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DISCUSSION PAPER

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Mechanism Design for Unequal Societies

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Abstract

We study optimal mechanisms for a utilitarian designer who seeks to assign a finite number of goods to a group of ex ante heterogeneous agents with unit demand. The agents have heterogeneous marginal utilities of money, which may naturally arise in environments where agents have different wealth levels or financing conditions. We show that the utilitarian optimal allocation rule deviates from the ex post efficient allocation rule in two ways, namely by (1) allocating the good to agents with lower willingnesses to pay in certain situations and (2) by potentially keeping some units of the good unallocated. We also highlight how our mechanism can be implemented as an auction with minimum bids and bidding subsidies.

Keywords: optimal mechanism design, redistribution, inequality, auctions

JEL Classification: D44, D47, D61, D63, D82

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1 Introduction

Consider the following canonical mechanism design problem with a twist: The designer owns a limited number of indivisible goods and a finite number of ex ante heterogeneous agents with different marginal utilities of money are vying for the allocation of this good. This optimization problem is faced by many entities in the real world: In frequency auctions or airport slot allocation mechanisms, incumbents with large amounts of collateral have significantly easier financing conditions than potential entrants. In government real estate auctions, corporations compete with private citizens. The assignment of kindergarten spots is usually conducted by small-scale kindergarten providers and the pool of applicants contains families with significantly different wealth levels.[1]

It seems reasonable to assume that the authorities in these examples aim to maximize utilitarian welfare. For example, the goals of frequency auctions in the US are, among others, the "efficient use of the spectrum" and "the rapid deployment of new systems".[2] Similarly, many kindergarten providers such as municipal entities are non-profit organizations. Moreover, these entities typically face constraints on transfers. For example, local kindergarten providers generally operate under tight budget constraints — thus, kindergarten spot allocations must satisfy an ex ante budget balance condition.

We derive the utilitarian optimal mechanism for small-market allocation problems in which agents have different marginal utilities of money and the designer faces an ex ante budget constraint. We show that the utilitarian optimal allocation rule deviates from the ex post efficient allocation rule in two ways, namely by allocating the good to agents with lower willingnesses to pay in certain situations and by potentially leaving some units of the good unallocated. When the agents are ex ante heterogeneous with respect to their marginal utilities of money, there thus is a tension between ex post and ex ante optimality. Finally, we show how these optimal mechanisms can be implemented by slightly altering popular allocation mechanisms such as auctions.

When agents have different marginal utilities of money, the designer has a redistributive motive. How the optimal allocation of resources is impacted by such considerations has been studied in several papers that preceed our work. Weitzman [1977] analyses when a simple rationing scheme in which all consumers get the same amount of a good is preferable to a market price rule. Condorelli [2013] provides a methodological contribution that enables the derivation of optimal mechanisms for generalized social welfare functions in small markets. We apply this methodology, but our two particular setups are not discussed by Condorelli [2013]. Our preference framework resembles Dworczak

[1] In Germany, for instance, kindergarten spot allocation decisions are largely made by municipal entities, parent associations, and non-profit organizations — as detailed by Fritz [2021]. Moreover, local authorities condition their decision on several observables that are linked with income — see LH-Mainz [2018].

[2] For details, please see Crippen [2000].

et al. [2021] and Akbarpour et al. [2022]. By contrast, these authors study settings with a continuum of goods and agents while we study small markets and focus on different research questions.

We consider the following framework: A utilitarian designer owns a number of indivisible goods which can be allocated to a finite number of agents with unit demand. Following Dworczak et al. [2021], an agent's type is two-dimensional and consists of (i) her consumption utility, i.e. the increase of utility she attains by receiving the good, and (ii) her marginal utility of money. Both components of an agent's type are stochastic and private information. Crucially, the distribution of types may vary across agents. The designer's mechanism must respect incentive compatibility, individual rationality and must satisfy constraints on transfers. We consider two such constraints, namely (1) the requirement that the designer's budget must be balanced ex ante and (2) the constraint that no agent may receive payments from the mechanism in expectation.[3]

The utilitarian optimal mechanism condenses three considerations: First, the designer wishes to allocate the goods to the agents with the highest consumption utilities, ceteris paribus. Moreover, the designer potentially has two redistributive motives: "within" an agent and between agents. When the agents' marginal utilities of money are stochastic, the designer wants to redistribute "within" a given agent. The designer has incentives to pay transfers to an agent in states of the world in which her marginal utility of money is high and finance these payments by obtaining revenue from the agent when her marginal utility of money is low. The allocation rule will reflect this desire, because the assignment of a good to an agent is always associated with payments to and from the agent. When redistribution between agents is possible, the designer additionally wants to allocate the goods in a way that increases the revenue he can redistribute to agents with high marginal utilities of money.

The total effect of allocating one unit of the good on social welfare is given by the key statistic of our model: an agent's *inequality adjusted valuation*. In the optimal mechanism, the goods are allocated to the agents with the highest positive inequality adjusted valuations, which are not necessarily the agents with the highest willingnesses to pay.

To see why allocation by willingness to pay is not utilitarian optimal, consider the case in which the designer faces the ex ante budget balance condition. To fix ideas, suppose further that all agents have deterministic (but different) marginal utilities of money. Thus, the designer's optimal mechanism will be determined fully by his incentives to allocate the goods to agents with high consumption utilities and his desire to raise revenue for redistribution between agents. We say that an agent with a high marginal utility of money is poor, while an agent with a low marginal utility of money is rich.

Consider a situation in which a poor agent with a comparatively low willingness to

[3]The fact that we only require the budget to be balanced ex ante and not in every possible state of the world is without loss of generality, given the insights of Börgers and Norman [2009].

pay (but a high inequality adjusted valuation) receives the good, while a rich agent with a higher willingness to pay receives nothing. Why would the designer not deviate from this rule by assigning the good to the rich agent with a higher willingness to pay and compensate the other agent with a payment somewhere in between the willingnesses to pay of the agents? In a nutshell, committing to refrain from allocating the good to the rich agent with the higher willingness to pay, but lower inequality adjusted valuation, will enable the designer to generate more revenue from this agent while satisfying incentive compatibility. This is beneficial for the designer because he can redistribute this money to the poor agent.

The designer's desire to raise revenue will imply that some units of the good will be left unallocated under the ex ante budget balance condition. We refer to this outcome as rationing. Rationing is a part of the optimal mechanism because, as in Myerson [1981], this raises the revenue the designer obtains. We study how the probability that rationing occurs is affected by the distribution of types. We determine that increases of wealth inequality which mirror the development of real wages in the USA from 1990-2010 will lead to more rationing and thus, more allocative inefficiency in the optimal mechanism.

When agents are ex ante symmetric, the utilitarian optimal allocation rule coincides with the ex post efficient rule: The agents with the highest willingnesses to pay will receive the goods and there is no rationing. The intuition is the following: When all agents are ex ante symmetric, every agent is considered equally rich or poor ex ante. Moreover, the mapping from willingness to pay into consumption utility and marginal utility of money is the same for all agents. For these reasons, the designer applies the standard allocation rule.

In an extension, we study the case in which the designer faces the constraints that no agent may receive positive expected transfers from the mechanism. Under these constraints, redistribution between agents is no longer possible. Thus, the utilitarian optimal mechanism will only reflect the designer's desire to allocate the goods to agents with high consumption utilities and his incentives to redistribute within agents using the transfer rule. We show that the utilitarian optimal mechanism still deviates from the ex post efficient allocation rule when agents are ex ante asymmetric.

To guide the practical implementation of our ideas, we provide auction rules which implement the respective utilitarian optimal mechanisms in section 5. Under the ex ante budget balance condition, agents with high marginal utilities of money receive bidding subsidies which allow them to compete against agents with easier financing conditions. Moreover, the respective auction features bidder-specific minimum bids.

The rest of our paper proceeds as follows: We review the related literature in section 2. In sections 3 and 4, we outline our framework and characterize the utilitarian optimal mechanisms. Thereafter, we formalize the auction rules that implement the optimal mechanisms in section 5. Finally, we provide numerical illustrations in section 6 and conclude thereafter.

2 Related Literature

Our work relates to three strands of literature. Firstly, our research has strong connections to the contributions that characterize optimal mechanisms in non-quasilinear settings. Secondly, our work relates to the contributions from various fields which investigate the role of heterogeneous, but constant, marginal utilities of money. Thirdly, some of our key ideas complement insights from social choice theory and public finance.

One of the earliest extensions of the standard quasilinear framework was Maskin and Riley [1984], who pin down the optimal auction in a setting with risk-averse buyers. Saitoh and Serizawa [2008], Hashimoto and Saitoh [2010], and Kazumura et al. [2020] characterize, among others, the set of mechanisms that retain certain desiderata in non-quasilinear settings, such as the VCG features. Eisenhuth [2019] studies the revenue-maximizing auction when agents are loss averse and the reference point is endogenous to the choice of the mechanism. Pai and Vohra [2014] and Kotowski [2020] analyse, among others, allocation problems where buyers face heterogeneous budget constraints.

Within this literature, the paper that is most closely related to our own is Huesmann [2017], who examines the problem of assigning indivisible goods to a unit mass of agents with two different wealth levels. An agent's wealth, which enters the utility function concavely, is private information. In contrast to our work, Huesmann [2017] assumes that the utility an agent receives when consuming the good is the same across all agents and non-stochastic. As stated above, Huesmann [2017] considers a setting with a continuum of agents, whereas we model a finite number of agents to understand the local allocation problems we have in mind. Our key result, namely that it may be utilitarian optimal to allocate the good to an agent with the lower willingness to pay in certain situations, is unobtainable in the framework of Huesmann [2017]. In addition, we show how a designer may account for wealth inequality by implementing an auction with bidding subsidies and minimum bids.

The second related strand of literature consists of papers that study settings where agents differ in their marginal utilities of money, but there are no wealth effects. Esteban and Ray [2006] study a lobbying framework where different lobby groups have different wealth levels and the costs of lobbying fall in wealth. Kang and Zheng [2019] and Kang and Zheng [2022] characterize the set of interim-pareto-optimal mechanisms when agents have heterogeneous pareto weights.

Within this literature, the papers that are closest to ours are Dworczak et al. [2021] and Akbarpour et al. [2022]. Our modeling technique, in particular the utility function with two dimensional types, is based on Dworczak et al. [2021]. However, both these papers consider a setting with a continuum of goods to be allocated and a continuum of agents to allocate the goods to. By contrast, we model settings with a finite number of agents and goods to be allocated.[4] Our two main contributions are the closed form expressions governing the utilitarian optimal allocation decision for any realization of types in our settings. By construction, these have no counterpart in Dworczak et al. [2021] and Akbarpour et al. [2022], because such characterizations are not required in these papers due to the large markets assumption. Moreover, we derive a bulk of our results under the constraint that no agent can receive positive transfers from the mechanism in expectation, which is not considered by either of these two papers. In addition, our results regarding the implementation of the utilitarian optimal mechanisms via auctions are exclusive to our paper.

Condorelli [2013] outlines a method for determining the optimal allocation of goods under generalized objectives of the planner in small-market situations, subject to incentive compatibility and individual rationality. We apply the linear programming approach outlined by Condorelli [2013] to the two particular allocation problems of our paper, both of which are not discussed in Condorelli [2013]. In Condorelli [2013], allocation is based on exogenously given priority functions. The inequality adjusted valuations in our model can be understood as endogenous counterparts of these priority functions. These inequality adjusted valuations are determined through the interplay of the incentive compatibility, the individual rationality, and the transfer constraints in our settings.

The idea of assigning different agents heterogeneous welfare weights based on their economic standing was already voiced by Diamond and Mirrlees [1971] and Atkinson and Stiglitz [1976]. Our paper is also related to Weitzman [1977], who analyses when a simple rationing scheme in which all consumers get the same amount of a good is preferable to a market price mechanism. The idea of using the public provision of goods as a redistributive tool is also reflected in the work of Besley and Coate [1991] and Gahvari and Mattos [2007]. The authors study a market for an indivisible and rivalrous good such as healthcare. A state with utilitarian objectives will provide an intermediate quality of the good at no costs, which a redistributive act under lump-sum taxation.[5]

3 Framework

We consider a finite but arbitrary number of agents $i \in \{1, 2, ..., N\}$ with demand for an indivisible good. Every agent can consume at most one unit of the good. Initially $m < N$ units of this good are owned by the mechanism designer and are to be allocated among the agents. Following Dworczak et al. [2021], the agents' behavior is described by

[4] In the examples we have mentioned, the number of goods to be allocated and the number of agents vying for the allocation of the goods are small. In such local markets, feasibility constraints have to hold for every possible type realization.

[5] Kang [2022] studies the optimal provision of a public good when the quality of said good affects the prices of goods on a private market.

the utility function $u_i = v_i^K x_i^K + v_i^M x_i^M$, where v_i^K represents the valuation for the good (which we also refer to as the consumption utility). The term x_i^K is a binary variable that describes whether or not the agent has received the good. What sets this specification apart from most of the literature is that the marginal utility of money (v_i^M) may vary across agents. The amount of money received or paid by the agent in the mechanism is denoted by x_i^M. Both v_i^K and v_i^M are assumed to be private information.

The joint distribution of these variables, namely F_i, is common knowledge. We impose that this distribution is continuous, has bounded support, and that the agents' types are drawn independently. The marginal densities of v_i^K and v_i^M are denoted by $f_i^K(\cdot)$ and $f_i^M(\cdot)$, respectively. We assume that an agent's willingness to pay, namely $r_i = v_i^K/v_i^M$, is independently and continuously distributed on an interval $[\underline{r}_i, \bar{r}_i]$ with $0 \leq \underline{r_i} \leq \bar{r}_i$. The cdf of r_i will be denoted by $G_i(r_i)$.

The mechanism designer is utilitarian and maximizes ex ante welfare given by

$$\sum_{i=1}^{N} \mathbb{E}[v_i^K x_i^K + v_i^M x_i^M] \tag{1}$$

subject to incentive compatibility, individual rationality and potential constraints on the transfer rules. Everything else equal, moving money between the agents thus impacts social welfare. We denote the allocation rule by x_i and the transfer rule by t_i. In line with the standard definitions of the literature we say that a mechanism is (Bayesian) incentive compatible if and only if for all agents i and possible types (v_i^K, v_i^M)

$$\begin{aligned}&\mathbb{E}_{-i}[v_i^K x_i(v_i^K, v_i^M, v_{-i}^K, v_{-i}^M) + v_i^M t_i(v_i^K, v_i^M, v_{-i}^K, v_{-i}^M)]\\ \geq\ &\mathbb{E}_{-i}[v_i^K x_i(\hat{v}_i^K, \hat{v}_i^M, v_{-i}^K, v_{-i}^M) + v_i^M t_i(\hat{v}_i^K, \hat{v}_i^M, v_{-i}^K, v_{-i}^M)]\end{aligned} \tag{2}$$

holds for all other possible type reports $(\hat{v}_i^K, \hat{v}_i^M)$.

We define $\underline{U}_i$ as the value an agent i obtains from her outside option. Because utility functions are linear in both components, we normalize the value of the outside option to 0. Participation in a mechanism is individually rational if and only if for all agents and possible types (v_i^K, v_i^M), the following constraint is satisfied:

$$\mathbb{E}_{-i}[v_i^K x_i(v_i^K, v_i^M, v_{-i}^K, v_{-i}^M) + v_i^M t_i(v_i^K, v_i^M, v_{-i}^K, v_{-i}^M)] \geq 0 \tag{3}$$

In section 4.1., we restrict attention to mechanisms that satisfy ex ante budget balance. We say that a mechanism satisfies ex ante budget balance if and only if

$$\sum_{i=1}^{N} \mathbb{E}[t_i(v_i^K, v_i^M, v_{-i}^K, v_{-i}^M)] \leq 0. \tag{4}$$

Restricting attention to a budget balance condition that is expressed in ex ante and not in ex post terms is without loss, given the insights of Börgers and Norman [2009].[6]

In section 4.2., we impose that the expected transfer any agent obtains from the mechanism cannot be positive, i.e. that the following no-subsidy constraints hold for all agents:

$$\mathbb{E}[t_i(v_i^K, v_i^M, v_{-i}^K, v_{-i}^M)] \leq 0 \tag{5}$$

Note that we do not require transfers to be negative for every possible realization of types.

Before deriving the optimal mechanism, we establish some preliminary results. Note firstly that an agent's willingness to pay (r_i) will fully describe her behavior. As derived in Dworczak et al. [2021], an agent's utility function can be rewritten as follows, where the factor $\lambda_i(r_i)$ can be understood as a Pareto weight:

$$\mathbb{E}[v_i^K x_i^K + v_i^M x_i^M] = \mathbb{E}_{r_i}[\underbrace{\mathbb{E}[v_i^M | r_i]}_{\lambda_i(r_i)}(r_i x_i^K + x_i^M)] \tag{6}$$

Because the statistic r_i fully pins down an agent's behaviour, any attempt at treating two agents with the same r_i (but potentially heterogeneous realizations of v_i^M and v_i^K) differently can not be successful. Thus, restricting attention to mechanisms that elicit only r_i is without loss of optimality. This intuition is formalized in the following proposition due to Dworczak et al. [2021]:

Proposition 1 (Dworczak et al. [2021], Theorem 8) *If a mechanism is feasible (respectively, optimal) in the two dimensional model [i.e. eliciting v_i^K and v_i^M], then there exists a payoff-equivalent mechanism eliciting only r_i that is feasible (respectively, optimal) in the one dimensional model [i.e. eliciting only r_i].*

In light of this result, we restrict attention to mechanisms that only elicit r_i. By the revelation principle, we are also free to restrict our attention to direct mechanisms subject to incentive compatibility constraints. We define $X_i(r_i) = \mathbb{E}_{-i}[x_i(r_i, r_{-i})]$, $T_i(r_i) = \mathbb{E}_{-i}[t_i(r_i, r_{-i})]$, and $U_i(r_i) = r_i X_i(r_i) + T_i(r_i)$. As derived in Dworczak et al. [2021], characterizing incentive compatibility follows the familiar formulation of the literature:

Lemma 1 (Incentive Compatibility) *A mechanism $\{x_i(r_i, r_{-i}), t_i(r_i, r_{-i})\}_{i=1}^N$ is incentive compatible if and only if*

1. *$X_i(r_i)$ is non-decreasing in r_i (Monotonicity)*

[6] This is because we study independent types and assume that the agents' marginal utilities of money are constant. Proposition 2 in Börgers and Norman [2009] thus establishes the following: For any ex ante budget balanced mechanism in our framework, there exists an ex post budget balanced mechanism with the same allocation rule and interim expected payments (and thus the same utilitarian welfare).

2. $r_iX_i(r_i) + T_i(r_i) = U_i(\underline{r}_i) + \int_{\underline{r}_i}^{r_i} X_i(s)ds$ *(Integrability)*

This result follows after rescaling the utility functions and then applying the standard arguments of the literature. By integrability, the expected transfer of an agent is given by

$$\mathbb{E}[T_i(r_i)] = U_i(\underline{r}_i) - \int_{\underline{r}_i}^{\bar{r}_i} X_i(r_i) \underbrace{J_i(r_i)}_{r_i - \frac{1-G_i(r_i)}{g_i(r_i)}} dG_i(r_i) \tag{7}$$

where $J_i(r_i)$ denotes the virtual valuation of an agent as defined in Myerson [1981]. Defining $\Lambda_i = \mathbb{E}[v_i^M]$ and $\Pi_i(s) := \frac{\int_s^{\bar{r}_i} \lambda_i(r_i)dG_i(r_i)}{g_i(s)}$, standard arguments from the literature allow us to rewrite utilitarian social welfare as follows:

$$\sum_i \mathbb{E}[\lambda_i(r_i)(r_iX_i(r_i) + T_i(r_i))] = \sum_i \left(\Lambda_i U_i(\underline{r}_i) + \int_{\underline{r}_i}^{\bar{r}_i} X_i(s)\Pi_i(s)dG_i(s) \right) \tag{8}$$

Before moving forward, we impose the following assumption for the remainder of the paper:

Assumption 1 *For all agents, the virtual valuation function crosses 0 at most once, i.e. for every agent i, there exists an $\hat{r}_i \in [\underline{r}_i, \bar{r}_i]$ s.t. $J_i(r_i) \leq 0 \ \forall r_i < \hat{r}_i$ and $J_i(r_i) \geq 0 \ \forall r_i \geq \hat{r}_i$.*

Regularity of the virtual valuation functions as in Myerson [1981] is sufficient, but not necessary, for assumption 1 to hold. This assumption is useful because, together with monotonicity of $X_i(r_i)$, it is sufficient to ensure that the expected revenue that is raised from an agent will always be weakly positive.

4 Optimal mechanisms

4.1 Ex ante budget balance

In this section, we assume that the designer faces the ex ante budget constraint. We define $\Lambda^* = \max_i\{\Lambda_i\}$ and set $i^* \in arg \max_i \Lambda_i$. Noting the way in which we have rewritten the utilitarian social welfare function, the maximization problem can be stated as:

$$\max_{\{x_i(r_i,r_{-i})\in[0,1],U_i(\underline{r}_i)\}_{i=1}^N} \sum_i \left(\Lambda_i U_i(\underline{r}_i) + \int \Pi_i(r_i)x_i(r_i,r_{-i})dG(r_i,r_{-i})\right)$$

$$\text{s.t. } \sum_i \left(U_i(\underline{r}_i) - \int J_i(r_i)x_i(r_i,r_{-i})dG(r_i,r_{-i})\right) \leq 0 \qquad \text{(Budget)}$$

$$\sum_i x_i(r_i,r_{-i}) \leq m \qquad \text{(Feasibility)}$$

$$X_i(r_i) \text{ non-decreasing} \qquad \text{(Monotonicity)}$$

$$U_i(\underline{r}_i) \geq 0 \qquad \text{(IR)}$$

Then, the optimization problem boils down to choosing the optimal allocation rule and the optimal utility levels for the lowest type of each agent.

Note firstly that the ex ante budget balance requirement must bind in any optimal mechanism. Otherwise, $U_i^*(\underline{r}_i)$ could be increased, leading to a rise in social welfare. Moreover, the IR constraints of all agents $i \notin arg\max_i \Lambda_i$ must also bind in the optimal mechanism. If any such constraint were slack, the designer could decrease $U_i(\underline{r}_i)$ for some agent $i \notin arg\max_i \Lambda_i$ to raise $U_{i^*}(\underline{r}_{i^*})$ by the same amount. This change would satisfy all constraints and improve social welfare because $\Lambda^* > \Lambda_i$, a contradiction. Taken together, these two arguments imply that the budget constraint can be rewritten as follows:

$$\sum_i U_i(\underline{r}_i) = U_{i^*}(\underline{r}_{i^*}) = \sum_i \int J_i(r_i)x_i(r_i,r_{-i})dG(r_i,r_{-i})$$

Plugging this expression into the objective function implies that our maximization problem, when ignoring the IR and the monotonicity constraints, becomes the following:

$$\max_{\{x_i(r_i,r_{-i})\in[0,1],U_i(\underline{r}_i)\}_{i=1}^N} \sum_i \left(\int \big(\Pi_i(r_i) + \Lambda^* J_i(r_i)\big)x_i(r_i,r_{-i})dG(r_i,r_{-i})\right)$$

$$\text{s.t. } \sum_i x_i(r_i,r_{-i}) \leq m \qquad \text{(Feasbility)}$$

Key components of this functional are the functions $\Pi_i(r_i) + \Lambda^* J_i(r_i)$, which we label now:

Definition 1 (Inequality adjusted valuation - I) *We define $\varphi_i(r_i) := \Pi_i(r_i) + \Lambda^* J_i(r_i)$ as the inequality adjusted valuation of agent i under the ex ante budget balance requirement.*

Having established this, we characterize the optimal mechanism:

Proposition 2 (Optimal Mechanism - I) *Suppose that φ_i is weakly increasing for all agents i. Then, the optimal mechanism assigns the goods to the m agents with the highest positive inequality adjusted valuations. If the number of agents with positive inequality adjusted valuations is below the number of goods, some units of the good remain unallocated.*

The structure of the relaxed problem outlined above implies that its solution, which can be found using the linear programming approach outlined by Condorelli [2013], is the following: In any state of the world, the goods are assigned to the agents with the highest inequality adjusted valuations $\varphi_i(r_i)$, provided they are positive.

In the appendix, we show that the solution to the relaxed problem (in which the monotonicity and IR constraints were ignored) constitutes a solution to the original problem under the stated assumptions. To build intuition, consider the monotonicity constraint first. The assumption that $\varphi_i(r_i)$ is increasing guarantees that the monotonicity constraint w.r.t $X_i(r_i)$ will be satisfied in the solution of the relaxed problem. This is because, in the solution to the relaxed problem, the probability of receiving the good is increasing in an agent's inequality adjusted valuation. Now consider the IR constraints. For all agents i with $\Lambda_i \neq \Lambda^*$, the IR constraints bind. Moreover, assumption 1 (together with monotonicity of $X_i(r_i)$ for all agents) implies that the IR constraint of an agent i^* will be satisfied as well, because this agent is guaranteed positive ex ante transfers. Thus, all IR constraints will also be satisfied in the solution of the relaxed problem.

To build further intuition for our results, we present a decomposition of the inequality adjusted valuation φ_i into three components that highlight the trade-offs the designer faces:

$$\varphi_i(r_i) = \underbrace{\Lambda_i r_i}_{\text{Efficient allocation}} + \underbrace{(\Lambda^* - \Lambda_i) J_i(r_i)}_{\text{Ex ante transfers}} + \underbrace{\frac{\int_{r_i}^{\bar{r}_i} \big(\lambda_i(s) - \Lambda_i\big) dG_i(s)}{g_i(r_i)}}_{\text{Ex interim uncertainty}} \tag{9}$$

The inequality adjusted valuation captures the total effect of allocating a good to an agent i with type r_i on social welfare.[7]

[7] They represent endogenous counterparts of the exogenously given priority functions found in Condorelli [2013]. In the standard case when $\lambda_i(r_i) = 1$ holds for any agent i and any r_i, the inequality adjusted valuation of any agent is always equal to r_i.

To fix ideas, suppose that the marginal utility of money of any agent i, namely v_i^M, is non-stochastic. Then, $\lambda_i(r_i) = \Lambda_i$ holds for all r_i, which means that the third term of the function $\varphi_i(r_i)$ becomes zero and the inequality adjusted valuation of any agent just contains the first two terms. When allocating the good to an agent i, this raises social welfare by this agent's consumption utility (v_i^K). This is captured by the first term.

The second component of $\varphi_i(r_i)$ captures how allocating the good to an agent affects welfare by raising the revenue available to the designer for redistribution between agents. Raising revenue from agents with $\Lambda_i < \Lambda^*$ through allocation of the good, as captured by the virtual valuation $J_i(r_i)$, is beneficial for the designer. This revenue will be redistributed ex ante to the poorest agent, increasing welfare by Λ^* at the cost of the ex ante marginal utility of money of the agent from which the revenue was generated, namely Λ_i. The total effect of this ex ante movement of money on social welfare is captured by the second term.

Now suppose that an agent's marginal utility of money is stochastic. Then, an allocation decision affects social welfare even beyond generating consumption utility and raising revenue for redistribution between agents. This is because an allocation choice affects the transfer schedule of an agent and these shifts are not neutral when the agent's marginal utility of money is stochastic. Intuitively, this stochasticity endows the designer with a desire to transfer money to the agent when this agent's inferred marginal utility of money, namely $\lambda_i(r_i)$, is high and vice versa. These incentives are captured by the third component of the inequality adjusted valuation.

Naturally, it is of interest to investigate when our allocation rule simplifies to the ex post efficient allocation rule which allocates the good to the m agents with the highest valuations r_i. Our model simplifies to the standard framework when $\lambda_i(r) = 1$ for all agents and thus yields the ex post efficient allocation rule in that case. More interestingly, our allocation rule also simplifies to the standard allocation in the well studied i.i.d. environment, as is shown in the next corollary:

Corollary 1 (Ex ante symmetry) *Suppose that either of the following conditions is met:*

1. $\lambda_i(r_i) = 1$ *for all* i *and* r_i
2. *The pair* (v_i^K, v_i^M) *is i.i.d. for all agents* i *and* $\varphi_i(r_i)$ *is strictly increasing*

Then, the utilitarian optimal allocation rule is equal to the ex post efficient allocation rule, i.e. the good is assigned to the m *agents with the highest valuations* r_i.

Suppose that (v_i^K, v_i^M) is i.i.d. among all agents. From an ex ante standpoint, the mapping from willingness to pay into consumption utility and marginal utility of money is thus the same for all agents. Moreover, every agent is considered equally rich or poor ex ante. Thus, the designer finds himself unwilling to engage in any kind of redistribution between agents. For these reasons, the designer applies the standard allocation rule.

Now, we study the incidence of rationing (i.e. when some units of the good are left unallocated) in the optimal mechanism. Intuitively, rationing is a part of the optimal solution because, as in Myerson [1981], it positively impacts the amount of money a designer can raise. In the optimal mechanism, rationing will occur if more than $N - m$ agents have an inequality adjusted valuation that is negative. We say that an agent is subject to rationing when some units of the good are not allocated but this agent still has demand for the good. Agents with the highest marginal utility of money (Λ^*) will never be subject to rationing:

Corollary 2 (Rationing) *Any agent i with $\Lambda_i = \Lambda^*$ is never subject to rationing. All agents with $\Lambda_i \neq \Lambda^*$ will be subject to rationing if $\underline{r}_i = 0$.*

Rationing is a key source of allocative inefficiency in our model and hence plays an important role for social welfare. Thus, it is instructive to understand how wealth inequality affects the incidence of rationing. To understand the quantitative magnitude of rationing in a given setting, we consider the probability that rationing occurs, i.e. the fraction of possible type realizations for which rationing would occur. To fix ideas, we assume that the marginal utilities of money are fixed for any agent, but can vary across agents. This allows us to obtain the following results:

Proposition 3 (Inequality and the probability of rationing) *Assume that the marginal utility of money of all agents is non-stochastic. Then, it holds that:*

1. $\frac{\partial Pr(\varphi_i(r_i)<0)}{\partial \Lambda^*} \geq 0$ *holds for all agents $i \neq i^*$. Thus, when Λ^* increases, the probability with which rationing occurs weakly increases.*

2. *Suppose* $\frac{\partial[(1-F_i^K(v_i^K))/f_i^K(v_i^K)]}{\partial v_i^K} \leq 0$. *Then,* $\frac{\partial Pr(\varphi_i(r_i)<0)}{\partial \Lambda_i} < 0$ *holds, and a decrease of Λ_i will raise the probability with which rationing will occur.*

This result shows that increases of wealth inequality that mirror the development of real wages in the USA over the years 1990-2010 will lead to an increase in the probability of rationing. To see why, note that real wages of men have stagnated at the 10^{th} percentile and 50^{th} percentile, while the real wages of the 90^{th} percentile have risen by 22% over this period.[8] Within our model, this can be viewed as a decrease of Λ_i for the wealthier members of the distribution, while all other Λ_i's are left unchanged. Result 2 shows that these developments will lead to a greater probability that any rich agent would be subject to rationing in a utilitarian optimal mechanism, and thus to more allocative ineffiency.

[8] For details, please see Donovan and Bradley [2019].

4.2 No-subsidy constraints

In this subsection, we impose that the expected transfer any agent receives from the mechanism must be weakly negative. These constraints make it impossible for the planner to engage in ex ante redistribution between the agents. Plugging in the integrability condition, these constraints may be expressed as follows:

$$\mathbb{E}[T_i(r_i)] = U_i(\underline{r}_i) - \int_{\underline{r}_i}^{\bar{r}_i} X_i(r_i)J_i(r_i)dG_i(r_i) \leq 0 \qquad \forall i \tag{10}$$

Thus, the maximization problem can be stated as:

$$\begin{aligned}
\max_{\{x_i(r_i,r_{-i})\in[0,1],U_i(\underline{r}_i)\}_{i=1}^N} & \sum_i \left(\Lambda_i U_i(\underline{r}_i) + \int \Pi_i(r_i)x_i(r_i,r_{-i})dG(r_i,r_{-i})\right) \\
\text{s.t. } & U_i(\underline{r}_i) - \int_{\underline{r}_i}^{\bar{r}_i} X_i(r_i)J_i(r_i)dG_i(r_i) \leq 0 \quad \forall i && \text{(No-subsidy)} \\
& \sum_i x_i(r_i,r_{-i}) \leq m && \text{(Feasibility)} \\
& X_i(r_i) \text{ non-decreasing} \quad \forall i && \text{(Monotonicity)} \\
& U_i(\underline{r}_i) \geq 0 \quad \forall i && \text{(IR)}
\end{aligned}$$

In the optimal solution to the above problem, all no-subsidy constraints must bind. Suppose, for a contradiction, that the no-subsidy constraint is slack for some agent i in the optimal solution. Then, the designer could increase $U_i(\underline{r}_i)$ in compliance with this constraint. This change would not violate any other constraint and would raise social welfare, a contradiction.

Because all no-subsidy constraints must bind in the optimal solution, our problem involves the maximization of the following functional, subject to the remaining constraints:

$$\max_{\{x_i(r_i,r_{-i}),U_i(\underline{r}_i)\}_{i=1}^N} \sum_i \left(\int \big(\Pi_i(r_i) + \Lambda_i J_i(r_i)\big)x_i(r_i,r_{-i})dG(r_i,r_{-i})\right) \tag{11}$$

The structure of this relaxed problem is almost the same as in the previous section, with the only difference being that the arguments in the objective function are now slightly different. To that end, we define the *inequality adjusted valuation* for this particular setting now:

Definition 2 (Inequality adjusted valuation - II) *We define $\gamma_i(r_i) := \Pi_i(r_i)+\Lambda_i J_i(r_i)$ as the inequality adjusted valuation of agent i under the no-subsidy constraints.*

Note that the function γ_i in this setting differs from the inequality adjusted valuation of the previous setting (φ_i) only in the factor with which an agent's virtual valuation $J_i(r_i)$

is multiplied.[9] Previously, this was Λ^*, and now it is Λ_i. This reflects the following logic: In the previous setting, any money that was raised from an agent i with $\Lambda_i \neq \Lambda^*$ was redistributed to agent i^* ex ante (raising social welfare by Λ^*), while any such money will now be refunded to agent i ex ante, given that the no subsidy constraints must bind in the optimal mechanism.

The optimal mechanism, which revolves around these inequality adjusted valuations, is characterized by the following proposition:

Proposition 4 (Optimal Mechanism - II) *Suppose that γ_i is weakly increasing for all agents i. When ex ante transfers must be weakly negative, the optimal mechanism assigns the goods to the m agents with the highest $\gamma_i(r_i)$. All units of the good are always allocated.*

To understand the result, consider a relaxed version of our maximization problem, in which the functional in equation (11) is maximized, subject only to the feasibility constraint. In the solution to this relaxed problem, the agents with the highest positive inequality adjusted valuations will receive the good.

In the appendix, we show that the solution to the relaxed problem (in which the monotonicity and IR constraints were ignored) constitutes a solution to the original problem under the stated assumptions. The intuition is as before: The assumption that $\gamma_i(r_i)$ is increasing guarantees that the monotonicity constraint w.r.t $X_i(r_i)$ will be satisfied in the solution of the relaxed problem. Now consider the IR constraints. Because the no-subsidy constraints bind, the utility of an agent i with type $\underline{r}_i$ is given by $U_i(\underline{r}_i) = \int_{\underline{r}_i}^{\bar{r}_i} X_i(r_i) J_i(r_i) dG_i(r_i)$. Assumption 1 guarantees that the right-hand side of this expression is always strictly positive when the allocation rule $X_i(r_i)$ satisfies monotonicity, which we have verified. Thus, all IR constraints will be satisfied in the solution of the relaxed problem. In contrast to the previous section, the assumption that $\gamma_i(r_i)$ is weakly increasing implies that the functions $\gamma_i(r_i)$ are always positive, which means that there will be no rationing.

To gain further intuition for the determinants of the optimal mechanism, consider the following decomposition of $\gamma_i(r)$:

$$\gamma_i(r_i) = \Pi_i(r_i) + \Lambda_i J_i(r_i) = \underbrace{\Lambda_i r_i}_{\text{Efficient allocation}} + \underbrace{\frac{\int_{r_i}^{\bar{r}_i} \big(\lambda_i(s) - \Lambda_i\big) dG_i(s)}{g_i(r_i)}}_{\text{Ex interim uncertainty}} \tag{12}$$

This inequality adjusted valuation is very similar to the function $\varphi_i(r_i)$ from the previous section, with only one difference: The second term of the previous inequality adjusted valuation $\varphi_i(r_i)$ is now absent. This holds because this second term reflected the effect of an allocation decision on the extent of redistribution between agents, which is now

[9] In the standard case when $\lambda_i(r) = 1$, the inequality adjusted valuation $\gamma_i(r_i)$ equals r_i as before.

prohibited. Thus, these considerations will not affect the optimal allocation decision, i.e. they do not enter the inequality adjusted valuation.

We now provide sufficient conditions under which assumption 1 holds and the inequality adjusted valuations we have studied (γ_i and φ_i) are weakly increasing:

Remark 1 *Suppose that the following holds true for an agent i: (i) $\frac{\partial J_i(r_i)}{\partial r_i} \geq 0$, (ii) $\lambda_i(r_i)$ is weakly decreasing in r_i for all $r_i \in [\underline{r}_i, \bar{r}_i]$, and (iii) $\bar{v}_i^M \leq 2\underline{v}_i^M$. Then, both γ_i and φ_i are weakly increasing. Moreover, if point (i) holds true for all agents, assumption 1 is satisfied.*

Remark 1 states that monotonicity of the inequality adjusted valuations is satisfied in an environment with three characteristics. First, the virtual valuation must be weakly increasing as in Myerson [1981]. Secondly, a high willingness to pay is most likely to be supported, ceteris paribus, by a relatively low expected valuation for money. As Dworczak et al. [2021] point out, this assumption is "fairly natural: Generating an increasing $\lambda_i(r_i)$ would require a very strong positive correlation between v_i^K and v_i^M." Thirdly, the spread of possible marginal utilities of money for any given agent must be sufficiently small. Intuitively, the last requirement requires that the designer knows an agent's marginal utility of money with relatively high precision and that the remaining stochasticity plays no major role. When the inequality adjusted valuations are non-monotonic, the optimal mechanism can be derived using the ironing procedure put forth by Condorelli [2013], which is based on the classical approach of Myerson [1981].

5 Implementation via auctions

5.1 Ex ante budget balance

Now, we move on to describe how the utilitarian optimal mechanism in the presence of an ex ante budget balance constraint can be implemented as an auction. The environment is the following: There is one good to be allocated and all agent's types r_i are continuously distributed on $[0, \bar{r}_i]$. The inequality adjusted valuations of all agents are strictly increasing and the supports of the inequality adjusted valuations are given by $[\underline{\varphi}_i, \bar{\varphi}_i]$, where the agents are ordered such that $\bar{\varphi}_1 \leq ... \leq \bar{\varphi}_{N-1} \leq \bar{\varphi}_N$.

In the auction we describe, agents only make payments if they win the auction — the winning agent pays their bid, as in a first-price auction. All other agents pay nothing.

To describe the auction rules, we define $\Phi_i(r_i) := Pr\{\max_{j \neq i}\{\varphi_j(r_j)\} \leq \varphi_i(r_i)|r_i\}$ as the interim allocation probabilities specified by the utilitarian optimal mechanism. Note further that the inequality adjusted valuations can be negative in this setting. To that end, define r_i^{min} as the agent's type that satisfies $\varphi_i(r_i^{min}) = 0$. We define the following functions:

$$\beta_i(r_i) := \begin{cases} r_i - (1/\Phi_i(r_i))\big[\int_{\underline{r}_i}^{r_i} \Phi_i(s)ds\big] & r_i > r_i^{min} \\ r_i^{min} & r_i \leq r_i^{min} \end{cases} \tag{13}$$

Let $b = (b_1, ..., b_N)$ denote the vector of the agents' bids. For all agents, we define functions $\tilde{\beta}_i(r_i)$ that, roughly speaking, map bids into the associated types if all agents bid according to $\beta_i(r_i)$. For the agents $j \in \{1, ..., N-1\}$, this function is:

$$\tilde{\beta}_j(b_j) = \begin{cases} r_j^{min} & b_j \leq \beta_j(r_j^{min}) \\ \beta_j^{-1}(b_j) & b_j \in (\beta_j(r_j^{min}), \beta_j(\bar{r}_j)) \\ \bar{r}_j & b_j \geq \beta_j(\bar{r}_j) \end{cases} \tag{14}$$

For agent N, this function is given by:

$$\tilde{\beta}_N(b_N) = \begin{cases} r_N^{min} & b_N \leq \beta_N(r_N^{min}) \\ \beta_N^{-1}(b_N) & b_N \in (\beta_N(r_N^{min}), \beta_N(\tilde{r}_N)) \\ \tilde{r}_N & b_N \geq \beta_N(\tilde{r}_N) \end{cases} \tag{15}$$

Finally, agent i wins the auction if and only if:

$$\varphi_i(\tilde{\beta}_i(b_i)) > 0 \ \wedge \ \varphi_i(\tilde{\beta}_i(b_i)) \geq \max_{j \in \{1,2,...,N\}} \{\varphi_j(\tilde{\beta}_j(b_j))\} \tag{16}$$

The following proposition formalizes that this auction has a Bayes-Nash equilibrium in which our allocation rule is implemented.

Proposition 5 *Suppose that φ_i is strictly increasing for all agents. In the auction described above, the profile of bidding functions $(b_1(r_1), ..., b_N(r_N)) = (\beta_1(r_1), ..., \beta_N(r_N))$ constitutes a Bayes-Nash equilibrium, in which the utilitarian optimal allocation rule is implemented.*

By the results of Milgrom and Segal [2002], these bidding functions constitute an equilibrium because the social choice function they induce satisfies the integrability condition and implements the desired allocation rule. In contrast to a standard first-price auction, our auction employs bidding subsidies and minimum bids. Thus, the bidder who bids the highest amount will not necessarily win the auction.

For every agent i, her minimum bid is given by r_i^{min}, since these solve $\varphi_i(r_i^{min}) = 0$. A bidder i has a positive chance of receiving the good if and only if her bid is above the bidder-specific minimum bid r_i^{min}. Note that all agents $i \notin arg\max_j \Lambda_j$ have strictly positive minimum bids, while all agents $i^* \in arg\max_j \Lambda_j$ have a minimum bid equal to

0.[10]

To illustrate the above insights, we calculate the auction rules, equilibrium bidding functions and expected payments for the following example: Suppose agent 1 has a valuation of the good $v_1^K \sim U[0,1]$ and a deterministic utility of money $\Lambda_1 = 1$, such that $r_1 \sim U[0,1]$. Agent 2 has a valuation of the good $v_2^K \sim U[0,2]$ and a deterministic utility of money $\Lambda_2 = 2$ such that $r_2 \sim U[0,1]$.

Agents 1 and 2 have minimum bids equal to $b_1^{min} = \frac{1}{3}$ and $b_2^{min} = 0$, respectively. When agent 1 submits a bid below her minimum bid, she will never win the auction. If agent 1 bids above this minimum bid, she will win the auction if and only if[11]:

$$b_1 \geq b_2 + \underbrace{\frac{1}{3}\left(\sqrt{b_2^2 + b_2} - 2b_2 + 1\right)}_{\text{bidding subsidy}} \tag{17}$$

Moreover, agent 2 receives a bidding subsidy. This bidding subsidy is non-constant and equal to $\frac{1}{3}$ at bids $b_2 = 0$ and $b_2 = \frac{1}{3}$. It is slightly larger at bids b_2 in between these values. Then, the equilibrium bidding functions are given by:

$$b_1(r_1) = 0.5r_1 + \frac{1}{6} \tag{18}$$

$$b_2(r_2) = r_2\left(1 - \frac{r_2 + 1}{2r_2 + 1}\right) \tag{19}$$

The interim expected payments of the bidders can be calculated by multiplying the probability of winning with one's bid. For bidder 1, these equal $\frac{1}{12}(9r_1^2 - 1)$ when $r_1 \geq \frac{1}{3}$ and 0 otherwise due to her minimum bid. They equal $\frac{r_2^2}{3}$ for bidder 2.

5.2 No subsidy constraints

In this subsection, we describe how our optimal mechanism in the presence of the no-subsidy constraints can be implemented as an auction. As before, there is just one good to be allocated. Assume further that $\gamma_i(r_i)$ is strictly increasing and define the support of possible inequality adjusted valuations of an agent i as $[\underline{\gamma}_i, \bar{\gamma}_i]$. We order the N agents such that $\bar{\gamma}_1 \leq ... \leq \bar{\gamma}_{N-1} \leq \bar{\gamma}_N$. Moreover, we define $\tilde{r}_N$ as the type of agent N that solves $\gamma_N(\tilde{r}_N) = \bar{\gamma}_{N-1}$.

In the auction we describe, agents only make payments if they win the auction — the winning agent pays their bid, as in a first-price auction. To describe which agent will win the auction, we define $\Gamma_i(r_i) = Pr\{\max_{j \neq i}\{\gamma_j(r_j)\} \leq \gamma_i(r_i)|r_i\}$ as the interim allocation

[10] Formally, this holds because the inequality adjusted valuation of any agent i^* is zero when this agent's type is $r_{i^*} = 0$. For any other agent j, the inequality adjusted valuation at $r_j = 0$ is strictly negative.

[11] The corresponding calculations may be found in appendix A.11.

probabilities under the utilitarian optimal allocation rule, and the following functions:

$$\tau_i(r_i) := \begin{cases} r_i - (1/\Gamma_i(r_i))\left[\int_{\underline{r}_i}^{r_i} \Gamma_i(s)ds\right] & r_i > 0 \\ 0 & r_i = 0 \end{cases} \tag{20}$$

As before, let $b = (b_1, ..., b_N)$ denote the vector of the agents' bids. Now, we construct functions $\tilde{\tau}_i(r_i)$ that, loosely speaking, allow the auctioneer to infer an agent's type (r_i) from her bid if all agents bid according to the rule $\tau_i(r_i)$. These functions satisfy the following:

$$\tilde{\tau}_j(b_j) = \begin{cases} \tau_j^{-1}(b_j) & b_j \leq \tau_j(\bar{r}_j) \\ \bar{r}_j & b_j > \tau_j(\bar{r}_j) \end{cases} \quad \forall j \in \{1, ..., N-1\} \quad ; \quad \tilde{\tau}_N(b_N) = \begin{cases} \tau_N^{-1}(b_N) & b_N \leq \tau_N(\tilde{r}_N) \\ \tilde{r}_N & b_N > \tau_N(\tilde{r}_N) \end{cases} \tag{21}$$

Having defined the functions $\tilde{\tau}_i(b_i)$, we close the definition of the auction rules by specifying that an agent i wins the auction if and only if:

$$\gamma_i(\tilde{\tau}_i(b_i)) \geq \max_{j \in \{1,,2,...,N\}} \{\gamma_j(\tilde{\tau}_j(b_j)\} \tag{22}$$

This auction has a Bayes-Nash equilibrium in which the utilitarian optimal allocation rule for this setting is implemented.

Proposition 6 *Suppose that $\gamma_i(r_i)$ is strictly increasing for all agents. In the aforementioned auction, the profile of bidding functions $(b_1(r_1), ..., b_N(r_N)) = (\tau_1(r_1), .., \tau_N(r_N))$ is a Bayes-Nash equilibrium in which the utilitarian optimal allocation rule is implemented.*

This auction implements our allocation rule via multiplicative bidding subsidies. To see this, consider a setting with two agents $i \in \{1, 2\}$, where $v_i^K \sim U[0, 1]$. The agents have deterministic marginal utilities of money: Define Λ_1 (Λ_2) as agent 1's (2's) marginal utility of money. In the outlined auction, each agent will optimally bid according to the rule $\tau_i(r_i) = 0.5r_i$. The expected ex interim payments of the bidders equal their bid multiplied with their ex interim winning probability, which equates to $0.5r_i \cdot \Lambda_i r_i = 0.5\Lambda_i r_i^2$. For a given vector of bids (b_1, b_2), agent 1 receives the good if and only if:[12]

$$\gamma_1(\tau_1^{-1}(b_1)) \geq \gamma_2(\tau_2^{-1}(b_2)) \iff (\Lambda_1/\Lambda_2)b_1 \geq b_2 \tag{23}$$

Suppose that agent 1 has tougher financing conditions than agent 2, i.e. $\Lambda_1 > \Lambda_2$. Then, agent 1's bids will be scaled up by a factor greater than 1. Thus, agent 1 will receive bidding subsidies in these auctions. The larger the inequality between the agents, i.e. the higher Λ_1/Λ_2, the greater will be the bidding subsidies received by the agent 1.

[12]The corresponding calculations may be found in appendix A.11.

6 Numerical illustrations

To further emphasize the key points of our paper, we provide some numerical illustrations. Assume that there are two agents $i = 1, 2$ with $v_i^K \sim U[0,1]$ and $v_1^M \sim Pareto(k = 3, x_{\min} = 1.5)$ while $v_2^M \sim Pareto(k = 3, x_{\min} = 1)$. Note that the support of the Pareto distribution is $[x_{\min}, \infty)$. Therefore, agent 2 has support on lower values of v^M than agent 1. This can naturally arise in a setting where agent 2 is ex ante more wealthy than agent 1 or has easier financing conditions, but preferences are not fully known ex ante. In this example, both inequality adjusted valuations will be non-decreasing.

In the following figure, we show how the utilitarian optimal allocation rules deviate from the ex post efficient rule.[13] The yellow line illustrates the ex post efficient allocation rule. The blue line represents the utilitarian optimal allocation rule under ex ante budget balance. The red line represents the utilitarian optimal allocation rule under the no-subsidy constraints. All three allocation rules can be understood as follows: For a given r_1 (on the x-axis), agent 2 is allocated the good under a given allocation rule if and only if her willingness to pay r_2 is such that the point (r_1, r_2) lies above the line corresponding to the allocation rule.

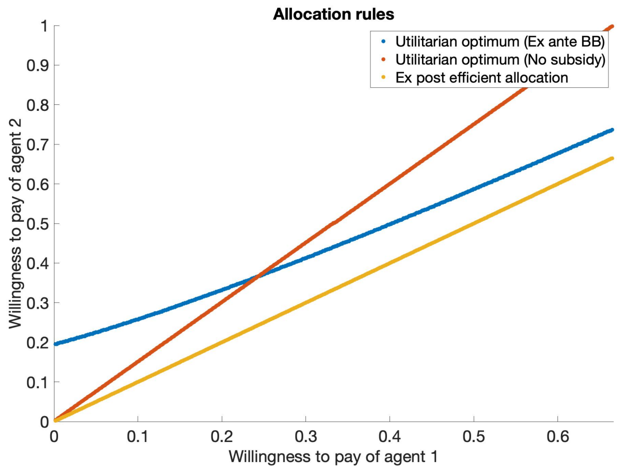

Figure 1: Utilitarian optimal allocation rule vs. ex post efficient allocation rule

In either utilitarian optimal mechanism, agent 1 (who is perceived to have more difficult financing conditions) receives the good more often than under the ex post efficient allocation rule. This result is driven by two forces. Firstly, when agent 1 reports low values of r_1, this is often driven by a high marginal utility of money, not by a low consumption utility. Achieving allocative efficiency necessitates controlling for this. Secondly, the designer can realize his preference to redistribute money from agent 2 to agent 1 under the ex ante budget balance requirement. When agent 2 has a negative virtual valuation, allocating

[13]The algebra involved in calculating the inequality adjusted valuations may be found in appendix A.12.

the good to this agent reduces the revenue that is raised from her, which is undesirable to the designer. This effect leads to greater differences between the utilitarian optimal and the ex post efficient allocation rules when r_2 is low because the virtual valuation of agent 2 is negative when r_2 is low. Even when $r_1 = r_2 = 0$, agent 1 would surely receive the good under the ex ante budget balance requirement, because allocating the good to agent 2 would negatively impact the revenue that can be raised from the latter.

Finally, it remains to discuss the discrepancies between the utilitarian optimal allocation rules under the no-subsidy constraints and under the ex ante budget balance constraint. Any differences inbetween these rules are exclusively driven by the fact that the designer can redistribute towards agent 2 under ex ante budget balance, but not under the no-subsidy constraints. At low values of r_2, agent 2's virtual valuation is negative, thus favoring allocation of the good towards agent 1 when the budget has to be balanced. At high values of r_2, the opposite holds, thus motivating the assignment of the good to agent 2. These arguments explain why, under ex ante budget balance, the good is less frequently allocated to agent 2 when r_2 is high and vice versa.

7 Conclusion

We have derived the utilitarian optimal mechanism for an assignment problem in which a designer initially owns a fixed number of indivisible goods which are to be distributed among a finite number of agents. In contrast to the usual assumption made in the literature, we work with heterogeneous marginal utilities of money. We have formalized this feature by adapting the model of Dworczak et al. [2021] to our framework. In addition to the standard incentive compatibility and individual rationality constraints, the designer also faces additional constraints on transfers, namely i) a requirement that the designer's budget must be balanced ex ante or (ii) constraints stating that no agent can receive positive transfers from the mechanism in expectation.

We derive the utilitarian optimal mechanism using methodologies developed in Condorelli [2013] and Dworczak et al. [2021]. It revolves around a key statistic which we call the *inequality adjusted valuation.* The inequality adjusted valuation, whose exact form depends on the constraints on transfers that are imposed, condenses three critical considerations: First, the designer has a desire to allocate the goods to the agents with the highest consumption utilities, ceteris paribus. Second, when there is ex-interim uncertainty about the agent's marginal utilities of money, the designer wants to pay transfers to an agent when her marginal utility of money is above its average and vice versa. The allocation rule will reflect this, because assignment of the good will always be associated with payments. Thirdly, when redistribution is possible under the ex ante budget balance requirement, the allocation rule will accomodate the designer's wish to raise revenue for redistribution towards agents with high willingnesses to pay.

We have shown that the utilitarian optimal mechanism allocates the goods the agents with the highest positive inequality adjusted valuations. These agents do not necessarily have the highest willingnesses to pay. Thus, ex ante asymmetries between the agents in the form of heterogeneity in the marginal utilities of money creates a tension between ex post efficiency and ex ante optimality. Under the ex ante budget balance condition, there are states of the world in which some units of the good are left unallocated. Such outcomes, which are a byproduct of the designer's revenue motive, may be exacerbated by high levels of inequality.

In the real world applications we have discussed, incorporating our ideas may be beneficial even beyond raising instantaneous social welfare. In the kindergarten example, accounting for wealth inequality may foster equality of opportunity by promoting equal access to education. In the auction examples we discussed, applying our insights may be quite pro-competitive. This is because our mechanism reduces the advantage that incumbents with easy financing conditions have in traditional auction mechanisms.

Appendices

A Proofs

A.1 Proof of proposition 1

See Dworczak et al. [2021].

A.2 Proof of lemma 1

The result follows directly from rescaling the agents' utility functions and applying the standard results from the literature.

A.3 Proof of proposition 2

Part 1: Deriving and solving a relaxed problem

The optimal mechanism needs to solve:

$$\begin{aligned}
\max_{\{x_i(r_i,r_{-i}),U_i(\underline{r}_i)\}_{i=1}^N} \; & \sum_i \left(\Lambda_i U_i(\underline{r}_i) + \int \Pi_i(r_i)x_i(r_i,r_{-i})dG(r_i,r_{-i})\right) \\
\text{s.t.} \; & \sum_i \left(U_i(\underline{r}_i) - \int J_i(r_i)x_i(r_i,r_{-i})dG(r_i,r_{-i})\right) \leq 0 && \text{(Budget)} \\
& 0 \leq x_i(r_i,r_{-i}) \leq 1 && \text{(Prob)} \\
& \sum_i x_i(r_i,r_{-i}) \leq m && \text{(Feas)} \\
& X_i(r_i) \text{ non-decreasing} && \text{(Mono)} \\
& U_i(\underline{r}_i) \geq \underline{U}_i = 0 && \text{(IR)}
\end{aligned}$$

Note that the budget constraint must bind in the solution to this problem. Suppose, for a contradiction, that it is slack. Then, $U_i(\underline{r}_i)$ could be increased for some agent i. This would be in line with all constraints and would raise social welfare, implying that the starting mechanism could not have been optimal.

Based on this, note that the IR constraints for all agents $j \notin arg\max_i \Lambda_i$ must also bind. Suppose, for a contradiction, that there is one such constraint that does not bind. Then, $U_{i^*}(\underline{r}_{i^*})$ could be increased at the cost of a one-for-one decrease in $U_j(\underline{r}_j)$. Because $\Lambda^* > \Lambda_j$ by definition, this would raise welfare without violating any other constraints.

Taking these two results together implies that the budget constraint can be rewritten as follows:

$$\sum_i \left(U_i(\underline{r}_i) - \int J_i(r_i)x_i(r_i,r_{-i})dG(r_i,r_{-i})\right) = 0 \iff$$

$$\sum_i U_i(\underline{r}_i) = U_{i^*}(\underline{r}_{i^*}) = \sum_i \left(\int J_i(r_i) x_i(r_i, r_{-i}) dG(r_i, r_{-i}) \right) = 0 \tag{A.1}$$

Plugging this into the objective function and ignoring the remaining IR constraints and the monotonicity constraints implies that our optimization problem boils down to the following:

$$\max_{\{x_i(r_i,r_{-i}),U_i(\underline{r}_i)\}_{i=1}^N} \sum_i \left(\int \big(\Pi_i(r_i) + \Lambda^* J_i(r_i)\big) x_i(r_i, r_{-i}) dG(r_i, r_{-i}) \right)$$

$$\text{s.t. } 0 \leq x_i(r_i, r_{-i}) \leq 1 \tag{Prob}$$

$$\sum_i x_i(r_i, r_{-i}) \leq m \tag{Feas}$$

Following the insights of Condorelli [2013], the solution to this relaxed problem assigns the goods to the m agents with the highest φ_i, provided they are positive.

Part 2: Showing that the above solution also solves the general optimization problem

Finally, it remains to show that the monotonicity constraints and IR constraint of i^* will hold in the solution of this relaxed problem.

Monotonicity requires that $X_i(r_i)$ is non-decreasing. Note our key assumption that $\varphi_i(r_i)$ is increasing in r_i. For values of r_i where the inequality adjusted valuation is negative, monotonicity holds. Now consider values of r_i where $\varphi_i(r_i) > 0$. Agent i will receive the good if and only the agent's inequality adjusted valuation is among the m highest. Since $\varphi_i(r_i)$ is increasing in r_i, this implies that the probability of allocation, which is given by cannot be falling.

Note that the IR constraints of all agents i^* must also be satisfied, since the integrals $\int J_i(r_i)x_i(r_i, r_{-i})dG(r_i, r_{-i})$ must all be positive, given that $X_i(r_i)$ is monotone (this holds true by assumption 1. For the full derivation, please see the arguments on pages 26 and 27).

A.4 Proof of corollary 1

The proof of 1. is immediate after plugging in $\lambda_i(r_i)$. To argue why condition 2 implies the standard allocation rule, we note that when (v_i^K, v_i^M) is i.i.d. for all agents i, we have $\varphi_i(r) = \varphi_j(r) = \varphi(r)$ for all i, j.

By assumption, $\varphi_i(r_i)$ is strictly increasing. This implies that $\varphi(r)$ is a strictly increasing transformation of r and therefore $r_i \geq r_j$ if and only if $\varphi(r_i) \geq \varphi(r_j)$. Further, due to the i.i.d assumption on (v_i^K, v_i^M), it holds that $\Lambda^* = \Lambda_i$ for all i. Mathematically, this implies that $\varphi_i(r_i) \geq 0$ if and only if $r_i \geq 0$.

A.5 Proof of corollary 2

Consider the inequality adjusted valuation $\varphi_i(r_i)$ at the lowest possible realization $\underline{r}_i$:

$$\begin{aligned}
\varphi_i(\underline{r}_i) &= \Pi_i(\underline{r}_i) + \Lambda^* J_i(\underline{r}_i) && \text{(A.2)}\\
&= \frac{\int_{\underline{r}_i}^{\bar{r}_i} \lambda(s) dG_i(s)}{g_i(\underline{r}_i)} + \Lambda^* \left(\underline{r}_i - \frac{1 - G_i(\underline{r}_i)}{g_i(\underline{r}_i)} \right) && \text{(A.3)}\\
&= \Lambda^* \underline{r}_i + \frac{\Lambda_i - \Lambda^*}{g_i(\underline{r}_i)} && \text{(A.4)}
\end{aligned}$$

We note that for agent i^*, this expression is weakly positive if and only if $\underline{r}_i \geq 0$. Therefore, the only reason to not allocate the good to agent i^* would be that she has a negative valuation for the good. For the other agents, the expression is weakly positive if and only if

$$\Lambda^* \underline{r}_i + \frac{\Lambda_i - \Lambda^*}{g_i(\underline{r}_i)} \geq 0 \tag{A.5}$$

which will generally subject them to rationing unless $\underline{r}_i$ is sufficiently large.

A.6 Proof of Proposition 3

Point 1:

Firstly, we need to show that:

$$\frac{\partial Pr(\varphi_i(r) < 0)}{\partial \Lambda^*} \geq 0 \tag{A.6}$$

Consider any agent i and note the following:

$$\frac{\partial \varphi_i(r)}{\partial \Lambda^*} = J_i(r) \;\; \forall r \in (\underline{r}_i, \bar{r}_i) \tag{A.7}$$

To understand the effect of an increase in Λ^* on the probability with which agent i is rationed, note firstly that the random variable is r_i.

Firstly, consider realizations of r_i where $\varphi_i(r_i) < 0$ a priori. Because $\varphi_i(r_i) < 0$, it must hold that $J_i(r_i) < 0$ at these realizations of r_i. For these realizations, an increase in Λ^* will thus reduce $\varphi_i(r_i)$, keeping this negative for all these realizations of r_i.

Secondly, consider realizations of r_i where $\varphi_i(r_i) \geq 0$ and $J_i(r_i) \geq 0$ holds true. For these realizations of r_i, the increase in Λ^* will imply a weak increase of $\varphi_i(r_i)$, such that $\varphi_i(r_i) \geq 0$ will still hold after the increase in Λ^*.

Thirdly and finally, consider realizations of r_i where $\varphi_i(r_i) \geq 0$ and $J_i(r_i) < 0$ holds true. For these realizations of r_i, the increase in Λ^* will imply a (weak) decrease of φ_i, which can potentially push those into the negative region, even though they were positive ex ante. This working channel has a weakly positive effect on the probability that this agent is rationed.

This, the above arguments imply that $Pr(\varphi_i(r_i) < 0) < 0$ is weakly rising in Λ^*

It remains to argue why the increase in $Pr(\varphi_i < 0)$ implies an increase in the probability with which rationing occurs. To see this, note that the inequality adjusted valuations of all agents are independent. Because $Pr(\varphi_i(r_i) < 0)$ is weakly increasing in Λ^* for all agents, the probability that at least $N - m$ inequality adjusted are negative is weakly increasing in Λ^*.

Point 2:

It was shown previously that an increase of $Pr(\varphi_i(r) < 0)$ will imply an increase in the probability with which rationing occurs. We thus only have to show:

$$\frac{\partial[(1 - F_i^K(v_i^K))/f_i(v_i^K)]}{\partial v_i^K} \leq 0 \implies \frac{\partial Pr(\varphi_i(r_i) < 0)}{\partial \Lambda_i} < 0 \tag{A.8}$$

Consider any agent i. The derivative of φ_i with respect to Λ_i is:

$$\frac{\partial \varphi_i(r_i)}{\partial \Lambda_i} = r_i + (\Lambda^* - \Lambda_i)\frac{\partial J_i(r_i)}{\partial \Lambda_i} + (-1)J_i(r_i) \tag{A.9}$$

$$= r_i + (\Lambda^* - \Lambda_i)\frac{\partial J_i(r_i)}{\partial \Lambda_i} + (-1)\left[r_i - \frac{1 - G_i(r_i)}{g_i(r_i)}\right] \tag{A.10}$$

$$= (\Lambda^* - \Lambda_i)\frac{\partial J_i(r_i)}{\partial \Lambda_i} + \frac{1 - G_i(r_i)}{g_i(r_i)} \tag{A.11}$$

One can show that $\frac{\partial J_i(r_i)}{\partial \Lambda_i} \geq 0$ holds true if $\frac{\partial[(1-F_i^K(v_i^K))/f_i^K(v_i^K)]}{\partial v_i^K} \leq 0$. When the virtual valuation is weakly increasing in Λ_i, the inequality adjusted valuation will uniformly increase in Λ_i. A uniform increase in $\varphi_i(r_i)$ as a result of a change in in Λ_i implies that the probability with which rationing occur will fall.

A.7 Proof of proposition 4

Part 1: Deriving the relaxed problem

The full optimization problem is given by the following:

$$\max_{\{x_i(r_i,r_{-i}),U_i(\underline{r}_i)\}_{i=1}^N} \sum_i \left(\Lambda_i U_i(\underline{r}_i) + \int \Pi_i(r_i)x_i(r_i,r_{-i})dG(r_i,r_{-i})\right)$$

$$\text{s.t. } U_i(\underline{r}_i) - \int J_i(r_i)x_i(r_i,r_{-i})dG_i(r_i,r_{-i}) \leq 0 \ \forall i \tag{Transfers}$$

$$0 \leq x_i(r_i,r_{-i}) \leq 1 \tag{Prob}$$

$$\sum_i x_i(r_i,r_{-i}) \leq m \tag{Feas}$$

$$X_i(r_i) \text{ non-decreasing} \tag{Mono}$$

$$U_i(\underline{r}_i) \geq 0 \tag{IR}$$

The transfer constraints must bind for all agents. Otherwise, $U_i(\underline{r}_i)$ could be increased for a given agent i, which is in line with all constraints and would raise welfare.

Plugging in this result into our objective function yields the following:

$$\sum_i \left(\Lambda_i U_i(\underline{r}_i) + \int \Pi_i(r_i) x_i(r_i, r_{-i}) dG(r_i, r_{-i})\right) =$$

$$\sum_i \left(\Lambda_i \int J_i(r_i) x_i(r_i, r_{-i}) dG_i(r_i, r_{-i}) + \int \Pi_i(r_i) x_i(r_i, r_{-i}) dG(r_i, r_{-i})\right) =$$

$$\sum_i \left(\int \big(\Pi_i(r_i) + \Lambda_i J_i(r_i)\big) x_i(r_i, r_{-i}) dG(r_i, r_{-i})\right) \tag{A.12}$$

Ignoring the monotonicity and the IR constraints, our optimization problem thus boils down to maximizing the aforementioned function, subject to the feasibility and the probability constraints.

This is a standard linear programming problem. Mirroring the insights of Condorelli [2013], the goods will be allocated to the m agents with the highest $\gamma_i(r_i)$ for any given realization of types in the solution of this relaxed problem.

<u>Part 2:</u> Showing that the solution to the relaxed problem solves the full optimization problem.

It remains to show that both the monotonicity and the IR constraints will be satisfied in the solution to this relaxed problem. Monotonicity will be satisfied because γ_i is increasing in r_i and the allocation probability $X_i(r_i)$ is increasing in γ_i.

Now consider the IR constraint of agent i. We show the following: Under assumption 1 and for a monotone $X_i(r_i)$, the integral $\int_{\underline{r}_i}^{\bar{r}_i} J_i(r_i) g_i(r_i) dr_i$ will always be strictly positive.

Firstly, note the following:

$$\int_{\underline{r}_i}^{\bar{r}_i} J_i(r_i) g_i(r_i) dr_i = \left[\int_{\underline{r}_i}^{\bar{r}_i} \left(r - \frac{1 - G_i(r_i)}{g_i(r_i)}\right) g_i(r_i) dr_i\right] =$$

$$\left[\int_{\underline{r}_i}^{\bar{r}_i} \Big(g_i(r_i) r_i - \big(1 - G_i(r_i)\big)\Big) dr_i\right] = \left[\int_{\underline{r}_i}^{\bar{r}_i} \big(g_i(r_i) r_i + G_i(r_i)\big) dr_i - \int_{\underline{r}_i}^{\bar{r}_i} (1) dr_i\right] \tag{A.13}$$

Note that:

$$\frac{\partial G_i(r_i) r_i}{\partial r_i} = g_i(r_i) r_i + G_i(r_i) \tag{A.14}$$

Plugging this in yields that:

$$\int_{\underline{r}_i}^{\bar{r}_i} J_i(r_i) g_i(r_i) dr_i = \left[\big[G_i(r_i) r_i\big]_{\underline{r}_i}^{\bar{r}_i} - \int_{\underline{r}_i}^{\bar{r}_i} (1) dr_i\right] = \left[\bar{r}_i - \int_{\underline{r}_i}^{\bar{r}_i} (1) dr_i\right] = \underline{r_i} \tag{A.15}$$

Thus, this term is always weakly positive as long as $\underline{r}_i \geq 0$.

By assumption 1, there exists an $\hat{r}$ such that:

$$\forall r_i \geq \hat{r}_i: \;\; J_i(r_i) \geq 0 \tag{A.16}$$

$$\forall r_i < \hat{r}_i: \;\; J_i(r_i) \leq 0 \tag{A.17}$$

By this assumption and because $g_i(r_i)$ is always positive, it holds that:

$$\forall r_i \geq \hat{r}_i: \;\; J_i(r_i)g_i(r_i) \geq 0 \tag{A.18}$$

$$\forall r_i < \hat{r}_i: \;\; J_i(r_i)g_i(r_i) \leq 0 \tag{A.19}$$

Consider a monotone $X_i(r_i)$. By monotonicity of X_i and the above arguments, we have that:

$$\forall r_i \geq \hat{r}_i: \;\; X_i(r_i) \geq X_i(\hat{r}_i) \implies X_i(r_i)J_i(r_i)g_i(r_i) \geq X_i(\hat{r}_i)J_i(r_i)g_i(r_i) \tag{A.20}$$

$$\forall r_i < \hat{r}_i: \;\; X_i(r_i) \leq X_i(\hat{r}_i) \implies X_i(r_i)J_i(r_i)g_i(r_i) \geq X_i(\hat{r}_i)J_i(r_i)g_i(r_i) \tag{A.21}$$

Thus, we have that:

$$\int_{\underline{r}_i}^{\bar{r}_i} X_i(r_i)J_i(r_i)g_i(r_i)dr_i = \int_{\underline{r}_i}^{\hat{r}_i} X_i(r_i)J_i(r_i)g_i(r_i)dr_i + \int_{\hat{r}_i}^{\bar{r}_i} X_i(r_i)J_i(r_i)g_i(r_i)dr_i \geq$$

$$\int_{\underline{r}_i}^{\hat{r}_i} X_i(\hat{r}_i)J_i(r_i)g_i(r_i)dr_i + \int_{\hat{r}_i}^{\bar{r}_i} X_i(\hat{r}_i)J_i(r_i)g_i(r_i)dr_i = X_i(\hat{r}_i)\int_{\underline{r}_i}^{\bar{r}_i} J_i(r_i)g_i(r_i)dr_i = X_i(\hat{r}_i)\underline{r}_i \geq 0 \tag{A.22}$$

The latter holds since $X_i(\hat{r}_i)$ is a weakly positive constant.

This result implies that the IR constraints will also be satisfied in the solution of the relaxed problem. Thus, we are done.

<u>Part 3:</u> No rationing

Finally, it remains to show that all units of the good will always be allocated. Sufficient for this is to show that γ_i is always strictly positive.

Thus, we need to show that $\gamma_i(r_i) \geq 0$ for all i. To show this, it suffices to show that $\gamma_i(\underline{r}_i) \geq 0$, together with our assumption that γ_i is increasing in r_i. Thus, note that:

$$\gamma_i(r_i) = \Lambda_i r_i + \frac{\int_{r_i}^{\bar{r}_i} \big(\lambda_i(s) - \Lambda_i\big) dG_i(s)}{g_i(r_i)} \tag{A.23}$$

$$\implies \gamma_i(\underline{r}_i) = \Lambda_i \underline{r}_i + \frac{\int_{\underline{r}_i}^{\bar{r}_i} \big(\lambda_i(s) - \Lambda_i\big) dG_i(s)}{g_i(r_i)} = \Lambda_i \underline{r}_i \geq 0 \tag{A.24}$$

A.8 Proof of Remark 1

Part 1:

Note first that the inequality adjusted valuation φ_i can be written as follows:

$$\varphi_i(r_i) = \Lambda^* J_i(r_i) + \frac{1 - G_i(r_i)}{g_i(r_i)} \mathbb{E}\big[\lambda_i(s)|s \geq r_i\big] \tag{A.25}$$

Taking the derivative of φ_i w.r.t r_i yields:

$$\frac{\partial \varphi_i}{\partial r_i} = \Lambda^* \frac{\partial J_i(r_i)}{\partial r_i} + \left[\frac{\partial}{\partial r_i}\left(\frac{1 - G_i(r_i)}{g_i(r_i)}\right)\right] \mathbb{E}\big[\lambda_i(s)|s \geq r_i\big] +$$
$$\left[\frac{1 - G_i(r_i)}{g_i(r_i)}\right] \frac{\partial \mathbb{E}\big[\lambda_i(s)|s \geq r_i\big]}{\partial r_i} \tag{A.26}$$

The critical term is the third term. We evaluate this term in more detail. Note firstly that:

$$\mathbb{E}\big[\lambda_i(s)|s \geq r_i\big] = \int_{r_i}^{\bar{r}_i} \lambda_i(s) g_i(s) \big[1 - G_i(r_i)\big]^{-1} d(s) \tag{A.27}$$

The derivative of this w.r.t r_i is:

$$\frac{\partial \mathbb{E}\big[\lambda_i(s)|s \geq r_i\big]}{\partial r_i} =$$

$$-\lambda_i(r_i) g_i(r_i) \big[1 - G_i(r_i)\big]^{-1} + \int_{r_i}^{\bar{r}_i} \lambda_i(s) g_i(s) \big[1 - G_i(r_i)\big]^{-2} (-1)(-g_i(r_i)) d(s) \tag{A.28}$$

$$= -\lambda_i(r_i) \left[\frac{1 - G_i(r_i)}{g_i(r_i)}\right]^{-1} + \left[\frac{1 - G_i(r_i)}{g_i(r_i)}\right]^{-1} \int_{r_i}^{\bar{r}_i} \lambda_i(s) g_i(s) \big[1 - G_i(r_i)\big]^{-1} d(s) \tag{A.29}$$

$$= \left[\frac{1 - G_i(r_i)}{g_i(r_i)}\right]^{-1} \Big[\mathbb{E}\big[\lambda_i(s)|s \geq r_i\big] - \lambda_i(r_i)\Big] \tag{A.30}$$

Plugging in then yields that:

$$\frac{\partial \varphi_i}{\partial r_i} = \Lambda^* \frac{\partial J_i(r_i)}{\partial r_i} + \left[\frac{\partial}{\partial r_i}\left(\frac{1-G_i(r_i)}{g_i(r_i)}\right)\right]\mathbb{E}\big[\lambda_i(s)|s \geq r_i\big] + \Big[\mathbb{E}\big[\lambda_i(s)|s \geq r_i\big] - \lambda_i(r_i)\Big] \tag{A.31}$$

$$= \Lambda^* \frac{\partial J_i(r_i)}{\partial r_i} + \left[\frac{\partial}{\partial r_i}\left(\frac{1-G_i(r_i)}{g_i(r_i)}\right) - 1\right]\mathbb{E}\big[\lambda_i(s)|s \geq r_i\big] + \Big[2\mathbb{E}\big[\lambda_i(s)|s \geq r_i\big] - \lambda_i(r_i)\Big] \tag{A.32}$$

$$= \frac{\partial J_i(r_i)}{\partial r_i}\Big[\Lambda^* - \mathbb{E}\big[\lambda_i(s)|s \geq r_i\big]\Big] + \Big[2\mathbb{E}\big[\lambda_i(s)|s \geq r_i\big] - \lambda_i(r_i)\Big] \tag{A.33}$$

This derivative is hence strictly positive if and only if:

$$\frac{\partial J_i(r_i)}{\partial r_i}\Big[\Lambda^* - \mathbb{E}\big[\lambda_i(s)|s \geq r_i\big]\Big] \geq \lambda_i(r_i) - 2\mathbb{E}\big[\lambda_i(s)|s \geq r_i\big] \tag{A.34}$$

Now we can find sufficient conditions which this might hold. To understand sufficient conditions, note that the support of v_i^M is $[\underline{v}_i^M, \bar{v}_i^M]$. Then, it holds that $\lambda_i(s) \in [\underline{v}_i^M, \bar{v}_i^M]$ and thus $\mathbb{E}[\lambda_i(s)|s \geq r_i] \in [\underline{v}_i^M, \bar{v}_i^M]$.

The assumption that $\lambda_i(s)$ is weakly decreasing in s implies that:

$$\frac{\partial \mathbb{E}\big[\lambda_i(s)|s \geq r_i\big]}{\partial r_i} = \left[\frac{1-G_i(r_i)}{g_i(r_i)}\right]^{-1}\Big[\mathbb{E}\big[\lambda_i(s)|s \geq r_i\big] - \lambda_i(r_i)\Big] \leq 0 \tag{A.35}$$

This implies that, for any agent, we have that:

$$\mathbb{E}\big[\lambda_i(s)|s \geq r_i\big] \leq \mathbb{E}\big[\lambda_i(s)|s \geq \underline{r}_i\big] = \Lambda_i \leq \Lambda^* \tag{A.36}$$

The following three regularity conditions are thus sufficient to ensure that the inequality in equation (A.34) is satisfied.

- $\forall i : J_i'(r_i) \geq 0$
- $\forall i : \lambda_i'(s) \leq 0 \implies \Lambda^* - \mathbb{E}\big[\lambda_i(s)|s \geq r_i\big] \geq 0$
- $\forall i : \bar{v}_i^M \leq 2\underline{v}_i^M \implies \lambda_i(r_i) - 2\mathbb{E}[\lambda_i(s)|s \geq r_i] \leq \bar{v}_i^M - 2\underline{v}_i^M \leq 0$

Under these three conditions, the following holds for any agent i:

$$\frac{\partial J_i(r_i)}{\partial r_i}\Big[\Lambda^* - \mathbb{E}\big[\lambda_i(s)|s \geq r_i\big]\Big] \geq 0 \geq \lambda_i(r_i) - 2\mathbb{E}[\lambda_i(s)|s \geq r_i] \tag{A.37}$$

<u>Part 2:</u>

If $J_i(r_i)$ is weakly monotonic, it will cross 0 at most once. Suppose there exists an r_i^* such that $J_i(r_i^*) = 0$. Because J_i is weakly increasing, setting $\hat{r}_i = r_i^*$ satisfies our requirements. If such a point does not exist, set $\hat{r}_i = 1$.

<u>Part 3:</u>

The inequality adjusted valuation γ_i reads as follows:

$$\gamma_i = \Lambda_i J_i(r_i) + \Pi_i(r_i) = \Lambda_i\left(r_i - \frac{1-G_i(r_i)}{g_i(r_i)}\right) + \frac{1-G_i(r_i)}{g_i(r_i)}\mathbb{E}[\lambda_i(s)|s > r_i]$$

Taking the derivative of γ_i w.r.t r_i yields:

$$\frac{\partial \gamma_i}{\partial r_i} = \Lambda_i \frac{\partial J_i(r_i)}{\partial r_i} + \left[\frac{\partial}{\partial r_i}\left(\frac{1-G_i(r_i)}{g_i(r_i)}\right)\right]\mathbb{E}\big[\lambda_i(s)|s \geq r_i\big] + \left[\frac{1-G_i(r_i)}{g_i(r_i)}\right]\frac{\partial \mathbb{E}\big[\lambda_i(s)|s \geq r_i\big]}{\partial r_i}$$

Plugging in our previous results yields that:

$$\frac{\partial \gamma_i}{\partial r_i} = \Lambda_i \frac{\partial J_i(r_i)}{\partial r_i} + \left[\frac{\partial}{\partial r_i}\left(\frac{1-G_i(r_i)}{g_i(r_i)}\right)\right]\mathbb{E}\big[\lambda_i(s)|s \geq r_i\big] + \Big[\mathbb{E}\big[\lambda_i(s)|s \geq r_i\big] - \lambda_i(r_i)\Big] \tag{A.38}$$

Thus:

$$\frac{\partial \gamma_i}{\partial r_i} \geq 0 \iff \frac{\partial J_i(r_i)}{\partial r_i}\Big[\Lambda_i - \mathbb{E}\big[\lambda_i(s)|s \geq r_i\big]\Big] \geq \lambda_i(r_i) - 2\mathbb{E}\big[\lambda_i(s)|s \geq r_i\big] \tag{A.39}$$

By equation A.35, the LHS is strictly positive under our assumptions. By the third assumption, the RHS is negative - we are done.

A.9 Proof of proposition 5

We work with the following functions:

$$\Phi_i(r_i) := Pr\{\max_{j\neq i}\{\varphi_j(r_j)\} \leq \varphi_i(r_i)\} \tag{A.40}$$

$$\beta_i(r_i) = \begin{cases} r_i - \frac{\int_{\underline{r}_i}^{r_i}\Phi_i(s)ds}{\Phi_i(r_i)} & r_i > r_i^{min} \\ r_i^{min} & r_i \leq r_i^{min} \end{cases} \tag{A.41}$$

We can show continuity of this function at r_i^{min} by applying L'Hopital's rule.

We further define the following functions for all agents $i \leq N-1$:

$$\tilde{\beta}_i(b_i) = \begin{cases} r_i^{min} & b_i \leq \beta_i(r_i^{min}) \\ \beta_i^{-1}(b_i) & b_i \in (\beta_i(r_i^{min}), \beta_i(\bar{r}_i)) \\ \bar{r}_i & b_i \geq \beta_i(\bar{r}_i) \end{cases} \tag{A.42}$$

For agent N, this function is:

$$\tilde{\beta}_N(b_N) = \begin{cases} r_N^{min} & b_N \leq \beta_N(r_N^{min}) \\ \beta_N^{-1}(b_N) & b_N \in (\beta_N(r_N^{min}), \beta_N(\tilde{r}_N)) \\ \tilde{r}_N & b_N \geq \beta_N(\tilde{r}_N) \end{cases} \tag{A.43}$$

We have defined $\tilde{r}_N$ such that $\varphi_N(\tilde{r}_N) = \varphi_{N-1}$. The allocation rule is:

$$\varphi_i(\tilde{\beta}(b_i)) \geq \max_{j \in \{1,2,\dots,N\}} \{\varphi_j(\tilde{\beta}(b_j)\} \tag{A.44}$$

The implied auction has a Bayes-Nash equilibrium where all agents bid according to the rule $b_i(r_i) = \beta_i(r_i)$.

As before, one can show that the functions $\beta_i(r_i)$ are strictly increasing on the intervals for which the above formulation requires them to be (to yield a well-defined inverse).

Part 2: Auction equilibrium - assuming that $\bar{\varphi}_{N-1} = \bar{\varphi}_N$.

Under this assumption, the structure of the functions $\tilde{\beta}$ is identical for all agents.

We first show that the mechanism we have described induces a social choice function $c(r_i, r_{-i}) = \big(x(r), t(r)\big)$ that is incentive compatible.

When all agents $i \leq N-1$ bid according to the rule $b_i(r_i) = \beta_i(r_i) \leq \beta_i(\bar{r}_i)$, it holds that $\tilde{\beta}_i(\beta_i(r_i)) = \beta_i^{-1}(\beta_i(r_i)) = r_i$ for any $r_i \in [r_i^{min}, \bar{r}_i]$. For any such agent with $r_i \leq r_i^{min}$, $\tilde{\beta}(b_i) = r_i^{min}$, which implies that the agent will never receive the good.

The induced interim allocation probabilities $X_i(r_i)$ will be equal to $\Gamma_i(r_i)$, which is monotone.

To see this, consider any agent i. When $r_i \leq r_i^{min}$, the agent will never receive the good in the auction (since agents only receive the good when bidding strictly above the minimum bid). When $r_i \geq r_i^{min}$, then $\tilde{\beta}_i(\beta_i(r_i)) = r_i$, so the allocation probability is given by $\Phi_i(r_i)$.

Moreover, the implied transfer rule will satisfy the integrability constraint. To see this, note that any agent that bids b_i makes the expected payment $b_i \Phi_i(\tilde{\beta}_i(b_i))$. When bidding according to $b_i(r_i) = \beta_i(r_i)$, it thus holds that:

$$\beta_i(0) = r_i^{min} \implies U_i(\underline{r}_i) = U_i(0) = 0$$

Since the interim expected transfers are given by $T_i(r_i) = -\beta_i(r_i)\Phi_i(r_i)$, the integrability condition is satisfied for the implied social choice function because:

$$r_i X_i(r_i) + T_i(r_i) = U_i(\underline{r}_i) + \int_{\underline{r}_i}^{r_i} X_i(s) ds$$

$$\iff$$

$$r_i \Phi_i(r_i) - \tau_i(r_i)\Phi_i(r_i) = \int_{\underline{r}_i}^{r_i} \Phi_i(s) ds \iff \tau_i(r_i) = r_i - \frac{\int_{\underline{r}_i}^{r_i} \Phi_i(s) ds}{\Phi_i(r_i)}$$

This establishes that the social choice function induced when all agents bid according to $b_i(r_i) = \tau_i(r_i)$ is incentive compatible.

Having established this, we now show that it is an equilibrium when all agents bid according to the rule $\beta_i(r_i)$ defined above.

Consider any agent with $r_i \leq \bar{r}_i$. Any such agent would have no incentives to bid anything else in the interval $b_i \in (\beta_i(r_i^{min}), \beta_i(\bar{r}_i))$, since this would surely violate incentive compatibility.

Bidding anything in the interval $b_i \in [0, \beta_i(r_i^{min})]$ yields zero chance of receiving the good and hence 0 utility - thus, this deviation cannot be profitable either. Bidding anything above $\beta_i(\bar{r}_i)$ is dominated by bidding $\beta_i(\bar{r}_i)$, which cannot be profitable by previous arguments.

<u>Part 3:</u> Auction equilibrium - assuming that $\bar{\varphi}_{N-1} < \varphi_N(r_N)$.

First, note that the ex-interim allocation probabilities will also be equal to $\Phi_i(r_i)$ for any agents.

To see this, consider an agent $i \leq N-1$ and suppose that all agents bid according to $\beta_i(r_i)$. If $r_i \leq r_i^{min}$m the interim allocation probability is 0, as specified by $\Phi_i(r_i)$. Now suppose $r_i \in (r_i^{min}, \bar{r}_i]$, such that $\tilde{\beta}(\beta(r_i)) = r_i$. By the law of total probability, we can write the interim allocation probability as follows:

$$X_i(r_i) = Pr\big[\varphi_i(r_i) \geq \max_{j \in \{1,2,\ldots,N\}} \{\varphi_j(\tilde{\tau}_i(b_j)\}\big] =$$

$$Pr\big[\varphi_i(r_i) \geq \max_{j \in \{1,2,\ldots,N\}} \{\varphi_j(\tilde{\beta}_i(b_j))\} \wedge r_N \geq \tilde{r}_N\big] + \big[\varphi_i(r_i) \geq \max_{j \in \{1,2,\ldots,N\}} \{\varphi_j(\tilde{\beta}_i(b_j)\} \wedge r_N < \tilde{r}_N\big]$$

$$=$$

$$\big[\varphi_i(r_i) \geq \max_{j \in \{1,2,\ldots,N\}} \{\varphi_j(r_j)\} \wedge r_N < \tilde{r}_N\big] := \Gamma_i(r_i)$$

In these arguments, it was used that it does not make a difference for this probability whether the inequality adjusted valuations of other agents is strictly negative or just 0.

This allocation probability is monotone. Moreover, previous results establish that the transfer rule will satisfy integrability. Thus, when all agents bid according to $b_i(r_i) = \beta_i(r_i)$, the resulting social choice function will be incentive compatible.

Consider an agent $i \leq N-1$. By incentive compatibility, there can be no deviations into the region $(\beta_i(r_i^{min}), \beta_i(\bar{r}_i)]$. Any deviation above this is dominated by deviating to $\beta_i(\bar{r}_i)$. Any deviation to a bid below $\beta_i(r_i^{min})$ will yield 0 utility and thus cannot be profitable either.

Now consider agent N. In general, any bid $b_N > \beta_N(\tilde{r}_N)$ is dominated by bidding $b_N = \beta_N(\tilde{r}_N)$. Similarly, any bid $b_N \leq \beta_N(r_N^{min})$ cannot be a profitable deviation.

If $r_N \leq \tilde{r}_N$, incentive compatibility implies that no other bid in $(\beta_N(r_N^{min}), \beta_N(\tilde{r}_N)]$ can yield a better outcome - thus, it must be optimal for such agents to bid according to $\beta_N(r_N)$.

If $r_N > \tilde{r}_N$, the best possible bid will be $b_N = \tilde{r}_N$.

For an agent with $r_N = \tilde{r}_N$, the utility of bidding $\beta_N(\tilde{r}_N)$ is:

$$U_N(\beta_N(\tilde{r}_N); \tilde{r}_N) = \big(\tilde{r}_N - \beta_N(\tilde{r}_N)\big)\Phi_N(\beta_N(\tilde{r}_N))$$

This must be weakly greater than the utility of any other bid in $[0, \beta_N(\tilde{r}_N)]$ by incentive compatibility. For an agent with $r_N > \tilde{r}_N$, the utility of bidding $\beta_N(\tilde{r}_N)$ is:

$$U_N(\beta_N(\tilde{r}_N); r_N) = \big(r_N - \beta_N(\tilde{r}_N)\big)\Phi_N(\beta_N(\tilde{r}_N))$$

The utility of making any other bid $\beta_N(\hat{r}_N) < \beta_N(\tilde{r}_N)$ is:

$$U_N(\beta_N(\tilde{r}_N); r_N) = \big(r_N - \beta_N(\hat{r}_N)\big)\Phi_N(\beta_N(\hat{r}_N))$$

For an agent with $\tilde{r}_N$, we have:

$$\frac{\tilde{r}_N - \beta_N(\tilde{r}_N)}{\tilde{r}_N - \beta_N(\hat{r}_N)} \geq \frac{\Phi_N(\beta_N(\hat{r}_N))}{\Phi_N(\beta_N(\tilde{r}_N))}$$

Consider the following function:

$$LHS(r_N) := \frac{r_N - \beta_N(\tilde{r}_N)}{r_N - \beta_N(\hat{r}_N)}$$

$$\implies \frac{\partial LHS(r_N)}{\partial r_N} = \frac{(r_N - \beta_N(\hat{r}_N))(1) - (r_N - \beta_N(\tilde{r}_N))(1)}{(r_N - \beta_N(\hat{r}_N))^2} = \frac{\beta_N(\tilde{r}_N) - \beta_N(\hat{r}_N)}{(r_N - \beta_N(\hat{r}_N))^2} > 0$$

This implies that any agent with $r_N > \tilde{r}_N$ would also prefer the bid $\beta_N(\tilde{r}_N)$ over any other bid $\beta_N(\hat{r}_N) < \beta_N(\tilde{r}_N)$. Thus, we are done.

A.10 Proof of proposition 6

We work with the following functions:

$$\Gamma_i(r_i) := Pr\{\max_{j \neq i}\{\gamma_j(r_j)\} \leq \gamma_i(r_i)\} \tag{A.45}$$

$$\tau_i(r_i) := \begin{cases} r_i - (1/\Gamma_i(r_i))\big[\int_{\underline{r}_i}^{r_i} \Gamma_i(s)ds\big] & r_i > 0 \\ 0 & r_i = 0 \end{cases} \tag{A.46}$$

Moreover, note that:

$$\tilde{r}_i(b_i) = \begin{cases} \tau_i^{-1}(b_i) & b_i \leq \tau_i(\bar{r}_i) \\ \bar{r}_i & b_i > \tau_i(\bar{r}_i) \end{cases} \tag{A.47}$$

$$\tilde{r}_N(b_N) = \begin{cases} \tau_N^{-1}(b_N) & b_N \leq \tau_N(\tilde{r}_N) \\ \tilde{r}_N & b_N > \tau_N(\tilde{r}_N) \end{cases} \tag{A.48}$$

Agent i wins the auction if and only if:

$$\gamma_i(\tilde{\tau}(b_i)) \geq \max_{j\in\{1,2,\ldots,N\}} \{\gamma_j(\tilde{\tau}_i(b_j))\} \tag{A.49}$$

<u>Part 1a:</u> Showing that $\tau_i(r_i)$ is strictly increasing on $[0, \bar{r}_i]$ for $i \leq N-1$ and on $[0, \tilde{r}_N]$ for agent N when $\gamma_i(r_i)$ is strictly increasing and continuous.

Firstly, recall that the inequality adjusted valuations were defined as follows:

$$\varphi_i(r_i) = \Lambda_i r_i + \frac{\int_{r_i}^{\bar{r}_i} \big(\lambda_i(s) - \Lambda_i\big) dG_i(s)}{g_i(r_i)}$$

When $r_i = \underline{r}_i = 0$, $\gamma_i(\underline{r}_i) = 0$. Because the function γ_i is continuous, the allocation probability $\Gamma_i(r_i)$ will be strictly positive for any $r_i > 0$.

On $r_i \in (0, \bar{r}_i)$, the derivative of the function τ_i w.r.t. r_i is:

$$\frac{\partial \tau_i(r_i)}{\partial r_i} = 1 - \frac{\big(\Gamma_i(r_i)\big)\big(\Gamma_i(r_i)\big) - \big(\int_{\underline{r}_i}^{r_i} \Gamma_i(s)ds\big)\big(\Gamma_i'(r_i)\big)}{\big(\Gamma_i(r_i)\big)^2} = \frac{\big(\int_{\underline{r}_i}^{r_i} \Gamma_i(s)ds\big)\big(\Gamma_i'(r_i)\big)}{\big(\Gamma_i(r_i)\big)^2}$$

This derivative is strictly positive under the stated specifications. For all agents $i \in \{1, ..., N-1\}$, the function $\gamma_i(r_i)$ will be strictly increasing in r_i, which triggers a strict increase in the allocation probability $\Gamma_i(r_i)$.

For agent N, the function $\gamma_N(r_N)$ is also strictly increasing - but this will only strictly increase the allocation probability when $\gamma_N(r_N) < \bar{\gamma}_{N-1}$, because the allocation probability is 1 for any type above this.

When $r_i = 0$, then $\tau_i(r_i) = 0$ holds for all agents. For any type $r_i > 0$, we have:

$$\tau_i(r_i) = r_i - \frac{\int_0^{r_i} \Gamma_i(s)ds}{\Gamma_i(r_i)} = \frac{\int_0^{r_i} s\Gamma_i'(s)ds}{\Gamma_i(r_i)} > 0$$

This function will be strictly positive as $\Gamma_i'(s) > 0$ holds for all $s \in (\underline{r}_i, r_i)$. This prooves that the function $\tau_i(r_i)$ is strictly increasing for $r_i \in [0, \bar{r}_i]$ or $r_N \in [0, \tilde{r}_N]$, respectively.

<u>Part 1b:</u>

For agent N, the function $\tau_N(r_N)$ equals $\tau_N(\tilde{r}_N)$ (i.e. is flat) for any $r_N \geq \tilde{r}_N$. To see this, note that $\Gamma_N(r_N) = 1$ for any $r_N \geq \tilde{r}_N$. Thus, the function $\tau_N(r_N)$ becomes:

$$\tau_N(r_N) = r_N - \frac{\int_{\underline{r}_N}^{r_N} \Gamma_N(s)ds}{1} = \int_{\underline{r}_N}^{r_N} s\Gamma_N'(s)ds = \int_{\underline{r}_N}^{\tilde{r}_N} s\Gamma_N'(s)ds$$

<u>Part 1c:</u> The function τ_i is continuous at $r_i = 0$.

To see this, note that this function can be written as follows:

$$\tau_i(r_i) = \frac{\int_0^{r_i} s\Gamma_i'(s)ds}{\Gamma_i(r_i)}$$

Both terms converge to zero from the top. Applying L'Hopital's rule yields that:

$$\lim_{r_i \downarrow 0} \tau_i(r_i) = \lim_{r_i \downarrow 0} \frac{\int_0^{r_i} s\Gamma_i'(s)ds}{\Gamma_i(r_i)} = \lim_{r_i \downarrow 0} \frac{r_i\Gamma_i'(r_i)}{\Gamma_i'(r_i)} = 0$$

<u>Part 2:</u> Auction equilibrium - assuming that $\bar{\gamma}_{N-1} = \bar{\gamma}_N$.

Under this assumption, the structure of the functions $\tilde{\tau}$ is identical for all agents.

We first show that the mechanism we have described induces a social choice function $c(r_i, r_{-i}) = \big(x(r), t(r)\big)$ that is incentive compatible.

When all agents bid according to the rule $b_i(r_i) = \tau_i(r_i) \leq \tau_i(\bar{r}_i)$, it holds that $\tilde{\tau}_i(\tau_i(r_i)) = r_i$. Then, the interim allocation probabilities induced by this mechanism, namely $X_i(r_i)$, will be equal to $\Gamma_i(r_i)$, which is monotone under our assumptions.

Moreover, the implied transfer rule will satisfy the integrability constraint. To see this, note that any agent that bids b_i makes the expected payment $-b_i\Gamma_i(\tilde{\tau}_i(b_i))$. When bidding according to $b_i(r_i) = \tau_i(r_i)$, it thus holds that:

$$\tau_i(0) = 0 \implies U_i(\underline{r}_i) = U_i(0) = 0$$

Moreover, the interim expected transfers are given by $T_i(r_i) = -\tau_i(r_i)\Gamma_i(r_i)$

The integrability condition is thus satisfied for the implied social choice function because:

$$r_iX_i(r_i) + T_i(r_i) = U_i(\underline{r}_i) + \int_{\underline{r}_i}^{r_i} X_i(s)ds$$

$$\iff$$

$$r_i\Gamma_i(r_i) - \tau_i(r_i)\Gamma_i(r_i) = \int_{\underline{r}_i}^{r_i} \Gamma_i(s)ds \iff \tau_i(r_i) = r_i - \frac{\int_{\underline{r}_i}^{r_i} \Gamma_i(s)ds}{\Gamma_i(r_i)}$$

This establishes that the social choice function induced when all agents bid according to $b_i(r_i) = \tau_i(r_i)$ is incentive compatible.

Thus, it is an equilibrium that all agents bid according to this rule. Consider an agent i and suppose that all other agents $-i$ bid according to $b_{-i}(r_{-i}) = \tau_{-i}(r_{-i})$.

By the intermediate value theorem, we have the following: For any bid $b_i \in [0, \tau_i(\bar{r}_i)]$, there exists an r_i such that $\tau_i(r_i) = b_i$. Because the social choice function $c(r)$ is incentive compatible, there can be no profitable deviation in the range $[0, \tau_i(\bar{r}_i)]$.

This is because any such bid $\hat{b}_i$ would be associated with an $\hat{r}_i$ such that $\tau_i(\hat{r}_i) = \hat{b}_i$. Thus, this deviation would generate the outcome $c(\hat{r}_i, r_{-i})$ as defined by the social choice function, which cannot make agent i better off.

Now consider possible deviations into the range $b_i > \tau_i(\bar{r}_i)$. The allocation probability would be the same as when bidding $b_i = \tau_i(\bar{r}_i)$, but the payment would be higher upon winning the auction - thus, this deviation can also not be strictly profitable.

Part 3: Auction equilibrium - assuming that $\bar{\gamma}_{N-1} < \gamma_N(r_N)$.

Suppose that all agents bid according to $\tau_i(r_i)$. The interim allocation probabilities $X_i(r_i)$ will also be equal to $\Gamma_i(r_i)$ for any agents.

To see this, consider an agent $i \leq N-1$. By the law of total probability, we have that:

$$X_i(r_i) = Pr\big[\gamma_i(r_i) \geq \max_{j\in\{1,2,\ldots,N\}}\{\gamma_j(\tilde{\tau}_i(b_j)\}\big] =$$

$$Pr\big[\gamma_i(r_i) \geq \max_{j\in\{1,2,\ldots,N\}}\{\gamma_j(\tilde{\tau}_i(b_j)\} \wedge r_N \geq \tilde{r}_N\big] + \big[\gamma_i(r_i) \geq \max_{j\in\{1,2,\ldots,N\}}\{\gamma_j(\tilde{\tau}_i(b_j)\} \wedge r_N < \tilde{r}_N\big]$$

$$=$$

$$\big[\gamma_i(r_i) \geq \max_{j\in\{1,2,\ldots,N\}}\{\gamma_j(r_j)\} \wedge r_N < \tilde{r}_N\big] := \Gamma_i(r_i)$$

This is monotone. The arguments for the interim allocation probabilities of agent N are analogous.

Moreover, previous results establish that the transfer rule will satisfy integrability. Thus, when all agents bid according to $b_i(r_i) = \tau_i(r_i)$, the resulting social choice function will be incentive compatible.

Consider any agent i. By incentive compatibility, there can be no profitable deviations into the region $[0, \tau_i(\bar{r}_i)]$ and any deviation above this is dominated by deviating to $\tau_i(\bar{r}_i)$.

Now consider agent N. In general, any bid $b_N > \tau_N(\tilde{r}_N)$ is dominated by bidding $b_N = \tau_N(\tilde{r}_N)$.

If $r_N \leq \tilde{r}_N$, incentive compatibility implies that no other bid in $[0, \tau_N(\tilde{r}_N)]$ can yield a better outcome.

If $r_N > \tilde{r}_N$, the best possible bid will be $b_N = \tau_N(\tilde{r}_N)$.

To see this, note the following: For an agent with $r_N = \tilde{r}_N$, the utility of bidding $\tau_N(\tilde{r}_N)$ is:

$$U_N(\tau_N(\tilde{r}_N); \tilde{r}_N) = \big(\tilde{r}_N - \tau_N(\tilde{r}_N)\big)\Gamma_N(\tau_N(\tilde{r}_N))$$

This must be weakly greater than the utility of any other bid in $[0, \tau_N(\tilde{r}_N)]$ by incentive compatibility. For an agent with type $r_N > \tilde{r}_N$, the utility of bidding $\tau_N(\tilde{r}_N)$ is:

$$U_N(\tau_N(\tilde{r}_N); r_N) = \big(r_N - \tau_N(\tilde{r}_N)\big)\Gamma_N(\tau_N(\tilde{r}_N))$$

The utility of making any other bid $\tau_N(\hat{r}_N) < \tau_N(\tilde{r}_N)$ is:

$$U_N(\tau_N(\hat{r}_N); r_N) = \big(r_N - \tau_N(\hat{r}_N)\big)\Gamma_N(\tau_N(\hat{r}_N))$$

The inequality $r_N - \tau_N(\hat{r}_N) > r_N - \tau_N(\tilde{r}_N) \geq 0$ must hold. For agent N with type $\tilde{r}_N$, the following must thus hold:

$$\frac{\tilde{r}_N - \tau_N(\tilde{r}_N)}{\tilde{r}_N - \tau_N(\hat{r}_N)} \geq \frac{\Gamma_N(\tau_N(\hat{r}_N))}{\Gamma_N(\tau_N(\tilde{r}_N))}$$

Consider the following function:

$$LHS(r_N) := \frac{r_N - \tau_N(\tilde{r}_N)}{r_N - \tau_N(\hat{r}_N)}$$

$$\implies \frac{\partial LHS(r_N)}{\partial r_N} = \frac{(r_N - \tau_N(\hat{r}_N))(1) - (r_N - \tau_N(\tilde{r}_N))(1)}{(r_N - \tau_N(\hat{r}_N))^2} = \frac{\tau_N(\tilde{r}_N)) - \tau_N(\hat{r}_N))}{(r_N - \tau_N(\hat{r}_N))^2} > 0$$

This implies that any agent with $r_N > \tilde{r}_N$ would also prefer the bid $\tau_N(\tilde{r}_N)$ over any other bid $\tau_N(\hat{r}_N) < \tau_N(\tilde{r}_N)$

A.11 Calculations of bidding subsidies for the specific examples

Part 1: No subsidy example:

Consider the simple two-agent case with deterministic marginal utilities of money. Assume $v_i^K \sim U[0,1]$ for both agents. Suppose that agent 1 is poorer, i.e. $\Lambda_1 > \Lambda_2$. The inequality adjusted valuations become $\gamma_1 = \Lambda_1 r_1$ and $\gamma_2 = \Lambda_2 r_2$. Thus, we have $\bar{\gamma}_i = 1$ for both agents. Note also that $r_i \sim U[0, 1/\Lambda_i]$ holds for both agents.

To define the bidding subsidies, we first need to compute $\Gamma_i(r_i) = Pr\big(\gamma_i(r_i) \geq \gamma_j(r_j)\big)$. Noting that $Pr(r_i \leq x) = \Lambda_i x$ holds for both agents., we can write that:

$$\Gamma_i(s) = Pr\big(\Lambda_i s \geq \Lambda_j r_j\big) = Pr\big(r_j \leq (\Lambda_i/\Lambda_j)s\big) = \Lambda_i s$$

Thus, we have that:

$$\tau_i(r_i) = r_i - \frac{\int_0^{r_i} \Lambda_i s ds}{\Lambda_i r_i} = r_i - \frac{\big[0.5s^2\big]_0^{r_i}}{r_i} = r_i - \frac{0.5(r_i)^2}{r_i} = 0.5r_i$$

Moreover, note further that:

$$\tau_i^{-1}(y) = 2y$$

Agent i receives the good under our allocation rule if and only if:

$$\gamma_i(\tau_i^{-1}(b_i)) \geq \gamma_j(\tau_j^{-1}(b_j)) \iff \Lambda_i(2b_i) \geq \Lambda_j(2b_j) \iff \frac{\Lambda_i}{\Lambda_j} b_i \geq b_j$$

Part 2: Ex ante budget balance example:

Agent 1 has a valuation $v^K \sim U[0,1]$ and deterministic utility of money $\Lambda_1 = 1$. Thus, $r_1 \sim U[0,1]$, $\varphi_1(r_1) = 3r_1 - 1$ and $\varphi_1(r_1) \sim [-1, 2]$. Agent 2 has a valuation

$v^K \sim U[0,2]$ and deterministic utility of money $\Lambda_2 = 2$. Thus $r_2 \sim [0,1]$, $\varphi_2(r_2) = 2r_2$ and $\varphi_2 \sim U[0,2]$. Then, we calculate $\Phi_i(r_i) = Pr\{\max_{j\neq i}\{\varphi_j(r_j)\} \leq \varphi_i(r_i)|r_i\}$. We derive $\Phi_1(r_1) = 0.5(3r_1 - 1)$ and $\Phi_2(r_2) = \frac{2r_2+1}{3}$. Then plugging things in, we get

$$\beta_1(r_1) = r_1 - \frac{\int_{1/3}^{r_1} 0.5(3s-1)ds}{0.5(3r_1-1)} = 0.5r_1 + \frac{1}{6} \tag{A.50}$$

$$\beta(r_2) = r_2 - \frac{\int_0^{r_1} \frac{2s+1}{3}ds}{\frac{2r_2+1}{3}} = \frac{r_2^2}{2r_2+1} \tag{A.51}$$

Taking the inverse of $\beta_1(r_1)$ for $r_1 \in [1/3, 1]$ yields:

$$y = 0.5\beta_1^{-1}(y) + \frac{1}{6} \iff \beta_1^{-1}(y) = 2y - \frac{1}{3} \tag{A.52}$$

The inverse of $\beta_2(r_2)$, which is defined for all r_2, must solve the following:

$$y = \frac{\beta_2^{-1}(y)^2}{2\beta_2^{-1}(y)+1} \iff (\beta_2^{-1}(y))^2 - 2y\beta_2^{-1}(y) - y = 0 \tag{A.53}$$

Applying the quadratic formula yields the following solutions for the inverse:

$$\beta_2^{-1}(y) = \frac{2y +/- \sqrt{4y^2+4y}}{2} = \frac{2y +/- 2\sqrt{y^2+y}}{2} = y +/- \sqrt{y^2+y} \tag{A.54}$$

We know that this solution must be in the positive domain. Thus, this inverse must solve:

$$\beta_2^{-1}(y) = y + \sqrt{y^2+y} \tag{A.55}$$

Thus, our allocation rule for $b_1 > 1/3$ is:

$$\varphi_1(\beta_1^{-1}(b_1)) > \varphi_2(\beta_2^{-1}(b_2)) \iff 3\left(2b_1 - \frac{1}{3}\right) - 1 > 2\big(b_2 + \sqrt{b_2^2+b_2}\big) \tag{A.56}$$

$$\iff$$

$$6b_1 - 2 > 2b_2 + 2\sqrt{b_2^2+b_2} \iff b_1 > (1/3)b_2 + (1/3)\sqrt{b_2^2+b_2} + 1/3 \tag{A.57}$$

A.12 Mathematical results underlying the numerical results for the uniform v^K, Pareto v^M example

Let $v^K \sim U[0,1]$ and $v^M \sim Pareto(k, x_{min})$ where v^K and v^M are independent random variables. It is well known that for the ratio of two independent random variables the density is given by:

$$g(r) = \int_{-\infty}^{\infty} |v^M| f^K(rv^M) f^M(v^M) dv^M \tag{A.58}$$

We note that $v^M \geq 0$ and adjust the integral boundaries to our setting. Then

$$g(r) = \int_{x_{min}}^{1/r} v^M k x_{min}^k v^{M-(k+1)} dv^M \tag{A.59}$$

$$= \frac{k}{1-k} r^{k-1} x_{\min}^k - \frac{k}{1-k} x_{\min} \tag{A.60}$$

In the next step we derive $\lambda(r) = \mathbb{E}[v^M|r]$. For this we start by deriving

$$Pr\{v^M \leq u, v^K/v^M \leq r\} = \int_{x_{min}}^{u} \int_0^{v^M r} f^M(v^M) f^K(v^K) dv^K dv^M \tag{A.61}$$

$$= \frac{k}{k-1} x_{min}^k r (x_{min}^{-k+1} - u^{-k+1}) \tag{A.62}$$

For $u \leq 1/r$. We restrict to $u \leq 1/r$ as this constitutes the relevant region in which the joint density of v^M and r is strictly positive. We determine this joint density through differentiating twice and determine

$$f(v^M, r) = k x_{min}^k v^{M-k} \tag{A.63}$$

Based on this, we can check the marginal density of r, namely $g(r)$. This is:

$$g(r) = \int_{x_{min}}^{1/r} f(v^M, r) dv^M = \left[\frac{1}{-k+1} k x_{min}^k v^{M-k+1}\right]_{x_{min}}^{1/r} = \frac{k}{1-k} x_{min}^k \left[r^{k-1} - (x_{min})^{1-k}\right] \tag{A.64}$$

We then derive the conditional density function

$$f(v^M|r) = \frac{(1-k) x_{min}^k v^{M-k}}{r^{k-1} x_{min}^k - x_{min}} \tag{A.65}$$

Then we determine $\lambda(r)$

$$\lambda(r) = \int_{x_{min}}^{1/r} v^M \frac{(1-k) x_{min}^k v^{M-k}}{r^{k-1} x_{min}^k - x_{min}} dv^M \tag{A.66}$$

$$= \frac{k-1}{k-2} \frac{x_{min}^{2-k} - r^{k-2}}{x_{min}^{1-k} - r^{k-1}} \tag{A.67}$$

Integrating up the marginal density $g(r)$ then yields $G(r)$.

$$G(r) = \int g(r) dr = \frac{k}{1-k} x_{min}^k \left[(1/k) r^k - (x_{min})^{1-k} r\right] \tag{A.68}$$

We can check the marginal density of v^M, which is:

$$f_{V^M}(y) = \int_0^{1/y} f(y,r)dr = \int_0^{1/y} \left[kx_{min}^k y^{-k}\right] dr \tag{A.69}$$

$$= \left[kx_{min}^k y^{-k} r\right]_0^{1/y} = kx_{min}^k y^{-k}(y)^{-1} = \frac{k(x_{min})^k}{y^{k+1}} \tag{A.70}$$

Based on this, we can compute Λ_i, namely:

$$\Lambda = \int_{x_{min}}^{\infty} y f_{v^M}(y)dy = \int_{x_{min}}^{\infty} k(x_{min})^k (y)^{-k} dy = \left[k(x_{min})^k \frac{1}{-k+1}(y)^{-k+1}\right]_{x_{min}}^{\infty} \tag{A.71}$$

$$= -k(x_{min})^k \frac{1}{-k+1}(x_{min})^{-k+1} = \frac{k}{k-1} x_{min} \tag{A.72}$$

This would not have been necessary, given our knowledge of v^M that we specify a priori, but serves as a nice check of our previous calculations.

One can show that all inequality adjusted valuations satisfy the monotonicity assumptions in this example:

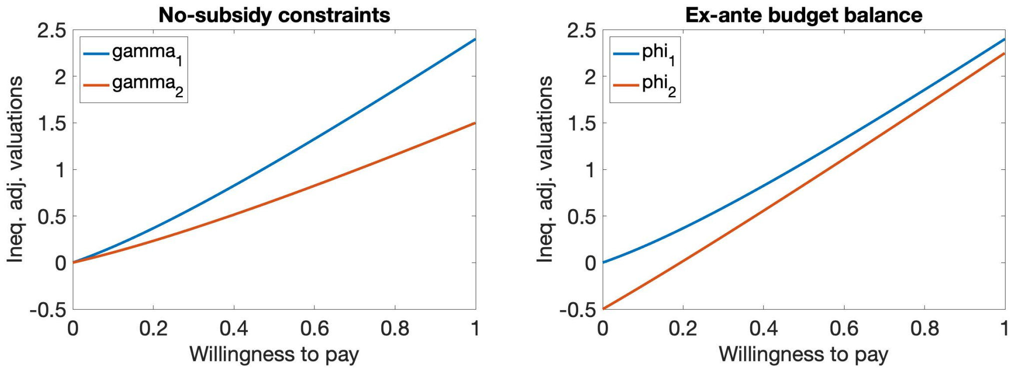

Figure 2: Utilitarian optimal allocation rule vs. ex post efficient allocation rule

References

Mohammad Akbarpour, Piotr Dworczak, and Scott Duke Kominers. Redistributive allocation mechanisms. Available at SSRN, 2022. doi: 10.2139/ssrn.3609182.

Anthony Barnes Atkinson and Joseph E Stiglitz. The design of tax structure: direct versus indirect taxation. Journal of public Economics, 6(1-2):55–75, 1976.

Timothy Besley and Stephen Coate. Public provision of private goods and the redistribution of income. The American Economic Review, 81(4):979–984, 1991.

Tilman Börgers and Peter Norman. A note on budget balance under interim participation constraints: the case of independent types. Economic Theory, 39(3):477–489, 2009. doi: 10.1007/s00199-008-0347-7.

Daniele Condorelli. Market and non-market mechanisms for the optimal allocation of scarce resources. Games and Economic Behavior, 82:582–591, 2013. doi: 10.1016/j.geb.2013.08.008.

Dan L. Crippen. The budget and economic outlook. United States Congressional Budget Office, 2000.

Peter A Diamond and James A Mirrlees. Optimal taxation and public production i: Production efficiency. The American Economic Review, 61(1):8–27, 1971.

Sarah Donovan and David Bradley. Real wage trends, 1979 to 2018. Congressional research service, 2019.

Piotr Dworczak, Scott Duke Kominers, and Mohammad Akbarpour. Redistribution through markets. Econometrica, 88(7), 2021. doi: 10.2139/ssrn.3143887.

Roland Eisenhuth. Reference-dependent mechanism design. Economic Theory Bulletin, 7(1):77–103, 2019. doi: 10.1007/s40505-018-0144-9.

Joan Esteban and Debraj Ray. Inequality, lobbying, and resource allocation. American Economic Review, 96(1):257–279, 2006. doi: 10.1257/000282806776157533.

Dana Fritz. Der kindergarten: Diese unterschiede über die verschiedenen kindergarten-konzeptionen sollten sie kennen. 2021. doi: https://www.kita.de/wissen/kindergarten/#2_Die_Traegerschaft_eines_Kindergartens(11.01.2022).

Firouz Gahvari and Enlinson Mattos. Conditional cash transfers, public provision of private goods, and income redistribution. American Economic Review, 97(1):491–502, 2007. doi: 10.1257/aer.97.1.491.

Kazuhiko Hashimoto and Hiroki Saitoh. Domain expansion of the pivotal mechanism. Social Choice and Welfare, 34(3):455–470, 2010. doi: 10.1007/s00355-009-0411-7.

Katharina Huesmann. Public assignment of scarce resources under income effects. 2017.

Mingshi Kang and Charles Z Zheng. Necessity of auctions for redistributive optimality. Technical report, Working Paper, 2019.

Mingshi Kang and Charles Z Zheng. Optimal design for redistributions among endogenous buyers and sellers. 2022.

Zi Yang Kang. The public option and optimal redistribution. 2022.

Tomoya Kazumura, Debasis Mishra, and Shigehiro Serizawa. Mechanism design without quasilinearity. Theoretical Economics, 15(2):511–544, 2020. doi: 10.3982/TE2910.

Maciej H Kotowski. First-price auctions with budget constraints. Theoretical Economics, 15(1):199–237, 2020. doi: 10.3982/TE2982.

LH-Mainz. Vergabeverfahren von kita-plätzen, landeshauptstadt mainz. 2018. doi: https://fragdenstaat.de/anfrage/vergabeverfahren-von-kita-platzen/(11.01.2022).

Eric Maskin and John Riley. Optimal auctions with risk averse buyers. Econometrica: Journal of the Econometric Society, pages 1473–1518, 1984.

Paul Milgrom and Ilya Segal. Envelope theorems for arbitrary choice sets. Econometrica, 70(2):583–601, 2002.

Roger B Myerson. Optimal auction design. Mathematics of operations research, 6(1): 58–73, 1981.

Mallesh M Pai and Rakesh Vohra. Optimal auctions with financially constrained buyers. Journal of Economic Theory, 150:383–425, 2014. doi: 10.1016/j.jet.2013.09.015.

Hiroki Saitoh and Shigehiro Serizawa. Vickrey allocation rule with income effect. Economic Theory, 35(2):391–401, 2008. doi: 10.1007/s00199-007-0235-6.

Martin L Weitzman. Is the price system or rationing more effective in getting a commodity to those who need it most? The Bell Journal of Economics, pages 517–524, 1977.

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