The Macroeconomics of Demographic Change

Essays on Economic Modelling

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To Wiebke

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1 Introduction

In the vast majority of countries populations are aging, and demographic change will continue well into the 21^{st} century. Demographic change is characterized by falling fertility rates and rising life-expectancies which leads to an increased share of the elderly population and a corresponding decreased share of the young population. This process of population aging will affect societies in various ways. In industrialized countries, the probably most obvious consequence of population aging is the pressure for reform of social security systems which is mirrored in the lively policy debate especially within European countries. While obvious, this is by no means the only consequence.

From an economy wide perspective demographic change will affect virtually all markets, e.g., consumption goods markets, labor markets and capital markets. To give some examples: As a consequence of changes in the age structure, society's aggregate consumption bundle is projected to change (Lührmann 2005). This will lead to changes in sectoral demand and will therefore have feedback effects on factor markets, especially on labor markets (Börsch-Supan 2003). Labor markets will be affected by demographic change in at least two more ways: first, to the extent that overall productivity of workers of different age groups differ, it will require adjustments in the organization of firms (Skirbekk 2004). Second, labor may become a scarce factor relative to capital. The relative price of labor, measured as the sum of gross wages and non-wage labor costs may therefore increase (Cutler, Poterba, Sheiner, and Summers 1990). At the same time, however, net wages are likely to decline as a consequence of increases in social security contribution rates.

While this list of possible effects of demographic change on factor markets is by far not exhaustive (Börsch-Supan 2004), this thesis focuses on the capital market effects of demographic change and aims at quantifying them. As mentioned, capital is predicted to be an abundant factor relative to labor in aging societies since the aggregate labor force is projected to decline. Loosely speaking, there is already a number of equipment and machines in existence that will not be entirely destroyed (depreciated) over the coming years, but who shall use them? This, in other words, implies that capital is predicted to be worth relatively less in the future than it is today and therefore that the rate of return to capital may decline. Several articles in the popular press and popular books have attributed rises in stock market prices in the 1990s to the size of the *baby boom* cohorts

(Dent 1993). These high-fertility rate cohorts will retire in the next thirty years, and finance their consumption by withdrawing financial savings from capital markets. Since future younger cohorts are much smaller in size - due to the *baby bust* -, the future composite effect of asset demands is projected to lead to a decrease in the demand for capital and therefore to a decrease in its price. With these effects in mind, what will happen to asset markets when the baby boom generation retires? Is an "asset meltdown", a substantial decrease in asset values, ahead? The academic literature has not reached a consensus on this question (Abel 2001; Poterba 2001; Brooks 2002a).

Such capital market effects are common to most industrialized countries since virtually all are affected by the impact of demographic change. However, extent and timing of demographic change differ substantially across countries, even within the group of industrialized countries. As the above sketch of the capital market consequences of demographic change suggests, differential aging will generate differences in saving rates, investment rates, and rates of return to capital across countries. To the extent that capital is internationally mobile, population aging will therefore induce capital flows between countries that arbitrage away international return differentials (Cutler, Poterba, Sheiner, and Summers 1990; Lucas 1992). These capital flows will modify the effects of population aging and pension reform in each county vis-à-vis a world of closed economies. How large will these capital flows be? Results in Börsch-Supan (1996), Feroli (2002), Henriksen (2002), Brooks (2003) and Domeij and Floden (2004) suggest that they may be quite large. Would international capital flows from demographically old countries to demographically young countries with higher returns to capital mitigate the decrease in asset prices in demographically older economies?

How will this situation change if countries reform their social security systems? European countries such as France, Germany and Italy have not yet, or so far only partially, reformed their pension systems. Reform proposals or actual reforms that already passed legislation all aim at shifts from so-called pay-as-you-go (PAYG) pension systems that dominate in most countries to (partially) funded pension systems. The design of PAYG pension systems is such that the contributions to the pension system by the current work force are used to pay the pensions of the current pensioners. In the other extreme, i.e., in a fully funded pension system, retirement savings of the current work force are invested and retirees receive their own savings including interest as retirement income. Future demographic change puts PAYG financed pension systems under severe pressure because the number of contributors shrinks relative to the number of pensioners. In order to keep replacement rates, defined here as the average net pension income of current pensioners relative to the average net labor income of current workers, at current levels, contribution rates would have to increase. But since the projected increase in the pensioner to workers ratio over the next 30 to 50 years is immense, this is an unfavorable option since it implies not only strong decreases in net wage incomes of the current work force and therefore low or even negative implicit returns on pension savings, but also strong increases of non-wage labor costs because roughly half of social security contributions are paid by employers. By now, the majority of academic researchers and policy makers believes that a shift towards more pre-funding is a more favorable option.

If public transfers of retirement income decrease, households have to make up for the resulting pension gap by own savings. This could lead to crowding-out of other forms of saving and in the limit, in the case of perfect crowding out, no additional savings would be created. Suppose that crowding-out is imperfect, what will then be the capital market consequences? It leads to a higher demand for assets which may drive up their price, at least temporarily. At the same time, these assets may be used productively and hence the factor capital may become even more abundant relative to labor. This is predicted to further decrease the rate of return to capital beyond the pure effects induced by demographic change. The effect may be smaller if capital is internationally mobile, since part of the additional retirement savings may be invested abroad. In addition, households could react to this decrease in pension income by retiring later and (or) by working more hours. Such an increase in labor supply would work in the opposite direction than the increase in asset demand just described, because it decreases the price of labor, the wage rate, relative to the rate of return to capital.

In order to quantify these effects a model will be developed that can be used to simulate the economic consequences of projected demographic change and of fundamental pension reforms. A particularly useful tool to model the macroeconomic implications of demographic change and to analyze the questions outlined above is the so-called Auerbach-Kotlikoff Overlapping Generations (AK-OLG) model (Samuelson 1958; Diamond 1965; Auerbach, Kotlikoff, and Skinner 1983; Auerbach and Kotlikoff 1987). The name "overlapping generations" stems from the model feature that in each period the model represents many different households of different age. From one period to the next, older households pass away and newborns enter the model and hence generations *overlap*, as with real world demographic developments. In the earlier model versions, researchers used stylized demographic profiles. A feature common to newer developments in the literature, such as the model used here, is the use of actual demographic data and, as in the particular application of this thesis, for a group of different countries (Bommier and Lee 2003; INGENUE 2001; Fehr, Jokisch, and Kotlikoff 2004).

This thesis covers three papers, Ludwig (2004a), Ludwig (2005) and Börsch-Supan, Ludwig, and Winter (2004), all dealing with different aspects of such a large scale AK-OLG model. The key features of the AK-OLG simulation model are described in Chapter

2. Three economic sectors are modelled in each country. Final goods are produced using capital goods and labor as input factors in the firm sector. Consumption of these goods takes place in the household sector. Households also supply capital and labor as inputs to the firm sector. The particular feature of an overlapping generations model is that it is *micro-founded*. This means, that, as in the real world, the aggregate behavior of the household sector is derived from the individual behavior of many different households. In the present context, households differ with respect to their age (and their date of birth). Finally, there is a government sector. In addition to these national markets within each country, economies are linked by an international capital market.

In order to quantify the effects of future demographic change, the model is used to simulate time series of aggregate data, such as aggregate production, asset demand, labor supply and the rate of return to capital. Since demographic change is a slow moving process that continues well into the future, these time series must be computed over a long forecasting horizon. This and the model feature that generations overlap make it not an easy task to solve the model. What solving the model requires is to determine *equilibrium* quantities and prices on all markets for all time periods such that markets clear. Hence, one hast to determine, e.g., equilibrium wage rates such that labor supply equals labor demand in each period. Chapter 3, based on Ludwig (2004a), presents a particular solution method of the OLG model by improving and extending upon other methods used in the literature.

While any economic model is an abstract description of reality, Chapter 4, based on Ludwig (2005), asks the question how good the model actually is in matching selected features of real world data. Modelling an economy gives a researcher at least two degrees of freedom for choice. One is on functional forms, i.e., on the description on how firms and households behave. These are regarded as fixed and given from the model description in Chapter 2. The other is on parameters that determine the values of these functions, i.e., parameters that, in the end, describe (the amount of) aggregate demand and supply, respectively. Values of particular model parameters are chosen such that the model exactly matches some dimensions of the data. This selection procedure for model parameters is referred to as *calibration* in the literature and Chapter 4 develops a new calibration procedure for large-scale OLG models. With parameter values at hand, the performance of the model is then evaluated in other dimensions than those used to determine these parameter values. This exercise provides important information for future model developments.

In Chapter 5, based on Börsch-Supan, Ludwig, and Winter (2004), the model is finally applied to the questions raised above. It is used to assess the impact of demographic change in France, Germany and Italy as a particular group of countries most severely affected by population aging. In addition, all three are countries that are about to reform their largely PAYG financed public pension systems. These countries are embedded in the global economy by being linked through an international capital market. It will be shown that the open economy setup significantly alters results relative to the counterfactual assumption of closed economies. It is also shown that substantial international capital flows are induced between countries through the impact of differential demographic change. Further, flexible labor markets will be shown to play an important role.

Finally, Chapter 6 concludes with a summary of results and a discussion of some selected modelling aspects.

2 A Multi-Country Simulation Model

This chapter describes the features of the macroeconomic simulation model used throughout all chapters of this thesis. The description is based on Börsch-Supan, Ludwig, and Winter (2004). While a much simpler version than the one introduced here will be used in Chapter 3, the richness of the full model will be particularly relevant for the questions addressed in Chapters 4 and 5.

2.1 Introduction

This Chapter presents a dynamic macroeconomic model that allows for an analysis of the macroeconomic effects of population aging and of shifts from PAYG pension systems to (partially) funded pension systems, induced by the pressure of population aging on public pension budgets. The dynamic macroeconomic model used is based on a version of a standard large-scale Overlapping Generations (OLG) Model (Samuelson 1958; Diamond 1965) introduced by Auerbach and Kotlikoff (1987, Chapter 3). The model described and applied in this thesis has been developed in a series of papers. An early closed economy version of the model is described in Börsch-Supan, Heiss, Ludwig, and Winter (2003) and a corresponding open economy version of the model was presented by Börsch-Supan, Ludwig, and Winter (2002). In this thesis, the most recent version of the simulation model is used. The description of the key model features is based on Börsch-Supan, Ludwig, and Winter (2004).

Overlapping generations models have been used extensively to study the effects of population aging on social security systems, a purpose for which they are well suited since they are based on households' and firms' optimal reactions to demographic change and public policy measures. Kotlikoff (1998) provides a review of the earlier literature. The work in Börsch-Supan, Ludwig, and Winter (2002) and Börsch-Supan, Ludwig, and Winter (2004) are among the main contributions to two recent developments in the literature: Models have, on the one hand, been augmented with realistic demographic profiles and, on the other hand, existing closed economy models have been extended to multi-country simulation models.

The international dimension was introduced to the OLG literature through Buiter's (1981) seminal contribution. Buiter studies the impact of differential discount rates on capital formation, international capital flows, and welfare in a stylized two-country model. In his model, differences of the representative households' preferences, represented by different discount rates across the two countries, drive differences in capital formation. This results in international capital flows if the two economies are linked by an international market. The low-discount rate country, i.e. the country with a relatively low preference for early consumption, features higher savings and therefore runs a current-account surplus (in the steady state of the model). A similar analysis can be conducted if countries differ with regard to output growth rates (Frenkel, Razin, and Yuen 1996; Obstfeld and Rogoff 1998).

One of the first assessments of the effects of future demographic developments on macroeconomic outcomes using quantitative simulation models has been conducted by Cutler, Poterba, Sheiner, and Summers (1990). They focus on the United States and use a Barro-Ramsey-Cass-Koopmans type representative agent model (Ramsey 1928; Cass 1965; Koopmans 1965; Barro 1974) and augment it with data on projected labor supply shares as a summary statistic to capture future demographic change. Large-scale Auerbach-Kotlikoff models do however allow a much more detailed representation of demographic developments in a large set of countries.

Therefore, several authors have more recently moved away from the stylized two-country-two-generations setup used by Buiter (1981) and simultaneously developed large-scale multi-country OLG models to quantify the effects of population aging and pension reform on international capital flows (Attanasio and Violante 2000; INGENUE 2001; Börsch-Supan, Ludwig, and Winter 2002; Brooks 2003). While Attanasio and Violante (2000) focus on how the Latin American demographic transition affects international capital markets, Brooks (2003) examines capital flows in a multi-regional OLG model. Similar analyses has been conducted by Feroli (2002), Henriksen (2002), and Domeij and Floden (2004).

The model used here improves upon the above mentioned literature along a number of dimensions. While some of these models share the one or the other feature, the model used here is much more precisely formulated along all these dimensions. First, the model is at an annual frequency which allows for a very detailed and realistic description of macroeconomic dynamics. Second, realistic demographic data enter the model which, due to the annual frequency, enables a careful distinction between the effects of population aging and population shrinkage and the resulting effects on macroeconomic aggregates across countries. Third, a PAYG pension system is explicitly modelled and the issue of pension reform with its associated changes in saving patterns and its implications for international capital flows is addressed. Fourth, not only final goods consumption but also consumption of leisure enters household's objective functions. This opens an additional channel to households to react to the differential effects of demographic change and of fundamental pension reforms. Finally, adjustment costs to capital formation are considered and hence the impact of aging and pension reforms on the price of capital can be analyzed (Abel 2001).

Similar analyses on the "triangular" relationship between demographic change, fundamental pension reform and international capital markets are contained in INGENUE (2001) and Fehr, Jokisch, and Kotlikoff (2003, 2004). While the model of the INGENUE team does not share a number of the above mentioned features, the models of Fehr, Jokisch, and Kotlikoff are similar. However, significant differences exist with regard to calibration (Chapter 4) and with regard to the economic questions being addressed (Chapter 5).

2.2 The Model

The model has three building blocks: a demographic projection, a stylized pension system, and a macroeconomic overlapping generations model which generates the general equilibrium of internationally linked economies. Each of these building blocks are described in the following sub-sections.

2.2.1 The Demographic Model

Detailed demographic projections form the background of the analysis. Demography is taken as exogenous and represents the main driving force of the simulation model.[1] In each country i, the size of population of age j in period t, $N_{t,j,i}$, is given recursively by

$$N_{t,j,i} = \begin{cases} \sum_{j=15}^{50} f_{t-1,j,i} N_{t-1,j,i} & \text{for } j = 0 \\ N_{t-1,j-1,i}(s_{t-1,j-1,i} + m_{t-1,j-1,i}) & \text{for } j > 0, \end{cases}$$

where $s_{t,j,i}$ denotes the age-specific conditional survival rate, $m_{t,j,i}$ the net migration ratio, and $f_{t,j,i}$ the age-specific fertility rate. Demographic projections are based on the assumptions underlying the United Nations' demographic projections (United Nations 2002). Population data are given at an annual frequency for the period 1950-2050 for age-groups of five. Further input data such as age-group specific mortality rates, life expectancy, and aggregate migration is only given at quinquennial frequency. Population data are interpolated between age groups and time intervals. The "backfit" of the population model to

[1] Assuming exogenous demographic processes is of course a simplifying assumption since, in the long run, neither fertility nor mortality and of course not migration is exogenous to economic growth.

the UN population data for the time period 1950-2050 shows that the interpolated population data match well with the actual data (results not shown). Furthermore, population is forecasted beyond the UN horizon of 2050 and demographic processes are assumed to stabilize after year 2200 by assuming constant mortality and fertility rates.

Individuals in the model economies enter economic life at the age of 20 which is denoted by $a = 1$. The maximum age as implied by the demographic projections is 104 years. Accordingly the maximum economic age, denoted by Z, is 85. To simplify calculations of the economic model, it is assumed that all migration takes place at the initial age of 20. This simplifying assumption allows to treat all "newborns" - immigrants and natives - in the economic model alike, see below.[2]

2.2.2 The Pension Model

Each region i is assumed to have a two-tier pension system. The first tier represents a conventional public pay-as-you-go (PAYG) system characterized by a country-specific contribution and replacement rate. More precisely, for each region i, the exogenous policy variable is the time-specific gross replacement rate, $\gamma_{t,i}$, defined as the ratio of average gross pension to average gross wage income at time t. The budget of the PAYG pension system is balanced at any time t and determines the contribution rate, $\tau_{t,i}$, by

$$\tau_{t,i} \sum_{a=1}^{Z} w^g_{t,a,i} l^d_{t,a,i} N_{t,a,i} = \sum_{a=1}^{Z} p_{t,a,i}(1 - l^d_{t,a,i}) N_{t,a,i}, \tag{2.1}$$

where pension benefits $p_{t,a,i}$ of a household of age a in time period t are calculated by

$$p_{t,a,i} = \gamma_{t,i} \lambda_{t,a,i} w^g_{t,a,i} \tag{2.2}$$

On the revenue side, $w^g_{t,a,i}$ denotes age-specific gross wages. Net wages are given by $w^n_{t,a,i} = w^g_{t,a,i}(1 - 0.5\tau_{t,i})$ under the assumption that half of contributions are paid by the employee and the other half by the employer. This latter half will be taken into account when firms maximize profits. $l^d_{t,a,i}$ denotes labor supply resulting from optimal household decisions. The use of superscript d will be explained below.

On the benefit side of the budget equation, pensions are defined by the general replacement rate and by a "point system" that credits $\lambda_{t,a,i}$ times the gross wage earned at age a. This is an approximation to the actual computation of pension benefits. Benefits are

[2]Both groups, newborns and immigrants, enter the economic model with zero assets. Furthermore, there are no skill differences between the two groups as analyzed by, e.g., Razin and Sadka (1999) and Storesletten (2000).

not taxed and interactions with other social protection systems are ignored. It is further assumed that all persons in each region participate in the same pension system.

In order to solve the pension system equation 2.1 for each country, net replacement rates are assumed constant over time at current levels. The associated path of the contribution rate is then endogenously calculated taking data on the pension systems replacement rates as given.

The second tier of the stylized pension system represents pre-funded private pensions. This funded component is not modeled explicitly. Rather, it consists of voluntary private savings resulting from the households' optimal life-cycle decisions.

2.2.3 The Overlapping Generations Model

The two core elements of the macroeconomic general equilibrium model are the production and the household sector. They are presented separately here, although they are linked through several channels, in particular through the household's labor supply and savings decisions. The production sector in each country consists of a representative firm that uses a CES production function given by

$$Y_{t,i} = F(\Omega_i, K_{t,i}, L_{t,i}) = \Omega_i \left(\alpha_i K_{t,i}^{-\theta_i} + (1-\alpha_i) L_{t,i}^{-\theta_i}\right)^{-1/\theta_i}, \tag{2.3}$$

where $K_{t,i}$ denotes the capital stock and $L_{t,i}$ the labor supply of country i at time t.[3] Labor supply is measured in efficiency units and α_i denotes the capital share. The elasticity of substitution between the factors of production, capital and labor, is given by $\zeta_i = 1/(1+\theta_i)$.

Production efficiency of a household of age a at time t in country i has a factorial structure with three elements, relating to age, time and country. On the micro level, where households are distinguished by their age, labor productivity changes over the life-cycle according to age-specific productivity parameters ϵ_a. Hence, the age-specific gross wage is $w^g_{t,a,i}\epsilon_a$ and the aggregate labor supply is $L_{t,i} = \sum_{a=1}^{Z} \epsilon_a l_{t,a,i} N_{t,a,i}$, where $l_{t,a,i}$ denotes a single household's labor supply. Second, aggregate and individual labor supply ($L_{t,i}$ and $l_{t,a,i}$) are measured in efficiency units relative to a time endowment $E_{t,i}$. The actual age specific labor supply which corresponds to what is observed in the data is therefore given by $L^d_{t,a,i} = l_{t,a,i}N_{t,a,i}/E_{t,i}$. Superscript d is used to denote "$d$etrended" effective labor supply, see also equation 2.1. The time endowment grows over time at a constant rate of g_i. This "growth in time endowment" specification is equivalent to the standard labor augmenting

[3] While estimation of the elasticity of substitution between capital and labor however results in a coefficient close to one (Chapter 4), which suggests to use the simpler case of a Cobb-Douglas production function, the more general CES production function is relevant for the analysis in Chapter 3.

technological change specification for the production sector and has useful properties for the specification of the household sector, see below. Third, Ω_i is the technology level of country i.

Investment is assumed to be subject to convex adjustment costs with a proportionality factor ψ_i (Hayashi 1982). The dynamic problem of the firm is given by

$$\max_{\{K_{t+1,i}\}_{t=1}^{\infty},\{L_{t,i}\}_{t=1}^{\infty},\{I_{t,i}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} d_{t,i}^{f} \left[F(\Omega_i, K_{t,i}, L_{t,i}) - I_{t,i} - C(I_{t,i}) - w_{t,i}^{g}(1+\tau_{t,i}/2)L_{t,i}\right] \tag{2.4}$$

s.t.

$$C(I_{t,i}, K_{t,i}) = \frac{\psi}{2}\frac{I_{t,i}^2}{K_{t,i}}$$

$$I_{t,i} = K_{t+1,i} - K_{t,i}(1-\delta_i)$$

$K_{1,i}$ given

where $d_{t,i}^{f}$ is the firm's discount factor defined as $d_{t,i}^{f} = \prod_{s=1}^{t}(1+r_s)^{-1}$ and δ_i is the rate of depreciation of capital. The adjustment cost formulation of $C(I_{t,i}, K_{t,i})$ features the standard quadratic term, and the term $1/(1+\tau_{t,i}/2)$ raises gross wages $w_{t,i}^{g}$ by employer's contribution to social security.

The first order conditions resulting from profit maximization give the following expressions for equilibrium wages and interest rates and for the equilibrium price of capital:

$$w_{t,i}^{g}(1+0.5\tau_{t,i}) = \frac{1-\alpha_i}{\Omega_i^{\theta_i}}\left(\frac{Y_{t,i}}{L_{t,i}}\right)^{1+\theta_i}, \tag{2.5}$$

$$q_{t,i} = 1 + \psi_i \frac{I_{t,i}}{K_{t,i}}, \tag{2.6}$$

and

$$r_{t,i} = \frac{\frac{\alpha_i}{\Omega_i^{\theta_i}}\left(\frac{Y_{t,i}}{K_{t,i}}\right)^{1+\theta_i} + (1-\delta_i)\Delta q_{t,i} + \frac{\psi_i}{2}\left(\frac{I_{t,i}}{K_{t,i}}\right)^2}{q_{t-1,i}} - \delta_i. \tag{2.7}$$

The variable $q_{t,i}$ in equation 2.6 denotes the Lagrangian factor of gross investment, the total marginal costs of investment, which, in this formulation, also equals Tobin's q (Tobin 1969; Hayashi 1982). Equation 2.7 is the familiar arbitrage condition for the rate of return on financial and physical investment: The return on financial investment, $r_{t,i}$, must be equal to the return on one unit of physical investment at a price of $q_{t-1,i}$ in each country. The latter equals the marginal product of capital plus capital gains on non-depreciated capital plus the reduction in marginal adjustment costs minus depreciation. If $\psi_i = 0$, i.e.

if there are no adjustment costs to capital, then equation 2.7 reduces to the standard static condition $r_{t,i} = \alpha_i / \Omega_i^{\theta_i} (Y_{t,i}/K_{t,i})^{1+\theta_i} - \delta_i$, compare Chapters 3 and 4.

In order to determine aggregate consumption, savings and wealth, optimal household behavior derived from intertemporal utility maximization is considered next. By choosing an optimal consumption path, each cohort born in time period t maximizes at any point in time $t+a$ and age a the sum of discounted future utility. The within-period utility function exhibits constant relative risk aversion, and preferences are additive and separable over time. Cohort t's maximization problem at $a = 1$ is given by

$$\max_{\{C_{t,a,i}, l_{t,a,i}\}_{a=1}^{Z}} U = \sum_{a=1}^{Z} \beta_i^{a-1} \pi_{t,a,i} U\left(C_{t,a,i}, l_{t,a,i}\right), \tag{2.8}$$

where β_i is the pure time discount factor. In addition to pure discounting, households discount future utility with their unconditional survival probability, $\pi_{t,a,i} = \prod_{j=1}^{a} s_{t,j-1,i}$. $C_{t,a,i}$ denotes consumption and $l_{t,a,i}$ labor supply of the household. Remember that the latter is measured in efficiency units relative to time endowment, $E_{t,i}$, which increases over time. It is assumed that the period specific utility function is of the standard CES form given by

$$U\left(C_{t,a,i}, l_{t,a,i}\right) = \frac{1}{1-\sigma_i}\left(\left(\left[\phi_{a,i} C_{t,a,i}^{-\gamma_i} + (1-\phi_{a,i})(E_{t+a} - l_{t,a,i})^{-\gamma_i}\right]^{-\frac{1}{\gamma_i}}\right)^{1-\sigma_i} - 1\right).$$

σ_i is the coefficient of relative risk aversion, $\phi_{a,i}$ the consumption share parameter, i.e. the weight of consumption relative to leisure in household's utility and $\xi_i = 1/(1+\gamma_i)$ denotes the intra-temporal substitution elasticity between consumption and leisure.

Consumption share parameters, $\phi_{a,i}$, vary across country and age. The functional form of $\phi_{a,i}$ in each country is given by

$$\phi_{a,i} = \begin{cases} \overline{\phi}_i & \text{for } a \leq A^l \\ \phi_{a,i} = \overline{\phi}_i - \Delta\phi_i(a - A^l) & \text{for } A^l < a < A^h \\ \phi_{a,i} = \underline{\phi}_i = \overline{\phi}_i - \Delta\phi_i(A^h - A^l) & \text{for } a \geq A^h, \end{cases} \tag{2.9}$$

i.e., the consumption share parameter is assumed to be constant for ages $a \leq A^l$, then linearly decreases and is assumed to remain constant again for ages $a > A^h$. This parsimonious functional form is used in Chapter 5 to capture the strong decreases of labor supply shares observed in the data for ages above $A^l = 54$.

A complication arises because households face the risk of prematurely dying with positive wealth. For simplification, the assumption of perfect annuity markets is made which implies that accidental bequests are distributed implicitly, as in the life-insurance framework by

Yaari (1965). As shown in Chapter 5, differences in simulation outcomes between models with accidental bequests and perfect insurance are small on the aggregate level.

Denoting total wealth by $A_{t,a,i}$, maximization of the household's intertemporal utility is subject to a dynamic budget constraint given by

$$A_{t,a+1,i} = \frac{1}{s_{t,a,i}} \left(A_{t,a,i}(1 + r_{t+a,i}) + l_{t,a,i} w^n_{t,a,i} + (E_{t+a} - l_{t,a,i}) p_{t,a,i} - C_{t,a,i} \right) \tag{2.10}$$

The term $1/s_{t,a,i}$ reflects how the accidental bequests, resulting from total savings at the end of the period, are dissipated through the annuity market.[4] Income consists of asset income, net wages, and pensions. Note that we do not distinguish explicitly between workers and pensioners - each cohort consists of one representative household. Therefore, each representative household receives some pension income, $p_{t,a,i}$, because a fraction of the corresponding cohort is already retired. This fraction increases as the cohort ages, until the legal retirement age. This process is governed by the parameter $\lambda_{t,a,i}$ as defined in equation 2.2.

The corresponding present value budget constraint is given by

$$\sum_{a=1}^{Z} \pi_{t,a,i} y^n_{t,a,i} \prod_{j=1}^{a} (1 + r_{t+j-1,i})^{-1} - \sum_{a=1}^{Z} \pi_{t,a,i} C_{t,a,i} \prod_{j=1}^{a} (1 + r_{t+j-1,i})^{-1} \geq 0, \tag{2.11}$$

where the short hand notation $y^n_{t,a,i} = l_{t,a,i} w^n_{t,a,i} + (E_{t+a} - l_{t,a,i}) p_{t,a,i}$ is adopted to denote non-asset net income. Furthermore, maximization is subject to the constraint that leisure may not exceed time endowment (and may not be negative)

$$0 \leq l_{t,a,i} \leq E_{t+a}. \tag{2.12}$$

The solution to the intertemporal optimization problem is characterized by two first-order conditions. First, the inter-temporal Euler equation describes the consumption growth rate of each household given by

$$C_{t,a+1,i} = C_{t,a,i} \left((1 + r_{t+a+1,i}) \beta_i \frac{v_{t,a+1,i}}{v_{t,a,i}} \right)^{1/\sigma_i}, \tag{2.13}$$

where $v_{t,a,i} = (\phi_i + (1-\phi_i) lcr_{t,a,i}^{-\gamma_i})^{-(1+\gamma-\sigma)/\gamma}$. $lcr_{t,a,i}$ is the leisure-consumption ratio defined by the intra-temporal Euler equation which relates current period consumption to current period leisure choice by

$$E_{t+a} - l_{t,a,i} = \left(\frac{1-\phi_i}{\phi_i} \frac{1}{w^n_{t,a,i} + \mu_{t,a,i} - p_{t,a,i}} \right)^{1/(1+\gamma_i)} C_{t,a,i} = lcr_{t,a,i} C_{t,a,i}, \tag{2.14}$$

[4] The timing convention is as in Rios-Rull (1996, 2001).

where $\mu_{t,a,i} \geq 0$ is the shadow value of leisure.

As Auerbach and Kotlikoff (1987, p. 35) point out, this specification results in a trending steady state labor force participation if technological progress affects the technology level Ω_i and if the elasticity of substitution between consumption and leisure is not equal one, i.e., if $\gamma_i \neq 0$. Altig et al. (2001) avoid this problem by assuming a "growth in time endowment" specification: technological change affects the time endowment of households rather than the technology level of the economy. In their specification, each cohort is endowed with more time than the previous one but time endowment is constant across the life-cycle of each individual cohort. Across the life-cycle, technological change is implemented by growth in life-cycle wages across age. The specification chosen here differs in that it is assumed that $E_{t,i}$ measures efficiency of all cohorts which increases according to $E_{t+1,i} = E_{t,i}(1+g_i)$ such that households get more efficient as time passes by. Accordingly, the wage profile is assumed to be flat across the life-cycle (apart from the effect due to age-specific productivity, which results in the familiar hump-shaped profile). For the production sector, this specification is the standard labor augmenting technological change specification. For the household sector, it is not only labor that gets more efficient but also leisure, hence households get more efficient in using their time.

For given factor prices (i.e., wages and interest rates), shadow wage rates and the parameters of the public pension system (i.e., contribution and replacement rates), the life-time consumption paths of all generations can be computed using the Euler equations 2.13 and 2.14 and the budget constraints.

The dynamic general equilibrium of the model economy is defined sequentially.[5]

Definition 1: A competitive equilibrium of the economy is defined as a sequence of disaggregate variables, $\{C_{t,a,i}, l_{t,a,i}, A_{t,a,i}\}$, aggregate variables, $\{C_{t,i}, L_{t,i}, K_{t,i}\}$, wage rates, $\{w_{t,i}\}$ in each country i and a common world interest rate, $\{r_t\}$ such that

- The allocations are feasible, i.e.

$$
\begin{aligned}
Y_{t,i} + r_t F_{t,i} &= S^n_{t,i} + C_{t,i} + D_{t,i} = S^g_{t,i} + C_{t,i} \\
&= \sum_{a=1}^{Z} ((s_{t-a,a,i} A_{t+1-a,a+1,i} - A_{t-a,a,i}) N_{t,a,i} + C_{t-a,a,i} N_{t,a,i}) + \\
&+ \left(\delta_i - (1-\delta_i) \frac{\Delta q_{t,i}}{q_{t-1,i}} \right) q_{t-1,i} K_{t,i},
\end{aligned}
$$

[5] The definition of equilibrium as sequential coincides with the computational solution method, see Chapter 3. It can be numerically computed since the model economy converges to a steady state and becomes a well-behaved system with a small number of equations.

where $F_{t,i}$ is the amount of foreign assets and $D_{t,i}$ is depreciation of capital and $S^n_{t,i}(S^g_{t,i})$ is net (gross) savings.

- Factor prices equal their marginal productivities as given in equations 2.5 and 2.7.
- Firms and households behave optimally, i.e., firms maximize profits subject to the constraints given in equation 2.4 and households maximize life-time utility given in equation 2.8 subject to the constraints in equations 2.10 through 2.12.
- All markets clear. Market clearing on national markets requires that

$$S^n_{t,i} = \sum_{a=1}^{Z} S^n_{t-a,a,i} N_{t-a,a,i}, \quad C_{t,i} = \sum_{a=1}^{Z} C_{t-a,a,i} N_{t-a,a,i}$$
$$A_{t,i} = \sum_{a=1}^{Z} A_{t-a,a,i} N_{t-a,a,i} \quad L_{t,i} = \sum_{a=1}^{Z} l_{t-a,a,i} N_{t-a,a,i}.$$

Market clearing on the international capital market and the assumption of perfect capital mobility across regions requires that the rate of return to capital is equalized across all countries,

$$r_{t,i} = r_t, \tag{2.15}$$

and that the sum of all foreign assets, defined as the difference between home assets and the home capital stock, $F_{t,i} = A_{t,i} - q_{t-1,i} K_{t,i}$, across all world regions equals zero, i.e.,

$$\sum_{i=1}^{R} F_{t,i} = 0,$$

where R is the total number of regions.

Hence, in equilibrium, world output is equal to

$$Y_t = \sum_{i=1}^{R} Y_{t,i} = \sum_{i=1}^{R} S^n_{t,i} + C_{t,i} + D_{t,i}$$

and international capital flows are defined by the difference between gross savings and investment,

$$CA_{t,i} = S^g_{t,i} - q_{t,i} I_{t,i},$$

where $q_{t,i} I_{t,i}$ is physical investment valued in terms of consumption units which, in turn, is given by

$$q_{t,i} I_{t,i} = q_{t,i} \left(K_{t+1,i} - (1-\delta_i) K_{t,i} \right) = q_{t,i} K_{t+1,i} - q_{t-1,i} K_{t,i} + \left(\delta_i - (1-\delta_i) \frac{\Delta q_{t,i}}{q_{t-1,i}} \right) q_{t-1,i} K_{t,i}.$$

The last term on the right-hand side of the above equation reflects depreciation net of capital gains.[6]

The time line of the model is as described in Börsch-Supan, Ludwig, and Winter (2004) and has four periods: a phase-in period, a calibration period (1960-2003), a projection period (2004-2100), and a phase-out period. First, calculations start 10 years before the calibration period begins. At the start of this albeit short phase-in period, households start from initial conditions that are constructed such that households born before 1950 behave as households of their age-class already living in 1950. This is similar to imposing a steady state, but not exactly identical since the condition is only imposed for one year. The time between 1960 and 2003 is used as the sample period for estimation of structural model parameters. Projections run from 2004 through 2100. The phase-out period after 2100 has two parts: a transition to a steady population, and an additional 100-year period towards a steady state of the economic model. Mortality rates are assumed constant beyond 2100. Furthermore, fertility rates are assumed to adjust during the period from 2100 until 2200 such that the total number of newborns is constant each year which implies that stable populations are reached in 2200. Finally, simulations run for an additional 100 years until the model reaches a final steady state in 2300.

[6] Throughout, implicit use of the simplifying assumption that all migration is concentrated at age a=1 was made. Since initial wealth at $a = 1$ is assumed zero, transfers of assets due to migration does not need to be taken into account.

3 Solution Method

This chapter describes the solution method of the model introduced in the previous chapter and is based on Ludwig (2004a). Solving the model requires determination of equilibrium sequences of aggregate and disaggregate variables. Standard block Gauss-Seidel iterations used by tatonnement methods for solving large scale deterministic heterogeneous agent models as the OLG model of Chapter 2 are modified. The composite method between *first-* and *second-order tatonnement* methods developed here is shown to considerably improve convergence both in terms of speed as well as robustness relative to conventional first-order tatonnement methods. In addition, the relative advantage of the modified algorithm increases in the size and complexity of the economic model. Therefore, the algorithm allows significant reductions in computational time when solving large models. The algorithm is particularly attractive since it is easy to implement - it only augments conventional and intuitive tatonnement iterations with standard numerical methods.[1]

3.1 Introduction

In this Chapter Gauss-Seidel iterations used to solve large-scale deterministic heterogeneous agent models are modified. Such models are increasingly used for the analysis of economic questions. The model introduced in the previous chapter is an example. Standard procedures use domain truncation methods and resort to general methods for solving large systems of (nonlinear) equations. Three types of such conventional solution methods can be distinguished: (i) Newton based methods such as the L-B-J method[2], (ii) the

[1]I thank Alan Auerbach, Axel Börsch-Supan, Wouter Den Haan, Ken Judd, Michel Juillard, Michael Reiter, Gabriele Steidl and Joachim Winter as well as several seminar participants at the University of Mannheim, at the 2004 SED Annual Meeting in Florence and the 2004 EEA Annual Meeting in Madrid for helpful comments. Financial support from the Deutsche Forschungsgemeinschaft, SFB 504, at the University of Mannheim and from the Gesamtverband der Deutschen Versicherungswirtschaft is gratefully acknowledged.

[2]See Laffargue (1990), Boucekkine (1995), Juillard (1996) and Juillard, Laxton, McAdam, and Pioro (1998).

Fair-Taylor (extended path) method[3] and (iii) tatonnement methods[4], see Judd, Kubler, and Schmedders (2000). These conventional methods have in common that they solve the model for each time t element of all endogenous variables.[5]

The analysis here is concerned with traditional methods. Conventional *first-order tatonnement* methods are commonly used to solve large-scale overlapping generations (OLG) models. While more general applications of the algorithm developed in this chapter are possible, the analysis focuses on a standard OLG model as introduced in Chapter 2. As a composite of first- and second-order tatonnement methods, the algorithm developed here is a straightforward modification of such conventional methods. The analysis shows that this hybrid method greatly improves convergence relative to standard first-order methods.

While L-B-J and Fair-Taylor methods regard any perfect foresight general equilibrium model simply as a system of (non-linear) equations including aggregate and disaggregate variables and iterate over this entire system, tatonnement methods break variables into *aggregate* and *disaggregate* variables. Outer loops then proceed via block Gauss-Seidel algorithms using aggregate variables only, whereas inner loops are used to solve for disaggregate variables in a (separate) disaggregate model. Outer loops work as follows: Let $P = S^{-1}(Q)$ denote a sequence of factor prices corresponding to sequences of factor supplies Q, where S^{-1} denotes the inverse supply function. Equilibrium of tatonnement methods is defined as a fixed point, $Q = D(S^{-1}(Q))$, where D denotes the demand function. $S^{-1}(Q)$ and $D(P)$ are solved by inner loops of the disaggregate model and by aggregating individual decisions. The fixed point problem suggests to execute the iteration $Q^{k+1} = D(P^{k+1}) = D(S^{-1}(Q^k))$, which is the familiar *hog-cycle* process, where k is the iteration number.[6] Depending on the functional form of S relative to D such iterations may however not converge. These convergence problems force researchers to rely on *ad hoc* dampening factors such that the iteration rewrites as $Q^{k+1} = Q^k - w(Q^k - D(S^{-1}(Q^k))) = Q^k - w(Q^k - \tilde{Q}^k)$, where w is the dampening factor, the relative weight w attached to Q^k and $\tilde{Q}^k$ respectively.[7]

Such modifications of standard Gauss-Seidel iterations have also been referred to as *fast* Gauss-Seidel (FGS) iterations (Hughes Hallet 1984).[8] Since only values of $D(S^{-1}(Q^k))$ and no additional information on the functional form of D, respectively S, are used to solve

[3]See Fair and Taylor (1983).

[4]See Auerbach and Kotlikoff (1987).

[5]More recently, Judd (2002) has proposed an alternative route. Rather than explicitely solving for each time t element, Judd suggests to use prior information about the time path of the endogenous variables and to approximate it by a functional form with a low-dimensional parameter vector. Judd's method can be regarded as a more modern approach.

[6]Since P^{k+1} and not P^k is used to form an update of Q^{k+1} the iterations performed are non-linear Block-Gauss-Seidel iterations.

[7]Note that dampening factors play a similar role as adaptive expectations in the familiar cobweb model.

[8]For convergent problems, w may also be set such as to accelerate convergence.

the fixed point problem these methods belong to the class of first-order iterative methods. While intuitive, convergence of these methods is slow (linear at best) and they may not converge at all even after various dampening factors have been tried out. As an alternative to using *ad hoc* dampening factors, *optimal* dampening factors can be determined. However, they are difficult to determine even for linear models, see, e.g., Hagemann and Young (1981) and Judd (1999). Therefore, various adaptive techniques to update dampening factors as the iteration proceeds have been suggested in the literature (Hagemann and Young 1981; Hughes Hallet 1982).

As an alternative to first-order iterations, *second-order tatonnnement* methods may be used. Fixed point problems such as $Q = D(S^{-1}(Q))$ can be transformed to a root-finding problem which suggests to iterate as $Q^{k+1} = Q^k - [J(Q^k)]^{-1}(Q^k - D(S^{-1}(Q^k))) = Q^k - [J(Q^k)]^{-1}G(Q^k)$ where $G(Q^k)$ is a system of simultaneous non-linear equations, Q^k is the root of these equations and $J(Q^k)$ is the Jacobi matrix. Such systems may be solved using standard non-linear equation solvers, see, e.g., Feroli (2002) and Domeij and Floden (2004) for applications in an OLG context using relatively simple models. Since the dimension of the Jacobi matrix is $mT \times mT$, second-order methods become costly in terms of running time and memory as the dimension of T or m, and therefore the complexity of the economic model, increases.

Against this background, this analysis suggests to use a composite of standard first-order iterations and second-order methods by combining Gauss-Seidel iterations with Quasi-Newton methods[9]. The algorithm will therefore by referred to as Gauss-Seidel-Quasi-Newton method (GSQN). By economic insight the dimension of the Jacobi matrix is reduced for the system $G(Q^k)$ of non-linear equations characterizing steady state situations. Since certain transformations of economic variables in Q (and P) are constant in the steady state of economic models, the exact Jacobi matrix is shown to be given by $J = W^{-1} \otimes I$ where W is of dimension $m \times m$. Since m is generally quite small - for a standard one sector closed economy general equilibrium growth model with endogenous capital formation and endogenous labor supply m equals 2 - the Jacobi matrix can easily be determined by standard finite difference methods in fast steady state iterations. For transition iterations, the matrix is used as an approximate Jacobi matrix and updated by Broyden's method as the iterations proceeds. Accordingly, the matrix W may be interpreted as an approximate Jacobi matrix or as a matrix of multiple dampening factors (Hughes Hallet 1984). The attractiveness of GSQN stems from its simplicity: the intuitive appeal and relatively low computational demands of tatonnement iterations are combined with standard Newton based methods that are implementable at little extra cost.

[9]An extensive treatment of similar methods can be found in Ortega and Rheinboldt (2000).

As an illustration of the GSQN procedure, two economic models are used. The first is a simple static hog-cycle model that is only used to shed light on the strong *economic* restrictions implicit in one-parameter fixed dampening. The second model is a simplified version of the large-scale dynamic multi-country overlapping generations (OLG) model of Chapter 2. It is used for simulations to compare the relative performance of the fast Gauss-Seidel algorithm (FGS) with the Gauss-Seidel-Quasi-Newton (GSQN) algorithm under various combinations of structural model parameters. In addition, the dimension m is increased from $m = 1$ (closed economy model with exogenous labor supply) to $m = 4$ (three country model with endogenous labor supply). Previewing results, the simple modifications suggested here quite considerably improve convergence when compared to standard FGS. For the latter, only relatively low values of the dampening factor such as $w = 0.1$ lead to convergence for all cases considered. For higher values of w, robustness of FGS is found to decrease sharply: for $w = 0.3$, FGS does not converge for up to 40 percent of cases. In contrast, GSQN converges for all these simulations. For transition calculations, average convergence speeds of GSQN are about two times higher than those of FGS with $w = 0.1$ when $m = 1$ and about *seven* times higher when $m = 4$. Hence, GSQN considerably improves convergence both in terms of speed and in terms of robustness relative to standard FGS. The increase of the relative advantage of GSQN relative to FGS as m increases is due to the fact that the restrictions on the true Jacobi matrix of the system of equations $G(Q^k)$ imply constant (and equal) elements along the diagonal and off-diagonal elements to be equal to zero. As the dimension m increases, the loss of information implied by these restrictions becomes more and more costly. Therefore, GSQN is of particular advantage for large and therefore more complex models.

The remainder of this Chapter proceeds as follows: Section 3.2 provides some general definitions and a brief review of tatonnement methods. Section 3.3 develops the suggested modification of the conventional Gauss-Seidel algorithm, GSQN. Section 3.4 contains the above mentioned economic example used to illustrate GSQN and its differences to FGS. Differences of the OLG model relative to Chapter 2 are also described. Section 3.5 compares the relative performances of FGS and GSQN for the stylized OLG model. Section 3.6 concludes.

3.2 Tatonnement Methods

Let $Y = \{y_i\}_{i=1}^{n}$ where $y_i = \{y_{i,t}\}_{t=0}^{T}\ \forall i$ be a list of all endogenous variables of the economic model. For example, y_i includes wage rates and interest rates as aggregate variables (a_i) as well as disaggregate variables (b_i) such as consumption and assets of individual households, etc. Note that $b_i = \{\{b_{i,e,t}\}_{e=0}^{E_i}\}_{t=0}^{T}$ where the number of disaggregate units e may differ across i. Collect $A = (a_1, a_2, ...)$ and $B = (b_1, b_2, ...)$. For further reference, split A as $A = (Q, P)$ where Q are aggregate factor supply variables such as the aggregate capital stock and aggregate labor supply of an economy and P are the associated factor price variables such as aggregate interest and wage rates and let $Q = (q_1, ..., q_m) \in \mathbb{R}^{mT}$ as well as $P = (p_1, ..., p_m) \in \mathbb{R}^{mT}$, compare Section 3.1. Further, let $Z = (z_1, z_2, ...)$ be a list of exogenous variables such as population data of cohorts living at time t. Note that some z_i may be disaggregate variables as well. Deterministic perfect foresight heterogeneous agent models can be written in a general form as

$$\begin{aligned} F(Y, Z) &= 0 \\ y_{i,0} = \bar{y}_{i,0},\ i = 0, 1, ..., n_i,\ n_i &< n \\ y_{i,t} \text{ bounded for all } i&, \end{aligned} \tag{3.1}$$

where $F(Y, Z)$ are nT equations of non-linear functions that represent equilibrium. Since Z are exogenous they will be dropped from here on. The equations in (3.1) include Euler equations, asset accumulation equations, market clearing conditions as well as any other equations that define equilibrium. Domain truncation has been applied in equation (3.1) since the time horizon starts in period $t = 0$ departing from some initial conditions and is restricted to T.

Solution methods such as Fair-Taylor and L-B-J directly solve systems of equations such as (3.1) for each element in $y_{i,t}$ by Gauss-Seidel iterations or Newton based methods respectively. In contrast, tatonnement methods break the system of equations in (3.1) into a factor *supply* and a factor *demand* model. Both require inner loops to solve and to aggregate individual decision problems.

A perfect foresight OLG model of the form given in equation (3.1) can be re-written as

$$\begin{aligned} \textit{Supply model: } P &= S^{-1}(Q) \\ \textit{Demand model: } Q &= D(P) \\ \textit{Aggregation: } P = S^{-1}(Q) &= \Sigma^s(B^s(Q)) \\ \text{and } Q = D(P) &= \Sigma^d(B^d(P)), \end{aligned} \tag{3.2}$$

where S^{-1} is the inverse aggregate supply function and D is the aggregate demand function. B^s (B^d) are supply (demand) side disaggregate variables and $B = (B^s, B^d)$. The aggregators, Σ^d and Σ^s, are only used to indicate that aggregate demand and supply functions are derived from individual decisions of heterogeneous agents and will be ignored from here on.

Combining the first two lines of equation (3.2) leads to the definition of equilibrium of a heterogeneous agent model as a fixed point given by

$$Q = D(S^{-1}(Q)), \tag{3.3}$$

where $Q - D(S^{-1}(Q))$ are m equations of non-linear functions.

The fixed-point equation in (3.3) suggests to use standard (block) Gauss-Seidel iterations to solve for Q and hence to iterate over the system[10]

$$\begin{aligned} P^{k+1} &= S^{-1}(Q^k) \\ Q^{k+1} &= D(P^{k+1}). \end{aligned}$$

This simple form ignores updates of disaggregate variables. A more general case will be discussed in Section 3.3.4.

The above equation system can be more concisely written as a Gauss-Seidel fixed point iteration

$$Q^{k+1} = D(S^{-1}(Q^k)). \tag{3.4}$$

It is well-known that such iterations may not converge. Therefore, a dampening factor may be applied. Gauss-Seidel iterations with one-parameter fixed dampening with factor $0 < w < 1$ iterate on

$$Q^{k+1} = Q^k - w\left(Q^k - D(S^{-1}(Q^k))\right), \tag{3.5}$$

compare Auerbach and Kotlikoff (1987, pp. 46-50). In case the fixed-point iteration in equation (3.4) is convergent, w may be used to accelerate convergence in which case $w > 1$. But even if such fixed-point iterations converge, convergence is slow and linear at best.

An alternative to fixed point iterations is to transform equation (3.3) into a root-finding problem as

$$G(Q) = Q - H(Q) = Q - D(S^{-1}(Q)) = 0,$$

where $H(\cdot)$ is introduced as a shorthand notation for $D(S^{-1})(\cdot)$.

Applying a first-order Taylor series approximation to equation (3.2) leads to the familiar Newton updating formula of Q given by

$$Q^{k+1} = Q^k - J^{-1}[Q^k]G(Q^k), \tag{3.6}$$

where $J[Q^k]$ is the Jacobi matrix of the system of equations in (3.2) evaluated at Q^k. Recently, several authors have used general purpose root-finding methods to solve such problems in the OLG context, e.g., Feroli (2002) and Domeij and Floden (2004) for relatively simple models. However, as the complexity of the economic model and therefore

[10] The - generally less efficient - (block) Gauss-Jacobi method may be used as an alternative in which case $P^{k+1} = S^{-1}(Q^k)$ and $Q^{k+1} = D(P^k)$. Hence, rather than using P^{k+1} resulting from the first block, Gauss-Jacobi uses P^k resulting from previous iterations to form an update of Q in the second block.

the dimension of m and T increases, such methods become costly (despite sparsity of the Jacobi matrix, see below).

Rewriting equation (3.5) as

$$Q^{k+1} = Q^k - w\left(Q^k - D(S^{-1}(Q^k))\right) = Q^k - wI_{(mT\times mT)}G(Q^k)$$

then makes it obvious that Gauss-Seidel iterations with one-parameter fixed dampening restrict the elements of the true Jacobi matrix $J[Q^k]$ to $w^{-1}I_{(mT\times mT)}$. These restrictions may be summarized as follows: first, the iteration matrix is constant across all iteration steps k, second, elements along the diagonal are restricted to be equal and third, off-diagonal elements are restricted to zero. An economic interpretation of such restrictions for a stylized hog-cycle model is given below in Section 3.4.1.

3.3 The Gauss-Seidel-Quasi-Newton Method

Here, an alternative to pure first- or second-order tatonnement methods is suggested by reducing the dimension of the Jacobi matrix in equation (3.6). For further reference and in order to highlight the restrictions implied by standard first-order methods, it will be useful to derive explicit expressions for the elements of the Jacobi matrix. Recall that $Q = \{q_i\}_{i=1}^m$, where $q_i = \{q_{i,t}\}_{t=1}^T$. Due to the specific form of the functions $G = \{g_i(Q)\}_{i=1}^m$ where $g_i(Q) = \{g_{i,t}(Q)\}_{t=0}^T$ in equation (3.2), the elements of the Jacobi matrix given by

$$J[Q^k] = \begin{bmatrix}
\frac{\partial g_{1,0}(Q^k)}{\partial q_{1,0}^k} & \frac{\partial g_{1,0}(Q^k)}{\partial q_{1,1}^k} & \dots & \frac{\partial g_{1,0}(Q^k)}{\partial q_{1,T}^k} & \frac{\partial g_{1,0}(Q^k)}{\partial q_{2,0}^k} & \frac{\partial g_{1,0}(Q^k)}{\partial q_{2,1}^k} & \dots & \frac{\partial g_{1,0}(Q^k)}{\partial q_{2,T}^k} & \dots \\
\frac{\partial g_{1,1}(Q^k)}{\partial q_{1,0}^k} & \frac{\partial g_{1,1}(Q^k)}{\partial q_{1,1}^k} & \dots & \frac{\partial g_{1,1}(Q^k)}{\partial q_{1,T}^k} & \frac{\partial g_{1,1}(Q^k)}{\partial q_{2,0}^k} & \frac{\partial g_{1,1}(Q^k)}{\partial q_{2,1}^k} & \dots & \frac{\partial g_{1,1}(Q^k)}{\partial q_{2,T}^k} & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\frac{\partial g_{2,0}(Q^k)}{\partial q_{1,0}^k} & \frac{\partial g_{2,0}(Q^k)}{\partial q_{1,1}^k} & \dots & \frac{\partial g_{2,0}(Q^k)}{\partial q_{1,T}^k} & \frac{\partial g_{2,0}(Q^k)}{\partial q_{2,0}^k} & \frac{\partial g_{1,0}(Q^k)}{\partial q_{2,1}^k} & \dots & \frac{\partial g_{2,0}(Q^k)}{\partial q_{2,T}^k} & \dots \\
\frac{\partial g_{2,1}(Q^k)}{\partial q_{1,0}^k} & \frac{\partial g_{2,1}(Q^k)}{\partial q_{1,1}^k} & \dots & \frac{\partial g_{2,1}(Q^k)}{\partial q_{1,T}^k} & \frac{\partial g_{2,1}(Q^k)}{\partial q_{2,0}^k} & \frac{\partial g_{1,1}(Q^k)}{\partial q_{2,1}^k} & \dots & \frac{\partial g_{2,1}(Q^k)}{\partial q_{2,T}^k} & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{bmatrix}$$

can be re-written as

$$\begin{bmatrix}
1 - \frac{\partial h_{1,0}(Q^k)}{\partial q_{1,0}^k} & -\frac{\partial h_{1,0}(Q^k)}{\partial q_{1,1}^k} & \dots & -\frac{\partial h_{1,0}(Q^k)}{\partial q_{1,T}^k} & -\frac{\partial h_{1,0}(Q^k)}{\partial q_{2,0}^k} & -\frac{\partial h_{1,0}(Q^k)}{\partial q_{2,1}^k} & \dots \\
-\frac{\partial h_{1,1}(Q^k)}{\partial q_{1,0}^k} & 1 - \frac{\partial h_{1,1}(Q^k)}{\partial q_{1,1}^k} & \dots & -\frac{\partial h_{1,1}(Q^k)}{\partial q_{1,T}^k} & -\frac{\partial h_{1,1}(Q^k)}{\partial q_{2,0}^k} & -\frac{\partial h_{1,1}(Q^k)}{\partial q_{2,1}^k} & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
-\frac{\partial h_{2,0}(Q^k)}{\partial q_{1,0}^k} & -\frac{\partial h_{2,0}(Q^k)}{\partial q_{1,1}^k} & \dots & -\frac{\partial h_{2,0}(Q^k)}{\partial q_{1,T}^k} & 1 - \frac{\partial h_{2,0}(Q^k)}{\partial q_{2,0}^k} & -\frac{\partial h_{1,0}(Q^k)}{\partial q_{2,1}^k} & \dots \\
-\frac{\partial h_{2,1}(Q^k)}{\partial q_{1,0}^k} & -\frac{\partial h_{2,1}(Q^k)}{\partial q_{1,1}^k} & \dots & -\frac{\partial h_{2,1}(Q^k)}{\partial q_{1,T}^k} & -\frac{\partial h_{2,1}(Q^k)}{\partial q_{2,0}^k} & 1 - \frac{\partial h_{1,1}(Q^k)}{\partial q_{2,1}^k} & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{bmatrix}$$

and may be partitioned as

$$\begin{bmatrix} J^k_{1,1} & J^k_{1,2} & \dots & J^k_{1,m} \\ J^k_{2,1} & J^k_{2,2} & \dots & J^k_{2,m} \\ \dots & \dots & \dots & \dots \\ J^k_{m,1} & J^k_{m,2} & \dots & J^k_{m,m} \end{bmatrix} \tag{3.7}$$

according to all the endogenous variables q_i. Hence, each sub-matrix $J_{i,j}$, for $i, j = 1, ..., m$ is of dimension $T \times T$ with each element given by

$$\begin{aligned} J_{i,j,t,\Delta_t} &= \begin{cases} 1 - \frac{\partial h_{i,t}(Q^k)}{\partial q^k_{j,t+\Delta_t}} & \text{for } \Delta_t = 0 \text{ and } i = j \\ -\frac{\partial h_{i,t}(Q^k)}{\partial q^k_{j,t+\Delta_t}} & \text{else} \end{cases} \\ \text{for} &\quad t = 0, ..., T, \\ \text{where} &\quad -t \leq \Delta_t \leq T - t \end{aligned} \tag{3.8}$$

For a heterogeneous agent model with finite life-times of each individual agent, $-\frac{\partial h_{i,t}(Q^k)}{\partial q^k_{j,t+\Delta_t}} = 0$ for Δ_t sufficiently large. Hence $J[G(Q^k)]$ is sparse. Despite, it is generally quite costly to determine all non-zero elements of the Jacobi matrix $J[G(Q^k)]$ as T (and m) become large.

3.3.1 The Steady State

Suppose now that variables in Q are transformed such that they are constant in the steady state. E.g., q_1 could be a time series of the capital to output ratio and q_2 of the labor supply ratio in a closed economy growth model with endogenous labor supply ($m = 2$). Further, note that domain truncation imposes a restriction on the equation system which is mirrored by a Jacobi matrix of finite dimension and hence by the restriction on Δ_t in equation (3.8) requiring that $-t \leq \Delta_t \leq T - t$. This restriction is invalid if the economy is in steady state. For such a model the restriction on Δ_t is $-T_0 - t \leq \Delta_t \leq T_0 - t$, where $T_0 \leq T$, since, as noted above, $-\frac{\partial h_{i,t}(Q^k)}{\partial q^k_{j,t+\Delta_t}}$ equals zero for Δ_t sufficiently large. Further, since the elements of each $\{q_i\}_{i=1}^m$ are constant in the steady state, the partial derivatives in equation (3.8) will be constant across time as well. The corresponding representation of the elements of the actual Jacobi matrix in equation (3.8) is given by

$$J^{T_0}_{i,j,\Delta_t} = \begin{cases} 1 - \frac{\partial h_i(Q^k)}{\partial q^k_{j,\Delta_t}} & \text{for } \Delta_t = 0 \text{ and } i = j \\ -\frac{\partial h_i(Q^k)}{\partial q^k_{j,\Delta_t}} & \text{else} \end{cases} \quad \text{where } -T_0 - t \leq \Delta_t \leq T_0 - t, \tag{3.9}$$

which only depends on Δ_t and not on the time period t itself.

Therefore each of the m^2 different sub-matrices of the Jacobi matrix defined in equation (3.7) can be written as

(3.10)
$$J_{i,j}^{T_0} = \left(\sum_{\Delta_t=-T_0-t}^{T_0-t} D_{\Delta_t} - \frac{\partial h_i(Q^k)}{\partial q_{j,\Delta_t}^k} \right) \cdot I_{T\times T}, \text{where } \begin{cases} D_{\Delta_t} = 1 & \text{for } \Delta_t = 0 \text{ and } i = j \\ D_{\Delta_t} = 0 & \text{else} \end{cases}.$$

For steady state situations of the economic model the exact Jacobi matrix is accordingly given by

$$\text{(3.11)}\quad \hat{J} = [W^{-1}]_{(m\times m)} \otimes I_{(T\times T)} = \begin{bmatrix} \omega_{1,1} I_{(T\times T)} & \omega_{1,2} I_{(T\times T)} & \dots & \omega_{1,m} I_{(T\times T)} \\ \omega_{2,1} I_{(T\times T)} & \omega_{2,2} I_{(T\times T)} & \dots & \omega_{2,m} I_{(T\times T)} \\ \dots & & & \\ \omega_{m,1} I_{(T\times T)} & \omega_{m,2} I_{(T\times T)} & \dots & \omega_{m,m} I_{(T\times T)} \end{bmatrix}.$$

This structure of the Jacobi matrix is very different from a scaled identity matrix and considerably relaxes the restrictions imposed by standard Gauss-Seidel iterations. Note that m is generally small and hence W is of low dimension.

To summarize: The Jacobi matrix of a root-finding problem of an economic model as represented in equation (3.6) is generally quite large. For example, for a closed economy model with endogenous capital formation and endogenous labor supply ($m = 2$) that is solved for $T = 300$ years - a standard time horizon for OLG models solved at an annual frequency -, the Jacobi matrix consists of $(mT)^2 = 360,000$ elements.[11] However, in the steady state of the model and if the elements in Q are defined such that they are constant, the actual Jacobi matrix reduces to the Kronecker product of the low-dimensional W^{-1}-matrix and an identity matrix. Hence, for the above example, the exact Jacobi matrix effectively consists of only $m^2 = 4$ elements. This Jacobi matrix can easily be determined by standard finite difference methods in the first tatonnement iteration and can be updated by Broyden's method as the iteration proceeds, see Section 3.3.3 below. Hence, the final Jacobi matrix derived in steady state iterations, $J^{*,ss}$ is asymptotically optimal and convergence will be super-linear (Press, Teukolsky, Vetterling, and Flannery 1992, Chapter 9).

3.3.2 The Transition

Since T is quite large, transition calculations may take considerable time to compute. Against this background, the idea behind the implementation of GSQN for transition

[11] The full Jacobi matrix of the seven-country model with endogenous labor supply and adjustment costs to capital formation ($m = 7$) solved for $T = 350$ years used in Börsch-Supan, Ludwig, and Winter (2004) has 24,010,000 elements.

calculations is to use the Jacobi matrix derived during (fast) steady state calculations as an initial approximate Jacobi matrix for transition calculations and to update it by Broyden's method as the iteration proceeds, see Section 3.3.3 below.[12] The exact implementation of the algorithm during transition calculations depends on the restrictions on the structure of the equation system imposed by (initial and final) steady states or (and) arbitrary initial conditions. Four different models can be distinguished:

- **Model 1**: The economy starts from an initial steady state and converges to a final steady state. The final steady state has been calculated.
- **Model 2**: The economy starts from an initial steady state and converges to a final steady state. The initial steady state has been calculated.
- **Model 3**: The economy starts from an initial steady state and converges to a final steady state. Both steady states have been calculated.
- **Model 4**: The economy starts from arbitrary initial conditions and converges to a final steady state. The initial conditions are known and the final steady state has been calculated.

Permanent structural changes are implicit in the definitions of all models. However, for temporary changes, the economy starts from the same steady state as it converges to and hence such a specification is nested in model 3.

In terms of equations the four different models can be written as follows. For ease of presentation it is assumed that $m = 1$. Recall that the variables in Q are transformed such that they are constant in the steady state.

- **Model 1**:

$$\begin{aligned} q_0 &= q_1 \\ q_1 &= h_1(Q) \\ q_2 &= h_2(Q) \\ &\dots \\ q_T &= q^{fss}, \end{aligned}$$

  where fss stands for final steady state.

[12]Note that applying different dampening factors for different time periods t is not reasonable since it would create artificial kinks in the time paths of Q.

- **Model 2**:

$$\begin{aligned} q_0 &= q^{iss} \\ q_1 &= h_1(Q) \\ q_2 &= h_2(Q) \\ &\dots \\ q_T &= q_{T-1} \end{aligned}$$

where iss stands for initial steady state.

- **Model 3**:

$$\begin{aligned} q_0 &= q^{iss} \\ q_1 &= h_1(Q) \\ q_2 &= h_2(Q) \\ &\dots \\ q_T &= q^{fss} \end{aligned}$$

- **Model 4**:

$$\begin{aligned} q_0 &= \bar{q}_0 \\ q_1 &= h_1(Q) \\ q_2 &= h_2(Q) \\ &\dots \\ q_T &= q^{fss} \end{aligned}$$

For Models 1 to 2 it is assumed that the final (or initial) steady state is calculated during the transition solution while the initial (or final) steady state is already known from steady state calculations. The GSQN Jacobi matrix derived during final (initial) steady state calculations, J^{fss} (J^{iss}), is then used as initial Jacobi matrix and updated by Broyden's method using the information contained in $Q^k_{i,t^{ss}} = \{q_{i,t^{ss}}\}_{i=1}^m$ and $G(Q^k_{i,t^{ss}}) = \{g(q_{i,t^{ss}})\}_{i=1}^m$ where $t^{ss} = 1$ ($t^{ss} = T$), i.e., the information contained in the initial (final) steady state period (compare Section 3.3.3 below).[13] For Model 3 it is assumed that both steady states were calculated during steady state calculations. GSQN is then implemented by using the

[13]Updating J^k by Broyden's method is not necessary, but using the additional information contained in each iteration step k is more efficient than using a constant approximate Jacobi matrix throughout.

iteration matrix derived during these steady state calculations (either initial or final), J^{ss}, throughout all transition iterations. The procedure for Model 4 is equivalent to Model 3.

To summarize: While the Jacobi matrix determined by the suggested method is asymptotically optimal as Q^k approaches Q^{ss} for steady state calculations, it is a good approximation for transition calculations. The matrix W may therefore be interpreted either as an approximate Jacobi matrix or as an $m \times m$ matrix of multiple dampening factors that vary with the iteration number k (Hughes Hallet 1984).

3.3.3 Implementation of Gauss-Seidel-Quasi-Newton Iterations

This section summarizes the implementation steps of GSQN. It thereby makes explicit that an application of GSQN just requires to augment intuitive tatonnement iterations with standard and well-established numerical methods.

It is well-known that if $G(Q)$ is continuously differentiable over a convex set D containing the equilibrium values Q^* with $G(Q^*) = 0$, then there exists an open set C about Q^* such that equation (3.6) converges at least linearly from any $Q^0 \in C$. If in addition the Lipschitz condition $\|Q^k - Q^*\| \leq \; d|Q^{k-1} - Q^*\|$ holds for $Q^0 \in C$ and some $d > 0$, the rate of convergence becomes quadratic. However, if the starting values Q^0 are not within C, then Newton iterations such as equation (3.6) may not be convergent. In order to obtain an iteration scheme that converges for almost any starting value, it is therefore reasonable to augment the Newton iteration by a line search method to get

$$Q^{k+1} = Q^k - s^k \hat{J}^{-1}[Q^k] G(Q^k), \tag{3.12}$$

where s^k is a standard variable step-size parameter and $\hat{J}$ is the GSQN (approximate) Jacobi matrix.

Recall that $\hat{J}^{-1}[Q^k] = W^{-1}_{(m \times m)}[Q^k] \otimes I_{(T \times T)}$. A fast algorithm for line searches is by backtracking, see e.g., Press et al. (1992). It relies on a quadratic approximation of the (unknown) objective function given by $g(Q^k) = \frac{1}{2} G(Q^k)' G(Q^k)$ and determines a step that minimizes this quadratic approximation. If the resulting step is not acceptable, then the algorithm iterates over a cubic approximation of the objective function until an acceptable step is found.

However, since $\hat{J}^k$ is not the exact Jacobian, it is not guaranteed that the line search algorithm will give a descent step direction. Hence, the Jacobian will be re-initialized (by finite difference methods) in case the line search algorithm does not return a suitable step (after a maximum of only three line search iterations or when reaching a minimum value for s^k). For transition calculations, both line search algorithm and even more re-initializing the Jacobian can be costly in terms of computational time. Therefore, restarts of iterations reset the Jacobi matrix to the initial Jacobi matrix if line searches fail during transition iterations.

Moreover, it will be useful to re-initialize the Jacobian if the updated Jacobian $\hat{J}^k$ fails to satisfy two conditions: (i) if $\hat{J}^k$ is ill conditioned[14] and (ii) if some of the elements of $\hat{J}^k$ do not satisfy certain criteria reflecting prior knowledge regarding their value. E.g., for the applications considered in Section 3.5, it is required that the diagonal elements of J^k are positive. Condition (i) is standard and condition (ii) would automatically be fixed in the next iteration step by the methods just described (it would result in a divergent process and hence the Jacobian would be re-computed in the next iteration step). Making use of prior information is therefore not necessary but may save iteration steps.

While the application of Broyden's method is well-established, it is useful to more concisely summarize the GSQN algorithm as follows:

1. Chose some initial value Q^0 and a stopping criterion ϵ. For steady state calculations, Q^0 consists of time series of any - but reasonable - constant values and for transition calculations $Q^0 = Q^{*,s}$, i.e., the equilibrium values from steady state calculations (or other constant or non-constant values, e.g., obtained during previous transition calculations).

2. Initialize the Jacobian, $\hat{J}^0 = [W^{-1}]^0 \otimes I$. Use finite difference methods for steady state calculations and $\hat{J}^0 = \hat{J}^{*,s}$ for transition calculations, i.e. the last approximate Jacobi matrix of steady state iterations (or any other initial matrix such as a scaled identity matrix).

3. For iteration k, determine Q^{k+1} by

$$Q^{k+1} = Q^k - s^k \hat{J}^{-1}[Q^k]G(Q^k), \text{ for } s^k = 1$$

   and evaluate $G(Q^k)$ as well as

$$g(Q^k) = \frac{1}{2}G(Q^k)'G(Q^k)$$

   - If $g(Q^k) < g(Q^{k-1})$ continue with step 4, else start a line-search algorithm. Use a standard backtracking algorithm for line search that stops if $g(Q^k) < g(Q^{k-1})$, if $s^k = s^{\min}$ or a maximum number of line search iterations of only three is reached. A good choice for s^{min} is 0.1, see Press et al. (1992) for details.

[14]For models where the Jacobian is ill-conditioned at equilibrium, J^k would not be further updated in case Q^k approaches Q^*. In case iterations are divergent, J^k would only be scaled by line search methods.

- If the line search algorithm is successful then continue with step 4, else re-initialize $\hat{J}^k$ by finite difference methods in steady state iterations and by setting $\hat{J}^k = \hat{J}^0$, re-evaluate $G(Q^k)$ as well as $g(Q^k)$ and continue with step 4.

4. If $\max(||G(Q^k)/Q^k||) < \epsilon$[15] then stop and report success, else if $\Delta Q^k > \eta$, where η is some small number, determine $[\hat{W}^{-1}]^{k+1}$ by Broyden's method as

$$[\hat{W}^{-1}]^{k+1} = [\hat{W}^{-1}]^k + \frac{(\Delta G_{t^{ss}}(Q^k_{t^{ss}}) - [\hat{W}^{-1}]^k \Delta Q^k_{t^{ss}})(Q^k_{t^{ss}})'}{(Q^k_{t^{ss}})' Q^{k-1}_{t^{ss}}},$$

where $\Delta Q^k_{t^{ss}} = Q^k_{t^{ss}} - Q^{k-1}_{t^{ss}}$ and $\Delta G_{t^{ss}}(Q^k_{t^{ss}}) = G_{t^{ss}}(Q^k_{t^{ss}}) - G_{t^{ss}}(Q^{k-1}_{t^{ss}})$. Do not update if $\Delta Q^k_{t^{ss}} \leq \eta$. t^{ss} denotes the steady state period of the model with $t^{ss} = 1$ ($t^{ss} = T$) for the initial (final) steady state, compare Section 3.3.2 and $Q_{t^{ss}} = (q_{1,t^{ss}}, q_{2,t^{ss}}, ..., q_{m,t^{ss}})$.
If

- $[\hat{W}^{-1}]^{k+1}$ is ill-conditioned or
- $[\hat{W}^{-1}]^{k+1}$ does not satisfy prior information regarding its structure

then re-initialize $[\hat{W}^{-1}]^{k+1}$, otherwise proceed. Re-initialize $[\hat{W}^{-1}]^{k+1}$ by first-differences in steady state iterations and by resetting $[\hat{W}^{-1}]^{k+1} = [\hat{W}^{-1}]^0$ for transition iterations. Define $\hat{J}^{k+1} = [\hat{W}^{-1}]^{k+1} \otimes I_{T\times T}$ and continue with step 3.

3.3.4 Further Considerations

For ease of presentation, suppose throughout this section that there is no growth and hence that all variables are constant in the steady state. The assumption underlying equation (3.4) is that disaggregate variables need not be updated as the iteration proceeds. This is restrictive and will not be the case for most applications. Often, important feedback effects exist between disaggregate and aggregate variables in each iteration loop.

To formalize such relationships, rewrite the system of equations in (3.4) to the modified system

$$\begin{aligned} P^{k+1} &= S^{-1}(Q^k, B^{d,k}) \\ B^{s,k+1} &= B^s(P^{k+1}, Q^k, B^{d,k}) \\ Q^{k+1} &= D(P^{k+1}, B^{s,k+1}, B^{d,k}) \\ B^{d,k+1} &= B^d(P^{k+1}, B^{s,k+1}, B^{d,k}). \end{aligned} \tag{3.13}$$

[15] Throughout the analysis, I use the relative error tolerance only.

For ease of presentation, the fact that only subsets of the disaggregate variables B^s and B^d may only be important for the above mentioned circular relationships is ignored here. Due to the block Gauss-Seidel structure, disaggregate variables of the supply model, B^s, can be substituted out and the modified system can be more concisely written as

$$\begin{aligned} Q^{k+1} &= H_1(Q^k, B^{d,k}) \\ B^{d,k+1} &= H_2(Q^{k+1}, B^{d,k}). \end{aligned} \tag{3.14}$$

As an example for such disaggregate variables in an OLG context consider the shadow value of leisure in a model with endogenous labor supply, compare Section 3.4.2 below. For given aggregate wages and disaggregate shadow wages households determine how much labor to supply. In case constraints are violated, e.g. if leisure exceeds time endowment or if labor supply is positive even though shadow wage rates are positive, then shadow wages need to be updated, compare Auerbach and Kotlikoff (1987, p.31 and p.47). Hence, there is a feedback effect between aggregate and disaggregate variables.[16]

Instead of applying Quasi-Newton methods to the entire system of equations in 3.14, GSQN proceeds as follows. First, computational stability increases if disaggregate variables are related to aggregate variables, e.g., shadow wages are linked to the overall wage level. Let $p \subset P$ denote aggregate net wages and $\{b_e^d\}_{e=0}^E \subset B^d$ denote disaggregate shadow wage rates. Then by

$$\left\{ r_e = \frac{b_e^d}{p} \right\}_{e=0}^{E}$$

the time path of shadow wages of each age-group n is related to the overall wage level p.

Define by $R = (r_1, r_2, ...)$ where $r_i = \{\{r_{i,n,t}\}_{n=0}^{N_i}\}_{t=0}^{T}$ (the number of disaggregate units n may again differ across i) the set of all variables that involve transformations of B^d and Q (or P) respectively. Substituting out variables P from these relationships, the above system of equations then rewrites as

$$\begin{aligned} B^{d,k} &= V^{-1}(Q^k, R^k) \\ Q^{k+1} &= H_1(Q^k, B^{d,k}) \\ B^{d,k+1} &= H_2(Q^{k+1}, B^{d,k}) \\ R^{k+1} &= V(B^{d,k+1}, Q^{k+1}), \end{aligned}$$

[16] As an alternative to updating shadow wages as outer loops proceed, the household model may be solved accurately - up to some tolerance bound - by a standard shooting algorithm requiring a number of inner loop iterations per household and per outer loop. Yet, this is not efficient since accuracy of inner loops will increase automatically as the number of outer loops increases.

where V are all non-linear functions that transform Q and B^d to R. Note that updating of the transformed variables R without dampening translates into dampened updates of the original variables B^d.

Second, dampening of updated expressions for Q^k proceeds as before. However, the circular relationships between aggregate and disaggregate variables add further "noise" to the updating of the Jacobi matrix of the reduced sub-system of non-linear equations given by

$$G_1(Q^k) = Q^k - H_1(Q^k, B^{d,k}) = 0,$$

due to the presence of the variables B^d. The elements of the corresponding dampening factor matrix are given by

$$W^{-1} = \left\{ \sum_{\Delta_t=-T_0-t}^{T_0-t} D_{\Delta_t} - \frac{\partial h_{1,i}(Q^k, B^{d,k})}{\partial q^k_{j,\Delta_t}} - \sum_{l=1}^{L} \sum_{e=0}^{E_l} \frac{\partial h_{1,i}(Q^k, B^{d,k})}{\partial b^{d,k}_{j,\Delta_t,l,e}} \frac{\partial b^{d,k}_{j,\Delta_t,l,e}}{\partial q^k_{j,\Delta_t}} \right\}^m_{i,j=1},$$

$$\text{where} \begin{cases} D_{\Delta_t} = 1 & \text{for } \Delta_t = 0 \text{ and } i = j \\ D_{\Delta_t} = 0 & \text{else} \end{cases}.$$

Here, L denotes the number of relevant disaggregate variables and E_l the dimension of disaggregate variable l.

For most applications the additional terms in the above expression will be small and will not be determined during finite difference evaluations of the Jacobi matrix. Broyden's updating automatically takes into account these additional terms. The larger the additional terms, the more reasonable it will therefore be to start with any initial guess of a Jacobi matrix rather than to determine it by finite difference methods and to update it by Broyden's method as before. Dealing with disaggregate variables as described adds an additional channel through which GSQN combines first-order with second-order methods.

3.4 Economic Examples

This section describes two economic examples to illustrate the GSQN algorithm. As a first example, the familiar hog-cycle model is used to highlight the restrictions implicit to first-order iterative schemes such as Gauss-Seidel. The second example is a conventional large scale open-economy OLG model as introduced in Chapter 2. The OLG model is used for a simulation analysis regarding the relative performances of FGS and GSQN, respectively. Results of this simulation analysis are presented in Section 3.5.

3.4.1 The Hog-Cycle Model

The hog-cycle model is used to highlight the restrictions implicit in first-order iterations such as Gauss-Seidel. To this end, the relationship between the approximate Jacobi matrix and the actual Jacobi matrix implied by the the economic model is reversed: the question asked here is what kind of restrictions must be imposed on the economic model such that the Jacobi matrix implied by the fixed dampening factor is the actual Jacobi matrix of the economic model.

3.4.1.1 One-good model

The familiar static one-good hog-cycle model consists of a demand and a supply relationship. Suppose that

$$\begin{aligned} p &= s^{-1}(q) \\ q &= d(p) \end{aligned}$$

describes these economic relationships. As before these equations may be more concisely written as

$$\begin{aligned} q &= d(s^{-1}(q)) \Leftrightarrow \\ g(q) &= q - d(s^{-1}(q)) = q - h(q) = 0 \end{aligned}$$

and the (1×1) Jacobi matrix is given by

$$J = 1 - \frac{\partial h(q)}{\partial q} = \frac{\partial d(p)}{\partial p} \frac{\partial s^{-1}(q)}{\partial q},$$

which - among other things - depends on q. But a constant dampening factor w restricts the Jacobi matrix to be independent of q which will be the case if the inverse supply function, $s^{-1}(q)$, and the demand function, $d(p)$, are linear.[17]

Suppose that

$$\begin{aligned} p = s^{-1}(q) &= a_0 + a_1 q \\ q = d(p) &= b_0 + b_1 p \end{aligned}$$

then

$$J = 1 - \frac{\partial h(q)}{\partial q} = 1 - \frac{\partial d(p)}{\partial p} \frac{\partial s^{-1}(q)}{\partial q} = 1 - b_1 a_1.$$

[17] Linearity of both curves is only a sufficient condition. For example, J will also be independent of q if $p = s^{-1}(q) = \sqrt{a_0 + a_1 q}$ and $q = d(p) = b_0 + b_1 p^2$.

Then the restriction implied by $J = w^{-1}$ is of course only correct if the relationship between the slopes of the demand and supply curves satisfies

$$a_1 = \frac{1 - w^{-1}}{b_1}.$$

3.4.1.2 Two-goods model

Suppose that the above model is extended to a two good model and that (inverse) supply and demand functions are linear and of the following form

$$\begin{aligned} s_1^{-1}(q_1, q_2) = p_1 &= a_{10} + a_{11}q_1 + a_{12}q_2 \\ s_2^{-1}(q_1, q_2) = p_2 &= a_{20} + a_{21}q_1 + a_{22}q_2 \\ d_1(p_1, p_2) = q_1 &= b_{10} + b_{11}p_1 + b_{12}p_2 \\ d_2(p_1, p_2) = q_2 &= b_{20} + b_{21}p_1 + b_{22}p_2. \end{aligned}$$

The corresponding functions $h_1(q_1, q_2)$ and $h_2(q_1, q_2)$ are accordingly given by

$$\begin{aligned} h_1(q_1, q_2) &= b_{10} + b_{11}(a_{10} + a_{11}q_1 + a_{12}q_2) + b_{12}(a_{20} + a_{21}q_1 + a_{22}q_2) \\ h_2(q_1, q_2) &= b_{20} + b_{21}(a_{10} + a_{11}q_1 + a_{12}q_2) + b_{22}(a_{20} + a_{21}q_1 + a_{22}q_2) \end{aligned}$$

and the Jacobi matrix of the system of equations $Q - S^{-1}(D(Q)) = 0$, where $Q = (q_1, q_2)$ becomes

$$J = \begin{bmatrix} 1 - \frac{\partial h_1(q_1,q_2)}{\partial q_1} & -\frac{\partial h_1(q_1,q_2)}{\partial q_2} \\ -\frac{\partial h_2(q_1,q_2)}{\partial q_1} & 1 - \frac{\partial h_2(q_1,q_2)}{\partial q_2} \end{bmatrix} = \begin{bmatrix} 1 - (b_{11}a_{11} + b_{12}a_{21}) & -(b_{11}a_{12} + b_{12}a_{21}) \\ -(b_{21}a_{11} + b_{22}a_{21}) & 1 - (b_{22}a_{22} + b_{21}a_{12}) \end{bmatrix}.$$

If $a_{11} \neq 0$, $a_{22} \neq 0$, $b_{11} \neq 0$ and $b_{22} \neq 0$, then the off-diagonal elements of J will only be zero iff

$$a_{12} = a_{21} = b_{12} = b_{21} = 0.$$

This condition implies that cross-price elasticities of demand are equal to zero and that supplier's prices for good i are independent of supply of good $j \neq i$.

If these conditions hold, equality of off-diagonal elements of J further implies restrictions on the relationship between demand and supply curves for each good - just as in the above one-good example -, but also on the relationship across the two goods, since then

$$\begin{aligned} 1 - a_{11}b_{11} &= 1 - a_{22}b_{22} = w^{-1} \Leftrightarrow \\ a_{11}b_{11} &= a_{22}b_{22}. \end{aligned}$$

A number of lessons can be learned from these simple examples. First, it is obvious that all these conditions imply strong restrictions on both technology and preferences and will likely not hold even for these very simple linear models. Second, the restrictions are less likely to hold if the size of the model increases, i.e., if additional markets are added. Furthermore, assume that an explicit representation of the demand and supply functions does not exist for a linear model as the one considered above. Newton based methods immediately converge for linear models once the Jacobi matrix of the system is known. If it needs to be evaluated, then GSQN would require $m + 2$ iterations to calculate the equilibrium (one iteration to calculate the initial values Q^{k+1} for a given starting value Q^k, m iterations to calculate the Jacobi matrix and one more iteration to calculate the final solution). In contrast, FGS only needs 2 iterations if the economic model meets the restrictions implicit in one-parameter dampening. Therefore, third, in the unlikely event that the restrictions imposed by FGS are (approximately) valid, FGS will converge faster than GSQN. This is the more unlikely the larger is the economic model.

3.4.2 The Structure of the OLG Model

The structure of the OLG model is as introduced in Chapter 2. However, the following simplifying assumptions are made:

- Assumptions made on the production technology:
  - There is no age-specific productivity, hence $\epsilon_a = 0\,\forall a$.
  - Rather than assuming the growth in time endowment specification, standard labor augmenting technological change is assumed.
  - Adjustment costs to capital are not present, hence $\psi_i = 0\,\forall i$;
- Assumptions made on preferences:
  - The consumption function is Cobb-Douglas, hence $\xi_i = 1\,\forall i$. This allows the standard assumption of labor augmenting technological change, since for this preference specification the consumption to aggregate wages ratio will not trend in the steady state of the model.
  - The consumption share parameter is constant for all ages, hence $\Delta\phi_i = 0$ and $\phi_{a,i} = \phi_i\,\forall a$.
- There is not pension system, hence $\tau_t = \gamma_t = 0\,\forall t$.

- A stylized demographic model is used. Demographic transition scenarios are characterized by falling fertility rates and rising mortality rates that differ across the model regions. An additional baby bust is assumed in some regions resulting in additional variation across regions. Details are described in Ludwig (2004a).

- All parameters are restricted to be identical across countries, hence

$$\Omega_i = 1,\ g_i = g,\ \alpha_i = \alpha,\ \delta_i = \delta,\ \zeta_i = \zeta,\ \beta_i = \beta,\ \sigma_i = \sigma,\ \phi_i = \phi \quad \forall i = 1, ..., R$$

The model is solved for four alternative scenarios in increasing order of computational complexity. Solution of the model is by iteration over the capital-output ratio which, according to the assumptions on technology, is constant in the steady state.[18] In terms of notation of Section 3.3, the variables P and Q depend on the models used. These models and the associated definition of P and Q may be summarized as follows:

- Exogenous labor supply / closed economy:
  $P = \{r_t\}_{t=1}^T$ and $Q = \{k_t^y\}_{t=1}^T$

- Endogenous labor supply / closed economy:
  $P = (\{r_t\}_{t=1}^T, \{w_t\}_{t=1}^T)$ and $Q = (\{k_t^y\}_{t=1}^T, \{l_t\}_{t=1}^T)$

- Endogenous labor supply / two-country open economy:
  $P = (\{r_t\}_{t=1}^T, \{w_{t,1}\}_{t=1}^T, \{w_{t,2}\}_{t=1}^T)$ and $Q = (\{k_t^y\}_{t=1}^T, \{l_{t,1}\}_{t=1}^T, \{l_{t,2}\}_{t=1}^T)$

- Endogenous labor supply / three-country open economy:
  $P = (\{r_t\}_{t=1}^T, \{w_{t,1}\}_{t=1}^T, \{w_{t,2}\}_{t=1}^T, \{w_{t,3}\}_{t=1}^T)$ and
  $Q = (\{k_t^y\}_{t=1}^T, \{l_{t,1}\}_{t=1}^T, \{l_{t,2}\}_{t=1}^T, \{l_{t,3}\}_{t=1}^T)$

In addition, shadow wage rates are updated by the procedure described in Section 3.3.4. Calibration of structural parameters is described in Section 3.5.

3.5 Results for the OLG Model

This section compares the relative performance of FGS and GSQN applying the simplified version of the large-scale OLG model presented in Section 2. The analysis is grouped into two subsections. First, a steady state analysis is carried out to determine starting values of Q, $Q^{*,ss}$ and of J, $J^{*,ss}$, to be used for the transition analysis. Second, the performance

[18]From equation 2.7 is follows that the rate of return, r_t, is constant and equal across country units, if the capital output ratio, $\frac{K_t}{Y_t}$, is equal, since calibration parameters, $\alpha_i, \delta_i, \theta_i$, are equal across countries and since $\Omega_i = 1\,\forall i$.

of the two algorithms is compared for the demographic transition scenarios described in Ludwig (2004a). The transition analysis is carried out by using $Q^0 = Q^{*,ss}$ and $J^0 = J^{*,ss}$ as starting values. In terms of notation of Section 3.3.2, results reported below refer to model 2. This means that, first, an initial steady state of the model is calculated and next, steady state results are used as initial conditions for the transition calculations.

The structural model parameters of the above OLG model are given by

$$\Psi = (\Omega_0, g, \alpha, \delta, \zeta, \sigma, \beta, \phi).$$

In order to compare the performance of the algorithms, three different parameter values of a subset of these structural parameters, $\Psi_1 = (\alpha, \zeta, \beta, \sigma)$, are combined with each other which results in $3^4 = 81$ different parameterizations of the OLG model per model simulation, see Table 3.1.[19] These parameterizations reflect standard parameterizations chosen for OLG models in the literature. For steady state simulations, the starting value of the capital to output ratio is constant at three for the closed economy scenario with exogenous labor supply ($m = 1$). For all other models ($m > 1$), the steady state capital to output ratio resulting from previous models with $m - 1$ endogenous variables is used. The same procedure is adopted for the choice of starting values regarding the labor supply ratio: it is assumed constant at 0.5 for the closed economy model with endogenous labor supply ($m = 2$) and equilibrium labor supply shares resulting from previous computations are used for all subsequent models with $m > 2$. In addition, two alternative dampening factors $w_1 = 0.1$ and $w_2 = 0.3$ will be compared for FGS.

Table 3.1: Calibration parameters

Parameter		**Value**	
capital share α	0.3	0.4	0.5
substitution elasticity ζ	0.8	1	1.2
coefficient of relative risk aversion σ	1	2	3
discount factor β	0.99	0.98	0.97
growth rate g		0.015	
depreciation rate δ		0.05	
consumption share ϕ		0.6	

[19]Except for Ω_0 which is normalized in each iteration step by requiring the model to match arbitrary GDP levels of 100 for all countries.

The convergence criterion ϵ is set to $1e^{-4}$ for steady state and to $1e^{-3}$ for transition calculations[20]. This is an arbitrary choice. The relative advantage of GSQN increases the lower the convergence criterion since it asymptotically converges at a super-linear rate whereas FGS converges at a linear rate. No convergence may occur under two cases: first, when Q^k is divergent or exhibits cyclical behavior and second, when $\max(|G(Q^{k^{max}})/Q^{k^{max}}|) > \epsilon$ for some maximum number of iteration steps k^{max}. To rule out the latter case, k^{max} is set to 200.[21] The convergence properties of the two algorithms are evaluated along two dimensions, number of cases without convergence as well as running time (average and median) as the time it takes for convergent runs (in seconds). Since running time per iteration step differs between the two algorithms, results for the number of iterations it takes until convergence are only reported for sake of completeness.[22]

3.5.1 The Steady State Analysis

Convergence results for the steady state analysis of the model are reported in Table 3.2. The table is organized in four panels in increasing order of m. The first two rows of each section show results for FGS with $w = 0.1$ and $w = 0.3$ respectively. The third row shows results for GSQN. The last two columns of each row show the relative cases without convergence respectively. GSQN always converges whereas FGS may not converge for the higher value of the dampening factor ($w = 0.3$). The fourth and fifth row of each panel show the relation between running time (and number of iterations) between FGS for each $w = 0.1, 0.3$ and GSQN for the convergent runs of FGS respectively. For example, column one shows the average running time it takes for convergent simulations of FGS divided by the average running time of those GSQN simulations for which FGS also converges.

Average convergence speeds for $w = 0.1$ are about three times lower than GSQN when $m = 1$ and about 1.8 times lower when $m = 4$. This reduction in the relative performance of GSQN is due to the additional computations required for GSQN to calculate the Jacobi matrix. One might regard these differences as marginal. However, the resulting good initial estimates of J contribute to quite considerable differences in convergence speeds during the transition analysis, see below. For FGS with $w = 0.3$ the algorithm fails to converge in quite many cases (about 3.7 percent for $m = 1$ up to 27 percent when $m > 1$, but

[20] Setting a lower convergence criterion for steady state simulations is reasonable since a higher degree of accuracy is required (steady state solutions are fixed during transition simulations), but of course not necessary.

[21] For the scenarios considered here, this is sufficiently high since all non-convergent cases reported below are due to cyclical or divergent behavior.

[22] Running time per outer loop differs between FGS and GSQN since GSQN requires additional iterations for evaluation of the Jacobi matrix and line searches, compare Section 3.3.3.

convergence speed (of the convergent runs) is higher than for FGS with $w = 0.1$. Hence, a higher value of the fixed dampening factor trades robustness for speed. If it converges, FGS with $w = 0.3$ is even faster on average than GSQN for $m = 4$. The table also shows median running times since some cases of difficulties in convergence may be driven by outliers, but results do not look much different according to this criterion.

3.5.2 The Transition Analysis

While these steady state results already show that FGS is clearly inferior, this may not seem very compelling since there are no non-convergent cases of FGS with $w = 0.1$ and absolute overall speed is high since steady state solutions are fast to compute. But of course convergence speeds slow down a lot if larger models are used during transition iterations. Hence, computational speed may become relevant after all.

Results for transition calculations are shown in Table 3.3 where steady state solutions for $Q^0 = Q^{*,ss}$ and $\hat{J}^0 = \hat{J}^{*,ss}(W^{*,ss})$ are used as initial conditions throughout. A standard weighting matrix W derived in steady state simulations is e.g. given by

$$W_{4\times 4} = \begin{bmatrix} 0.205 & -0.608 & -0.426 & -0.191 \\ 0.024 & 0.928 & -0.051 & -0.023 \\ 0.021 & -0.062 & 0.956 & -0.020 \\ 0.017 & -0.051 & -0.036 & 0.984 \end{bmatrix} \tag{3.15}$$

which is far from a scaled identity matrix as in FGS. This multiple dampening factor matrix results from a standard parametrization of the OlG model with $\alpha = 0.4$, $\zeta = 1$, $\beta = 0.99$ and $\sigma = 2$. For this standard parametrization, all three algorithms converged. However, while GSQN took only 22.8 seconds, FGS took 64.353 (193.177) for $w = 0.3$ ($w = 0.1$).

The summary of results on transition iterations reported in Table 3.3 shows the following: First, GSQN and FGS with $w = 0.1$ always converge but the number of non-convergent cases of FGS with $w = 0.3$ quite significantly increases to roughly 38 percent for $m > 1$. Second, compared to FGS with $w = 0.1$, GSQN is roughly 3 to 7 times faster than FGS and this speed advantage strictly increases in the number m of endogenous variables Q. Third, the user may be lucky when using FGS with $w = 0.3$ for $m = 1$ since the algorithm might converge even faster than GSQN (if it converges). But for values of $m > 1$ GSQN is 3 to 5 times faster for those cases when FGS with $w = 0.3$ converges.

These results are striking and suggest to use GSQN with good starting values derived from steady state solutions of the simulation model or earlier transition iterations since GSQN is so much superior and since it is so easy to implement. The most important aspect of GSQN is that these significant increases in running times relative to standard FGS are achieved at low costs since GSQN just combines traditional fixed-point iterations

Table 3.2: Convergence of FGS and GSQN for the steady state

	Running time		Iteration number		No convergence
	Mean	Median	Mean	Median	Fraction
	Closed economy, exogenous labor supply ($m = 1$)				
$FGS(w = 0.1)$	4.16	3.97	20.31	20.00	0.00%
$FGS(w = 0.3)$	2.18	1.31	10.69	6.50	**3.7%**
$GSQN$	1.36	1.29	5.25	5.00	0.00%
$FGS(w = 0.1)/GSQN$	**3.07**	3.07	3.87	4.00	
$FGS(w = 0.3)/GSQN$	1.61	1.02	2.04	1.30	
	Closed economy, endogenous labor supply ($m = 2$)				
$FGS(w = 0.1)$	15.37	15.89	55.23	58.00	0.00%
$FGS(w = 0.3)$	6.72	4.98	24.10	18	**27.16%**
$GSQN$	3.87	2.79	6.59	6.00	0.00%
$FGS(w = 0.1)/GSQN$	**3.98**	5.69	8.38	9.67	
$FGS(w = 0.3)/GSQN$	1.57	1.78	3.66	3.00	
	Two-country model, endogenous labor supply ($m = 3$)				
$FGS(w = 0.1)$	23.81	23.42	44.49	44.00	0.00%
$FGS(w = 0.3)$	11.25	8.53	21.03	16.00	**24.69%**
$GSQN$	9.46	6.31	3.37	3.00	0.00%
$FGS(w = 0.1)/GSQN$	**2.52**	3.71	13.20	14.67	
$FGS(w = 0.3)/GSQN$	1.09	1.35	6.24	5.33	
	Three-country model, endogenous labor supply ($m = 4$)				
$FGS(w = 0.1)$	38.65	38.46	48.36	48.00	0.00%
$FGS(w = 0.3)$	14.81	12.93	18.56	16.00	**24.69%**
$GSQN$	21.52	15.08	3.83	3.00	0.00%
$FGS(w = 0.1)/GSQN$	**1.80**	2.55	12.64	16.00	
$FGS(w = 0.3)/GSQN$	0.64	0.86	4.85	5.33	

Notes: FGS: Conventional fast Gauss-Seidel algorithm with one-parameter dampening. GSQN: Gauss-Seidel-Quasi-Newton algorithm. This table shows steady state convergence results of FGS and GSQN for four different scenarios with 81 model simulations each. The last two rows of each section show the relative performance of FGS and GSQN for convergent runs of FGS only.

with simple Newton based methods. Therefore, existing sub-routines may be used for implementation.

3.6 Conclusions

The analysis conducted in this Chapter suggests to use Gauss-Seidel-Quasi-Newton (GSQN) instead of conventional fast Gauss-Seidel (FGS) iterations for solving heterogeneous agent models. Standard Quasi-Newton based methods (Broyden's method) are used to determine elements of a low-dimensional approximation of a Jacobi matrix for Gauss-Seidel iterations which considerably improves convergence both in terms of speed as well as robustness of the iterations. This approximate Jacobi matrix may also be interpreted as a matrix of multiple dampening factors (Hughes Hallet 1984). By this, GSQN is a composite method of standard first-order and second-order tatonnnement methods.

The particular attractiveness of the algorithm stems from the combination of low computational costs of conventional tatonnement methods with the speed of Newton based methods. It only requires augmenting these intuitive tatonnement methods with well-established and simple numerical methods.

The simulation analysis shows, that GSQN increases convergence speed by a factor of two to seven relative to FGS for transition simulations. This relative speed advantage strictly increases in the number of aggregate endogenous variables, m, required for tatonnement iterations. Therefore, GSQN enables a researcher to solve a larger simulation model within the same time frame as FGS needs for a smaller model. This allows the researcher to investigate much more interesting scenarios. Furthermore, computational speed is relevant for estimation and sensitivity analysis, see Chapter 4.

The idea behind the algorithm - constructing a composite between fixed-point iterations and Quasi-Newton methods - can be applied to other economic models and solution procedures. As shown in Ludwig (2004b), the same idea can be used in fixed-point iterations to dampen coefficients that characterize polynomials used to solve rational expectations models by standard projection methods.

Table 3.3: Convergence of FGS and GSQN for the transition

	Running time		Iteration number		No convergence
	Mean	Median	Mean	Median	Fraction
	Closed economy, exogenous labor supply ($m = 1$)				
FGS($w = 0.1$)	22.78	23.20	16.60	17.00	0.00%
FGS($w = 0.3$)	11.60	6.95	8.49	5.00	**3.7%**
GSQN	8.08	6.99	5.85	5.00	0.00%
FGS($w = 0.1$)/GSQN	**2.82**	3.32	2.84	3.40	
FGS($w = 0.3$)/GSQN	1.47	0.99	1.45	1.00	
	Closed economy, endogenous labor supply ($m = 2$)				
FGS($w = 0.1$)	50.89	43.16	28.31	28.00	0.00%
FGS($w = 0.3$)	36.13	21.43	17.98	10.00	**39.51%**
GSQN	11.84	9.26	5.84	5.00	0.00%
FGS($w = 0.1$)/GSQN	**4.30**	4.66	4.85	5.60	
FGS($w = 0.3$)/GSQN	2.87	2.31	3.08	2.00	
	Two-country model, endogenous labor supply ($m = 3$)				
FGS($w = 0.1$)	151.44	120.81	40.96	40.00	0.00%
FGS($w = 0.3$)	89.86	57.30	22.94	15.00	**37.04%**
GSQN	22.73	18.28	5.60	5.00	0.00%
FGS($w = 0.1$)/GSQN	**6.66**	6.61	7.31	8.00	
FGS($w = 0.3$)/GSQN	3.57	3.14	4.09	3.00	
	Three-country model, endogenous labor supply ($m = 4$)				
FGS($w = 0.1$)	242.19	191.12	42.95	42.00	0.00%
FGS($w = 0.3$)	172.37	92.11	28.32	16.00	**38.27%**
GSQN	34.45	27.57	5.86	5.00	0.00%
FGS($w = 0.1$)/GSQN	**7.03**	6.93	7.32	8.40	
FGS($w = 0.3$)/GSQN	4.56	3.34	4.83	3.20	

Notes: FGS: Conventional fast Gauss-Seidel algorithm with one-parameter dampening. GSQN: Gauss-Seidel-Quasi-Newton algorithm. This table shows transition convergence results of FGS and GSQN for four different scenarios with 81 model simulations each when results derived from steady state calculations are used as starting values. The last two rows of each section show the relative performance of FGS and GSQN for convergent runs of FGS only.

4 Matching Moments

This chapter is based on Ludwig (2005) and takes Auerbach-Kotlikoff OLG models to the data by feeding realistic demographic data into the simulation model. Despite their widespread use for the analysis of economic questions, a formal and systematic calibration methodology has not yet been developed for Auerbach-Kotlikoff (Auerbach and Kotlikoff 1987) overlapping generations (AK-OLG) models. Calibration as estimation in macroeconomics involves choosing free parameters by matching moments of simulated models with those of the data. This Chapter maps this approach into the framework of AK-OLG models. Furthermore, the back-fitting properties of three different versions of a prototype AK-OLG model are evaluated along a number of dimensions of US data for the time period 1960-2003.[1]

4.1 Introduction

Since almost a quarter of a century, Auerbach-Kotlikoff type overlapping generations (AK-OLG) models (Auerbach, Kotlikoff, and Skinner 1983; Auerbach and Kotlikoff 1987) have been applied to the analysis of economic questions. Kotlikoff (1998) provides a review of the (earlier) literature and summarizes avenues of future research. Among the more recent developments in the AK-OLG literature are the inclusion of realistic demographic profiles and the extension towards multi-country versions of these models (Bommier and Lee 2003; INGENUE 2001; Börsch-Supan, Ludwig, and Winter 2004; Fehr, Jokisch, and Kotlikoff

[1] I thank Donald Andrews, Alan Auerbach, Axel Börsch-Supan, Juan-Carlos Conesa, Wouter Den Haan, Florian Heiss, Stefan Hoderlein, Michel Juillard, Albert Marcet, Nico Voigtländer, Viktor Winschel, Joachim Winter and several seminar participants at the University of Mannheim, at the IMF Research Department, at Universitat Pompeu Fabra and at the SEC's 9^{th} International Conference in Seattle, Washington for helpful comments. Financial support from the Deutsche Forschungsgemeinschaft, SFB 504, at the University of Mannheim, the Volkswagenstiftung and the Gesamtverband der Deutschen Versicherungswirtschaft is gratefully acknowledged.

2004).[2] Such extensions have moved AK-OLG models from being mere analytical models applied to public finance questions into the direction of forecasting tools.

These recent developments necessitate a careful evaluation of AK-OLG models with regard to their fit to long time series of macroeconomic data. This in turn requires a formal procedure to determine values of structural model parameters, which is referred to as calibration.[3] The purpose of this chapter is twofold: First, a new, systematic calibration procedure is developed for large-scale AK-OLG models in outside steady state situations. The suggested approach is to estimate structural model parameters by a formal matching of moments procedure. Second, the fit of a prototype AK-OLG model to long time series of macroeconomic data is evaluated and the relative performance of different model features is compared. Model evaluation relates to the alternative interpretation of calibration that has been used in the literature as a way of *testing* an economic model.[4] To the best of my knowledge, this analysis is the first to provide such detailed investigation of both these aspects for AK-OLG models.

Standard calibration procedures of AK-OLG models stratify the set of all structural model parameters into two sets, *predetermined* and *free* parameters. *Predetermined* parameter values are set by reference to (estimates of) other studies. Values of *free* parameters are determined by informally matching moments.

The use of *predetermined* parameters has been criticized with the notion that statistical inference depends on the structure of the econometric model. Parameter values are therefore not easily transferable from one particular model to another (Hansen and Heckman 1996). Furthermore, and as emphasized by Gregory and Smith (1990), estimation of the subset of free parameters depends on the values of predetermined parameters. While not desirable, it is often unavoidable to rely on predetermined parameters. Here, the selection of predetermined parameters is regarded as exogenous but the sensitivity of the effects of errors in it can be shown to be low.[5]

The standard procedure of informally matching moments to determine values of *free* parameters used in the AK-OLG literature is a mix of the following two approaches. The

[2] It is beyond the scope of this chapter to provide a similar review on the more recent literature as in Kotlikoff (1998). Among other model features that have recently been added are, e.g., within generation heterogeneity and idiosyncratic as well as aggregate uncertainty (Imrohoroglu, Imrohoroglu, and Joines 1995; Conesa and Krueger 1999; Altig et al. 2001; Krueger and Kubler 2003).

[3] Kim and Pagan (1995) provide a review of the literature on "calibration as estimation" (Gregory and Smith 1990).

[4] This alternative interpretation of calibration is more in line with the interpretation of calibration by Kydland and Prescott (1982). Canova and Ortega (1999) provide a review of the literature on "calibration as testing" (Gregory and Smith 1991).

[5] Results on such a sensitivity analysis are available from the author upon request.

first is to focus exclusively on observations of a base year.[6] Obviously the procedure has the drawback that observations in any time period are just realizations of an (unknown) stochastic data generating process and/or are measured with error. The second approach calibrates the model such as to (informally) match long term averages of statistical data. While this second procedure to a large extent overcomes the deficiencies of the first, growth rates of variables are usually regarded as predetermined. Being informal, both approaches do further not take account of the sampling uncertainty of structural model parameters.

One reason for the lack of more sophisticated econometric techniques in AK-OLG calibration is certainly conceptually grounded in the deterministic nature of these models. Accordingly, observations of a base year suffice to determine values of structural model parameters. This chapter deviates from this view by augmenting deterministic dynamic AK-OLG models with additional random components as in the early work on CGE models by Jorgenson (1984) and Mansur and Whalley (1984). *Free* structural model parameters are estimated using a method of moments methodology that sets to zero the average discrepancy (discrepancy function) between actual and predicted (simulated) values along pre-specified dimensions. This is by no means a trivial task since a number of the moment conditions do not have closed form solutions and the estimation method therefore has to rely on numerical simulation.[7] Adopting the terminology of Gregory and Smith (1990), the suggested calibration procedure can therefore be understood as a restricted method of simulated moments procedure, where the restrictions stem from the choice of predetermined parameters.

Model evaluation is by means of two approaches. First, graphical inspection is used to study the discrepancies between the time paths of actual and simulated data. While this way of testing the model provides most information, it has been criticized in the literature as being too *informal* (Hansen and Heckman 1996) since a formal metric to evaluate the distance between actual and simulated data is not provided. In order to provide such a formal metric, this chapter adopts the framework of Christiano and Eichenbaum (1992) who map estimation and testing of a Real Business Cycle (RBC) model into a modification of Hansen's (1982) GMM framework in an elegant way. While more emphasis will be put on model evaluation by graphical means, the formal criteria are regarded as a useful complement of the graphical analysis that validate its findings from a statistical perspective.

[6] Abdelkhalek and Dufour (1998), within the context of a different type of CGE model, justify this procedure by noting that long time series on economic variables are often not available, e.g., for developing countries.

[7] The implied costs of the estimation procedure may be another reason for the lack of more sophisticated econometric calibration techniques in the AK-OLG literature.

The remainder of the chapter is structured as follows. Section 4.2 presents the calibration methodology. Section 4.3 describes the implementation of the methodology using a version of the model described in Chapter 2. Section 4.4 presents the results. Finally, Section 4.5 draws conclusions from these findings.

4.2 The Calibration Procedure

Computable general equilibrium models such as AK-OLG models can be represented by the following function F

$$y_t = F(Y, X, \Psi^c) \quad \forall t = T_1, ..., T_2. \tag{4.1}$$

$X = \{\{x_{t,i}\}_{i=1}^{m}\}_{t=T_1}^{T_2}$ is a collection of exogenous and $Y = \{\{y_{t,i}\}_{i=1}^{n}\}_{t=T_1}^{T_2}$ is a collection of endogenous variables. This general representation allows lagged and future endogenous (exogenous) variables to enter the model. They are determined (given) for a simulation period of length $T_1 + T_2 + 1$ starting from the initial date $T_1 < 1$ and ending at the final date $T_2 \geq T$, whereas data are only observed for the period $1, ..., T$.

$\Psi^c \in \Gamma \subset \mathbb{R}^c$ denotes the $c \times 1$ vector of structural model parameters which are referred to as calibration parameters. Define by Ψ^p the vector of p *predetermined* parameters and by Ψ^e the vector of e *estimated* parameters, where $\Psi = [(\Psi^p)', (\Psi^e)']'$. While Ψ^p and Ψ^e are not fundamentally different from a theoretical viewpoint, they are treated differently in standard calibration of CGE models. Predetermined parameters, Ψ^p, are set by reference to other studies and are usually elasticity parameters that describe behavioral functions, whereas estimated parameters, Ψ^e, are usually scale or share parameters (Abdelkhalek and Dufour 1998). Note that, in the extreme cases, either of the two vectors may be empty. Hence, if $p = 0$ all parameters are determined by estimation and if $e = 0$ all parameters are predetermined.

To simplify notation, the above equation can be rewritten as

$$y_t = f(y_t, x_t, \Psi) = h_t(\Psi^e) \quad \forall t = T_1, ..., T_2, \tag{4.2}$$

such that only contemporaneous variables enter the right-hand side of the equation.

4.2.1 A Modified GMM Framework

Structural model parameters, Ψ^e, are estimated by unconditional matching of moments as in Christiano and Eichenbaum (1992). e moment conditions will be used to estimate the elements of Ψ^e (exactly identified case of GMM estimation). In anticipation of further results, it is however useful to start with the more general case of GMM estimation where the total number of moment conditions, r, exceeds the number of parameters, e.

Let

$$u_t^e(\Psi^e) = y_t^e - h_t^e(\Psi^e) \tag{4.3}$$

be an $e \times 1$ vector. Assume that q additional moment conditions are given and define by

$$u_t^q(\Psi^e) = y_t^q - h_t^q(\Psi^e) \tag{4.4}$$

a $q \times 1$ vector, where $r = e + q$. Further define the overall GMM error as $u_t = [(u_t^e)', (u_t^q)']'$.

Under the assumption that the model is correctly specified, the restrictions on the GMM error can be written as

$$\mathbb{E}[u_t(\Psi^{e,0})] = 0, \tag{4.5}$$

where $\Psi^{e,0}$ denotes the vector of true values.

Denote the sample averages of u_t as

$$g_T(\Psi^e) \equiv \frac{1}{T}\sum_{t=1}^{T} u_t(\Psi^e), \qquad g_T(\Psi^e) = [g_T^e(\Psi^e)', g_T^q(\Psi^e)']', \tag{4.6}$$

where $T < T_2$ is the sample size. Hansen's 1982 GMM estimator $\widehat{\Psi}_T^e$ is then defined as

$$\widehat{\Psi}_T^e = \arg\min_{\Psi^e} g_T(\Psi^e) W g_T(\Psi^e) \tag{4.7}$$

for some weighting matrix W.

Calibration as unconditional moment estimation of Ψ^e and testing of the model by informal methods can be understood as restricted GMM estimation with the restriction on W given by

$$W = \begin{bmatrix} I_{e\times e} & 0_{e\times q} \\ 0_{q\times e} & 0_{q\times q} \end{bmatrix}, \tag{4.8}$$

compare, e.g., Marcet (1994). In other words, while e moment conditions are used to estimate e structural model parameters, the remaining q moment conditions are used to test the model. By the above restriction on W, tests of the model based on g_T^q are necessarily informal.

Following Christiano and Eichenbaum (1992) a formal framework for testing the model - without leaving the "philosophy" of calibration of exactly matching e moments to estimate Ψ^e - is developed as follows. Define a $q \times 1$ vector of additional model parameters, Ψ^q, and by $\Psi = [(\Psi^e)', (\Psi^q)']'$ the $r \times 1$ vector collecting all parameters. Further, rewrite the GMM errors in equation 4.4 as

$$u_t^q(\Psi) = \underbrace{y_t^q - \Psi^q}_{u_{t,1}^q(\Psi^q)} - \underbrace{(h_t^q(\Psi^e) - \Psi^q)}_{u_{t,2}^q(\Psi)}, \quad \forall t = 1, ..., T \quad (4.9)$$

and define the sample averages of the GMM errors $u_{t,1}^q(\Psi^q)$ and $u_{t,2}^q(\Psi)$ as

$$g_{T,1}^q(\Psi^q) \equiv \frac{1}{T}\sum_{t=1}^{T} u_{t,1}^q(\Psi^q) \qquad g_{T,2}^q(\Psi) \equiv \frac{1}{T}\sum_{t=1}^{T} u_{t,2}^q(\Psi). \quad (4.10)$$

Notice that $g_{T,1}^q(\Psi^q)$ measures the average discrepancy between actual variables, y_t^q, from the parameters Ψ^q - i.e., Ψ^q are the sample averages of y_t^q -, whereas $g_{T,2}^q(\Psi)$ measures the average discrepancy between simulated variables, $h_t^q(\Psi^e)$, and the parameters Ψ^q.

The GMM estimator of the $r \times 1$ vector $\widehat{\Psi}$ is now derived from the $r \times 1$ moment conditions $g_T(\Psi) = [(g_T^e(\Psi^e))', (g_{T,1}^q(\Psi^q))']'$ and defined by

$$g_T(\widehat{\Psi}_T) = 0, \quad (4.11)$$

i.e., the weighting matrix corresponding to the representation in equation 4.7 is an identity matrix, $W = I_{r\times r}$. The role of $g_{T,2}^q(\Psi)$ is addressed below.

Assume, as in the seminal contribution by Hansen (1982), that u_t, are strictly stationary for all possible Ψ. Then $\widehat{\Psi}_T$ is asymptotically normally distributed,

$$\sqrt{T}(\widehat{\Psi}_T - \Psi^0) \sim N(0, V), \quad (4.12)$$

where

$$V = D^{-1}S(D')^{-1} \quad (4.13)$$

and

$$D = \mathbb{E}\left[\frac{\partial g_T(\Psi)}{\partial \Psi'}|_{\Psi=\Psi^0}\right] = 0. \quad (4.14)$$

S is the positive semi-definite spectral density at frequency 0 of $u_t(\Psi^0)$ defined by

$$S = \sum_{l=-\infty}^{\infty} C_l \ \text{ where } \ C_l = \mathbb{E}[u_t(\Psi)u_{t-l}(\Psi)']. \quad (4.15)$$

Inference is based on replacing D and S with estimators, hence

$$\widehat{V}_T = \widehat{D}_T^{-1}\widehat{S}_T(\widehat{D}_T')^{-1} \quad (4.16)$$

and $\widehat{\Psi}$ can be treated approximately as

$$\widehat{\Psi}_T \sim N\left(\Psi^0, \text{var}(\widehat{\Psi})\right), \qquad \text{var}(\widehat{\Psi}) = \widehat{V}/T. \tag{4.17}$$

Considering formal tests of the model, define by $f^s(\Psi^0)$ a function that maps $\mathbb{R}^r$ into the $s \times 1$ vector 0_s. Then $f^s(\Psi^0) = 0_s$ presents s hypothesis each of which potentially involves all elements of Ψ^0. As shown in Christiano and Eichenbaum (1992), the statistic

$$J = f(\widehat{\Psi})' \text{var} f(\widehat{\Psi})^{-1} f(\widehat{\Psi}), \tag{4.18}$$

where

$$\text{var} f(\widehat{\Psi}) = f'(\widehat{\Psi}) \text{var}(\widehat{\Psi}) f'(\widehat{\Psi})' \tag{4.19}$$

is asymptotically χ^2-distributed with s degrees of freedom, also see Eichenbaum, Hansen, and Singleton (1988); Christiano and Den Haan (1996). For example, tests involving all the additional q parameters can be mapped into this framework if $f(\widehat{\Psi}) = g_{T,1}(\widehat{\Psi})$, hence $s = q$. Equation 4.18 takes into account the joint sampling uncertainty of the model parameter estimates and the moments of the data and represents a formal theory of inference that may serve as a useful complement of the informal and mostly graphical model evaluation procedure.

4.2.2 The Case of Non-Stationarity

The assumption of strict stationarity of u_t is restrictive since economic models often evolve variables that are trending over time as is also the case for the economic model described in Section 4.3. Cases with trending variables have been considered by Eichenbaum and Hansen (1990) and by Ogaki (1993, 1999). An obvious solution to the non-stationarity is to transform variables of the economic model such that the transformed variables used in the econometric application are stationary as in the study by Hansen and Singleton (1982). However, it may not always be feasible to rewrite an economic model as such.

An alternative has been discussed by Eichenbaum and Hansen (1990) and by Ogaki (1993). Eichenbaum and Hansen consider two types of trends, a deterministic and a stochastic trend. For the economic application in this chapter, the deterministic trend specification is of relevance. Suppose that a variable Z_t satisfies

$$Z_t = Z_0 \exp(\gamma^z t + u_t^z),$$

and hence that

$$z_t = \ln Z_t = z_0 + \gamma^z t + u_t^z,$$

i.e., the log of the variable follows a deterministic linear trend. As Eichenbaum and Hansen show, consistent estimation is possible if z_0, γ^z, and Ψ are jointly estimated.

The theoretical framework of Andrews and McDermott (1995) offers an alternative to de-trending in the presence of deterministic trends. Using triangular-array rather than traditional sequential asymptotic theory, Andrews and McDermott establish that consistent estimation is possible if the deterministic trend of the data has a particular structure relative to the economic model. Under such circumstances, model parameters and the asymptotic variance-covariance matrix can be estimated with the same procedures as in the case of strictly stationary regressors described above. The framework of Andrews and McDermott is convenient since it allows for a more general specification of the trend and is therefore applied here.

4.2.3 Interpretation of the MM Error

The MM error, u_t, measures the discrepancy between observed and model predicted values. In a deterministic model as the one introduced in Section 4.3, the error may be due to three aspects: (i) while the model is deterministic, real world data are generated by an unknown stochastic process and u_t reflects stochastic shocks, (ii) real world data are measured with error and u_t reflects this measurement error and (iii) u_t reflects specification error.

The issue of missing intrinsic stochastic components in the economic model is addressed here by first filtering observed time series of data using the Hodrick-Prescott procedure to decompose observed data z_t into a cyclical component r_t and a trend component τ_t (Hodrick and Prescott 1997). The discrepancy functions u_t are described using the deterministic components of the time series, τ_t, that reflect the smooth growth component of aggregate data.[8]

Let $\{Z_t\}_{t=1}^{T}$ be the observed time series of an aggregate economic variable, e.g., GDP and let $z_t = \ln(Z_t)$. The Hodrick-Prescott filter decomposes z_t into r_t and τ_t by solving the following programming problem

$$\min_{\{\tau_t\}_{t=1}^{T}} \left\{ \sum_{t=2}^{T} (z_t - \tau_t)^2 + \lambda \sum_{t=2}^{T} \left[(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1}) \right]^2 \right\}$$

for some predetermined parameter λ. For $\lambda \to \infty$, $\tau_t \to \tau_0 + \gamma t$ which is the least squares fit of a liner trend model. Since z_t is defined here as the log of the original variable, $\lambda \to \infty$ results in exponential growth of the trend component of the original variable Z_t. As Hodrick and Prescott point out, the linear trend specification is not an appropriate description of

[8] Note that this approach is just opposite to conventional procedures in the RBC literature where the cyclical component of the data is used for inference.

the data since the growth component varies "smoothly" over time. This feature of actual trends corresponds to the features of simulated trends of the model presented in Section 4.3. The appropriate λ-value for annual data recommended in the literature is 100.

To the extent that the de-trending procedure returns the "true" value of the deterministic component of the economic variable z_t of interest, the remaining interpretation for the MM error, u_t, is as specification error. However, there might be significant measurement errors of the original observed values of trending variables, Z_t. Let $Z_t^* = Z_t \exp \epsilon_t$ be the measured variable and let ϵ_t be the measurement error with the property that $\mathbb{E}\epsilon_t = 0$. As shown by King and Rebelo (1993), the solution of the above non-linear programming problem is a linear lag polynomial $\tau_t = (1 - h(L))z_t$, where $h(L)$ is the lag polynomial. Therefore, since the log of the measured variable is given by $z_t^* = z_t + \epsilon_t$, the measurement error also enters the de-trended variable linearly.[9]

In the presence of linear measurement error, the MM error writes as

$$u_t(\Psi) = y_t^* - h(\Psi) = y_t + \epsilon_t^y - f(y_t + \epsilon_t^y, x_t + \epsilon_t^x, \Psi) + \mu_t$$

Here, ϵ_t^y and ϵ_t^x are $r \times 1$ vectors of measurement error and μ_t is an $r \times 1$ vector of specification errors as before.

The presence of measurement error is problematic since under the assumption that $\mathbb{E}\epsilon_t^y = \mathbb{E}\epsilon_t^x = \mathbb{E}\mu_t = 0$, that is, under the assumption that measurement and specification errors are on average zero, the expected value of the MM error, $\mathbb{E}u_t$, may no longer be zero at Ψ^0.

For the economic model introduced in Section 4.3, equation f is, however, linear in y_t and x_t, hence

$$u_t(\Psi) = y_t^* - h(\Psi) = y_t + \epsilon_t^y - A(\Psi)y_t + B(\Psi)x_t + \mu_t + A(\Psi)\epsilon_t^y + B(\Psi)\epsilon_t^x$$

for some matrices $A(\Psi)$ and $B(\Psi)$. Therefore, the framework considered in this analysis allows for an interpretation of the error terms as linear specification error and as linear measurement error.

4.3 Implementation of the Calibration Methodology

The structure of the OLG model is as introduced in Chapter 2. However, the following simplifying assumptions are made:

- Assumptions made on the production technology:

[9] Moreover, measurement error might be induced by the de-trending procedure itself, see, e.g., Conova (1998).

  - Adjustment costs to capital are not present, hence $\psi_i = 0 \, \forall i$;

- Assumptions made on preferences:
  - The consumption share parameter is constant for all ages, hence $\Delta\phi_i = 0$ and $\phi_{a,i} = \phi_i \, \forall a$.

4.3.1 Moment Conditions

The total set of structural model parameters can be collected in the following vectors

$$\begin{aligned} \text{Production Sector: } \Psi^{PS} &= [\{\delta_i\}_{i=1}^R, \{\alpha_i\}_{i=1}^R, \{g_i\}_{i=1}^R, \{\Omega_i\}_{i=1}^R]' \\ \text{Household Sector: } \Psi^{HS} &= [\{\beta_i\}_{i=1}^R, \{\theta_i\}_{i=1}^R, \{\xi_i\}_{i=1}^R, \{\phi_i\}_{i=1}^R]'. \end{aligned}$$

However, not all of these parameters will be estimated by matching of moments. Since the open economy version of the model only serves as an illustration of the additional effects of openness, see below, the following simplifying assumptions are imposed:

$$\begin{aligned} &\delta_i = \delta_1; \alpha_i = \alpha_1; g_i = g \ \forall i \text{ and} \\ &\beta_i = \beta_1; \theta_i = \theta_1; \xi_i = \xi_1 \ \forall i. \end{aligned}$$

In other words, most of the parameters are estimated only for country $i = 1$.

In addition, a subset, Ψ^p, of the remaining calibration parameters are regarded as predetermined (i.e., as fixed by reference to other studies). Specifically, the elasticity parameters $1/\theta_1$ and ξ_1 are treated as predetermined since estimated values of these parameters would be outside ranges regarded as reasonable in the literature.

To summarize, predetermined parameters, Ψ^p, and estimated (free) parameters, Ψ^e, are given as follows:

$$\begin{aligned} &\Psi^p = [\theta_1, \xi_1]' \\ &\Psi^e = [\delta_1, \alpha_1, g_1, \{\Omega_i\}_{i=1}^R, \beta_1, \{\phi_i\}_{i=1}^R]' \end{aligned}$$

According to these assumptions, only the structural model parameters Ω_i and ϕ_i vary across countries. These parameters determine the effective "size" of each country in terms of technology levels (aggregate output, GDP) and in terms of the size of the aggregate labor force.

Remark Despite simplification, there is also a deeper role for the restrictions imposed in the open economy version of the simulation model that is due to an inconsistency between capital stock data and theoretical relationships of the above model. The market clearing condition in the open economy version of the model, equation 2.15, and the "no arbitrage" rule between financial and physical investment, equation 2.7, imply

$$\frac{Y_{t,j}}{K_{t,j}} = \frac{\alpha_i \frac{Y_{t,i}}{K_{t,i}} - \delta_i - \delta_j}{\alpha_j} \quad i \neq j$$

a restriction that may not hold. Augmenting the simulation model with adjustment costs on physical capital investment is unlikely to solve this inconsistency and it could only be reasonably addressed by a model with additional components, e.g., with some market imperfection on the international capital market. Under the assumptions made here,

$$\frac{Y_{t,j}}{K_{t,j}} = \frac{Y_{t,i}}{K_{t,i}} \quad i \neq j.$$

This restriction is exploited below for the estimation of Ω_j, for $j > 1$.

4.3.1.1 Moment Conditions Underlying the Estimates of Ψ^e

Moment conditions for estimation of the structural model parameters Ψ^e follow directly from the above relationships of the theoretical model. Notice that lower case letters denote the log of the HP-filtered data. Recall that the estimation framework builds on the theoretical results established by Andrews and McDermott (1995) and therefore allows estimation using trending data. Also recall that the error terms, u_t may consist of two components, specification and measurement error, that both enter the logs of the HP-filtered data linearly.

The capital accumulation equation contained in the system of equations in 2.4, implies that δ_1 can be estimated by

$$\mathbb{E}\left[d_{t,1} - k_{t,1} - \ln \delta_1\right] = 0,$$

and α_1 by transforming equation 2.5 as

$$\mathbb{E}\left[w_{t,1} + y_{t,1} - l_{t,1} - \ln(1-\alpha_1)\right] = 0.$$

The moment conditions underlying the estimates of Ω_i, the levels of total factor productivity, are derived from rewriting the production function, equation 2.3, in logs

$$\mathbb{E}\left[y_{t,i} - \ln \Omega_i - \alpha_i k_{t,i} - (1-\alpha_i)(l_{t,i} + g_i t)\right] = 0 \ \forall i = 1, ..., R.$$

The moment condition underlying the estimate g_1, the trend growth rate of efficiency units, is derived by taking first differences of the above equation as

$$\mathbb{E}\left[\gamma^Y_{t,1} - \alpha_1 \gamma^K_{t,1} - (1-\alpha_1)(\gamma^L_{t,1} + g_1)\right] = 0.$$

Since no closed form solution exists, estimation of structural model parameters of the household sector requires simulation. While the above moment conditions for the production sector imply stationarity of the MM error u_t at $\Psi^{e,0}$, this may not be the case for the household sector. For instance, as shown below, the endogenous labor supply model fails to replicate the growth rate of actual labor supply. Matching simulated to actual labor supply on average would then result in a non-stationary MM error even at $\Psi^{e,0}$. To address this, suitable normalization is required.

The moment condition underlying the estimate of the discount factor, β_1, is by matching the simulated to the actual average capital output ratio,

$$\mathbb{E}\left[k_{t,1} - y_{t,1} - \left\{\ln\left(\sum_{t,a}^{Z} H^s_{t-a,a,1}(\Psi, X)N_{t-a,a,1}\right) - y^s_{t,1}\right\}\right] = 0, \tag{4.20}$$

where $H^s_{t,a,1}$ are simulated age-specific holdings of home assets by households of age a living in country 1 and $y^s_{t,1}$ is simulated output of country 1. Normalization by output insures stationarity of u_t at $\Psi^{e,0}$ if the model fails to match growth rates.

Identification of ϕ_i is by similar conditions on labor supply. Stationarity of u_t is achieved by deterministically de-trending. The moment conditions are accordingly given by

$$\mathbb{E}\left[l_{t,i} - \gamma_i^L t - \left\{\ln\left(\sum_{a=1}^{Z} \frac{l^s_{t-a,a,i}(\Psi, X)N_{t-a,a,i}}{E_{t,i}}\right) - \gamma_i^{L,s} t\right\}\right] = 0 \; \forall i = 1, ..., R. \tag{4.21}$$

Division by $E_{t,i}$ is necessary since individual simulated labor supply is measured in efficiency units, see Section 4.3. Growth rates of labor supply, γ_i^L, are elements of Ψ^q, see below.

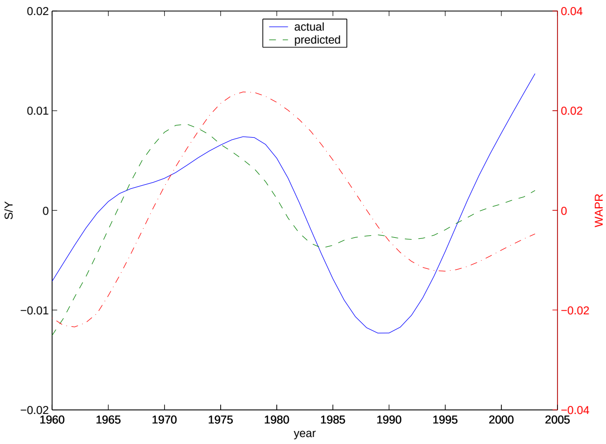
4.3.1.2 The parameters Ψ^q

Testing of the model within the calibration framework of Section 4.2 requires specification of the additional parameter vector Ψ^q. In the RBC literature, an obvious choice for Ψ^q are second moments, e.g., variance ratios of consumption to output. In the context of a deterministic model, this approach is not particularly meaningful. The basic idea of measuring variances and covariances - as being summary statistics that provide information on the time paths of variables - can however be nicely mapped into the AK-OLG framework where the statistics of interest are the relationships between the dynamics of aggregate variables and the dynamics of demographic change.

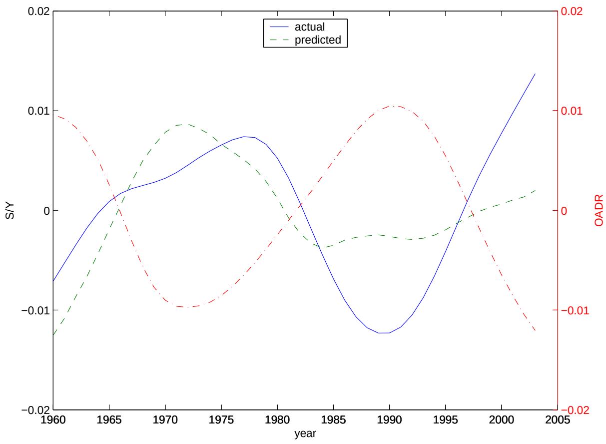
Figure 4.1 shows the time paths of the saving rate (solid line, left scale) and demographic measures such as the working age population ratio in Panel (a) and the old age dependency ratio in Panel (b) (dashed-dotted lines, right scale). The working age population ratio is defined as the ratio of the population in prime work age (aged 15 to 64) to total population and the old age dependency ratio is defined as the ratio of the old age population (aged 65 and older) to the working age population. All variables are shown as deviations from their deterministic trends. The graphs illustrate the positive relationship between the working age population ratio and the saving rate observed in the data and the strong negative relationship between the old-age dependency ratio and the saving rate. It is convenient to express such relationships in terms of correlations between the demographic measures and the macroeconomic variables of interest.

Figure 4.1: Saving rates and population statistics (deviations from trend)

a. saving rate and working age population



b. saving rate and old age dependency



Notes: Each panel of this figure shows, on the left scale, actual (solid line) and predicted (Model III, dashed line) values of saving rates and population statistics (dashed-dotted line) on the right scale. Population statistics shown are the working age population ratio in Panel (a) and the old age dependency ratio in Panel (b). All series are shown as deviations from their deterministic trends for the period 1960-2003.
Source: Own calculations, based on demographic projections of the United Nations (2002).

The figure also shows the predicted (and de-trended) saving rate for the open economy version of the model (dashed line), also see below. The predicted de-trended saving rate tracks the actual de-trended saving rate quite well with an exception being the period 1985-1995 where the decrease of the saving rate (relative to the trend) is under-predicted. Since the correlation statistic of two variables x and y normalizes the covariance by the standard deviations of both variables, this deviation would not be reflected in the correlation statistic. It is therefore convenient to express this additional information on the variation of the variables over the sample period in terms of the standard deviation of the de-trended variable of interest.

Furthermore, the simulation model may fail to match growth rates or levels of variables not used for estimation of Ψ^e. One way to summarize this is to look at the deviations of predicted growth rates of capital and labor supply.

These considerations motivate the definition of Ψ^q as

$$\begin{aligned}
\Psi^q &= [\gamma^K, \{\gamma^L\}_{i=1}^R, \sigma(x), \rho(x,z)]' \\
z &= WAPR, OADR \\
x &= \frac{K}{Y}, \frac{S}{Y} \quad \text{for the closed economy version of the model} \\
x &= \frac{K}{Y}, \frac{S}{Y}, \frac{I}{Y} \quad \text{for the open economy version of the model,}
\end{aligned}$$

where $WAPR$ and $OADR$ denote the working age population ratio and the old-age dependency ratio, respectively. $\sigma(x)$ denotes the standard deviation of variable x and $\rho(x,z)$ denotes the correlation coefficient between variables x and z.

The additional moment conditions used to estimate Ψ^q are therefore given by

$$\mathbb{E}\left[w_t - \Psi^q\right] = 0 \tag{4.22}$$

for $w_t = [\gamma_t^K, \{\gamma_t^L\}_{i=1}^R, \sigma(x_t), \rho(x_t, z_t)]$ and x_t, z_t defined as above.

4.3.2 Data

Below, different model versions of the simulation model will be used, see Section 4.4. The analysis focuses mostly on the US. In addition, a two-country open economy version of the model will be simulated. The second country thereby represents a country aggregate of all OECD countries other than the US.

For the US, national income and product accounts (NIPA) data are used taken from the Bureau of Economic Analysis (BEA). While model simulations start in 1950, the first ten years are discarded and structural model parameters are estimated using sample data for the years 1960-2003. Throughout, data for the entire economy are used. Since there is no real role for a government in the model, it is therefore implicitly assumed that the government is a substitute to the private sector. All real data are calculated using the GDP deflator.

The capital stock is defined as the sum of fixed capital held by the private and the public sector and private inventories. Depreciation is calculated to be consistent with the data on the capital stock and the investment flow satisfying the capital accumulation equation, see equation 2.4. Consumption is calculated as the sum of private consumption and government consumption. In the closed economy version of the model, no additional correction to the data by deducting consumption of imported goods and services is made. The reason for not doing this correction is that the model comparison in Section 4.4 would be flawed if different data sets were used in the open and closed economy scenarios. However, this also implies that actual data on investment and savings differ in the closed economy models, whereas simulated data on these variables are equal by definition of the closed economy. Finally, output is defined as the sum of investment and total consumption (including government consumption) which corresponds to actual GDP as observed in the data.

As a measure of aggregate gross wages, data on total compensation of employees is used which includes supplements to wages and salaries. Labor supply is measured as actual labor supply multiplied by an index for the total amount of hours worked. The wage rate is calculated as total wages divided by the weighted labor supply data.

The open economy version of the model focuses on OECD countries. Data on GDP and labor supply for these countries are taken from the World Development Indicators (World Bank 2003). Some minor adjustments are made to ensure consistency between the US data and the data used for the other model economies. This data is summed across all countries to obtain the data for the country aggregate "other OECD countries", see Section 4.4.

For sake of consistency between the demographic and the economic model, especially with regard to mortality rates that enter the household's objective function, demographic projections are explicitly calculated. They are based on the United Nations World Population Projections (United Nations 2002). The demographic model is calibrated such as to match the data. The resulting demographic data are taken as exogenous in the estimation exercises conducted in Section 4.4. Since the fit of the demographic model is good, results of errors in the imputation procedure on simulation outcomes are found to be low (results not shown).

Pension payments are calculated as the sum of the NIPA data on pension payments for old-age, survivors and disability insurance, railroad retirement, pension benefit guarantee and pension and disability insurance of veterans. The pension system's overall contribution rate is calculated by dividing pension payments through the data on wages and salary accruals. The pension system's net replacement is determined using the pension system's budget constraint in equation 2.1 for the exogenous labor supply model. Across all simulations, net replacement rates are held constant at the resulting level and contribution rates are endogenously calculated.[10] For the remaining countries, the public pension system's gross replacement rates are calculated using data from Palacios and Pallarès-Miralles

[10] This reversal in the procedure ensures that replacement rates continuously rise as implied by the exogenous labor supply scenario also in the endogenous labor supply scenario. This may not be the case if replacement rates were calculated endogenously in both scenarios holding contribution rates fixed.

(2000). Contribution rates provided in Blöndal and Scarpetta (1999) are used to calculate net replacement rates.

4.4 Results

In what follows, three different sub-models of the AK-OLG model of Section 4.3 are analyzed. Model I is an exogenous labor supply, closed economy model, hence $R = 1$ and $\phi_1 = \xi_1 = 1$. Model II is an endogenous labor supply, closed economy model and Model III is an endogenous labor supply, open economy model. In the open economy version, $R = 2$ countries (regions) will only be considered which simplifies computations. The second model region consists of all OECD countries other than the US. Table 4.1 summarizes these model properties.

Table 4.1: Properties of Models I-III

Property	Model I	Model II	Model III
Endogenous Labor Supply	No	Yes	Yes
Open Economy	No	No	Yes
Implication			
R	1	1	2
ϕ_1	1		
ξ_1	1		

Notes: This table summarizes properties of Models I-III and the implied restrictions on parameter values.

Remark Notice that the closed economy as well as the exogenous labor supply assumptions are both counterfactual. Also, the open economy model makes the assumption of perfect capital mobility which is counterfactual at least for the years 1960-1980. In other words, resulting parameter estimates are inconsistent. However, as the results in Table 4.2 show, values of estimated parameters do not differ much across the different sub-models. It can also be shown that values of estimated parameters vary monotonically as one moves between these extreme assumptions. Therefore, if the model is otherwise correctly specified and if the values of the predetermined parameters represent the true deep parameters, estimated parameter values of the different sub-models provide narrow bounds of consistent point estimates.

4.4.1 First Results: The Role of Technology

As a first step, Model I (closed economy, exogenous labor supply) is analyzed in two versions. First, it is assumed that productivity follows the constant trend growth assumption and that total factor productivity (*TFP*) $\Omega_{i,t}$ is held constant over time. In slight abuse of notation relative to Section 2, a time subscript t is added to the *TFP*-Level here. However, the constant trend growth assumption is not the most reasonable description of actual technological change. Therefore, a second version is analyzed where the assumption that $\Omega_{t,i} = \Omega_i$ for $t < 1$ and $t > T$, is maintained, i.e., out of sample, the *TFP* level is held constant, but where, in sample, $\Omega_{t,i}$ is replaced with the actual "Solow-Residual" (equivalent) resulting from the growth regressions, $SR_{t,i}$, i.e., $\Omega_{t,i} = SR_{t,i}$ for $1 \leq t \leq T$. Notice that $SR_{t,i}$ is a stationary variable in this model which explains the above use of the word "equivalent". Feeding $SR_{t,i}$ explicitly into the simulation model accounts for the effects of potential changes in aggregate productivity, like a productivity slowdown, that are ruled out by the constant growth assumption of $E_{t,i}$. The Solow-Residual (equivalent) is defined as

$$SR_{t,i} = \frac{Y_{t,i}}{K_{t,i}^{\alpha_i} L_{t,i}^{1-\alpha_i}}.$$

Recall that $L_{t,i}$ is efficient labor which is trending over time.

Feeding the actual Solow-Residual, $SR_{t,i}$, into the model implies that output during the simulation period is given by

$$Y_{t,i}^s = SR_{t,i}(K_{t,i}^s)^{\alpha_i}(L_{t,i}^s)^{1-\alpha_i},$$

where $Y_{t,i}^s$, $K_{t,i}^s$ and $L_{t,i}^s$ denote simulated output, capital and labor, respectively. This also implies that simulated wages are given by

$$w_{t,i}^{g,s}(1 + 0.5\tau_{t,i}) = (1 - \alpha_i)SR_{t,i}(K_{t,i}^s/L_{t,i}^s)^{\alpha_i}$$

and simulated interest rates by

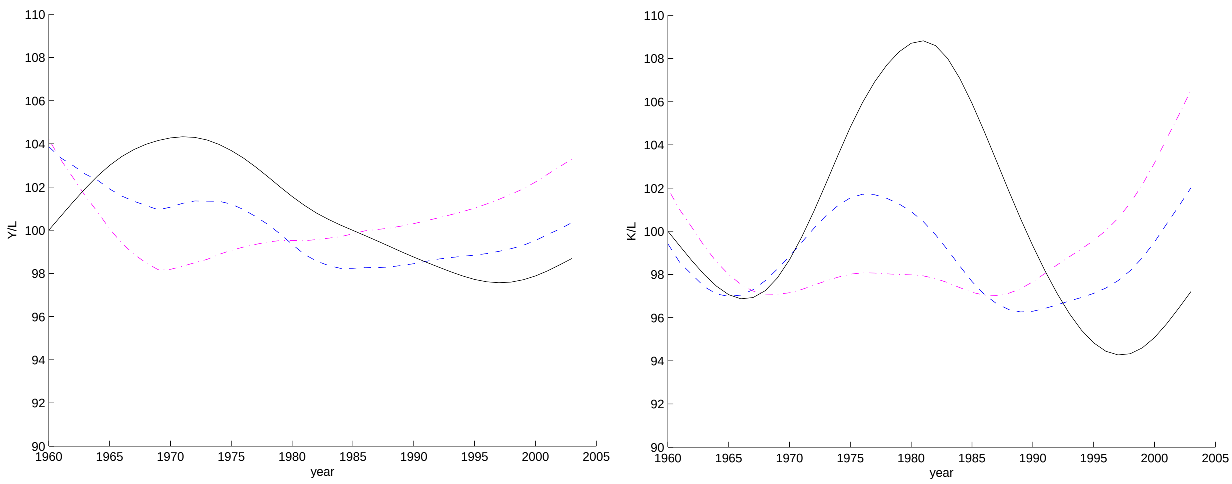
$$r_{t,i}^s = \alpha_i SR_{t,i}(L_{t,i}^s/K_{t,i}^s)^{1-\alpha_i} - \delta_i.$$

The additional argument $SR_{t,i}$ hence affects the time paths of households labor and asset income and thereby alters their labor supply, consumption and savings decisions relative to the constant trend growth assumption.

Figure 4.2 shows results on actual and predicted output and the capital stock per efficient unit of labor, $Y_{t,1}/L_{t,1}$ and $K_{t,1}/L_{t,1}$, respectively, for the two versions of Model I. While the model fails to match the time paths of both variables, the "Solow-Residual" model version does a much better job, especially with regard to tracking the observed swings of the capital-output ratio. For this reason, the remainder of the analysis focuses on models where, in sample, the constant technology level is replaced with the actual Solow-Residual (equivalent).

Figure 4.2: The role of technology: Output and capital stock per efficient unit of labor for Model I

a. output b. capital stock



Notes: This graph shows actual values (solid line) and predicted (Model I) values (dashed dotted line: constant TFP, dashed line: Solow-Residual) of output and of the capital stock per efficient unit of labor. *Source:* Own calculations, based on demographic projections of the United Nations (2002).

4.4.2 Main Results: The Roles of Endogenous Labor Supply and Openness

4.4.2.1 Parameter Estimates of Ψ^e

Table 4.2 contains predetermined and estimated parameter values of the vector of structural model parameters $\Psi^c = [(\Psi^p)', (\Psi^e)']'$ for Models I through III. Values of predetermined parameters, Ψ^p, are chosen in accordance with the literature. The value of the elasticity parameter ξ corresponds to the value chosen by Altig, Auerbach, Kotlikoff, Smetters, and Walliser (2001). Values of estimated parameters, Ψ^e, are within ranges considered as reasonable in the literature. The point estimates of the discount factor, β, correspond to the value of the discount rate of 0.011 estimated by Hurd (1989). Notice, however, that the estimated value depends on the value of the predetermined parameter θ, the coefficient of relative risk aversion. A higher (lower) θ-value implies a higher (lower) discount factor (results not shown).

Table 4.2: Structural Model Parameters Ψ^c for Models I-III

Ψ^p	Model I	Model II	Model III
θ: coefficient of relative risk aversion	2	2	2
ξ: intra-temporal substitution elasticity	1	0.8	0.8
ϕ_1: consumption share parameter	1		
Ψ^e		Models I-III	
δ: depreciation rate		0.037	
		(0.002)	
α: capital share parameter		0.329	
		(0.004)	
g: growth rate		0.017	
		(0.002)	
Ω_1: technology level		0.077	
		(0.002)	
Ψ^e	Models I	Model II	Model III
Ω_2: technology level			0.062
			(0.004)
β: discount factor	0.991	0.996	0.989
	(0.004)	(0.005)	(0.005)
ϕ_1: consumption share parameter		0.608	0.610
		(0.009)	(0.009)
ϕ_2: consumption share parameter			0.570
			(0.007)

Notes: This table shows predetermined parameter values, Ψ^p, and estimated parameter values, Ψ^e, of the structural model parameters Ψ^c for Models I-III. Standard errors are calculated using the Hansen-Hodrick-White (HHW) estimator with bandwidth parameter $b = 4$.
Source: Own calculations, based on demographic projections of the United Nations (2002).

Standard errors of the estimated parameters Ψ^e are based on the un-weighted, truncated kernel Hansen-Hodrick-White (HHW) estimator of $\widehat{S}_T$ given by

$$\widehat{S}_T = \sum_{i=-T+1}^{T-1} k(i)\widehat{C}_i$$

with

$$\widehat{C}_l = \frac{1}{T-J}\sum_{t=l}^{T} u_t(\widehat{\Psi^e}_T)u_{t-l}(\widehat{\Psi^e}_T)'$$

and with the Bartlett kernel defined as

$$k(i) = \begin{cases} \left(1-\frac{|i|}{b}\right)^{\upsilon}, & 0 \leq |i/b| \leq 1 \\ 0, & |i/b| > 1. \end{cases}$$

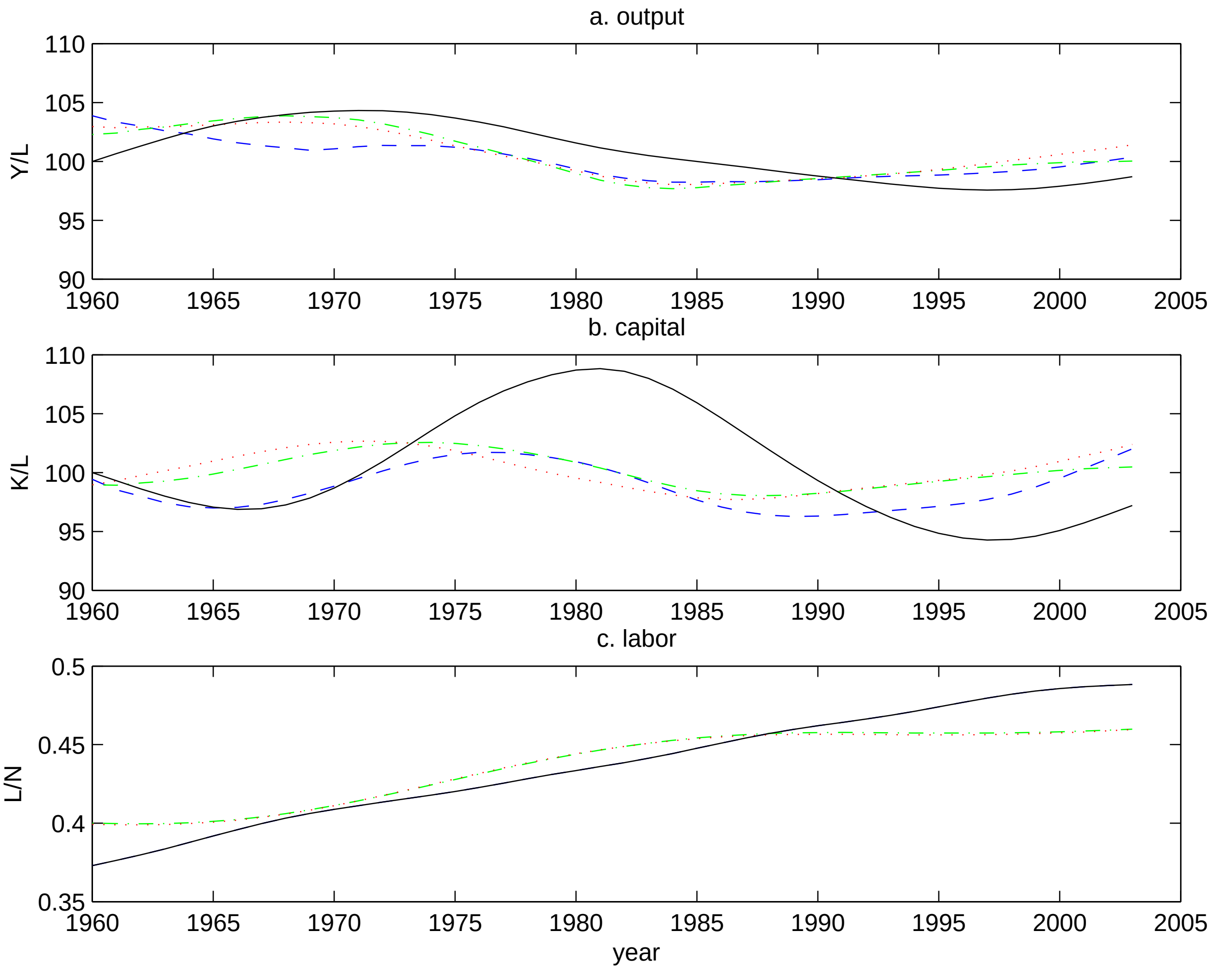
for $\upsilon = 0$ and for a fixed bandwidth of $b = 4$ years (Hansen and Hodrick 1980; White 1984). Results obtained with the alternative Newey-West (Newey and West 1987) kernel estimator with $\upsilon = 1$ are similar. The advantage of the HHW-Estimator over the NW-Estimator estimators is that it does use all the information in $\widehat{C}_t$ until the truncation point. The disadvantage is that positive definiteness of the resulting estimate of $\widehat{S}_T$ is not guaranteed. Here, this was not the case for $b = 4$. Consistency of $\widehat{S}_T$ requires that the truncation point, the bandwidth parameter b, approaches infinity at the appropriate rate as T goes to infinity (Andrews and Monahan 1992). Automatic selection criteria for the optimal bandwidth b that optimize asymptotic efficiency criteria have been developed by Andrews (1991) and Newey and West (1994). However, as discussed by Christiano and Den Haan (1996), neither of these procedures is entirely automatic since they require exogenous parameter selection at a different stage. Therefore, results obtained for a fixed bandwidth are reported here. The parameters are estimated with high precision, see Table 4.2.

4.4.2.2 Informal Model Evaluation

Figures 4.3 and Figures 4.4 summarize simulation results obtained for Models I-III if the Solow-Residual (equivalent) replaces the constant TFP level. As before, the solid lines represent the data and the dashed lines represent results for Model I (closed economy, exogenous labor supply). Simulation results for Model II (closed economy, endogenous labor supply) are represented by the dashed-dotted lines and results for Model III (open economy, endogenous labor supply) are represented by the dotted lines.

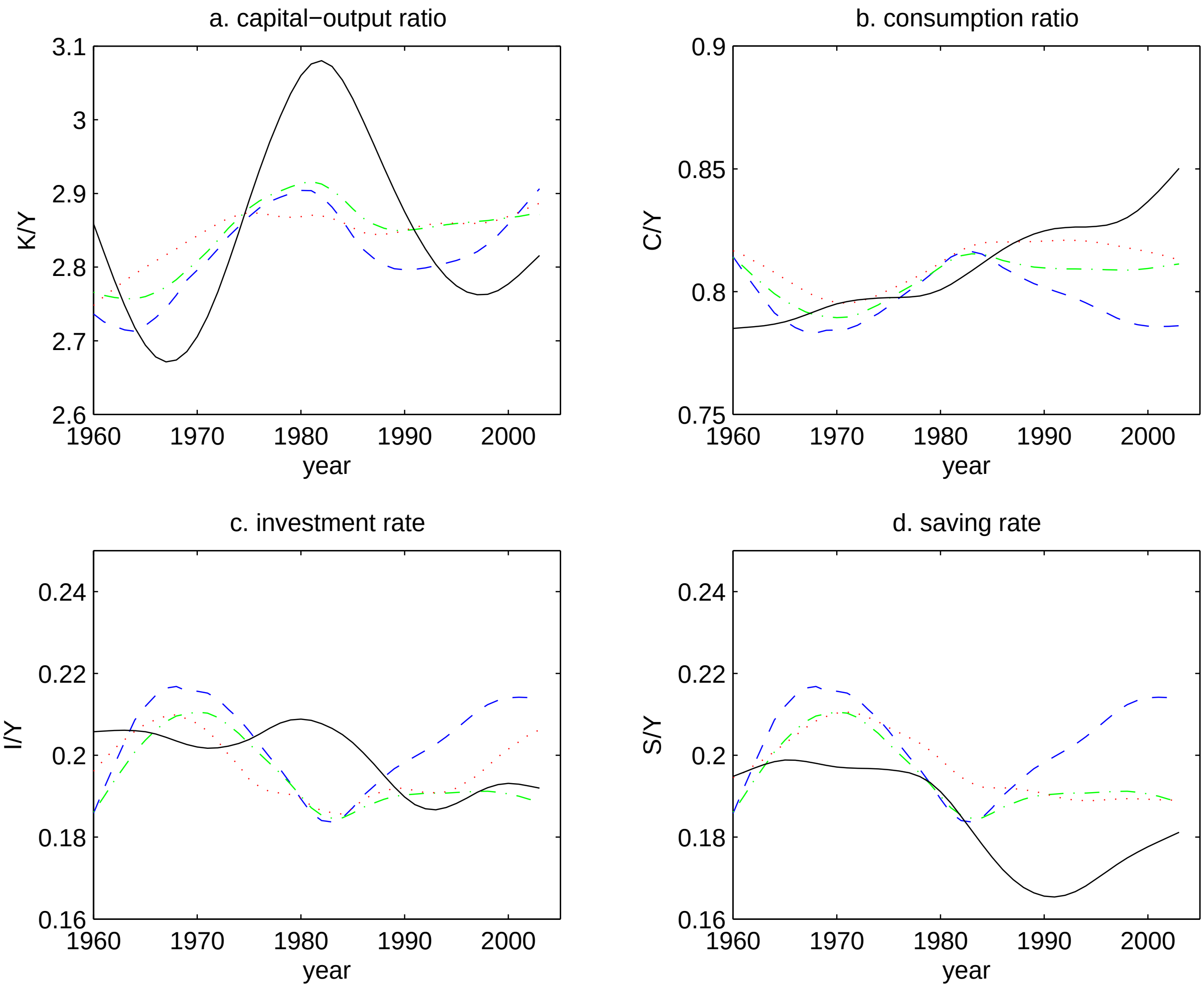
Results can be summarized as follows: First, the endogenous labor supply model fails to match the average growth rate of actual labor supply, see Panel c of Figure 4.3 depicting

Figure 4.3: The roles of endogenous labor supply and openness: Output and capital stock per efficient unit of labor and labor supply for Models I, II and III



Notes: This graph shows actual (solid line) and predicted (Models I-III) values (dashed line: Model I, dashed-dotted line: Model II, dotted line: Model III) of output and the capital stock per efficient unit of labor and of labor supply.
Source: Own calculations, based on demographic projections of the United Nations (2002).

Figure 4.4: The roles of endogenous labor supply and openness: Capital stock, consumption investment and savings as percentage of GDP for Models I, II and III



Notes: This graph shows actual values (solid line) and predicted (Models I-III) values (dashed line: Model I, dashed-dotted line: Model II, dotted line: Model III) of the capital-output ratio, the consumption rate, the investment rate and the saving rate.
Source: Own calculations, based on demographic projections of the United Nations (2002).

actual and predicted labor supply shares. Results on predicted labor supply shares between Models II and III are indistinguishable. As further shown in Table 4.3 below, the model at the same time overestimates the trend growth rate of labor supply in the second country. The failure of the model to match the data along the labor supply dimension is not related to the predetermined parameter ξ (results not shown). It can therefore be concluded that the above way of modelling labor supply is an imperfect approximation of actual labor supply decisions.

Second, while Model I to some extent matches the timing of swings (but not their amplitudes) of the actual capital-output ratio, this is no longer the case for Models II and III prior to about 1980, see Panel (b) of Figure 4.3. Both, modelling endogenous labor supply and openness also "smoothes out" the variation of the capital-output ratio; see also Panel (a) of Figure 4.4. Third, Models II and III seem to track de-trended output a bit closer, see Panel (a) of Figure 4.3.

Fourth, Model I appears to lead the data by about ten years with respect to the fall of the saving rate observed in the early 80s and the subsequent rise observed in the 90s, see Panel (d) of Figure 4.4. The drop of the saving rate also appears too early in Model II, whereas for Model III the decline of savings appears at the same time as observed in the data, see also Figure 4.1. For both Models II and III predicted saving rates remain roughly constant throughout the 80s and 90s. Fifth, and in correspondence with these findings, Models II and III do a slightly better job in tracking the persistent increase in the consumption-output ratio, see Panel (b) of Figure 4.4. Finally, non of the models matches the time path of the investment ratio, see Panel (c) of Figure 4.4.

4.4.2.3 Formal Model Evaluation

Results on the moments of the data collected in Ψ^q and their simulated counterparts $h^q(\Psi^e)$ are shown in Table 4.3. These results more or less confirm the findings obtained in the graphical analysis. For instance, since all models fail to match the actual variation of the capital-output ratio, the predicted standard deviation of the de-trended capital output ratio is lower than in the data (and it decreases across models). All models replicate the positive (and significant) correlation between the capital-output ratio and the working age population ratio. The correlation between the old-age dependency ratio and the capital-output ratio is found to be insignificant in the data which is replicated by Model II (although with the wrong sign).

All models are found to replicate the sample variation of the saving rate. The correlation between the saving rate and the working age population ratio is found to be insignificant which is replicated by Models I (although with the wrong sign) and II but not by Model III

Table 4.3: Parameters Ψ^q, simulated values $h^q(\Psi^e)$ and J-Statistics for Models I-III

Parameter	Ψ^q	$h^q(\Psi^e)$		
	Data	Model I	Model II	Model III
γ^K	0.033	0.034	0.031	0.031
	(0.003)	(0.004)	(0.005)	(0.004)
γ_1^L	0.017		0.014	0.014
	(0.001)		(0.001)	(0.001)
γ_2^L	0.009			0.013
	(0.001)			(0.001)
$\sigma(K/Y)$	0.122	0.044	0.035	0.022
	(0.022)	(0.019)	(0.019)	(0.022)
$\rho(K/Y, WAPR)$	0.667	0.941	0.946	0.916
	(0.299)	(0.250)	(0.277)	(0.294)
$\rho(K/Y, OADR)$	0.154	-0.590	-0.291	-0.642
	(0.225)	(0.227)	(0.243)	(0.255)
$\sigma(S/Y)$	0.007	0.009	0.006	0.005
	(0.001)	(0.002)	(0.001)	(0.002)
$\rho(S/Y, WAPR)$	0.344	-0.311	0.150	0.641
	(0.256)	(0.203)	(0.230)	(0.237)
$\rho(S/Y, OADR)$	-0.900	-0.586	-0.717	-0.838
	(0.269)	(0.234)	(0.264)	(0.270)
$\sigma(I/Y)$	0.005			0.006
	(0.001)			(0.002)
$\rho(I/Y, WAPR)$	0.672			-0.481
	(0.320)			(0.315)
$\rho(I/Y, OADR)$	-0.385			-0.484
	(0.282)			(0.283)
J-Statistic				
J_7: Ψ^q elements of Model I		288.536	262.050	94.306
		[0.000]	[0.000]	[0.000]
J_3: S/Y		15.934	2.471	6.620
		[0.001]	[0.485]	[0.088]

Notes: The upper part of this table shows estimated values of the model parameters Ψ^q and their simulated counterparts $h^q(\Psi^e)$ for Models I-III. Standard errors are reported in parentheses. The lower part shows results of two J-Statistics: J_7 is the J-Statistic based on the (7×1) vector $[\gamma^K, \sigma(x), \rho(x, WAPR), \rho(x, OADR)]'$ for $x = K/Y, S/Y$. J_3 is the J-Statistic based on the (3×1) vector $[\sigma(S/Y), \rho(S/Y, WAPR), \rho(S/Y, OADR)]'$. p-values are reported in brackets.
Source: Own calculations, based on demographic projections of the United Nations (2002).

(but with the correct sign). All models match the significant negative correlation between the saving rate and the old age dependency ratio.

Results of formal J-Tests are reported in the lower part of Table 4.3. J_7 is the J-Statistic based on the all moments relevant for Model I, hence the (7×1) vector $[\gamma^K, \sigma(x), \rho(x, WAPR), \rho(x, OADR)]'$ for $x = K/Y,\ S/Y$. Unsurprisingly, all models are rejected according to this criterion. The J_3-Statistic is based on all moments of the saving rate, that is, on the (3×1) vector $[\sigma(S/Y), \rho(S/Y, WAPR), \rho(S/Y, OADR)]'$. According to the findings of this statistical criterion, Models II and III cannot be rejected with regard to the moments of the actual saving rate at the 0.48 and the 0.08 level of significance, respectively.

4.5 Conclusions

This chapter develops a systematic calibration procedure for large-scale Auerbach-Kotlikoff-OLG (AK-OLG) models in outside steady state situations. Structural model parameters are estimated by matching first moments of model predicted, in some cases simulated, values to long time series of aggregate data. It is found that the procedure works well and that resulting parameter values are within ranges considered as reasonable in the literature. As an illustration, three versions of a prototype AK-OLG model are evaluated using informal graphical analysis and by formal statistical criteria that complement the graphical analysis.

The illustrative AK-OLG model developed in this chapter is an open economy AK-OLG model that features realistic demographic profiles. While it is well-suited for the questions addressed in this chapter along these two dimensions, a number of aspects which have been regarded as important in the literature are missing: For example, the model does not account for bequest motives, within age group heterogeneity, idiosyncratic and/or aggregate uncertainties, human capital formation and a detailed representation of the government sector (Imrohoroglu, Imrohoroglu, and Joines 1995; Conesa and Krueger 1999; Altig, Auerbach, Kotlikoff, Smetters, and Walliser 2001; Krueger and Kubler 2003). Against this background, results derived from the model evaluation procedure must be tentative. They nevertheless allow the following insights: First, allowing the actual Solow-Residual resulting from growth regressions to enter the simulation model rather than assuming TFP to grow linearly at a constant rate significantly improves the performance of the exogenous labor supply version of the simulation model. Second, modelling endogenous labor supply decisions as resulting from pure life-time utility maximization over consumption and leisure fails to match the data. Third, the endogenous labor supply and open economy versions of the model are shown to match the saving rate quite well. Finally, all models fail to match the time paths of investment and consumption.

What explains these discrepancies between actual and simulated data? Certainly, a good proportion of the discrepancies may be due to the features missing in the model, and the failure of the model also reflects the inadequacy of the life-cycle theory of consumption and savings (Attanasio 1999). The above mentioned results point to three distinct but related aspects which may provide guidance for future model developments: First, the results suggest that the way in which technological progress is modelled matters. This is not only important for the back-fitting implications but also for the analysis of future macroeconomic developments and of future public policy. Second, better models of the labor market are needed and third, improved ways of modelling the open economy and physical capital investment are required.

A related aspect that is not addressed in the above analysis is the role of capital depreciation. The constant depreciation rate assumption made above may explain why a model that is augmented with the actual Solow-Residual still fails to match a large proportion of the observed fluctuations of the capital-output ratio. The importance of both, non-constant technology and non-constant depreciation may also point into the direction of missing intrinsic model uncertainty that could be modelled by adding technology shocks and shocks to depreciation. Such extensions, however, imply a huge increase in the computational costs required to solve such models (Krueger and Kubler 2003).

A few final comments on the econometric methodology are in order. The econometric methodology applied here is with a classical statistical perspective. In other words, calibration parameters are regarded as an unknown but fixed number. The uncertainty reflected in the estimated variance-covariance matrix is due to sampling uncertainty. Apart from exogenously fixing values of predetermined model parameters, which is equivalent to assuming degenerated priors in a Bayesian sense, prior information on model parameters is not incorporated. For the last decade, the RBC literature has seen numerous developments of Bayesian approaches to estimate and test dynamic macroeconomic models. Bayesian methods regard parameter values themselves as random variables and express inference in statements of probability regarding their value. They are the standard procedure to combine uncertainty about prior distributions of parameter values with the uncertainty implied by the data.

In the context of the above application, Bayesian methods would "kill three birds with one stone": First, they do not require the artificial distinction between predetermined and estimated parameters made above. Second, they incorporate uncertainty over all model parameters and allow for use of prior knowledge on parameter values derived from other studies. Finally, the literature more recently developed methods not only to compare models to the data but also to compare different sub-models. The Bayesian approach is attractive in this context since model uncertainty is handled in the same manner as any other uncertainty in the model even if models are not nested (Fernández-Villaverde and Rubio-Ramírez 2002). Embedding the above analysis in a Bayesian framework is subject to future research.

5 Aging, Pension Reform and International Capital Markets

This chapter is based on Börsch-Supan, Ludwig, and Winter (2004). A quantitative analysis of the effects of population aging and pension reform on international capital markets is presented. First, demographic change alters the time path of aggregate savings within each country. Second, this process may be amplified when a pension reform shifts old-age provision towards more pre-funding. Third, while the patterns of population aging are similar in most countries, timing and initial conditions differ substantially. Hence, to the extent that capital is internationally mobile, population aging will induce capital flows between countries. All three effects influence the rate of return to capital and interact with the demand for capital in production and with labor supply. In order to quantify these effects, the computational general equilibrium model introduced in Chapter 2 is used. As in Chapter 4 detailed long-term demographic projections are used in this multi-country overlapping generations model, but now for seven world regions. The simulation results suggest that capital flows from fast-aging regions to the rest of the world will be substantial. Furthermore it is shown that closed-economy models of pension reform miss quantitatively important effects of international capital mobility.[1]

5.1 Introduction

In the vast majority of countries, populations are aging, and demographic change will continue well into the 21st century. While the fact of population aging is common to most

[1]Axel Börsch-Supan and Joachim Winter have co-authored the paper covered in this chapter. We thank Alan Auerbach, Ralph Bryant, Hans Fehr, Alexia Fürnkranz-Prskawetz, Ulrich Grosch, Florian Heiss, Heinz Hermann, Gary Hufbauer, Ulf von Kalkreuth, Florence Legros, Melanie Lührmann, Shinichi Nishiyama, Howard Rosen, Tarmo Valkonen for their helpful remarks on this line of research, two anonymous referees for their comments, and Holger Herz and Max Flötotto for their excellent research assistance. We also received helpful feedback at many conferences and seminar presentations. This ongoing research project is supported by the Volkswagenstiftung, the Deutsche Forschungsgemeinschaft, the Land of Baden Württemberg, the Gesamtverband der deutschen Versicherungswirtschaft, and the US Social Security Administration.

countries, extent and timing differ substantially, even within the industrialized countries. It is well known that within each country, demographic change alters the time path of aggregate savings. In a world of closed economies, differential aging will generate additional international differences in saving rates, investment, and rates of return. These differences are likely to be accentuated when some countries implement fundamental pension reforms - that is, shifts to-wards more pre-funding, induced by the effects of population aging on public pension budgets. In reality, closed economies do not exist but capital markets are global. To the extent that capital is internationally mobile, population aging will therefore induce capital flows between countries, and these capital flows will modify the effects of population aging and pension re-form in each county vis-à-vis a world of closed economies. This Chapter focuses on these effects of population aging and pension reform on international capital markets and other key macroeconomic variables.

A quantitative analysis of capital and labor market effects is presented and, in particular, of capital flows induced by differential aging processes across countries and by pension reforms. In order to quantify these effects, the computational general equilibrium model introduced in Chapter 2 is used. As in Chapter 4 detailed long-term demographic projections are used in this multi-country overlapping generations model, but now for seven world regions. Although all countries and regions are modeled symmetrically as large open economies, the presentation focuses on Continental Europe as one of the world regions most severely affected by aging which, at the same time, has pension systems dominated by still relatively generous pay-as-you-go (PAYG) financed public pensions.

The “triangular” relationship between population aging, pension reform and international capital markets receives increasing attention in the academic literature, see Börsch-Supan, Ludwig, and Winter (2002), INGENUE (2001), Fehr, Jokisch, and Kotlikoff (2003, 2004). Here, a rich modeling framework is used which allows to address different strands of the academic literature. First, the analysis is related to several recent papers that compare implications for capital flows predicted by OLG models with actual current account data, see, e.g., Brooks (2003), Feroli (2002), Henriksen (2002), Domeij and Floden (2004). Their analyses show that calibrated OLG models may explain a good fraction of the low frequency movements of international capital flows as observed in the data. In addition, it is shown in this chapter that the existence of PAYG pension systems in different world regions adds an additional indirect channel to the interaction between capital flows and demographic change. This channel is of particular importance if countries severely affected by the impact of population aging such as the continental European countries reform their pension systems.

Second, the analysis adds to the discussion about the so-called ”asset market meltdown hypothesis”. Several articles in the popular press have attributed recent turbulences in stock market prices to population aging and raised the fear that an asset market meltdown might occur when the baby boom generation decumulates its assets. In the academic literature, there is no consensus on the asset market meltdown hypothesis (see e.g. Poterba (2001), Abel (2001), and Brooks (2002a). Here it is shown that closed-economy models often used in the academic literature miss the important fact of international capital flows. Due to international diversification, the dynamics of capital accumulation and rates of return are different from what would be predicted by closed-economy models. One of the main goals of the analysis is to analyze and quantify these mechanisms.

Third, the analysis sheds light on the effects of international diversification on savings behavior and its interaction with pension reforms. This topic has received increasing attention as the pension reform debate progresses. Deardorff (1985) contains an early analysis, and Reisen (2000) provides a comprehensive overview of these issues. Reisen argues that there are pension-improving benefits of global asset diversification. In a theoretical paper, Pemberton (1999) highlights the importance of international externalities caused by the effects of national pension and savings policies on the world interest rate. Pemberton (2000) goes a step further and shows that an intergenerational Pareto improvement through coordinated pension reforms is possible. This policy issue will not be tackled here; instead the welfare analysis is restricted to the direct welfare effects of population aging, pension reform, and capital mobility.

Finally, from an economic modeling perspective, similar to Chapter 4 the analysis sheds new light on the various interactions among different features of calibrated OLG models. To this end, a sensitivity analysis is presented that subsequently switches off features of the model. This approach allows, for instance, to compare the effects of demographic change and of fundamental pension reforms in a model with and without endogenous labor supply. For results on additional simulation outcomes with regard to structural model parameters, the reader is referred to Börsch-Supan, Ludwig, and Winter (2004).

The simulations predict substantial capital flows due to population aging. Population aging results in decreases of the capital-to-output ratio when the baby boomers decumulate their assets. International capital flows follow this trend. The countries most affected by aging such as the European Union will initially be capital exporters, while countries less affected by aging like the United States und other OECD regions will import capital. Current account positions are projected to reverse when the baby boom generations decumulate assets. Fast-aging economies are therefore projected to become capital import countries after about 2030. Pension reforms with higher degrees of pre-funding are likely to induce more capital exports. They also increase labor supply considerably, while the effects on the rate of return to capital are small. While it declines in response to population aging, there is no devastating "asset meltdown".

The remainder of this chapter is structured as follows. Section 5.2 presents empirical evidence on, and theoretical explanations for, the effects of population aging on international capital flows. Section 5.3 presents the version of the model used here and discusses some differences in the calibration methodology relative to the methodology developed in Chapter 4. Section 5.4 contains the ex ante simulation results for several pension policy and capital mobility scenarios applying a version of the model presented in Chapter 2. Section 5.5 presents an extensive sensitivity analysis. Section 5.6 concludes.

5.2 Some Facts about Population Aging and International Capital Flows

Throughout the world, demographic processes are determined by the demographic transition which is characterized by falling mortality rates followed by a decline in birth rates, resulting in population aging and reducing the population growth rate (in some countries, even turning it negative). While the patterns of demographic change are similar in most countries, extent and timing differ substantially. Europe and some Asian countries have almost passed the closing stages of the demographic transition process while Latin America is only at the beginning stages (Bloom and Williamson 1998). North America is in between. So far, characteristics of a demographic transition process cannot be identified in Africa - fertility is at the highest level worldwide, and even though child mortality is declining, life expectancy is still very low (United Nations 2002).[2]

In order to capture projected differences in demographic change across the world (particularly within the European Union) and differences in the generosity of public pensions systems, seven world regions are distinguished in the benchmark scenario: (i) France, (ii) Germany , (iii) Italy as three European countries severely affected by population aging, (iv) the remainder of the European Union, (v) North America (the US and Canada), (vi) the remaining OECD countries, and (vii) all other countries in the world. While France, Germany, and Italy are treated as separate countries in the simulations, the presentation simplifies by aggregating them, except for Sections 5.4.4 and 5.4.5, where results at the individual countries' level are presented.

Figure 5.1, based on United Nations (2002), shows for these regions the effects of demographic change on two important demographic measures, the working age population ratio (the number of persons aged 15 to 65 as a percentage of total population) and the old-age dependency ratio (the number of persons older than 65 as a percentage of the working age population). A number of lessons can be learned from these graphs. First, all of the

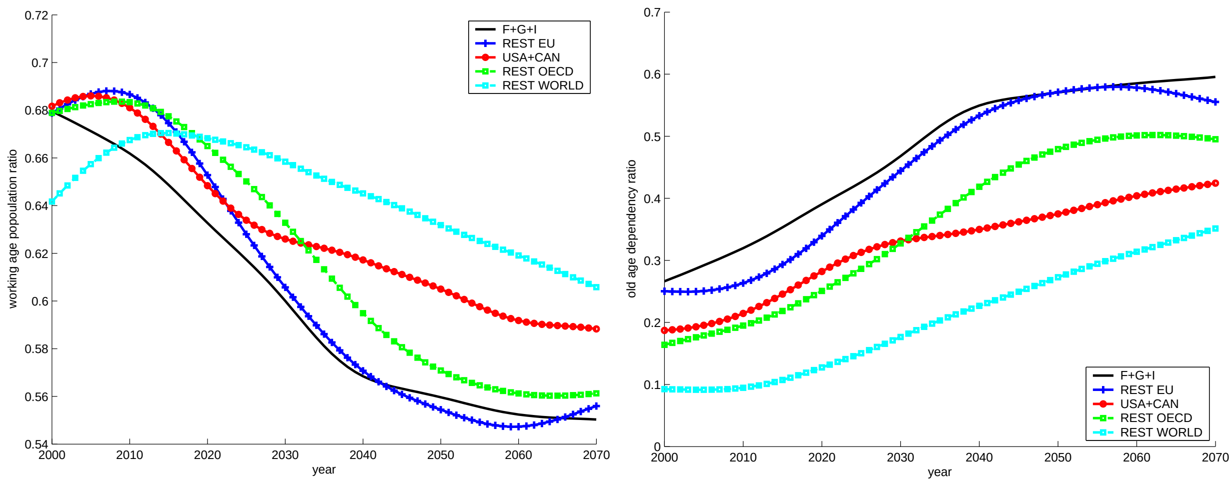
[2] Only in part due to the enormous impact of AIDS.

world regions that are considered are affected by the consequences of demographic change - increasing life expectancies and falling fertility rates - resulting in decreasing working age population ratios and increasing old-age dependency ratios. Second, while working-age population ratios are more or less identical in 2000 for the OECD countries, the decrease in working age population ratio is strongest for the European Union countries, especially the three-country group of France, Germany, and Italy. Third, the latter group has the highest level of the old-age dependency ratio. Forth, there are significant differences in the timing and the pattern of demographic change across regions. As shown below and as the results of Chapter 4 suggest, these different patterns have profound implications for the evolution of saving rates, rates of return and international capital flows.

Figure 5.1: Projections of working age and old-age population ratios for different world regions

Working age population rate

Old age dependency ratio



Notes: These figures show projections of the working-age population ratio - the number of people aged 15 to 65 as a percentage of total population - and the old-age dependency ratio - the number of people older than 65 as a percentage of the working age population - for five different world regions. G+F+I: Germany, France and Italy; REST EU: the remaining countries of the European Union; USA+CAN: the United States and Canada; REST OECD: the remaining OECD countries; REST WORLD: the remaining world countries.

Source: Own calculations, based on demographic projections of the United Nations (2002).

From a macroeconomic point of view, population aging will change the balance between capital and labor, in particular in industrialized countries. Labor supply will be scarce whereas capital will be relatively abundant. This will drive up wages relative to the rate of return on capital, reducing households' incentive to save (if the interest elasticity of saving is positive).

Differences in timing of demographic change across countries and regions induce international capital flows. Theoretical arguments that establish this link build on the well-known life-cycle theory of consumption and savings by Modigliani, Ando and Brumberg (Modigliani and Brumberg 1954, Ando and Modigliani 1963). The aggregation of individual, cohort-specific life-cycle savings profiles leads to a decrease of national saving rates in an aging economy. In a general equilibrium model of forward-looking individuals, it is not only the current demographic structure that alters the time path of aggregate savings, but also future demographic developments. There are two main channels for effects of demographic change on domestic capital formation. First, decreasing labor supply reduces the demand for investment goods since less capital is needed. Second, in a closed economy, a decline in national savings leads to a decline in investment by definition. In an open economy, the link between these two aggregates is broken to the extent that capital is internationally mobile, see below.

Empirical evidence on how demographic change has affected saving behavior across countries in the past is reviewed by Poterba (2001). Following earlier work by Higgins (1998) and others, Lührmann (2003) investigates whether demographic factors have influenced international capital flows in the past. She uses a broad panel of 141 countries that covers the period 1960-1997 to investigate the effects of demographics on international capital flows. She confirms that cross-country capital flows are indeed influenced by demographic variables. Moreover, she shows that relative differences in the age structure across countries are the most important determinants of capital flows. In addition, Lührmann (2003) shows that future changes in the age structure of countries are important determinants of current saving and investment decisions, a finding that confirms forward-looking household behavior.

For quantitative projections of international capital flows induced by population aging, the degree of capital mobility is crucial. This is essentially an empirical question, and there has been no shortage of research on this issue since the famous puzzle of Feldstein and Horioka (1980). In their original contribution, Feldstein and Horioka have shown that national saving and investment rates are highly correlated in virtually all OECD countries. While the coefficient has fallen over time, it is still remarkably high. These findings have been interpreted as an indication that capital is imperfectly mobile. However, there is no lack of alternative explanations for the observed correlation. For example, high correlations between saving and investment rates are consistent with perfect capital mobility in a growth model with demographic change and technological progress, as pointed out by Obstfeld (1986); see also Baxter and Crucini (1993), Taylor (1994), Obstfeld and Rogoff (1998, 2000).

Even if capital is fully mobile, this does not necessarily imply that households do actually diversify their portfolios optimally. There is a large empirical literature on "home bias" in international portfolio choice (e.g., French and Poterba 1991), and it is not yet fully understood why households do not optimally diversify their portfolios across countries. Portes and Rey (2004) suggest that information asymmetries across countries are a major source of home bias effects and that capital flows are affected by both geographic and informational proximity. Applied to pension reform policies, this literature suggests that households might be more willing to invest their retirement savings in "similar" countries such as the EU or OECD countries than in, say, developing countries. For the lack of a better model of capital mobility, a symmetric model is not constructed here, but instead several capital mobility scenarios are built from the point of view taken by the three largest economies in Continental Europe (France, Germany, and Italy). Then the polar cases of France, Germany, and Italy as a closed capital market and of perfect capital mobility within increasingly large regions (entire EU, entire OECD, and entire world) are considered. This approach allows to understand the effects of capital mobility on savings, investment, and rates of return in the future even though the true effect might be smaller.

5.3 The Structure of the OLG Model

The model is as described in Chapter 2, but there are some differences in model calibration relative to the methodology described in Chapter 4. The methodology followed here is more conventional in that calibration is done by reference to other studies and by informally matching of moments. The reason for this discrepancy is due to the fact that the paper by Börsch-Supan, Ludwig, and Winter (2004) was written prior to the development of the calibration methodology discussed in Chapter 4. However, since the estimated parameter values in Chapter 4 are close to what is regarded as reasonable in the literature, and since calibration here is at least partially done by reference to these parameter values, the resulting differences are not large.

The model is calibrated for the seven benchmark regions mentioned earlier: (i) France, (ii) Germany (iii) Italy, (iv) the remainder of the European Union, (v) North America, (vi) the remaining OECD countries, and (vii) all other countries in the world. In the benchmark model, capital mobility is restricted to the OECD area but capital flows freely within this area.

The timeline of the model is as described in Chapter 4. Due to the use of the World Development Indicators Data (World Bank 2001), the calibration period runs from 1960 to

2001 only. While the projection period runs until 2200, results are displayed only through the year 2070 to show the main period of population aging.

In order to solve the pension system for each country, net replacement rates are assumed constant over time at current levels. Then, the associated time path of the contribution rate is calculated. The pension systems are calibrated with data on gross replacement rates taken from Palacios and Pallarès-Miralles (2000). Net replacement rates are calculated as gross replacement divided by 1 minus taxes and employee's social security contributions taken from OECD (2001). Further net replacement rates are normalized such as to match the average net replacement rates of a standard pensioner of roughly 70 percent in Germany.

Further parameters of the model are the households' preference parameters, the parameters of the production function, and values of the age-specific productivity profile. For the latter, the cohort-corrected non-linear regression estimates presented in Fitzenberger et al. (2001) are applied. The representative age-wage profile peaks at the age of 52 and then decreases slightly.

As in Chapter 4, apart from three exceptions, technological and preference parameters are assumed to be constant and equal across all countries. More precisely, it is assumed that

$$g_i = g;\ \alpha_i = \alpha;\ \zeta_i = \zeta;\ \delta_i = \delta;\ \psi_i = \psi;\ \beta_i = \beta;\ \sigma_i = \sigma;\ \xi_i = \xi,\ \forall i = 1, ..., R$$

Parameter values of these parameters are standard in the literature and summarized in Table 5.1. The growth rate of productivity, g, is set to 1.5 percentage points which is in between the value of 1.4 percentage points suggested by Cutler et al. (1990) and the estimate in Chapter 4. The capital share parameter, α, is usually set to $0.3 - 0.4$. Here, it is set to 0.35, slightly lower than the value estimated in Chapter 4. The annual depreciation rate, δ, is assumed to be 5 percentage points per year which is higher than the estimate in Chapter 4 but closer to a standard estimate for, e.g., Germany. As in Chapter 4, a Cobb-Douglas production function is imposed. The elasticity of substitution between capital and labor, ζ, is therefore set to one. Börsch-Supan, Ludwig, and Winter (2004) consider the more general CES production for the sensitivity analysis and show that deviations from Cobb-Douglas do not affect results much.

The adjustment cost parameter, ψ, deserves more discussion. In a model without depreciation but with capital taxation, and with a lower growth rate, g, of 1 percentage point, the value for ψ equal to 10 as chosen by Altig et al. (2001) results in a steady state q-value of 1.04. The empirical study by Oliner, Rudebusch, and Sichel (1995) results in an equilibrium q-value of 1.13. In the model used here, with a productivity growth rate of 1.5 percentage points and a depreciation rate of 5 percent-age points, the value of $\psi = 1.5$ chosen results in a steady state q-value of 1.0975 which is just in between these two values.

Table 5.1: Calibration parameters

Parameter	**Value**
depreciation rate δ	0.05
capital share α	0.35
growth rate g	0.015
substitution elasticity ζ	1
adjustment cost parameter ψ	1.5
coefficient of relative risk aversion σ	2
discount factor β	0.99
intra-temporal substitution elasticity ξ	0.8
technology level Ω_i	0.05-0.07
consumption share parameters $\bar{\phi}_{a,i}$	0.535-0.665
increment of consumption share parameter $\Delta\phi_{a,i}$	0.015 0.02

As shown in the sensitivity analysis, adjustment costs do not affect results much, but allow the analysis of the effects of demographic change on the price of capital.

The discount factor in all countries, β, is set to 0.99 which corresponds to the estimate of an annual discount rate of 0.011 by Hurd (1989). With this choice - and given all the other parameter values - the model produces an average capital to output ratio for the region "European Union" of about 2.9 for the calibration period 1960-2001. While comparable capital-output ratios for a large cross-section of countries are not available, a value of 2.9 is reasonable for many countries OECD (2003). The coefficient of relative risk aversion is set to 2 which is within the standard range of 1 to 4. As in Chapter 4 the value for the intra-temporal substitution elasticity, ξ,ObstfeldRogoff00 is set to 0.8.

As in Chapter 4 levels of total factor productivity, Ω_i, vary across countries and are calibrated such that the model replicates output data in each country for the period 1960-2001. The consumption share parameter $\phi_{a,i}$ is assumed to decrease across the life-cycle according to the simple "step" function given in equation 2.9. A^l, the lower age boundary of the consumption share parameter, is set to 54 beyond which empirically observed labor supply starts to decrease and A^h, the upper age, is set to 80 since labor supply is essentially zero in all countries beyond the age of 80. While age boundaries are held constant across all countries, $\bar{\phi}_i$ and $\Delta\phi_i$ are calibrated such that the simulation model matches aggregate labor supply as well as labor supply profiles across ages on average in each country for the period 1960-2001, compare Chapter 4. This parsimonious parametrization results in a decent fit of empirically observed labor supply profiles across age, compare Börsch-Supan, Ludwig, and Winter (2004).

A final remark concerns the initial values of the model for the year 2002 under the different capital mobility scenarios. Conceptually, it is problematic to simulate a calibrated macroeconomic model under policy scenarios other than the one under which it was calibrated. As in Chapter 4, the world for which the model is calibrated changes with the number of regions considered in the capital mobility scenarios. On the one hand, it would make sense to adjust the calibration parameters each time the number of regions considered is changed, see Chapter 4. On the other hand, this would change households' reactions to changes in policy and it would therefore be more difficult to interpret results with respect to a reform of the public pension system. Since households reactions to policy reform are of key importance here, parameter values are held constant across all capital mobility scenarios which contrasts with the approach of Chapter 4.

5.4 Simulation Results for Alternative Pension and Capital Mobility Scenarios

In this section, the results of the macroeconomic simulation model are presented. For tractability, focus is on the three-country continental European region consisting of France, Germany, and Italy as a region with a very severe aging problem and with pension systems in an ongoing reform process.

To separate the direct effects of population aging on capital markets and potential feedback effects from pension reform, projections are presented for the two counterfactual pension policy scenarios described above: (a) the "old system scenario" which maintains these countries' current generous public pension systems, and (b) the "reform scenario" which introduces a transition to a funded pension system by freezing contribution rates in these three countries. The other regions' pension systems remain unchanged. By comparing these polar scenarios, it can be shown that a good portion of the capital market effects of population aging arise even without a fundamental pension reform.

Accordingly, the figures below have three panels. Panel (a) corresponds to the "old system scenario", panel (b) shows the "reform scenario", and panel (c) shows the differences between these two scenarios, i.e. the effect if the three large continental European countries simultaneously implement a fundamental pension reform of the type described.

Moreover, each figure displays four lines, representing four capital mobility scenarios. The first scenario corresponds to a closed economy where all investment of France, Germany, and Italy takes place within France, Germany, and Italy. The other three capital mobility scenarios open this closed economy sequentially up: France, Germany, and Italy diversify their investments (i) across all countries of the European Union, (ii) across all OECD countries, and (iii) across the entire world. As noted earlier, the benchmark scenario assumes that capital mobility is restricted to the OECD area.

The presentation of results proceeds in several steps. Throughout, focus is on the economic consequences of aging and of fundamental pension reforms on the aggregate region of European countries consisting of Germany, France, and Italy. First, the two channels of reaction of households to both, demographic change and pension reform are analyzed. Accordingly it is shown how labor supply and savings patterns are affected by demographic change and by pension reforms. Next an analysis of the firm sector is carried out with respect to the evolution of wage rates and the return to capital as well as its price, Tobin's q. This is followed by description on how international capital flows, resulting from differences between national saving and investment, are affected by demographic change.

While the results show substantial differences of international capital flow patterns between countries of the European Union and other world regions, there are also significant differences between countries within the different world aggregates. To highlight this aspect, further results on saving patterns and international capital flows for the three European countries France, Germany, and Italy are presented. The discussion of simulation results is concluded by a brief welfare analysis for households living in Germany.

Before presenting the results of the simulation model, it is useful to briefly describe the main mechanisms that are at work simultaneously in such a complex general equilibrium model. Consider a two-region world were there is an old region (e.g. the France-Germany-Italy region) and a relatively younger region (e.g. all non-European OECD countries). Assume that the younger region also has a less generous PAYG pension system, i.e., lower PAYG contribution and replacement rates. Further assume that both economies are closed. What are the effects of demographic change on saving rates and rates of return in such a stylized world with PAYG financed pension systems?

First, there is a *direct level* effect. The younger region has a relatively larger work force, a lower capital-labor ratio, and hence a higher rate of return. Accordingly, the saving rate is higher in that region. Over time and as a result of demographic change, the work force shrinks in both economies. Hence, capital-output ratios increase and both, rates of return and the saving rates, decrease. This effect is therefore referred to as the *direct trend* effect of demographic change. The effect is stronger for the older economy.

Second, there are indirect effects due to the existence of PAYG financed pension systems. PAYG financed pension systems “crowd out” private savings by providing old-age pension income and by taxing labor income. Hence, this *indirect level* effect works in the opposite direction than the direct effect of demographic change. Relative to a situation without PAYG financed pension systems, the indirect effect decreases the differences in saving rates and rates of return between the two economies. However, as the simulation results presented below suggest, the direct level effect dominates. Moreover, over time, old-age dependency ratios increase and therefore contribution rates to the PAYG pension system increase as well (taking PAYG replacement rates as given as done in the old system scenario). This *indirect trend* effect is stronger in the older region which is more severely affected by the impact of demographic change and has a more generous PAYG pension system. As a result, the decrease of savings rates is relatively stronger in the older region and hence the decrease in the rate of return is less strong.

Consider now a case in which capital is mobile between the two economies. Due to the dominance of the direct level effect, the rate of return is initially higher for the older region as it would be if it was a closed economy. This increases savings relative to the closed economy case. However, due to the indirect trend effect the decrease in the rate of return is stronger than under the closed economy scenario. The decrease in saving rates is therefore stronger as well. These interactions between demographic change and the PAYG pension system are important for the interpretation of the main results that follow.

5.4.1 Labor Supply, Contribution and Replacement Rates

Immediately evident effects of population aging are reflected in the amount of labor supply and the balance of the pension systems. During the entire observation period, labor supply shares in the three European countries Germany, France, and Italy decrease from current levels of slightly below 42 percent to roughly 36 percent in 2050. The economic dependency ratio defined as the ratio of pensioners to workers, is projected to increase from roughly 50 percent in 2002 to about 80 percent in 2050.

As a result of the decrease in labor supply shares and the resulting increase in the economic dependency ratio, the contribution rate to the PAYG pension system increases sharply under the "old system scenario", i.e. if current generous pension systems were maintained. These contribution rates are equilibrium contribution rates such that the budget of the pension system of each country is balanced at every point in time and implicitly include tax subsidies to the pension system. The time patterns of net replacement and contribution rates for Germany, France, and Italy that result from the procedure are summarized in Table 5.2.

If current generous replacement rates were maintained, the model predicts increases in the equilibrium contribution rate in Germany from its current levels of roughly 28 percent to 41 percent in 2050 - more than a 50 percent increase. The stylized pension reform freezes contribution rates at the level reached in 2006, roughly at 29 percent. As a result of this reform, average pension levels decrease: the net pension replacement rate is projected to

Table 5.2: Predicted contribution and replacement rates of PAYG pension systems

	France			Germany			Italy		
	2000	2030	2050	2000	2030	2050	2000	2030	2050
Pure PAYG									
Contribution rates	0.275	0.356	0.375	0.268	0.375	0.415	0.325	0.476	0.534
Net replacement rates	0.654	0.654	0.654	0.7	0.7	0.7	0.646	0.646	0.646
Freezing Reform									
Contribution rates	0.275	0.295	0.295	0.268	0.294	0.294	0.325	0.34	0.34
Net replacement rates	0.654	0.549	0.513	0.7	0.568	0.504	0.646	0.489	0.415

Notes: Figures shown in the table refer to the open economy scenario "OECD".

decrease from 70 percent in 2000 to about 50 percent in 2050. Hence, for Germany, the model predicts a one-third transition towards pre-funding until 2050. Results for the other countries are similar, compare Table 5.2.

Households respond to these decreases in pension benefit levels not only by increasing savings, but also by increasing labor supply. Despite the restriction on preferences - decreasing consumption shares and increasing preference for leisure as described in Section 5.3 above - the stylized pension reform would lead to quite substantial increases in aggregate labor supply. Labor supply shares are predicted to increase by more than 6.5 percent or 2.5 percentage points until 2050. This increase is roughly the same for all capital mobility scenarios. For instance, labor supply shares in the France-Germany-Italy region increase from about 36 to 38.5 percent in the year 2050. As a consequence, the economic dependency ratio is projected to decrease by almost 6 percentage points. Endogenous labor supply reaction is therefore a helpful mechanism to dampen the effects of population aging. As Börsch-Supan, Ludwig, and Winter (2004) further show, this effect holds over the entire range of the crucial elasticity parameters in the OLG model.

5.4.2 Savings and Capital Stock

Panel (a) of Figure 5.2 shows the aggregate average saving rate of Germany, France, and Italy for the four capital mobility scenarios. In the year 2000, savings rates are substantially higher in the open economy scenarios than in the closed Germany-France-Italy region. This is in line with the higher rates of return (see next subsection) generated in an open economy which diversifies a great deal of the demographic effects that create lower saving rates (and rates of return) in economies with a large share of older persons.

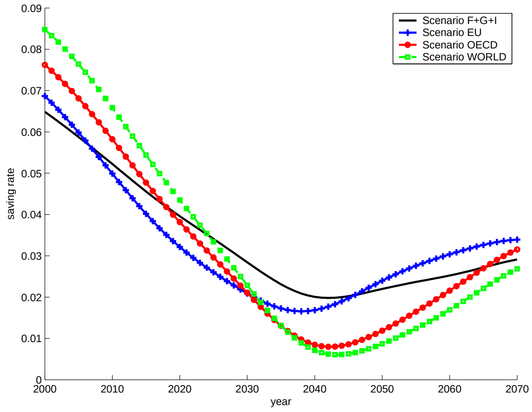
This direct level effect is superseded by the demographic changes during the 2000 to 2070 prediction window. Saving rates decrease until 2050 across all capital mobility scenarios since the baby boom generation decumulates assets. Saving rates are projected to rebound after the year 2050. The decrease of the savings rate caused by population aging - the difference between the value in 2000 and the minimum reached just after 2040 - is roughly 4.5 percentage points if capital mobility is restricted at most to the EU region (scenarios “F+G+I” and “EU”). If capital is fully mobile within the OECD or the entire world, this decrease is 6.5 or 8 percentage points, respectively. This larger decrease in the open economy scenarios is explained by the indirect trend effect described above. The diversification advantages of worldwide capital mobility thus decline, and saving rates respond accordingly.

Projected aggregate saving rates under a fundamental pension reform are substantially higher and the effect of a pension reform is stronger in the OECD / World open-economy scenarios (the saving rate is projected to increase by about one percentage point in the EU scenario as compared to almost 2 percentage points in the OECD / World scenarios). An increase in national savings leads to an increase in the capital stock and thereby to a decrease in the rate of return to capital which then further crowds out savings. In those scenarios with a larger international capital market, substantially more saving is generated since - as shown below - the rate of return decreases by much less. These projections show that optimal life-cycle behavior generates additional saving under a fundamental pension reform - in this model, it is not the case that additional retirement saving induced by a pension reform crowds out other saving totally, as has often been claimed.

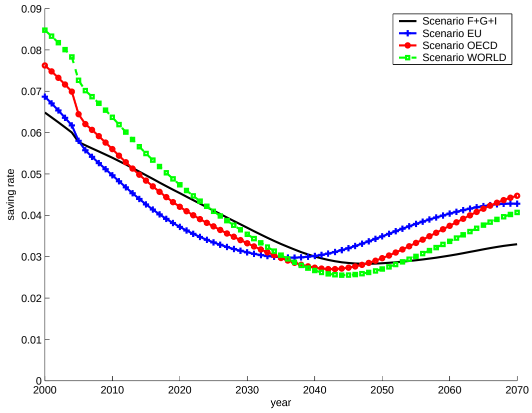
Accumulated aggregate savings result in Europe’s capital stock and the related capital-to-output ratios. As a consequence of decreasing labor supply, the capital-to-output ratio increases from a current level of about 3 until it reaches a level of about 3.25 around 2040 and then decreases slightly since baby boomers decumulate assets (capital mobility scenario “OECD”, figures not shown). This decrease is much more pronounced if the international capital market is restricted to the EU area only. The simultaneous fundamental pension reform of France, Germany, and Italy leads to substantial increases in the capital-to-output ratio if capital mobility is restricted to these countries or the EU area. The increase is much lower if this constraint is relaxed which suggests that the additional savings shown in Figure 4 are largely invested abroad.

Figure 5.2: Saving rates

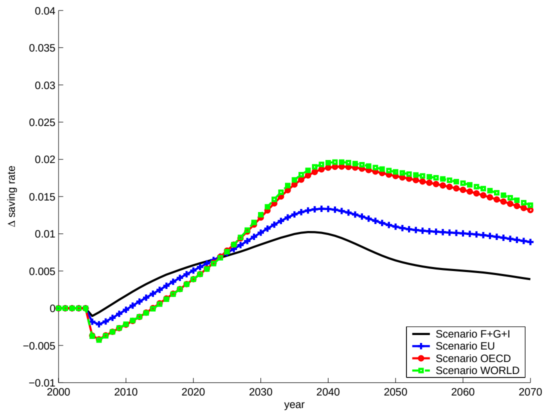
a. Pure PAYG



b. Freezing



c. Difference



Notes: These figures show the projected aggregate saving rate of households living in Germany, France and Italy. Scenario G+I+F: perfect capital mobility within Germany, France and Italy; Scenario EU: perfect capital mobility within the European Union; Scenario OECD: perfect capital mobility with the OECD; Scenario WORLD: perfect capital mobility across all world regions.
Source: Own calculations, based on demographic projections of the United Nations (2002).

5.4.3 The Rate of Return and the Price of Capital

Much of the political and academic debate on the capital market consequences of demographic change and of pension reforms is centered around the rate of return to capital which is analyzed next. First, the same level effects can be observed as described in the previous section. It is noteworthy, that the demographic effect is larger than a second level effect. Since the PAYG systems are slimmer in the aggregate rest-of-the-world region than in France, Germany and Italy, the capital stock accumulated for retirement savings is larger which depresses rates of return.

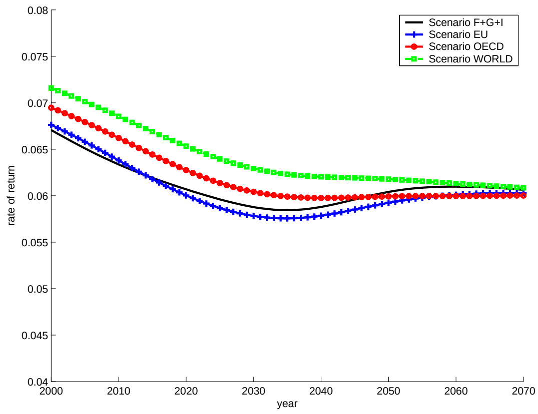
Second, as a consequence of population aging and the resulting increases in capital to output ratios, the model predicts the rate of return of return to capital to decrease by roughly 0.8 percentage points if capital moves freely within the OECD, see Figure 5.3. This decrease is less than would be associated with a "meltdown of asset prices". Third, while the rate of return decreases across all capital mobility scenarios, substantial gains would be possible by shifting investments to demographic younger countries since the model predicts higher returns if free capital mobility across all world regions is allowed for. However, as demographic processes are highly correlated across countries (compare Figure 5.1), differences in demographic processes across countries more or less only affect the level of the rate of return. Furthermore, diversification advantages decrease across time since the above mentioned indirect trend effects are at work as well.

As Panels (b) and (c) of Figure 5.3 suggest, there would be an additional decrease in the rate of return to capital if Germany, France, and Italy would simultaneously reform their pension systems in a fundamental way by about 0.25 percentage points until 2070 if capital was freely mobile within these countries only. Due to the increase in labor supply, this long-run decrease in the rate of return is lower than a model with exogenous labor supply would suggest, see Section 5.5. Moreover, and in line with the earlier results, the decrease in the rate of return is negligibly small if capital moves freely across OECD countries (or the entire world). In contrast to a model of exogenous labor supply, the present model even predicts an increase in the rate of return until 2030-2040 (as a result of the endogenous labor supply reaction). While saving rates immediately start to increase after the reform, labor supply increases as well. As a net effect, this initially leads to a decrease in the capital to output ratio and an associated initial increase in the rate of return to capital.

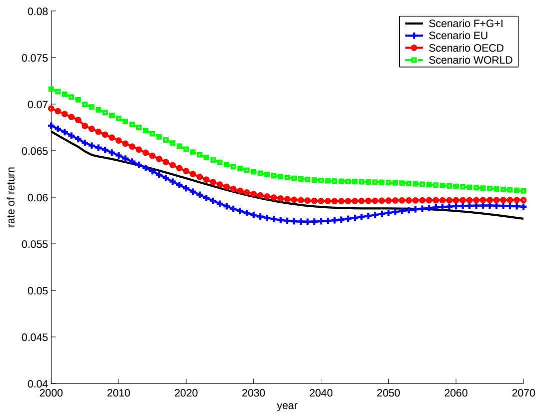
Tobin's q, the price of capital, also decreases as a consequence of population aging but its level is higher in the demographically younger regions. As a consequence of fundamental pension reforms, q-values are predicted to increase slightly since the demand for assets increases which leads to an increase in the investment to capital ratio. Results on Tobin's q for different world regions are summarized in Figure 5.4. Here, capital mobility is restricted to the OECD area.

Figure 5.3: Rate of return

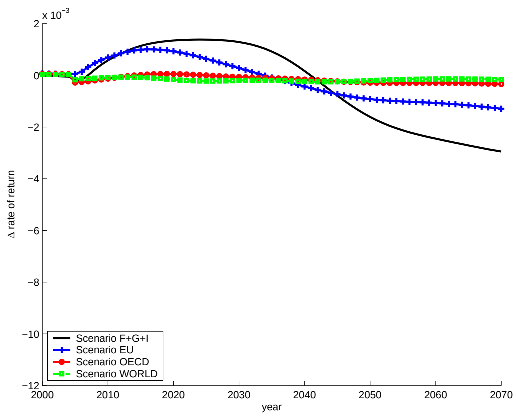
a. Pure PAYG



b. Freezing



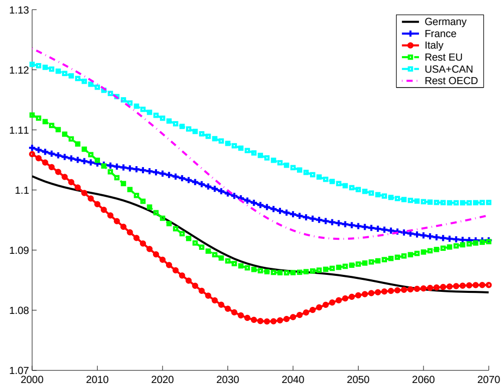
c. Difference



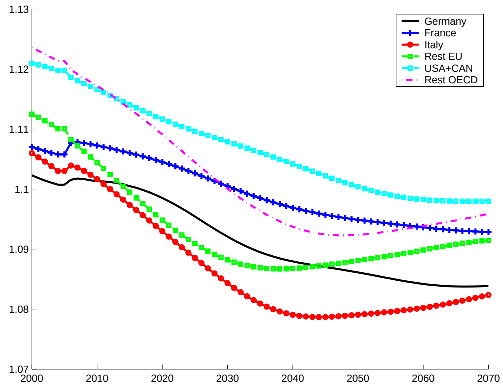
Notes: These figures show the projected rate of return of the aggregate capital stock in Germany, France and Italy. Scenario G+I+F: perfect capital mobility within Germany, France and Italy; Scenario EU: perfect capital mobility within the European Union; Scenario OECD: perfect capital mobility with the OECD; Scenario WORLD: perfect capital mobility across all world regions.
Source: Own calculations, based on demographic projections of the United Nations (2002).

Figure 5.4: Tobin's q

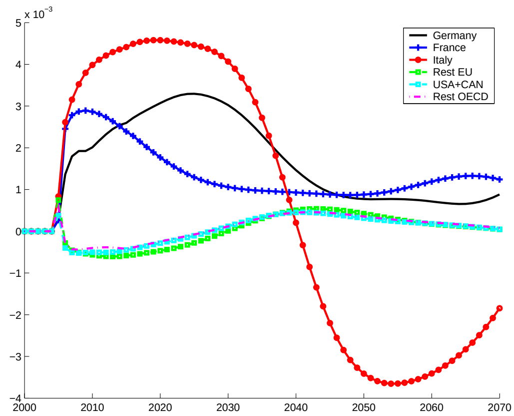
a. Pure PAYG



b. Freezing



c. Difference



Notes: These figures show the projected Tobin-q in different world regions. Rest EU: All countries of the European Union except France, Germany and Italy. USA+CAN: United States and Canada. Rest OECD: All other OECD countries.
Source: Own calculations, based on demographic projections of the United Nations (2002).

5.4.4 International Capital Flows

International capital outflows from France, Germany and Italy to other OECD countries roughly follow the pattern of savings and decrease steadily until 2050, see Figure 5.5. In the OECD (World) capital mobility scenario, they are initially positive at about 1.5 (2.7) percent-age points and turn negative to -1 (-1.2) percentage points in 2050 (Figure 5.5, Panel a). Hence, the model predicts reversals in current account positions for fast aging countries such as France, Germany, and Italy.

With the exception of Tobin's q the analysis mostly concentrated on France, Germany and Italy as a country aggregate. However, as the results on Tobin's q have already indicated, there are substantial differences across countries, even within Continental Europe. To highlight this aspect, savings patterns and international capital flows within the region of EU countries are analyzed if the international capital market is restricted to the OECD area.

Figure 5.6, panel (a), shows saving rates for France, Germany and Italy, the remaining EU countries and the EU average. The time pattern of German saving rates roughly equals the EU average and is projected to decrease from current levels of 7 percent to about 2 percent in 2050. In France, as the demographic youngest among the three regions, decreases in savings rate only last until 2030 and the overall decrease is smaller than in other EU countries. Italy, faced with the strongest population aging process within Europe, is the other extreme: Italian household's saving rates are projected to become negative in 2050. This also explains the pattern of Tobin's q under a fundamental pension reform shown above in panel (c) of Figure 5.4.

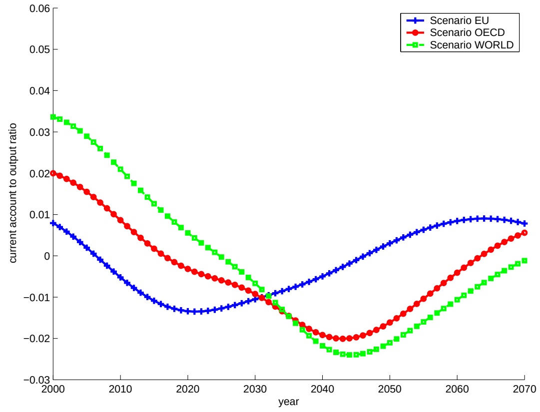
5.4.5 Welfare Analysis

Figure 5.7 shows the effects of the fundamental pension reform on remaining lifetime utility for different cohorts. Following Altig et al. (2001) the change in remaining lifetime utility is measured as the equivalent variation of full lifetime income. The index measures the present value of remaining life-time resources relative to current full life-time resources a household would have to receive (pay) under the new system to make him indifferent between the old and the new system. Therefore, an index number greater (smaller) than one has to be inter-preted as loss (gain) in remaining life-time utility.

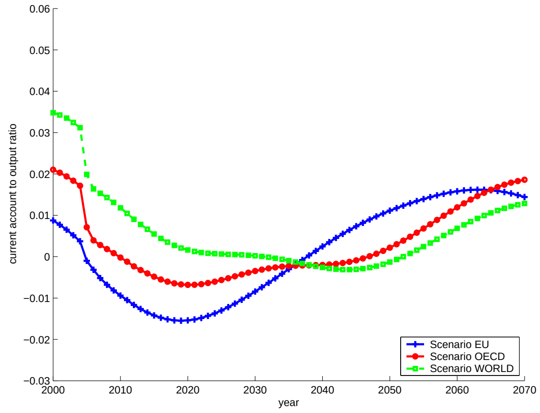
The results show that remaining life-time utility of relatively many generations decrease as a consequence of the fundamental pension reform. Cohorts born between the years 1928 and 1982 are those who experience losses in remaining lifetime utility. Welfare losses are slightly higher if capital is restricted to be mobile only within the EU. While substantial welfare gains are possible in the long run in all capital mobility scenarios, the figure also

Figure 5.5: Current account to output ratio

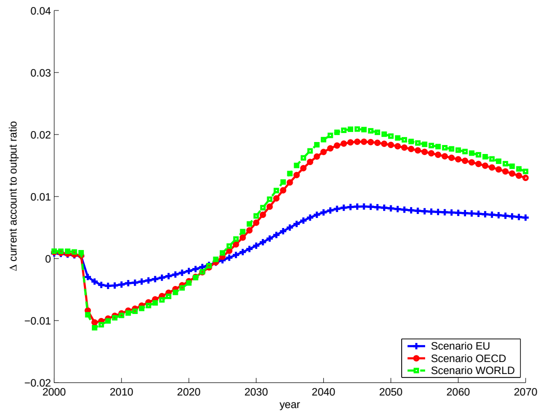
a. Pure PAYG



b. Freezing



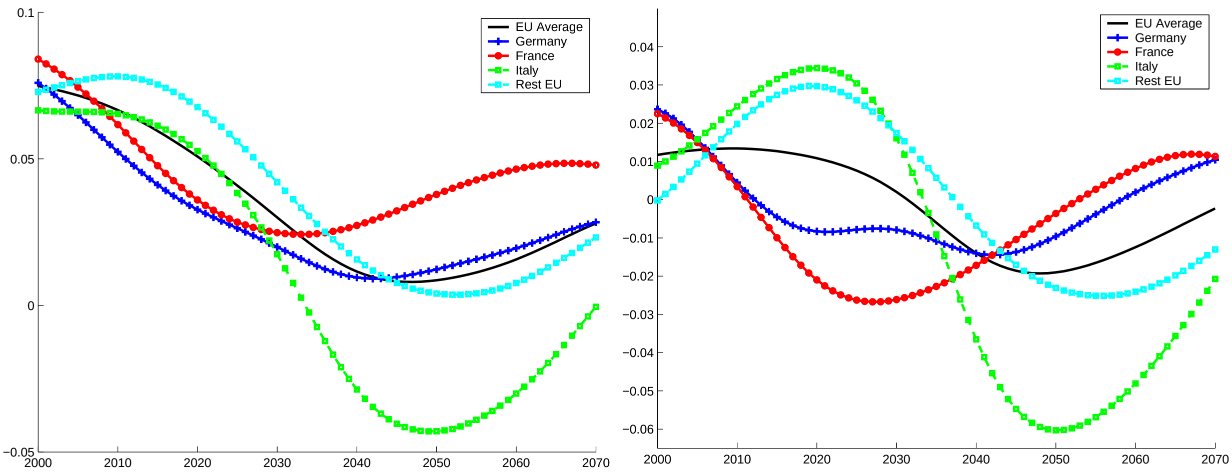
c. Difference



Notes: These figures show the projected current account to output ratio in France, Germany and Italy. Scenario F+G+I: perfect capital mobility within France, Germany and Italy; Scenario EU: perfect capital mobility within the European Union; Scenario OECD: perfect capital mobility within the OECD; Scenario WORLD: perfect capital mobility across all world regions.
Source: Own calculations, based on demographic projections of the United Nations (2002).

Figure 5.6: Saving rates and capital flows in the European Union

Saving rate | Current account to output ratio



Notes: This figures show the projected saving rates and the current account to output ratios within countries of the European Union if capital mobility is restricted to the OECD area. EU Average: Average of all EU countries; Rest EU: all EU countries excluding France, Germany and Italy.
Source: Own calculations, based on demographic projections of the United Nations (2002).

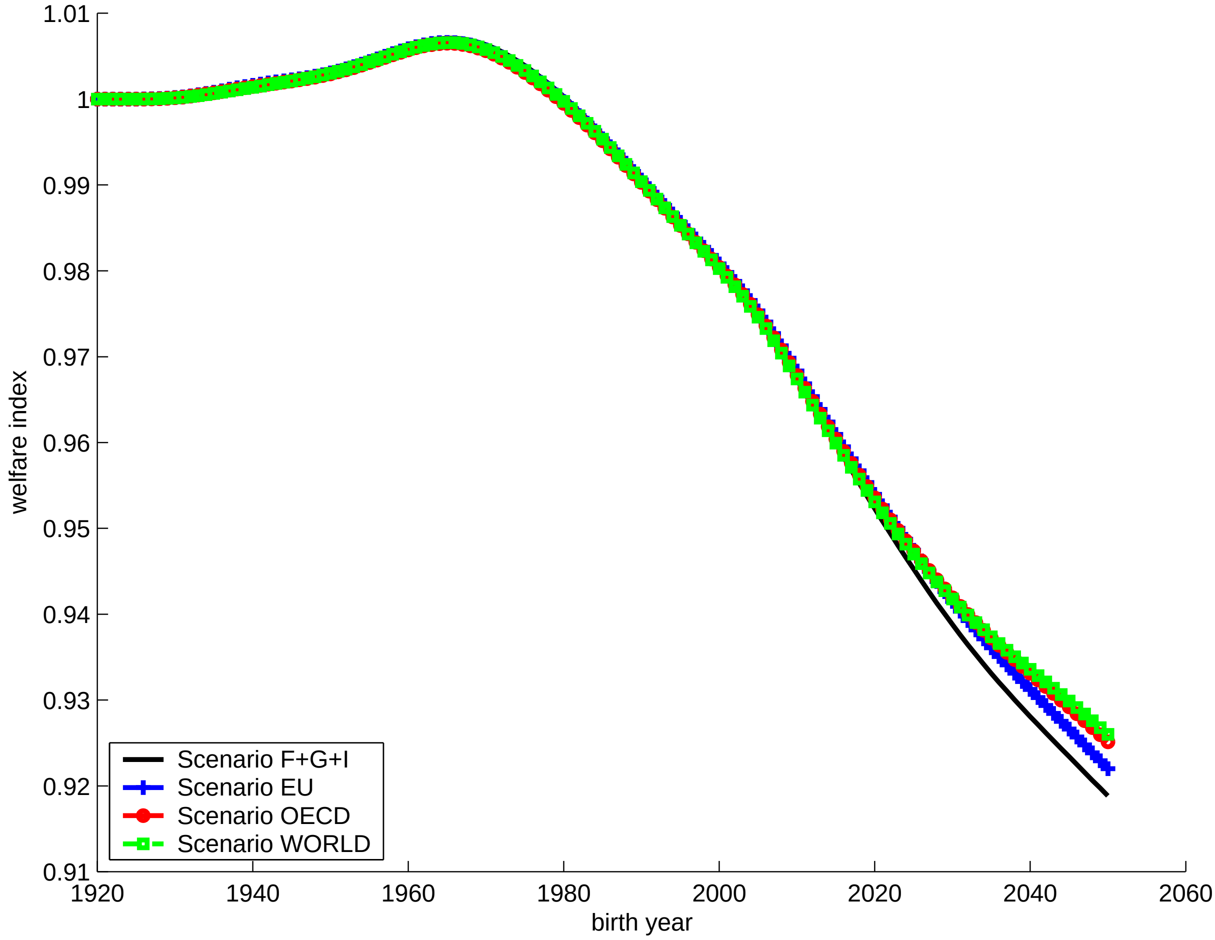
illustrates that fewer cohorts experience losses if the capital mobility regions is widened. However, differences between the capital mobility scenarios are not large.

5.5 Sensitivity Analysis

The existing literature has mostly concentrated on sensitivity analysis of simulation results with regard to values of structural (deep) model parameters, see, e.g., Altig, Auerbach, Kotlikoff, Smetters, and Walliser (2001), Börsch-Supan, Heiss, Ludwig, and Winter (2003). Results on such a standard sensitivity analysis are presented in Börsch-Supan, Ludwig, and Winter (2004). Here, only the results of the less standard sensitivity analysis with regard to different sub-models will be presented. What difference does it make whether labor supply is endogenous or exogenous? Whether investment incurs adjustment costs? Whether perfect annuity markets absorb all accidental bequests? Whether part of retirement income is provided by a PAYG pension system? In order to shed light on these questions, the benchmark model is recomputed and in addition three alternative models by subsequently switching off features of the benchmark model. Table 5.3 provides an overview of the various alternative models analyzed below.

The benchmark model has the following features: (a) adjustment costs, (b) perfect annuity markets, (c) endogenous labor supply, and (d) existence of a PAYG pension system.

Figure 5.7: Welfare index



Notes: This figure shows the projected welfare index for the Germany. Scenario F+G+I: perfect capital mobility within France, Germany and Italy; Scenario EU: perfect capital mobility within the European Union; Scenario OECD: perfect capital mobility with the OECD; Scenario WORLD: perfect capital mobility across all world regions..

Source: Own calculations, based on demographic projections of the United Nations (2002).

Table 5.3: Models used for sensitivity analysis

	Model A1	**Model A2**	**Model A3**	**Model A4**
Adjustment costs	**No**	**No**	**No**	**No**
Annuity markets	Perfect	**Acc. Beqeusts**	Perfect	Perfect
Labor supply	Endogenous	Endogenous	**Exogenous**	**Exogenous**
PAYG pension system	Yes	Yes	Yes	**No**

First, adjustment costs are eliminated (Model AI). As it turns out, the existence of adjustment costs has little influence on the results. Since models without adjustment costs are substantially easier to solve, the sensitivity analysis continues with models that do not feature adjustment costs to capital.

Model AII then does away with the assumption of perfect annuity markets and allows for accidental bequests. As will be shown, this assumption does not affect results in any significant way either, and therefore the simpler model model will be used for the remainder of the sensitivity analysis which imposes perfect annuity markets and abstracts from adjustment costs.

Model AIII makes labor supply exogenous. As opposed to the previous two assumptions, this is a serious restriction. It will be shown that the results for the pension reform scenario are strongly affected by ignoring endogenous labor supply since now only the capital accumulation channel remains for households to adjust their behavior in reaction to the policy change.

Finally, Model AIV ignores the fact that PAYG pension systems exist in almost all countries of the world as it is done in the models by Brooks (2003), Feroli (2002) and Henriksen (2002). Comparing models III and IV allows to disentangle the direct effects of population aging on macroeconomic aggregates from the indirect effects which are generated through the channel of PAYG pension system changes in response to population aging. These effects are confounded in the analyses by Brooks (2003), Feroli (2002) and Henriksen (2002) and are separated here for didactical purposes.

For simplicity, the sensitivity analysis uses a three-region rather than a seven-region model as in the previous section. To this end world regions are summarized as follows: (i) France, Germany and Italy, (ii) all other EU countries, and (iii) all other OECD countries. Due to Jensen's inequality, results for region (i) might differ from those shown in the previous section: here, first input data across three countries, $\sum X_i$, are summarized and then $f(\sum X_i)$ is calculated, whereas before the average outcome, $1/R \sum f(X_i)$ was presented. As shown below, this approximation is of minor importance for the simulation results. Moreover, unless simulation outcomes between the benchmark and the alternative models differ significantly between the two pension reform scenarios, only the "Pure PAYG" scenario is presented.

5.5.1 The Role of Adjustment Costs

First, the role of adjustment costs is analyzed. Their first role is to dampen the adjustment process of investment. Second, the presence of adjustment costs leads to differences capital-output ratios across countries even under a Cobb-Douglas technology. Third, modelling adjustment costs allows to study cross-national differences in the price of capital and its evolution over time.

Figure 5.8 compares simulation results for saving rates, rates of return to capital and current account to output ratios in a model with and without adjustment costs. As the figures illustrate, the time path of these variables are virtually identical in the “pure PAYG” pension system. The same holds for the “freezing reform” scenario (not shown). It can therefore be concluded that the simulation results are not affected much by the presence of adjustment costs.

5.5.2 The Role of Perfect Annuity Markets

Figure 5.9 compares the OLG model featuring perfect annuity markets with a model in which annuity markets cannot perfectly absorb the longevity risk. Households face the risk of pre-maturely dying with positive wealth. For simplicity, the dissipation of bequests is modelled as an equal distribution to all persons still living in each model region.

The result from Figure 5.9 is similar as in Figure 5.8: there are no discernible differences between the projections apart from level effects. Since households face the risk of pre-maturely dying with a positive amount of wealth, their preference for early consumption increases. Hence households have a flatter life-cycle saving profile and accumulate less wealth over the life-cycle than they do in a world with perfect annuity markets. Therefore, predicted levels of saving rates (rates of return) are lower (higher). It can be concluded that modelling annuity markets and accidental bequests is not an important issue for the study of aggregate saving rates, rates of return and international capital flows.

5.5.3 The Case of Exogenous Labor Supply

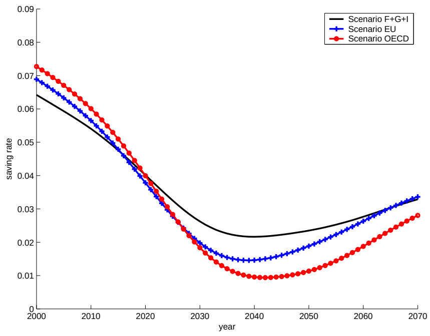
Figure 11 compares the time paths for the rate of return, the saving rate and the current account between models of endogenous and exogenous labor supply. As the figure illustrates, the time pattern is only slightly different under the ”pure PAYG” pension system scenario. With exogenous labor supply, the time path of the aggregate saving rate fluctuates a bit more since households do not endogenously adjust their labor supply to changes in demographic processes and resulting changes in interest rates and wage rates and hence cannot ”smooth” their savings pattern as much.

Differences are much larger when a pension reform occurs. The adjustment paths under the new policy are depicted in panels b of the Figure 5.10. If labor supply is endogenous, households simultaneously adjust their labor supply and their saving behavior to the change in policy. If labor supply is assumed to be exogenously fixed, however, households can only react with their saving behavior but not with changing their labor supply. The saving rate

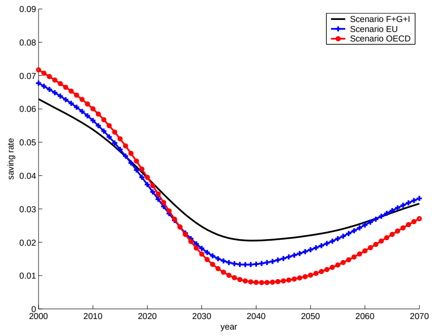
Figure 5.8: The influence of modeling adjustment costs (Pure PAYG)

a. Saving rates

Benchmark Model

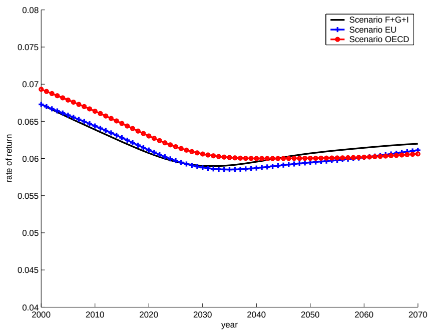


Alternative model I

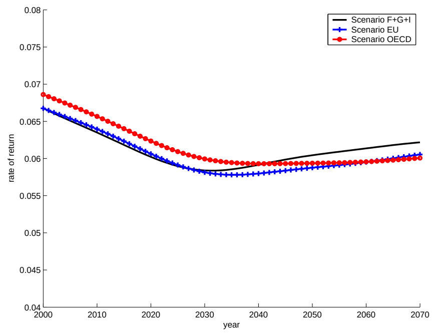


b. Rates of return to capital

Benchmark Model

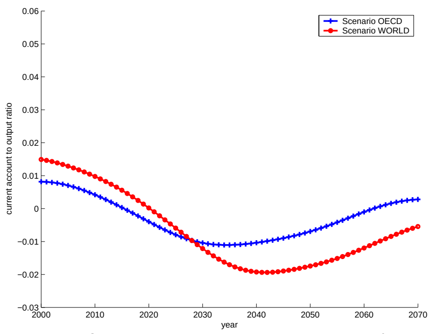


Alternative model I

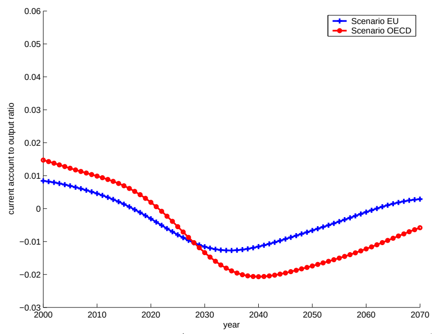


c. Current account to output ratio

Benchmark Model



Alternative model I



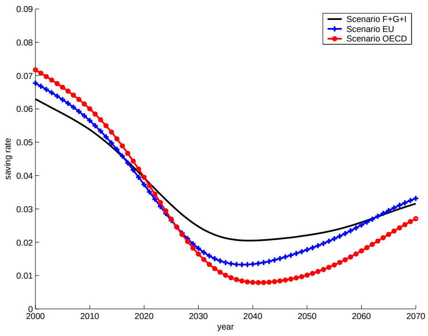
Notes: These figures show projections for the Benchmark Model and the Alternative Model I. Scenario G+I+F: perfect capital mobility within Germany, France and Italy; Scenario EU: perfect capital mobility within the European Union; Scenario OECD: perfect capital mobility within the OECD

Source: Own calculations, based on demographic projections of the United Nations (2002).

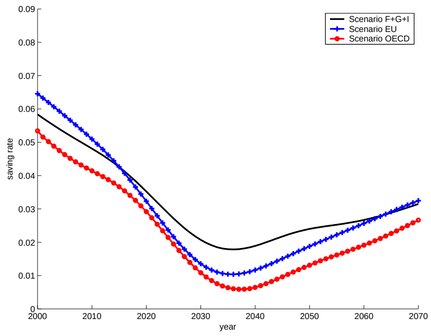
Figure 5.9: The influence of imposing perfect annuity markets (Pure PAYG)

a. Saving rates

Alternative Model I

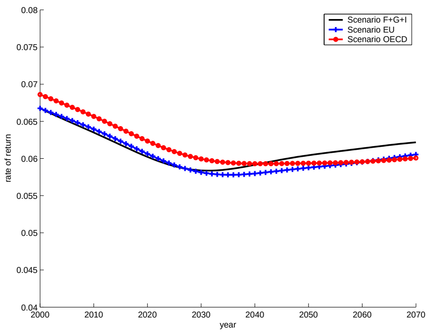


Alternative model II

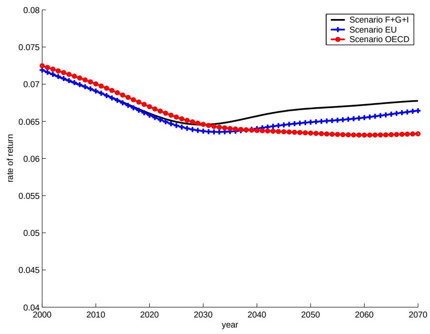


b. Rates of return to capital

Alternative model I

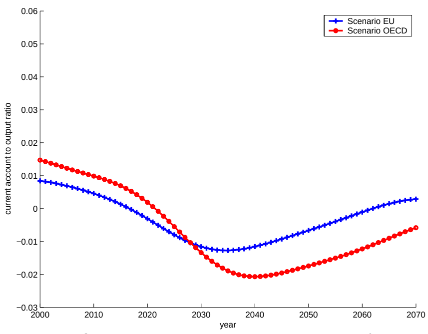


Alternative model II

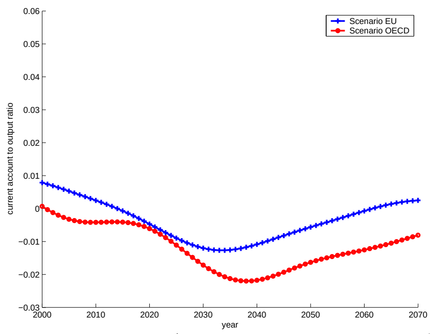


c. Current account to output ratio

Alternative model I



Alternative model II



Notes: These figures show projections for the Alternative Model I and the Alternative Model II. Scenario G+I+F: perfect capital mobility within Germany, France and Italy; Scenario EU: perfect capital mobility within the European Union; Scenario OECD: perfect capital mobility within the OECD

Source: Own calculations, based on demographic projections of the United Nations (2002).

therefore immediately jumps to the higher level after the announcement of the reform and does not adjust gradually, see Figure 5.10, Panel b. Moreover, the overall increase in the saving rate is considerably higher under the fixed labor supply assumption.

This difference in behavior directly translates into substantial differences in the time paths for the rate of return to capital. As Panel a of Figure 5.11 shows, the impact of aging on the rate of return to capital is higher if households are constrained and cannot adjust their labor supply. This result has already been demonstrated in Chapter 4. Panel b of Figure 5.11 then shows the substantial differences that results from the reaction of savings to the change in policy as described above. If labor supply is endogenous, the rate of return initially increases since households increase their labor supply as a reaction to the change in policy. This effect is absent in case labor supply is exogenous. Hence, the rate of return to capital immediately decreases. As a result, the overall decrease of the rate of return to capital is also much larger.

Finally, this is also reflected in the relative size of international capital flows, see Figure 5.12. Opening capital markets around the world creates substantially higher flows if the adaptation channel of labor supply responses does not work.

5.5.4 The Absence of a Pension System

So far, the effects of demographic change were analyzed in a world which is characterized by the existence of fairly large PAYG pension systems. As populations age, these PAYG systems require higher contribution rates and/or provide lower replacement rates. These changes in the pension systems create indirect effects on saving rates, the rates of return and international capital flows in addition to the direct effects that are generated through household and firm maximization even in the absence of mandatory PAYG pensions. This section shows how large these “direct” effects are.

Here these effects are simulated in a model with exogenous labor supply, because in this specification, the effects on saving rates, the rates of return and international capital flows are most clearly seen. Allowing households to react to population aging also via labor supply adjustments will dampen the effects on the capital market variables, as was shown above. It is also clear from the comparisons between panels a and b in Figures 5.10 through 5.12 that labor supply will increase if no PAYG systems exist. As Börsch-Supan, Ludwig, and Winter (2004) further show, labor supply increases as a reaction to the reform of the pension system for the entire range of elasticity parameters considered in their analysis.

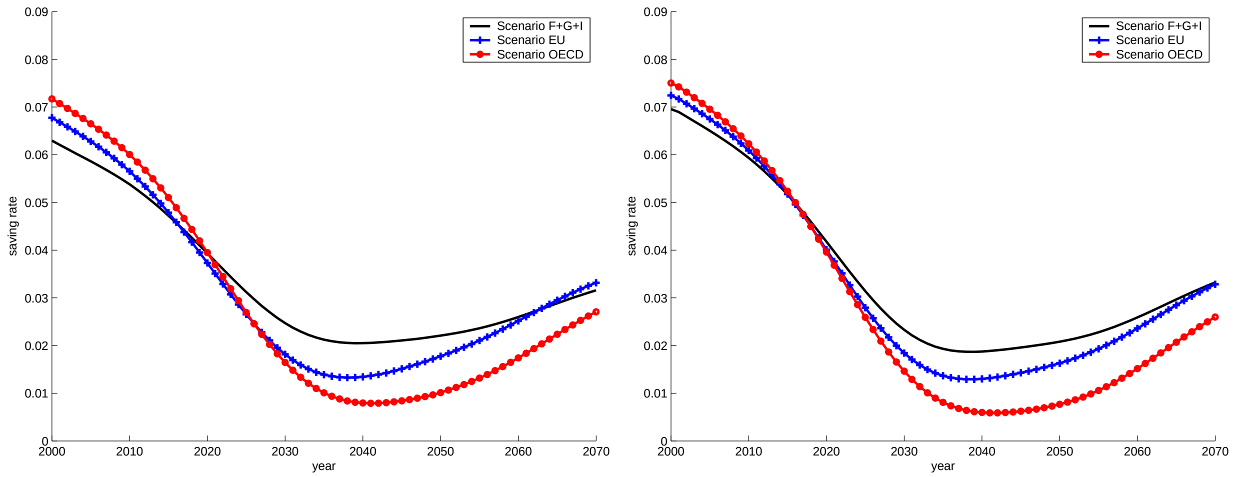
The exercise is, of course, a counterfactual one and purely analytical. It separates various effects, but does not provide realistic estimates of a world without PAYG systems. Model IV is also not re-calibrated, but the same parameters are used as always in this section.

Figure 5.10: The influence of modeling endogenous labor supply: Saving rates

a. Pure PAYG

Alternative Model I

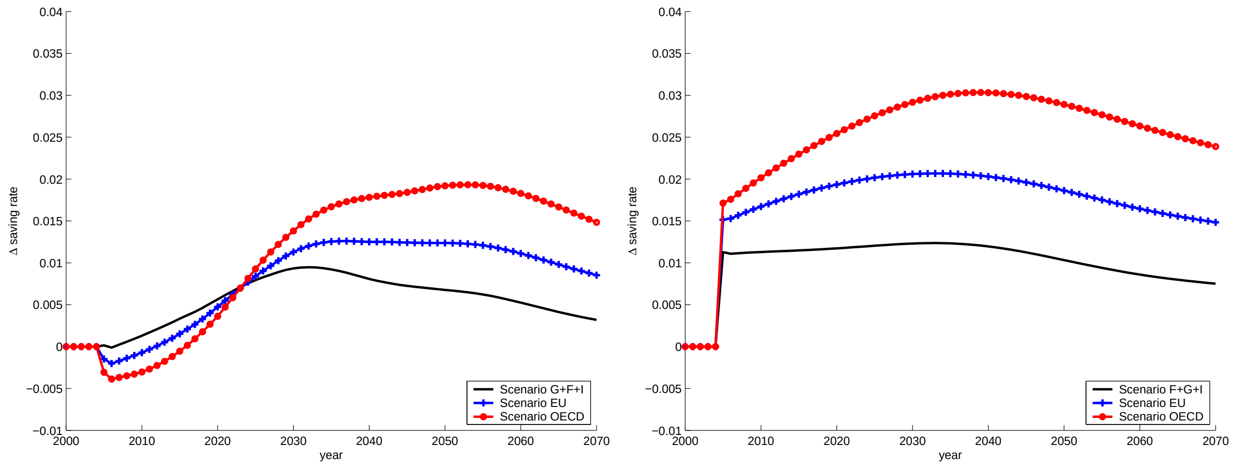
Alternative model III



b. Difference between Freezing and Pure PAYG

Alternative model I

Alternative model III

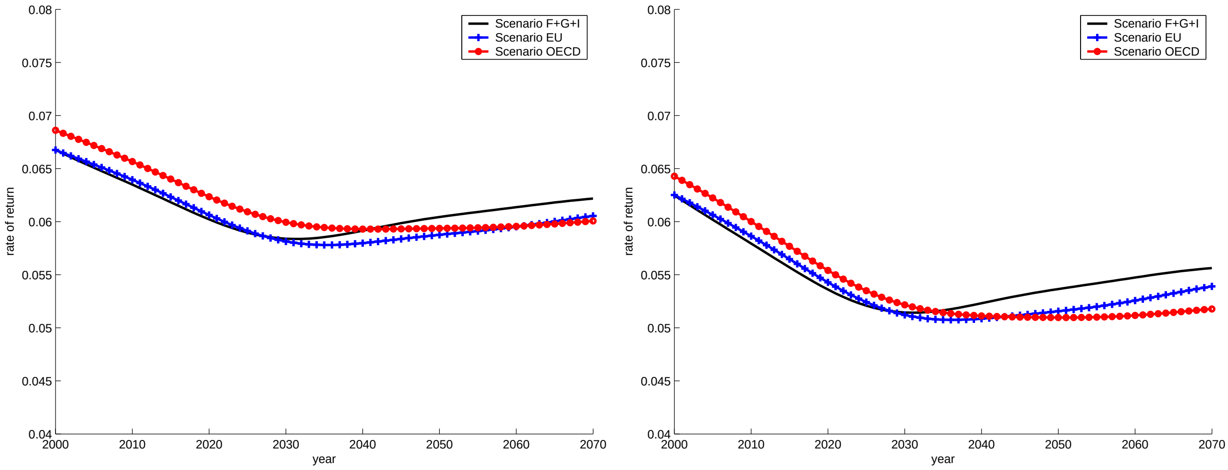


Notes: These figures show projections of the saving rate for Alternative Model I and Alternative Model III. Scenario G+I+F: perfect capital mobility within Germany, France and Italy; Scenario EU: perfect capital mobility within the European Union; Scenario OECD: perfect capital mobility within the OECD
Source: Own calculations, based on demographic projections of the United Nations (2002).

Figure 5.11: The influence of modeling endogenous labor supply: Rates of return to capital

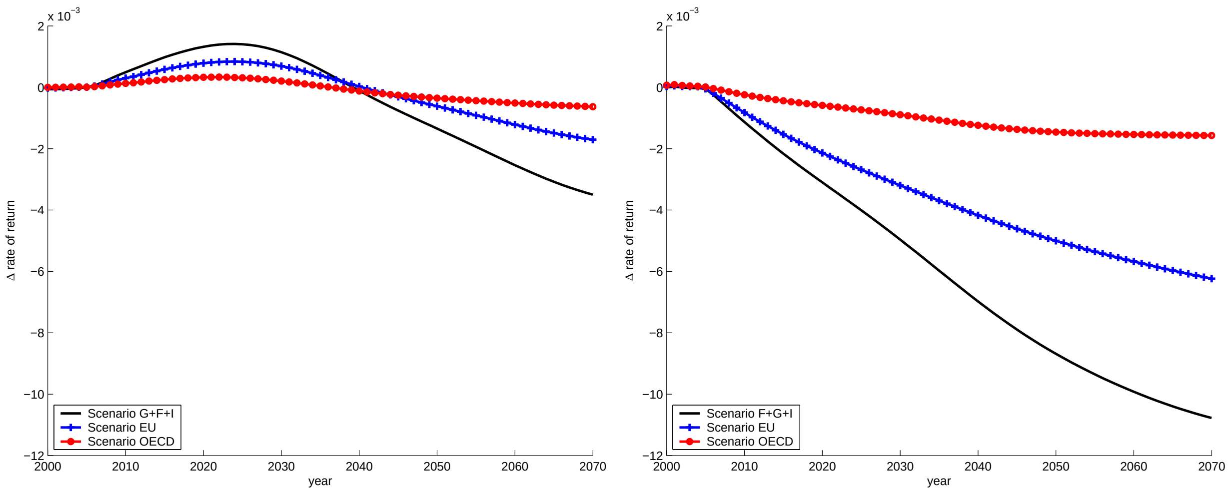
a. Pure PAYG

Alternative Model I | Alternative model III



b. Difference between Freezing and Pure PAYG

Alternative model I | Alternative model III



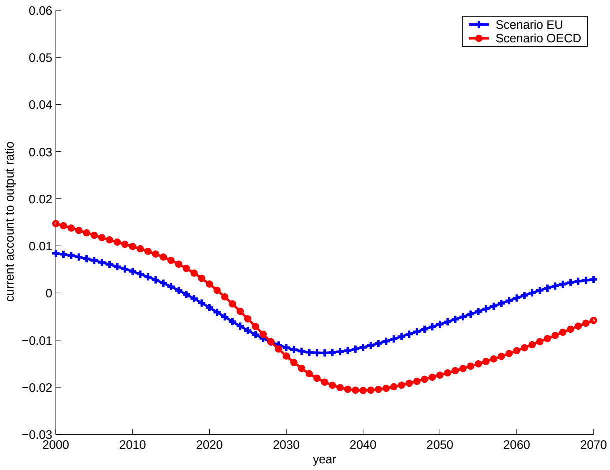
Notes: These figures show projections of the rate of return to capital for Alternative Model I and Alternative Model III. Scenario G+I+F: perfect capital mobility within Germany, France and Italy; Scenario EU: perfect capital mobility within the European Union; Scenario OECD: perfect capital mobility within the OECD

Source: Own calculations, based on demographic projections of the United Nations (2002).

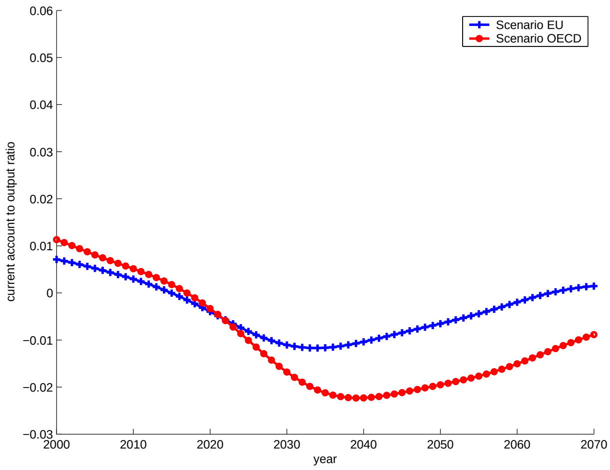
Figure 5.12: The influence of modeling endogenous labor supply: Current account to output ratios

a. Pure PAYG

Alternative Model I

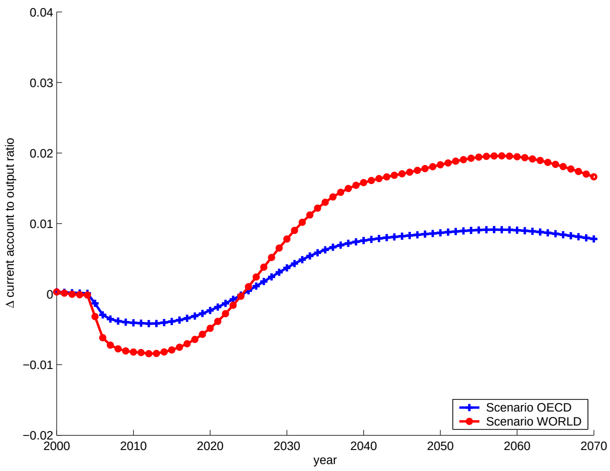


Alternative model III

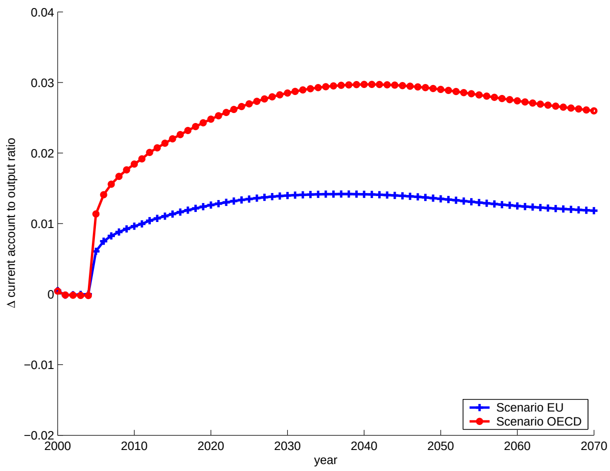


b. Difference between Freezing and Pure PAYG

Alternative model I



Alternative model III



Notes: These figures show projections of the current account to output ratios for Alternative Model I and Alternative Model III. Scenario G+I+F: perfect capital mobility within Germany, France and Italy; Scenario EU: perfect capital mobility within the European Union; Scenario OECD: perfect capital mobility within the OECD

Source: Own calculations, based on demographic projections of the United Nations (2002).

The argument is less that there is little point in calibrating a highly counterfactual model to historical data. More important is the aspect that re-calibration would introduce yet another confounding effect in this multidimensional sensitivity analysis.

Results on the time path of the saving rates, rates of return to capital, and international capital flows in this counterfactual world are shown in Figure 5.13. Here, the focus is on the rate of return effects first. The level effect of the open economy scenario labeled Scenario OECD is much higher than in a model which models PAYG pension systems. Since in the absence of a PAYG system, all retirement income has to be generated by savings, capital stocks are higher, decreasing the returns to capital across all regions.

Second, the long-run decrease in the rate of return to capital (i.e., between the years 2000 and 2070) is lower in the open economy scenario and if the existence of PAYG-financed pension systems is ignored. This is the pure (and intuitive) effect of demographic change: while virtually all OECD countries are affected by demographic change, countries outside the Euro-pean Union are younger and hence the rate of return to capital is higher and decreases more slowly in these countries.

There is, however, the additional indirect effect already described above. In a world with PAYG systems (left panel), the rate of return in the open economy scenarios is lower than in the closed economy scenario after about 2030, while it is reversed if all retirement income has to be provided through own savings. The indirect trend effect therefore masks the pure demographic effect. Since PAYG pension systems are less generous in countries outside Europe, households have to save more for retirement which decreases the rate of return (indirect level effect). In addition, crowding out of private savings is stronger in the European countries than in the region labeled "Rest OECD". This indirect trend effect dominates the direct "pure demographic" trend effect. Therefore, the rate of return to capital decreases more in the demographically younger countries than it would in a world without PAYG pension systems which eventually leads to the reversal of the rate of return levels (around the year 2030).

5.6 Conclusions

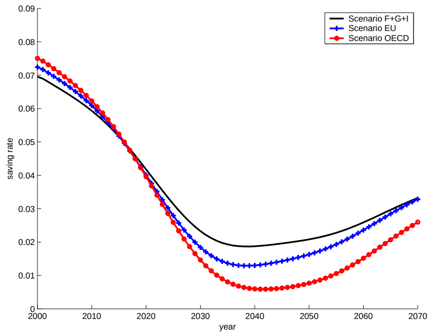
In this chapter, a quantitative analysis of the effects of population aging and pension reform on international capital markets was presented, using several modifications of the computational general equilibrium multi-country overlapping generations model presented in Chapter 2, viewed from a perspective of the three large continental European countries with large pay-as-you-go pensions systems: Germany, France and Italy.

The first part of the analysis focused on substantive results. Population aging works through various mechanisms. First, demographic change alters the time path of aggregate

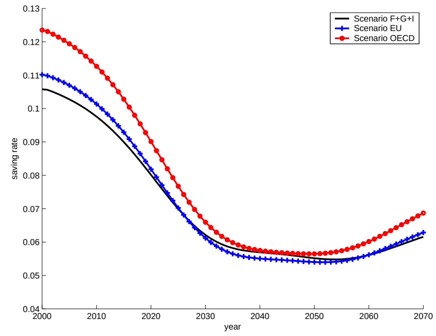
Figure 5.13: The influence of modeling PAYG pension systems

a. Saving rates

Alternative Model I

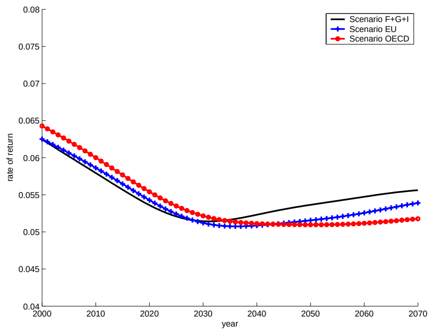


Alternative model II

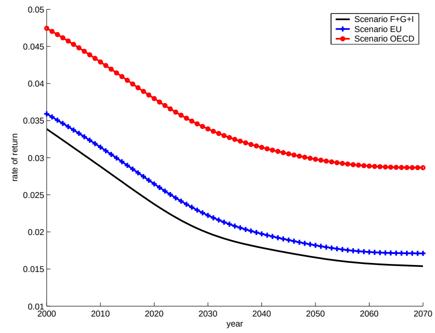


b. Rates of return to capital

Alternative model I

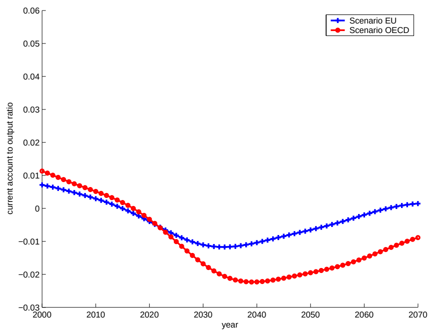


Alternative model II

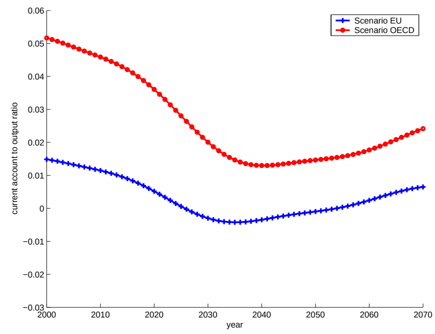


c. Current account to output ratio

Alternative model I



Alternative model II



Notes: These figures show projections for the Alternative Model III and the Alternative Model IV (both with exogenous labor supply). Scenario G+I+F: perfect capital mobility within Germany, France and Italy; Scenario EU: perfect capital mobility within the European Union; Scenario OECD: perfect capital mobility within the OECD

Source: Own calculations, based on demographic projections of the United Nations (2002).

savings within each country. Second, this process may be amplified when a (population-aging induced) pension reform shifts old-age provision from pure pay-as-you-go towards more pre-funding. Even with no reform, the core parameters of pay-as-you-go pensions need to adapt, changing saving behavior. Third, while the patterns of population aging are similar in most countries, timing and initial conditions differ substantially. Hence, to the extent that capital is internationally mobile, population aging will induce capital flows between countries.

All three effects influence the rate of return to capital and interact with the demand for capital in production and with labor supply. The simulations predict substantial capital flows due to population aging. Population aging results in decreases of the capital-to-output ratio when the baby boomers decumulate their assets. The coun36 tries most affected by aging such as the European Union will initially be capital exporters, while countries less affected by aging like the United States und other OECD regions will import capital. This pattern is reversed in about the year 2020 when baby boomers decumulate assets and the fast aging economies therefore become capital import regions. Pension reforms with higher degrees of pre-funding are likely to induce more capital exports. They also increase labor supply considerably, while the effects on the rate of return to capital are small. While the rate of return to capital declines in response to population aging, there is no devastating “asset meltdown”.

The timing pattern of these adjustments is complex, and one has to carefully distinguish level effects from changes over time. In the initial year of the projections (2000), savings rates in the Germany-France-Italy region are substantially higher in the open economy scenarios than under a closed economy assumption. This is in line with higher rates of return in economies with a smaller share of older persons. Open economies are able to diversify a great deal of the demographic effects that depress savings and the rate of return to capital.

This level effect is superseded by the demographic changes during the 2000 to 2070 prediction window. Saving rates decrease until 2050 across all capital mobility scenarios since the baby boom generation de-cumulates assets. Saving rates are projected to rebound after the year 2050. Since PAYG pension systems partially crowd out private savings, decreases of saving rates are stronger in the older regions. As a result, the decrease in the rate of return would be lower in these regions than in regions with less generous pension systems if these regions were closed economies. Diversification advantages of worldwide capital mobility thus decline, and saving rates respond accordingly. It should be stressed that population projections are reliable one generation ahead, while the projection error increases substantially thereafter. Consequently, results for the post-2030 period should be interpreted with care.

The second part of the analysis provides an extensive sensitivity analysis of simulation results. Here, the sensitivity analysis focused only on the influence of modelling strategies - such as modelling adjustment costs, perfect annuity markets, endogenous labor supply and explicit pay-as-you-go systems in an overlapping generations context. The interested reader is referred to Börsch-Supan, Ludwig, and Winter (2004) for results of an additional sensitivity analysis with regard to structural model parameters similar to the analysis in Chapter 4. As Börsch-Supan, Ludwig, and Winter show the politically probably most contentious conclusion, the absence of a serious asset meltdown, is robust with respect to the choice of these elasticity parameters.

Whether adjustment costs and perfect annuity markets are assumed or not has only second-order effects on the time paths of simulated macroeconomic aggregates. Results are robust with respect to the choice of such modelling strategies. Assuming exogenous labor supply, however, is a serious restriction, as well as ignoring the existence of large pay-as-you-go pension systems. Therefore the channel of labor supply adjustments to the challenges of population aging is an important one, and the complex overlapping generations structure is necessary needed to model pay-as-you-go pensions in order to generate realistic projections of the macroeconomic effects of population aging.

6 Concluding Remarks

The chapters of this thesis cover three papers, Ludwig (2004a), Ludwig (2005) and Börsch-Supan, Ludwig, and Winter (2004), all dealing with different aspects of a large scale Auerbach-Kotlikoff type Overlapping Generations (AK-OLG) model (Auerbach and Kotlikoff 1987). As concluding remarks, a number of comments on the simplifying assumptions made throughout these chapters are in order. First, the model used is a one-good model which implies that the only transaction with other countries takes place in the form of physical capital investment. Other traded goods flowing between countries are not modelled. An obvious extension is a multiple goods model, which would allow for an analysis of traded good flows between countries and how they are affected by the consequences of demographic change. In addition, exchange rates in terms of relative prices between countries implied by such a multiple goods approach may dampen some of the capital market effects that were focused on in Chapter 5.

Second, there are no frictions, neither in the labor nor in the capital goods markets. This assumption is less critical for countries with market based systems as in the United States but probably more critical for the continental European economies. Accounting for market frictions may be more important for an analysis concerned with short-run developments rather than the long-run effects this thesis is concerned with. As argued in Chapter 4, adding capital market frictions (Lührmann 2003; Berkel 2004) will be an unavoidable and important feature of future open economy AK-OLG models. The analysis of Chapter 4 also suggests that a fully flexible labor market model is not well-suited to match the time path of actual labor supply shares.

Third, demographic processes are assumed to be exogenous to economic development, but in the long run neither fertility nor mortality are exogenous to economic growth (Barro and Becker 1989; Boldrin and Jones 2002; Greenwood, Seshadri, and Vandenbroucke 2005). Migration reacts to international income differences also in the short-run. Against this background a number of researchers, e.g., Storesletten (2000), have asked the question whether higher immigration could solve the fiscal problems associated with aging and the retiring of baby boomers and its accompanying impact on economic growth.

Berkel, Börsch-Supan, Ludwig, and Winter (2004) address the relationship between the age-structure and growth from a slightly different angle: What would happen if, for some reason, fertility rates increase again? The key difference between increases in migration and increases in fertility is that migrants usually enter in prime work years and therefore immediately become part of the productive work force whereas children have to be raised and educated first. Therefore, if there were a permanent increase in fertility, children that become productive 20 years after the initial increase in fertility would have to feed those who continue to follow and hence it will take a long time until positive effects on economic growth show up. As further shown in Berkel et al., endogenous human capital formation, which is - fourth - absent from the model setup presented here, may make a difference since a higher share of a young population may lead to permanent increases in the rate of economic growth. This interaction is not yet fully understood and the focus of ongoing research.

Fifth, heterogeneity of households within each age group is ignored. In other words, only an average behavior of households of a given age is taken into account. Since the analysis conducted here does not focus on the distributional consequences of demographic change and of fundamental pension reforms, this assumption may not be critical at first sight. However, as shown in Chapter 4, the failure of the model to match some features of the data may be due to assuming away heterogeneity within age groups.

Sixth, and among the most critical assumptions, is the assumption that households are fully rational and forward looking. The entire analysis follows the rational expectations paradigm introduced to the economic literature by Muth (1961) and Radner (1972) and popularized by Lucas (1976). While departures from this paradigm make it very complicated for an economic modeler to grasp behavior of model agents, there are certainly good reasons to question the assumption made on ultra-rationality of agents. And indeed, a large body of empirical literature has shown that households do not smooth consumption as much as the conventional life-cycle theory predicts (Attanasio 1999). Of particular importance for the issues addressed in this thesis is the question whether households are really as forward looking and willing to plan for retirement as assumed (Imrohoroglu, Imrohoroglu, and Joines 2003). Furthermore, and as the results of Chapter 4 suggest, it is an open research question whether an enriched AK-OLG framework incorporating additional features and constraints into the pure life-cycle model, but otherwise making the assumption of full rationality of agents as in (Altig et al. 2001) would help closing the gap between actual and predicted values of aggregate flow variables, or whether departures from ultra-rationality, for example by assuming rules-of-thumb behavior (Campbell and Mankiw 1991; Mankiw 2000) are required.

Finally, to keep the analysis tractable, any uncertainty about future developments is ignored. In other words, households are not only modelled as fully rational, but they also have perfect foresight about future economic and demographic developments. In the context of overlapping generations models and questions related to social security reform, the effect of idiosyncratic uncertainties has been addressed, e.g., by Imrohoroglu, Imrohoroglu, and Joines (1995) and Conesa and Krueger (1999). Both studies find that PAYG financed pension systems and the implied insurance device against income risks may be welfare enhancing. Krueger and Kubler (2002) ask a similar question in an incomplete markets economy with aggregate uncertainties. Their analysis suggests that, in the absence of a crowding-out effect of social security on private capital, the introduction of a PAYG pension system may be Pareto improving. Yet, the crowding-out effect of private capital is shown to overturn these effects under reasonable assumptions on preferences. All of these papers however miss the potentially important implications of political uncertainty associated with an unfunded pension system.

For mathematical reasons, adding uncertainty to such complex models as those used in this thesis would render the analysis impossible with current techniques and computer power. This is an area of ongoing research (Krueger and Kubler 2003). Related with the aspect of adding aggregate uncertainty to the simulation model is the relationship between aging and the equity premium, that is, the return differential between risky and risk-free assets. Since older households prefer relatively risk-free investments, the overall preference for relatively risk-free assets may increase in aging societies. As a consequence, the relative price of risk-free assets would increase which would reduce their return and thereby increase the equity premium. The relationship between life-cycle savings behavior and the equity premium has recently received a lot of attention in the academic literate (Storesletten, Telmer, and Yaron 2001; Brooks 2002b; Constantidines, Donaldson, and Mehra 2002; Börsch-Supan, Ludwig, and Sommer 2003; Ludwig 2004c; Gomes and Michaelides 2005).

While these aspects would certainly extend the focus of the analysis and would shed additional light on the various interactions at work in aging societies, results found in this thesis that are of course subject to the simplifying assumptions made can finally be summarized as follows: Capital flows resulting from differential demographic change across countries will be quite substantial. Aging economies, such as the continental European countries will initially remain capital export regions, but as baby boom generations retire and finance consumption using up their retirement savings, these regions are projected to become capital import regions. This trend towards capital imports reaches its peak in about 2040-2050.

The decrease in the rate of return to capital induced by population aging will be significant, but there is not a devastating asset meltdown ahead. Closed economy models overestimate the decrease of the rate of return to capital. However, despite significant differences in patterns of demographic change, investing capital globally does not make too much of a difference, since the major demographic trends are highly correlated across countries. The interaction with pension systems is different: Closed economy models of fundamental pension reform miss important effects of international capital mobility and overestimate the additional decrease in the rate of return to capital caused by the higher degree of pre-funding.

The analysis further highlights the importance of flexible labor markets within the context of aging societies. It is not only by increasing retirement savings how households may react to demographic change and to fundamental reforms of pension systems, but also by retiring later and/or by working more hours. To the extend that not only savings but also labor supply increases as a consequence of a reduction in PAYG financed pensions, the additional decrease in the rate of return to capital will be postponed and much smaller in magnitude. This aspect has so far not been sufficiently considered in the asset market meltdown literature.

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Eidesstattliche Erklärung

Hiermit erkläre ich, dass ich diese Dissertation selbständig angefertigt und mich anderer als der in ihr angegebenen Hilfsmittel nicht bedient habe, insbesondere, dass aus anderen Schriften Entlehnungen, soweit sie in dieser Disseratation nicht ausdrücklich als solche gekennzeichnet und mit Quellenangaben versehen sind, nicht stattgefunden haben.

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