// NO.24-032 | 05/2024

DISCUSSION PAPER

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Search on a Grid: Directed Consumer Search With Correlated Products

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Search on a Grid: Directed Consumer Search with Correlated Products*

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April 2, 2024

Abstract

The consumer search literature mostly considers independently distributed products. In contrast, I study a model of directed search with infinitely many products whose valuations are correlated through shared attributes. I propose a tractable, systematic, history-dependent scoring system based on nests of correlated products that leverages the predictability of the optimal search process along different paths. This scoring system generates an optimal search policy conceptually equivalent to the familiar optimal policy with independently distributed search products. The policy instructs the consumer to inspect unrelated products until an attribute the realization of which surpasses the added informational value of inspecting two new attributes is found. The search paths emerging from this policy match recent evidence of consumer learning through search, and can rationalize backtracking to a previously abandoned attribute.

Keywords: consumer search, directed search, learning

JEL Codes: D83

*I thank the Department of Economics at University of Mannheim and the IOEK department and Junior Research Group for Competition and Innovation at ZEW for guidance and support. In particular, I thank Martin Peitz, Bernhard Ganglmair, Nicolas Schutz, Volker Nocke, Anton Sobolev, and Atabek Atayev; I also thank Daniel Savelle, Axel Gautier, José Moraga-Gonzales, Justus Preusser, Sander Onderstal, Marco Haan, Julia Reimer, Heiko Karle, Michelangelo Rossi, David Ronayne, Willy Lefez, Chiara Farronato, and Marcel Preuss. Funding by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through CRC TR 224 (Project B04) is gratefully acknowledged.

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1. Introduction

The consumer search literature has highlighted the role of search frictions as determinants of market outcomes.[1] The effect of these frictions, however, has so far been studied almost exclusively under the assumption of products' value being independently distributed.[2] I propose a framework for consumer search that incorporates correlation across products mechanically through shared attributes. I consider consumers that value products based on their attributes (Lancaster, 1966): initially, consumers observe all products and respective attributes, but they do not know how much they value them. For example, laptops may differ in their processing speed and graphical capabilities, which depend on the processor and the graphic card that are installed.

Consumers decide which products to search for and inspect, and then, based on their findings, adapt their strategy accordingly for their next search. The reasoning is as follows: if two products share an attribute, consumers value them identically with respect to that attribute. Through the search process, consumers learn their preferences for attributes and, depending on what they learn about specific attributes, can redirect their subsequent search because they know which products share the same attributes and which ones do not. The result of any given inspection makes consumers update her expectations for the remaining products based on which attributes they share. This, in turn, instructs the next inspection. The proposed framework allows inspection of one product to affect the expected return of inspecting a different one, and allows for the direction of search to be endogenously determined rather than being predetermined.

In many circumstances, these learning dynamics represent well consumer search behavior: if a consumer learns that she dislikes a certain attribute in a product, she would rationally try to avoid other products that share that attribute. For example, Hodgson and Lewis (2020) shows evidence of "spatial learning" in search: consumers inspect more differentiated products early and get closer to the eventually purchased option as search progresses. I show that this multi-attribute structure generates a version of Weitzman (1979)'s optimal search in an environment with correlated products that matches this dynamic. Further, I show that "backtracking" to a previously inspected and abandoned attribute can be optimal in this environment.

The main contribution of this paper is to bring together observable attributes instructing the search process,[3] and shared attributes that allow the consumer to adapt as they search, and to score the value of inspecting new products by nesting them based on the attributes they share. Novel to the directed consumer search literature is an optimal search

[1] Consumers generally do not make consumption choices with perfect information; evidence of this can be found in the empirical industrial organization literature (Sovinsky Goeree, 2008) and in the marketing literature (Mayzlin and Shin, 2011).

[2] Prominent exceptions are Ke and Lin (2022), Bao et al. (2022), and Hodgson and Lewis (2020).

[3] Already a core component of contributions by Choi et al. (2018), Haan et al. (2018), and Greminger (2022).

order which, unlike in almost all existing papers on directed consumer search, cannot be pinned down before the search process has started but that can still be characterized by an index policy.[4]

The ordered consumer search literature pioneered by Weitzman (1979) still largely relies on its seminal result that characterizes the optimal process for a consumer costly searching among n independently distributed boxes. Each box is characterized by a reservation value, a score representing the value that would make the consumer indifferent between opening the box and keeping a certain reward equal to the score. The optimal search order has the consumer opening boxes from the highest to the lowest score. The consumer optimally stops when no unopened box has a score higher than the highest past realization.

Weitzman (1979)'s seminal optimal policy relies on the assumption that boxes are independently distributed. I relax this assumption and propose a tractable, history-dependent scoring system that incorporates the value of searching beyond the target of inspection. The score is determined accounting for the paths that would be optimally taken by the consumer after the realization they refer to and, therefore, reflect the full value of inspecting new attributes and the respective continuation value. Through this scoring system, I show that a dynamic, adaptive version of Weitzman (1979)'s optimal search policy can be characterized in this environment.

The proposed approach builds on recent contributions by Ke and Lin (2022), Bao et al. (2022), and the aforementioned Hodgson and Lewis (2020). Ke and Lin (2022) provides conditions under which correlation in search leads to complementarity of the products available. Bao et al. (2022), instead, studies Bayesian updating when the consumer cannot distinguish the role of each attribute in the *ex post* utility each product grants.[5]

While in their frameworks search order cannot be predetermined either, the set-up does not allow for generalizations, nor it generates an applicable optimal search policy. In contrast, my approach and the intuition behind it can be reasonably applied to a wide variety of search environments. Nesting correlated products and obtaining statistics representing the value of searching optimally inside these nests allows to seamlessly carry over the value of the relevant information learned as search progresses. The resulting optimal search policy allows for more nuanced predictions regarding rational consumer search behavior.

The rest of the paper is structured as follows: in Section 2, I present the framework. I characterize the optimal search process with multiple attributes and the learning process they imply in an environment with infinitely many products in Section 3. I explore several

[4]Conditional search order is famously at the core of a recent contribution by Doval (2018): The paper extends Weitzman (1979)'s search process by allowing the consumer to consider all uninspected products as viable outside options.

[5]Other prominent examples of correlation in search can be found in Shen (2015) and Armstrong and Zhou (2011), that embed the search process in a Hotelling framework so that, in both settings, the available products are perfectly negatively correlated.

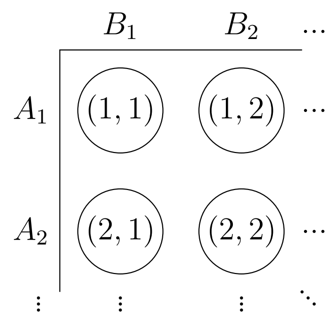

Figure 1: Products in the same row (resp. column) share attribute A_i (resp. B_j).

extensions in Section 4, among which I obtain equilibrium pricing for correlated products when a single agent prices the whole menu: I show that only a uniform pricing scheme can ever be optimal in this environment with infinitely many products. In the same section, I explore some limitations to my approach before concluding in Section 5.

2. Framework

I consider an environment with products differentiated with respect to two attributes. A product (i, j) is identified by attributes $A_i \in A$ and $B_j \in B$. There are infinitely many variants A_i, B_j, and infinitely many products (i, j). Each variant A_i can be found combined with all variants B_j, $i, j \in \{1, 2, 3, ...\}$, and *vice versa*. One can visualize the products as displayed in a grid, with the rows representing the A variants, the columns representing the B variants, and the cells representing products defined by a specific combination of A and B as depicted in Figure 1. Products are only differentiated horizontally through their attribute compositions and are otherwise identical in quality.

A representative, risk-neutral consumer (she) has unit demand, is aware of the available products and their attribute composition, and can inspect the products in any order she likes. The consumer has no prior knowledge of her preferences over the available attributes; she learns the realization of each attribute separately by inspecting a product characterized by it. In line with existing models,[6] I assume that *ex post* utility generated by a generic product (i, j) takes the form:

$$u(A_i, B_j) = A_i + B_j = u_{i,j}.$$

I assume each attribute to be an i.i.d random variable distributed according to a cumulative distribution function F: given a generic attribute $y \in A \cup B$, $F(y)$ is assumed to have support $[0, \hat{y}]$ for some positive $\hat{y}$, and to be twice-differentiable everywhere on it. The assumption that attributes enter $u_{i,j}$ additively crucially implies that there are no

[6] For example: Choi et al. (2018) and Greminger (2022).

complementarities between attributes: once an attribute is discovered, its realized value affects all products that are defined by it in the same way.

In this environment, I study the optimal sequential search process with free recall: a consumer can always go back to a previously inspected product at no additional cost. The cost of inspecting a product is indexed by the constant $s \in (0, 2\,E[y])$. The consumer's outside option is normalized to $u_0 = 0$.

Finally, I consider Subgame Perfect Equilibria for a game with the following timing:

1. The consumer observes the infinitely large product menu, chooses between searching and her outside option, and, if she searches, what to inspect.

2. After each inspection, the consumer chooses between stopping and inspecting a different product (and what to inspect next) until she either purchases an inspected product or leaves without making a purchase.

3. Optimal Search with Multi-Attribute Products

For illustrative purposes, consider first the simpler case of Figure 2. The two products available share one attribute (A_1) and are independent along the other (B_j, $j \in \{1,2\}$). Suppose that the consumer already inspected $(1,1)$: she has learned her valuation for A_1, shared by both products, and B_1. She still does not know her valuation for B_2. At this stage, it is clear that choosing between stopping at $(1,1)$ and costly inspecting $(1,2)$ is governed by the standard myopic search process illustrated in Weitzman (1979).[7] In particular, $u_{1,1}$ is known, and $(1,2)$'s value is only unknown in B_2.

The value of inspecting $(1,2)$ at this point can be expressed using Weitzman (1979)'s familiar reservation value. The certain equivalent that makes a consumer indifferent between that value and costly discovering realization B_2 is the value z that solves:

$$s = \int_z^{\hat{y}} (B_2 - z)\, dF(B_2).$$

Then, the reservation value of inspecting $(1,2)$ when A_1 is known is simply:[8]

$$r_{1,2} = A_1 + z.$$

Following Weitzman (1979), the consumer would inspect $(1,2)$ if and only if $B_1 < z$, or $u_{1,1} < r_{1,2}$. We cannot go backwards and apply the same myopic logic to the choice of inspecting $(1,1)$: because the reservation value of each individual product depends on the other, we cannot apply Pandora's search algorithm.

[7]This intuition can be found, for example, in Ke and Lin (2022).
[8]Notice that this is the same utility structure studied in Choi et al. (2018).

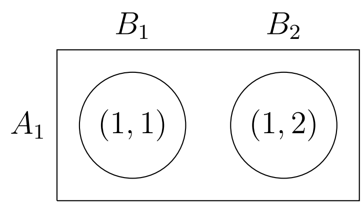

Figure 2: Two products available

Suppose however that the products were in a bigger box, and that the consumer had to decide whether to open one box containing $(1,1)$ and a nested box containing $(1,2)$, or nothing at all.[9] I henceforth refer to this larger box as "compound", and the smaller one containing $(1,2)$ as "nested". Further, I henceforth refer to a compound box built around product (i,j) as $X_{i,j}$.

If the consumer opens the box, she discovers $u_{1,1}$, the implied reservation value $r_{1,2}$, and searches accordingly. Because we know how search takes place inside this box, $X_{1,1}$ can be scored in a way that reflects not just the value of inspecting $(1,1)$ but also the value of the information learned through the possibility of correcting towards $(1,2)$. When applied to each product separately, this intuition generates an environment in which products sharing attributes can be appropriately scored to reflect the information they carry.

The consumer could also want to inspect $(1,2)$ first. We can imagine another compound box, $X_{1,2}$, containing $(1,2)$ and a nested box containing $(1,1)$. The two are *ex ante* identical before either is opened and, once one is opened, the other becomes the smaller nested box contained in the one inspected first. Henceforth, I make the assumption that the consumer inspects unknown attributes in increasing order of their index when indifferent, which is without loss of generality.

3.1. Searching Inside a Compound Box

A generic compound box $X_{i,j}$ contains products (i,j), immediately available, and all products $(i,j' \neq j)$, $(i' \neq i, j)$ inside smaller boxes that must be opened by paying an additional search cost s. In words, $X_{i,j}$ contains all products defined by A_i, B_j, or both. Notice that all products are contained in multiple boxes.

The assumption that the consumer inspects new attributes in ascending order of their index implies that, if the consumer wants to inspect two new attributes and has never inspect a single attribute before, she will inspect (i,i), $i \in \{1,2,3,...\}$. It also implies that, if she does so, all attributes $A_{j<i}$, $B_{j<i}$ must have been inspected already and are, therefore, known before (i,i) is. As the indices are a simple label, this is without loss of generality. Formally, we can define $I \subset A \times B$ to be the set of already inspected attributes

[9]This approach was inspired by the work contained in Anderson et al. (2021); I thank Daniel Savelle for his many helpful comments.

at any given point of the search process.

Suppose the consumer is about to open $X_{i,i}$ and pay the relative search cost s. Let $A^H = \max\{y \in A \cap I\}$, $B^H = \max\{y \in B \cap I\}$ be the highest past realization of the previously inspected $A_{j<i}$, $B_{j<i}$. If $\max\{A^H, B^H\}$ is low enough, the choice of opening $X_{i,i}$ is unaffected by all realizations that took place before. On the other hand, if either or both A^H and B^H are high enough, the choice is predictably affected by said realization.

The consumer is aware that inside $X_{i,i}$ she will find product (i,i) and will have the option to stop or inspect products $(i, j \neq i)$, $(j \neq i, i)$. How would she do so? All attributes $A_{j<i}$, $B_{j<i}$ have already been inspected and are known. Suppose the consumer already opened the box. The consumer can choose between

- stopping at (i,i), generating utility $u_{i,i} = A_i + B_i$, or
- searching again keeping A_i and
  - inspect a product defined by $B_{j<i}$, whose realization is already known, after paying cost s: $u_{i,j<i} = A_i + B_j - s$,
  - search a product defined by $B_{j>i}$, whose realization is unknown after paying cost s: $E[u_{i,j>i}] = A_i + E[B_j] - s$,
- searching again keeping B_i and:
  - search a product defined by $A_{j<i}$, whose realization is already known, after paying cost s: $u_{i,j<i} = A_j + B_i - s$,
  - search a product defined by $A_{j>i}$, whose realization is unknown after paying cost s: $E[u_{j>i,i}] = E[A_j] + B_i - s$.

After opening $X_{i,i}$, these choices can be ranked according to the classic result of Weitzman (1979).[10] In particular, after $X_{i,i}$ has been opened, the remaining options inside the compound box are independent of each other because all attributes are assumed to be i.i.d. Therefore, we can assign a score to all by finding the certain equivalent of each. Stopping and inspecting a product whose realization is fully known trivially has certain equivalent matching the known ex post utility: $r_{i,i} = u_{i,i}$, $r_{i,j<i} = u_{i,j<i} - s$, $r_{j<i,i} = u_{j<i,i} - s$. Keeping the classic nomenclature, I refer to this as “reservation values” of these options.

The unknown nested boxes, instead, are only unknown in one attribute after $X_{i,i}$ has been opened. This is the same object whose reservation value is provided in Choi et al. (2018).[11] In particular, because all attributes $y \sim F(y)$ with support $[0, \hat{y}]$, the certain

[10] The result relies on the fact, proven in Appendix A, that if the consumer optimally decides to open a nested box with unknown content in a newly opened compound box, she never stops and never deviates, making the process of searching forward equivalent to a myopic one.

[11] In Appendix A I show that once a nested boxes is optimally opened, the consumer either stops or opens more nested boxes depending on the current realized payoff, making this branch of the optimal search policy myopic in nature.

equivalent of spending a search cost s to discover the realization of any unknown attribute y is z that solves:

$$s = \int_{z}^{\hat{y}} (y - z)\, dF(y), \tag{1}$$

and therefore: $r_{i,j>i} = A_i + z$, $r_{j>i,i} = z + B_j$.

Notice that the choice between moving forward towards $(i, j > i)$ or $(j > i, i)$ and going backward to any known $(i, i' < i)$, $(i' < i, i)$ is resolved again simply by applying Weitzman (1979)'s optimal search policy: if there is at least one product $(i, j < i)$ (or $(j < i, i)$) such that $u_{i,j<i} - s > A_i + z$ (or $u_{j<i,i} - s > z + B_i$), no nested box with score $r_{i,j>i} = A_i + z$ (or $r_{j>i,i} = B_i + z$) would be opened, and the product generating the highest $u_{i,j<i} - s$ (or $u_{j<i,i} - s$) would be inspected and selected. This happens if $A^H > z + s$ (or $B^H > z + s$). Otherwise, all products $(i, j < i)$ (or $(j < i, i)$) would be ignored. Because A^H, B^H are the highest past realizations, they are known before $X_{i,i}$ is opened. Therefore, the consumer opens $X_{i,i}$ knowing already whether she would go forwards (that is, open nested boxes $(i, j > i)$ or $(j > i, i)$) or backwards (that is, inspecting a product $(i, j < i)$ or $(j < i, i)$) if she decides to search again.

This observation implies that unopened compound boxes that are constructed around products not sharing attributes are *de facto* independent for all values of A^H, B^H at the moment of making the choice of opening a new compound box. The possible configurations in which compound boxes can be found is illustrated in Figure 3. Depending on A^H and B^H, the effective path inside each unrelated box does not cross. We can then compute the expected value of searching box $X_{i,i}$ in isolation by tracing the optimal search therein for different values of A^H, B^H. This, in turn, means that each configuration as in Figure 3 can be solved independently and then combined to obtain the actual, history dependent optimal search policy.

3.2. Four Independent Configurations

3.2.a. Configuration 1

If $\max\{A^H, B^H\} < z + s$, the consumer will either stop at (i, j) or open nested boxes with unknown content if she chooses to search inside the newly opened compound box. In this configuration the consumer searching inside a compound box always keeps the highest between A_i, B_j and either stops if the lowest is above z or opens nested boxes paying search cost s. In the latter case, the consumer will keep doing so until she finds something that beats z paying a search cost for each inspection.

To leverage the independence of compound boxes locked in a given configuration, it is necessary to obtain the distribution of values the consumer expect to find inside of it. Let w be the expected payoff of a consumer opening a compound box in this configuration and searching optimally inside of it, and $H(w)$ be its CDF. Given optimal search inside

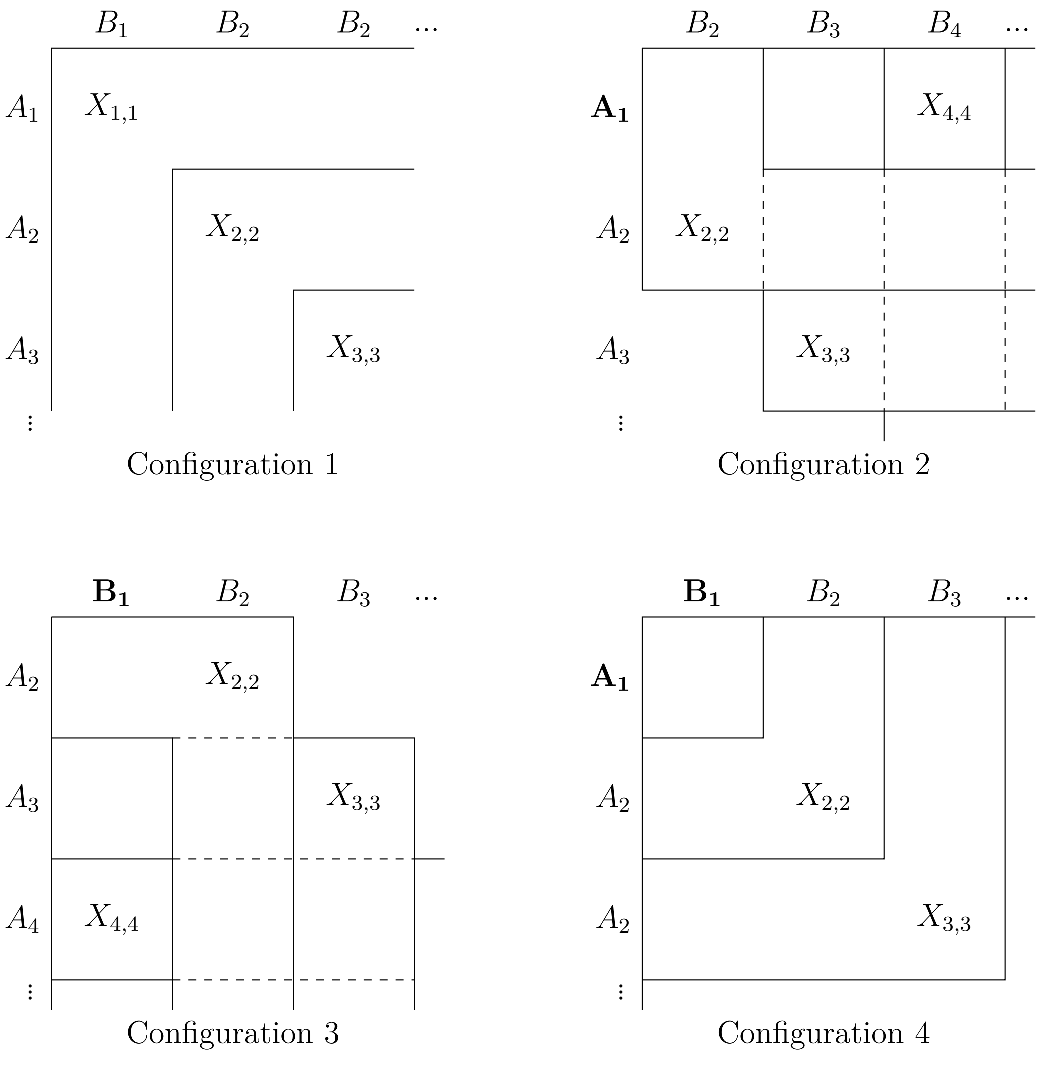

Figure 3: Possible configurations of compound boxes. Configuration 1 represents boxes when $\max\{A^H, B^H\} < z + s$; Configurations 2 and 3 represent boxes when $A_1 = A^H > z + s > B^H$ and $B_1 = B^H > z + s > A^H$ respectively and, therefore, reroute search towards themselves; Configuration 4 represents boxes in which $A_1 = A^H > z + s$ and $B_1 = B^H > z + s$.

the compound box, $w_{i,i}$ of opening box $X_{i,i}$ is:

$$w_{i,i} = \max\{A_i, B_i\} + \max\{z, \min\{A_i, B_i\}\}.$$

To compute $H(w)$, we must consider how different realizations for A_i and B_i interact. Fix a generic value B_i. If $B_i < z$, it is kept if and only if $B_i > A_i$. Otherwise, A_i is kept. In this case, $w = \max\{A_i, B_i\} + z$. If $B_i > z$, it is always kept over nested boxes $(i, j > i)$; A_i is also kept if its realization is above z, or: $w = B_i + \max\{A_i, z\}$.

Because $F_a(A) \equiv F_b(B) \equiv F(y)$ have support $[0, \hat{y}]$, $H(w)$ has support $[z, 2\hat{y}]$ and can be expressed as:

$$H(w) = \int_0^z F_a \left(\int_0^B F_b\,(w - z)\; dF_a(A) + \int_B^{\hat{y}} F_b\,(w - z)\; dF_a(A) \right) dF_b(B) + \\ + \int_z^{\hat{y}} F_a \left(\int_0^z F_b\,(w - z)\; dF_a(A) + \int_z^{\hat{y}} F_b\,(w - A)\; dF_a(A) \right) dF_b(B).$$

Suppose that the all compound boxes are "locked" in this configuration as in Figure 3 (top left corner).[12] The optimal search in this simplified case can be obtained through definition of a value function as shown in McCall (1970) and Kohn and Shavell (1974). In particular, we want to find $\underline{W}$ that solves the dynamic programming problem:

$$\underline{W} = -s + \max\{w, E[\underline{W}]\}, \tag{2}$$

where w follows the cumulative distribution function $H(w)$, and $\underline{W}$ is the maximum return the consumer would obtain after opening a compound box (and searching optimally therein if she stopped there). In this case, the optimal process sees the consumer stopping and keeping $w \geq E[\underline{W}]$ and searching if $w < E[\underline{W}]$. Because compound boxes locked in a configuration are effectively independent objects, this problem bears the same solution as Weitzman (1979). In particular, the relevant threshold value above which a box is kept is $\underline{W}$ that solves:

$$s = \int_{\underline{W}}^{2\hat{y}} (w - \underline{W})\, dH(w). \tag{3}$$

3.2.b. Configurations 2, 3, and 4

The same procedure allows to obtain static reservation value associated with boxes locked in different configurations. Consider first configurations 2 and 3: if $\max\{A^H, B^H\} > z + s > \min\{A^H, B^H\}$, the consumer will not open nested boxes along one attribute but would do so along the other. W.L.O.G., assume $A^H > z + s > B^H$ so that after opening $X_{i,i}$, the consumer would always go back to a product $(j < i, i)$ rather than opening nested boxes $(j > i, i)$ (but could still open nested boxes $(i, j > i)$ as per the top right corner of

[12] That is, imagine boxes to be unchangeable and such that the value of its content always follows w without possibility of being updated.

Figure 3). In particular:

- if $B_i > z$, the consumer chooses between keeping (i, i), $u_{i,i} = A_i + B_i$, and returning to $(j < i, i)$, $u_{j<i,i} = A^H + B_i - s$
- if $B_i < z$, instead, the consumer chooses between $(j < i, i)$ and inspecting nested boxes $(i, j > i)$, $r_{i,j>i} = A_i + z$.

Let $w_a(A^H)$ be the expected payoff of a consumer opening a compound box in configuration 2 and $H_a(w_a(A^H))$ be its CDF.[13] Assuming again that unopened compound boxes are locked in this configuration, their static reservation value is $\underline{W}_a(A^H)$ that solves:

$$s = \int_{\underline{W}_a(A^H)}^{\hat{y}} (w^a - \underline{W}_a(A^H))\, dH_a(w^a). \tag{4}$$

The same exact exercise leads to $\underline{W}_b(B^H)$ (bottom left corner of Figure 3), relevant when $A^H < z + s < B^H$.

Consider now configuration 4 (bottom right corner of Figure 3): if $\min\{A^H, B^H\} > z+s$, the consumer will not open any nested box. In particular:

- if $B_i > B^H - s$ and $A_i > A^H - s$, the consumer stops,
- if $B_i > B^H - s$ and $A_i < A^H - s$, the consumer inspects and keeps (i', i), $u_{i',i} = A^H + B_i - s$,
- if $B_i < B^H - s$ and $A_i > A^H - s$, the consumer inspects and keeps (i, i'), $u_{i,i'} = A_i + B^H - s$,
- if $B_i < B^H - s$ and $A_i < A^H - s$, instead, the consumer chooses between (i', i) and (i, i'), depending on which has the highest utility.

Labeling $w_{a,b}(A^H, B^H)$ and $H_{a,b}(w_{a,b}(A^H, B^H))$ the expected payoff and CDF of boxes locked in this configuration, their reservation value is $\underline{W}_{a,b}(A^H, B^H)$ that solves:

$$s = \int_{\underline{W}_{a,b}(A^H,B^H)}^{\hat{y}} (w^a - \underline{W}_{a,b}(A^H, B^H))\, dH_{a,b}(w^{a,b}). \tag{5}$$

The final step requires to combine these thresholds to account for the fact that compound boxes are not locked in any given configuration but, rather, can move from one configuration to the next depending on the realizations A^H and B^H found along the search process.

[13] A closed form expression for this and all subsequent CDFs can be found in Appendix A.

3.3. Optimal Search Process

The values $\underline{W}$, $\underline{W}_a(A^H)$, $\underline{W}_b(B^H)$, and $\underline{W}_{a,b}(A^H, B^H)$ can be appropriately combined to obtain the reservation values of unopened compound boxes when they are not locked in any given configuration. Once again, which of this values is relevant depends on past realizations: if some $A_{j<i} > z + s$ and/or some $B_{j<i} > z + s$ is found, this affects the value of all future boxes because by construction all compound boxes contain at least one product defined by all attributes.

The relevant value of the unopened compound boxes can evolve only in one direction, from configuration 1 to 4, and never backwards. Indeed, once $A_{j<i} > z + s$ is found, it can never be forgotten: once the relevant reservation value of the current configuration of $X_{i,i}$ changes from $\underline{W}$ to $\underline{W}_a(A^H)$, it can never revert to $\underline{W}$ or change to $\underline{W}_b(B^H)$. From this point onward, it can only stay at $\underline{W}_a(A^H)$ or change to $\underline{W}_{a,b}(A^H, B^H)$. Moreover, once configuration 4 is reached, all unopened compound boxes will keep this configuration.

Suppose all closed boxes reached configuration 4. This implies that $\min\{A^H, B^H\} > z + s$. Suppose the consumer has observed these A^H and B^H and must choose whether to open the next box. If boxes were to be locked, with any future A and B realization not being able to affect the next, the value of all closed boxes would be $\underline{W}_{a,b}(A^H, B^H)$. However, this does not capture the search dynamics appropriately.

Suppose the next box were to be opened and that $A_i > A^H$ was be found. The next compound box would have a different reservation value, $\underline{W}_{a,b}(A_i, B^H)$. The expected value of future boxes given the current values A^H, B^H can be obtained recursively. Let $\underline{W}^*_{a,b}(A^H, B^H)$ be the expected equivalent of costly opening the next box on the search path. This can be rewritten as a linear combination of expected $\underline{W}_{a,b}$ values:

$$\begin{aligned}\underline{W}^*_{a,b}(A^H, B^H) =& \underline{W}_{a,b}(A^H, B^H) \int_0^{B^H} \int_0^{A^H} dF_a(A) dF_b(B) + \\ &+ \int_0^{B^H} \int_{A^H}^{\hat{y}} \underline{W}_{a,b}(A, B^H)\, dF_a(A) dF_b(B) + \\ &+ \int_{B^H}^{\hat{y}} \int_0^{A^H} \underline{W}_{a,b}(A^H, B)\, dF_a(A) dF_b(B) + \\ &+ \int_{B^H}^{\hat{y}} \int_{A^H}^{\hat{y}} \underline{W}_{a,b}(A, B)\, dF_a(A) dF_b(B).\end{aligned}$$

Consider the choice of the consumer. If she opens the next compound box, $X_{i+1,i+1}$, she knows that she will stop only if $w_{i+1,i+1}$ is higher than the new value $\underline{W}_{a,b}(A^H, B^H)$, which in expectation is equal to $\underline{W}^*_{a,b}(A^H, B^H)$ before $X_{i+1,i+1}$ is opened.

Notice that $\underline{W}^*_{a,b}(A^H, B^H)$ is strictly higher than $\underline{W}_{a,b}(A^H, B^H)$ because $\underline{W}_{a,b}(A^H, B^H)$ is increasing in A^H and B^H. This threshold captures not only the value of inspecting the next box, that by itself would have had reservation value $\underline{W}_{a,b}(A^H, B^H)$, but also that of the updating that opening the box might lead to. In words, the value of opening the next box is the expected "certain equivalent" of opening the next, which in itself depends on the outcome of the inspection.

We can repeat the same exact exercise with the other configurations. For configuration 2, we write:

$$\begin{aligned}\underline{W}_a^*(A^H) =& \underline{W}_a(A^H)\int_0^{z+s}\int_0^{A^H} dF_a(A)dF_b(B)+\\ &+\int_0^{z+s}\int_{A^H}^{\hat{y}} \underline{W}_a(A)\, dF_a(A)dF_b(B)+\\ &+\int_{z+s}^{\hat{y}}\int_0^{A^H} \underline{W}_{a,b}(A^H,B)\, dF_a(A)dF_b(B)+\\ &+\int_{z+s}^{\hat{y}}\int_{A^H}^{\hat{y}} \underline{W}_{a,b}(A,B)\, dF_a(A)dF_b(B).\end{aligned}$$

which is both higher than $\underline{W}_a(A^H)$ and the hypothetical $\underline{\tilde{W}}_a(A^H)$ one would compute ignoring the possibility that the next box could change in value. An equivalent formulation can be found for configuration 3.

Finally, for configuration 1, we can write:

$$\begin{aligned}\underline{W}^* =& \underline{W}\int_0^{z+s}\int_0^{z+s} dF_a(A)dF_b(B)+\\ &+\int_0^{z+s}\int_{z+s}^{\hat{y}} \underline{W}_a(A)\, dF_a(A)dF_b(B)+\\ &+\int_{z+s}^{\hat{y}}\int_0^{z+s} \underline{W}_b(B)\, dF_a(A)dF_b(B)+\\ &+\int_{z+s}^{\hat{y}}\int_{z+s}^{\hat{y}} \underline{W}_{a,b}(A,B)\, dF_a(A)dF_b(B).\end{aligned}$$

By taking into account all possible configurations, and all the ways in which this configurations can evolve into one another, we can then write the reservation value of all unopened boxes as:

$$\mathcal{W}(A^H,B^H) = \begin{cases} \underline{W}^* & \text{if} \quad \max\{A^H,B^H\} < z+s, \\ \underline{W}_a^*(A^H) & \text{if} \quad A^H > z+s > B^H, \\ \underline{W}_B^*(B^H) & \text{if} \quad B^H > z+s > A^H, \\ \underline{W}_{a,b}^*(A^H,B^H) & \text{if} \quad \min\{A^H,B^H\} > z+s. \end{cases}$$

The values above reflect the value of inspecting any given compound box given all information learned so far and anticipating how the game could change given future realizations: $\mathcal{W}(A^H,B^H)$ incorporates the value of searching along all possible paths, defined by the number of attributes found above $z+s$. Once a path is taken, that path can never be left. Each path is built through the optimal search process when boxes are in the appropriate state, which is pinned down by the optimal search process inside the compound box as per its current configuration. Because each branch of the search path is optimized, the whole process is too.

While the optimal search order cannot be determined *ex ante* because of the learning component, whenever the consumer must choose what to do there is no ambiguity regarding the value of her possible options. Therefore:

Proposition 1. *Let* $A^H = \max\{y \in A \cap I\}$*,* $B^H = \max\{y \in B \cap I\}$ *be the highest discovered realization for* A *and* B*. Optimal search is characterized as follow:*

- ***Compound box selection:*** *compound boxes are opened until the expected payoff according to the optimal search policy inside of it,* $w_{i,j}$*, is higher than the reservation value of all unopened compound boxes,* $\mathcal{W}(A^H, B^H)$*.*
- ***Search inside the selected compound box:*** *Given selection of compound box* $X_{i,j}$*,* (i,j) *is kept if* $u_{i,j} > \max\{r_{i,k\neq j}, r_{k\neq i,j}\}$*; otherwise, the next box opened is the nested box with the highest* $r_{i,k}$ *or* $r_{k,i}$*.*
- ***Stopping rule:*** *Boxes (compound or nested) are opened until all unopened (compound or nested) boxes have updated reservation value below the highest realized payoff.*

Proof. All calculations and closed form equations for the four relevant CDFs can be found in Appendix A. ■

The multi-attribute structure proposed here allows one to score search options appropriately by leveraging the fact that at any given point compound boxes can be thought of as effectively independent object along the search path. This, in turn, generates an environment in which the standard intuition behind optimal search can be adapted. This process can be thought of as a consumer sampling unrelated products until at least an attribute worth keeping is found. When this happens, the consumer ignores all remaining compound boxes and searches inside the one that let her find that first attribute to keep in order to find an appropriate other one to pair with it, be it previously discovered or not.

Notice that the structure of the compound boxes reflect the internal consistency of the search process: opening a compound box always carries more information than a nested box inside a previously opened one. Therefore, if a compound box is selected, the attribute that is kept when searching inside of it must have had a realization high enough to compensate for the lower informational value of not inspecting two new attributes.

4. Extensions

4.1. Optimal Pricing by a Single Seller

Suppose a multiproduct monopoly seller (he) were to price the infinitely many products defined as above. He is aware of distribution F and search costs s. This seller can influence the search pattern over available products through prices, which are set before the search process starts, cannot be changed, and are observed costlessly by the consumer before she starts searching. Assuming all production costs to be equal to zero, we are interested in finding the subgame perfect equilibrium pricing under the following, updated timing:

1. The consumer and the seller observe distribution F and search cost s.
2. The seller commits to a vector of posted prices $p_{i,j}$ for all products (i, j).
3. The consumer observes all products and their relative prices.
4. The consumer makes searching and purchasing decision.

In order to solve for the optimal pricing scheme in this complex environment, we leverage once again the structure of compound boxes. The structure presented above can be readily adapted to incorporate prices. In particular, the value associated with each product must be reduced by the posted price; these new values can be used to score compound boxes appropriately and accounting for the price of all products on the relevant search paths. In other words, prices affect the value of opening any compound box; the effect cascades to the reservation values $\mathcal{W}$, which allows to solve for optimal pricing.

Consider the compound box $X_{1,1}$ built around product $(1, 1)$ priced at $p_{1,1}$; the box contains all products $(1, j)$, priced at $p_{1,j}$, and all products $(i, 1)$, priced at $p_{i,1}$. Suppose the consumer opened $X_{1,1}$ and decided to search in it keeping attribute A_1. Then, she would inspect next the product $(1, j)$ that satisfies:

$$\max_j (A_1 + z - p_{1,j}) \geq A_1 + B_1 - p_{1,1},$$

Three things are worth noticing: first, if $p_{1,j}$ is not uniform, the consumer would always select to inspect products $(1, j)$ in increasing order of price. Second, for $(1, 1)$ to be inspected before all other $(1, j)$ products, it must have been the cheapest of them. Third, if $p_{1,1} \neq p_{1,j}$, $(1, j)$ would be inspected next if and only if:

$$B_1 \leq z - (p_{1,j} - p_{1,1}) < z.$$

The same structure governs inspection of products $(j, 1)$.

In principle, all products $(1,j)$ could be priced differently. Suppose that prices where increasing in j and always strictly below z. Then, if the consumer decided to inspect $(1,2)$ after discovering A_1, B_1, he would expect to either keep it if it beats the reservation value of $(1,3)$, or keep searching, and so on for all subsequent inspections. The total value associated with this path given vector of prices $\mathbf{p_{1,k}}$ of all products $(1,k>1)$ is then:

$$y(\mathbf{p_{1,j}}) = \sum_{k=1}^{\infty} F(z-(p_{1,k+1}-p_{1,k}))^k \int_{z-(p_{1,k+1}-p_{1,k})}^{\hat{y}} (y-p_{1,k+1})\, dF(y).$$

To see the effect of prices, it is useful to compute the expected value of a compound box when the products therein have prices posted. Consider the generic compound box $X_{i,j}$ in the first configuration (for simplicity and for illustrative purposes). Let $\Delta_{i,k} \equiv p_{i,k+1} - p_{i,k}$ and $\Delta_{k,j} \equiv p_{k+1,j} - p_{k,j}$; further, let:

$$\bar{y}_{i,k} = E[y|y > z - \Delta_{i,k}], \quad \bar{y}_{k,j} = E[y|y > z - \Delta_{k,j}].$$

Then:

$$\begin{aligned} E[w_{i,j}(\mathbf{p_{i,j}})] =& [1-F(z-\Delta_{i,i+1})][1-F(z-\Delta_{j+1,j})](\bar{y}_{i,i+1} - \bar{y}_{j+1,j} - p_{i,j}) \\ &+ [1-F(z-\Delta_{i,i+1})]F(z-\Delta_{j+1,j})(\bar{y}_{i,i+1} + y(\mathbf{p_{i,k}})) \\ &+ F(z-\Delta_{i,i+1})[1-F(z-\Delta_{j+1,j})](y(\mathbf{p_{k,j}}) + \bar{y}_{j+1,j}) \\ &+ F(z-\Delta_{i,i+1})F(z-\Delta_{j+1,j})\,(y(\mathbf{p_{i,k}}) + y(\mathbf{p_{k,j}})\}), \end{aligned}$$

While a high price that does not make a product never worth inspecting makes it more profitable to sell, it also pushes the product attached to it further away from the optimal starting point of the consumer. Suppose all products were priced a some uniform level p^u and one was slightly more expensive. Then, not only the more expensive product would have lower value in any search path in which it could be found, but all compound boxes that contain it would also have a lower $E[w(\mathbf{p})]$, which translates to a lower reservation value. None of the boxes associated with this product, then, would ever be inspected as there are infinite better alternative for the consumer.

Another difficulty relates to the updating process described in the pages above. Attributes can still have realizations that reroute search towards themselves, and in a way that is much more cumbersome to keep track of when prices are accounted for. Moreover, because the relationship between the different possible scores $\mathcal{W}_{\cdot}$ depends on the specific realization or realizations that triggered the update, the updating could lead to all unopened boxes to become less valuable than they originally were, which could lead the consumer to end his search prematurely.

Both concerns can be addressed, and the following result emerges:

Proposition 2. *Consider a multiproduct seller pricing infinite products defined by two infinite sets of i.i.d. attributes. There exist a unique, uniform equilibrium pricing vector such that* $p_{i,j} = p^* = \underline{W}^*$, $\forall (i,j)$.

Proof. All calculations can be found in Appendix B. ■

Proposition 2 states that the only possible equilibrium features uniform pricing. In principle, given the reservation value of a compound box, different products could be priced differently to capitalize on the information learned through inspection. In Appendix B, I show that this cannot be optimal. The intuition is as follows: suppose that compound box $X_{1,1}$'s products were priced according to $p_{1,1} = p$ for some $p > 0$ and $p_{1,j} = p_{i,1} = p+\delta$ for some $\delta > 0$.[14] Plugging in these prices in the score of the compound box, one finds:

$$\begin{aligned} E[w_{1,1}(\mathbf{p_{1,1}})] =& [1 - F(z-\delta)]^2 (2\bar{y}_\delta - p) \\ &+ 2F(z-\delta)[1 - F(z-\delta)](\bar{y}_\delta + z - (p+\delta)) \\ &+ F(z-\delta)^2 \left(\underline{y}_\delta + z - (p+\delta)\right), \end{aligned}$$

where $\underline{y}_\delta$ is the expected value of the highest of two realizations below $z - \delta$.

Studying $E[w_{1,1}(\mathbf{p_{1,1}})]$ reveals that any positive δ would be detrimental to the expected profit of the seller. On one hand, the probability that the consumer finds a realization that induces her to keep searching after inspecting $(1,1)$ shrinks as δ increases because $F(z-\delta)$ is decreasing in δ. On the other hand, the participation constrain implied by the fact that the consumer must decide to open the first box becomes tighter as δ increases.

To see why, notice that the expected value of opening a compound box net of prices is equivalent to that of opening the same box when search costs are higher, and in particular $s' > s$ such that $z' = z - \delta$. It follows that $\delta > 0$ makes starting the search process less valuable, which tightens the consumer participation constraint and, therefore, how high prices that do not discourage search can be.

That $p^* = \underline{W}^*$, the initial reservation value of any compound boxes, follows from the updating dynamic detailed in the previous section. In particular, it follows from the fact that all updates increase the value of subsequent boxes rather than shrink it: the lowest value of a compound box after any updating can be shown to be $\underline{W}^*_{a,b}(A^H, B^H) \geq \underline{W}^*_{a,b}(z+s, z+s) = \underline{W}^*$. Therefore, the highest prices that the monopolist can set is the highest price that does not prevent search from taking place and, in particular, $p^* = \underline{W}^*$.

4.2. More than Two Attributes

In the baseline model, two attribute products are scored by building fictitious boxes including the product itself and closed boxes with all other products sharing attributes

[14] In the Appendix, I show that if an equilibrium with differential prices exists, it must have prices following this structure.

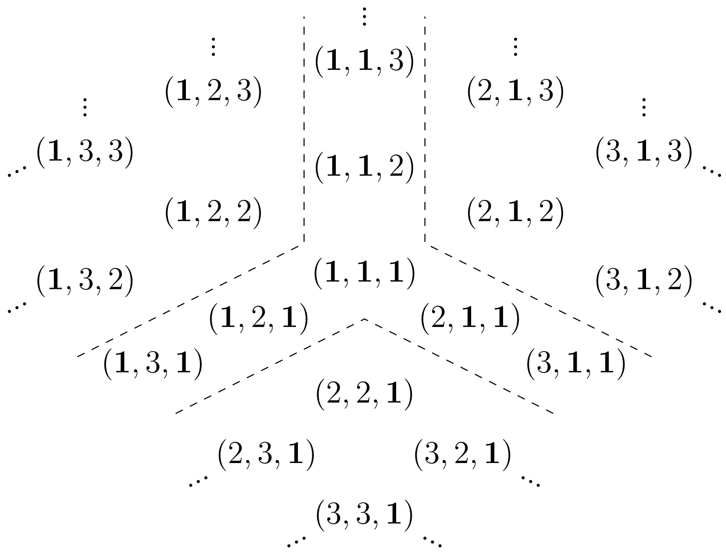

Figure 4: Graphical representation of a three attribute compound box centered around $(1,1,1)$. Products that share two attributes with $(1,1,1)$ can be displayed along the edges of a cube (north for products sharing A_1, B_1, south-west for products sharing A_1, C_1, south east for products sharing B_1, C_1); Products that share one attribute with $(1,1,1)$ can be displayed along the sides of the cube (north-west for products sharing only A_1, north-east for products sharing only B_1, south for products sharing only C_1).

with it. The same logic can be applied to three attribute products. With three attributes, two kinds of closed boxes must be included in the compound box with the product it represents. On one hand, all products sharing exactly two attributes with the central product can be represented as small nested boxes equivalent to the ones contained in the two attributes case.

On the other, products that share only one attribute with the central one are unknown in two dimensions, and must be placed in two-dimensional boxes equivalent to the compound boxes of the baseline model. These "intermediate" boxes themselves contain infinite small nested boxes as well. One can imagine multiple grids representing two attribute products side by side to resemble a cube, with the intermediate boxes representing search along one of the sides, and the small boxes representing search along one of the edges as in Figure 4.

We can conceptualize the same process to find the optimal search path for a consumer searching in this environment. First, it is necessary to rethink the structure of a generic nested box $X_{i,i,i}$. If $X_{i,i,i}$ is built around product (i,i,i), it contains all products that share at least one attribute with it. Therefore, it can be represented as the three edges of a cube and the sides delimited by them. Each side can be thought of as a two-dimensional grid in which "intermediate" compound boxes akin to the ones defined in the main model can be found. The choice of opening these boxes is governed by the same $\mathcal{W}_\cdot$ functions defined above. The choice of opening a different three-dimensional box, instead, requires computing the reservation value of the possible "locked" configurations this box can come out of. All configurations will always be made of three edges and three sides. Whether the edges stretch forward, towards undiscovered attributes, or backward, to known past realizations, depends once again on whether single attributes are found above or below $z + s$, or combinations of two attributes above or below $\mathcal{W}$.

4.3. Purchase Without Inspection

It is assumed throughout the paper that consumers must expend a search cost to inspect any product. Because products in this environment share attributes some uninspected products could be fully revealed without being inspected. If search is understood as the physical action of finding a product, this distinction is immaterial. If, however, one were to interpret search as the time and effort necessary to ascertain the quality of the match of a product, it would be sensible to suggest that products uninspected but nonetheless known in their realization should not need search costs to be expended. This alternative interpretation affects the search dynamic in a straightforward manner.

If taking a product whose attribute have been fully independently discovered is free, the only optimal search process would be one that involves searching new attributes in pairs until the highest realization for each attribute is such that they, together surpass the value of all uninspected products. This can be accomplished by modifying the way reservation values update after each observation. The lowest realization that reroutes search towards itself inside all unopened compound boxes is (without loss of generality) $A_1 > z$ rather than $A_1 > z + s$. With this change, the choice of keeping an attribute is always dominated because all products sharing an attribute with an inspected product would be contained, at zero additional cost, in all unopened compound boxes, and affects them all through the same updating detailed above.

4.4. Limitations and Directions Forward

4.4.a. On the Eventual Purchase Theorem

The eventual purchase theorem (henceforth, EPT), first proposed in Armstrong (2017) and Choi et al. (2018), states that the outcome of a search process to find one out of independently distributed products can be obtained through a simple statistic. In particular, the product i that is ultimately selected by a consumer will be the one with the highest statistic:

$$W_i = \min\{r_i, u_i\},$$

or, the highest minimum between reservation and match value of a product.

Obtaining a similar statistic in this environment comes with a few challenges. First, a product is kept as long as it's match value surpasses that of different objects, namely the closed compound boxes and the closed reachable nested boxes. These objects are associated with different scores, one reflecting the value of the implied search paths that

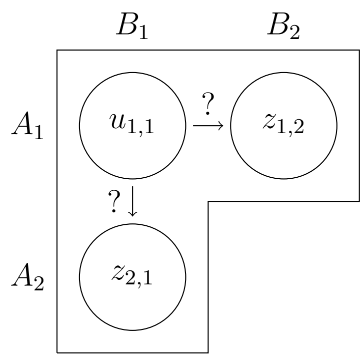

Figure 5: Three products available.

would follow from it, one reflecting the value of only its inspection. A statistic like the one governing the EPT, then, should account for both.

Another difficulty is the threshold over which attributes are kept. In the main analysis I show that an attribute is kept as long as it has realization above $\mathcal{W}$. Besides the obvious difficulty caused by $\mathcal{W}$ having an history-dependent value, the more challenging issue comes from the fact that search in the selected compound box follows different directions for different past realizations. An appropriate statistic, then, should be able to account for all possible directions the optimal search policy could instruct to take inside any given compound box.

4.4.b. Finite number of products

The structure proposed for the infinitely many products does not immediately translate to the finite product menu case. Consider a box like the one in Figure 5: Since pinning down the optimal search path inside this box, and its value besides, is not as straightforward as before, scoring this box requires significant nuance. To see why, Suppose A_1 and $B_1 < A_1$ both had very low realizations, and suppose the consumer optimally inspects $(1, 2)$ next. She now might want to inspect $(2, 1)$ if $z + B_1 > A_1 + \max\{B_1, B_2\}$. This affects the value of inspecting $(1, 2)$, and therefore the threshold dictating whether the consumer would stop at $(1, 1)$ instead.

The complexity of the problem is apparent even with this simple example, but it is worth stressing out that in a finite grid environment every additional product generates several possible search paths that must be scored and compared. In turn, this implies that every finite product menu requires building every possible search path by backward induction in order to select the optimal one. While solvable for any given configuration, a general optimization problem with this structure might not be immediately in reach.

Another problematic difference with the infinitely large boxes of the baseline model follows from the fact that even if the issue above was resolved, and a compound box structure could be feasibly built, when a compound box is opened and discarded in a finite grid, the following boxes "shrink" by one variant per attribute. Suppose $X_{1,1}$

contained products characterized by n variants of A and m variants of B. Further, suppose realizations A_1 and B_1 were very low. Then, $X_{2,2}$ would effectively contain products characterized by $n-1$ variants of A and $m-1$ variants of B. Assuming consumers search in increasing order of the index, then, the size of each subsequent compound box $X_{i,i}$ would have $n+1-i$ variants of A and $m+1-i$ variants of B.

The implication of this last remark is that while thinking about boxes as locked in some configuration achieves the same conceptual independence between objects, now every subsequent choice is "discounted" by the value associated with one more variant for each attribute. Effectively, this means that the choice of searching now and searching again later can never be the same. While in principle this could be accounted for, as the structure resembles of that of Weitzman (1979), combining the resulting locked reservation values to generate adaptive ones to take the place of $\mathcal{W}$ quickly leads to a computationally intractable problem.

4.4.c. Different distributions

In principle, removing the assumption of attributes following the same distribution can be accommodated. One can imagine a variant of the model above in which all A attributes were i.i.d and all B attributes were too, but the two sets followed a different distribution. This does not affect the analysis significantly. Far more challenging is accounting for different distributions across different variants of the same attribute in the general framework. The reason stems from the way compound boxes are constructed: with different distributions come different reservation values z for the same search cost s, which means that the expected value of searching along one dimension is not straightforward to compute.

A possible solution might be to use the EPT as characterized by Armstrong (2017) and Choi et al. (2018) to pin down said value, and the value of all other dimensions. A general solution of this more complex problem, however, becomes quickly intractable, and is therefore left for future research.

5. Conclusion

The framework's predicted search patterns align well with recent evidence of spatial learning in search: Hodgson and Lewis (2020) reports evidence of search for digital cameras to be characterized by a learning process consistent with the one in this framework. Consumers are shown to inspect a broader set of attributes early only to close in on their preferred alternatives in later stages, getting closer and closer to the product they ultimately choose to purchase. This pattern cannot be easily reconciled with standard search models, but is well in line with the prediction of this framework. Further, the model presented here can more easily rationalize the pervasive tendency of consumers to retrace their steps while searching for products.

The learning process detailed in this paper has implications on our understanding of search costs in markets with horizontally differentiated products. The consumer in my framework searches more than one that does not update her expectations to account for the information learned. This is immediate when one considers that a realization that would induce a consumer to stop searching before accounting for the updating proposed here might not induce her to stop afterwards. Since the thresholds governing the choice of stopping become higher with better realizations, they would suggest that searching becomes "less costly" as it progresses. Studying search frictions in markets ignoring the learning component presented in this paper, then, would lead to an underestimation of search frictions precisely because the cut-off rule with learning is by construction higher than an alternative one without. Search friction estimation should then strive to incorporate learning when appropriate to.

Future research should aim at generating resulting for environments that are not constrained by the assumption of attributes coming in infinitely many varieties, and that go beyond the two-attributes structure used throughout this paper. The resulting insight would allow to better estimate search frictions empirically, and to generate more sound considerations with regard to the way consumers approach searching with complex, multi-dimensional products. In particular, extending the framework to address its current limitations could greatly improve our understanding of online consumers' search decisions and sellers' strategic reactions.

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Appendix

A. Optimal Search Policy - Proof of Proposition 1

Step 1: The first compound box Let $X_{1,1}$ be the compound box containing $(1,1)$ and infinitely many nested boxes containing $(1, j > 1)$, $(j > 1, 1)$. Suppose the consumer had already opened the box. Her current payoff is $k = \max\{u_0, A_1 + B_1\}$. To determine how she would act afterwards, consider the value function:

$$V(k) = \max\{k, -s + E[V(\max\{k, A_j + B_1\})], \\ - s + E[V(\max\{k, A_1 + B_j\})]\}.$$

Suppose $V(k) = k$. Then:

$$k > -s + E[V(\max\{k, A_j + B_1\})] = -s + E[\max\{V(A_i j B_1), k\}],$$
$$s > E[\max\{V(A_j + B_1) - k, 0\}] = \int_k^{\hat{y}} (V(A_j + B_1) - k)\, dF(y).$$

$$k > -s + E[V(\max\{k, A_1 + B_j\})] = -s + E[\max\{V(A_1 + B_j), k\}],$$
$$s > E[\max\{V(A_1 + B_j) - k, 0\}] = \int_k^{\hat{y}} (V(A_1 + B_j) - k)\, dF(y).$$

Therefore, there exist values r_A, r_B such that if $k > \max\{r_A, r_B\}$, $V(k) = k$. Suppose that $-s + E[V(\max\{k, A_j + B_1\})] > \max\{k, -s + E[V(\max\{k, A_1 + B_j\})]\}$. Then:

$$V(k) = -s + E[\max\{V(A_1 + B_j), V(k)\}],$$
$$s = E[\max V(A_1 + B_j, V(k))] \quad \rightarrow \quad V(k) = r_A$$

Suppose now that $-s+E[V(\max\{k, A_1+B_j\})] > \max\{k, -s+E[V(\max\{k, A_k+B_1\})]\}$. Then:

$$V(k) = -s + E[\max\{V(A_j + B_1), V(k)\}],$$
$$s = E[\max V(A_j + B_1, V(k))] \quad \rightarrow \quad V(k) = r_B$$

To compute r_A and r_B, the optimal policy conditional on $V(k) = r_A$ and $V(k) = r_B$ respectively must be defined. Assuming that the consumer inspects products in increasing order of their indices when indifferent, I make the following:

Claim 1. *If* $V(\max\{u_0, A_1 + B_1\}) = r_A$*,* $V(\max\{u_0, A_1 + B_1, A_1 + B_2\}) = \max\{A_1 + B_2, r_A\}$*; if* $V(\max\{u_0, A_1+B_1\}) = r_B$*,* $V(\max\{u_0, A_1+B_1, A_2+B_1\}) = \max\{A_2+B_1, r_B\}$*.*

By contradiction, suppose that $V(\max\{u_0, A_1 + B_1\}) = r_A$. Then:

$$V(\max\{u_0, A_1 + B_1, A_1 + B_2\}) = \max\{u_0, A_1 + B_1, A_1 + B_2, \\ - s + E[V(\max\{k, A_j + B_1\})], -s + E[V(\max\{k, A_1 + B_j\})]\}.$$

This is immediate: because $V(\max\{u_0, A_1 + B_1\}) = r_A$, it must hold:

$$-s + E[V(\max\{k, A_1 + B_j\})] > \max\{u_0, A_1 + B_1, -s + E[V(\max\{k, A_i + B_1\})]\}.$$

To see why, suppose $\max\{u_0, A_1 + B_1\} = A_1 + B_1$. Then, by the same argument as above, for $V(\max\{u_0, A_1 + B_1\}) = r_A$ it must be that $k < r_A$. If $\max\{u_0, A_1 + B_1\} > A_1 + B_2$, the same condition applies. Otherwise, if $\max\{u_0, A_1 + B_1\} < A_2 + B_2$, then for $V(\max\{u_0, A_1 + B_1, A_2 + B, 2\}) = r_A$ to be true, $A_2 + B_2$ must also be below r_A. Because $V(\max\{u_0, A_1 + B_1\}) = r_A$, it must be that $A_1 + B_1 < r_A$. It follows that $V(\max\{u_0, A_1 + B_1, A_1 + B_2\}) = \max\{A_1 + B_2, r_A\}$.

In words: if it is optimal to inspect $A_1 + B_2$ after opening the compound box, it must also be optimal to inspect $A_1 + B_3$ if the consumer does not want to stop searching. Therefore, the optimal policy conditional on $V(\max\{u_0, A_1 + B_1\}) = r_A$ is a myopic policy in which the current highest realization is compared to the value of inspecting the next product. The consumer is indifferent between stopping at $(1, j)$ and inspecting $(1, j+1)$, $j \geq 1$, if:

$$A_1 + B_j = - s + A_1 + B_j \int_0^{B_j} dF(y) + \int_{B_j}^{\hat{y}} B_{j+1} \, dF(y),$$
$$s = \int_{B_j}^{\hat{y}} (B_{j+1} - B_j) \, dF(y).$$

Let z be the value of B_j that satisfies the condition above. It follows that $r_A = A_1 + z$.[15] In the same fashion, from $V(\max\{u_0, A_1 + B_1\}) = r_B$ one obtains that $r_B = z + B_1$. It follows that the value function representing the choice of opening the compound box $X_{1,1}$ and searching optimally in it is:

$$V(u_0) = \max\left\{u_0, \max\left\{u_0, \max\{A_1, B_1\} + \max\{z, \min\{A_1, B_1\}\}\right\}\right\},$$

because the consumer would always select $A_1 + z$ over $B_1 + z$ if and only if $A_1 > B_1$, and will stop at $(1, 1)$ if $\min\{A_1, B_1\} > z$ She would also take her outside option, u_0, if she opens the box and none of these options had value above it. The consumer is indifferent

[15] as found in McCall (1970) and Kohn and Shavell (1974)

between opening the compound box and not opening if:

$$\begin{aligned} u_0 =& - s + u_0 \int_0^{u_0} dF(y) + \int_{u_0}^{2\hat{y}} w \, dH(w), \\ s =& \int_{u_0}^{2\hat{y}} (w - u_0) \, dH(w), \end{aligned}$$

where $w = \max\{A_1, B_1\} + \max\{z, \min\{A_1, B_1\}\} \in (z, 2\hat{y})$, and its CDF satisfies:

$$\begin{aligned} H(w) =& \int_0^z F_a \left(\int_0^B F_b \left(w - z \right) \, dF_a(A) + \int_B^{\hat{y}} F_b \left(w - z \right) \, dF_a(A) \right) \, dF_b(B) + \\ &+ \int_z^{\hat{y}} F_a \left(\int_0^z F_b \left(w - z \right) \, dF_a(A) + \int_z^{\hat{y}} F_b \left(w - A \right) \, dF_a(A) \right) \, dF_b(B). \end{aligned}$$

Keeping the standard nomenclature, I refer to the value u_0 that satisfies the above equation as the reservation value of the the compound box.

Result 1. *Let $\underline{W}$ be the reservation value of $X_{1,1}$ and z the reservation value of any $y \in A \cup B$. The optimal policy with only one compound box $X_{1,1}$ is:*

- *Open $X_{1,1}$ if $u_0 < \underline{W}$, otherwise keep u_0,*
- *if $u_0 > \max\{A_1, B_1\} + \max\{z, \min\{A_1, B_1\}\}$, stop and keep the outside option, otherwise:*
  - *if $\max\{z, \min\{A_1, B_1\}\} = \min\{A_1, B_1\}$, stop and keep $A_1 + B_1$,*
  - *if $\max\{z, \min\{A_1, B_1\}\} = z$, inspect $(1, j)$ until $B_j \geq z$ is found if $A_1 > B_1$, and inspect $(i, 1)$ until $A_i \geq z$ is found if $A_1 < B_1$.*

Step 2: Uncorrelated compound boxes Let $\tilde{X}_{i,i}$, $i \geq 1$, be the compound box containing (i, i) and infinitely many compound boxes containing $(i, j > i)$, $(j > i, i)$. We want to show that, in this environment, the optimal search policy follows a myopic optimal policy such that if $u_0 < \underline{W}$, the consumer starts searching and stops after finding a product with *ex post* utility higher than the reservation value of all closed boxes.

Suppose the consumer opened $X_{1,1}$. Let $k = \max\{u_0, A_1 + B_1\}$; consider the value function:

$$\begin{aligned} V(k) = \max\{& k, -s + E[V(\max\{k, A_j + B_1\})], \\ & - s + E[V(\max\{k, A_1 + B_j\})], -s + E[V(\max\{k, A_j + B_j\})]\}. \end{aligned}$$

Compared the value function of the last paragraph, we must now also compare the first three options with the last one. Suppose once again that $V(k) = k$. Then, it holds:

$$k > -s + E[V(\max\{k, A_i + B_j\})] = -s + E[\max\{V(A_i + B_1), k\}],$$
$$s > E[\max\{V(A_i + B_1) - k, 0\}].$$

which once again implies that there exist a value $R_{2,2}$ such that if $k > \max\{r_A, r_B, R_{2,2}\}$, $V(k) = k$.

We must verify that r_A andr_B are still the value associated with a myopic policy. This is once again immediate: if $r_A > \max\{k, r_B, R_{2,2}\}$ (resp., $r_B > \max\{k, r_A, R_{2,2}\}$), then $V(\max\{u_0, A_1 + B + 1, A_1 + B_2\}) = \max\{A_1 + B_1, r_A\}$ (resp., $V(\max\{u_0, A_1 + B + 1, A_1 + B_2\}) = \max\{A_1 + B_1, r_B\}$). In words: if keeping A_1 or B_1 and inspecting B_2 or A_2 has value higher than searching $(2, 2)$, keeping the same attribute and inspecting A_3 or B_3 must also have a higher value. This implies that once an attribute is optimally kept, it is never abandoned.

To fully characterize the search process, we must compute the optimal policy after opening $X_{2,2}$. To do so, we prove the following:

Claim 2. *If* $V(\max\{u_0, A_1 + B_1\}) = R_{2,2}$*:*

$$V(\max\{u_0, A_1 + B_1, A_2 + B_2\}) = \max\{A_2 + B_2, -s + E[V(\max\{A_2 + B_2, A_j + B_2\})], \\ - s + E[V(\max\{A_2 + B_2, A_2 + B_j\})], -s + E[V(\max\{A_2 + B_2, A_j + B_j\})]\},$$

where once again A_j*,* B_j *are unsampled attributes.*

Since currently all compound boxes are uncorrelated, opening $X_{2,2}$ does not generate any new information about the content of $X_{1,1}$. This will not be the case when we prove the statement in the final step of the proof. For now: since we know that optimally keeping an attribute leads to an myopic optimal policy, the result follows from the same observation than before. In particular:

- If $V(\max\{u_0, A_1 + B_1, A_2 + B_2\}) = -s + E[V(\max\{A_2 + B_2, A_2 + B_j\})] = A_2 + z$, or $V(\max\{u_0, A_1 + B_1, A_2 + B_2\}) = -s + E[V(\max\{A_2 + B_2, A_i + B_2\})] = B_2 + z$ the consumer will open nested boxes myopically forever,
- If $V(\max\{u_0, A_1 + B_1, A_2 + B_2\}) = A_2 + B_2$, the consumer would stop,
- If $V(\max\{u_0, A_1 + B_1, A_2 + B_2\}) = -s + E[V(\max\{A_2 + B_2, A_i + B_j\})]$, the consumer would open the next compound box.

It must hold that $V(\max\{u_0, A_1 + B_1, A_2 + B_2\}) \neq \max\{u_0, A_1 + B_1, A_1 + z, B_1 + z\}$ because $V(\max\{u_0, A_1 + B_1\}) = R_{2,2}$. In particular, to see that $V(\max\{u_0, A_1 + B_1, A_2 + B_2\}) \neq \max\{A_1 + z, B_1 + z\}$ – that is, that it is sub optimal to go back to $X_{1,1}$ after opening $X_{2,2}$, suppose by contradiction that the optimal policy makes the consumer go back to $A_1 + z$ or $B_1 + z$ after opening $X_{2,2}$ with probability $q \in (0, 1]$ and proceed optimally afterwards. We already established that after inspecting $(1, 2)$ or $(2, 1)$ the consumer would optimally keep searching keeping either A_1 or B_1 fixed. If this is the case, since $R_{2,2} > r_A$ because the consumer opened $X_{2,2}$ instead of opening nested boxes, it must hold:

$$\begin{aligned} R_{2,2} =& q r_A + (1 - q)(R_{2,2} - r_A), \\ R_{2,2} q =& 2q r_A - r_A, \\ R_{2,2} =& 2 r_A - \frac{r_A}{q} > r_A, \\ q >& 1. \end{aligned}$$

which is a contradiction.

It follows that the optimal policy after opening $X_{2,2}$ is to myopically select between $\max\{A_2, B_2\}+\max\{z, \min\{A_2, B_2\}\}$ and $R_{3,3}$. Since this was the same policy the consumer followed at $(1, 1)$, the consumer is once again following a myopic policy. Therefore, $R_{2,2} = \underline{W}$, which proves the claim.

Result 2. *Let $\underline{W}_{i,i} = \underline{W}$ be the reservation value of uncorrelated compound boxes $\tilde{X}_{i,i}$ and z the reservation value of any $y \in A \cup B$. The optimal policy with infinitely many $\tilde{X}_{i,i}$ is:*

- *Open $\tilde{X}_{1,1}$ if $u_0 < \underline{W}$, otherwise keep u_0,*
- *if $\max\{A_1, B_1\} + \max\{z, \min\{A_1, B_1\}\} > \underline{W}$:*
  - *if $\max\{z, \min\{A_1, B_1\}\} = \min\{A_1, B_1\}$, stop and keep $A_1 + B_1$,*
  - *if $\max\{z, \min\{A_1, B_1\}\} = z$, inspect $(1, j)$ until $B_j > z$ is found if $A_1 > B_1$ and inspect $(i, 1)$ until $A_i > z$ is found if $A_1 < B_1$,*
- *if $\max\{A_1, B_1\} + \max\{z, \min\{A_1, B_1\}\} < \underline{W}$, open $\tilde{X}_{2,2}$ and go back to the second point.*

Step 3: General model We now remove the assumption of compound boxes being uncorrelated. This implies that the consumer can move freely on the grid of products inspecting one or two new attributes as she sees fit. We want to show that the optimal search process still follows a process that can fully characterized with threshold rules.

Suppose for now that the consumer never optimally goes back to a previously discarded attribute. That is, suppose that $k = \max\{u_0, A_1 + B_1, A_1 + B_2\}$. Then:

$$\begin{aligned} V(k) = \max\{&k, -s + E[V(\max\{k, A_1 + B_j\})], \\ &- s + E[V(\max\{k, A_j + B_2\})], -s + E[V(\max\{k, A_j + B_j\})]\}. \end{aligned}$$

Notice that this value function does not allow the consumer to inspect combinations of discovered attributes. This will be addressed shortly. For now, we want to show that:

Claim 3. *If the consumer cannot backtrack to a combination of discovered attributes,* $V(\max\{u_0, A_1 + B_1\}) = r_A$ *implies:*

$$V(\max\{u_0, A_1 + B_1, A_1 + B_2\}) = \max\{A_1 + B_2, -s + E[V(\max\{A_1 + B_2, A_1 + B_j\})],$$

where B_j *are unsampled* B *attributes.*

In words: we want to show that the if the consumer cannot backtrack to a combination of undiscovered attributes, keeping an attribute and still searching follows an optimal policy that is still myopic.

From the last paragraph, we know that if $V(\max\{u_0, A_1 + B_1\}) = r_A$, it must hold:

$$\begin{aligned} V(\max\{u_0, A_1 + B_1, A_1 + B_2\}) \neq \max\{&u_0, A_1 + B_1, \\ &- s + E[V(\max\{k, A_i + B_1\})], -s + E[V(\max\{k, A_i + B_j\})]. \end{aligned}$$

We must now prove that $r_A > R_{2,2}$ implies that $V(\max\{u_0, A_1 + B_1, A_1 + B_2\}) \neq -s + E[V(\max\{k, A_i + B_2\})]$, that is, inspecting $(2, 2)$ after $(1, 2)$ cannot be optimal. Suppose by contradiction that the optimal policy was such that, after inspecting $(1, 2)$, the consumer would optimally inspect $(2, 2)$ with probability $q \in (0, 1)$ and then follow the optimal policy from there. Then, it must hold:

$$\begin{aligned} r_A =& qR_{2,2} + (1 - q)(r_A - R_{2,2}), \\ r_A q =& 2qR_{2,2} - R_{2,2}, \\ r_A =& 2R_{2,2} - \frac{R_{2,2}}{q} > R_{2,2}, \\ q >& 1. \end{aligned}$$

Which is clearly a contradiction: if $(1, 2)$ is optimally picked over $(2, 2)$, it can never be optimal to inspect $(2, 2)$ afterwards. This proves the claim.

When compound boxes share products, each combination of known attributes, inspected or not, becomes effectively an outside option. Suppose that the consumer opened $X_{1,1}$, $X_{2,2}$, and $X_{3,3}$. Then, the highest available payoff for the consumer is:

$$\begin{aligned} k = \max\{&u_0, A_1 + B_1, A_2 + B_2, A_3 + B_3, \\ &A_1 + B_2 - s, A_1 + B_3 - s, A_2 + B_3 - s, \\ &A_2 + B_1 - s, A_3 + B_1 - s, A_3 + B_2 - s\}. \end{aligned}$$

Only a subset of these can ever be relevant. Consider $A_2 + B_3 - s$ and $A_1 + B_3 - s$. If the consumer decides to backtrack to one of these two available payoffs after opening $X_{3,3}$

and observing the realization B_3, it is clear that she woulds select the former if $A_2 > A_1$ and the latter otherwise. Importantly, this information is known to the consumer before opening the compound box $X_{3,3}$.

We must now prove that if the consumer decides to open nested boxes, he never backtracks. Suppose the consumer optimally opened in sequence: $X_{1,1}$, $X_{2,2}$, $(2,3)$. We want to show that the consumer does not backtrack to $(1,2)$ or $(2,1)$ nor to the newly discovered $(1,3)$. For the first two, notice that the sequencing implies:

$$V(\max\{u_0, A_1 + B_1, A_2 + B_2\}) = r_{A_2} > \max\{R_{3,3} = R_{2,2}, A_1 + B_2 - s, A_2 + B_1 - s\},$$

where r_{A_2} is the continuation value of the optimal policy after keeping A_2.

For the last one, notice that $V(\max\{u_0, A_1 + B_1\}) = R_{2,2}$ implies $R_{2,2} > r_{A_1}$, where r_{A_1} is the continuation value of the optimal policy after keeping A_1. But then it holds:

$$r_{A_2} > R_{3,3} = R_{2,2} > r_{A_1} \iff A_2 > A_1 \iff A_2 + B_3 > A_1 + B_3 - s.$$

Since keeping $(2,3)$ must better than backtracking to known $(1,3)$, backtracking is never optimal. Therefore, once again keeping an attribute leads to a myopic optimal policy: once an attribute is kept, it is never dropped.

The choice of backtracking then must be pinned down by the highest past realization. Suppose the consumer opens $X_{3,3}$. If she chooses to keep A_3, she chooses between:

- $A_3 + B_3$, readily available,
- $A_3 + z$, opening nested boxes,
- $A_3 + B_1 - s$ or $A_3 + B_2 - s$, backtracking.

Without loss of generality, suppose $B_2 > B_1$. Suppose further that $B_3 < z$. Then, the consumer chooses $\max\{A_3 + z, A_3 + B_1 - s\}$. Notice that:

$$\max\{A_3 + z, A_3 + B_1 - s\} = A_3 + B_1 - s \quad \iff \quad B_1 \geq z + s.$$

Because the realization B_1 is known before opening the box, the consumer is already aware of whether she would backtrack or go forward. Moreover, if $V(k) = A_3 + z$, it is clear that the consumer would never choose to backtrack afterwards. This confirms that the optimal policy conditional on inspecting a single attribute is myopic.

We can finally prove the main statement. In words, we now use the predictability of the search process when the highest past realization of A attributes, A^H, and B attributes, B^H, are above or below $z + s$ to define these thresholds.

Formally, suppose the consumer needs to decide whether to open $X_{i,i}$. let $k = \max\{u_0, \max_{j<i}\{u_{j,j}\}, A^H + B^H - s\}$ be the current highest sure payoff for the consumer.

Define:

$$V(k) = \max\{k, -s + E[V(\max\{k, A_j + B_k\})], -s + E[V(\max\{k, A_k + B_j\}),$$
$$- s + E[V(\max\{k, A_k + B_k\})]], \quad \forall k > i.$$

We already established that there exist a value $\hat{k}$ such that if $k > \hat{k}$, $V(k) = k$. Further, we proved above that if $V(k) = -s + E[V(\max\{k, A_k + B_i\})$ or $V(k) = -s + E[V(\max\{k, A_i + B_k\})$, the optimal policy has the consumer only opening nested boxes keeping the same attribute A_i or B_i fixed. Finally, we established that if A^H and/or B^H are above $z+s$, either one or both $-s+E[V(\max\{k, A_k+B_i\})$ and $-s+E[V(\max\{k, A_i+B_k\})$ will be dominated by backtracking. With these considerations we can define the value $\mathcal{W}$ such that the consumer opens compound box $X_{i,i}$ if and only if $\mathcal{W} > \max\{k, A_{i-1} + B_i, A_i + B_{i-1}\}$.

If $\max\{A^H, B^H\} < z + s$, the myopic (but incorrect) value of inspecting the next compound box is the same is in the last paragraph: $\underline{W}$. Suppose $X_{i,i}$ is opened next and $A_i > z + s$. The next box will have a different reservation value. The correct reservation value must account for the possibility of discovering attributes that change the search from that point onward. To do so, we must first compute the value of inspecting a compound box when $\max\{A^H, B^H\} > z + s$. We first do so assuming, as in the last paragraph, that boxes are uncorrelated but in different configurations depending on the number of attributes above $z + s$ that were found. Then, we combine them appropriately to produce the correct reservation values.

Suppose an attribute A^H was found above $z + s$. The consumer would go back to it rather than opening nested boxes unknown in their A component. Let $w_a(A^H)$ be the expected payoff of opening a compound box locked in this configuration. Let this box be $X_{i,i}$. Suppose $B_i < z$ is found. Then, the consumer must choose between opening nested boxes with score $r_{i,j>i} = A_i + z > A_i + B_i = u_{i,i}$ and backtracking to a box with score $r_{i,j<i} = A^H - s + B_i$. Therefore, the consumer inspects nested boxes if $A_i > A^H - s - (z - B_i)$, and backtrack otherwise. If $B_i > z$, the consumer stops at (i, i) if $A_i > A^H - s$ and backtrack otherwise. The CDF then can be obtained by fixing the value B_i and then integrating for it as it was done for configuration 1:

$$\begin{aligned}
H_a(w_a) = \int_0^z F_a \Bigg(\int_0^{A^H - s - (z-B)} & F_b \left(w_a - (A^H - s)\right) dF_a(A) + \\
+ \int_{A^H - s - (z-B)}^{\hat{y}} & F_b\left(w_a - z\right) dF_a(A) \Bigg) dF_b(B) + \\
+ \int_z^{\hat{y}} F_a \Bigg(\int_0^{A^H - s} & F_b \left(w_a - (A^H - s)\right) dF_a(A) + \\
+ \int_{A^H - s}^{\hat{y}} & F_b\left(w_a - A\right) dF_a(A) \Bigg) dF_b(B).
\end{aligned}$$

The equivalent formulation for $H_b(w_b)$, relevant if $A^H > z + s > B^H$ is:

$$
\begin{aligned}
H_b(w_b) = \int_0^z F_b \Bigg(\int_0^{B^H - s - (z-A)} & F_a \left(w_b - (B^H - s)\right) dF_b(B) + \\
&+ \int_{B^H - s - (z-A)}^{\hat{y}} F_a \left(w_b - z\right) dF_b(B) \Bigg) dF_a(A) + \\
&+ \int_z^{\hat{y}} F_b \Bigg(\int_0^{B^H - s} F_a \left(w_b - (B^H - s)\right) dF_b(B) + \\
&+ \int_{B^H - s}^{\hat{y}} F_a \left(w_b - B\right) dF_b(B) \Bigg) dF_a(A).
\end{aligned}
$$

Suppose now $\min\{A^H, B^H\} > z$: the consumer will not inspect any nested box of which he does not already know the value of. Now, the relevant thresholds determining whether something is kept or not are $A^H - s$ and $B_H - s$: (i, i) is only kept if both $A_i > A^H - s$ and $B_i > B^H - s$. Otherwise, the highest between $A^H - s + B_i$ and $A_i + B^H - s$ is kept. Therefore:

$$
\begin{aligned}
H_{a,b}(w_{a,b}) = \int_0^{B^H - s} F_a \Bigg(\int_0^{A^H - B^H + B} & F_b \left(w_{a,b} - (A^H - s)\right) dF_a(A) + \\
&+ \int_{A^H - B^H + B}^{\hat{y}} F_b \left(w_{a,b} - (B^H - s)\right) dF_a(A) \Bigg) dF_b(B) + \\
&+ \int_{B^H - s}^{\hat{y}} F_a \Bigg(\int_0^{A^H - s} F_b \left(w_{a,b} - (A^H - s)\right) dF_a(A) + \\
&+ \int_{A^H - s}^{\hat{y}} F_b \left(w_{a,b} - A\right) dF_a(A) \Bigg) dF_b(B).
\end{aligned}
$$

When boxes are assumed to be locked in any of these configurations, the value function governing search is exactly the same as the one in the last paragraph since all boxes are independent. Then, we construct myopic reservation values:

$$
s = \int_{\underline{W}_\kappa}^{2\hat{y}} (w_\kappa - \underline{W}_\kappa) dH_\kappa(w_\kappa).,
$$

with $\kappa \in \{\{a\}, \{b\}, \{a, b\}\}$.

Let $\underline{W}^*_{a,b}(A^H, B^H)$ be the expected equivalent of costly opening the next box on the search path:

$$\begin{aligned}\underline{W}^*_{a,b}(A^H,B^H) = \underline{W}_{a,b}(A^H,B^H)\int_0^{B^H}\int_0^{A^H} & dF_a(A)dF_b(B)+\\ +\int_0^{B^H}\int_{A^H}^{\hat{y}} \underline{W}_{a,b}(A,B^H)\, & dF_a(A)dF_b(B)+\\ +\int_{B^H}^{\hat{y}}\int_0^{A^H} \underline{W}_{a,b}(A^H,B)\, & dF_a(A)dF_b(B)+\\ +\int_{B^H}^{\hat{y}}\int_{A^H}^{\hat{y}} \underline{W}_{a,b}(A,B)\, & dF_a(A)dF_b(B).\end{aligned}$$

Consider the choice of the consumer. If she chooses to open the next compound box, $X_{i+1,i+1}$, she knows that she will stop only if $w_{i+1,i+1} > \underline{W}_{a,b}(\max\{A_{i+1},A^H\},\max\{B_{i+1},B^H\})$. All future boxes will have this updated value. Suppose the consumer does open the box. The choice of opening the box after it will follow the same logic, with possibly new values of A^H, B^H. Notice that the value of this follow-up search is already incorporated in $\underline{W}^*_{a,b}(A^H,B^H)$, as it accounts for possible upward changes in the value of future boxes. Therefore, $\underline{W}^*_{a,b}(A^H,B^H)$ represents the value of inspecting the next box and following up optimally given the new information acquired with the new box, and fully capture the value of the search process from that point onward.

In the same way, for configuration 2 we write:

$$\begin{aligned}\underline{W}^*_a(A^H) =& \underline{W}_a(A^H)\int_0^{z+s}\int_0^{A^H} dF_a(A)dF_b(B)+\\ &+\int_0^{z+s}\int_{A^H}^{\hat{y}} \underline{W}_a(A)\, dF_a(A)dF_b(B)+\\ &+\int_{z+s}^{\hat{y}}\int_0^{A^H} \underline{W}_{a,b}(A^H,B)\, dF_a(A)dF_b(B)+\\ &+\int_{z+s}^{\hat{y}}\int_{A^H}^{\hat{y}} \underline{W}_{a,b}(A,B)\, dF_a(A)dF_b(B).\end{aligned}$$

An equivalent formulation can be found for configuration 3. Finally, for configuration 1, we write:

$$\begin{aligned}\underline{W}^* =& \underline{W}\int_0^{z+s}\int_0^{z+s} dF_a(A)dF_b(B)+\\ &+\int_0^{z+s}\int_{z+s}^{\hat{y}} \underline{W}_a(A)\, dF_a(A)dF_b(B)+\\ &+\int_{z+s}^{\hat{y}}\int_0^{z+s} \underline{W}_b(B)\, dF_a(A)dF_b(B)+\\ &+\int_{z+s}^{\hat{y}}\int_{z+s}^{\hat{y}} \underline{W}_{a,b}(A,B)\, dF_a(A)dF_b(B).\end{aligned}$$

Therefore:

$$\mathcal{W}(A^H, B^H) = \begin{cases} \underline{W}^* & \text{if} \quad \max\{A^H, B^H\} < z + s, \\ \underline{W}^*_a(A^H) & \text{if} \quad A^H > z + s > B^H, \\ \underline{W}^*_B(B^H) & \text{if} \quad B^H > z + s > A^H, \\ \underline{W}^*_{a,b}(A^H, B^H) & \text{if} \quad \min\{A^H, B^H\} > z + s. \end{cases}$$

represents the history dependent value of the search process from that point onward accounting for all possible search paths conditional on them arising during the search process itself.

B. Optimal Monopoly Pricing - Proof of Proposition 2

The proof of Proposition 2 comes in three steps. First, I show that if a non-uniform equilibrium price vector exists, it must be such that lower uniform prices are set for exactly one product characterizing all attributes, and higher uniform prices are set for all other products. Next, I show that all such price vectors are dominated by a uniform price vector. Finally, I show that the optimal uniform price vectore is the one as per Proposition 2.

Optimal differentiated price vector Suppose the seller wanted to set differential prices for his infinitely many products. First, it is obvious that at least one product must be priced differently than all others. For notational clarity, I define $p_1 < p_2 < p_3$ as a set of three price levels. I show that any optimal differential price vector must be such that a set of products sharing no attributes with each other must be priced at p_1 and all other products must be priced at either p_2 or p_3, but there cannot be any vector with more than two price levels.

First suppose that more than one product sharing an attribute A_i has price set at p_1. The geometry of the product space implies that there must be one attribute B_j for which the same applies. For example, if $(1, 1)$ and $(1, 2)$ were priced at p_1, $(1, 2)$ and $(2, 2)$ would also need to be. Then, the consumer would optimally start her search process from $(1, 2)$ because compound box $X_{1,2}$ contains the most cheap products. If the consumer then wanted to open a new compound box, she would optimally select $X_{3,3}$ and proceed along the diagonal.

If p_1 is such that the consumer would want to open $X_{1,2}$ but not $X_{3,3}$ without updating, the seller would have the incentive to set a lower p_1 to all products on the diagonal and increase the price of $(1, 2)$; on the other hand, if the consumer is willing to open $X_{3,3}$ without updating, then $p_{1,2} = p_1$ implies that with positive probability the consumer will choose to keep either A_1 or B_2 and purchase $(1, 1)$ or $(2, 2)$ at a lower price that he would have been willing to. Therefore, the seller would have the incentive to increase $p_{1,2}$ to re-establish the canonical order of search. This intuition extends to any number $n > 1$

of products for each attribute, and to all attributes. Therefore, at most one product per attribute can be optimally set to be cheaper than the others.

Suppose now that a strict subset of attributes has all associated products priced at either p_1 or p_2, while all other attributes follow the pricing detailed above. If the selected price is p_1, all products with such attributes are cheaper than all others, and are therefore more valuable to the consumer. If the consumer is willing to exhaust these products and still inspect the attributes with differentiated prices, with positive probability the seller sells at a lower price than the consumer was willing to pay. If the selected price is p_2, all such attributes would be pushed to the end of the search order and never reached because the consumer has infinite better alternatives available.[16]

Finally, suppose that exactly one product per attribute is priced at p_1 and all others are priced at either p_2 or p_3. Suppose first that a finite subset of attributes has products priced at either p_1 or p_2 and all other attributes have products priced at either p_1 or p_3. A consumer that optimally decides to start searching will search first the compound box or boxes in which the most cheap products can be found. If he is willing to keep searching the boxes until only the ones with the highest number of expensive products and stop without updating, having the latter group cannot be optimal, and all products should belong to the former group. If the consumer is still interested in searching, instead, all products should belong to the latter group.

Suppose now that all attributes are such that one product is priced at p_1, a finite subset of products is priced at p_2 and all others are priced at p_3. If the consumer optimally elected to keep an attribute after inspecting a product priced at p_1, she would select to inspect the ones priced at p_2 first. If after exhausting them she would stop, all other products should also have been priced at p_2. Otherwise, all products should have been priced at p_3. The result immediately extends to any number of price levels larger than two. The result follows.

Optimality of uniform prices Next, I show that for any vector of differential prices structured as above, there exist an uniform price vector that preserves probability of trade and returns strictly higher expected profit. As discussed in the main text (and detailed in the next part of the proof), probability of trade conditional on the consumer starting to search depends on the probability of finding realizations such that the resulting updating of unopened compound boxes makes the consumer stop searching and not purchase anything. The highest uniform price is such that:

$$\mathcal{W}(\mathbf{p^{unif}}) = \mathcal{W} - p^{unif} = 0, \quad \forall (i, j).$$

Where $\mathcal{W}$ is the initial value of inspecting a closed nested box net of prices. From the

[16]This implies that the pricing scheme with infinitely many products has infinitely many payoff-equivalent equilibria in which the seller sets a high price for all products defined by a finite subset of attributes which are never reached.

discussion above and from the proof of proposition 1, all updating to $\mathcal{W}$ is upward and, therefore, probability of trade in case of the highest uniform price is 1. Therefore, it must be shown that no pricing scheme with differential prices can generate a higher expected profit than $\mathcal{W}$.

To do so, it is sufficient to show that when differential prices are set, the value of inspecting any closed compound box is lower than with uniform prices. Suppose this is the case: the threshold above which a compound box is kept when locked in configuration 1 would then be lower. This cascades into a reduction of the overall value of searching and, therefore, reduces expected profits of the firm.

Notice that, as per the main text, without prices it holds:

$$w_{i,i} = \max\{A_i, B_i\} + \max\{z, \min\{A_i, B_i\}\},$$

when all products are priced uniformly, purchasing any of the products inside the compound box is equivalent. When they are not, instead, the price spread affects when consumers would keep searching instead of stopping at (i,i). Suppose $p_{i,i} = p$ and $p_{i,j\neq i} = p_{j\neq i,i} = p+\delta$ for some $\delta > 0$. Then, a consumer would elect to inspect nested boxes if $z - \delta > \min\{A_i, B_i\}$, because now the prices associated with the nested boxes is higher. Notice that this implies that $w_{i,i}$ when differential prices are set is equivalent to $w_{i,i}$ with uniform prices when search costs are higher, or:

$$\max\{A_i, B_i\} + \max\{z - \delta, \min\{A_i, B_i\}\} = \max\{A_i, B_i\} + \max\{z', \min\{A_i, B_i\}\},$$

where z' solves:

$$s' = \int_{z'}^{\hat{y}} (y - z')\, dF(y)$$

and because $z' = z - \delta$ and z is decreasing in s.

The result follows: since differential prices reduces the value of search when there are infinitely many attributes, this in turn means that any pricing vector with differential prices limits the value of search compared to an equivalent one with uniform prices. Therefore, any differential pricing vector that makes the consumer indifferent between searching or not (which makes it equivalent to $p^* = \mathcal{W}$) must generate lower expected profits than its equivalent counterpart.

Optimal uniform prices vector Finally, it must be shown that $p^* = \mathcal{W}$ is indeed optimal. To do so, it sufficient to show that $\mathcal{W}_{a,b}(z+s, z+s)$ is the lowest updated value a compound box can ever have and that $\mathcal{W}_{a,b}(z+s, z+s) \geq \mathcal{W}$.

For the former, recall that it holds:

$$\mathcal{W}_{a,b}(A^H,B^H)=\underline{W}_{a,b}(A^H,B^H)\int_0^{B^H}\int_0^{A^H} dF_a(A)dF_b(B)+ \\ +\int_0^{B^H}\int_{A^H}^{\hat{y}}\underline{W}_{a,b}(A,B^H)\,dF_a(A)dF_b(B)+\int_{B^H}^{\hat{y}}\int_0^{A^H}\underline{W}_{a,b}(A^H,B)\,dF_a(A)dF_b(B)+ \\ +\int_{B^H}^{\hat{y}}\int_{A^H}^{\hat{y}}\underline{W}_{a,b}(A,B)\,dF_a(A)dF_b(B)$$

$$\mathcal{W}_a(A^H)=\underline{W}_a(A^H)\int_0^{z+s}\int_0^{A^H} dF_a(A)dF_b(B)+\int_0^{z+s}\int_{A^H}^{\hat{y}}\underline{W}_a(A)\,dF_a(A)dF_b(B)+ \\ +\int_{z+s}^{\hat{y}}\int_0^{A^H}\mathcal{W}_{a,b}(A^H,B)\,dF_a(A)dF_b(B)+\int_{z+s}^{\hat{y}}\int_{A^H}^{\hat{y}}\mathcal{W}_{a,b}(A,B)\,dF_a(A)dF_b(B)$$

(which has an equivalent counterpart for $\mathcal{W}_b(B^H)$), and

$$\mathcal{W}=\underline{W}\int_0^{z+s}\int_0^{z+s} dF_a(A)dF_b(B)+\int_0^{z+s}\int_{z+s}^{\hat{y}}\mathcal{W}_a(A)\,dF_a(A)dF_b(B)+ \\ +\int_{z+s}^{\hat{y}}\int_0^{z+s}\mathcal{W}_b(B)\,dF_a(A)dF_b(B)+\int_{z+s}^{\hat{y}}\int_{z+s}^{\hat{y}}\mathcal{W}_{a,b}(A,B)\,dF_a(A)dF_b(B),$$

and notice that if $A^H=z+s$ (equivalently, $B^H=z+s$), it holds:

$$H_{a,b}=H_a(=H_b)=H,$$

$$\underline{W}_{a,b}(z+s,z+s)=\underline{W}_a(z+s)(=\underline{W}_b(z+s))=\underline{W}.$$

and that all are weakly increasing in A^H and/or B^H, with $\underline{W}$ being constant in both and the other being strictly increasing in either or both.

Therefore:

$$\mathcal{W}_{a,b}(z+s,z+s)=\underline{W}_{a,b}(z+s,z+s)\int_0^{z+s}\int_0^{z+s} dF_a(A)dF_b(B)+ \\ +\int_0^{z+s}\int_{z+s}^{\hat{y}}\underline{W}_a(A)\,dF_a(A)dF_b(B)+ \\ +\int_{z+s}^{\hat{y}}\int_0^{z+s}\underline{W}_b(B)\,dF_a(A)dF_b(B)+\int_{z+s}^{\hat{y}}\int_{z+s}^{\hat{y}}\underline{W}_{a,b}(A,B)\,dF_a(A)dF_b(B)= \\ =\underline{W}\int_0^{z+s}\int_0^{z+s} dF_a(A)dF_b(B)+\int_0^{z+s}\int_{z+s}^{\hat{y}}\underline{W}_a(A)\,dF_a(A)dF_b(B)+ \\ +\int_{z+s}^{\hat{y}}\int_0^{z+s}\underline{W}_b(B)\,dF_a(A)dF_b(B)+\int_{z+s}^{\hat{y}}\int_{z+s}^{\hat{y}}\underline{W}_{a,b}(A,B)\,dF_a(A)dF_b(B)=\mathcal{W},$$

which proves the result.

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