\Phi * V^{\varepsilon_k} * u^\sigma - \nabla\Phi * u^\sigma)\Big), \\ \mathbf{u}(0,x) &= 0.\end{aligned} \tag{2.48}$$

Therefore for any $t \in (0, T_*)$ we obtain that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}|D^\alpha\mathbf{u}|^2\,dx + \sigma\int_{\mathbb{R}^d}|\nabla D^\alpha\mathbf{u}|^2\,dx \\ &= -\int_{\mathbb{R}^d} D^\alpha\Big((\mathbf{u}+u^\sigma)\nabla(p_\lambda - p)(V^{\varepsilon_p} * (\mathbf{u}+u^\sigma))\Big)\cdot\nabla D^\alpha\mathbf{u}\,dx \\ &- \int_{\mathbb{R}^d} D^\alpha\Big(\mathbf{u}\nabla p(V^{\varepsilon_p} * (\mathbf{u}+u^\sigma))\Big)\cdot\nabla D^\alpha\mathbf{u}\,dx \\ &- \int_{\mathbb{R}^d} D^\alpha\Big(u^\sigma\nabla\int_0^1 p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{u})V^{\varepsilon_p} * \mathbf{u}\,dz\Big)\cdot\nabla D^\alpha\mathbf{u}\,dx \\ &- \int_{\mathbb{R}^d} D^\alpha\Big(u^\sigma(\nabla p(V^{\varepsilon_p} * u^\sigma) - \nabla p(u^\sigma))\Big)\cdot\nabla D^\alpha\mathbf{u}\,dx \\ &+ \int_{\mathbb{R}^d} D^\alpha\Big(\mathbf{u}\nabla\Phi * V^{\varepsilon_k} * (\mathbf{u}+u^\sigma)\Big)\cdot\nabla D^\alpha\mathbf{u}\,dx \\ &+ \int_{\mathbb{R}^d} D^\alpha\Big(u^\sigma\nabla\Phi * V^{\varepsilon_k} * \mathbf{u}\Big)\cdot\nabla D^\alpha\mathbf{u}\,dx \\ &+ \int_{\mathbb{R}^d} D^\alpha\Big(u^\sigma(\nabla\Phi * V^{\varepsilon_k} * u^\sigma - \nabla\Phi * u^\sigma)\Big)\cdot\nabla D^\alpha\mathbf{u}\,dx \\ &=: K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + K_7.\end{aligned}$$

Since $\|V^{\varepsilon_p} * (\mathbf{u}+u^\sigma)\|_{L^\infty((0,T)\times\mathbb{R}^d)}$ is bounded, we can choose λ small enough in order to ensure that $K_1 = 0$. For K_2 we get that

$$\begin{aligned}K_2 &\le \frac{\sigma}{13}\int_{\mathbb{R}^d}|\nabla D^\alpha\mathbf{u}|^2\,dx + C\|D^\alpha\Big(\mathbf{u}\nabla p(V^{\varepsilon_p} * (\mathbf{u}+u^\sigma))\Big)\|^2_{L^2(\mathbb{R}^d)} \\ &\le \frac{\sigma}{13}\int_{\mathbb{R}^d}|\nabla D^\alpha\mathbf{u}|^2\,dx + C\|\mathbf{u}\|^2_{L^\infty(\mathbb{R}^d)}\|D^\alpha\nabla p(V^{\varepsilon_p} * (\mathbf{u}+u^\sigma))\|^2_{L^2(\mathbb{R}^d)} \\ &+ C\|D^\alpha\mathbf{u}\|^2_{L^2(\mathbb{R}^d)}\|\nabla p(V^{\varepsilon_p} * (\mathbf{u}+u^\sigma))\|^2_{L^\infty(\mathbb{R}^d)} \\ &\le \frac{\sigma}{13}\int_{\mathbb{R}^d}|\nabla D^\alpha\mathbf{u}|^2\,dx + C\|\mathbf{u}\|^2_{L^\infty(\mathbb{R}^d)}(C + C\|\nabla D^\alpha\mathbf{u}\|^2_{L^2(\mathbb{R}^d)}) + C\|D^\alpha\mathbf{u}\|^2_{L^2(\mathbb{R}^d)} \\ &\le \frac{\sigma}{13}\int_{\mathbb{R}^d}|\nabla D^\alpha\mathbf{u}|^2\,dx + C\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)} + C(\varepsilon_k + \varepsilon_p)\|\nabla D^\alpha\mathbf{u}\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

where we used that $\|\mathbf{u}\|^2_{L^\infty(0,T_*;H^s(\mathbb{R}^d))} \le \varepsilon_k + \varepsilon_p$. Then we take $\varepsilon_k + \varepsilon_p$ small enough such that

$$K_2 \le \frac{\sigma}{12}\|\nabla\mathbf{u}\|^2_{H^s(\mathbb{R}^d)} + C\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)}.$$

Next, we analyze K_3.

$$\begin{aligned}
K_3 = &-\int_{\mathbb{R}^d}\int_0^1 u^\sigma \nabla D^\alpha\Big(p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{u})V^{\varepsilon_p} * \mathbf{u}\Big)\cdot \nabla D^\alpha \mathbf{u}\, dz\, dx \\
&- \int_{\mathbb{R}^d}\int_0^1 \bigg(D^\alpha\Big(u^\sigma \nabla\Big(p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{u})V^{\varepsilon_p} * \mathbf{u}\Big)\Big) \\
&\qquad - u^\sigma \nabla D^\alpha\Big(p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{u})V^{\varepsilon_p} * \mathbf{u}\Big)\bigg)\cdot \nabla D^\alpha \mathbf{u}\, dz\, dx \\
=:&\, K_{31} + K_{32}.
\end{aligned}$$

So,

$$\begin{aligned}
K_{32} &\le \|\nabla D^\alpha \mathbf{u}\|_{L^2(\mathbb{R}^d)} \sup_{0<z<1} \|\nabla u^\sigma\|_{L^\infty(\mathbb{R}^d)} \|D^{\alpha-1}\nabla\Big(p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{u})V^{\varepsilon_p} * \mathbf{u}\Big)\|_{L^2(\mathbb{R}^d)} \\
&+ \|\nabla D^\alpha \mathbf{u}\|_{L^2(\mathbb{R}^d)} \sup_{0<z<1} \|D^\alpha u^\sigma\|_{L^2(\mathbb{R}^d)} \|\nabla\Big(p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{u})V^{\varepsilon_p} * \mathbf{u}\Big)\|_{L^\infty(\mathbb{R}^d)} \\
&\le \frac{\sigma}{36}\|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)} + C\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)}.
\end{aligned}$$

Now we handle K_{31}.

$$\begin{aligned}
K_{31} = &-\int_{\mathbb{R}^d}\int_0^1 u^\sigma p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{u})V^{\varepsilon_p} * \nabla D^\alpha \mathbf{u}\cdot \nabla D^\alpha \mathbf{u}\, dz\, dx \\
&- \int_{\mathbb{R}^d}\int_0^1 u^\sigma \bigg(\nabla D^\alpha\Big(p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{u})V^{\varepsilon_p} * \mathbf{u}\Big) \\
&\qquad - p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{u})V^{\varepsilon_p} \nabla D^\alpha * \mathbf{u}\bigg)\cdot \nabla D^\alpha \mathbf{u}\, dz\, dx \\
=:&\, K_{311} + K_{312}.
\end{aligned}$$

Then we obtain that

$$\begin{aligned}
K_{312} &\le \|\nabla D^\alpha \mathbf{u}\|_{L^2(\mathbb{R}^d)} \|u^\sigma\|_{L^\infty(\mathbb{R}^d)} \|D^\alpha V^{\varepsilon_p} * \mathbf{u}\|_{L^2(\mathbb{R}^d)} \sup_{0\le z\le 1} \|\nabla p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{u})\|_{L^\infty(\mathbb{R}^d)} \\
&+ \|\nabla D^\alpha \mathbf{u}\|_{L^2(\mathbb{R}^d)} \|u^\sigma\|_{L^\infty(\mathbb{R}^d)} \|V^{\varepsilon_p} * \mathbf{u}\|_{L^\infty(\mathbb{R}^d)} \sup_{0\le z\le 1} \|\nabla D^\alpha p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{u})\|_{L^2(\mathbb{R}^d)} \\
&\le \|\nabla D^\alpha \mathbf{u}\|_{L^2(\mathbb{R}^d)} \|u^\sigma\|_{L^\infty(\mathbb{R}^d)} \|D^\alpha V^{\varepsilon_p} * \mathbf{u}\|_{L^2(\mathbb{R}^d)} \sup_{0\le z\le 1} \|\nabla p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{u})\|_{L^\infty(\mathbb{R}^d)} \\
&+ \|\nabla D^\alpha \mathbf{u}\|_{L^2(\mathbb{R}^d)} \|u^\sigma\|_{L^\infty(\mathbb{R}^d)} \|V^{\varepsilon_p} * \mathbf{u}\|_{L^\infty(\mathbb{R}^d)} (C + C\|D^\alpha \nabla \mathbf{u}\|_{L^2(\mathbb{R}^d)}) \\
&\le \frac{\sigma}{37}\|\nabla \mathbf{u}\|^2_{H^s(\mathbb{R}^d)} + C\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)} + C(\varepsilon_k + \varepsilon_p)\|\nabla \mathbf{u}\|^2_{H^s(\mathbb{R}^d)}.
\end{aligned}$$

So, we can take $\varepsilon_k + \varepsilon_p$ small enough such that

$$K_{312} \le \frac{\sigma}{36}\|\nabla \mathbf{u}\|^2_{H^s(\mathbb{R}^d)} + C\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)}.$$

Then consider K_{311}.

$$
\begin{aligned}
K_{311} &= -\int_{\mathbb{R}^d}\int_0^1 u^\sigma p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{u})W^{\varepsilon_p} * W^{\varepsilon_p} * \nabla D^\alpha \mathbf{u} \cdot \nabla D^\alpha \mathbf{u}\, dz\, dx \\
&= -\int_{\mathbb{R}^d}\int_0^1 W^{\varepsilon_p} * \Big(u^\sigma p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{u})\nabla D^\alpha \mathbf{u}\Big) \cdot W^{\varepsilon_p} * \nabla D^\alpha \mathbf{u}\, dz\, dx \\
&= -\int_{\mathbb{R}^d}\int_0^1 u^\sigma p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{u})\Big|W^{\varepsilon_p} * \nabla D^\alpha \mathbf{u}\Big|^2 dz\, dx \\
&\quad - \int_{\mathbb{R}^d}\int_0^1 \Big(W^{\varepsilon_p} * \Big(u^\sigma p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{u})\nabla D^\alpha \mathbf{u}\Big) \\
&\qquad\qquad - u^\sigma W^{\varepsilon_p} * \Big(p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{u})\nabla D^\alpha \mathbf{u}\Big)\Big) \cdot W^{\varepsilon_p} * \nabla D^\alpha \mathbf{u}\, dz\, dx \\
&\quad - \int_{\mathbb{R}^d}\int_0^1 u^\sigma \Big(W^{\varepsilon_p} * \Big(p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{u})\nabla D^\alpha \mathbf{u}\Big) \\
&\qquad\qquad - p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{u})W^{\varepsilon_p} * \nabla D^\alpha \mathbf{u}\Big) \cdot W^{\varepsilon_p} * \nabla D^\alpha \mathbf{u}\, dz\, dx \\
&\le 0 + K_{3111} + K_{3112}.
\end{aligned}
$$

So, using Lemma 2.15 we obtain that

$$
\begin{aligned}
K_{3111} &\le \|\nabla D^\alpha \mathbf{u}\|_{L^2(\mathbb{R}^d)} \sup_{0<z<1} \|W^{\varepsilon_p} * \Big(u^\sigma p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{u})\nabla D^\alpha \mathbf{u}\Big) \\
&\qquad\qquad - u^\sigma W^{\varepsilon_p} * \Big(p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{u})\nabla D^\alpha \mathbf{u}\Big)\|_{L^2(\mathbb{R}^d)} \\
&\le C\|\nabla D^\alpha \mathbf{u}\|_{L^2(\mathbb{R}^d)}\varepsilon_p\|\nabla u^\sigma\|_{L^\infty(\mathbb{R}^d)} \sup_{0<z<1} \|p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{u})\nabla D^\alpha \mathbf{u}\|_{L^2(\mathbb{R}^d)} \\
&\le C\varepsilon_p\|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}
$$

and

$$
\begin{aligned}
K_{3112} &\le \|W^{\varepsilon_p} * \nabla D^\alpha \mathbf{u}\|_{L^2(\mathbb{R}^d)} \sup_{0<z<1} \|W^{\varepsilon_p} * \Big(p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{u})\nabla D^\alpha \mathbf{u}\Big) \\
&\qquad\qquad - p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{u})W^{\varepsilon_p} * \nabla D^\alpha \mathbf{u}\|_{L^2(\mathbb{R}^d)} \\
&\le C\|\nabla D^\alpha \mathbf{u}\|_{L^2(\mathbb{R}^d)}\varepsilon_p\|\nabla D^\alpha \mathbf{u}\|_{L^2(\mathbb{R}^d)} \sup_{0<z<1} \|\nabla p'(V^{\varepsilon_p} * u^\sigma + zV^{\varepsilon_p} * \mathbf{u})\|_{L^\infty(\mathbb{R}^d)} \\
&\le C\varepsilon_p\|\nabla D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}
$$

Taking ε_p small enough such that $C\varepsilon_p \le \frac{\sigma}{36}$, it follows that

$$
K_3 \le \frac{\sigma}{12}\|\nabla \mathbf{u}\|^2_{H^s(\mathbb{R}^d)} + C\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)}.
$$

Since $\nabla p(V^{\varepsilon_p} * u^\sigma) - \nabla p(u^\sigma) = \nabla \int_0^1 p'(zV^{\varepsilon_p} * u^\sigma + (1-z)u^\sigma)(V^{\varepsilon_p} * u^\sigma - u^\sigma)\,dz$, we deduce that

$$\begin{aligned}
K_4 = &-\int_{\mathbb{R}^d} D^\alpha\Big(u^\sigma(\nabla p(V^{\varepsilon_p} * u^\sigma) - \nabla p(u^\sigma))\Big) \cdot \nabla D^\alpha \mathbf{u}\,dx \\
&\le \frac{\sigma}{12}\int_{\mathbb{R}^d} |\nabla D^\alpha \mathbf{u}|^2\,dx + C\|D^\alpha\Big(u^\sigma(\nabla p(V^{\varepsilon_p} * u^\sigma) - \nabla p(u^\sigma))\Big)\|^2_{L^2(\mathbb{R}^d)} \\
&\le \frac{\sigma}{12}\int_{\mathbb{R}^d} |\nabla D^\alpha \mathbf{u}|^2\,dx \\
&+ C\|D^\alpha u^\sigma\|^2_{L^2(\mathbb{R}^d)}\|\nabla \int_0^1 p'(zV^{\varepsilon_p} * u^\sigma + (1-z)u^\sigma)(V^{\varepsilon_p} * u^\sigma - u^\sigma)\,dz\|^2_{L^\infty(\mathbb{R}^d)} \\
&+ C\|u^\sigma\|^2_{L^\infty(\mathbb{R}^d)}\|D^\alpha\nabla \int_0^1 p'(zV^{\varepsilon_p} * u^\sigma + (1-z)u^\sigma)(V^{\varepsilon_p} * u^\sigma - u^\sigma)\,dz\|^2_{L^2(\mathbb{R}^d)} \\
&\le \frac{\sigma}{12}\int_{\mathbb{R}^d} |\nabla D^\alpha \mathbf{u}|^2\,dx + C\|V^{\varepsilon_p} * u^\sigma - u^\sigma\|^2_{L^\infty(\mathbb{R}^d)} + C\|\nabla(V^{\varepsilon_p} * u^\sigma - u^\sigma)\|^2_{L^\infty(\mathbb{R}^d)} \\
&+ C\int_0^1 \|D^{\alpha+1}\Big(p'(zV^{\varepsilon_p} * u^\sigma + (1-z)u^\sigma)(V^{\varepsilon_p} * u^\sigma - u^\sigma)\Big)\|_{L^2(\mathbb{R}^d)}\,dz \\
&\le \frac{\sigma}{12}\int_{\mathbb{R}^d} |\nabla D^\alpha \mathbf{u}|^2\,dx + C\|V^{\varepsilon_p} * u^\sigma - u^\sigma\|^2_{H^s(\mathbb{R}^d)} \\
&+ C\int_0^1 \Big(\|D^{\alpha+1}p'(zV^{\varepsilon_p} * u^\sigma + (1-z)u^\sigma)\|^2_{L^2(\mathbb{R}^d)}\|V^{\varepsilon_p} * u^\sigma - u^\sigma\|^2_{L^\infty(\mathbb{R}^d)} \\
&\qquad + \|p'(zV^{\varepsilon_p} * u^\sigma + (1-z)u^\sigma)\|^2_{L^\infty(\mathbb{R}^d)}\|D^{\alpha+1}(V^{\varepsilon_p} * u^\sigma - u^\sigma)\|^2_{L^2(\mathbb{R}^d)}\Big)\,dz \\
&\le \frac{\sigma}{12}\int_{\mathbb{R}^d} |\nabla D^\alpha \mathbf{u}|^2\,dx + C\|V^{\varepsilon_p} * u^\sigma - u^\sigma\|^2_{H^{s+1}(\mathbb{R}^d)} \\
&\le \frac{\sigma}{12}\int_{\mathbb{R}^d} |\nabla D^\alpha \mathbf{u}|^2\,dx + C\varepsilon_p^2.
\end{aligned}$$

Next, we estimate K_5.

$$\begin{aligned}
K_5 &= \int_{\mathbb{R}^d} D^\alpha\Big(\mathbf{u}\nabla\Phi * V^{\varepsilon_k} * (\mathbf{u} + u^\sigma)\Big) \cdot \nabla D^\alpha \mathbf{u}\,dx \\
&\le \frac{\sigma}{12}\int_{\mathbb{R}^d} |\nabla D^\alpha \mathbf{u}|^2\,dx + C\|D^\alpha\Big(\mathbf{u}\nabla\Phi * V^{\varepsilon_k} * \mathbf{u}\Big)\|^2_{L^2(\mathbb{R}^d)} + C\|D^\alpha\Big(\mathbf{u}\nabla\Phi * V^{\varepsilon_k} * u^\sigma\Big)\|^2_{L^2(\mathbb{R}^d)} \\
&\le \frac{\sigma}{12}\int_{\mathbb{R}^d} |\nabla D^\alpha \mathbf{u}|^2\,dx + C\|D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)}\|\nabla\Phi * V^{\varepsilon_k} * \mathbf{u}\|^2_{L^\infty(\mathbb{R}^d)} + C\|\mathbf{u}\|^2_{L^\infty(\mathbb{R}^d)}\|D^\alpha V^{\varepsilon_k} * \nabla\Phi * u^\sigma\|^2_{L^2(\mathbb{R}^d)} \\
&+ C\|D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)}\|\nabla\Phi * V^{\varepsilon_k} * u^\sigma\|^2_{L^\infty(\mathbb{R}^d)} + C\|\mathbf{u}\|^2_{L^\infty(\mathbb{R}^d)}\|D^\alpha V^{\varepsilon_k} * \nabla\Phi * u^\sigma\|^2_{L^2(\mathbb{R}^d)} \\
&\le \frac{\sigma}{12}\int_{\mathbb{R}^d} |\nabla D^\alpha \mathbf{u}|^2\,dx + C\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)}.
\end{aligned}$$

Now we estimate K_6 and K_7.

$$\begin{aligned}
K_6 &\le \frac{\sigma}{12}\int_{\mathbb{R}^d} |\nabla D^\alpha \mathbf{u}|^2\,dx + C\|D^\alpha(u^\sigma\nabla\Phi * V^{\varepsilon_k} * \mathbf{u})\|^2_{L^2(\mathbb{R}^d)} \\
&\le \frac{\sigma}{12}\int_{\mathbb{R}^d} |\nabla D^\alpha \mathbf{u}|^2\,dx + C\|D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)}\|\nabla\Phi * V^{\varepsilon_k} * \mathbf{u}\|^2_{L^\infty(\mathbb{R}^d)} + C\|u^\sigma\|^2_{L^\infty(\mathbb{R}^d)}\|D^\alpha V^{\varepsilon_k} * \nabla\Phi * \mathbf{u}\|^2_{L^2(\mathbb{R}^d)} \\
&\le \frac{\sigma}{12}\int_{\mathbb{R}^d} |\nabla D^\alpha \mathbf{u}|^2\,dx + C\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)}.
\end{aligned}$$

$$
\begin{aligned}
K_7 &= \int_{\mathbb{R}^d} D^\alpha\Big(u^\sigma(\nabla\Phi * V^{\varepsilon_k} * u^\sigma - \nabla\Phi * u^\sigma)\Big) \cdot \nabla D^\alpha \mathbf{u}\, dx \\
&\le \frac{\sigma}{12}\int_{\mathbb{R}^d} |\nabla D^\alpha \mathbf{u}|^2\, dx + C\|D^\alpha\Big(u^\sigma(\nabla\Phi * V^{\varepsilon_k} * u^\sigma - \nabla\Phi * u^\sigma)\Big)\|^2_{L^2(\mathbb{R}^d)} \\
&\le \frac{\sigma}{12}\int_{\mathbb{R}^d} |\nabla D^\alpha \mathbf{u}|^2\, dx + C\|D^\alpha \mathbf{u}\|^2_{L^2(\mathbb{R}^d)}\|\nabla\Phi * V^{\varepsilon_k} * u^\sigma - \nabla\Phi * u^\sigma\|^2_{L^\infty(\mathbb{R}^d)} \\
&+ C\|u^\sigma\|^2_{L^\infty(\mathbb{R}^d)}\|D^\alpha\Big(\nabla\Phi * V^{\varepsilon_k} * u^\sigma - \nabla\Phi * u^\sigma\Big)\|^2_{L^2(\mathbb{R}^d)} \\
&\le \frac{\sigma}{12}\int_{\mathbb{R}^d} |\nabla D^\alpha \mathbf{u}|^2\, dx + C\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)}.
\end{aligned}
$$

From estimates of $K_1 - K_7$ we obtain that

$$
\frac{d}{dt}\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)} + \frac{\sigma}{2}\|\nabla \mathbf{u}\|^2_{H^s(\mathbb{R}^d)} \le C\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)} + C(\varepsilon_k + \varepsilon_p)^2.
$$

By Grönwall's inequality it follows that

$$
\|\mathbf{u}(t)\|_{H^s(\mathbb{R}^d)} \le e^{CT} CT(\varepsilon_k + \varepsilon_p)^2, \ \forall t \in [0, T_*].
$$

Therefore, we can construct a solution $\mathbf{u}$ to the following partial differential equation

$$
\begin{aligned}
\partial_t \mathbf{u} - \sigma\Delta \mathbf{u} &= \nabla \cdot \Big((\mathbf{u} + u^\sigma)\nabla p_\lambda(V^{\varepsilon_p} * (\mathbf{u} + u^\sigma)) - u^\sigma \nabla p(u^\sigma)\Big) \\
&\quad - \nabla \cdot \Big((\mathbf{u} + u^\sigma)\mathbf{u}\nabla\Phi * V^{\varepsilon_k} * (\mathbf{u} + u^\sigma) - u^\sigma\nabla(\Phi * u^\sigma)\Big), \\
\mathbf{u}(0, x) &= 0.
\end{aligned}
$$

on $(0, T)$ for any $T < \infty$.

So, we obtain that

$$
\|\mathbf{u}\|^2_{L^\infty(0,T;H^s(\mathbb{R}^d))} \le C(\varepsilon_k + \varepsilon_p)^2, \ s > \frac{d}{2} + 2.
$$

Then by Sobolev's embedding theorem we get that

$$
\|u^\sigma - u^{\varepsilon,\sigma}\|_{L^\infty(0,T;W^{2,\infty}(\mathbb{R}^d))} \le C\|\mathbf{u}\|_{L^\infty(0,T;H^s(\mathbb{R}^d))} \le C(\varepsilon_k + \varepsilon_p).
$$

Chapter 3

Connection between PDEs and Particle Systems

In this chapter we explain why the systems of stochastic differential equations (1.12) and (1.14) have densities with respect to the Lebesgue measure and how to identify them with the solutions to the partial differential equations (1.13) and (1.15) respectively. We present the proof only for (1.14) and (1.15). The same arguments work for (1.12) and (1.13).

3.1 Preliminaries and Useful Tools

First, we need to derive some important estimates for Φ^ε.

Lemma 3.1. *There exist a positive constants $C > 0$ such that*

$$\|\nabla\Phi^\varepsilon\|_{L^\infty(\mathbb{R}^d)} \le \frac{C}{\varepsilon^{d-1}}, \qquad \|D^2\Phi^\varepsilon\|_{L^\infty(\mathbb{R}^d)} \le \frac{C}{\varepsilon^{d}}, \qquad \|D^3\Phi^\varepsilon\|_{L^\infty(\mathbb{R}^d)} \le \frac{C}{\varepsilon^{d+1}}.$$

Proof. Since $\nabla\Phi = C_d\frac{x}{|x|^d}$, we obtain that there exists a constant $C > 0$ such that

$$\begin{aligned}
\int_{|y|<\varepsilon} \frac{1}{|x-y|^{d-1}}dy &= \int_{\substack{|y|<\varepsilon \\ |x-y|>\varepsilon}} \frac{1}{|x-y|^{d-1}}dy + \int_{\substack{|y|<\varepsilon \\ |x-y|<\varepsilon}} \frac{1}{|x-y|^{d-1}}dy \\
&\le C\left(\frac{1}{\varepsilon^{d-1}}Vol(|y|<\varepsilon) + \int_{|x-y|<\varepsilon} \frac{1}{|x-y|^{d-1}}d(x-y)\right) \\
&\le C\left(\frac{\varepsilon^d}{\varepsilon^{d-1}} + Vol(B_1(0))\int_0^\varepsilon \frac{r^{d-1}}{r^{d-1}}dr\right) \\
&\le C\varepsilon.
\end{aligned}$$

So, we obtain that there exist a constant C such that

$$\begin{aligned}
\|\nabla\Phi^\varepsilon\|_{L^\infty(\mathbb{R}^d)} &= \left\|\int_{\mathbb{R}^d} \nabla\Phi(x-y)V^\varepsilon(y)dy\right\|_{L^\infty(\mathbb{R}^d)} \\
&\leq \left\|\int_{\mathbb{R}^d} \left|\frac{C_d}{|x-y|^{d-1}}V^\varepsilon(y)\right| dy\right\|_{L^\infty(\mathbb{R}^d)} \\
&= C_d\frac{1}{\varepsilon^d}\left\|\int_{|y|<\varepsilon} \frac{1}{|x-y|^{d-1}}\left|V\left(\frac{y}{\varepsilon}\right)\right| dy\right\|_{L^\infty(\mathbb{R}^d)} \\
&\leq C_d\frac{\|V\|_{L^\infty(\mathbb{R}^d)}}{\varepsilon^d}\int_{|y|<\varepsilon} \frac{1}{|x-y|^{d-1}}dy \\
&\leq C_d\frac{\|V\|_{L^\infty(\mathbb{R}^d)}}{\varepsilon^d}C\varepsilon \\
&\leq \frac{C}{\varepsilon^{d-1}},
\end{aligned}$$

$$\begin{aligned}
\|D^2\Phi^\varepsilon\|_{L^\infty(\mathbb{R}^d)} &\leq \left\|\int_{\mathbb{R}^d} \left|\frac{C_d}{|x-y|^{d-1}}\nabla V^\varepsilon(y)\right| dy\right\|_{L^\infty(\mathbb{R}^d)} \\
&= C_d\frac{1}{\varepsilon^{d+1}}\left\|\int_{|y|<\varepsilon} \frac{1}{|x-y|^{d-1}}\left|\nabla V\left(\frac{y}{\varepsilon}\right)\right| dy\right\|_{L^\infty(\mathbb{R}^d)} \\
&\leq C_d\frac{\|\nabla V\|_{L^\infty(\mathbb{R}^d)}}{\varepsilon^{d+1}}\int_{|y|<\varepsilon} \frac{1}{|x-y|^{d-1}}dy \\
&\leq C_d\frac{\|\nabla V\|_{L^\infty(\mathbb{R}^d)}}{\varepsilon^{d+1}}C\varepsilon \\
&\leq \frac{C}{\varepsilon^{d}},
\end{aligned}$$

$$\begin{aligned}
\|D^3\Phi^\varepsilon\|_{L^\infty(\mathbb{R}^d)} &\leq \left\|\int_{\mathbb{R}^d} \left|\frac{C_d}{|x-y|^{d-1}}D^2 V^\varepsilon(y)\right| dy\right\|_{L^\infty(\mathbb{R}^d)} \\
&= C_d\frac{1}{\varepsilon^{d+2}}\left\|\int_{|y|<\varepsilon} \frac{1}{|x-y|^{d-1}}\left|D^2 V\left(\frac{y}{\varepsilon}\right)\right| dy\right\|_{L^\infty(\mathbb{R}^d)} \\
&\leq C_d\frac{\|D^2 V\|_{L^\infty(\mathbb{R}^d)}}{\varepsilon^{d+2}}\int_{|y|<\varepsilon} \frac{1}{|x-y|^{d-1}}dy \\
&\leq C_d\frac{\|D^2 V\|_{L^\infty(\mathbb{R}^d)}}{\varepsilon^{d+2}}C\varepsilon \\
&\leq \frac{C}{\varepsilon^{d+1}}.
\end{aligned}$$

$\square$

3.1.1 Existence of Density Functions

For $i \in \{1, \dots, N\}$ consider

$$\begin{cases} d\hat{X}_t^{i,\sigma} = \nabla\Phi * w^\sigma(t, \hat{X}_t^{i,\sigma})dt - \nabla p(w^\sigma(t, \hat{X}_t^{i,\sigma}))dt + \sqrt{2\sigma}dB_t^i, \\ \hat{X}_0^{i,\sigma} = \zeta^i, \end{cases} \tag{3.1}$$

where w^σ is a weak solution to

$$\begin{cases} \partial_t w^\sigma = \sigma\Delta w^\sigma - \nabla \cdot (w^\sigma \nabla c^\sigma) + \nabla \cdot (w^\sigma \nabla p(w^\sigma)), \\ -\Delta c^\sigma = w^\sigma(t, x), \\ w^\sigma(0, x) = u_0^\sigma(x), \qquad x \in \mathbb{R}^d, t > 0. \end{cases} \tag{3.2}$$

Using Section 2.3.1 from [Nualart, 2006] we want to prove that (3.1) is absolutely continuos with respect to the Lebesgue measure. So, we need to ensure that $\nabla\Phi * u^\sigma(t, \cdot) - \nabla p(u^\sigma(t, \cdot))$ is globally Lipschitz continuous for any $t \in [0, T]$ and that $\nabla\Phi * u^\sigma(t, \cdot) - \nabla p(u^\sigma(t, \cdot))$ has at most linear growth for any $t \in [0, T]$.

Global Lipschitz continuity for all $x, y \in \mathbb{R}^d$,

$$\begin{aligned} &|\nabla\Phi * w^\sigma(t, x) - \nabla p(w^\sigma(t, x)) - (\nabla\Phi * w^\sigma(t, y) - \nabla p(w^\sigma(t, y)))| \\ &\le |\nabla\Phi * w^\sigma(t, x) - \nabla\Phi * w^\sigma(t, y)| + |\nabla p(w^\sigma(t, x)) - \nabla p(w^\sigma(t, y))| \\ &\le \Big(\|D^2\Phi * w^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} + \|p'\|_{L^\infty(0,\|w^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))})} \|D^2 w^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \\ &\quad + \|p''\|_{L^\infty(0,\|w^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))})} \|\nabla w^\sigma\|^2_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \Big) |x - y|. \end{aligned}$$

The boundedness for all $x \in \mathbb{R}^d$,

$$\begin{aligned} &|\nabla\Phi * w^\sigma(t, x) - \nabla p(w^\sigma(t, x))| \\ &\le \|\nabla\Phi * w^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} + \|\nabla p(w^\sigma)\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \\ &\le \|\nabla\Phi * w^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} + \|\nabla w^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \|p'\|_{L^\infty(0,\|w^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))})}. \end{aligned}$$

So we obtain that the solution of (3.1) has a density function $u^\sigma \in L^\infty(0, T; L^1(\mathbb{R}^d))$ with respect to Lebesgue measure on $\mathbb{R}^d \times \mathbb{R}_+$.

3.1.2 Connection between Density Functions and PDEs

Now we want to prove that u^σ can be identified with w^σ.

Let $\varphi \in C_0^\infty([0,T]\times\mathbb{R}^d)$. Using Itô's formula and taking the expectation of (3.1) we obtain that

$$\begin{aligned}\int_{\mathbb{R}^d}\varphi(t,x)u^\sigma(t,x)dx &= \int_{\mathbb{R}^d}\varphi(0,x)u_0^\sigma(t,x) + \int_{\mathbb{R}^d}\int_0^t \partial_s\varphi(s,x)u^\sigma(s,x)dsdx \\ &+ \int_{\mathbb{R}^d}\int_0^t \nabla\Phi * w^\sigma(s,x)\cdot\nabla\varphi(s,x)u^\sigma(s,x)dsdx \\ &- \int_{\mathbb{R}^d}\int_0^t \nabla p(w^\sigma(s,x))\cdot\nabla\varphi(s,x)u^\sigma(s,x)dsdx + \int_{\mathbb{R}^d}\int_0^t \sigma\Delta\varphi(s,x)u^\sigma(s,x)dsdx\end{aligned}$$

Therefore, $u^\sigma \in L^\infty(0,T;L^1(\mathbb{R}^d))$ is a weak solution to the linear partial differential equation

$$\begin{cases}\partial_t u^\sigma = \sigma\Delta u^\sigma - \nabla\cdot(u^\sigma\nabla\Phi * w^\sigma) + \nabla\cdot(u^\sigma\nabla p(w^\sigma)), \\ u^\sigma(0,x) = u_0^\sigma(x), \qquad x\in\mathbb{R}^d, t>0.\end{cases} \tag{3.3}$$

where $w^\sigma \in L^\infty(0,T;W^{1,\infty}(\mathbb{R}^d))$ is the unique solution to (3.2).

Now we prove that the weak solution to (3.3) is unique. We show that if $u_0^\sigma(x)\equiv 0$, then $u^\sigma(t,x)\equiv 0$ which implies uniqueness of the solution to (3.3).

Let $0 \le u^\sigma \in L^\infty(0,T;L^1(\mathbb{R}^d))$ be a solution to

$$\int_0^T\int_{\mathbb{R}^d} u^\sigma(\partial_t\varphi + \sigma\Delta\varphi + \nabla\Phi * w^\sigma\cdot\nabla\varphi - \nabla p(w^\sigma)\cdot\nabla\varphi)\,dx\,dt = 0, \qquad \forall\varphi\in C_0^\infty([0,T)\times\mathbb{R}^d). \tag{3.4}$$

First, we prove that $u^\sigma \in L^q((0,T)\times\mathbb{R}^d)$ for $q\in(1,\frac{d}{d-1})$. Fix $M>0$ and define $g:[0,\infty)\to[0,\infty)$ such that

$$g(v) := \begin{cases} v^{q-1}, \text{ for } 0\le v\le M^{\frac{1}{q-1}}, \\ M, \text{ for } v > M^{\frac{1}{q-1}}.\end{cases}$$

Since $u^\sigma \in L^\infty(0,T;L^1(\mathbb{R}^d))$, it follows that $g(u^\sigma)\in L^{\frac{1}{q-1}}((0,T)\times\mathbb{R}^d)$. Now consider a sequence of functions $(\bar g_n)_{n\in\mathbb{N}} \subset C_0^\infty((0,T)\times\mathbb{R}^d)$ such that

$$\bar g_n \to g(u^\sigma) \text{ in } L^{\frac{1}{q-1}}((0,T)\times\mathbb{R}^d), \tag{3.5}$$

$$\|\bar g_n\|_{L^\infty((0,T)\times\mathbb{R}^d)} \le \bar C,\ \forall n\in\mathbb{N}, \tag{3.6}$$

$$\bar g_n \overset{*}{\rightharpoonup} g(u^\sigma) \text{ in } L^\infty((0,T)\times\mathbb{R}^d) \tag{3.7}$$

where $\bar C$ is a positive constant which depends on M. From (3.5), (3.6) and interpolation inequality we obtain that

$$\bar g_n \to g(u^\sigma) \text{ in } L^r((0,T)\times\mathbb{R}^d) \text{ for } r\in[\frac{1}{q-1},\infty). \tag{3.8}$$

Let us consider the backward heat equation

$$\begin{aligned}
&\partial_t \varphi_n + \sigma \Delta \varphi_n = \bar{g}_n \text{ in } (0,T)\times\mathbb{R}^d,\\
&\varphi_n(\cdot,T) = 0 \text{ in } \mathbb{R}^d.
\end{aligned}$$

Since $C_0^\infty([0,T)\times\mathbb{R}^d)$ is dense in $\{\varphi \in C_b^{2,1}([0,T]\times\mathbb{R}^d) \mid \varphi(T,\cdot)\equiv 0\}$, it implies that (3.4) holds for any $\varphi \in C_b^{2,1}([0,T]\times\mathbb{R}^d)$ such that $\varphi(T,\cdot)\equiv 0$. Choosing φ_n in (3.4) and using Gagliardo–Nirenberg–Sobolev inequality we have that

$$\begin{aligned}
\int_0^T\int_{\mathbb{R}^d} u^\sigma \bar{g}_n \,dx\,dt &= \int_0^T\int_{\mathbb{R}^d} \left(-u^\sigma \nabla\Phi * w^\sigma\cdot\nabla\varphi_n + u^\sigma\nabla p(w^\sigma)\cdot\nabla\varphi_n\right)dx\,dt\\
&\le \int_0^T \|u^\sigma\|_{L^1(\mathbb{R}^d)}\|\nabla\varphi_n\|_{L^\infty(\mathbb{R}^d)}\Big(\|\nabla\Phi * w^\sigma\|_{L^\infty(\mathbb{R}^d)} + \|\nabla p(w^\sigma)\|_{L^\infty(\mathbb{R}^d)}\Big)\,dt\\
&\le C\int_0^T \|\nabla\varphi_n\|_{L^\infty(\mathbb{R}^d)}\,dt\\
&\le C\int_0^T \|\varphi_n\|^{1-\theta}_{L^{\frac{1}{q-1}}(\mathbb{R}^d)}\|D^2\varphi_n\|^{\theta}_{L^{\tilde{q}}(\mathbb{R}^d)}\,dt
\end{aligned}$$

where $\tilde{q} = \frac{q}{q-1} > d$, $\theta = \frac{q-1+\frac{1}{d}}{\frac{1}{q}+q+\frac{2}{d}-2} \in (\frac{1}{2},1)$ and C appeared in this subsection is independent of n and M.

Hölder's inequality yields that

$$\int_0^T\int_{\mathbb{R}^d} u^\sigma\bar{g}_n\,dx\,dt \le C\|\varphi_n\|^{1-\theta}_{L^{\frac{1}{q-1}}((0,T)\times\mathbb{R}^d)}\|D^2\varphi_n\|^{\theta}_{L^{\tilde{q}}((0,T)\times\mathbb{R}^d)}.$$

Then using parabolic regularity theory we have that

$$\int_0^T\int_{\mathbb{R}^d} u^\sigma\bar{g}_n\,dx\,dt \le C\|\bar{g}_n\|^{1-\theta}_{L^{\frac{1}{q-1}}((0,T)\times\mathbb{R}^d)}\|\bar{g}_n\|^{\theta}_{L^{\tilde{q}}((0,T)\times\mathbb{R}^d)}.$$

Taking $n\to\infty$ on both sides of the above inequality together with (3.7) and (3.8), we obtain that

$$\int_0^T\int_{\mathbb{R}^d} u^\sigma g(u^\sigma)\,dx\,dt \le C\|g(u^\sigma)\|^{1-\theta}_{L^{\frac{1}{q-1}}((0,T)\times\mathbb{R}^d)}\|g(u^\sigma)\|^{\theta}_{L^{\tilde{q}}((0,T)\times\mathbb{R}^d)} \le C\|g(u^\sigma)\|^{\theta}_{L^{\tilde{q}}((0,T)\times\mathbb{R}^d)}.$$

Young's inequality yields

$$\int_0^T\int_{\mathbb{R}^d} u^\sigma g(u^\sigma)\,dx\,dt \le C + \frac{1}{2}\int_0^T\int_{\mathbb{R}^d} |g(u^\sigma)|^{\tilde{q}}\,dx\,dt$$

From the definition of $g(u^\sigma)$ and $\tilde{q} = \frac{q}{q-1}$ we have that

$$\begin{aligned} 0 &\leq \int_{\{u^\sigma \leq M^{\frac{1}{q-1}}\}} (u^\sigma)^q \, dx \, dt + M \int_{\{u^\sigma > M^{\frac{1}{q-1}}\}} u^\sigma \, dx \, dt \\ &\leq C + \frac{1}{2} \int_{\{u^\sigma \leq M^{\frac{1}{q-1}}\}} (u^\sigma)^q \, dx \, dt + \frac{1}{2} M^{\frac{q}{q-1}} \int_{\{u^\sigma > M^{\frac{1}{q-1}}\}} 1 \, dx \, dt \end{aligned}$$

Moreover,

$$0 \leq \frac{1}{2} \int_{\{u^\sigma \leq M^{\frac{1}{q-1}}\}} (u^\sigma)^q \, dx \, dt \leq C + M \int_{\{u^\sigma > M^{\frac{1}{q-1}}\}} \Big(\frac{1}{2} M^{\frac{1}{q-1}} - u^\sigma \Big) \, dx \, dt \leq C.$$

Therefore, we are allowed us to take the limit $M \to \infty$ in the above inequality and so deduce

$$\|u^\sigma\|_{L^q((0,T)\times\mathbb{R}^d)} \leq C.$$

Let us now recall (3.4). By the density argument we have that (3.4) holds for $\varphi \in W^{2,1}_{\tilde{q}}((0,T)\times\mathbb{R}^d)$ for $\tilde{q} > d$ and $\varphi(\cdot, T) \equiv 0$ on $\mathbb{R}^d$. Now let $g \in C_0^\infty((0,T) \times \mathbb{R}^d)$ and consider

$$\begin{cases} \partial_t \varphi + \sigma \Delta \varphi + \nabla \Phi * w^\sigma \cdot \nabla \varphi - \nabla p(w^\sigma) \cdot \nabla \varphi = g \text{ in } (0,T) \times \mathbb{R}^d, \\ \varphi(T, \cdot) \equiv 0 \text{ in } \mathbb{R}^d. \end{cases}$$

Then we have that $\varphi \in W^{2,1}_{\tilde{q}}((0,T) \times \mathbb{R}^d)$ and therefore

$$\int_0^T \int_{\mathbb{R}^d} u^\sigma g \, dx \, dt = 0.$$

Since g is arbitrary, we get that

$$u^\sigma = 0 \text{ a.e. in } (0,T) \times \mathbb{R}^d.$$

Hence, $u^\sigma(t,x) = w^\sigma(t,x)$ a.e. in $(0,T) \times \mathbb{R}^d$.

3.2 Intermediate Particle Model vs. Final Particle Model

Step 1 Itô's formula

Since the diffusion coefficient $\sqrt{2\sigma}$ is the same one for (1.12) and (1.14), using Itô's formula we deduce that

$$\begin{aligned} \left| \bar{X}_t^{i,\varepsilon,\sigma} - \hat{X}_t^{i,\sigma} \right|^2 &= \int_0^t 2(\bar{X}_s^{i,\varepsilon,\sigma} - \hat{X}_s^{i,\sigma}) \Big(-\nabla p_\lambda \Big(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma}) \Big) + \nabla p \Big(u^\sigma(s, \hat{X}_s^{i,\sigma}) \Big) \Big) \, ds \\ &\quad + \int_0^t 2(\bar{X}_s^{i,\varepsilon,\sigma} - \hat{X}_s^{i,\sigma}) \Big(\nabla \Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma}) - \nabla \Phi * u^\sigma(s, \hat{X}_s^{i,\sigma}) \Big) \, ds. \end{aligned}$$

Step 2 Estimates of the aggregation term

Using triangle inequality, we obtain that

$$\begin{aligned}|\nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}) - \nabla\Phi * u^{\sigma}(s,\hat{X}_s^{i,\sigma})| &\leq |\nabla\Phi * u^{\sigma}(s,\hat{X}_s^{i,\sigma}) - \nabla\Phi^{\varepsilon_k} * u^{\sigma}(s,\hat{X}_s^{i,\sigma})| \\ &+ |\nabla\Phi^{\varepsilon_k} * u^{\sigma}(s,\hat{X}_s^{i,\sigma}) - \nabla\Phi^{\varepsilon_k} * u^{\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})| \\ &+ |\nabla\Phi^{\varepsilon_k} * u^{\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}) - \nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})|\end{aligned}$$

Step 2(a) Estimation of $|\nabla\Phi * u^{\sigma}(s,\hat{X}_s^{i,\sigma}) - \nabla\Phi^{\varepsilon_k} * u^{\sigma}(s,\hat{X}_s^{i,\sigma})|$

Since

$$\begin{aligned}&|\nabla\Phi * u^{\sigma}(s,\hat{X}_s^{i,\sigma}) - \nabla\Phi^{\varepsilon_k} * u^{\sigma}(s,\hat{X}_s^{i,\sigma})| \\ &= \left|\nabla\Phi * u^{\sigma}(s,\hat{X}_s^{i,\sigma}) - (V^{\varepsilon_k} * (\nabla\Phi * u^{\sigma}))(s,\hat{X}_s^{i,\sigma})\right| \\ &= \left|\int_{\mathbb{R}^d} V^{\varepsilon_k}(y)(\nabla\Phi * u^{\sigma}(s,\hat{X}_s^{i,\sigma}) - \nabla\Phi * u^{\sigma}(s,\hat{X}_s^{i,\sigma} - y))dy\right| \\ &= \left|\int_{\mathbb{R}^d} V^{\varepsilon_k}(y)|y| \left(\frac{\nabla\Phi * u^{\sigma}(s,\hat{X}_s^{i,\sigma}) - \nabla\Phi * u^{\sigma}(s,\hat{X}_s^{i,\sigma} - y)}{|y|}\right) dy\right|,\end{aligned}$$

we obtain

$$\begin{aligned}|\nabla\Phi * u^{\sigma}(s,\hat{X}_s^{i,\sigma}) - \nabla\Phi^{\varepsilon_k} * u^{\sigma}(s,\hat{X}_s^{i,\sigma})| &\leq \|D^2\Phi * u^{\sigma}(s,\cdot)\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} V^{\varepsilon_k}(y)|y|dy \\ &\leq \|\Phi * D^2 u^{\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\varepsilon_k \int_{\mathbb{R}^d} V(x)|x|dx.\end{aligned}$$

Step 2(b) Estimation of $|\nabla\Phi^{\varepsilon_k} * u^{\sigma}(s,\hat{X}_s^{i,\sigma}) - \nabla\Phi^{\varepsilon_k} * u^{\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})|$

Here we take the L^∞ norm out and deduce that

$$\begin{aligned}&|\nabla\Phi^{\varepsilon_k} * u^{\sigma}(s,\hat{X}_s^{i,\sigma}) - \nabla\Phi^{\varepsilon_k} * u^{\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})| \\ &= |\Phi * \nabla u^{\sigma} * V^{\varepsilon_k}(s,\hat{X}_s^{i,\sigma}) - \nabla\Phi * u^{\sigma} * V^{\varepsilon_k}(s,\bar{X}_s^{i,\varepsilon,\sigma})| \\ &\leq \|\Phi * D^2 u^{\sigma} * V^{\varepsilon_k}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \left|\bar{X}_s^{i,\varepsilon,\sigma} - \hat{X}_s^{i,\sigma}\right| \\ &\leq \|\Phi * D^2 u^{\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \left|\bar{X}_s^{i,\varepsilon,\sigma} - \hat{X}_s^{i,\sigma}\right|\end{aligned}$$

Step 2(c) Estimation of $|\nabla\Phi^{\varepsilon_k} * u^{\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}) - \nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})|$

We estimate this term directly with the L^∞ norm

$$\begin{aligned}|\nabla\Phi^{\varepsilon_k} * u^{\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}) - \nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})| &\leq \|\Phi * V^{\varepsilon_k} * (\nabla(u^{\sigma} - u^{\varepsilon,\sigma}))\,\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \\ &\leq \|\Phi * (\nabla(u^{\sigma} - u^{\varepsilon,\sigma}))\,\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\end{aligned}$$

Combining steps $2(a) - 2(c)$, we obtain that

$$\begin{aligned}
&\left|\int_0^t 2(\bar{X}_s^{i,\varepsilon,\sigma} - \hat{X}_s^{i,\sigma})\Big(\nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}) - \nabla\Phi * u^\sigma(s,\hat{X}_s^{i,\sigma})\Big)\,ds\right| \\
&\leq 2\|\Phi * D^2 u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\varepsilon_k \int_{\mathbb{R}^d} V(x)|x|dx \int_0^t \left|\bar{X}_s^{i,\varepsilon,\sigma} - \hat{X}_s^{i,\sigma}\right| ds \\
&+ 2\|\Phi * D^2 u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \int_0^t \left|\bar{X}_s^{i,\varepsilon,\sigma} - \hat{X}_s^{i,\sigma}\right|^2 ds \\
&+ 2\|\Phi * \Big(\nabla(u^\sigma - u^{\varepsilon,\sigma})\Big)\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \int_0^t \left|\bar{X}_s^{i,\varepsilon,\sigma} - \hat{X}_s^{i,\sigma}\right| ds
\end{aligned} \tag{3.9}$$

Step 3 Estimates of the diffusion term

First, we use triangle inequality as we did for the aggregation term and so get that

$$\begin{aligned}
\left|\nabla p\Big(u^\sigma(s,\hat{X}_s^{i,\sigma})\Big) - \nabla p_\lambda\Big(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\right| &\leq \left|\nabla p\Big(u^\sigma(s,\hat{X}_s^{i,\sigma})\Big) - \nabla p\Big(u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\right| \\
&+ \left|\nabla p\Big(u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big) - \nabla p\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\right| \\
&+ \left|\nabla p\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big) - \nabla p_\lambda\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\right| \\
&+ \left|\nabla p_\lambda\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big) - \nabla p_\lambda\Big(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\right|
\end{aligned}$$

Step 3(a) Estimation of $\left|\nabla p\Big(u^\sigma(s,\hat{X}_s^{i,\sigma})\Big) - \nabla p\Big(u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\right|$

Since

$$\begin{aligned}
&\nabla p\Big(u^\sigma(s,\hat{X}_s^{i,\sigma})\Big) - \nabla p\Big(u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big) \\
&= p'\Big(u^\sigma(s,\hat{X}_s^{i,\sigma})\Big)\nabla u^\sigma(s,\hat{X}_s^{i,\sigma}) - p'\Big(u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\nabla u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma}) \\
&= p'\Big(u^\sigma(s,\hat{X}_s^{i,\sigma})\Big)\nabla u^\sigma(s,\hat{X}_s^{i,\sigma}) - p'\Big(u^\sigma(s,\hat{X}_s^{i,\sigma})\Big)\nabla u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma}) \\
&+ p'\Big(u^\sigma(s,\hat{X}_s^{i,\sigma})\Big)\nabla u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma}) - p'\Big(u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\nabla u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma}) \\
&= p'\Big(u^\sigma(s,\hat{X}_s^{i,\sigma})\Big)\Big(\nabla u^\sigma(s,\hat{X}_s^{i,\sigma}) - \nabla u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big) \\
&+ \left(p'\Big(u^\sigma(s,\hat{X}_s^{i,\sigma})\Big) - p'\Big(u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\right)\nabla u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma}),
\end{aligned}$$

we obtain that

$$\begin{aligned}
\left|\nabla p\Big(u^\sigma(s,\hat{X}_s^{i,\sigma})\Big) - \nabla p\Big(u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\right| &\leq \left|p'\Big(u^\sigma(s,\hat{X}_s^{i,\sigma})\Big)\Big(\nabla u^\sigma(s,\hat{X}_s^{i,\sigma}) - \nabla u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\right| \\
&+ \left|\Big(p'\Big(u^\sigma(s,\hat{X}_s^{i,\sigma})\Big) - p'\Big(u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\Big)\,\nabla u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\right|.
\end{aligned}$$

Now we study all terms from the above inequality

- $\left|p'\Big(u^\sigma(s,\hat{X}_s^{i,\sigma})\Big)\right|$ is bounded by $\|p'\|_{L^\infty(0,\|u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))})}$
- $\left|\nabla u^\sigma(s,\hat{X}_s^{i,\sigma}) - \nabla u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\right| \leq \|D^2 u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}|\hat{X}_s^{i,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}|$
- $\left|\nabla u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\right| \leq \|\nabla u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}$

- With the help of the mean value theorem we obtain that

$$\begin{aligned}\Big|p'\Big(u^\sigma(s,\hat{X}_s^{i,\sigma})\Big)-p'\Big(u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\Big| &\le \|p''\|_{L^\infty(0,\|u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))})}\Big|u^\sigma(s,\hat{X}_s^{i,\sigma})-u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big| \\ &\le \|p''\|_{L^\infty(0,\|u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))})}\|\nabla u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}|\hat{X}_s^{i,\sigma}-\bar{X}_s^{i,\varepsilon,\sigma}|\end{aligned}$$

Therefore,

$$\begin{aligned}\Big|\nabla p\Big(u^\sigma(s,\hat{X}_s^{i,\sigma})\Big)-\nabla p\Big(u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\Big| &\le \|p'\|_{L^\infty(0,\|u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))})}\|D^2u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}|\hat{X}_s^{i,\sigma}-\bar{X}_s^{i,\varepsilon,\sigma}| \\ &\quad+\|p''\|_{L^\infty(0,\|u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))})}\|\nabla u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}|\hat{X}_s^{i,\sigma}-\bar{X}_s^{i,\varepsilon,\sigma}| \\ &= C_{3a}|\hat{X}_s^{i,\sigma}-\bar{X}_s^{i,\varepsilon,\sigma}|,\end{aligned}$$

where

$$C_{3a}=\|p'\|_{L^\infty(0,\|u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))})}\|D^2u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}+\|p''\|_{L^\infty(0,\|u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))})}\|\nabla u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}.$$

Step 3(b) Estimation of $\Big|\nabla p\Big(u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)-\nabla p\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\Big|$

Using similar computation as in Step 3(a), we obtain that

$$\begin{aligned}&\nabla p\Big(u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)-\nabla p\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\\ &=p'\Big(u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\nabla u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})-p'\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\\ &=p'\Big(u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\nabla u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})-p'\Big(u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\\ &+p'\Big(u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})-p'\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\\ &=p'\Big(u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\Big(\nabla u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})-\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\\ &+\bigg(p'\Big(u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)-p'\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\bigg)\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}).\end{aligned}$$

This implies that

$$\begin{aligned}\Big|\nabla p\Big(u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)-\nabla p\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\Big| &\le \Big|p'\Big(u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\Big(\nabla u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})-\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\Big| \\ &\quad+\bigg|\bigg(p'\Big(u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)-p'\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\bigg)\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\bigg|.\end{aligned}$$

Since

- $\Big|p'\Big(u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\Big|\le\|p'\|_{L^\infty(0,\|u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))})}$,
- $\Big|\nabla u^\sigma(s,\bar{X}_s^{i,\varepsilon,\sigma})-\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big|\le\|\nabla\left(u^\sigma-u^{\varepsilon,\sigma}\right)\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}$,

- by the mean value theorem

$$
\begin{aligned}
&\Big|p'\Big(u^{\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big) - p'\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\Big| \\
&\leq \|p''\|_{L^\infty(0,\max\{\|u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))},\|u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\})}\Big|u^{\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}) - u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big| \\
&\leq \|p''\|_{L^\infty(0,\max\{\|u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))},\|u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\})}\|u^\sigma - u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))},
\end{aligned}
$$

- $|\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})| \leq \|\nabla u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}$,

we get that

$$
\begin{aligned}
&\Big|\nabla p\Big(u^{\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big) - \nabla p\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\Big| \\
&\leq \|p'\|_{L^\infty(0,\|u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))})}\|\nabla\left(u^\sigma - u^{\varepsilon,\sigma}\right)\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \\
&+ \|\nabla u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\|p''\|_{L^\infty(0,\max\{\|u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))},\|u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\})}\|u^\sigma - u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \\
&=: C_{3b}
\end{aligned}
$$

Step 3(c) Estimation of $\Big|\nabla p\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big) - \nabla p_\lambda\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\Big|$

Since

$$
\begin{aligned}
&\nabla p\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big) - \nabla p_\lambda\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big) \\
&= p'\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}) - p'_\lambda\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}) \\
&= \left(p'\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big) - p'_\lambda\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\right)\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}),
\end{aligned}
$$

and

- $\Big|p'\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big) - p'_\lambda\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\Big| = 0$ by the definition of p_λ if $\|u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \leq \frac{1}{\lambda}$,
- $\Big|\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big| \leq \|\nabla u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}$,

we obtain that

$$
\Big|\nabla p\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big) - \nabla p_\lambda\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\Big| = 0 \quad \text{if} \quad \|u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \leq \frac{1}{\lambda}
$$

Step 3(d) Estimation of $\Big|\nabla p_\lambda\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big) - \nabla p_\lambda\Big(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\Big|$

Since

$$\begin{aligned}
&\nabla p_\lambda\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)-\nabla p_\lambda\Big(V^{\varepsilon_p}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\\
&=p'_\lambda\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})-p'_\lambda\Big(V^{\varepsilon_p}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)V^{\varepsilon_p}*\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\\
&=p'_\lambda\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})-p'_\lambda\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)V^{\varepsilon_p}*\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\\
&+p'_\lambda\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)V^{\varepsilon_p}*\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})-p'_\lambda\Big(V^{\varepsilon_p}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)V^{\varepsilon_p}*\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}),
\end{aligned}$$

we obtain that

$$\begin{aligned}
&\nabla p_\lambda\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)-\nabla p_\lambda\Big(V^{\varepsilon_p}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\\
&=p'_\lambda\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\Big(\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})-V^{\varepsilon_p}*\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\\
&+\Bigg(p'_\lambda\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)-p'_\lambda\Big(V^{\varepsilon_p}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\Bigg)V^{\varepsilon_p}*\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})
\end{aligned}$$

Now we study all terms from the above inequality as we did before

- $p'_\lambda\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\le\|p'\|_{L^\infty(0,\|u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))})}$.
- Since $\int_{\mathbb{R}^d}V^{\varepsilon_p}(y)dy=1$, we obtain that

$$\begin{aligned}
&\Big|\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})-V^{\varepsilon_p}*\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big|\\
&=\Big|\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})-\int_{\mathbb{R}^d}V^{\varepsilon_p}(y)\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}-y)dy\Big|\\
&=\Big|\int_{\mathbb{R}^d}V^{\varepsilon_p}(y)\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})dy-\int_{\mathbb{R}^d}V^{\varepsilon_p}(y)\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}-y)dy\Big|\\
&=\Big|\int_{\mathbb{R}^d}V^{\varepsilon_p}(y)\Big(\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})-\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}-y)\Big)dy\Big|\\
&=\Big|\int_{\mathbb{R}^d}V^{\varepsilon_p}(y)\Big(\frac{\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})-\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}-y)}{|y|}\Big)|y|dy\Big|.
\end{aligned}$$

  Therefore, it yields

$$\begin{aligned}
\Big|\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})-V^{\varepsilon_p}*\nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big|&\le\|D^2u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\int_{\mathbb{R}^d}V^{\varepsilon_p}(y)|y|dy\\
&=\|D^2u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\varepsilon_p\int_{\mathbb{R}^d}\frac{1}{\varepsilon^d}V(\frac{y}{\varepsilon_p})|\frac{y}{\varepsilon_p}|dy\\
&=\|D^2u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\varepsilon_p\int_{\mathbb{R}^d}V(x)|x|dx.
\end{aligned}$$

- With the help of the mean value theorem we obtain that

$$\begin{aligned}
&\Big|p'_\lambda\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)-p'_\lambda\Big(V^{\varepsilon_p}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\Big|\\
&\le\|p''\|_{L^\infty(0,\|u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))})}\Big|u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})-V^{\varepsilon_p}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big|.
\end{aligned}$$

Since $\int_{\mathbb{R}^d} V^{\varepsilon_p}(y)dy = 1$, we obtain that

$$\begin{aligned}
&\Big|u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}) - V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big| \\
&= \Big|u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}) - \int_{\mathbb{R}^d} V^{\varepsilon_p}(y) u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma} - y)dy\Big| \\
&= \Big|\int_{\mathbb{R}^d} V^{\varepsilon_p}(y) u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})dy - \int_{\mathbb{R}^d} V^{\varepsilon_p}(y) u_s^{\varepsilon,\sigma}(\bar{X}_s^{i,\varepsilon,\sigma} - y)dy\Big| \\
&= \Big|\int_{\mathbb{R}^d} V^{\varepsilon_p}(y)\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}) - u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma} - y)\Big)dy\Big|.
\end{aligned}$$

This implies that

$$\begin{aligned}
\Big|u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}) - V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big| &\le \|\nabla u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \int_{\mathbb{R}^d} V^{\varepsilon_p}(y)|y|dy \\
&= \|\nabla u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \varepsilon_p \int_{\mathbb{R}^d} \frac{1}{\varepsilon_p^d} V(\frac{y}{\varepsilon_p})|\frac{y}{\varepsilon_p}|dy \\
&= \|\nabla u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \varepsilon_p \int_{\mathbb{R}^d} V(x)|x|dx.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\Big|p'_\lambda\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big) - p'_\lambda\Big(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\Big| \\
&\le \|p''\|_{L^\infty(0,\|u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))})} \|\nabla u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \varepsilon_p \int_{\mathbb{R}^d} V(x)|x|dx.
\end{aligned}$$

- $\Big|V^{\varepsilon_p} * \nabla u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big| \le \|\nabla u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}$.

Therefore, we obtain that

$$\begin{aligned}
&\Big|\nabla p_\lambda\Big(u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big) - \nabla p_\lambda\Big(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\Big| \\
&\le \|p'\|_{L^\infty(0,\|u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))})} \|D^2 u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \varepsilon_p \int_{\mathbb{R}^d} V(x)|x|dx \\
&+ \|p''\|_{L^\infty(0,\|u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))})} \|\nabla u^{\varepsilon,\sigma}\|^2_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \varepsilon_p \int_{\mathbb{R}^d} V(x)|x|dx =: C_{3d}
\end{aligned}$$

Combining all previous estimates from steps $3(a) - 3(d)$ we deduce that

$$\begin{aligned}
&\mathbb{E} \sup_{i=1,\dots,N} \Big[\int_0^t 2(\bar{X}_s^{i,\varepsilon,\sigma} - \hat{X}_s^{i,\sigma})\bigg(-\nabla p_\lambda\Big(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big) + \nabla p\Big(u^\sigma(s,\hat{X}_s^{i,\sigma})\Big)\bigg)\, ds\Big] \\
&\le 2C_{3a}\int_0^t \mathbb{E}\Big[\sup_{i=1,\dots,N} |\bar{X}_s^{i,\varepsilon,\sigma} - \hat{X}_s^{i,\sigma}|^2\Big]\, ds \\
&+ 2C_{3b}\int_0^t \mathbb{E}\Big[\sup_{i=1,\dots,N} |\bar{X}_s^{i,\varepsilon,\sigma} - \hat{X}_s^{i,\sigma}|\Big] ds \\
&+ 2C_{3d}\int_0^t \mathbb{E}\Big[\sup_{i=1,\dots,N} |\bar{X}_s^{i,\varepsilon,\sigma} - \hat{X}_s^{i,\sigma}|\Big] ds,
\end{aligned}$$

where

$$
\begin{aligned}
C_{3a} &= \|p'\|_{L^\infty(0,\|u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))})}\|D^2u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} + \|p''\|_{L^\infty(0,\|u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))})}\|\nabla u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))},\\
C_{3b} &= \|p'\|_{L^\infty(0,\|u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))})}\|\nabla\left(u^\sigma - u^{\varepsilon,\sigma}\right)\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\\
&\quad + \|\nabla u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\|p''\|_{L^\infty(0,\max\{\|u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))},\|u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\})}\|u^\sigma - u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))},\\
C_{3d} &= \|p'\|_{L^\infty(0,\|u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))})}\|D^2u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\varepsilon_p\int_{\mathbb{R}^d}V(x)|x|dx\\
&\quad + \|p''\|_{L^\infty(0,\|u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))})}\|\nabla u^{\varepsilon,\sigma}\|^2_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\varepsilon_p\int_{\mathbb{R}^d}V(x)|x|dx.
\end{aligned}
$$

Step 4 Estimation of $\mathbb{E}\left[\sup_{i=1,\dots,N}\left|\bar{X}_t^{i,\varepsilon,\sigma} - \hat{X}_t^{i,\sigma}\right|^2\right]$

Since the following conditions are true

- $\|u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}, \|\nabla u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}, \|D^2u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} < \infty$
- $\|u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}, \|\nabla u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}, \|D^2u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} < \infty$
- $\exists C > 0$ such that
  - $\|u^\sigma - u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \le C(\varepsilon_k + \varepsilon_p)$,
  - $\|\nabla\left(u^\sigma - u^{\varepsilon,\sigma}\right)\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \le C(\varepsilon_k + \varepsilon_p)$,
  - $\|\Phi * \left(\nabla(u^\sigma - u^{\varepsilon,\sigma})\right)\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \le C(\varepsilon_k + \varepsilon_p)$
- $\|p'_\lambda\|_{L^\infty(0,\|u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))})}, \|p''_\lambda\|_{L^\infty(0,\|u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))})} < \infty$
- $\|u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \le \frac{1}{\lambda}$
- $\|\Phi * D^2u^\sigma\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} < \infty$,

combining the estimates of Step 2 and Step 3, we obtain that there exists a constant $C > 0$ such that

$$
\begin{aligned}
\mathbb{E}\left[\sup_{i=1,\dots,N}\left|\bar{X}_t^{i,\varepsilon,\sigma} - \hat{X}_t^{i,\sigma}\right|^2\right] &\le C\int_0^t \mathbb{E}\left[\sup_{i=1,\dots,N}\left|\bar{X}_s^{i,\varepsilon,\sigma} - \hat{X}_s^{i,\sigma}\right|^2\right] ds\\
&\quad + C(\varepsilon_k + \varepsilon_p)\int_0^t \mathbb{E}\left[\sup_{i=1,\dots,N}\left|\bar{X}_s^{i,\varepsilon,\sigma} - \hat{X}_s^{i,\sigma}\right|\right] ds.
\end{aligned}
$$

Using Young's inequality we obtain that

$$
\mathbb{E}\left[\sup_{i=1,\dots,N}\left|\bar{X}_t^{i,\varepsilon,\sigma} - \hat{X}_t^{i,\sigma}\right|^2\right] \le C\int_0^t \mathbb{E}\left[\sup_{i=1,\dots,N}\left|\bar{X}_s^{i,\varepsilon,\sigma} - \hat{X}_s^{i,\sigma}\right|^2\right] ds + C(\varepsilon_k + \varepsilon_p)^2.
$$

Grönwall's inequality yields

$$
\sup_{t\in[0,T]}\mathbb{E}\left[\sup_{i=1,\dots,N}\left|\bar{X}_t^{i,\varepsilon,\sigma} - \hat{X}_t^{i,\sigma}\right|^2\right] \le C(\varepsilon_k + \varepsilon_p)^2\ \exp\Big(CT\Big).
$$

Chapter 4

Regularized Particle Model vs. Intermediate Particle Model

4.1 Convergence in Expectation

In this section we prove Theorem 1.4. Since the diffusion coefficient $\sqrt{2\sigma}$ is the same one for (1.11) and (1.12), using Itô's formula we deduce that

$$
\begin{aligned}
&\left|X_t^{N,i,\varepsilon,\sigma} - \bar{X}_t^{i,\varepsilon,\sigma}\right|^2 = \\
&\int_0^t 2(X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma})\Big(\frac{1}{N}\sum_{j=1}^N \nabla\Phi^{\varepsilon_k}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma}) - \nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma})\Big) ds \\
&+ \int_0^t 2(X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma})\Big(- \nabla p_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma})\Big) + \nabla p_\lambda\Big(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma})\Big)\Big)\, ds
\end{aligned}
$$

4.1.1 Estimates of the Aggregation Term

First, we rewrite $\frac{1}{N}\sum_{j=1}^N \nabla\Phi^{\varepsilon_k}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma}) - \nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma})$ as

$$
\begin{aligned}
&\frac{1}{N}\sum_{j=1}^N \nabla\Phi^{\varepsilon_k}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma}) - \nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma}) \\
&= \frac{1}{N}\sum_{j=1}^N \nabla\Phi^{\varepsilon_k}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma}) - \frac{1}{N}\sum_{j=1}^N \nabla\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) \\
&+ \frac{1}{N}\sum_{j=1}^N \nabla\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) - \nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma}) \\
&=: A_1 + A_2
\end{aligned}
$$

Now we handle A_1

$$
\begin{aligned}
|A_1| &= \left| \frac{1}{N}\sum_{j=1}^{N} \nabla\Phi^{\varepsilon_k}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma}) - \frac{1}{N}\sum_{j=1}^{N} \nabla\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) \right| \\
&= \frac{1}{N}\left| \sum_{j=1}^{N} \Big(\nabla\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) - \nabla\Phi^{\varepsilon_k}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma}) \Big) \right|.
\end{aligned}
$$

Since by Lemma 3.1 there exists a constant $C > 0$ such that

$$
\|D^2\Phi^{\varepsilon_k}\|_{L^\infty(\mathbb{R}^d)} \le \frac{C}{\varepsilon_k^d},
$$

we obtain that

$$
|A_1| \le \frac{C}{N\varepsilon_k^d}\sum_{j=1}^{N} \Big(|\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}| + |X_s^{N,j,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}| \Big)
$$

Combining estimates for $|A_1|$ and Young's inequality yields

$$
\begin{aligned}
&2\Big|(X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma})\Big(\frac{1}{N}\sum_{j=1}^{N} \nabla\Phi^{\varepsilon_k}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma}) - \nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma})\Big)\Big| \\
&\le 2\left|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}\right| \frac{C}{N\varepsilon_k^d}\sum_{j=1}^{N}\Big(\left|\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}\right| + \left|X_s^{N,j,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}\right| \Big) + 2\left|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}\right| \; |A_2| \\
&= \frac{2C}{\varepsilon_k^d}\left|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}\right|^2 + \frac{2C}{N\varepsilon_k^d}\sum_{j=1}^{N}\left|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}\right| \; \left|X_s^{N,j,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}\right| + 2\left|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}\right| \; |A_2| \\
&\le \frac{2C}{\varepsilon_k^d}\left|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}\right|^2 + \frac{C}{\varepsilon_k^d}\left|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}\right|^2 + \frac{C}{N\varepsilon_k^d}\sum_{j=1}^{N}\left|X_s^{N,j,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}\right|^2 \\
&+ 2\kappa\left|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}\right|^2 + \frac{1}{2\kappa}|A_2|^2 \\
&= \frac{3C}{\varepsilon_k^d}\left|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}\right|^2 + \frac{C}{N\varepsilon_k^d}\sum_{j=1}^{N}\left|X_s^{N,j,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}\right|^2 + 2\kappa\left|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}\right|^2 + \frac{1}{2\kappa}|A_2|^2 \\
&= \Big(\frac{3C}{\varepsilon_k^d} + 2\kappa\Big)\left|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}\right|^2 + \frac{C}{N\varepsilon_k^d}\sum_{j=1}^{N}\left|X_s^{N,j,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}\right|^2 + \frac{1}{2\kappa}|A_2|^2.
\end{aligned}
$$

The constant κ will be determined later. After taking the expected value of both sides we get that

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^t 2(X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma})\Big(\frac{1}{N}\sum_{j=1}^{N} \nabla\Phi^{\varepsilon_k}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma}) - \nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma})\Big) ds\right] \\
&\le \int_0^t \Big(\frac{3C}{\varepsilon_k^d} + 2\kappa\Big)\mathbb{E}\left[\left|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}\right|^2\right] ds + \int_0^t \frac{C}{N\varepsilon_k^d}\sum_{j=1}^{N}\mathbb{E}\left[\left|X_s^{N,j,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}\right|^2\right] ds \\
&+ \int_0^t \frac{1}{2\kappa}\mathbb{E}[|A_2|^2] ds.
\end{aligned}
$$

Now we handle the term $|A_2|^2$. For simplicity we introduce

$$\hat{Z}_{ij} := \nabla\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) - \nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma}).$$

Therefore, we obtain that

$$\begin{aligned}|A_2|^2 &= \left|\frac{1}{N}\sum_{j=1}^{N}\left(\nabla\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) - \nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma})\right)\right|^2 \\ &= \left|\frac{1}{N}\sum_{j=1}^{N}\hat{Z}_{ij}\right|^2 = \frac{1}{N^2}\sum_{j=1}^{N}\sum_{k=1}^{N}\hat{Z}_{ij}\cdot\hat{Z}_{ik}\end{aligned}$$

For $i \neq j$ we know that $(B_t^i)_{t\geq 0}$ and $(B_t^j)_{t\geq 0}$ are independent Brownian motions. Recall that ζ_i and ζ_j are independent and identically distributed. Therefore, we obtain that $\bar{X}_s^{i,\varepsilon,\sigma}$ and $\bar{X}_s^{j,\varepsilon,\sigma}$ are also independent and identically distributed. It implies that

$$\begin{aligned}\mathbb{E}[\hat{Z}_j] &= \mathbb{E}[\nabla\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) - \nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma})] \\ &= \int_{\mathbb{R}^{2d}} \nabla\Phi^{\varepsilon_k}(x-y)\, u^{\varepsilon,\sigma}(s,x)u^{\varepsilon,\sigma}(s,y)dx\,dy - \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\nabla\Phi^{\varepsilon_k}(x-y)\, u^{\varepsilon,\sigma}(s,x)dx\right) u^{\varepsilon,\sigma}(s,y)dy \\ &= \int_{\mathbb{R}^d}(\int_{\mathbb{R}^d}\nabla\Phi^{\varepsilon_k}(x-y)\, u^{\varepsilon,\sigma}(s,x)dx)\, u^{\varepsilon,\sigma}(s,y)dy - \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\nabla\Phi^{\varepsilon_k}(x-y)\, u^{\varepsilon,\sigma}(s,x)dx\right) u^{\varepsilon,\sigma}(s,y)dy \\ &= 0.\end{aligned}$$

Recall that by Lemma 3.1 there exists a constant $C > 0$ such that

$$\|\nabla\Phi^{\varepsilon_k}\|_{L^\infty(\mathbb{R}^d)} \leq \frac{C}{\varepsilon_k^{d-1}}.$$

Moreover, for $j \neq k$, $j \neq i$ and $k \neq i$ it holds that

$$\begin{aligned}&\mathbb{E}[\hat{Z}_{ij}\cdot\hat{Z}_{ik}] \\ &= \mathbb{E}[(\nabla\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) - \nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma}))\cdot(\nabla\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{k,\varepsilon,\sigma}) - \nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma}))] \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\nabla\Phi^{\varepsilon_k}(x-y) - \nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s,x)) \\ &\qquad\cdot(\nabla\Phi^{\varepsilon_k}(x-z) - \nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s,x))u^{\varepsilon,\sigma}(s,x)u^{\varepsilon,\sigma}(s,y)u^{\varepsilon,\sigma}(s,z)\,dx\,dy\,dz \\ &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}(\nabla\Phi^{\varepsilon_k}(x-y) - \nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s,x))u^{\varepsilon,\sigma}(s,y)dy\Big) \\ &\quad\cdot\Big(\int_{\mathbb{R}^d}(\nabla\Phi^{\varepsilon_k}(x-z) - \nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s,x))u^{\varepsilon,\sigma}(s,z)dz\Big)\cdot u^{\varepsilon,\sigma}(s,x)\,dx \\ &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}\nabla\Phi^{\varepsilon_k}(x-y)u^{\varepsilon,\sigma}(s,y)dy - \int_{\mathbb{R}^d}\nabla\Phi^{\varepsilon_k}(x-w)u^{\varepsilon,\sigma}(s,w)dw\Big) \\ &\quad\cdot\Big(\int_{\mathbb{R}^d}\nabla\Phi^{\varepsilon_k}(x-z)u^{\varepsilon,\sigma}(s,z)dz - \int_{\mathbb{R}^d}\nabla\Phi^{\varepsilon_k}(x-v)u^{\varepsilon,\sigma}(s,v)dv\Big)\cdot u^{\varepsilon,\sigma}(s,x)\,dx \\ &= 0.\end{aligned}$$

Since $\nabla\Phi^{\varepsilon_k}(0)=0$, we get for $j=i$ and $k\neq i$ that

$$\begin{aligned}
&\mathbb{E}[\hat{Z}_{ii}\cdot\hat{Z}_{ik}]\\
&=\mathbb{E}[(-\nabla\Phi^{\varepsilon_k}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}))\cdot(\nabla\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{k,\varepsilon,\sigma})-\nabla\Phi^{\varepsilon_k}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}))]\\
&=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(-\nabla\Phi^{\varepsilon_k}*u^{\varepsilon,\sigma}(s,x))\ \cdot(\nabla\Phi^{\varepsilon_k}(x-z)-\nabla\Phi^{\varepsilon_k}*u^{\varepsilon,\sigma}(s,x))u^{\varepsilon,\sigma}(s,x)u^{\varepsilon,\sigma}(s,z)\,dx\,dz\\
&=\int_{\mathbb{R}^d}\Big(-\int_{\mathbb{R}^d}\nabla\Phi^{\varepsilon_k}(x-w)u^{\varepsilon,\sigma}(s,w)dw\Big)\\
&\quad\cdot\Big(\int_{\mathbb{R}^d}\nabla\Phi^{\varepsilon_k}(x-z)u^{\varepsilon,\sigma}(s,z)dz-\int_{\mathbb{R}^d}\nabla\Phi^{\varepsilon_k}(x-v)u^{\varepsilon,\sigma}(s,v)dv\Big)\cdot u^{\varepsilon,\sigma}(s,x)\,dx\\
&=0.
\end{aligned}$$

For $k=i$ and $j\neq i$ we have also that

$$\mathbb{E}[\hat{Z}_{ij}\cdot\hat{Z}_{ii}]=0.$$

For $k=i$ and $j=i$ we obtain that

$$\begin{aligned}
&\mathbb{E}[\hat{Z}_{ii}\cdot\hat{Z}_{ii}]\\
&=\mathbb{E}[(-\nabla\Phi^{\varepsilon_k}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}))\cdot(-\nabla\Phi^{\varepsilon_k}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}))]\\
&\leq\mathbb{E}[|\nabla\Phi^{\varepsilon_k}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})|^2]
\end{aligned}$$

Therefore,

$$\mathbb{E}[|A_2|^2]=\frac{1}{N^2}\mathbb{E}\left[\sum_{j=1}^N\sum_{k=1}^N\hat{Z}_{ij}\cdot\hat{Z}_{ik}\right]\leq\frac{N-1}{N^2}\max_{j\in\{1,\ldots,N\}}\mathbb{E}[|\hat{Z}_{ij}|^2]+\frac{1}{N^2}\mathbb{E}[|\hat{Z}_{ii}|^2]\leq\frac{C}{N\varepsilon_k^{2(d-1)}}\tag{4.1}$$

since for an arbitrary $j\in\{1,\ldots,N\}$ there exists a constant $C>0$ such that the following inequality holds:

$$\begin{aligned}
\hat{Z}_{ij}^2&=\Big(\nabla\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})-\nabla\Phi^{\varepsilon_k}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)^2\\
&\leq2|\nabla\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})|^2+2|\nabla\Phi^{\varepsilon_k}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})|^2\\
&\leq2\|\nabla\Phi^{\varepsilon_k}\|_{L^\infty(\mathbb{R}^d)}^2+2\left(\|u_0\|_{L^1(\mathbb{R}^d)}\|\nabla\Phi^{\varepsilon_k}\|_{L^\infty(\mathbb{R}^d)}\right)^2\\
&\leq C(\|u_0\|_{L^1(\mathbb{R}^d)}^2+1)\left(\frac{\varepsilon_k^2}{\varepsilon_k^{d+1}}\right)^2\\
&\leq\frac{C}{\varepsilon_k^{2(d-1)}}.
\end{aligned}$$

So, we obtain that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t 2(X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma})\Big(\frac{1}{N}\sum_{j=1}^N \nabla\Phi^{\varepsilon_k}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma}) - \nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)ds\right] \\
&\le \int_0^t \Big(\frac{3C}{\varepsilon_k^d} + 2\kappa\Big) \sup_{i=1,\dots,N} \mathbb{E}\left[\left|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}\right|^2\right] ds + \int_0^t \frac{C}{\varepsilon_k^d} \sup_{i=1,\dots,N} \mathbb{E}\left[|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}|^2\right] ds \\
&+ \frac{Ct}{2\kappa N \varepsilon_k^{2(d-1)}} \\
&= \Big(\frac{4C}{\varepsilon_k^d} + 2\kappa\Big)\int_0^t \sup_{i=1,\dots,N} \mathbb{E}\left[\left|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}\right|^2\right] ds + \frac{Ct}{2\kappa N \varepsilon_k^{2(d-1)}}
\end{aligned}$$

Choosing $\kappa = \frac{1}{2\varepsilon_k^d}$ we see that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t 2(X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma})\Big(\frac{1}{N}\sum_{j=1}^N \nabla\Phi^{\varepsilon_k}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma}) - \nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)ds\right] \\
&\le \frac{4C+1}{\varepsilon_k^d}\int_0^t \sup_{i=1,\dots,N} \mathbb{E}\left[\left|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}\right|^2\right] ds + \frac{Ct}{N\varepsilon_k^{d-2}}
\end{aligned} \tag{4.2}$$

4.1.2 Estimates of the Diffusion Term

First, we rewrite $-\nabla p_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma})\Big) + \nabla p_\lambda\Big(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)$ as

$$\begin{aligned}
&-\nabla p_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma})\Big) + \nabla p_\lambda\Big(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big) \\
&= \underbrace{-\nabla p_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma})\Big) + \nabla p_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big)}_{=:I_1} \\
&\underbrace{-\nabla p_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big) + \nabla p_\lambda\Big(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)}_{=:I_2} \\
&= I_1 + I_2.
\end{aligned}$$

Estimation of I_1

Applying chain rule we deduce that

$$\begin{aligned}
I_1 &= \nabla p_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big) - \nabla p_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma})\Big) \\
&= p_\lambda'\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big)\Big(\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big) \\
&- p_\lambda'\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma})\Big)\Big(\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma})\Big).
\end{aligned}$$

Since we can write I_1 as

$$\begin{aligned}
I_1 &= p_\lambda'\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\Big)\Big(\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\Big)\\
&- p_\lambda'\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma}-X_s^{N,j,\varepsilon,\sigma})\Big)\Big(\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\Big)\\
&+ p_\lambda'\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma}-X_s^{N,j,\varepsilon,\sigma})\Big)\Big(\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\Big)\\
&- p_\lambda'\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma}-X_s^{N,j,\varepsilon,\sigma})\Big)\Big(\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma}-X_s^{N,j,\varepsilon,\sigma})\Big),
\end{aligned}$$

we can factor out $\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma}) =: Q_1$ and $p_\lambda'\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma}-X_s^{N,j,\varepsilon,\sigma})\Big) =: Q_2$ in order to obtain that

$$\begin{aligned}
I_1 = &\left(p_\lambda'\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\Big) - p_\lambda'\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma}-X_s^{N,j,\varepsilon,\sigma})\Big)\right)Q_1\\
&+ Q_2\left(\Big(\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\Big) - \Big(\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma}-X_s^{N,j,\varepsilon,\sigma})\Big)\right).
\end{aligned}$$

Applying the mean value theorem we get that

$$\begin{aligned}
&\left|p_\lambda'\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\Big) - p_\lambda'\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma}-X_s^{N,j,\varepsilon,\sigma})\Big)\right|\\
&\le \frac{\|\nabla V\|_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{d+1}}\ \|p_\lambda''\|_{L^\infty(0,\|V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)})}\frac{1}{N}\sum_{j=1}^N \left|(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})-(X_s^{N,i,\varepsilon,\sigma}-X_s^{N,j,\varepsilon,\sigma})\right|\\
&\le \frac{\|\nabla V\|_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{d+1}}\ \|p_\lambda''\|_{L^\infty(0,\|V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)})}\frac{1}{N}\Big(N\left|\bar{X}_s^{i,\varepsilon,\sigma}-X_s^{N,i,\varepsilon,\sigma}\right| + \sum_{j=1}^N\left|X_s^{N,j,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma}\right|\Big)\\
&= \frac{\|\nabla V\|_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{d+1}}\ \|p_\lambda''\|_{L^\infty(0,\|V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)})}\Big(\left|\bar{X}_s^{i,\varepsilon,\sigma}-X_s^{N,i,\varepsilon,\sigma}\right| + \frac{1}{N}\sum_{j=1}^N\left|X_s^{N,j,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma}\right|\Big).
\end{aligned}$$

Now we study other terms from I_1.

- $|Q_1| = \left|\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\right|$ is bounded by $\frac{\|\nabla V\|_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{d+1}}$.
- $Q_2 = p_\lambda'\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma}-X_s^{N,j,\varepsilon,\sigma})\Big)$ is bounded by $\|p_\lambda'\|_{L^\infty(0,\|V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)})}$.
- The term $\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma}) - \frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma}-X_s^{N,j,\varepsilon,\sigma})$ is estimated by

$$\begin{aligned}
&\frac{1}{N}\frac{\|D^2V\|_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{d+2}}\sum_{j=1}^N |(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})-(X_s^{N,i,\varepsilon,\sigma}-X_s^{N,j,\varepsilon,\sigma})|\\
&\le \frac{\|D^2V\|_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{d+2}}\Big(\left|\bar{X}_s^{i,\varepsilon,\sigma}-X_s^{N,i,\varepsilon,\sigma}\right| + \frac{1}{N}\sum_{j=1}^N\left|X_s^{N,j,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma}\right|\Big)
\end{aligned}$$

Therefore, we obtain that

$$|I_1| \leq C_{I_1}\Big(\Big|\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}\Big| + \frac{1}{N}\sum_{j=1}^{N}\Big|X_s^{N,j,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}\Big|\Big),$$

where

$$C_{I_1} = \left(\frac{\|\nabla V\|_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{d+1}}\right)^2 \ \|p_\lambda''\|_{L^\infty(0,\|V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)})} + \|p_\lambda'\|_{L^\infty(0,\|V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)})}\frac{\|D^2 V\|_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{d+2}}.$$

So,

$$\begin{aligned}
&\int_0^t 2(X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}) I_1\, ds \\
&\leq 2C_{I_1}\int_0^t \Big|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}\Big|\Big(\Big|\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}\Big| + \frac{1}{N}\sum_{j=1}^{N}\Big|X_s^{N,j,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}\Big|\Big)ds \\
&\leq 2C_{I_1}\int_0^t \left(\Big|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}\Big|^2 + \frac{1}{N}\Big(\frac{N}{2}\Big|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}\Big|^2 + \frac{1}{2}\sum_{j=1}^{N}\Big|X_s^{N,j,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}\Big|^2\Big)\right)ds \\
&= 3C_{I_1}\int_0^t \Big|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}\Big|^2 ds + \frac{C_{I_1}}{N}\int_0^t \sum_{j=1}^{N}\Big|X_s^{N,j,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}\Big|^2 ds,
\end{aligned}$$

which yields

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t 2(X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}) I_1\, ds\right] \\
&\leq \mathbb{E}\left[3C_{I_1}\int_0^t \Big|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}\Big|^2 ds + \frac{C_{I_1}}{N}\int_0^t \sum_{j=1}^{N}\Big|X_s^{N,j,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}\Big|^2 ds\right] \\
&\leq 3C_{I_1}\int_0^t \mathbb{E}\left[\Big|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}\Big|^2\right] ds + \frac{C_{I_1}}{N}\int_0^t \mathbb{E}\left[\sum_{j=1}^{N}\Big|X_s^{N,j,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}\Big|^2\right] ds \\
&\leq 4C_{I_1}\int_0^t \sup_{i=1,\dots,N}\mathbb{E}\left[\Big|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}\Big|^2\right] ds.
\end{aligned}$$

Estimation of I_2

Next, we study I_2.

$$
\begin{aligned}
I_2 &= -\nabla p_\lambda\Big(\frac{1}{N}\sum_{j=1}^{N} V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big) + \nabla p_\lambda\Big(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma})\Big) \\
&= -p'_\lambda\Big(\frac{1}{N}\sum_{j=1}^{N} V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big)\Big(\frac{1}{N}\sum_{j=1}^{N} \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big) \\
&\quad + p'_\lambda\Big(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma})\Big)\Big(\nabla V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma})\Big) \\
&= p'_\lambda\Big(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma})\Big)\frac{1}{N}\sum_{j=1}^{N} \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) \\
&\quad - p'_\lambda\Big(\frac{1}{N}\sum_{j=1}^{N} V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big)\Big(\frac{1}{N}\sum_{j=1}^{N} \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big) \\
&\quad + p'_\lambda\Big(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma})\Big)\Big(\nabla V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma})\Big) \\
&\quad - p'_\lambda\Big(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma})\Big)\frac{1}{N}\sum_{j=1}^{N} \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})
\end{aligned}
$$

Then we split I_2 and get that

$$
\begin{aligned}
I_2 &= \Bigg(p'_\lambda(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma}) - p'_\lambda\Big(\frac{1}{N}\sum_{j=1}^{N} V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big)\Bigg)\frac{1}{N}\sum_{j=1}^{N} \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) \\
&+ \Bigg(\Big(\nabla V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma})\Big) - \frac{1}{N}\sum_{j=1}^{N} \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Bigg)p'_\lambda\Big(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma})\Big) \\
&= I_{2,1} + I_{2,2}.
\end{aligned}
$$

<u>*Estimation of* $\mathbb{E}\Big[\int_0^t 2(\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}) I_{2,1}\, ds\Big]$</u>

First, we estimate $\Big|\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}\Big|\ |I_{2,1}|$ as follows

$$
\begin{aligned}
&\Big|\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}\Big|\ |I_{2,1}| \\
&= \Big|\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}\Big|\Bigg|p'_\lambda(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma})) - p'_\lambda\Big(\frac{1}{N}\sum_{j=1}^{N} V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big)\Bigg| \\
&\quad \cdot \frac{1}{N}\Bigg|\sum_{j=1}^{N} \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Bigg| \\
&\leq \Big|\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}\Big| \frac{\|\nabla V\|_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{d+1}} \|p''_\lambda\|_{L^\infty(0,\|V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)})} \\
&\quad \cdot \Bigg|V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma}) - \frac{1}{N}\sum_{j=1}^{N} V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Bigg|
\end{aligned}
$$

Next, we define $Z_{ij} := V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma}) - V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})$ and obtain that

$$\left|\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}\right| |I_{2,1}| \leq \frac{\|\nabla V\|_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{d+1}} \|p_\lambda''\|_{L^\infty(0,\|V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)})} \left|\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}\right| \frac{1}{N}\Big|\sum_{j=1}^N Z_{ij}\Big|.$$

If we multiply the last term by $\sqrt{2\mu}\frac{1}{\sqrt{2\mu}}$ and apply Young's inequality, we obtain that

$$\begin{aligned}
&\left|\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}\right| \; |I_{2,1}| \\
&\leq \mu \frac{\|\nabla V\|^2_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{2(d+1)}} \|p_\lambda''\|^2_{L^\infty(0,\|V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)})} \left|\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}\right|^2 + \frac{1}{4\mu}\frac{1}{N^2}\sum_{j=1}^N\sum_{k=1}^N Z_{ij}Z_{ik},
\end{aligned}$$

since $\left|\sum_{j=1}^N Z_{ij}\right|^2 = (\sum_{j=1}^N Z_{ij})^2 = \sum_{j=1}^N\sum_{k=1}^N Z_{ij}Z_{ik}$.

Similarly as in (4.1) we deduce that

$$\mathbb{E}\left[\sum_{j=1}^N\sum_{k=1}^N Z_{ij}Z_{ij}\right] \leq CN\frac{\|V\|^2_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{2d}},$$

where we use the following estimate. For an arbitrary $j \in \{1, \ldots, N\}$ it holds that:

$$\begin{aligned}
Z_{ij}^2 &= \left(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma}) - V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\right)^2 \\
&\leq 2|V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma})|^2 + 2|V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})|^2 \\
&\leq C\frac{\|V\|^2_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{2d}}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^t 2(\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}) I_{2,1}\, ds\Big] \\
&\leq 2\mu\frac{\|\nabla V\|^2_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{2(d+1)}} \|p_\lambda''\|^2_{L^\infty(0,\|V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)})} \int_0^t \mathbb{E}\Big[\left|\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}\right|^2\Big]\, ds + \frac{1}{4\mu}\frac{1}{N^2}\int_0^t \mathbb{E}\Big[\sum_{j=1}^N\sum_{k=1}^N Z_{ij}Z_{ik}\Big]\, ds \\
&\leq 2\mu\frac{\|\nabla V\|^2_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{2(d+1)}} \|p_\lambda''\|^2_{L^\infty(0,\|V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)})} \int_0^t \mathbb{E}\Big[\left|\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}\right|^2\Big]\, ds + \frac{1}{4\mu}\frac{1}{N^2}CN\frac{\|V\|^2_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{2d}}\int_0^t 1\, ds \\
&= 2\mu\frac{\|\nabla V\|^2_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{2(d+1)}} \|p_\lambda''\|^2_{L^\infty(0,\|V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)})} \int_0^t \mathbb{E}\Big[\left|\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}\right|^2\Big]\, ds + \frac{C}{\mu}\frac{1}{N}\frac{\|V\|^2_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{2d}} t.
\end{aligned}$$

Now we choose $\mu = \frac{\varepsilon_p^{2(d+1)}}{2\|\nabla V\|^2_{L^\infty(\mathbb{R}^d)}\|p_\lambda''\|^2_{L^\infty(0,\|V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)})}}$ and obtain that

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^t 2(\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}) I_{2,1}\, ds\Big] \\
&\leq \int_0^t \mathbb{E}\Big[\left|\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}\right|^2\Big]\, ds + C\frac{\|\nabla V\|^2_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{2(2d+1)}} \|p_\lambda''\|^2_{L^\infty(0,\|V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)})} \frac{1}{N}\|V\|^2_{L^\infty(\mathbb{R}^d)} t.
\end{aligned}$$

Estimation of $\mathbb{E}\Big[\int_0^t 2(\bar{X}_s^{i,\varepsilon,\sigma}-X_s^{N,i,\varepsilon,\sigma})I_{2,2}\,ds\Big]$

The procedure is similar to the one we used for the estimation of $\mathbb{E}\Big[\int_0^t 2(\bar{X}_s^{i,\varepsilon,\sigma}-X_s^{N,i,\varepsilon,\sigma})I_{2,1}\,ds\Big]$. First, we observe that

$$\begin{aligned}
&\Big|\bar{X}_s^{i,\varepsilon,\sigma}-X_s^{N,i,\varepsilon,\sigma}\Big|\ |I_{2,2}|\\
&=\Big|\bar{X}_s^{i,\varepsilon,\sigma}-X_s^{N,i,\varepsilon,\sigma}\Big|\Big|\Big(\nabla V^{\varepsilon_p}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)-\frac{1}{N}\sum_{j=1}^N\nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\Big|\Big|p_\lambda'\Big(V^{\varepsilon_p}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\Big|\\
&\le\|p_\lambda'\|_{L^\infty(0,\|u^{\varepsilon,\sigma}\|_{L^\infty((0,T)\times\mathbb{R}^d)})}\Big|\bar{X}_s^{i,\varepsilon,\sigma}-X_s^{N,i,\varepsilon,\sigma}\Big|\ \frac{1}{N}\Big|\sum_{j=1}^N\Big(\nabla V^{\varepsilon_p}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})-\nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\Big)\Big|.
\end{aligned}$$

Now we define $\tilde{Z}_{ij}:=\nabla V^{\varepsilon_p}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})-\nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})$ and get that

$$\Big|\bar{X}_s^{i,\varepsilon,\sigma}-X_s^{N,i,\varepsilon,\sigma}\Big|\ |I_{2,2}|\le\|p_\lambda'\|_{L^\infty(0,\|u^{\varepsilon,\sigma}\|_{L^\infty((0,T)\times\mathbb{R}^d)})}\Big|\bar{X}_s^{i,\varepsilon,\sigma}-X_s^{N,i,\varepsilon,\sigma}\Big|\ \frac{1}{N}\Big|\sum_{j=1}^N\tilde{Z}_{ij}\Big|.$$

As we did before, we multiply the last term by $\sqrt{2\tilde{\mu}}\frac{1}{\sqrt{2\tilde{\mu}}}$, apply Young's inequality and deduce that

$$\Big|\bar{X}_s^{i,\varepsilon,\sigma}-X_s^{N,i,\varepsilon,\sigma}\Big|\ |I_{2,2}|\le\tilde{\mu}\|p_\lambda'\|^2_{L^\infty(0,\|u^{\varepsilon,\sigma}\|_{L^\infty((0,T)\times\mathbb{R}^d)})}\Big|\bar{X}_s^{i,\varepsilon,\sigma}-X_s^{N,i,\varepsilon,\sigma}\Big|^2+\frac{1}{4\tilde{\mu}}\frac{1}{N^2}\sum_{j=1}^N\sum_{j=k}^N\tilde{Z}_{ij}\cdot\tilde{Z}_{ik},$$

since $\Big|\sum_{j=1}^N\tilde{Z}_{ij}\Big|^2=(\sum_{j=1}^N\tilde{Z}_{ij})\cdot(\sum_{j=1}^N\tilde{Z}_{ij})=\sum_{j=1}^N\sum_{k=1}^N\tilde{Z}_{ij}\cdot\tilde{Z}_{ik}$.

Now we analyze the term $\sum_{j=1}^N\sum_{j=k}^N\tilde{Z}_{ij}\cdot\tilde{Z}_{ik}$. By using the same technique as in (4.1) we get that

$$\mathbb{E}\left[\sum_{j=1}^N\sum_{k=1}^N\tilde{Z}_{ij}\ \cdot\tilde{Z}_{ik}\right]\le CN\frac{\|\nabla V\|^2_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{2(d+1)}},$$

where we use the following estimate. For an arbitrary $j\in\{1,\dots,N\}$ the following inequality holds:

$$\begin{aligned}
|\tilde{Z}_{ij}|^2&=\Big|\nabla V^{\varepsilon_p}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})-\nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{,\varepsilon,\sigma})\Big|^2\\
&\le 2|\nabla V^{\varepsilon_p}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})|^2+2|\nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})|^2\\
&\le C\frac{\|\nabla V\|^2_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{2(d+1)}}.
\end{aligned}$$

Consequently, we obtain that

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^t 2(\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}) I_{2,2}\, ds\Big] \\
&\leq 2\tilde{\mu}\|p'_\lambda\|^2_{L^\infty(0,\|u^{\varepsilon,\sigma}\|_{L^\infty((0,T)\times\mathbb{R}^d)})} \int_0^t \mathbb{E}\Big[\Big|\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}\Big|^2\Big]\, ds + \frac{1}{4\tilde{\mu}}\frac{1}{N^2}\int_0^t \mathbb{E}\Big[\sum_{j=1}^N\sum_{k=1}^N \tilde{Z}_{ij}\cdot\tilde{Z}_{ik}\Big]\, ds \\
&\leq 2\tilde{\mu}\|p'_\lambda\|^2_{L^\infty(0,\|u^{\varepsilon,\sigma}\|_{L^\infty((0,T)\times\mathbb{R}^d)})} \int_0^t \mathbb{E}\Big[\Big|\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}\Big|^2\Big]\, ds + \frac{1}{4\tilde{\mu}}\frac{1}{N^2} CN \frac{\|\nabla V\|^2_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{2(d+1)}}\int_0^t 1\, ds \\
&= 2\tilde{\mu}\|p'_\lambda\|^2_{L^\infty(0,\|u^{\varepsilon,\sigma}\|_{L^\infty((0,T)\times\mathbb{R}^d)})} \int_0^t \mathbb{E}\Big[\Big|\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}\Big|^2\Big]\, ds + \frac{C}{\tilde{\mu}}\frac{1}{N}\frac{\|\nabla V\|^2_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{2(d+1)}} t.
\end{aligned}
$$

Now we choose $\tilde{\mu} = \frac{1}{2\|p'_\lambda\|^2_{L^\infty(0,\|u^{\varepsilon,\sigma}\|_{L^\infty((0,T)\times\mathbb{R}^d)})}}$ and so get that

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^t 2(\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}) I_{2,2}\, ds\Big] \\
&\leq \int_0^t \mathbb{E}\Big[\Big|\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}\Big|^2\Big]\, ds + 2\|p'_\lambda\|^2_{L^\infty(0,\|u^{\varepsilon,\sigma}\|_{L^\infty((0,T)\times\mathbb{R}^d)})}\frac{1}{N}\frac{\|\nabla V\|^2_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{2(d+1)}} t
\end{aligned}
$$

Combining the estimates for $\mathbb{E}\Big[\int_0^t 2(\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}) I_{2,1}\, ds\Big]$ and $\mathbb{E}\Big[\int_0^t 2(\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}) I_{2,2}\, ds\Big]$ together we obtain that

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^t 2(\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}) I_2\, ds\Big] \\
&= \mathbb{E}\Big[\int_0^t 2(\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}) I_{2,1}\, ds\Big] + \mathbb{E}\Big[\int_0^t 2(\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}) I_{2,2}\, ds\Big] \\
&\leq 2\int_0^t \mathbb{E}\Big[\Big|\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}\Big|^2\Big]\, ds + C_{I_2},
\end{aligned}
$$

where

$$
\begin{aligned}
C_{I_2} &= 2\frac{\|\nabla V\|^2_{L^\infty(\mathbb{R}^d)}}{N\varepsilon_p^{2(2d+1)}}\|p''_\lambda\|^2_{L^\infty(0,\|V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)})}\|V\|^2_{L^\infty(\mathbb{R}^d)} t + 2\|p'_\lambda\|^2_{L^\infty(0,\|u^{\varepsilon,\sigma}\|_{L^\infty((0,T)\times\mathbb{R}^d)})}\frac{\|\nabla V\|^2_{L^\infty(\mathbb{R}^d)}}{N\varepsilon_p^{2(d+1)}} t \\
&= \frac{2t\|\nabla V\|^2_{L^\infty(\mathbb{R}^d)}}{N\varepsilon_p^{2(d+1)}}\Bigg(\frac{1}{\varepsilon_p^{2d}}\|p''_\lambda\|^2_{L^\infty(0,\|V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)})}\|V\|^2_{L^\infty(\mathbb{R}^d)} + \|p'_\lambda\|^2_{L^\infty(0,\|u^{\varepsilon,\sigma}\|_{L^\infty((0,T)\times\mathbb{R}^d)})}\Bigg).
\end{aligned}
$$

Estimates for the whole diffusion term

From estimates for $\mathbb{E}\left[\int_0^t 2(X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma})I_1\, ds\right]$ and $\mathbb{E}\left[\int_0^t 2(\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma})I_2\, ds\right]$ it follows that

$$\mathbb{E}\Big[\int_0^t 2(X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma})\Big(-\nabla p_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma})\Big) + \nabla p_\lambda\Big(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma})\Big)\Big)\, ds\Big] \tag{4.3}$$

$$\begin{aligned}
&\leq 4C_{I_1}\int_0^t \sup_{i=1,\dots,N}\mathbb{E}\Big[\Big|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}\Big|^2\Big]ds + 2\int_0^t \mathbb{E}\Big[\Big|\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}\Big|^2\Big]\, ds + C_{I_2}\\
&= (4C_{I_1}+2)\int_0^t \sup_{i=1,\dots,N}\mathbb{E}\Big[\Big|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}\Big|^2\Big]ds + C_{I_2},
\end{aligned}$$

4.1.3 Grönwall's Argument

Since we take $\frac{2}{\lambda} \geq \frac{1}{\varepsilon_p^d}$, we obtain estimates of (4.2) and (4.3) which are independent of λ, i.e.

$$\begin{aligned}
\sup_{i=1,\dots,N}\mathbb{E}\left[\left|X_t^{N,i,\varepsilon,\sigma} - \bar{X}_t^{i,\varepsilon,\sigma}\right|^2\right] &\leq \left(\frac{4C+1}{\varepsilon_k^d}\right)\int_0^t \sup_{i=1,\dots,N}\mathbb{E}\left[\left|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}\right|^2\right]ds + \frac{Ct}{N\varepsilon_k^{d-2}}\\
&\quad + (4C_{I_1}+2)\int_0^t \sup_{i=1,\dots,N}\mathbb{E}\left[\left|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}\right|^2\right]ds + C_{I_2}\\
&\leq C_{G_1}\int_0^t \sup_{i=1,\dots,N}\mathbb{E}\left[\left|X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma}\right|^2\right]ds + C_{G_2},
\end{aligned}$$

where

$$\begin{aligned}
C_{G_1} &= \frac{4C+1}{\varepsilon_k^d} + 4C_{I_1} + 2\\
&= \frac{4C+1}{\varepsilon_k^d} + 4\left(\frac{\|\nabla V\|_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{d+1}}\right)^2 \|p''\|_{L^\infty(0,\|V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)})} + 4\|p'\|_{L^\infty(0,\|V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)})}\frac{\|D^2V\|_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{d+2}} + 2
\end{aligned}$$

and

$$\begin{aligned}
C_{G_2} &= \frac{Ct}{N\varepsilon_k^{d-2}} + C_{I_2}\\
&= \frac{Ct}{N\varepsilon_k^{d-2}} + \frac{2t\|\nabla V\|^2_{L^\infty(\mathbb{R}^d)}}{N\varepsilon_p^{2(d+1)}}\left(\frac{1}{\varepsilon_p^{2d}}\|p''\|^2_{L^\infty(0,\|V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)})}\|V\|^2_{L^\infty(\mathbb{R}^d)} + \|p'\|^2_{L^\infty(0,\|u^{\varepsilon,\sigma}\|_{L^\infty((0,T)\times\mathbb{R}^d)})}\right)
\end{aligned}$$

Now we apply Grönwall's inequality, which results in

$$\sup_{t\in[0,T]}\sup_{i=1,\dots,N}\mathbb{E}\left[\left|X_t^{N,i,\varepsilon,\sigma} - \bar{X}_t^{i,\varepsilon,\sigma}\right|^2\right] \leq C_{G_2}\ \exp\Big(C_{G_1}T\Big).$$

Since

$$\|p'\|_{L^\infty(0,\|V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)})} \le C(m)\|V^{\varepsilon_p}\|^{m-2}_{L^\infty(\mathbb{R}^d)} = \begin{cases} C, \text{ for } m=2, \\ C\frac{\|V\|^{m-2}_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{d(m-2)}}, \text{ for } m \ge 3, \end{cases}$$

$$\|p''\|_{L^\infty(0,\|V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)})} \le C(m)\|V^{\varepsilon_p}\|^{m-3}_{L^\infty(\mathbb{R}^d)} = \begin{cases} 0, \text{ for } m=2, \\ C\frac{\|V\|^{m-3}_{L^\infty(\mathbb{R}^d)}}{\varepsilon_p^{d(m-3)}}, \text{ for } m \ge 3, \end{cases}$$

we obtain that

$$\begin{aligned} C_{G_1} &\le C\bigg(1 + \frac{1}{\varepsilon_k^d} + \frac{1}{\varepsilon_p^{2(d+1)}}\ \frac{1}{\varepsilon_p^{d(m-3)}} + \frac{1}{\varepsilon_p^{d(m-2)}}\frac{1}{\varepsilon_p^{d+2}}\bigg) \le C\bigg(\frac{1}{\varepsilon_k^d} + \frac{1}{\varepsilon_p^{dm-d+2}}\bigg) \\ C_{G_2} &\le C\bigg(\frac{t}{N\varepsilon_k^{d-2}} + \frac{t}{N\varepsilon_p^{2(d+1)}}\bigg(\frac{1}{\varepsilon_p^{2d}}\frac{1}{\varepsilon_p^{2d(m-3)}} + 1\bigg)\bigg) \le C\bigg(\frac{t}{N\varepsilon_k^{d-2}} + \frac{t}{N\varepsilon_p^{2dm-2d+2}}\bigg) \end{aligned}$$

for $m \ge 3$ or $m = 2$. Therefore,

$$\sup_{t\in[0,T]}\sup_{i=1,\dots,N} \mathbb{E}\left[\left|X_t^{N,i,\varepsilon,\sigma} - \bar{X}_t^{i,\varepsilon,\sigma}\right|^2\right] \le C(T)\bigg(\frac{1}{N\varepsilon_k^{d-2}} + \frac{1}{N\varepsilon_p^{2dm-2d+2}}\bigg)\ \exp\bigg(\tilde{C}(T)\bigg(\frac{1}{\varepsilon_k^d} + \frac{1}{\varepsilon_p^{dm-d+2}}\bigg)\bigg).$$

It remains to prove that for $0 < \beta < 1$ it holds that

$$C(T)\bigg(\frac{1}{N\varepsilon_k^{d-2}} + \frac{1}{N\varepsilon_p^{2dm-2d+2}}\bigg)\ \exp\bigg(\tilde{C}(T)\bigg(\frac{1}{\varepsilon_k^d} + \frac{1}{\varepsilon_p^{dm-d+2}}\bigg)\bigg) \sim O(N^{-\beta}) \text{ as } N \to \infty.$$

Let $\varepsilon_k = \left(\frac{1}{\ln(N^{\alpha_k})}\right)^{\frac{1}{d}}$, $\varepsilon_p = \left(\frac{1}{\ln(N^{\alpha_p})}\right)^{\frac{1}{dm-d+2}}$ where $\alpha_k, \alpha_p > 0$. Then it follows that

$$\begin{aligned} \frac{1}{N\varepsilon_k^{d-2}}\exp\Big(\tilde{C}(T)\Big(\frac{1}{\varepsilon_k^d} + \frac{1}{\varepsilon_p^{dm-d+2}}\Big)\Big) &= \frac{1}{N}(\ln(N^{\alpha_k}))^{\frac{d-2}{d}}\exp\Big(\tilde{C}(T)(\ln(N^{\alpha_k}) + \ln(N^{\alpha_p}))\Big) \\ &= \frac{1}{N}(\alpha_k)^{\frac{d-2}{d}}(\ln(N))^{\frac{d-2}{d}}\exp\Big(\tilde{C}(T)\ln(N^{\alpha_k})\Big)\exp\Big(\tilde{C}(T)\ln(N^{\alpha_p})\Big) \\ &= (\alpha_k)^{\frac{d-2}{d}}N^{-1+\delta\frac{d-2}{d}+\tilde{C}(T)\alpha_k+\tilde{C}(T)\alpha_p} \end{aligned}$$

and

$$\begin{aligned} \frac{1}{N\varepsilon_p^{2dm-2d+2}}\exp\Big(\tilde{C}(T)\Big(\frac{1}{\varepsilon_k^d} + \frac{1}{\varepsilon_p^{dm-d+2}}\Big)\Big) &= \frac{1}{N}\Big(\ln(N^{\alpha_p})\Big)^{\frac{2dm-2d+2}{dm-d+2}}\exp\Big(\tilde{C}(T)(\ln(N^{\alpha_k}) + \ln(N^{\alpha_p}))\Big) \\ &= (\alpha_p)^{\frac{2dm-2d+2}{dm-d+2}}N^{-1+\delta\cdot\frac{2dm-2d+2}{dm-d+2}+\tilde{C}(T)\alpha_k+\tilde{C}(T)\alpha_p} \end{aligned}$$

Since $m = 2$ or $m \ge 3$, then it holds that $\frac{d-2}{d} \le \frac{2dm-2d+2}{dm-d+2}$.
Therefore, we need to ensure that

$$-1 + \delta \cdot \frac{2dm-2d+2}{dm-d+2} + \tilde{C}(T)\alpha_k + \tilde{C}(T)\alpha_p + \beta < 0.$$

This implies that

$$\alpha_k + \alpha_p < \frac{1 - \delta \cdot \frac{2dm-2d+2}{dm-d+2} - \beta}{\tilde{C}(T)}.$$

4.2 Convergence in Probability

Now we prove Theorem 1.5.

4.2.1 Preliminaries

Lemma 4.1. *Let $w \in L^\infty(0,T;W^{2,\infty}(\mathbb{R}^d))$ such that $\|D^2 w\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \le C$. Then we obtain that*

$$\||D^2V^\varepsilon| * w\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \text{ and } \||\nabla V^\varepsilon| * w\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}$$

are bounded uniformly in ε. Moreover, it implies that

$$\||D^2\Phi^\varepsilon| * w\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \text{ and } \||\nabla \Phi^\varepsilon| * w\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}$$

are bounded uniformly in ε as well.

Proof. We present the proof only for $|D^2V^\varepsilon| * w$. The arguments for the case $|\nabla V^\varepsilon| * w$ are the same.

$$\begin{aligned}
|D^2V^\varepsilon| * w &= \int_{\mathbb{R}^d} |D^2V^\varepsilon(x-y)| w(s,y)\,dy \\
&= \varepsilon^{-(d+2)} \int_{\mathbb{R}^d} |D^2V\Big(\frac{x-y}{\varepsilon}\Big)| w(s,y)\,dy \\
&= \varepsilon^{-2} \int_{B_1(0)} |D^2V(z)| w(s, x-\varepsilon z)\,dz \\
&= \langle |D^2V|, \varepsilon^{-2} w(s, x-\varepsilon\cdot)\rangle_{L^2(B_1(0))} \\
&= \langle \mathcal{L}(\mathcal{L}^{-1}|D^2V|), \varepsilon^{-2} w(s, x-\varepsilon\cdot)\rangle_{L^2(B_1(0))}
\end{aligned}$$

where $\mathcal{L}v = -\Delta v$ and $\mathcal{L}^{-1}|D^2V|$ is the unique solution $\tilde{v} \in C^2(B_1(0))$ to

$$\begin{aligned}
-\Delta \tilde{v} &= |D^2V| \text{ on } B_1(0), \\
\tilde{v} &= 0 \text{ on } \partial B_1(0).
\end{aligned}$$

Using integration by parts we obtain that

$$\begin{aligned}
|D^2V^\varepsilon| * w &= \langle \mathcal{L}^{-1}|D^2V|, (-\Delta)(\varepsilon^{-2} w(s, x-\varepsilon\cdot))\rangle_{L^2(B_1(0))} \\
&= \int_{B_1(0)} (\mathcal{L}^{-1}|D^2V|)(z)(-\Delta_z)(\varepsilon^{-2} w(s, x-\varepsilon z))\,dz \\
&= \int_{B_1(0)} (\mathcal{L}^{-1}|D^2V|)(z)(-\Delta w(s, x-\varepsilon z))\,dz \\
&\le \|D^2 w\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \|\mathcal{L}^{-1}|D^2V|\|_{L^1(B_1(0))}
\end{aligned}$$

The required assumption $\mathcal{L}^{-1}|D^2V| \in L^1(B_1(0))$ is satisfied if V is a C^2 function with a compact support.

Moreover, we get that

$$\begin{aligned}
\||D^2\Phi^\varepsilon| * w\|_{L^\infty(\mathbb{R}^d)} &= \left\| \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} D^2 V^\varepsilon(\cdot - y - z)\Phi(z)dz \right| w(t,y)dy \right\|_{L^\infty(\mathbb{R}^d)} \\
&\le \left\| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| D^2 V^\varepsilon(\cdot - y - z) \right| \Phi(z)dz\, w(t,y)dy \right\|_{L^\infty(\mathbb{R}^d)} \\
&= \left\| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| D^2 V^\varepsilon(\cdot - y - z) \right| w(t,y)dy\, \Phi(z)dz \right\|_{L^\infty(\mathbb{R}^d)} \\
&= \left\| \int_{\mathbb{R}^d} |D^2 V^\varepsilon| * w(t, \cdot - z)\, \Phi(z)dz \right\|_{L^\infty(\mathbb{R}^d)} \\
&= \left\| \Phi * \left(|D^2 V^\varepsilon| * w \right)(t,\cdot) \right\|_{L^\infty(\mathbb{R}^d)} \\
&\le C \left\| |D^2 V^\varepsilon| * w(t,\cdot) \right\|_{L^\infty(\mathbb{R}^d)}.
\end{aligned}$$

□

Now we define

$$\begin{aligned}
\tau(\omega) &= \inf \left\{ t \in (0,T) \middle| \max_{i\in\{1,\dots,N\}} \left| X_t^{N,i,\varepsilon,\sigma} - \bar{X}_t^{i,\varepsilon,\sigma} \right| \ge N^{-a} \right\} \\
S_t(\omega) &= N^{a\kappa} \max_{i\in\{1,\dots,N\}} \left| X_{t\wedge\tau}^{N,i,\varepsilon,\sigma} - \bar{X}_{t\wedge\tau}^{i,\varepsilon,\sigma} \right|^\kappa .
\end{aligned}$$

We know that

$$\sup_{0\le t\le T} \mathbb{P}\left(\max_{i\in\{1,\dots,N\}} \left| X_t^{N,i,\varepsilon,\sigma} - \bar{X}_t^{i,\varepsilon,\sigma} \right| > N^{-a} \right) = \sup_{0\le t\le T} \mathbb{P}\left(\max_{i\in\{1,\dots,N\}} \left| X_t^{N,i,\varepsilon,\sigma} - \bar{X}_t^{i,\varepsilon,\sigma} \right|^\kappa > N^{-a\kappa} \right). \tag{4.4}$$

Thanks to the stopping time τ it follows directly that $S_t \le 1$. This property implies that $S_t^2 \le S_t$, which we are going to use later. Since S_t is a stopped process, it follows that

$$\left\{ \omega \in \Omega \middle| \max_{i\in\{1,\dots,N\}} \left| X_t^{N,i,\varepsilon,\sigma}(\omega) - \bar{X}_t^{i,\varepsilon,\sigma}(\omega) \right|^\kappa > N^{-a\kappa} \right\} \subset \left\{ \omega \in \Omega \middle| S_t(\omega) = 1 \right\}.$$

So, together with Markov inequality we get that

$$\sup_{0\le t\le T} \mathbb{P}\left(\max_{i\in\{1,\dots,N\}} \left| X_t^{N,i,\varepsilon,\sigma} - \bar{X}_t^{i,\varepsilon,\sigma} \right| > N^{-a} \right) \le \sup_{0\le t\le T} \mathbb{P}\left(S_t = 1 \right) \le \sup_{0\le t\le T} \mathbb{E}\left[S_t \right].$$

The next step is to show the following version of the law of large numbers.

Lemma 4.2. *Let $(\bar{Y}^i)_{i\in\{1,\dots,N\}}$ be a collection of i.i.d. random variable with the density function v.*

Furthermore, let $U \in L^\infty(\mathbb{R}^d)$ *and* $U * v \in L^\infty(0,T;L^\infty(\mathbb{R}^d))$. *Define*

$$\mathcal{A}_\theta^i(U,v) := \left\{\omega \in \Omega \middle| \left|\frac{1}{N}\sum_{j=1}^N U(\bar{Y}^i - \bar{Y}^j) - U * v(\bar{Y}^i)\right| > \frac{1}{N^\theta}\right\},$$
$$\mathcal{A}_\theta^N(U,v) := \bigcup_{i=1}^N \mathcal{A}_\theta^i(U,v)$$

So we obtain that for arbitrary $\tilde{\kappa} \in \mathbb{N}$ *and* $\theta \in (0, \frac{1}{2})$

$$\mathbb{P}(\mathcal{A}_\theta^N(U,v)) \leq N \max_{i\in\{1,\dots N\}} \mathbb{P}(\mathcal{A}_\theta^i(U,v)) \leq N^{2\tilde{\kappa}(\theta-\frac{1}{2})+1} C(\tilde{\kappa}) \left(\|U\|^{2\tilde{\kappa}}_{L^\infty(\mathbb{R}^d)} + \|U * v\|^{2\tilde{\kappa}}_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\right)$$

Proof.

$$\mathbb{P}(\mathcal{A}_\theta^i(U,v)) \leq N^{2\tilde{\kappa}\theta}\mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N U(\bar{Y}^i - \bar{Y}^j) - U * v(\bar{Y}^i)\right|^{2\tilde{\kappa}}\right] = N^{2\tilde{\kappa}\theta}\mathbb{E}\left[\left(\frac{1}{N}\sum_{j=1}^N h(\bar{Y}^i,\bar{Y}^j)\frac{1}{N}\sum_{l=1}^N h(\bar{Y}^i,\bar{Y}^l)\right)^{\tilde{\kappa}}\right]$$

where $h(\bar{Y}^i,\bar{Y}^j) := U(\bar{Y}^i - \bar{Y}^j) - U * v(\bar{Y}^i)$. So, we deduce that

We now look at terms in $\mathbb{E}\left[\left(\sum_{j,l=1}^N h(\bar{Y}^i,\bar{Y}^j)h(\bar{Y}^i,\bar{Y}^l)\right)^{\tilde{\kappa}}\right]$ where one index $j \in \{1,\dots,N\}$ only appears once and $j \neq i$:

$$\begin{aligned}
&\mathbb{E}\left[h(\bar{Y}^i,\bar{Y}^j)\prod_{\substack{m=1\\ \ell_m\neq j}}^{2\tilde{\kappa}-1} h(\bar{Y}^i,\bar{Y}^{\ell_m})\right]\\
&= \int_{\mathbb{R}^d} v(x)\int_{\mathbb{R}^d} v(y)h(x,y)\,dy\,\mathbb{E}\left[\prod_{\substack{m=1\\ \ell_m\neq j}}^{2\tilde{\kappa}-1} h(x,\bar{Y}^{\ell_m})\right]dx\\
&= 0,
\end{aligned}$$

since

$$\begin{aligned}
\int_{\mathbb{R}^d} v(y)h(x,y)\,dy &= \int_{\mathbb{R}^d} v(y)(U(x-y) - U * v(x))\,dy\\
&= \int_{\mathbb{R}^d} v(y)U(x-y)\,dy - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v(y)U(x-z)v(z)\,dz\,dy\\
&= \int_{\mathbb{R}^d} v(y)U(x-y)\,dy - \int_{\mathbb{R}^d} U(x-z)v(z)\,dz\\
&= 0.
\end{aligned}$$

In order to estimate $\mathbb{E}\left[\left(\sum_{j,l=1}^N h(\bar{Y}^i,\bar{Y}^j)h(\bar{Y}^i,\bar{Y}^l)\right)^{\tilde{\kappa}}\right]$, we need an upper bound for the terms which do not vanish. These terms have the following form:

$$\begin{aligned}
&\mathcal{N} := \mathcal{N}_1 \cup \mathcal{N}_2,\\
&\mathcal{N}_1 := \Big\{ \prod_{j=1}^{2\tilde\kappa} h(\bar Y^i, \bar Y^{i_j})\ i_j \in \{1,\dots,N\} \text{ such that all appearing indices } i_j \text{ appear at least twice} \Big\},\\
&\mathcal{N}_2 := \Big\{ h(\bar Y^i, \bar Y^i) \prod_{j=1}^{2\tilde\kappa-1} h(\bar Y^i, \bar Y^{i_j}),\ i \neq i_j \in \{1,\dots,N\} \text{ such that all}\\
&\qquad\qquad \text{appearing indices } i_j \text{ appear at least twice} \Big\}.
\end{aligned}$$

Since for $N > \tilde\kappa$ big enough we get that

$$\begin{aligned}
|\mathcal{N}_1| &= \binom{N}{1} + \ldots + \binom{N}{\tilde\kappa},\\
|\mathcal{N}_2| &= \left(\binom{N}{1} + \ldots + \binom{N}{\tilde\kappa - 1}\right) \cdot N,
\end{aligned}$$

so the number of elements in $\mathcal{N}$ is bounded by

$$|\mathcal{N}| \le C(\tilde\kappa) N^{\tilde\kappa}.$$

Since $\mathbb{E}\Big(\prod_{j=1}^{2\tilde\kappa} h(\bar Y^i, \bar Y^{i_j})\Big) \le C\left(\|U\|^{2\tilde\kappa}_{L^\infty(\mathbb{R}^d)} + \|U * v\|^{2\tilde\kappa}_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\right)$, it follows that

$$\begin{aligned}
\mathbb{P}(\mathcal{A}^i_\theta(U)) &\le N^{2\tilde\kappa\theta}\mathbb{E}\left[\left(\frac{1}{N}\sum_{j=1}^N h(\bar Y^i, \bar Y^j)\frac{1}{N}\sum_{l=1}^N h(\bar Y^i, \bar Y^l)\right)^{\tilde\kappa}\right]\\
&\le N^{2\tilde\kappa\theta}\frac{1}{N^{2\tilde\kappa}} C(\tilde\kappa) N^{\tilde\kappa}\left(\|U\|^{2\tilde\kappa}_{L^\infty(\mathbb{R}^d)} + \|U * v\|^{2\tilde\kappa}_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\right)\\
&= N^{2\tilde\kappa(\theta - \frac{1}{2})} C(\tilde\kappa)\left(\|U\|^{2\tilde\kappa}_{L^\infty(\mathbb{R}^d)} + \|U * v\|^{2\tilde\kappa}_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\right)
\end{aligned}$$

This implies the desired result. □

4.2.2 Main Estimates

Now we come back to the estimates of $\mathbb{E}[S_t]$.

Since the initial data of $X_t^{N,i,\varepsilon,\sigma}$ and $\bar X_t^{i,\varepsilon,\sigma}$ are the same we deduce that

$$\begin{aligned}
X_t^{N,i,\varepsilon,\sigma} - \bar X_t^{i,\varepsilon,\sigma} &= \int_0^t \Big(\frac{1}{N}\sum_{j=1}^N \nabla\Phi^{\varepsilon_k}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma}) - \nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s, \bar X_s^{i,\varepsilon,\sigma})\Big) ds\\
&\quad + \int_0^t \Big(- \nabla p_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma})\Big) + \nabla p_\lambda\Big(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s, \bar X_s^{i,\varepsilon,\sigma})\Big)\Big)\, ds
\end{aligned}$$

and therefore

$$S_t = N^{a\kappa} \max_{i\in\{1,\dots,N\}} \left| X_{t\wedge\tau}^{N,i,\varepsilon,\sigma} - \bar{X}_{t\wedge\tau}^{i,\varepsilon,\sigma} \right|^\kappa \le C(I_1 + I_2 + II_1 + II_2)$$

where

$$I_1 = N^{a\kappa} \max_{i\in\{1,\dots,N\}} \int_0^{t\wedge\tau} \left| \frac{1}{N} \sum_{j=1}^N \Big(\nabla\Phi^{\varepsilon_k}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma}) - \nabla\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) \Big) \right|^\kappa ds,$$

$$I_2 = N^{a\kappa} \max_{i\in\{1,\dots,N\}} \int_0^{t\wedge\tau} \left| \frac{1}{N} \sum_{j=1}^N \nabla\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) - \nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma}) \right|^\kappa ds,$$

$$II_1 = N^{a\kappa} \max_{i\in\{1,\dots,N\}} \int_0^{t\wedge\tau} \left| \nabla p_\lambda\Big(\frac{1}{N} \sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma}) \Big) - \nabla p_\lambda\Big(\frac{1}{N} \sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) \Big) \right|^\kappa ds,$$

$$II_2 = N^{a\kappa} \max_{i\in\{1,\dots,N\}} \int_0^{t\wedge\tau} \left| \nabla p_\lambda\Big(\frac{1}{N} \sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) \Big) - \nabla p_\lambda\Big(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma}) \Big) \right|^\kappa ds.$$

Estimates of the Aggregation Term

Step 1 Estimation of $\mathbb{E}[I_1]$

Using Taylor's expansion we obtain that

$$\begin{aligned}
I_1 &:= N^{a\kappa} \max_{i\in\{1,\dots,N\}} \int_0^{t\wedge\tau} \left| \frac{1}{N} \sum_{j=1}^N \Big(\nabla\Phi^{\varepsilon_k}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma}) - \nabla\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) \Big) \right|^\kappa ds \\
&\le N^{a\kappa} \max_{i\in\{1,\dots,N\}} \int_0^{t\wedge\tau} \left| \frac{1}{N} \sum_{j=1}^N D^2\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) \Big(X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma} + \bar{X}_s^{j,\varepsilon,\sigma} \Big) \right|^\kappa ds \\
&+ C(\kappa,T) N^{a\kappa} \max_{i\in\{1,\dots,N\}} \int_0^{t\wedge\tau} \left| \frac{1}{N} \sum_{j=1}^N \|D^3\Phi^{\varepsilon_k}\|_{L^\infty(\mathbb{R}^d)} \left| X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma} + \bar{X}_s^{j,\varepsilon,\sigma} \right|^2 \right|^\kappa ds \\
&\le N^{a\kappa} \max_{i\in\{1,\dots,N\}} \int_0^{t\wedge\tau} \left| \frac{1}{N} \sum_{j=1}^N D^2\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) \Big(X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma} \Big) \right|^\kappa ds \\
&+ N^{a\kappa} \max_{i\in\{1,\dots,N\}} \int_0^{t\wedge\tau} \left| \frac{1}{N} \sum_{j=1}^N D^2\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) \Big(X_s^{N,j,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma} \Big) \right|^\kappa ds \\
&+ C(\kappa,T) \|D^3\Phi^{\varepsilon_k}\|_{L^\infty(\mathbb{R}^d)}^\kappa N^{a\kappa} \cdot 2^\kappa \int_0^{t\wedge\tau} \max_{i\in\{1,\dots,N\}} \left| X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma} \right|^{2\kappa} ds
\end{aligned}$$

With the help of triangle inequality and the definition of S_t we deduce that

$$I_1 \le C(t,\kappa)(I_{11} + I_{12} + I_{13})$$

where

$$I_{11} = \int_0^{t\wedge\tau} \max_{i\in\{1,\ldots,N\}} \left| \frac{1}{N}\sum_{j=1}^N D^2\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) \right|^\kappa S_s ds,$$

$$I_{12} = N^{a\kappa} \int_0^{t\wedge\tau} \max_{i\in\{1,\ldots,N\}} \left| \frac{1}{N}\sum_{j=1}^N D^2\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})(X_s^{N,j,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) \right|^\kappa ds,$$

$$I_{13} = \int_0^{t\wedge\tau} \|D^3\Phi^{\varepsilon_k}\|_{L^\infty(\mathbb{R}^d)}^\kappa \max_{i\in\{1,\ldots,N\}} \left| X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma} \right|^\kappa S_s ds.$$

Now we are going to estimate these three terms separately

Step 1 (a) Estimation of $\mathbb{E}[I_{11}]$

$$\begin{aligned}
\mathbb{E}\left[I_{11}\right] &= \mathbb{E}\left[\int_0^{t\wedge\tau} \max_{i\in\{1,\ldots,N\}} \left| \frac{1}{N}\sum_{j=1}^N D^2\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) \right|^\kappa S_s ds\right] \\
&\leq C(\kappa,T)\mathbb{E}\left[\int_0^{t\wedge\tau} \max_{i\in\{1,\ldots,N\}} \left| D^2\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}) \right|^\kappa S_s ds\right] \\
&+ C(\kappa,T)\mathbb{E}\left[\int_0^{t\wedge\tau} \max_{i\in\{1,\ldots,N\}} \left| \frac{1}{N}\sum_{j=1}^N D^2\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) - D^2\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}) \right|^\kappa S_s ds\right]
\end{aligned}$$

Recall Lemma 4.2 and consider

$$\mathcal{A}_{\theta_{(1)}}^{(1)}(s) := \mathcal{A}_{\theta_{(1)}}^N(D^2\Phi^{\varepsilon_k}(\cdot), u^{\varepsilon,\sigma}(s,\cdot)) \text{ with } (\bar{Y}^i)_{i\in\{1,\ldots,N\}} = (\bar{X}_s^{i,\varepsilon,\sigma})_{i\in\{1,\ldots,N\}}.$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\left[I_{11}\right] \leq C\|D^2\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}^\kappa \int_0^t \mathbb{E}\left[S_s\right] ds \\
&+ C(\kappa,T)\mathbb{E}\left[\int_0^{t\wedge\tau} \max_{i\in\{1,\ldots,N\}} \left| \frac{1}{N}\sum_{j=1}^N D^2\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) - D^2\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}) \right|^\kappa S_s ds\right]
\end{aligned}$$

Since $\Omega = \mathcal{A}_{\theta_{(1)}}^{(1)}(s) \cup (\mathcal{A}_{\theta_{(1)}}^{(1)}(s))^c$ it implies that

$$\begin{aligned}
\mathbb{E}\left[I_{11}\right] &\leq C(\kappa,T)\left(\|D^2\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}^\kappa + \frac{1}{N^{\theta_{(1)}\kappa}}\right)\int_0^t \mathbb{E}\left[S_s\right] ds \\
&+ C(\kappa,T)\left(\|D^2\Phi^{\varepsilon_k}\|_{L^\infty(\mathbb{R}^d)}^\kappa + \|D^2\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}^\kappa\right)\int_0^t \mathbb{P}(\mathcal{A}_{\theta_{(1)}}^{(1)}(s))ds
\end{aligned}$$

Step 1 (b) Estimation of $\mathbb{E}[I_{12}]$

Since

$$
\begin{aligned}
I_{12} &= N^{a\kappa} \int_0^{t\wedge\tau} \max_{i\in\{1,\dots,N\}} \left| \frac{1}{N} \sum_{j=1}^N D^2\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})(X_s^{N,j,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) \right|^\kappa \\
&\leq N^{a\kappa} \int_0^{t\wedge\tau} \max_{i\in\{1,\dots,N\}} \left(\frac{1}{N} \sum_{j=1}^N \left| D^2\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) \right| \right)^\kappa \max_{j\in\{1,\dots,N\}} \left| X_s^{N,j,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma} \right|^\kappa ds \\
&= \int_0^{t\wedge\tau} \max_{i\in\{1,\dots,N\}} \left(\frac{1}{N} \sum_{j=1}^N \left| D^2\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) \right| \right)^\kappa S_s ds.
\end{aligned}
$$

So, we obtain that

$$
\begin{aligned}
\mathbb{E}[I_{12}] \leq{}& C(\kappa,T)\mathbb{E}\left[\int_0^{t\wedge\tau} \max_{i\in\{1,\dots,N\}} \left| |D^2\Phi^{\varepsilon_k}| * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}) \right|^\kappa S_s ds \right] \\
&+ C(\kappa,T)\mathbb{E}\left[\int_0^{t\wedge\tau} \max_{i\in\{1,\dots,N\}} \left| \frac{1}{N} \sum_{j=1}^N |D^2\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})| - |D^2\Phi^{\varepsilon_k}| * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}) \right|^\kappa S_s ds \right]
\end{aligned}
$$

Recall again Lemma 4.2 and consider

$$
\mathcal{A}_{\theta_{(2)}}^{(2)}(s) := \mathcal{A}_{\theta_{(2)}}^N(|D^2\Phi^{\varepsilon_k}(\cdot)|, u^{\varepsilon,\sigma}(s,\cdot)) \text{ with } (\bar{Y}^i)_{i\in\{1,\dots,N\}} = (\bar{X}_s^{i,\varepsilon,\sigma})_{i\in\{1,\dots,N\}}.
$$

So we deduce that

$$
\begin{aligned}
&\mathbb{E}[I_{12}] \leq C(\kappa,T)\mathbb{E}\left[\int_0^{t\wedge\tau} \max_{i\in\{1,\dots,N\}} \left| |D^2\Phi^{\varepsilon_k}| * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}) \right|^\kappa S_s ds \right] \\
&+ C(\kappa,T)\mathbb{E}\left[\int_0^{t\wedge\tau} \max_{i\in\{1,\dots,N\}} \left| \frac{1}{N} \sum_{j=1}^N |D^2\Phi^{\varepsilon_k}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})| - |D^2\Phi^{\varepsilon_k}| * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}) \right|^\kappa S_s ds \right] \\
&\leq C(\kappa,T) \int_0^t \left(\| |D^2\Phi^{\varepsilon_k}| * u^{\varepsilon,\sigma} \|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}^\kappa + \frac{1}{N^{\theta_{(2)}\kappa}} \right) \mathbb{E}[S_s] ds \\
&+ C(\kappa,T) \left(\|D^2\Phi^{\varepsilon_k}\|_{L^\infty(\mathbb{R}^d)}^\kappa + \| |D^2\Phi^{\varepsilon_k}| * u^{\varepsilon,\sigma} \|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}^\kappa \right) \int_0^t \mathbb{P}(\mathcal{A}_{\theta_{(2)}}^{(2)}(s)) ds
\end{aligned}
$$

Step 1 (c) Estimation of $\mathbb{E}[I_{13}]$

Together with Lemma 3.1 and definition of the stopping time τ we obtain that

$$
\begin{aligned}
\mathbb{E}[I_{13}] &= \mathbb{E}\left[\int_0^{t\wedge\tau} \|D^3\Phi^{\varepsilon_k}\|_{L^\infty(\mathbb{R}^d)}^\kappa \max_{i\in\{1,\dots,N\}} \left| X_s^{N,i,\varepsilon,\sigma} - \bar{X}_s^{i,\varepsilon,\sigma} \right|^\kappa S_s ds \right] \\
&\leq C(\kappa,T) \int_0^t \mathbb{E}\left[\|D^3\Phi^{\varepsilon_k}\|_{L^\infty(\mathbb{R}^d)}^\kappa \frac{1}{N^{\kappa a}} S_s \right] ds
\end{aligned}
$$

So, together with Lemma 4.1 we obtain the following bound for $\mathbb{E}[I_1]$:

$$\begin{aligned}
\mathbb{E}\left[I_1\right] \leq C(\kappa,T)&\left(\|D^2\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}\|^\kappa_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} + \frac{1}{N^{\theta_{(1)}\kappa}}\right)\int_0^t \mathbb{E}\left[S_s\right]ds \\
&+ C(\kappa,T)\left(\|D^2\Phi^{\varepsilon_k}\|^\kappa_{L^\infty(\mathbb{R}^d)} + \|D^2\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}\|^\kappa_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\right)\int_0^t \mathbb{P}(\mathcal{A}^{(1)}_{\theta_{(1)}}(s))ds \\
&+ C(\kappa,T)\int_0^t\left(\||D^2\Phi^{\varepsilon_k}| * u^{\varepsilon,\sigma}\|^\kappa_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} + \frac{1}{N^{\theta_{(2)}\kappa}}\right)\mathbb{E}[S_s]ds \\
&+ C(\kappa,T)\left(\|D^2\Phi^{\varepsilon_k}\|^\kappa_{L^\infty(\mathbb{R}^d)} + \||D^2\Phi^{\varepsilon_k}| * u^{\varepsilon,\sigma}\|^\kappa_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\right)\int_0^t \mathbb{P}(\mathcal{A}^{(2)}_{\theta_{(2)}}(s))ds \\
&+ C(\kappa,T)\|D^3\Phi^{\varepsilon_k}\|^\kappa_{L^\infty(\mathbb{R}^d)}\frac{1}{N^{\kappa a}}\int_0^t \mathbb{E}\left[S_s\right]ds \\
&\leq C(\kappa,T)\left(1 + \|D^3\Phi^{\varepsilon_k}\|^\kappa_{L^\infty(\mathbb{R}^d)}\frac{1}{N^{\kappa a}}\right)\int_0^t \mathbb{E}\left[S_s\right]ds \\
&+ C(\kappa,T)\|D^2\Phi^{\varepsilon_k}\|^\kappa_{L^\infty(\mathbb{R}^d)}\int_0^t\Big(\mathbb{P}(\mathcal{A}^{(1)}_{\theta_{(1)}}(s)) + \mathbb{P}(\mathcal{A}^{(2)}_{\theta_{(2)}}(s))\Big)ds
\end{aligned}$$

Step 2 Estimation of $\mathbb{E}[I_2]$
Recall again Lemma 4.2 and consider

$$\mathcal{A}^{(3)}_{\theta_{(3)}}(s) := \mathcal{A}^N_{\theta_{(3)}}(\nabla\Phi^{\varepsilon_k}(\cdot), u^{\varepsilon,\sigma}(s,\cdot)) \text{ with } (\bar{Y}^i)_{i\in\{1,\ldots,N\}} = (\bar{X}^{i,\varepsilon,\sigma}_s)_{i\in\{1,\ldots,N\}}.$$

Since

$$I_2 = N^{a\kappa}\max_{i\in\{1,\ldots,N\}}\int_0^{t\wedge\tau}\left|\frac{1}{N}\sum_{j=1}^N \nabla\Phi^{\varepsilon_k}(\bar{X}^{i,\varepsilon,\sigma}_s - \bar{X}^{j,\varepsilon,\sigma}_s) - \nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s,\bar{X}^{i,\varepsilon,\sigma}_s)\right|^\kappa ds$$

we obtain that

$$\begin{aligned}
\mathbb{E}[I_2] &\leq C(\kappa,T)\left(N^{a\kappa}\frac{1}{N^{\theta_{(3)}\kappa}} + N^{a\kappa}\left(\|\nabla\Phi^{\varepsilon_k}\|^\kappa_{L^\infty(\mathbb{R}^d)} + \|\nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}\|^\kappa_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\right)\int_0^t \mathbb{P}(\mathcal{A}^{(3)}_{\theta_{(3)}}(s))ds\right) \\
&\leq C(\kappa,T)\left(N^{(a-\theta_{(3)})\kappa} + N^{a\kappa}\|\nabla\Phi^{\varepsilon_k}\|^\kappa_{L^\infty(\mathbb{R}^d)}\int_0^t \mathbb{P}(\mathcal{A}^{(3)}_{\theta_{(3)}}(s))ds\right)
\end{aligned}$$

Step 3 Estimation of the whole aggregation term
So we obtain that

$$\begin{aligned}
&\mathbb{E}\left[N^{a\kappa}\max_{i\in\{1,\ldots,N\}}\int_0^{t\wedge\tau}\left|\frac{1}{N}\sum_{j=1}^N \nabla\Phi^{\varepsilon_k}(X^{N,i,\varepsilon,\sigma}_s - X^{N,j,\varepsilon,\sigma}_s) - \nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s,\bar{X}^{i,\varepsilon,\sigma}_s)\right|^\kappa ds\right] \\
&\leq C(\kappa,T)\left(1 + \|D^3\Phi^{\varepsilon_k}\|^\kappa_{L^\infty(\mathbb{R}^d)}\frac{1}{N^{\kappa a}}\right)\int_0^t \mathbb{E}\left[S_s\right]ds \\
&+ C(\kappa,T)\|D^2\Phi^{\varepsilon_k}\|^\kappa_{L^\infty(\mathbb{R}^d)}\int_0^t\Big(\mathbb{P}(\mathcal{A}^{(1)}_{\theta_{(1)}}(s)) + \mathbb{P}(\mathcal{A}^{(2)}_{\theta_{(2)}}(s))\Big)ds + C(\kappa,T)N^{a\kappa}\|\nabla\Phi^{\varepsilon_k}\|^\kappa_{L^\infty(\mathbb{R}^d)}\int_0^t \mathbb{P}(\mathcal{A}^{(3)}_{\theta_{(3)}}(s))ds \\
&+ C(\kappa,T)N^{(a-\theta_{(3)})\kappa}
\end{aligned}$$

Therefore, choosing $\varepsilon_k = N^{-\beta_k}$ we obtain that

$$
\begin{aligned}
&\mathbb{E}\left[N^{a\kappa} \max_{i\in\{1,\dots,N\}} \int_0^{t\wedge\tau} \left|\frac{1}{N}\sum_{j=1}^N \nabla\Phi^{\varepsilon_k}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma}) - \nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\right|^\kappa ds\right] \\
&\leq C(\kappa,T)\left(1 + N^{\kappa((d+1)\beta_k - a)}\right)\int_0^t \mathbb{E}\left[S_s\right] ds \\
&+ C(\kappa,T)N^{d\kappa\beta_k}\int_0^t \left(\mathbb{P}(\mathcal{A}^{(1)}_{\theta_{(1)}}(s)) + \mathbb{P}(\mathcal{A}^{(2)}_{\theta_{(2)}}(s))\right) ds \\
&+ C(\kappa,T)N^{\kappa((d-1)\beta_k + a)}\int_0^t \mathbb{P}(\mathcal{A}^{(3)}_{\theta_{(3)}}(s)) ds \\
&+ C(\kappa,T)N^{\kappa(a-\theta_{(3)})}
\end{aligned}
\tag{4.5}
$$

Estimates of the Porous Media Term

Step 1 Preliminaries

First, we need to recall some computations from Subsection 4.1.2 .

$$
\begin{aligned}
&-\nabla p_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma})\Big) + \nabla p_\lambda\Big(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big) \\
&= \underbrace{-\nabla p_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma})\Big) + \nabla p_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big)}_{=:A_1} \\
&\underbrace{-\nabla p_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big) + \nabla p_\lambda\Big(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)}_{=:A_2} \\
&= A_1 + A_2.
\end{aligned}
$$

Recall that

$$
\begin{aligned}
A_1 &= \nabla p_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big) - \nabla p_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma})\Big) \\
&= p'_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big)\Big(\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big) \\
&- p'_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma})\Big)\Big(\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma})\Big)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
A_1 = {} & p'_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big)\Big(\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big) \\
& - p'_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma})\Big)\Big(\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big) \\
& + p'_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma})\Big)\Big(\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big) \\
& - p'_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma})\Big)\Big(\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma})\Big)
\end{aligned}
$$

Now we factor out $\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})$ and $p'_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma})\Big)$ in order to obtain that

$$
\begin{aligned}
A_1 = {} & \bigg(p'_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big) - p'_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma})\Big)\bigg) \\
& \quad \cdot \frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) \\
& + \bigg(\Big(\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big) - \Big(\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma})\Big)\bigg) \\
& \quad \cdot p'_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma})\Big) \\
= {} & A_{1,1} + A_{1,2}
\end{aligned}
$$

Moreover, recall that A_2 can be written as

$$
\begin{aligned}
A_2 &= -\nabla p_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\Big)+\nabla p_\lambda\Big(V^{\varepsilon_p}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\\
&= -p_\lambda'\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\Big)\Big(\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\Big)\\
&\quad + p_\lambda'\Big(V^{\varepsilon_p}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\Big(\nabla V^{\varepsilon_p}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\\
&= -p_\lambda'\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\Big)\Big(\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\Big)\\
&+ p_\lambda'(V^{\varepsilon_p}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big(\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\Big)\\
&- p_\lambda'(V^{\varepsilon_p}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big(\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\Big)\\
&+ p_\lambda'\Big(V^{\varepsilon_p}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\Big(\nabla V^{\varepsilon_p}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)
\end{aligned}
$$

Therefore, we obtain that A_2 is nothing but

$$
\begin{aligned}
A_2 &= \left(p_\lambda'(V^{\varepsilon_p}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})-p_\lambda'\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\Big)\right)\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\\
&+\left(\Big(\nabla V^{\varepsilon_p}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)-\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\right)p_\lambda'\Big(V^{\varepsilon_p}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\\
&= A_{2,1}+A_{2,2}
\end{aligned}
$$

Step 2 Main Estimates

We begin with the term A_1, namely

$$
A_1 = \nabla p_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\Big)-\nabla p_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma}-X_s^{N,j,\varepsilon,\sigma})\Big)
$$

First, we see that

$$
\int_0^{t\wedge\tau}|A_1|^\kappa ds \le C(\kappa,T)\left(\int_0^{t\wedge\tau}|A_{1,1}|^\kappa ds+\int_0^{t\wedge\tau}|A_{1,2}|^\kappa ds\right)
$$

Step 2(a) Estimates of $A_{1,1}$

Using Taylor's expantion of the second order of p_λ' at the point $\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})$ we deduce that

$$\begin{aligned}
&\int_0^{t\wedge\tau} |A_{1,1}|^\kappa ds \\
&\leq \int_0^{t\wedge\tau} \left| p''_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big) \cdot \frac{1}{N}\sum_{j=1}^N \Big(V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) - V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma})\Big)\right|^\kappa \\
&\qquad \cdot \left|\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\right|^\kappa ds \\
&+ \int_0^{t\wedge\tau} \|p'''_\lambda\|^\kappa_{L^\infty(\mathbb{R})} \left|\frac{1}{N}\sum_{j=1}^N \Big(V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) - V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma})\Big)\right|^{2\kappa} \\
&\qquad \cdot \left|\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\right|^\kappa ds \\
&=: B_1 + B_2
\end{aligned}$$

Now we study B_1. Using Taylor's expantion of the second order of V^{ε_p} at the point $\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}$ we deduce that

$$\begin{aligned}
B_1 &= \int_0^{t\wedge\tau} \left| p''_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big) \cdot \frac{1}{N}\sum_{j=1}^N \Big(V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) - V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma})\Big)\right|^\kappa \\
&\qquad \cdot \left|\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\right|^\kappa ds \\
&\leq C(\kappa,T)\int_0^{t\wedge\tau} \left| p''_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big)\right|^\kappa \\
&\qquad \cdot \left|\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big((\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}) - (\bar{X}_s^{j,\varepsilon,\sigma} - X_s^{N,j,\varepsilon,\sigma})\Big)\right|^\kappa \\
&\qquad \cdot \left|\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\right|^\kappa ds \\
&+ \int_0^{t\wedge\tau} \varepsilon_p^{-d\kappa|m-3|}\|D^2V^{\varepsilon_p}\|^\kappa_{L^\infty(\mathbb{R}^d)} \max_{i\in\{1,\dots,N\}} |\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}|^{2\kappa}\|\nabla V^{\varepsilon_p}\|^\kappa_{L^\infty(\mathbb{R}^d)} ds \\
&\leq C(\kappa,T)\int_0^{t\wedge\tau} \left| p''_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big)\right|^\kappa \\
&\cdot \left|\frac{1}{N}\sum_{j=1}^N \left|\nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\right| \max_{i\in\{1,\dots,N\}} |\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}|\right|^\kappa \left|\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\right|^\kappa ds \\
&+ \int_0^{t\wedge\tau} \varepsilon_p^{-d\kappa|m-3|}\|D^2V^{\varepsilon_p}\|^\kappa_{L^\infty(\mathbb{R}^d)} \max_{i\in\{1,\dots,N\}} |\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}|^{2\kappa}\|\nabla V^{\varepsilon_p}\|^\kappa_{L^\infty(\mathbb{R}^d)} ds \\
&\leq C(\kappa,T)\int_0^{t\wedge\tau} |\mathbf{a}\cdot\mathbf{b}\cdot\mathbf{e}|^\kappa\, ds \\
&+ C(\kappa,T)N^{\kappa(\beta_p d|m-3|+\beta_p(2d+3))}\int_0^t \max_{i\in\{1,\dots,N\}} |\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}|^{2\kappa} ds
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{a} &:= p_\lambda''\Big(\frac{1}{N}\sum_{j=1}^{N} V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big)\\
\mathbf{b} &:= \frac{1}{N}\sum_{j=1}^{N}\Big|\nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big| \max_{i\in\{1,\dots,N\}} |\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}|\\
\mathbf{e} &:= \frac{1}{N}\sum_{j=1}^{N} \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\\
\mathbf{A} &:= p_\lambda''\Big(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma})\Big)\\
\mathbf{B} &:= |\nabla V^{\varepsilon_p}| * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma}) \max_{i\in\{1,\dots,N\}} |\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}|\\
\mathbf{E} &:= \nabla V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s, \bar{X}_s^{i,\varepsilon,\sigma})
\end{aligned}$$

We now that

$$\mathbf{a}\cdot\mathbf{b}\cdot\mathbf{e} = \mathbf{A}\cdot\mathbf{B}\cdot\mathbf{E} + (\mathbf{a}-\mathbf{A})\mathbf{b}\cdot\mathbf{e} + (\mathbf{b}-\mathbf{B})\mathbf{A}\cdot\mathbf{e} + (\mathbf{e}-\mathbf{E})\mathbf{A}\cdot\mathbf{B}$$

Since

$$\begin{aligned}
&\mathbf{b} \le \|\nabla V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)} \max_{i\in\{1,\dots,N\}} |\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}|,\\
&\mathbf{B} \le \||\nabla V^{\varepsilon_p}| * u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \max_{i\in\{1,\dots,N\}} |\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}| \le C \max_{i\in\{1,\dots,N\}} |\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}|,\\
&\mathbf{e} \le \|\nabla V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)},\\
&\mathbf{E} \le \|\nabla V^{\varepsilon_p} * u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \le C,\\
&\mathbf{A} \le \|p_\lambda''(V^{\varepsilon_p} * u^{\varepsilon,\sigma})\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \le C,
\end{aligned}$$

we obtain that

$$
\begin{aligned}
&\int_0^{t\wedge\tau} |\mathbf{a}\cdot\mathbf{b}\cdot\mathbf{e}|^\kappa \, ds \\
&\leq C(\kappa,T)\int_0^{t\wedge\tau} \ |\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{E}|^\kappa \, ds + C(\kappa,T)\int_0^{t\wedge\tau} |(\mathbf{a}-\mathbf{A})\mathbf{b}\cdot\mathbf{e}|^\kappa ds \\
&+ C(\kappa,T)\int_0^{t\wedge\tau} |(\mathbf{b}-\mathbf{B})\mathbf{A}\cdot\mathbf{e}|^\kappa ds + C(\kappa,T)\int_0^{t\wedge\tau} |(\mathbf{e}-\mathbf{E})\mathbf{A}\cdot\mathbf{B}|^\kappa ds \\
&\leq C(\kappa,T)\int_0^{t\wedge\tau} \ |\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{E}|^\kappa \, ds \\
&+ C(\kappa,T)\int_0^{t\wedge\tau} \left| p''_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\Big) - p''_\lambda\Big(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\right|^\kappa \\
&\qquad \cdot \max_{i\in\{1,\dots,N\}} |\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}|^\kappa \|\nabla V^{\varepsilon_p}\|^{2\kappa}_{L^\infty(\mathbb{R}^d)} ds \\
&+ C(\kappa,T)\int_0^{t\wedge\tau} \left| \frac{1}{N}\sum_{j=1}^N \Big|\nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\Big| - |\nabla V^{\varepsilon_p}| * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\right|^\kappa \\
&\qquad \cdot \max_{i\in\{1,\dots,N\}} |\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}|^\kappa \|\nabla V^{\varepsilon_p}\|^{\kappa}_{L^\infty(\mathbb{R}^d)} ds \\
&+ C(\kappa,T)\int_0^{t\wedge\tau} \left| \frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma}) - \nabla V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\right|^\kappa \\
&\qquad \cdot \max_{i\in\{1,\dots,N\}} |\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}|^\kappa ds \\
&\leq C(\kappa,T)\int_0^{t\wedge\tau} \max_{i\in\{1,\dots,N\}} |\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}|^\kappa ds + B_{1,1} + B_{1,2} + B_{1,3}
\end{aligned}
$$

where

$$
\begin{aligned}
B_{1,1} &= C(\kappa,T)\int_0^{t\wedge\tau} \left| p''_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\Big) - p''_\lambda\Big(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)\right|^\kappa \\
&\qquad \cdot \max_{i\in\{1,\dots,N\}} |\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}|^\kappa \|\nabla V^{\varepsilon_p}\|^{2\kappa}_{L^\infty(\mathbb{R}^d)} ds \\
B_{1,2} &= C(\kappa,T)\int_0^{t\wedge\tau} \left| \frac{1}{N}\sum_{j=1}^N \Big|\nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\Big| - |\nabla V^{\varepsilon_p}| * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\right|^\kappa \\
&\qquad \cdot \max_{i\in\{1,\dots,N\}} |\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}|^\kappa \|\nabla V^{\varepsilon_p}\|^{\kappa}_{L^\infty(\mathbb{R}^d)} ds \\
B_{1,3} &= C(\kappa,T)\int_0^{t\wedge\tau} \left| \frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma}) - \nabla V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\right|^\kappa \\
&\qquad \cdot \max_{i\in\{1,\dots,N\}} |\bar{X}_s^{i,\varepsilon,\sigma} - X_s^{N,i,\varepsilon,\sigma}|^\kappa ds.
\end{aligned}
$$

Denote for $(\bar{Y}^i)_{i\in\{1,\dots,N\}} = (\bar{X}_s^{i,\varepsilon,\sigma})_{i\in\{1,\dots,N\}}$

$$\begin{aligned}
\mathcal{A}^{(4)}_{\theta_{(4)}}(s) &:= \mathcal{A}^N_{\theta_{(4)}}(V^{\varepsilon_p}(\cdot), u^{\varepsilon,\sigma}(s,\cdot)) \\
\mathcal{A}^{(5)}_{\theta_{(5)}}(s) &:= \mathcal{A}^N_{\theta_{(5)}}(|\nabla V^{\varepsilon_p}(\cdot)|, u^{\varepsilon,\sigma}(s,\cdot)) \\
\mathcal{A}^{(6)}_{\theta_{(6)}}(s) &:= \mathcal{A}^N_{\theta_{(6)}}(\nabla V^{\varepsilon_p}(\cdot), u^{\varepsilon,\sigma}(s,\cdot))
\end{aligned}$$

So,

$$\begin{aligned}
&\mathbb{E}\left[N^{\kappa a}\max_{i\in\{1,\dots,N\}} B_{1,1}\right] \\
&\leq C(\kappa,T)\mathbb{E}\Bigg[\max_{i\in\{1,\dots,N\}}\int_0^{t\wedge\tau}\|p_\lambda'''\|^\kappa_{L^\infty(\mathbb{R})}\left|\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}^{i,\varepsilon,\sigma}_s-\bar{X}^{j,\varepsilon,\sigma}_s)-V^{\varepsilon_p}*u^{\varepsilon,\sigma}(s,\bar{X}^{i,\varepsilon,\sigma}_s)\right|^\kappa \\
&\qquad\cdot\|\nabla V^{\varepsilon_p}\|^{2\kappa}_{L^\infty(\mathbb{R}^d)}S_s\,ds\Bigg] \\
&\leq C(\kappa,T)\Big(\varepsilon_p^{-d\kappa|m-4|}\|\nabla V^{\varepsilon_p}\|^{2\kappa}_{L^\infty(\mathbb{R}^d)}\frac{1}{N^{\theta_{(4)}\kappa}}\int_0^t\mathbb{E}[S_s]ds \\
&\qquad+\varepsilon_p^{-d\kappa|m-4|}\|V^{\varepsilon_p}\|^\kappa_{L^\infty(\mathbb{R}^d)}\|\nabla V^{\varepsilon_p}\|^{2\kappa}_{L^\infty(\mathbb{R}^d)}\int_0^t\mathbb{P}(\mathcal{A}^{(4)}_{\theta_{(4)}}(s))ds\Big) \\
&\leq C(\kappa,T)\left(N^{\beta_p d\kappa|m-4|}\cdot N^{2\beta_p(d+1)\kappa}\frac{1}{N^{\theta_{(4)}\kappa}}\int_0^t\mathbb{E}[S_s]ds+N^{\beta_p d\kappa|m-4|}\cdot N^{\kappa(2\beta_p(d+1)+\beta_p d)}\cdot\int_0^t\mathbb{P}(\mathcal{A}^{(4)}_{\theta_{(4)}}(s))ds\right) \\
&\leq C(\kappa,T)N^{\kappa\left(\beta_p d|m-4|+2\beta_p(d+1)-\theta_{(4)}\right)}\int_0^t\mathbb{E}[S_s]ds+C(\kappa,T)N^{\kappa(\beta_p d|m-4|+\beta_p(3d+2))}\int_0^t\mathbb{P}(\mathcal{A}^{(4)}_{\theta_{(4)}}(s))ds
\end{aligned}$$

Moreover, since $S_s\leq 1$ we obtain that

$$\begin{aligned}
&\mathbb{E}\left[N^{\kappa a}\max_{i\in\{1,\dots,N\}} B_{1,2}\right] \\
&= C(\kappa,T)\mathbb{E}\Bigg[N^{\kappa a}\max_{i\in\{1,\dots,N\}}\int_0^{t\wedge\tau}\left|\frac{1}{N}\sum_{j=1}^N\left|\nabla V^{\varepsilon_p}(\bar{X}^{i,\varepsilon,\sigma}_s-\bar{X}^{j,\varepsilon,\sigma}_s)\right|-|\nabla V^{\varepsilon_p}|*u^{\varepsilon,\sigma}(s,\bar{X}^{i,\varepsilon,\sigma}_s)\right|^\kappa \\
&\qquad\cdot\max_{i\in\{1,\dots,N\}}|\bar{X}^{i,\varepsilon,\sigma}_s-X^{N,i,\varepsilon,\sigma}_s|^\kappa\|\nabla V^{\varepsilon_p}\|^\kappa_{L^\infty(\mathbb{R}^d)}ds\Bigg] \\
&= C(\kappa,T)\|\nabla V^{\varepsilon_p}\|^\kappa_{L^\infty(\mathbb{R}^d)}\mathbb{E}\left[\max_{i\in\{1,\dots,N\}}\int_0^{t\wedge\tau}\left|\frac{1}{N}\sum_{j=1}^N\left|\nabla V^{\varepsilon_p}(\bar{X}^{i,\varepsilon,\sigma}_s-\bar{X}^{j,\varepsilon,\sigma}_s)\right|-|\nabla V^{\varepsilon_p}|*u^{\varepsilon,\sigma}(s,\bar{X}^{i,\varepsilon,\sigma}_s)\right|^\kappa\cdot S_s ds\right] \\
&\leq C(\kappa,T)\left(\|\nabla V^{\varepsilon_p}\|^\kappa_{L^\infty(\mathbb{R}^d)}\frac{1}{N^{\theta_{(5)}\kappa}}\int_0^t\mathbb{E}[S_s]ds+\|\nabla V^{\varepsilon_p}\|^{2\kappa}_{L^\infty(\mathbb{R}^d)}\int_0^t\mathbb{P}(\mathcal{A}^{(5)}_{\theta_{(5)}}(s))ds\right) \\
&\leq C(\kappa,T)N^{\kappa\left(\beta_p(d+1)-\theta_{(5)}\right)}\int_0^t\mathbb{E}[S_s]ds+C(\kappa,T)N^{\kappa(2\beta_p(d+1))}\int_0^t\mathbb{P}(\mathcal{A}^{(5)}_{\theta_{(5)}}(s))ds
\end{aligned}$$

$$
\begin{aligned}
&\mathbb{E}\left[N^{\kappa a}\max_{i\in\{1,\dots,N\}} B_{1,3}\right]\\
&\le C(\kappa,T)\mathbb{E}\Bigg[N^{\kappa a}\max_{i\in\{1,\dots,N\}}\int_0^{t\wedge\tau}\left|\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})-\nabla V^{\varepsilon_p}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\right|^{\kappa}\\
&\qquad\qquad\cdot\max_{i\in\{1,\dots,N\}}|\bar{X}_s^{i,\varepsilon,\sigma}-X_s^{N,i,\varepsilon,\sigma}|^{\kappa}ds\Bigg]\\
&\le C(\kappa,T)N^{-\theta_{(6)}\kappa}\int_0^t\mathbb{E}[S_s]ds+C(\kappa,T)\|\nabla V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)}^{\kappa}\int_0^t\mathbb{P}(\mathcal{A}^{(6)}_{\theta_{(6)}}(s))ds\\
&\le C(\kappa,T)N^{-\theta_{(6)}\kappa}\int_0^t\mathbb{E}[S_s]ds+C(\kappa,T)N^{\kappa\beta_p(d+1)}\int_0^t\mathbb{P}(\mathcal{A}^{(6)}_{\theta_{(6)}}(s))ds
\end{aligned}
$$

All together this implies that

$$
\begin{aligned}
&\mathbb{E}\left[N^{\kappa a}\max_{i\in\{1,\dots,N\}} B_1\right]\\
&\le C(\kappa,T)\left(1+N^{\kappa\left(\beta_p d|m-4|+2\beta_p(d+1)-\theta_{(4)}\right)}+N^{\kappa\left(\beta_p(d+1)-\theta_{(5)}\right)}+N^{-\theta_{(6)}\kappa}+N^{\kappa(\beta_p d|m-3|+\beta_p(2d+3)-a)}\right)\int_0^t\mathbb{E}[S_s]ds\\
&+C(\kappa,T)N^{\kappa(\beta_p d|m-4|+\beta_p(3d+2))}\int_0^t\mathbb{P}(\mathcal{A}^{(4)}_{\theta_{(4)}}(s))ds+C(\kappa,T)N^{\kappa(2\beta_p(d+1))}\int_0^t\mathbb{P}(\mathcal{A}^{(5)}_{\theta_{(5)}}(s))ds\\
&+C(\kappa,T)N^{\kappa\beta_p(d+1)}\int_0^t\mathbb{P}(\mathcal{A}^{(6)}_{\theta_{(6)}}(s))ds
\end{aligned}
$$

For the term B_2 we obtain that

$$
\begin{aligned}
B_2&\le C(\kappa,T)\int_0^{t\wedge\tau}\varepsilon_p^{-d|m-4|\kappa}\|\nabla V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)}^{3\kappa}\max_{i\in\{1,\dots,N\}}|\bar{X}_s^{i,\varepsilon,\sigma}-X_s^{N,i,\varepsilon,\sigma}|^{2\kappa}ds\\
&\le C(\kappa,T)N^{\kappa(\beta_p d|m-4|+3\beta_p(d+1))}\int_0^{t\wedge\tau}\max_{i\in\{1,\dots,N\}}|\bar{X}_s^{i,\varepsilon,\sigma}-X_s^{N,i,\varepsilon,\sigma}|^{2\kappa}ds
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}\left[N^{\kappa a}\max_{i\in\{1,\dots,N\}}\int_0^{t\wedge\tau}|A_{1,1}|^{\kappa}ds\right]\\
&\le C(\kappa,T)\mathbb{E}\left[N^{\kappa a}\max_{i\in\{1,\dots,N\}} B_1\right]+C(\kappa,T)\mathbb{E}\left[N^{\kappa a}\max_{i\in\{1,\dots,N\}} B_2\right]\\
&\le C(\kappa,T)\Big(1+N^{\kappa\left(\beta_p d|m-4|+2\beta_p(d+1)-\theta_{(4)}\right)}+N^{\kappa\left(\beta_p(d+1)-\theta_{(5)}\right)}+N^{-\theta_{(6)}\kappa}\\
&\qquad+N^{\kappa(\beta_p d|m-3|+\beta_p(2d+3)-a)}+N^{\kappa(\beta_p d|m-4|+3\beta_p(d+1)-a)}\Big)\int_0^t\mathbb{E}[S_s]ds\\
&+C(\kappa,T)N^{\kappa(\beta_p d|m-4|+\beta_p(3d+2))}\int_0^t\mathbb{P}(\mathcal{A}^{(4)}_{\theta_{(4)}}(s))ds+C(\kappa,T)N^{\kappa(2\beta_p(d+1))}\int_0^t\mathbb{P}(\mathcal{A}^{(5)}_{\theta_{(5)}}(s))ds\\
&+C(\kappa,T)N^{\kappa\beta_p(d+1)}\int_0^t\mathbb{P}(\mathcal{A}^{(6)}_{\theta_{(6)}}(s))ds
\end{aligned}
$$

Step 2(b) Estimates of $A_{1,2}$

Now let's estimate $\int_0^{t\wedge\tau}|A_{1,2}|^\kappa ds$

$$
\begin{aligned}
&\int_0^{t\wedge\tau}\left|p'_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma}-X_s^{N,j,\varepsilon,\sigma})\Big)\right|^\kappa \\
&\cdot\left|\Big(\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\Big)-\Big(\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma}-X_s^{N,j,\varepsilon,\sigma})\Big)\right|^\kappa ds \\
&\le \int_0^{t\wedge\tau}\left|p'_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma}-X_s^{N,j,\varepsilon,\sigma})\Big)-p'_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\Big)\right|^\kappa \\
&\cdot\left|\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})-\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma}-X_s^{N,j,\varepsilon,\sigma})\right|^\kappa ds \\
&+\int_0^{t\wedge\tau}\left|p'_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})\Big)\right|^\kappa \\
&\cdot\left|\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})-\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma}-X_s^{N,j,\varepsilon,\sigma})\right|^\kappa ds \\
&=: L_1+L_2
\end{aligned}
$$

So,

$$
\begin{aligned}
L_1 &\le \varepsilon_p^{-d\kappa|m-3|}\int_0^{t\wedge\tau}\left|\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})-\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma}-X_s^{N,j,\varepsilon,\sigma})\right|^\kappa \\
&\cdot\left|\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma}-\bar{X}_s^{j,\varepsilon,\sigma})-\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma}-X_s^{N,j,\varepsilon,\sigma})\right|^\kappa ds \\
&\le \varepsilon_p^{-d\kappa|m-3|}\|D^2V^{\varepsilon_p}\|^\kappa_{L^\infty(\mathbb{R}^d)}\cdot\|\nabla V^{\varepsilon_p}\|^\kappa_{L^\infty(\mathbb{R}^d)}\int_0^{t\wedge\tau}\max_{i\in\{1,\ldots,N\}}|\bar{X}_s^{i,\varepsilon,\sigma}-X_s^{N,i,\varepsilon,\sigma}|^{2\kappa}ds \\
&\le \int_0^{t\wedge\tau}N^{\beta_p d|m-3|\kappa}N^{(2d+3)\beta_p\kappa}\cdot N^{-2a\kappa}S_s^2 ds
\end{aligned}
$$

This implies that

$$
\mathbb{E}[N^{a\kappa}\max_{i\in\{1,\ldots,N\}}L_1]\le N^{(\beta_p d|m-3|+(2d+3)\beta_p-a)\kappa}\int_0^t\mathbb{E}[S_s]ds
$$

For the L_2 term we do the following estimates

$$
\begin{aligned}
&\mathbb{E}[N^{a\kappa} \max_{i\in\{1,\dots,N\}} L_2] \\
&\leq C(\kappa,T)\mathbb{E}\Bigg[N^{a\kappa} \max_{i\in\{1,\dots,N\}} \int_0^{t\wedge\tau} \|p'_\lambda(V^{\varepsilon_p} * u^{\varepsilon,\sigma})\|^\kappa_{L^\infty(\mathbb{R})} \\
&\quad \cdot \left|\frac{1}{N}\sum_{j=1}^N \Big(\nabla V^{\varepsilon_p}(\bar{X}^{i,\varepsilon,\sigma}_s - \bar{X}^{j,\varepsilon,\sigma}_s) - \nabla V^{\varepsilon_p}(X^{N,i,\varepsilon,\sigma}_s - X^{N,j,\varepsilon,\sigma}_s)\Big)\right|^\kappa ds\Bigg] \\
&+ C(\kappa,T)\mathbb{E}\Bigg[N^{a\kappa} \max_{i\in\{1,\dots,N\}} \int_0^{t\wedge\tau} \left|p'_\lambda\left(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}^{i,\varepsilon,\sigma}_s - \bar{X}^{j,\varepsilon,\sigma}_s)\right) - p'_\lambda\Big(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s,\bar{X}^{i,\varepsilon,\sigma}_s)\Big)\right|^\kappa \\
&\quad \cdot \left|\frac{1}{N}\sum_{j=1}^N \Big(\nabla V^{\varepsilon_p}(\bar{X}^{i,\varepsilon,\sigma}_s - \bar{X}^{j,\varepsilon,\sigma}_s) - \nabla V^{\varepsilon_p}(X^{N,i,\varepsilon,\sigma}_s - X^{N,j,\varepsilon,\sigma}_s)\Big)\right|^\kappa ds\Bigg] \\
&\leq C(\kappa,T)\mathbb{E}\Bigg[N^{a\kappa} \max_{i\in\{1,\dots,N\}} \int_0^{t\wedge\tau} \left|\frac{1}{N}\sum_{j=1}^N D^2V^{\varepsilon_p}(\bar{X}^{i,\varepsilon,\sigma}_s - \bar{X}^{j,\varepsilon,\sigma}_s)\cdot\Big((\bar{X}^{i,\varepsilon,\sigma}_s - X^{N,i,\varepsilon,\sigma}_s) - (\bar{X}^{j,\varepsilon,\sigma}_s - X^{N,j,\varepsilon,\sigma}_s)\Big)\right|^\kappa \\
&+ \|D^3V^{\varepsilon_p}\|^\kappa_{L^\infty(\mathbb{R}^d)} \max_{i\in\{1,\dots,N\}} \left|\bar{X}^{i,\varepsilon,\sigma}_s - X^{N,i,\varepsilon,\sigma}_s\right|^{2\kappa} ds\Bigg] \\
&+ C(\kappa,T)\varepsilon_p^{-d\kappa|m-3|}\|D^2V^{\varepsilon_p}\|^\kappa_{L^\infty(\mathbb{R}^d)} \int_0^t \mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}^{i,\varepsilon,\sigma}_s - \bar{X}^{j,\varepsilon,\sigma}_s) - V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s,\bar{X}^{i,\varepsilon,\sigma}_s)\right|^\kappa S_s\right] ds
\end{aligned}
$$

Since $S_s \leq 1$ we deduce that

$$
\begin{aligned}
&\mathbb{E}[N^{a\kappa} \max_{i\in\{1,\dots,N\}} L_2] \\
&\leq C(\kappa,T)\mathbb{E}\Bigg[\int_0^t \Big(\|D^2V^{\varepsilon_p} * u^{\varepsilon,\sigma}\|^\kappa_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} + \||D^2V^{\varepsilon_p}| * u^{\varepsilon,\sigma}\|^\kappa_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} \\
&+ \|D^3V^{\varepsilon_p}\|^\kappa_{L^\infty(\mathbb{R}^d)} \cdot \frac{1}{N^{\alpha\kappa}}\Big)S_s\Bigg] \\
&+ C(\kappa,T)\mathbb{E}\left[\int_0^{t\wedge\tau} \left|\frac{1}{N}\sum_{j=1}^N D^2V^{\varepsilon_p}(\bar{X}^{i,\varepsilon,\sigma}_s - \bar{X}^{j,\varepsilon,\sigma}_s) - D^2V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s,\bar{X}^{i,\varepsilon,\sigma}_s)\right|^\kappa S_s\right] \\
&+ C(\kappa,T)\mathbb{E}\left[\int_0^{t\wedge\tau} \left|\frac{1}{N}\sum_{j=1}^N |D^2V^{\varepsilon_p}(\bar{X}^{i,\varepsilon,\sigma}_s - \bar{X}^{j,\varepsilon,\sigma}_s)| - |D^2V^{\varepsilon_p}| * u^{\varepsilon,\sigma}(s,\bar{X}^{i,\varepsilon,\sigma}_s)\right|^\kappa S_s\right] \\
&+ C(\kappa,T)N^{\beta_p d\kappa|m-3|}\cdot N^{(d+2)\beta_p\kappa} \int_0^t \mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}^{i,\varepsilon,\sigma}_s - \bar{X}^{j,\varepsilon,\sigma}_s) - V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s,\bar{X}^{i,\varepsilon,\sigma}_s)\right|^\kappa S_s\right] ds
\end{aligned}
$$

Denote for $(\bar{Y}^i)_{i\in\{1,\dots,N\}} = (\bar{X}_s^{i,\varepsilon,\sigma})_{i\in\{1,\dots,N\}}$

$$\begin{aligned}
\mathcal{A}^{(7)}_{\theta_{(7)}}(s) &:= \mathcal{A}^N_{\theta_{(7)}}(D^2V^{\varepsilon_p}(\cdot), u^{\varepsilon,\sigma}(s,\cdot)),\\
\mathcal{A}^{(8)}_{\theta_{(8)}}(s) &:= \mathcal{A}^N_{\theta_{(8)}}(|D^2V^{\varepsilon_p}(\cdot)|, u^{\varepsilon,\sigma}(s,\cdot)),\\
\mathcal{A}^{(9)}_{\theta_{(9)}}(s) &:= \mathcal{A}^N_{\theta_{(9)}}(V^{\varepsilon_p}(\cdot), u^{\varepsilon,\sigma}(s,\cdot)),
\end{aligned}$$

Without los of generality we choose $\theta_{(7)} = \theta_{(8)} = 0$ and so obtain that

$$\begin{aligned}
&\mathbb{E}[N^{a\kappa} \max_{i\in\{1,\dots,N\}} L_2]\\
&\le C(\kappa,T)\left(1 + N^{(\beta_p(d+3)-a)\kappa} + N^{(\beta_p d|m-3|+(d+2)\beta_p-\theta_{(9)})\kappa}\right)\int_0^t \mathbb{E}[S_s]ds\\
&+ N^{\beta_p(d+2)\kappa}\int_0^t \mathbb{P}(\mathcal{A}^{(7)}_{\theta_{(7)}}(s))ds + N^{\beta_p(d+2)\kappa}\int_0^t \mathbb{P}(\mathcal{A}^{(8)}_{\theta_{(8)}}(s))ds\\
&+ N^{(\beta_p d|m-3|+(d+2)\beta_p)\kappa}\cdot N^{\beta_p d\kappa}\int_0^t \mathbb{P}(\mathcal{A}^{(9)}_{\theta_{(9)}}(s))ds
\end{aligned}$$

This implies that

$$\begin{aligned}
&\mathbb{E}\left[N^{\kappa a}\max_{i\in\{1,\dots,N\}}\int_0^{t\wedge\tau}|A_{1,2}|^\kappa ds\right]\\
&\le C(\kappa,T)\mathbb{E}[N^{a\kappa}\max_{i\in\{1,\dots,N\}} L_1] + \mathbb{E}[N^{a\kappa}\max_{i\in\{1,\dots,N\}} L_2]\\
&\le C(\kappa,T)\left(1 + N^{(\beta_p d|m-3|+(2d+3)\beta_p-a)\kappa} + N^{(\beta_p(d+3)-a)\kappa} + N^{(\beta_p d|m-3|+(d+2)\beta_p-\theta_{(9)})\kappa}\right)\int_0^t \mathbb{E}[S_s]ds\\
&+ N^{\beta_p(d+2)\kappa}\int_0^t \mathbb{P}(\mathcal{A}^{(7)}_{\theta_{(7)}}(s))ds + N^{\beta_p(d+2)\kappa}\int_0^t \mathbb{P}(\mathcal{A}^{(8)}_{\theta_{(8)}}(s))ds + N^{\kappa(\beta_p d|m-3|+(2d+2)\beta_p)}\int_0^t \mathbb{P}(\mathcal{A}^{(9)}_{\theta_{(9)}}(s))ds
\end{aligned}$$

Therefore we deduce that for A_1 it holds that

$$\begin{aligned}
&\mathbb{E}\left[N^{\kappa a}\max_{i\in\{1,\dots,N\}}\int_0^{t\wedge\tau}|A_1|^\kappa ds\right]\\
&\le C(\kappa,T)\Big(1 + N^{\kappa\left(\beta_p d|m-4|+2\beta_p(d+1)-\theta_{(4)}\right)} + N^{\kappa\left(\beta_p(d+1)-\theta_{(5)}\right)} + N^{-\theta_{(6)}\kappa}\\
&\qquad + N^{\kappa(\beta_p d|m-3|+\beta_p(2d+3)-a)} + N^{\kappa(\beta_p d|m-4|+3\beta_p(d+1)-a)}\\
&\qquad + N^{(\beta_p d|m-3|+(2d+3)\beta_p-a)\kappa} + N^{(\beta_p(d+3)-a)\kappa} + N^{(\beta_p d|m-3|+(d+2)\beta_p-\theta_{(9)})\kappa}\Big)\int_0^t \mathbb{E}[S_s]ds\\
&+ C(\kappa,T)N^{\kappa(\beta_p d|m-4|+\beta_p(3d+2))}\int_0^t \mathbb{P}(\mathcal{A}^{(4)}_{\theta_{(4)}}(s))ds + C(\kappa,T)N^{\kappa(2\beta_p(d+1))}\int_0^t \mathbb{P}(\mathcal{A}^{(5)}_{\theta_{(5)}}(s))ds\\
&+ C(\kappa,T)N^{\kappa\beta_p(d+1)}\int_0^t \mathbb{P}(\mathcal{A}^{(6)}_{\theta_{(6)}}(s))ds\\
&+ N^{\beta_p(d+2)\kappa}\int_0^t \mathbb{P}(\mathcal{A}^{(7)}_{\theta_{(7)}}(s))ds + N^{\beta_p(d+2)\kappa}\int_0^t \mathbb{P}(\mathcal{A}^{(8)}_{\theta_{(8)}}(s))ds + N^{\kappa(\beta_p d|m-3|+(2d+2)\beta_p)}\int_0^t \mathbb{P}(\mathcal{A}^{(9)}_{\theta_{(9)}}(s))ds
\end{aligned}\tag{4.6}$$

Step 2(c) Estimates of A_2

Now we are going to study A_2. Since

$$\begin{aligned}|A_{2,1}| &= \left|p'_\lambda(V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma}) - p'_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\Big)\right| \cdot \left|\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma})\right| \\ &\le \|p''_\lambda\|_{L^\infty(\mathbb{R})}\|\nabla V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)}\left|\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) - V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\right|\end{aligned}$$

we denote for $(\bar{Y}^i)_{i\in\{1,\dots,N\}} = (\bar{X}_s^{i,\varepsilon,\sigma})_{i\in\{1,\dots,N\}}$

$$\mathcal{A}^{(10)}_{\theta_{(10)}}(s) := \mathcal{A}^N_{\theta_{(10)}}(V^{\varepsilon_p}(\cdot), u^{\varepsilon,\sigma}(s,\cdot))$$

in order to get that

$$\begin{aligned}&\mathbb{E}\left[N^{a\kappa}\max_{i\in\{1,\dots,N\}}\int_0^{t\wedge\tau}|A_{2,1}|^\kappa ds\right] \\ &\le C(\kappa,T)\varepsilon_p^{-d\kappa|m-3|}N^{\beta_p(d+1)\kappa}\cdot N^{a\kappa}\cdot\int_0^t\mathbb{E}\left[\max_{i\in\{1,\dots,N\}}\left|\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) - V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\right|^\kappa\right]ds \\ &\le C(\kappa,T)N^{d|m-3|\beta_p\kappa}\cdot N^{(\beta_p(d+1)+a)\kappa}\left(N^{-\theta_{(10)}\kappa} + N^{\beta_p d\kappa}\int_0^t\mathbb{P}(\mathcal{A}^{(10)}_{\theta_{(10)}}(s))ds\right)\end{aligned}$$

Let's analyze $A_{2,2}$. We denote for $(\bar{Y}^i)_{i\in\{1,\dots,N\}} = (\bar{X}_s^{i,\varepsilon,\sigma})_{i\in\{1,\dots,N\}}$

$$\mathcal{A}^{(11)}_{\theta_{(11)}}(s) := \mathcal{A}^N_{\theta_{(11)}}(\nabla V^{\varepsilon_p}(\cdot), u^{\varepsilon,\sigma}(s,\cdot))$$

and so obtain that

$$\begin{aligned}&\mathbb{E}\left[N^{a\kappa}\max_{i\in\{1,\dots,N\}}\int_0^{t\wedge\tau}|A_{2,2}|^\kappa ds\right] \\ &\le \|p'_\lambda(V^{\varepsilon_p} * u^{\varepsilon,\sigma})\|^\kappa_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}N^{a\kappa} \\ &\quad\cdot\int_0^t\mathbb{E}\left[\max_{i\in\{1,\dots,N\}}\left|\frac{1}{N}\sum_{j=1}^N \nabla V^{\varepsilon_p}(\bar{X}_s^{i,\varepsilon,\sigma} - \bar{X}_s^{j,\varepsilon,\sigma}) - \nabla V^{\varepsilon_p} * u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\right|^\kappa\right]ds \\ &\le C(\kappa,T)N^{a\kappa}\left(N^{-\theta_{(11)}\kappa} + N^{\beta_p(d+1)\kappa}\int_0^t\mathbb{P}(\mathcal{A}^{(11)}_{\theta_{(11)}}(s))ds\right)\end{aligned}$$

So we obtain that

$$\begin{aligned}&\mathbb{E}\left[N^{a\kappa}\max_{i\in\{1,\dots,N\}}\int_0^{t\wedge\tau}|A_2|^\kappa ds\right] \\ &\le C(\kappa,T)\left(N^{\kappa(d|m-3|\beta_p+\beta_p(d+1)+a-\theta_{(10)})} + N^{\kappa(a-\theta_{(11)})}\right) \\ &+ C(\kappa,T)\left(N^{\kappa(d|m-3|\beta_p+\beta_p(2d+1)+a)}\int_0^t\mathbb{P}(\mathcal{A}^{(10)}_{\theta_{(10)}}(s))ds + N^{\kappa(\beta_p(d+1)+a)}\int_0^t\mathbb{P}(\mathcal{A}^{(11)}_{\theta_{(11)}}(s))ds\right) \qquad (4.7)\end{aligned}$$

Step 2(d) Final result for porous medium term

$$
\begin{aligned}
&\mathbb{E}\left[N^{\kappa a}\max_{i\in\{1,\dots,N\}}\int_0^{t\wedge\tau}|-\nabla p_\lambda\Big(\frac{1}{N}\sum_{j=1}^N V^{\varepsilon_p}(X_s^{N,i,\varepsilon,\sigma}-X_s^{N,j,\varepsilon,\sigma})\Big)+\nabla p_\lambda\Big(V^{\varepsilon_p}*u^{\varepsilon,\sigma}(s,\bar{X}_s^{i,\varepsilon,\sigma})\Big)|^\kappa ds\right]\\
&\le C(\kappa,T)\Big(1+N^{\kappa\big(\beta_p d|m-4|+2\beta_p(d+1)-\theta_{(4)}\big)}+N^{\kappa\big(\beta_p(d+1)-\theta_{(5)}\big)}+N^{-\theta_{(6)}\kappa}\\
&\qquad+N^{\kappa(\beta_p d|m-3|+\beta_p(2d+3)-a)}+N^{\kappa(\beta_p d|m-4|+3\beta_p(d+1)-a)}\\
&\qquad+N^{(\beta_p d|m-3|+(2d+3)\beta_p-a)\kappa}+N^{(\beta_p(d+3)-a)\kappa}+N^{(\beta_p d|m-3|+(d+2)\beta_p-\theta_{(9)})\kappa}\Big)\int_0^t\mathbb{E}[S_s]ds\\
&+C(\kappa,T)\bigg(N^{\kappa(\beta_p d|m-4|+\beta_p(3d+2))}\int_0^t\mathbb{P}(\mathcal{A}^{(4)}_{\theta_{(4)}}(s))ds+N^{\kappa(2\beta_p(d+1))}\int_0^t\mathbb{P}(\mathcal{A}^{(5)}_{\theta_{(5)}}(s))ds\\
&\qquad+N^{\kappa\beta_p(d+1)}\int_0^t\mathbb{P}(\mathcal{A}^{(6)}_{\theta_{(6)}}(s))ds+N^{\beta_p(d+2)\kappa}\int_0^t\mathbb{P}(\mathcal{A}^{(7)}_{\theta_{(7)}}(s))ds\\
&\qquad+N^{\beta_p(d+2)\kappa}\int_0^t\mathbb{P}(\mathcal{A}^{(8)}_{\theta_{(8)}}(s))ds+N^{\kappa(\beta_p d|m-3|+(2d+2)\beta_p)}\int_0^t\mathbb{P}(\mathcal{A}^{(9)}_{\theta_{(9)}}(s))ds\\
&\qquad+N^{\kappa(d|m-3|\beta_p+\beta_p(2d+1)+a)}\int_0^t\mathbb{P}(\mathcal{A}^{(10)}_{\theta_{(10)}}(s))ds+N^{\kappa(\beta_p(d+1)+a)}\int_0^t\mathbb{P}(\mathcal{A}^{(11)}_{\theta_{(11)}}(s))ds\bigg)\\
&+C(\kappa,T)\left(N^{\kappa(d|m-3|\beta_p+\beta_p(d+1)+a-\theta_{(10)})}+N^{\kappa(a-\theta_{(11)})}\right)
\end{aligned}
\tag{4.8}
$$

Final Estimate for S

$$
\begin{aligned}
&\mathbb{E}[S_t] \\
&\leq C(\kappa,T)\Bigg(1 + N^{\kappa((d+1)\beta_k - a)} + N^{\kappa\left(\beta_p d|m-4| + 2\beta_p(d+1) - \theta_{(4)}\right)} + N^{\kappa\left(\beta_p(d+1) - \theta_{(5)}\right)} + N^{-\theta_{(6)}\kappa} \\
&\qquad + N^{\kappa(\beta_p d|m-3| + \beta_p(2d+3) - a)} + N^{\kappa(\beta_p d|m-4| + 3\beta_p(d+1) - a)} \\
&\qquad + N^{(\beta_p d|m-3| + (2d+3)\beta_p - a)\kappa} + N^{(\beta_p(d+3) - a)\kappa} + N^{(\beta_p d|m-3| + (d+2)\beta_p - \theta_{(9)})\kappa}\Bigg) \cdot \int_0^t \mathbb{E}[S_s] ds \\
&\quad + C(\kappa,T)\Bigg(N^{d\kappa\beta_k} \int_0^t \mathbb{P}(\mathcal{A}^{(1)}_{\theta_{(1)}}(s)) ds + N^{d\kappa\beta_k} \int_0^t \mathbb{P}(\mathcal{A}^{(2)}_{\theta_{(2)}}(s)) ds \\
&\qquad\qquad + N^{\kappa((d-1)\beta_k + a)} \int_0^t \mathbb{P}(\mathcal{A}^{(3)}_{\theta_{(3)}}(s)) ds \\
&\qquad\qquad + N^{\kappa(\beta_p d|m-4| + \beta_p(3d+2))} \int_0^t \mathbb{P}(\mathcal{A}^{(4)}_{\theta_{(4)}}(s)) ds \\
&\qquad\qquad + N^{\kappa(2\beta_p(d+1))} \int_0^t \mathbb{P}(\mathcal{A}^{(5)}_{\theta_{(5)}}(s)) ds \\
&\qquad\qquad + N^{\kappa\beta_p(d+1)} \int_0^t \mathbb{P}(\mathcal{A}^{(6)}_{\theta_{(6)}}(s)) ds \\
&\qquad\qquad + N^{\beta_p(d+2)\kappa} \int_0^t \mathbb{P}(\mathcal{A}^{(7)}_{\theta_{(7)}}(s)) ds \\
&\qquad\qquad + N^{\beta_p(d+2)\kappa} \int_0^t \mathbb{P}(\mathcal{A}^{(8)}_{\theta_{(8)}}(s)) ds \\
&\qquad\qquad + N^{\kappa(\beta_p d|m-3| + (2d+2)\beta_p)} \int_0^t \mathbb{P}(\mathcal{A}^{(9)}_{\theta_{(9)}}(s)) ds \\
&\qquad\qquad + N^{\kappa(d|m-3|\beta_p + \beta_p(2d+1) + a)} \int_0^t \mathbb{P}(\mathcal{A}^{(10)}_{\theta_{(10)}}(s)) ds \\
&\qquad\qquad + N^{\kappa(\beta_p(d+1) + a)} \int_0^t \mathbb{P}(\mathcal{A}^{(11)}_{\theta_{(11)}}(s)) ds\Bigg) \\
&\quad + C(\kappa,T)\left(N^{\kappa(a - \theta_{(3)})} + N^{\kappa(d|m-3|\beta_p + \beta_p(d+1) + a - \theta_{(10)})} + N^{\kappa(a - \theta_{(11)})}\right)
\end{aligned}
\tag{4.9}
$$

Estimates of the Probabilities

Using Lemma 4.2 we obtain that

$$\mathbb{P}(\mathcal{A}^{(1)}_{\theta_{(1)}}(s)) \leq N^{2\tilde{\kappa}_{(1)}(\theta_{(1)}-\frac{1}{2})+1}C(\tilde{\kappa}_{(1)})\left(\|D^2\Phi^{\varepsilon_k}\|^{2\tilde{\kappa}_{(1)}}_{L^\infty(\mathbb{R}^d)} + \|D^2\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}\|^{2\tilde{\kappa}_{(1)}}_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\right)$$
$$\mathbb{P}(\mathcal{A}^{(2)}_{\theta_{(2)}}(s)) \leq N^{2\tilde{\kappa}_{(2)}(\theta_{(2)}-\frac{1}{2})+1}C(\tilde{\kappa}_{(2)})\left(\|D^2\Phi^{\varepsilon_k}\|^{2\tilde{\kappa}_{(2)}}_{L^\infty(\mathbb{R}^d)} + \||D^2\Phi^{\varepsilon_k}| * u^{\varepsilon,\sigma}\|^{2\tilde{\kappa}_{(2)}}_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\right)$$
$$\mathbb{P}(\mathcal{A}^{(3)}_{\theta_{(3)}}(s)) \leq N^{2\tilde{\kappa}_{(3)}(\theta_{(3)}-\frac{1}{2})+1}C(\tilde{\kappa}_{(3)})\left(\|\nabla\Phi^{\varepsilon_k}\|^{2\tilde{\kappa}_{(3)}}_{L^\infty(\mathbb{R}^d)} + \|\nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}\|^{2\tilde{\kappa}_{(3)}}_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\right)$$
$$\mathbb{P}(\mathcal{A}^{(4)}_{\theta_{(4)}}(s)) \leq N^{2\tilde{\kappa}_{(4)}(\theta_{(4)}-\frac{1}{2})+1}C(\tilde{\kappa}_{(4)})\left(\|V^{\varepsilon_p}\|^{2\tilde{\kappa}_{(4)}}_{L^\infty(\mathbb{R}^d)} + \|V^{\varepsilon_p} * u^{\varepsilon,\sigma}\|^{2\tilde{\kappa}_{(4)}}_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\right)$$
$$\mathbb{P}(\mathcal{A}^{(5)}_{\theta_{(5)}}(s)) \leq N^{2\tilde{\kappa}_{(5)}(\theta_{(5)}-\frac{1}{2})+1}C(\tilde{\kappa}_{(5)})\left(\|\nabla V^{\varepsilon_p}\|^{2\tilde{\kappa}_{(5)}}_{L^\infty(\mathbb{R}^d)} + \||\nabla V^{\varepsilon_p}| * u^{\varepsilon,\sigma}\|^{2\tilde{\kappa}_{(5)}}_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\right)$$
$$\mathbb{P}(\mathcal{A}^{(6)}_{\theta_{(6)}}(s)) \leq N^{2\tilde{\kappa}_{(6)}(\theta_{(6)}-\frac{1}{2})+1}C(\tilde{\kappa}_{(6)})\left(\|\nabla V^{\varepsilon_p}\|^{2\tilde{\kappa}_{(6)}}_{L^\infty(\mathbb{R}^d)} + \|\nabla V^{\varepsilon_p} * u^{\varepsilon,\sigma}\|^{2\tilde{\kappa}_{(6)}}_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\right)$$
$$\mathbb{P}(\mathcal{A}^{(7)}_{\theta_{(7)}}(s)) \leq N^{2\tilde{\kappa}_{(7)}(\theta_{(7)}-\frac{1}{2})+1}C(\tilde{\kappa}_{(7)})\left(\|D^2 V^{\varepsilon_p}\|^{2\tilde{\kappa}_{(7)}}_{L^\infty(\mathbb{R}^d)} + \|D^2 V^{\varepsilon_p} * u^{\varepsilon,\sigma}\|^{2\tilde{\kappa}_{(7)}}_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\right)$$
$$\mathbb{P}(\mathcal{A}^{(8)}_{\theta_{(8)}}(s)) \leq N^{2\tilde{\kappa}_{(8)}(\theta_{(8)}-\frac{1}{2})+1}C(\tilde{\kappa}_{(8)})\left(\|D^2 V^{\varepsilon_p}\|^{2\tilde{\kappa}_{(8)}}_{L^\infty(\mathbb{R}^d)} + \||D^2 V^{\varepsilon_p}| * u^{\varepsilon,\sigma}\|^{2\tilde{\kappa}_{(8)}}_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\right)$$
$$\mathbb{P}(\mathcal{A}^{(9)}_{\theta_{(9)}}(s)) \leq N^{2\tilde{\kappa}_{(9)}(\theta_{(9)}-\frac{1}{2})+1}C(\tilde{\kappa}_{(9)})\left(\|V^{\varepsilon_p}\|^{2\tilde{\kappa}_{(9)}}_{L^\infty(\mathbb{R}^d)} + \|V^{\varepsilon_p} * u^{\varepsilon,\sigma}\|^{2\tilde{\kappa}_{(9)}}_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\right)$$
$$\mathbb{P}(\mathcal{A}^{(10)}_{\theta_{(10)}}(s)) \leq N^{2\tilde{\kappa}_{(10)}(\theta_{(10)}-\frac{1}{2})+1}C(\tilde{\kappa}_{(10)})\left(\|V^{\varepsilon_p}\|^{2\tilde{\kappa}_{(10)}}_{L^\infty(\mathbb{R}^d)} + \|V^{\varepsilon_p} * u^{\varepsilon,\sigma}\|^{2\tilde{\kappa}_{(10)}}_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\right)$$
$$\mathbb{P}(\mathcal{A}^{(11)}_{\theta_{(11)}}(s)) \leq N^{2\tilde{\kappa}_{(11)}(\theta_{(11)}-\frac{1}{2})+1}C(\tilde{\kappa}_{(11)})\left(\|\nabla V^{\varepsilon_p}\|^{2\tilde{\kappa}_{(11)}}_{L^\infty(\mathbb{R}^d)} + \|\nabla V^{\varepsilon_p} * u^{\varepsilon,\sigma}\|^{2\tilde{\kappa}_{(11)}}_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\right)$$

Choosing $\varepsilon_k = N^{-\beta_k}$ and $\varepsilon_p = N^{-\beta_p}$ it follows that

$$\mathbb{P}(\mathcal{A}^{(1)}_{\theta_{(1)}}(s)) \leq N^{2\tilde{\kappa}_{(1)}((\theta_{(1)}-\frac{1}{2})+d\beta_k)+1}C(\tilde{\kappa}_{(1)})$$
$$\mathbb{P}(\mathcal{A}^{(2)}_{\theta_{(2)}}(s)) \leq N^{2\tilde{\kappa}_{(2)}((\theta_{(2)}-\frac{1}{2})+d\beta_k)+1}C(\tilde{\kappa}_{(2)})$$
$$\mathbb{P}(\mathcal{A}^{(3)}_{\theta_{(3)}}(s)) \leq N^{2\tilde{\kappa}_{(3)}((\theta_{(3)}-\frac{1}{2})+(d-1)\beta_k)+1}C(\tilde{\kappa}_{(3)})$$
$$\mathbb{P}(\mathcal{A}^{(4)}_{\theta_{(4)}}(s)) \leq N^{2\tilde{\kappa}_{(4)}((\theta_{(4)}-\frac{1}{2})+d\beta_p)+1}C(\tilde{\kappa}_{(4)})$$
$$\mathbb{P}(\mathcal{A}^{(5)}_{\theta_{(5)}}(s)) \leq N^{2\tilde{\kappa}_{(5)}((\theta_{(5)}-\frac{1}{2})+(d+1)\beta_p)+1}C(\tilde{\kappa}_{(5)})$$
$$\mathbb{P}(\mathcal{A}^{(6)}_{\theta_{(6)}}(s)) \leq N^{2\tilde{\kappa}_{(6)}((\theta_{(6)}-\frac{1}{2})+(d+1)\beta_p)+1}C(\tilde{\kappa}_{(6)})$$
$$\mathbb{P}(\mathcal{A}^{(7)}_{\theta_{(7)}}(s)) \leq N^{2\tilde{\kappa}_{(7)}((\theta_{(7)}-\frac{1}{2})+(d+2)\beta_p)+1}C(\tilde{\kappa}_{(7)})$$
$$\mathbb{P}(\mathcal{A}^{(8)}_{\theta_{(8)}}(s)) \leq N^{2\tilde{\kappa}_{(8)}((\theta_{(8)}-\frac{1}{2})+(d+2)\beta_p)+1}C(\tilde{\kappa}_{(8)})$$
$$\mathbb{P}(\mathcal{A}^{(9)}_{\theta_{(9)}}(s)) \leq N^{2\tilde{\kappa}_{(9)}((\theta_{(9)}-\frac{1}{2})+d\beta_p)+1}C(\tilde{\kappa}_{(9)})$$
$$\mathbb{P}(\mathcal{A}^{(10)}_{\theta_{(10)}}(s)) \leq N^{2\tilde{\kappa}_{(10)}((\theta_{(10)}-\frac{1}{2})+d\beta_p)+1}C(\tilde{\kappa}_{(10)})$$
$$\mathbb{P}(\mathcal{A}^{(11)}_{\theta_{(11)}}(s)) \leq N^{2\tilde{\kappa}_{(11)}((\theta_{(11)}-\frac{1}{2})+(d+1)\beta_p)+1}C(\tilde{\kappa}_{(11)})$$

Recall that

$$\|D^2\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}, \||D^2\Phi^{\varepsilon_k}| * u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}, \|\nabla\Phi^{\varepsilon_k} * u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))},$$
$$\|V^{\varepsilon_p} * u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}, \||\nabla V^{\varepsilon_p}| * u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}, \|\nabla V^{\varepsilon_p} * u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))},$$
$$\|D^2V^{\varepsilon_p} * u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}, \||D^2V^{\varepsilon_p}| * u^{\varepsilon,\sigma}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}$$

are all bounded and

$$\|\nabla\Phi^{\varepsilon_k}\|_{L^\infty(\mathbb{R}^d)} \le \frac{C}{\varepsilon_k^{d-1}}, \quad \|D^2\Phi^{\varepsilon_k}\|_{L^\infty(\mathbb{R}^d)} \le \frac{C}{\varepsilon_k^{d}},$$
$$\|V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)} \le \frac{C}{\varepsilon_p^{d}}, \quad \|\nabla V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)} \le \frac{C}{\varepsilon_p^{d+1}}, \quad \|D^2V^{\varepsilon_p}\|_{L^\infty(\mathbb{R}^d)} \le \frac{C}{\varepsilon_p^{d+2}}$$

Choice of Parameters

Now we need to choose required parameters in (4.9).

Step 1 Coefficient in front of $\mathbb{E}[S]$

Step 1 (a) Restrictions for a

Aggregation Term

For a given β_k we choose a such that

$$(d+1)\beta_k \le a. \tag{4.10}$$

Therefore, we obtain that $N^{(d+1)\beta_k - a}$ is bounded.

Porous Medium Term

For a given β_p we add another restrictions on a such that

$$\begin{aligned}
&\beta_p d|m-3| + \beta_p(2d+3) \le a,\\
&\beta_p d|m-4| + 3\beta_p(d+1) \le a,\\
&\beta_p(d+3) \le a.
\end{aligned}$$

We can see that $\beta_p d|m-4| + 3\beta_p(d+1) \le a$ is the strongest restriction, so we obtain that β_p should satisfy:

$$\beta_p d|m-4| + 3\beta_p(d+1) \le a. \tag{4.11}$$

Step 1 (b) Restrictions for $\theta_{(4)}, \theta_{(5)}, \theta_{(6)}, \theta_{(9)}$ Next, we see that

$$\begin{aligned} &\beta_p d|m-4| + 2\beta_p(d+1) \le \theta_{(4)} < \frac{1}{2}, \\ &\beta_p(d+1) \le \theta_{(5)} < \frac{1}{2}, \\ &0 \le \theta_{(6)} < \frac{1}{2}, \\ &\beta_p d|m-3| + (d+2)\beta_p \le \theta_{(9)} < \frac{1}{2}. \end{aligned} \tag{4.12}$$

should be satisfied.

Step 2 Constant terms

If the following conditions

$$\begin{aligned} &a < \theta_{(3)} < \frac{1}{2}, \\ &d|m-3|\beta_p + \beta_p(d+1) + a < \theta_{(10)} < \frac{1}{2}, \\ &a < \theta_{(11)} < \frac{1}{2}. \end{aligned} \tag{4.13}$$

are satisfied, then for an arbitrary $\gamma > 0$ we can choose $\kappa > 0$ such that

$$N^{\kappa(a-\theta_{(3)})} + N^{\kappa(d|m-3|\beta_p+\beta_p(d+1)+a-\theta_{(10)})} + N^{\kappa(a-\theta_{(11)})} \le CN^{-\gamma}.$$

Step 3 Probability terms

Let us analyze probability terms

Aggregation Term

$$\begin{aligned} &C(\kappa,T)\left(N^{d\kappa\beta_k}\int_0^t \mathbb{P}(\mathcal{A}^{(1)}_{\theta_{(1)}}(s))ds + N^{d\kappa\beta_k}\int_0^t \mathbb{P}(\mathcal{A}^{(2)}_{\theta_{(2)}}(s))ds + N^{\kappa((d-1)\beta_k+a)}\int_0^t \mathbb{P}(\mathcal{A}^{(3)}_{\theta_{(3)}}(s))ds\right) \\ &\le C(t,\tilde{\kappa}_{(1)}) \cdot N^{d\kappa\beta_k} \cdot N^{2\tilde{\kappa}_{(1)}((\theta_{(1)}-\frac{1}{2})+d\beta_k)+1} \\ &+ C(t,\tilde{\kappa}_{(2)}) \cdot N^{d\kappa\beta_k} \cdot N^{2\tilde{\kappa}_{(2)}((\theta_{(2)}-\frac{1}{2})+d\beta_k)+1} \\ &+ C(t,\tilde{\kappa}_{(3)}) N^{\kappa((d-1)\beta_k+a)} N^{2\tilde{\kappa}_{(3)}((\theta_{(3)}-\frac{1}{2})+(d-1)\beta_k)+1} \end{aligned}$$

Now we choose $\theta_{(1)}, \theta_{(2)}, \theta_{(3)}$ such that

$$\begin{aligned} &\theta_{(1)} = \theta_{(2)}, \\ &\theta_{(1)} - \frac{1}{2} + d\beta_k < 0, \\ &(\theta_{(3)} - \frac{1}{2}) + (d-1)\beta_k < 0. \end{aligned} \tag{4.14}$$

Then we choose $\tilde{\kappa}_{(1)}, \tilde{\kappa}_{(2)}, \tilde{\kappa}_{(3)}$ big enough such that

$$\begin{aligned}
&C(t, \tilde{\kappa}_{(1)}) \cdot N^{d\kappa\beta_k} \cdot N^{2\tilde{\kappa}_{(1)}((\theta_{(1)}-\frac{1}{2})+d\beta_k)+1} \\
&+ C(t, \tilde{\kappa}_{(2)}) \cdot N^{d\kappa\beta_k} \cdot N^{2\tilde{\kappa}_{(2)}((\theta_{(2)}-\frac{1}{2})+d\beta_k)+1} \\
&+ C(t, \tilde{\kappa}_{(3)}) N^{\kappa((d-1)\beta_k+a)} N^{2\tilde{\kappa}_{(3)}((\theta_{(3)}-\frac{1}{2})+(d-1)\beta_k)+1} \\
&\leq CN^{-\gamma}
\end{aligned}$$

Moreover, we establish now the range for β_k. Recall (4.10) and (4.14). and deduce that

$$\begin{aligned}
&(d+1)\beta_k \leq a \\
&\theta_{(1)} - \frac{1}{2} + d\beta_k < 0 \\
&(\theta_{(3)} - \frac{1}{2}) + (d-1)\beta_k < 0.
\end{aligned}$$

We choose $\theta_{(1)} = \theta_{(2)} = 0$. Since $a < \theta_{(3)} < \frac{1}{2}$, we deduce that if we choose

$$\beta_k < \frac{1}{4d}, \tag{4.15}$$

then we can choose a such that

$$(d+1)\beta_k \leq a < \frac{1}{2} - (d-1)\beta_k \tag{4.16}$$

and so ensure convergence in probability.

Porous Medium Term

Now we study probability terms for the porous medium part.

$$\begin{aligned}
\tilde{Q} :=& N^{\kappa(\beta_p d|m-4|+\beta_p(3d+2))} \int_0^t \mathbb{P}(\mathcal{A}^{(4)}_{\theta_{(4)}}(s))ds \\
&+ N^{\kappa(2\beta_p(d+1))} \int_0^t \mathbb{P}(\mathcal{A}^{(5)}_{\theta_{(5)}}(s))ds \\
&+ N^{\kappa\beta_p(d+1)} \int_0^t \mathbb{P}(\mathcal{A}^{(6)}_{\theta_{(6)}}(s))ds \\
&+ N^{\beta_p(d+2)\kappa} \int_0^t \mathbb{P}(\mathcal{A}^{(7)}_{\theta_{(7)}}(s))ds \\
&+ N^{\beta_p(d+2)\kappa} \int_0^t \mathbb{P}(\mathcal{A}^{(8)}_{\theta_{(8)}}(s))ds \\
&+ N^{\kappa(\beta_p d|m-3|+(2d+2)\beta_p)} \int_0^t \mathbb{P}(\mathcal{A}^{(9)}_{\theta_{(9)}}(s))ds \\
&+ N^{\kappa(d|m-3|\beta_p+\beta_p(2d+1)+a)} \int_0^t \mathbb{P}(\mathcal{A}^{(10)}_{\theta_{(10)}}(s))ds \\
&+ N^{\kappa(\beta_p(d+1)+a)} \int_0^t \mathbb{P}(\mathcal{A}^{(11)}_{\theta_{(11)}}(s))ds
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\tilde{Q} \leq & C(t, \tilde{\kappa}_{(4)}) N^{\kappa(\beta_p d|m-4| + \beta_p(3d+2))} N^{2\tilde{\kappa}_{(4)}((\theta_{(4)} - \frac{1}{2}) + d\beta_p) + 1} \\
& + C(t, \tilde{\kappa}_{(5)}) N^{\kappa(2\beta_p(d+1))} N^{2\tilde{\kappa}_{(5)}((\theta_{(5)} - \frac{1}{2}) + (d+1)\beta_p) + 1} \\
& + C(t, \tilde{\kappa}_{(6)}) N^{\kappa\beta_p(d+1)} N^{2\tilde{\kappa}_{(6)}((\theta_{(6)} - \frac{1}{2}) + (d+1)\beta_p) + 1} \\
& + C(t, \tilde{\kappa}_{(7)}) N^{\beta_p(d+2)\kappa} N^{2\tilde{\kappa}_{(7)}((\theta_{(7)} - \frac{1}{2}) + (d+2)\beta_p) + 1} \\
& + C(t, \tilde{\kappa}_{(8)}) N^{\beta_p(d+2)\kappa} N^{2\tilde{\kappa}_{(8)}((\theta_{(8)} - \frac{1}{2}) + (d+2)\beta_p) + 1} \\
& + C(t, \tilde{\kappa}_{(9)}) N^{\kappa(\beta_p d|m-3| + (2d+2)\beta_p)} N^{2\tilde{\kappa}_{(9)}((\theta_{(9)} - \frac{1}{2}) + d\beta_p) + 1} \\
& + C(t, \tilde{\kappa}_{(10)}) N^{\kappa(d|m-3|\beta_p + \beta_p(2d+1) + a)} N^{2\tilde{\kappa}_{(10)}((\theta_{(10)} - \frac{1}{2}) + d\beta_p) + 1} \\
& + C(t, \tilde{\kappa}_{(11)}) N^{\kappa(\beta_p(d+1) + a)} N^{2\tilde{\kappa}_{(11)}((\theta_{(11)} - \frac{1}{2}) + (d+1)\beta_p) + 1}
\end{aligned}
$$

Now choose $\theta_{(4)}, \theta_{(5)}, \theta_{(6)}, \theta_{(7)}, \theta_{(8)}, \theta_{(9)}, \theta_{(10)}, \theta_{(11)}$ such that

$$
\begin{aligned}
0 < & \theta_{(4)} < \frac{1}{2} - d\beta_p, \\
0 < & \theta_{(5)} < \frac{1}{2} - (d+1)\beta_p, \\
0 = & \theta_{(6)} < \frac{1}{2} - (d+1)\beta_p, \\
& \theta_{(7)} = \theta_{(8)} = 0, \\
0 = & \theta_{(7)} < \frac{1}{2} - (d+2)\beta_p, \\
0 < & \theta_{(9)} < \frac{1}{2} - d\beta_p, \\
0 < & \theta_{(10)} < \frac{1}{2} - d\beta_p, \\
0 < & \theta_{(11)} < \frac{1}{2} - (d+1)\beta_p.
\end{aligned}
$$

Then we choose $\tilde{\kappa}_{(4)} - \tilde{\kappa}_{(11)}$ big enough such that

$$
\begin{aligned}
\tilde{Q} \leq & C(t, \tilde{\kappa}_{(4)}) N^{\kappa(\beta_p d|m-4| + \beta_p(3d+2))} N^{2\tilde{\kappa}_{(4)}((\theta_{(4)} - \frac{1}{2}) + d\beta_p) + 1} \\
& + C(t, \tilde{\kappa}_{(5)}) N^{\kappa(2\beta_p(d+1))} N^{2\tilde{\kappa}_{(5)}((\theta_{(5)} - \frac{1}{2}) + (d+1)\beta_p) + 1} \\
& + C(t, \tilde{\kappa}_{(6)}) N^{\kappa\beta_p(d+1)} N^{2\tilde{\kappa}_{(6)}((\theta_{(6)} - \frac{1}{2}) + (d+1)\beta_p) + 1} \\
& + C(t, \tilde{\kappa}_{(7)}) N^{\beta_p(d+2)\kappa} N^{2\tilde{\kappa}_{(7)}((\theta_{(7)} - \frac{1}{2}) + (d+2)\beta_p) + 1} \\
& + C(t, \tilde{\kappa}_{(8)}) N^{\beta_p(d+2)\kappa} N^{2\tilde{\kappa}_{(8)}((\theta_{(8)} - \frac{1}{2}) + (d+2)\beta_p) + 1} \\
& + C(t, \tilde{\kappa}_{(9)}) N^{\kappa(\beta_p d|m-3| + (2d+2)\beta_p)} N^{2\tilde{\kappa}_{(9)}((\theta_{(9)} - \frac{1}{2}) + d\beta_p) + 1} \\
& + C(t, \tilde{\kappa}_{(10)}) N^{\kappa(d|m-3|\beta_p + \beta_p(2d+1) + a)} N^{2\tilde{\kappa}_{(10)}((\theta_{(10)} - \frac{1}{2}) + d\beta_p) + 1} \\
& + C(t, \tilde{\kappa}_{(11)}) N^{\kappa(\beta_p(d+1) + a)} N^{2\tilde{\kappa}_{(11)}((\theta_{(11)} - \frac{1}{2}) + (d+1)\beta_p) + 1} \\
& \leq CN^{-\gamma}.
\end{aligned}
$$

Recall (4.12) and (4.13). Therefore, we obtain the following restrictions for $\theta_{(4)} - \theta_{(11)}$

$$\begin{aligned}
\beta_p d|m-4| + 2\beta_p(d+1) \leq & \theta_{(4)} < \frac{1}{2} - d\beta_p, \\
\beta_p(d+1) \leq & \theta_{(5)} < \frac{1}{2} - (d+1)\beta_p, \\
0 = & \theta_{(6)} < \frac{1}{2} - (d+1)\beta_p, \\
& \theta_{(7)} = \theta_{(8)} = 0, \\
0 = & \theta_{(7)} < \frac{1}{2} - (d+2)\beta_p, \\
\beta_p d|m-3| + (d+2)\beta_p \leq & \theta_{(9)} < \frac{1}{2} - d\beta_p, \\
d|m-3|\beta_p + \beta_p(d+1) + a < & \theta_{(10)} < \frac{1}{2} - d\beta_p, \\
a < & \theta_{(11)} < \frac{1}{2} - (d+1)\beta_p.
\end{aligned} \tag{4.17}$$

Now we study all restrictions from (4.17). From conditions for $\theta_{(5)}, \theta_{(6)}, \theta_{(7)}, \theta_{(8)}$ we obtain that β_p should satisfy

$$\beta_p < \frac{1}{2(2d+2)}$$

Since the condition for $\theta_{(4)}$ is stronger than for $\theta_{(9)}$, it follows that

$$\beta_p < \frac{1}{2(d|m-4| + 3d + 2)}$$

should be true as well. Then we see that the restriction for $\theta_{(10)}$ is stronger than for $\theta_{(11)}$, therefore

$$a < \frac{1}{2} - (2d + 1 + d|m-3|)\beta_p$$

Recall (4.11) and obtain that

$$(d|m-4| + 3(d+1))\beta_p \leq a < \frac{1}{2} - (2d + 1 + d|m-3|)\beta_p. \tag{4.18}$$

Hence, β_p should satisfy

$$\beta_p < \frac{1}{2(d|m-4| + d|m-3| + 5d + 4)}, \tag{4.19}$$

which is the strongest restriction for β_p.

Now we combine (4.16) and (4.18) and so obtain that in order to find a valid a, the following inequality should be satisfied:

$$\max\{(d+1)\beta_k, (d|m-4|+3(d+1))\beta_p\} < a < \min\{\frac{1}{2}-(d-1)\beta_k, \frac{1}{2}-(2d+1+d|m-3|)\beta_p\}. \tag{4.20}$$

Therefore,

$$\mathbb{E}[S_t] \leq C\int_0^t \mathbb{E}[S_s]\,ds + CN^{-\gamma}.$$

Applying Grönwall's inequality we deduce that

$$\sup_{0\leq t\leq T} \mathbb{E}[S_t] \leq CN^{-\gamma}\exp(C\cdot T)$$

This implies the desired result.

Chapter 5

Propagation of Chaos

Remark 5.1. In order to prove Corollary 1.7, one needs to combine Theorem 1.3 with Theorem 1.4 and Theorem 1.5 respectively. For the logarithm scaling it is trivial. For the algebraic one we obtain that for $0 < \tilde{\alpha} < a$ there exists a constant $C > 0$ independent of N such that

$$\begin{aligned}
&\mathbb{P}\left(\max_{i\in\{1,\dots,N\}}\left|X_t^{N,i,\varepsilon,\sigma} - \hat{X}_t^{i,\sigma}\right| > N^{-\tilde{\alpha}}\right) \\
&\leq \mathbb{P}\left(\max_{i\in\{1,\dots,N\}}\left|X_t^{N,i,\varepsilon,\sigma} - \bar{X}_t^{i,\varepsilon,\sigma}\right| > N^{-\tilde{\alpha}}\right) + \mathbb{P}\left(\max_{i\in\{1,\dots,N\}}\left|\bar{X}_t^{i,\varepsilon,\sigma} - \hat{X}_t^{i,\sigma}\right| > N^{-\tilde{\alpha}}\right) \\
&\leq \mathbb{P}\left(\max_{i\in\{1,\dots,N\}}\left|X_t^{N,i,\varepsilon,\sigma} - \bar{X}_t^{i,\varepsilon,\sigma}\right| > N^{-a}\right) + \mathbb{E}\left[\max_{i\in\{1,\dots,N\}}\left|\bar{X}_t^{i,\varepsilon,\sigma} - \hat{X}_t^{i,\sigma}\right|^2 N^{2\tilde{\alpha}}\right] \\
&\leq CN^{-\gamma} + C(\varepsilon_k + \varepsilon_p)^2 N^{2\tilde{\alpha}}.
\end{aligned}$$

For $2\tilde{\alpha} \leq \min\{\beta_k, \beta_p\}$, $\varepsilon_k = N^{-\beta_k}$ and $\varepsilon_p = N^{-\beta_p}$ we deduce that

$$\mathbb{P}\left(\max_{i\in\{1,\dots,N\}}\left|X_t^{N,i,\varepsilon,\sigma} - \hat{X}_t^{i,\sigma}\right| > N^{-\tilde{\alpha}}\right) \leq C(\varepsilon_k + \varepsilon_p).$$

Now we prove that we can take the limit as $\sigma \to 0$ of (1.15) and so arrive at (1.16)

Proof of Theorem 1.8. We have proved before that

$$\|u^\sigma\|_{L^q((0,T)\times\mathbb{R}^d)} \leq C, \text{ for } q \in [1,\infty].$$

where C which appears in this proof is a positive constant independent of σ.

Now we multiply (1.15) by u^σ and integrate it over $\mathbb{R}^d$ in order to get that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}|u^\sigma|^2\,dx+\sigma\int_{\mathbb{R}^d}|\nabla u^\sigma|^2\,dx+m\int_{\mathbb{R}^d}(u^\sigma)^{m-1}\nabla(u^\sigma)^2\,dx &= \int_{\mathbb{R}^d}u^\sigma\nabla\Phi*u^\sigma\cdot\nabla u^\sigma\,dx\\
&=\frac{1}{2}\int_{\mathbb{R}^d}\nabla\Phi*u^\sigma\cdot\nabla(u^\sigma)^2\,dx\\
&=-\frac{1}{2}\int_{\mathbb{R}^d}\Delta\Phi*u^\sigma(u^\sigma)^2\,dx\\
&=\frac{1}{2}\int_{\mathbb{R}^d}(u^\sigma)^3\,dx\\
&\leq C.
\end{aligned}
$$

Combining this estimate with

$$
m\int_{\mathbb{R}^d}(u^\sigma)^{m-1}|\nabla u^\sigma|^2dx=\frac{4m}{(m+1)^2}\int_{\mathbb{R}^d}|\nabla(u^\sigma)^{\frac{m+1}{2}}|^2\,dx,
$$

we derive that

$$
\begin{aligned}
&\|u^\sigma\|_{L^\infty((0,T)\times\mathbb{R}^d)}\leq C,\\
&\sqrt{\sigma}\|\nabla u^\sigma\|_{L^2((0,T)\times\mathbb{R}^d)}\leq C,\\
&\|\nabla(u^\sigma)^{\frac{m+1}{2}}\|_{L^2((0,T)\times\mathbb{R}^d)}\leq C.
\end{aligned}
$$

Using $\|u^\sigma\|_{L^q((0,T)\times\mathbb{R}^d)}\leq C,$ for $q\in[1,\infty]$ and $\|\nabla(u^\sigma)^{\frac{m+1}{2}}\|_{L^2((0,T)\times\mathbb{R}^d)}\leq C$ we have that

$$
\begin{aligned}
\|\nabla(u^\sigma)^m\|_{L^2((0,T)\times\mathbb{R}^d)}&=m\|(u^\sigma)^{m-1}\nabla u^\sigma\|_{L^2((0,T)\times\mathbb{R}^d)}\\
&=m\|(u^\sigma)^{\frac{m-1}{2}}(u^\sigma)^{\frac{m-1}{2}}\nabla u^\sigma\|_{L^2((0,T)\times\mathbb{R}^d)}\\
&=\frac{2m}{m+1}\|(u^\sigma)^{\frac{m-1}{2}}\nabla(u^\sigma)^{\frac{m+1}{2}}\|_{L^2((0,T)\times\mathbb{R}^d)}\\
&\leq C\|\nabla(u^\sigma)^{\frac{m+1}{2}}\|_{L^2((0,T)\times\mathbb{R}^d)}\\
&\leq C.
\end{aligned}
\tag{5.1}
$$

So, we infer that

$$
\|\partial_t u^\sigma\|_{L^2(0,T;W^{-1,2}(\mathbb{R}^d))}\leq C. \tag{5.2}
$$

The inequalities (5.1) and (5.2) allow us to use Theorem 3 from [Chen et al., 2014] in growing d-dimensional balls B_R and use the diagonal argument to obtain that

$$
u^\sigma\to u \text{ in } L^{2m}((0,T)\times\mathbb{R}^d) \text{ as } \sigma\to 0.
$$

Moreover, $\|u^\sigma\|_{L^q((0,T)\times\mathbb{R}^d)} \le C$, for $q \in [1,\infty]$ implies that

$$u^\sigma \overset{*}{\rightharpoonup} u \text{ in } L^q(0,T;L^q(\mathbb{R}^d)),\ q \in (1,\infty). \tag{5.3}$$

Since $\|\nabla(u^\sigma)^m\|_{L^2((0,T)\times\mathbb{R}^d)} \le C$, it follows that

$$\nabla(u^\sigma)^m \rightharpoonup \zeta \text{ in } L^2((0,T)\times\mathbb{R}^d).$$

Using (5.2) it holds for any $\psi \in C_0^\infty((0,T)\times\mathbb{R}^d)$ that

$$\int_0^T\int_{\mathbb{R}^d} \nabla(u^\sigma)^m\psi\,dx\,dt = -\int_0^T\int_{\mathbb{R}^d}(u^\sigma)^m\nabla\psi\,dx\,dt$$

converges to

$$-\int_0^T\int_{\mathbb{R}^d} u^m\nabla\psi\,dx\,dt = \int_0^T\int_{\mathbb{R}^d}\nabla u^m\psi\,dx\,dt$$

as $\sigma \to 0$. Therefore,

$$\nabla(u^\sigma)^m \rightharpoonup \nabla u^m \text{ in } L^2((0,T)\times\mathbb{R}^d). \tag{5.4}$$

Now using (5.3) and (5.4), we can take $\sigma \to 0$ in the weak formulation of (1.15). So, for $\varphi \in C_0^\infty([0,T)\times\mathbb{R}^d)$ it holds that

$$\begin{aligned}
\int_0^T\int_{\mathbb{R}^d}\Big(u^\sigma\partial_t\varphi + u^\sigma\nabla\Phi * u^\sigma\cdot\nabla\varphi - \sigma\nabla u^\sigma\cdot\nabla\varphi - \nabla(u^\sigma)^m\cdot\nabla\varphi\Big)\,dx\,dt&\\
+\int_{\mathbb{R}^d}u_0(x)\varphi(0,x)\,dx = 0,&\\
\int_0^T\int_{\mathbb{R}^d}\Big(\nabla\Phi * u^\sigma\cdot\nabla\varphi - u^\sigma\varphi\Big)\,dx\,dt = 0.&
\end{aligned}$$

converges to

$$\begin{aligned}
\int_0^T\int_{\mathbb{R}^d}\Big(u\partial_t\varphi + u\nabla\Phi * u\cdot\nabla\varphi - \nabla u^m\cdot\nabla\varphi\Big)\,dx\,dt&\\
+\int_{\mathbb{R}^d}u_0(x)\varphi(0,x)\,dx = 0,&\\
\int_0^T\int_{\mathbb{R}^d}\Big(\nabla\Phi * u\cdot\nabla\varphi - u\varphi\Big)\,dx\,dt = 0.&
\end{aligned}$$

as $\sigma \to 0$.

□

Proof of Theorem 1.9. Now we show the propagation of chaos result. In this part the limit $N \to \infty$; $\varepsilon_k, \varepsilon_p, \lambda \to 0$ should be understood in the sense of Corollary 1.7 (a) or 1.7 (b).

We proved in Section 4.2 that $X_t^{N,i,\varepsilon,\sigma}$ converges in probability to $\bar{X}_t^{i,\varepsilon,\sigma}$ as $N \to \infty$. This implies that for $k \in \mathbb{N}$, $(X_t^{N,1,\varepsilon,\sigma}, \dots, X_t^{N,k,\varepsilon,\sigma})$ converges in probability to $(\bar{X}_t^{1,\varepsilon,\sigma}, \dots, \bar{X}_t^{k,\varepsilon,\sigma})$ as $N \to \infty$. We denote by $P^k_{N,\varepsilon,\sigma}(t)$ the joint distribution of $(X_t^{N,1,\varepsilon,\sigma}, \dots, X_t^{N,k,\varepsilon,\sigma})$ and by $P^k_{\varepsilon,\sigma}(t)$ the joint distribution of $(\bar{X}_t^{1,\varepsilon,\sigma}, \dots, \bar{X}_t^{k,\varepsilon,\sigma})$. Since convergence in probability implies convergence in distribution, we obtain that

$$\lim_{N\to\infty} P^k_{N,\varepsilon,\sigma}(t) = P^k_{\varepsilon,\sigma}(t) \text{ locally uniform in } t.$$

For the logarithmic scaling from Section 4.1 we obtain the same result since convergence in L^2 implies convergence in distribution.

The independence of $\{\bar{X}_t^{i,\varepsilon,\sigma}\}_{i\in\mathbb{N}}$ implies that

$$P^k_{N,\varepsilon,\sigma}(t) = P^{\otimes k}_{\varepsilon,\sigma}(t),$$

where $P_{\varepsilon,\sigma}(t)$ is the distribution of $\bar{X}_t^{i,\varepsilon,\sigma}$.

We proved in Section 3.2 that $\bar{X}_t^{i,\varepsilon,\sigma}$ converges in L^2 to $\hat{X}_t^{i,\sigma}$ as $\varepsilon_k, \varepsilon_p \to 0$ and $\lambda \to 0$. Since convergence in expectation implies convergence in distribution and $\{\hat{X}_t^{i,\sigma}\}_{i\in\mathbb{N}}$ are independent, we obtain that

$$\lim_{\varepsilon_p,\varepsilon_k,\lambda\to 0} P^{\otimes k}_{\varepsilon,\sigma}(t) = P^{\otimes k}_{\sigma}(t)$$

where $P_\sigma(t)$ is the distribution of $\hat{X}_t^{i,\sigma}$.

Applying Theorem 1.8 we obtain that

$$P^k_{N,\varepsilon,\sigma}(t) \text{ convergres weakly to } P^{\otimes k}(t)$$

as $N \to \infty$; $\varepsilon_k, \varepsilon_p, \lambda \to 0$ in the sense of Corollary 1.7 (a) or 1.7 (b) ; and $\sigma \to 0$ where $P(t)$ is a measure which is absolutely continuous with respect to the Lebesgue measure and has a probability density function $u(t,x)$ which solves (1.16).

□

Appendix A

Compact Embedding

Lemma A.1. *Let $q \in [1, d)$ and $k \in (0, q]$. For $r \geq 1$ consider the following space*

$$V_{k,q} := W^{1,q}(\mathbb{R}^d) \cap L^k(\mathbb{R}^d, |x|^r dx) := \left\{ f \in W^{1,q}(\mathbb{R}^d) \;\Big|\; \int_{R^d} f^k(x)|x|^r dx < \infty \right\}.$$

Then $V_{k,q}$ is compactly embedded in $L^{\tilde{q}}(\mathbb{R}^d)$ for $\tilde{q} \in [\max\{k, 1\}, \frac{dq}{d-q})$.

Proof. We adapt the proof of Lemma 1 from [Caffarelli et al., 2020]. Let $(f_n)_{n\in\mathbb{N}}$ be a uniformly bounded sequence in $V_{k,q}$. Firstly, we know that there exists a subsequence still denoted by $(f_n)_{n\in\mathbb{N}}$ such that

$$f_n \rightharpoonup f \in W^{1,q}(\mathbb{R}^d) \hookrightarrow L^{\frac{dq}{d-q}}(\mathbb{R}^d)$$

Let B_R be an open ball centered at $0 \in \mathbb{R}^d$ with radius $R > 0$. Since $W^{1,q}(B_R)$ is compactly embedded in $L^{\bar{q}}(B_R)$ for $\bar{q} \in \left[1, \frac{dq}{d-q}\right)$, there exists a subsequence still denoted by $(f_n)_{n\in\mathbb{N}}$ such that

$$f_n \to f \in L^{\bar{q}}(B_R) \text{ for any } \bar{q} \in \left[1, \frac{dq}{d-q}\right)$$

Thanks to the Cantor diagonal argument, we can choose $(f_n)_{n\in\mathbb{N}}$ which is independent of $\mathbb{R}$. Uniform boundedness of $(f_n)_{n\in\mathbb{N}}$ in $V_{g,k,q}$ and Fatou's lemma imply that $f \in V_{k,q}$.

Next we show that $|f_n - f|^k$ converges strongly in $L^1(\mathbb{R}^d)$. Since for n big enough we deduce that

$$\begin{aligned}\int_{\mathbb{R}^d} |f_n - f|^k dx &= \int_{B_R} |f_n - f|^k dx + \int_{\mathbb{R}^d \setminus B_R} |f_n - f|^k dx \\ &\leq \frac{\varepsilon}{2} + \frac{1}{R^r} \int_{\{|x| \geq R\}} |x|^r |f_n - f|^k dx \\ &\leq \varepsilon,\end{aligned}$$

by choosing R big enough. Interpolation between $L^{\frac{dq}{d-q}}(\mathbb{R}^d)$ and $L^{\max\{k,1\}}(\mathbb{R}^d)$ implies that for any $\tilde{q} \in \left[\max\{k, 1\}, \frac{dq}{d-q}\right)$ the sequence $(f_n)_{n\in\mathbb{N}}$ converges to $f \in L^{\tilde{q}}(\mathbb{R}^d)$. □

Bibliography

Arumugam, G. and Tyagi, J. [2021], ‘Keller-segel chemotaxis models: A review’, *Acta Applicandae Mathematicae* **171**(1), 1–82.

Blanchet, A., Carrillo, J. A. and Laurençot, P. [2009], ‘Critical mass for a patlak–keller–segel model with degenerate diffusion in higher dimensions’, *Calculus of Variations and Partial Differential Equations* **35**(2), 133–168.

Blanchet, A., Carrillo, J. and Masmoudi, N. [2008], ‘Infinite time aggregation for the critical two-dimensional patlak-keller-segel model, to appear in comm’, *Pure Appl. Math* .

Blanchet, A., Dolbeault, J. and Perthame, B. [2006], ‘Two-dimensional keller-segel model: Optimal critical mass and qualitative properties of the solutions.’, *Electronic Journal of Differential Equations (EJDE)[electronic only]* **2006**, Paper–No.

Boers, N. and Pickl, P. [2016], ‘On mean field limits for dynamical systems’, *Journal of Statistical Physics* **164**(1), 1–16.

Bolley, F., Canizo, J. A. and Carrillo, J. A. [2011], ‘Stochastic mean-field limit: non-lipschitz forces and swarming’, *Mathematical Models and Methods in Applied Sciences* **21**(11), 2179–2210.

Braun, W. and Hepp, K. [1977], ‘The vlasov dynamics and its fluctuations in the 1/n limit of interacting classical particles’, *Communications in mathematical physics* **56**(2), 101–113.

Bresch, D., Jabin, P.-E. and Wang, Z. [2019], ‘On mean-field limits and quantitative estimates with a large class of singular kernels: application to the patlak–keller–segel model’, *Comptes Rendus Mathematique* **357**(9), 708–720.

Brezis, H. and Brézis, H. [2011], *Functional analysis, Sobolev spaces and partial differential equations*, Vol. 2, Springer.

Caffarelli, L., Gualdani, M. and Zamponi, N. [2020], ‘Existence of weak solutions to a continuity equation with space time nonlocal darcy law’, *Communications in Partial Differential Equations* **45**(12), 1799–1819.

Chaintron, L.-P. and Diez, A. [2021], ‘Propagation of chaos: a review of models, methods and applications’, *arXiv preprint arXiv:2106.14812* .

Chen, L., Daus, E. S., Holzinger, A. and Jüngel, A. [2021a], 'Rigorous derivation of population cross-diffusion systems from moderately interacting particle systems', *Journal of Nonlinear Science* **31**(6), 1–38.

Chen, L., Daus, E. S. and Jüngel, A. [2019], 'Rigorous mean-field limit and cross-diffusion', *Zeitschrift für angewandte Mathematik und Physik* **70**(4), 1–21.

Chen, L., Göttlich, S. and Yin, Q. [2017], 'Mean field limit and propagation of chaos for a pedestrian flow model', *Journal of Statistical Physics* **166**(2), 211–229.

Chen, L., Gvozdik, V., Holzinger, A. and Li, Y. [2022], 'Rigorous derivation of the degenerate parabolic-elliptic keller-segel system from a moderately interacting stochastic particle system. part ii propagation of chaos', *Draft* .

Chen, L., Gvozdik, V. and Li, Y. [2022], 'Rigorous derivation of the degenerate parabolic-elliptic keller-segel system from a moderately interacting stochastic particle system. part i pde analysis', *Draft* .

Chen, L., Holzinger, A., Jüngel, A. and Zamponi, N. [2021b], 'Analysis and mean-field derivation of a porous-medium equation with fractional diffusion', *arXiv preprint arXiv:2109.08598* .

Chen, L., Li, X., Pickl, P. and Yin, Q. [2020], 'Combined mean field limit and non-relativistic limit of vlasov–maxwell particle system to vlasov–poisson system', *Journal of Mathematical Physics* **61**(6), 061903.

Chen, L., Liu, J.-G. and Wang, J. [2012], 'Multidimensional degenerate keller–segel system with critical diffusion exponent 2n/(n+2)', *SIAM Journal on Mathematical Analysis* **44**(2), 1077–1102.

Chen, X., Jüngel, A. and Liu, J.-G. [2014], 'A note on aubin-lions-dubinskii lemmas', *Acta applicandae mathematicae* **133**(1), 33–43.

Dobrushin, R. L. [1979], 'Vlasov equations', *Functional Analysis and Its Applications* **13**(2), 115–123.

Duerinckx, M. [2016], 'Mean-field limits for some riesz interaction gradient flows', *SIAM Journal on Mathematical Analysis* **48**(3), 2269–2300.

Evans, L. C. [2010], *Partial differential equations*, Vol. 19, American Mathematical Soc.

Fournier, N., Jourdain, B. et al. [2017], 'Stochastic particle approximation of the keller–segel equation and two-dimensional generalization of bessel processes', *Annals of Applied Probability* **27**(5), 2807–2861.

García, A. C. and Pickl, P. [2017], 'Microscopic derivation of the keller-segel equation in the sub-critical regime', *arXiv preprint arXiv:1703.04376* .

Gilbarg, D. and Trudinger, N. S. [2015], *Elliptic partial differential equations of second order*, Vol. 224, springer.

Godinho, D. and Quininao, C. [2015], Propagation of chaos for a subcritical keller–segel model, *in* 'Annales de l'IHP Probabilités et statistiques', Vol. 51, pp. 965–992.

Golse, F. [2016], On the dynamics of large particle systems in the mean field limit, *in* 'Macroscopic and large scale phenomena: coarse graining, mean field limits and ergodicity', Springer, pp. 1–144.

Ishida, S., Maeda, Y. and Yokota, T. [2013], 'Gradient estimate for solutions to quasilinear non-degenerate keller-segel systems on r n', *Discrete & Continuous Dynamical Systems-B* **18**(10), 2537.

Jabin, P.-E. [2014], 'A review of the mean field limits for vlasov equations', *Kinetic & Related Models* **7**(4), 661.

Jabin, P.-E. and Wang, Z. [2016], 'Mean field limit and propagation of chaos for vlasov systems with bounded forces', *Journal of Functional Analysis* **271**(12), 3588–3627.

Jabin, P.-E. and Wang, Z. [2017], Mean field limit for stochastic particle systems, *in* 'Active Particles, Volume 1', Springer, pp. 379–402.

Jabin, P.-E. and Wang, Z. [2018], 'Quantitative estimates of propagation of chaos for stochastic systems with w-1,inf kernels', *Inventiones mathematicae* **214**(1), 523–591.

Jeans, J. [1915], 'On the theory of star-streaming and the structure of the universe', *Monthly Notices of the Royal Astronomical Society* **76**, 70–84.

Kac, M. [1956], 'Foundations of kinetic theory in proceedings of the third berkeley symposium on mathematical statistics and probability, 1954–1955 3 (1956), 171–197'.

Keller, E. F. and Segel, L. A. [1970], 'Initiation of slime mold aggregation viewed as an instability', *Journal of theoretical biology* **26**(3), 399–415.

Keller, E. F. and Segel, L. A. [1971], 'Model for chemotaxis', *Journal of theoretical biology* **30**(2), 225–234.

Kipnis, C. and Landim, C. [1998], *Scaling limits of interacting particle systems*, Vol. 320, Springer Science & Business Media.

Lazarovici, D. and Pickl, P. [2017], 'A mean field limit for the vlasov–poisson system', *Archive for Rational Mechanics and Analysis* **225**(3), 1201–1231.

Leblé, T. and Serfaty, S. [2018], 'Fluctuations of two dimensional coulomb gases', *Geometric and Functional Analysis* **28**(2), 443–508.

Lieb, E. and Loss, M. [2001], 'Analysis 2nd edn (providence, ri: American mathematical society)'.

Luckhaus, S. and Sugiyama, Y. [2006], ‘Large time behavior of solutions in super-critical cases to degenerate keller-segel systems’, *ESAIM: Mathematical Modelling and Numerical Analysis* **40**(3), 597–621.

Majda, A. [1984], *Compressible fluid flow and systems of conservation laws in several space variables*, Vol. 53, Springer Science & Business Media.

Mayer, S. [2014], ‘The vlasov equation & mean field limit of a particle system’.

McKean, H. P. [1967], ‘Propagation of chaos for a class of non-linear parabolic equations’, *Stochastic Differential Equations (Lecture Series in Differential Equations, Session 7, Catholic Univ., 1967)* pp. 41–57.

Novotny, A. and Straskraba, I. [2004], *Introduction to the mathematical theory of compressible flow*, Vol. 27, OUP Oxford.

Nualart, D. [2006], *The Malliavin calculus and related topics*, Vol. 1995, Springer.

Oelschläger, K. [1985], ‘A law of large numbers for moderately interacting diffusion processes’, *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **69**(2), 279–322.

Oelschläger, K. [1990], ‘Large systems of interacting particles and the porous medium equation’, *Journal of differential equations* **88**(2), 294–346.

Oleinik, O. A. and Kruzhkov, S. N. [1961], ‘Quasi-linear second-order parabolic equations with many independent variables’, *Russian Mathematical Surveys* **16**(5), 105–146.

Patlak, C. S. [1953], ‘Random walk with persistence and external bias’, *The bulletin of mathematical biophysics* **15**(3), 311–338.

Philipowski, R. [2007], ‘Interacting diffusions approximating the porous medium equation and propagation of chaos’, *Stochastic processes and their applications* **117**(4), 526–538.

Serfaty, S. [2017], ‘Mean field limits of the gross-pitaevskii and parabolic ginzburg-landau equations’, *Journal of the American Mathematical Society* **30**(3), 713–768.

Serfaty, S. [2018], Systems of points with coulomb interactions, *in* ‘Proceedings of the International Congress of Mathematicians: Rio de Janeiro 2018’, World Scientific, pp. 935–977.

Serfaty, S. [2020], ‘Mean field limit for coulomb-type flows’, *Duke Mathematical Journal* **169**(15), 2887–2935.

Simon, J. [1986], ‘Compact sets in the spacel p (o, t; b)’, *Annali di Matematica pura ed applicata* **146**(1), 65–96.

Spohn, H. [2004], *Dynamics of charged particles and their radiation field*, Cambridge university press.

Sugiyama, Y. [2006], 'Global existence in sub-critical cases and finite time blow-up in super-critical cases to degenerate keller-segel systems', *Differential and Integral Equations* **19**(8), 841–876.

Sugiyama, Y. [2007], 'Application of the best constant of the sobolev inequality to degenerate keller-segel models', *Advances in Differential Equations* **12**(2), 121–144.

Sugiyama, Y. and Kunii, H. [2006], 'Global existence and decay properties for a degenerate keller–segel model with a power factor in drift term', *Journal of Differential Equations* **227**(1), 333–364.

Sugiyama, Y. et al. [2007], 'Time global existence and asymptotic behavior of solutions to degenerate quasi-linear parabolic systems of chemotaxis', *Differential and Integral Equations* **20**(2), 133–180.

Sznitman, A.-S. [1991], Topics in propagation of chaos, *in* 'Ecole d'été de probabilités de Saint-Flour XIX—1989', Springer, pp. 165–251.

Vlasov, A. A. [1938], 'On vibration properties of electron gas', *J. Exp. Theor. Phys. (in Russian)* **8 (3):291**.

Vlasov, A. A. [1968], 'The vibrational properties of an electron gas', *Soviet Physics Uspekhi* **10**(6), 721.

Wang, J., Chen, L. and Hong, L. [2016], 'Parabolic elliptic type keller-segel system on the whole space case', *Discrete & Continuous Dynamical Systems* **36**(2), 1061.

Wang, J., Li, Y. and Chen, L. [2019], 'Supercritical degenerate parabolic–parabolic keller–segel system: existence criterion given by the best constant in sobolev's inequality', *Zeitschrift für angewandte Mathematik und Physik* **70**(3), 1–18.

Wu, Z., Yin, J. and Wang, C. [2006], *Elliptic & parabolic equations*, World Scientific.

Yau, H.-T. [1991], 'Relative entropy and hydrodynamics of ginzburg-landau models', *Letters in Mathematical Physics* **22**(1), 63–80.