**Mannheimer Manuskripte zu Risikotheorie,
Portfolio Management und Versicherungswirtschaft**

Nr. 162

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Mannheim 07/2005

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ABSTRACT

Numerous insurance and investment products offered by the private financial industry contain embedded investment guarantees. In these products the contributions are invested on the capital market, which involves investment risk for the client. To protect the contribution payers against this risk the modern pension frames include performance guarantees. It is important that these guarantees are priced correctly because improper pricing and provision policy may lead to bankruptcy. This paper proposes a pricing model for maturity guarantees embedded in retirement saving plans. A deterministic guaranteed rate of return and periodic contributions are considered. Since maturity guarantees have long-term character, an approach is needed that models the long-run behaviour of the market prices of securities. For this reason this paper uses the geometric Brownian motion with Markov switching. For guarantee pricing WEBB's (2003) option pricing for Markov switching model is applied, which is based on the Esscher transform. The second part of the paper analyses the risk of shortfall in respect to three shortfall risk measures: the shortfall probability, the shortfall expectation and the mean expected loss.

Keywords: embedded options, shortfall risk, maturity guarantees, individual retirement accounts, option pricing, risk analysis, Esscher transform, Markov switching, Monte Carlo simulation

July 2005

All honour belongs to God

1. Introduction[1]

The demographic ageing process turned into a serious problem for numerous societies in the past decades. As a result retirement financial products, as individual saving plans, in which contributions are invested at the financial market are becoming more and more popular. Buying these products, people wish to participate in the return chances of the return markets. The risk, however, that the investment fails to cover the planned life standard in the old age can also be considerable. One of the aims of saving for the retirement is making up for the loss of income in the old age. Achieving this aim can be threatened if shortly before the retirement the stock market crashes and destroys, say, half of the savings. In such a scenario the financial well-being of the retiree might by in danger. In order to protect the contribution payers against this risk the modern pension frames include embedded guarantees. Examples of such guarantees are German individual retirement plans established in 2001 by the Retirement Savings Act (Altersvermögensgesetz). These plans are co-financed by the state if the provider guarantees that, at the maturity, the contributor will receive at least the sum of the premiums paid throughout the duration of the contract. This corresponds to the maturity guarantee with a deterministic rate of return.

It is important that performance guarantees are priced correctly because improper pricing and provision policy may lead to bankruptcy of the guarantor, as in the case of Equity Life in the UK. The interest on embedded guarantees has risen significantly over time (cf. FISCHER 1998, GRÜNDL, NIETRET and SCHMEISER 2004, HARDY 2001b, 2003, KLING, RUSS and SCHMEISER 2004, LACHANCE and MITCHELL 2003, MAURER and SCHLAG 2003, among others). This paper proposes a pricing model for maturity guarantees embedded in retirement saving plans. A deterministic guaranteed rate of return and periodic contributions is considered. Since maturity guarantees have long-term character, an approach is needed that models the long-run behaviour of securities returns. Thus, a geometric Brownian motion with Markov switching model for the prices of a risky asset (portfolio from equities and risky bonds) is used. The

[1] I owe special acknowledgments to my supervisor, Professor Peter Albrecht, for sharing his knowledge with me and for providing valuable comments. Please note that all opinions and errors are solely my own. Additionally, I would like to thank my friend Ingo Mistele for his encouragement when encouragement was needed.

pricing is accomplished by utilizing a Monte Carlo simulation. The second main topic of this paper is the analysis of the investment risk. The risk will be defined as the risk of shortfall with the value of the guarantee as the target. The risk will be quantified with the shortfall risk measures defined analogically as by ALBRECHT, MAURER and RUCKPAUL (2001).

This paper is organised as follows: Section 2 introduces the geometric Brownian motion with Markov switching, which describes the behaviour of the market prices of risky assets more precisely than the classic geometric Brownian motion. Section 3 describes an option pricing theory for stochastic models of the second section. The model is based on the Esscher transform martingale measure developed by GERBER and SHIU (1994a) and modified for the Markov switching by WEBB (2003). Section 4 discusses the cost of the maturity guarantees determined with the Monte Carlo simulation under Esscher martingale measure. Section 5 analyses the shortfall risk of the guarantee with respect to the shortfall risk measures: the shortfall probability, the shortfall expectation, and the mean expected loss. The last section presents the conclusion of the forth and the fifth section and gives a brief outlook for future research.

2. Geometric Brownian Motion with Regime Switching

The choice of a suitable stochastic process that describes the changes in the value of a risky portfolio is essential for the option pricing and the risk analysis. In general the well-established geometric Brownian motion is used (cf. PENNACCHI 1999, MAURER and SCHLAG 2003, LACHANCE and MITCHELL 2003, KLING, RUSS and SCHMEISER 2004, GRÜNDL, NIETRET and SCHMEISER 2004). The geometric Brownian motion is a stochastic process with a constant mean and a volatility parameter. The retirement saving contracts, however, may have durations of more than 30 years, and it is doubtful that during such a long period of time the parameters will not change. For instance, the monetary policy may change or the inflation level may rise significantly, which both have an impact on the interest rate. Furthermore, non-economical factors as international conflicts, political instability, or technical development influence the financial markets as well. To take this uncertainty into account, a model with stochastic distribution parameter could be used. This paper uses the geometric Brownian motion with Markov switching (cf. WEBB 2003) as the stochastic process for the security prices. It was shown, that compared to the classical geometric Brownian motion approach (cf.

PIASKOWSKI 2005, HARDY 2001a), the geometric Brownian motion with Markov switching is a better model for the behaviour of market prices of risky assets in the long run.

The class of Markov switching models (also known as regime switching or hidden Markov switching models) were initially introduced by HAMILTON (1989) to describe shocks in the development of economic time series. He analysed the development of the Gross Domestic Product in the USA and fund out, how the separate parameters for both normal growth and recessional periods can be estimated. However, it cannot be observed, which state the economy currently is in. Hamilton solved this problem by adding a latent random variable determining the state of the economy.

Assume that S_t is the price of a non-dividend-paying risky asset at time t, with the logarithmic returns driven by the stochastic process $X(t)$. Assume, as well, that $X(t)$ has stationary and independent increments and $X(0) = 0$ (c.f.GERBER and SHIU 1994a). Thus the development of the risky asset price S_t will be

$$S_t = S_0 e^{X(t)}, \text{ with } t \geq 0 \text{ and } S_0 \neq 0. \tag{2.1}$$

Let

$$F(x,t) = \Pr[X(t) \leq x] \tag{2.2}$$

be the cumulative distribution function of $X(t)$ and

$$M(z,t) = E\left[e^{zX(t)}\right] \tag{2.3}$$

its moment-generating function, which is continuous at $t = 0$, and

$$f(x,t) = \frac{d}{dx} F(x,t), \text{ with } t > 0 \tag{2.4}$$

its density. Under these assumptions the moment-generating function of $X(t)$ equals

$$M(z,t) = \int_{-\infty}^{\infty} e^{zX(t)} f(x,t)dx. \tag{2.5}$$

The classic assumption is, that $X(t)$ is the Wiener process (cf. HULL 2003)

$$dS_t = \mu dt + \sigma dW_t, \tag{2.6}$$

where μ and σ denote the drift and diffusion, respectively, S_t the price of the risky asset and W_t the standard Wiener process at time t. It is equivalent, to the assumption, that the logarithmic assert prices are driven by the geometric Brownian motion:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \tag{2.7}$$

Using Itô's lemma to solve the equation (2.7) the stochastic process for the development of the risky asset

$$S_t = S_{t_0} e^{(\mu - \frac{1}{2}\sigma^2)(t-t_0) + \sigma(W_t - W_{t_0})}, \quad 0 \le t_0 < t \quad \text{and} \quad S_{t_0} > 0 \tag{2.8}$$

can be determined. Therefore *X(t)* follows

$$X(t) = (\mu - \tfrac{1}{2}\sigma^2)(t - t_0) + \sigma(W_t - W_{t_0}) \tag{2.9}$$

and has mean $m = \mu - \frac{1}{2}\sigma^2$ and standard deviation σ for the unit time period.

The idea of the Markov switching model is to allow the parameters to depend on the state the process is currently in. Intuitively it corresponds to the recession and growth, where both phase have different means and/or volatilities. Under geometric Brownian motion, the log market price of the risky asset follows the geometric Brownian motion for the time interval $[t_n, t_{n+1})$, but at the discrete times t_n (n = 0, 1, …, N) stochastically "switches" among K states (geometric Brownian motion with other parameter) are allowed. The switches will be driven by the latent random variable z with a Markov probability (cf. WEBB 2003). For a given state $z_t = j$ the equation (2.7) will change into

$$\frac{dS_t}{S_t} = \mu(z_t = j)dt + \sigma(z_t = j)dW_t, \tag{2.10}$$

with the regime $z_t = j$ dependent drift $\mu(z_t{=}j)$ and diffusion $\sigma(z_t{=}j)$ and W_t the standard Wiener process. Hereafter $\mu(z_t{=}j)$ and $\sigma(z_t{=}j)$ will be designated with μ_j and σ_j, respectively. The Itô's solution of the equation (2.10) is given by

$$S_t = S_{t_0} e^{(\mu_j - \frac{1}{2}\sigma_j^2)(t-t_0) + \sigma_j(W_t - W_{t_0})}, \; 0 \le t_0 < t \text{ and } S_{t_0} > 0. \tag{2.11}$$

Equivalently *X(t)* follows

$$X(t) = (\mu_j - \tfrac{1}{2}\sigma_j^2)(t - t_0) + \sigma_j(W_t - W_{t_0}), \; 0 \le t_0 < t \tag{2.12}$$

with the regime dependant mean $m_j = \mu_j - \frac{1}{2}\sigma_j^2$ and standard deviation σ_j.

As stated above, the regime z_t is unobservable, the transition probabilities

$$p_{ji} = \Pr[z_t = j \mid z_{t-1} = i], \tag{2.13}$$

that the process switches from regime $z_{t-1} = i$ to $z_t = j$ can be, however, determined (cf. KIM 1994 for details). These conditional probabilities can be collected in the transition probability matrix

$$P_{K\times K}=\begin{bmatrix} p_{11} & \cdots & p_{1K} \\ \cdots & \ddots & \cdots \\ p_{K1} & \cdots & p_{KK} \end{bmatrix},\ \sum_{j=1}^{K} p_{ji}=1 \text{ and } 0\le p_{ji}\le 1 \text{ for } i,j=1,\dots,K, \tag{2.14}$$

with K – the number of possible regimes. As mentioned above, z is a Markov chain, so the transition probabilities from equitation (2.13) depend only on the previous state $z_{t-1} = i$.

Knowing the probability matrix $P_{K\times K}$ the unconditional probabilities the vector $\boldsymbol{\Pi}$:

$$\pi=(B'B)^{-1}B'e_{K+1},\quad B_{(K+1)\times K}=\begin{bmatrix} I_K-P \\ 1'_K \end{bmatrix},\ \sum_{i=1}^{K}\pi_i=1 \text{ and } 0\le\pi_i\le 1,\ \forall i=1,\dots,K, \tag{2.15}$$

of unconditional probabilities

$$\pi_i=\Pr(z_t=i), \tag{2.16}$$

that the process is in the regime $z_t = i$ can be determined. $\boldsymbol{1_K}$ denotes a K-element vector of ones, $\boldsymbol{e_K}$ denotes K-th column of unity matrix $\boldsymbol{I_K}$. For $K = 2$ that gives (cf. HAMILTON 1989)

$$\pi=\begin{bmatrix}\pi_1\\ \pi_2\end{bmatrix}=\begin{bmatrix}\dfrac{1-p_{22}}{2-p_{11}-p_{22}}\\ \dfrac{1-p_{11}}{2-p_{11}-p_{22}}\end{bmatrix}. \tag{2.17}$$

3. Option pricing

3.1. Design of the Guarantee

The German finance industry offers retirement saving plans in which the contributions are invested in risky portfolios (cf. MAURER and SCHLAG 2003). These saving plans are co-financed by the state, if the provider includes a guarantee that at least the sum of the charged premiums will be paid out to the investor at the maturity time. That results in a deterministic guarantee rate of 0% on the paid contributions. If the provider's investment strategy fails to generate this minimum return, it is required to finance the difference between the market value of the portfolio and the guaranteed amount. From economic point of view this guarantee is a European put option with the following payoff:

$$P_T=\max\{G_T-S_T;0\}, \tag{3.1}$$

with the guarantee value

$$G_T=\sum_{t=0}^{T-1}C_t e^{g(T-t)}, \tag{3.2}$$

where P_T is the value of the option, S_T is the market value of the risky portfolio at the maturity T, g is the guarantee's rate of return and C_t is the contribution paid in at the time t. The price of the put at the time $t = 0$ equals the expected value of the payoff at the maturity under the risk-free probability measure Q, discounted with the risk-free rate of return r

$$P_0 = e^{-rT} E_Q\left[\max\{G_T - S_T;0\}\right], \tag{3.3}$$

where $E_Q[\cdot]$ denotes the expected value under probability measure Q.

Under classic geometric Brownian motion the market is complete, so that there exists a unique martingale measure Q. However, if the Markov switching is allowed, the variance becomes stochastic and the market is not complete anymore. The option pricing in such economy is not straightforward, because there exists no unique equivalent probability measure and therefore no unique price of the option. Thus a reliable choice of the martingale measure has to be made. In this paper the Esscher transform will be used to find such a probability measure.

The Esscher transform is a well-approved tool among actuaries, which originally was developed by ESSCHER (1932) to transform a random variable, to give it a new distribution centred at a point of interest. The purpose of this is to enable more accurate approximations to be made at this point. Gerber and Shiu were the first to use the Esscher transform to price European (cf. GERBER and SHIU1994a) and American options (cf. GERBER and SHIU 1994b). The modification of the Gerber and Shiu's approach adapted for pricing a European option under the geometric Brownian motion with Markov switching was developed by WEBB (2003, chapter 5).

The advantage of the Esscher martingale probability measure is, that (a) the process under new martingale measure remains in the same class of models as the process under real-world probability measure P, which in the discussed case means, that the log prices under Esscher transform follow the geometric Brownian motion with Markov switching (cf. GERBER and SHIU1994a, comment of Michaud) (b) it converges to a well-known BLACK/SCHOLES (1973) option pricing formula for the case with one switching regime (K = 1) (cf. Webb 2003, corollary 5.4.4). (c) There is only one Esscher martingale probability measure, so there exists no problem with choosing among several probability measure, as it could happen if some other approach would be used (cf. WEBB 2003, theorem 5.2.2). (d) Finally, the Esscher transform approach is conform with maximising the expected utility with the utility function

$u(x) = {x^\gamma}/{\gamma}$ (0<γ<1) (cf. WEBB 2003). This means, intuitively, that the individual prefers to have more money than less. However, the wealth increase of 1 EUR, have the smaller additional utility, the more the individual posses. The individual with this utility function is risk averse.

3.2. Esscher Option Price for Geometric Brownian Motion with Markov Switching

Before the option pricing will be discussed, the briefly introduction of the Esscher transform will be made. Recall, please, that the log returns follow the stochastic process *X(t)* with stationary and independent increments and *X(*0*)* = 0. Furthermore the cumulative distribution function and density function of *X(t)* are given as in equations (2.2) and (2.4), respectively and there *X(t)* has the moment-generating function given by the equation (2.5).

Let h∈ℝ be an Esscher parameter, for which *M(h,t)* exists. The transformed stochastic process *X(t;h)* has again stationary and independent increments and the transformed density function

$$f(x,t;h) = \frac{e^{hx} f(x,t)}{\int_{-\infty}^{\infty} e^{hy} f(y,t)dy} = \frac{e^{hx} f(x,t)}{M(h,t)}. \tag{3.4}$$

The corresponding moment-generating function is

$$M(x,t;h) = \int_{-\infty}^{\infty} e^{zx} f(x,t;h)dx = \frac{M(z+h,t)}{M(h,t)}. \tag{3.5}$$

For option pricing under Esscher transform parameter h_Q has to be fund, such that the discounted stock price is a martingale with respect to the probability measure Q corresponding to h_Q

$$S_0 = E_Q\left[e^{-rT} S_t\right], \text{ for } t \geq 0, \tag{3.6}$$

with *r* risk-free discount rate. GERBER and SHIU (1994b) have shown, that h_Q is unique, so the corresponding equivalent martingale probability measure is unique as well. However note, that there may exist other (not Esscher) equivalent martingale measures. GERBER and SHIU (1994a) developed the pricing equation for a European call which from the put-call parity gives the price for the put

$$P_0 = e^{-rT} G_T F\left(\ln\frac{G_T}{S_0}, T; h_Q\right) - S_0 F\left(\ln\frac{G_T}{S_0}, T; h_Q + 1\right), \text{ for } t \geq 0, \tag{3.7}$$

where $F(x,t;h)$ denotes the Esscher transformed cumulative distribution function, G_T the exercise price at the maturity time T.

WEBB (2003) has derived the call price for the geometric Brownian motion with Markov switching. Implementing the put-call parity again gives the put price

$$P_{0,j_1} = P_0(s_t = j_1) = \sum_{j_2=1}^{K} \cdots \sum_{j_{N+1}=1}^{K} \underbrace{\prod_{i=1}^{N} p_{j_{i+1}j_i}^{(h)}}_{(*)}$$

$$\times \underbrace{\left[G_T e^{-rT} N\left(-d_j^-\right) - S_0 \exp\left[\sum_{k=1}^{N} \left[\left(\mu_{j_{k+1}} - r\right)\tau + h_{j_k} \sigma_{j_{k+1}}^2 \tau \right] \right] N\left(-d_j^+\right) \right]}_{(**)}, \tag{3.8}$$

with the Esscher transition probabilities

$$p_{ji}^{(h)} = \Pr(h; z_t = j \mid z_{t-1} = i) = \frac{p_{ji} \exp\left\{ h_i \left[\mu_j - \frac{1}{2}\sigma_j^2 \right]\tau + \frac{1}{2}\sigma_j^2 \tau h_i \right\}}{\sum_{j=1}^{M} p_{ji} \exp\left\{ h_i \left[\mu_j - \frac{1}{2}\sigma_j^2 \right]\tau + \frac{1}{2}\sigma_j^2 \tau h_i \right\}} \tag{3.9}$$

and parameters

$$d_j^{\pm} = \frac{\ln\left(S_0 / K \right) + \sum_{i=2}^{N+1} \left(\mu_{j_i} \pm \frac{1}{2}\sigma_{j_i}^2 \right)\tau + \sum_{i=1}^{N-1} \left(h_{j_i} - h_{j_N} \right) \sigma_{j_{i+1}}^2 \tau}{\left(\sum_{i=2}^{N+1} \sigma_{j_i}^2 \right)^{1/2} \sqrt{\tau}} + h_{j_N} \left(\sum_{i=2}^{N+1} \sigma_{j_i}^2 \right) \sqrt{\tau}, \tag{3.10}$$

where N denotes the amount of switches in the pricing horizon and $\tau = T/N$ the time period between two switches.

The Esscher parameter vector h can be computed numerically from the equations

$$\sum_{j=1}^{K} p_{ji} \exp\left\{ h_i \left(\mu_j - \frac{1}{2}\sigma_j^2 \right)\tau + \frac{1}{2}\sigma_j^2 \tau h_i^2 \right\} \left(\exp\left\{ \mu_j \tau + \sigma_j^2 \tau h_i \right\} - \exp\{ r\tau \} \right) = 0. \tag{3.11}$$

knowing, that h_i is a unique point from the interval

$$\left(\min_j \left(\frac{r - \mu_j}{\sigma_j^2} \right), \max_j \left(\frac{r - \mu_j}{\sigma_j^2} \right) \right) \tag{3.12}$$

(for further details cf. WEBB 2003, pp. 89-91).

The term (**) in the equation (3.8) gives the price of the put for the given switching path (combination of regimes) $j_1, j_2, \ldots, j_{N+1}$, which is weighted with the probability (*) that process will follow that path. Finally the price is computed for each possible switching path and added together. Note that the term (**) shows some analogy to the well-known BLACK/SCHOLES

(1973) formula for the European put option. (a) Terms d_j^+ and d_j^- correspond to d_1 and d_2, respectively. (b) The exponential term after S_0 is consistent with the Black/Scholes formula as well, because the stock return by Black and Scholes is equal to the risk free-rate r, which for the discounted asset price S_T gives

$$e^{-rT} S_T = e^{-rT} S_0 e^{rT} = S_0$$

Under the Esscher risk-neutral probability measure the equity return is equal to $(\mu_{j_n} + h_{j_{n-1}} \sigma_{j_n}^2)\tau$, so this term do not reduce to S_0.

Please, note as well, that the put price $P_{0,j1}$ depends on the initial state j_1. To determine the price independent on the initial regime, the switching path length N has to be increased, due to the fact that the influence of the initial state on the put price decreases, as the path length grows (cf. WEBB 2003, tables 7.1-7.3). This is, however, problematic, since the number of combinations grows exponentially as the path length increases. This is especially probematic by the long-term options, which are studies in this paper. For instance by two states ($K = 2$), 30 years maturity and one switch per year ($N = 30$) the number of combinations (K^N) equals 1,073,741,824. If the switch occurs every month ($N = 360$), the number of combinations rises to $2.3 \cdot 10^{108}$. Instead of increasing N, the approximate price $P_{0,app}$ could be determined through weighting the initial state dependent prices from equation (3.8) by the unconditionally Esscher probabilities, that the process stays in the i-th regime

$$P_{0,app} = \sum_{i=1}^{K} P_{0,i} \pi_i^{(h)} , \qquad (3.13)$$

where the unconditional probabilities $\pi_i^{(h)}$ could be compute with the equation (2.15) using the Esscher conditional probabilities $p_{ji}^{(h)}$ from equation (3.9) instead of the real-world probabilities p_{ji}.

Due to the fact, that contributions to retirement saving plans are paid periodically, there exists no closed-form solution of the option pricing and the formula (3.8) cannot be applied directly. It is however possible to simulate the option price with the Monte Carlo simulation with the return mean

$$\left(\mu_j - \tfrac{1}{2}\sigma_j^2\right)\tau + \tfrac{1}{2} h_i \sigma_j^2 \tau \qquad (3.14)$$

and the return standard deviation

$$\sigma_j \tau \qquad (3.15)$$

and the Esscher transition probabilities given in the equation (3.9) (for the simulation algorithm under Markov switching regime cf. HARDY 2003, pp. 98).

4. Results of the Price Simulation

4.1. Design of the Study

Before the results of the study on the cost of the return guarantee can be presented, the simulation design should be explained. An individual retirement account with a yearly contribution of 1200 EUR paid in advance and a fixed duration of T = 1, …, 40 years was assumed. The contract provider gave a guarantee that the paid premiums will generate a minimum return of g = -2%, 0%, 2% or 4% p.a. The client could choose between a bond and an equity fund, or he/she was allowed to divide the investment between both funds, so that 25%, 50%, or 75% of the contribution would be paid into the bond fund and the rest into the stock fund (shifts between the funds were not allowed). At the maturity of the contract, the provider would prove if the investment target was achieved, and if not, the difference between the value of the investment and the guaranteed amount would be paid to the client in addition to the portfolio market value (so the client would receive a maximum of the guaranteed amount and the value investment portfolio). If the client cancels the contract or dies before the maturity, he/she or his/hers inheritors receive the value of the portfolio. This means, that if the contract is cancelled before the maturity, no guarantee will be given. Please note, that such guarantee design corresponds to the maturity guarantee with a deterministic rate of return, and the protection only from the investment but not from the biometrical risk is given. The provider collects two fees to cover the costs and to make profit: the front-end-sales-charge of 3% of the bond fund units and 5% of the equity fund units and the administration charge. The administration charge will be approximated by subtracting 0.5% p.a. from the average return of the investment (cf. MAURER and SCHLAG 2003). Please note that the guarantee is given on the gross contribution, so e.g. in the in the one-year maturity case the guarantee of 2% on the pure stock portfolio is, indeed, a guarantee of 7.37% from the guarantor's point of view, because return from the investment has to cover both the front-end-sales-fee of 5% and the guarantee rate of 2%.

To estimate the distribution parameters of the returns, there was assumed, that the bond fund returns have the same distribution as the returns of the German Bond Performance Index

(REXP), and the equity fund returns have the same distribution as the returns of the German Stock Index (DAX). In both cases there are performance indices involved, which means that the whole income from the investment (dividends, coupon-payment etc.) will be reinvested in the portfolio underlying the index. To estimate the parameters for the simulation, five synthetic portfolios were built. It was assumed, that on 31/12/1974 the amount of 100 EUR was invested in each portfolio defined above (i.e. with a 100%/0%, 75%/25%, 50%/50%, 25%/75%, and 0%/100% REXP to DAX proportion, respectively) and that the portfolio was held until 31/12/2004. From the development of the value of these portfolios the monthly log-returns were determined. These log-returns were used to estimate the distribution parameters for the geometric Brownian motion with two Markov regimes applying the method described in PIASKOWSKI (2005) (for estimation of the Markov switching model cf. HOLZIG 1997 and KIM and NELSON 1999 as well). Statistic tests have shown that the mean for all portfolios (with exception of the portfolio with a 75%/25% bond-to-stocks-ratio) were regime-independent and all variances were dependent on the state (cf. table 4.1). The price was simulated under the Esscher martingale probability measure Q with the Monte Carlo simulation with 100,000 runs. Under the Esscher martingale measure, the mean and standard deviation for the stochastic process of the return are given by the equations (3.14), (3.15) and the transition probabilities are given by equation (3.9), respectively. As discount rate the risk-free rate of 0.44% per month was chosen, which is the average monthly money market rate (Monatsgeld) published by the Federal Bank of Germany (Deutsche Bundesbank) for the period of January 1975 to December 2004
(cf. http://www.bundesbank.de/statistik/statistik_zeitreihen.en.php).

In the following section, the cost of the guarantee simulated under the geometric Brownian motion with Markov switching will be discussed and compared to the price simulated under the classic geometric Brownian motion (the parameters for the geometric Brownian motion were estimated analogically as the ones for the Markov switching model, cf. table 4.2).

Please note that different contracts have different cash-flows and different guarantee value. So the put prices were divided by the net present value of the contributions paid during the contract, i.e.

$$\overline{P_0} = \frac{e^{-rT} E_Q\left[\max\{G_T - S_T ;0\}\right]}{\sum_{t=0}^{T-1} C_t e^{-rt}} \tag{4.1}$$

in order to enable the comparison among contracts with different durations and different guarantee rates. The value from the equation (4.1) will be referred to as "the normalised guarantee cost" hereafter.

Table 4.1 Parameters for geometric Brownian motion with Markov switching (on a monthly basis)

a) Transition probability matrix

bond/equity		100%0%		75%/25%		50%/50%		25%/75%		0%/100%	
Model		MS(1,2)		MS(2,2)		MS(1,2)		MS(1,2)		MS(1,2)	
Target regime		1	2	1	2	1	2	1	2	1	2
Initial regime		0.9947	0.0454	0.9463	0.0310	0.9624	0.0177	0.9636	0.0182	0.9604	0.0201
		0.0053	0.9546	0.0537	0.9690	0.0376	0.9823	0.0364	0.9818	0.0396	0.9799

b) mean vector

bond/equity		100%0%	75%/25%	50%/50%	25%/75%	0%/100%
Model		MS(1,2)	MS(2,2)	MS(1,2)	MS(1,2)	MS(1,2)
Regime	1	0.0061	0.0025	0.0063	0.0064	0.0066
	2		0.0085			

c) standard deviation vector

bond/equity		100%0%	75%/25%	50%/50%	25%/75%	0%/100%
Model		MS(1,2)	MS(2,2)	MS(1,2)	MS(1,2)	MS(1,2)
Regime	1	0.0186	0.0275	0.0491	0.0669	0.0839
	2	0.0095	0.014	0.0221	0.0301	0.0386

MS(m,s) denotes a stochastic process with Markov switching, with m – the number of means, s – the number of standard deviations. Both m and s can be equal to 1 or K – the number of regimes.

Table 4.2 Parameters for classic geometric Brownian motion (on a monthly basis)

bond/equity	100%/0%	75%/25%	50%50	25%/75%	0%/100%
mean	0.0060	0.0062	0.0063	0.0064	0.0066
std deviation	0.0110	0.0205	0.0342	0.0471	0.0593

4.2. Cost of the Guarantee

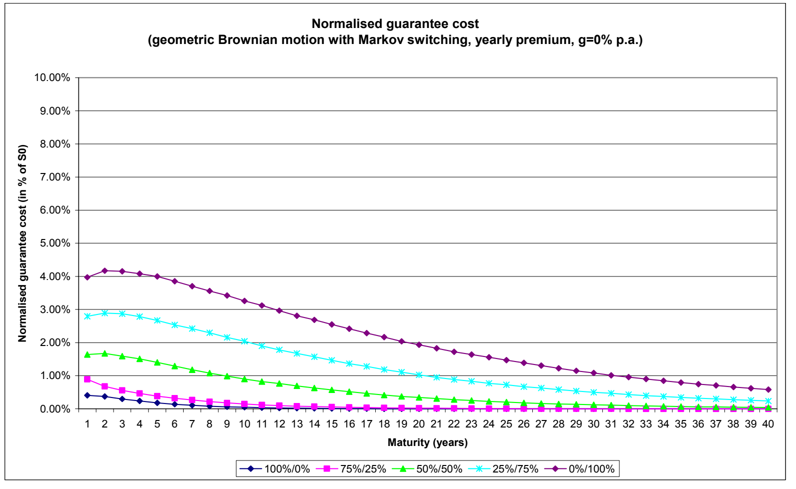
By analysis of the normalised guarantee cost it can be clearly seen that the behaviour of the option for $g \leq 2\%$ p.a. and $g = 4$ % p.a. is different to each other. Therefore the two groups will be discussed separately as the "low"-level guarantees (g = -2%, 0%, 2% p.a.) and the "high"-level guarantees (g = 4% p.a.). The normalised cost of the low-level guarantee (g = 0% p.a.) will be close to zero if the whole capital is invested in bonds and will be always under 1%, if the investment in equities does not exceed 25% (cf. graph 4.1). In case of following the more aggressive invest strategy, the cost of the normalised guarantee will rise. In case of the pure stock investment and the one year maturity, the normalised cost of the guarantee will be

3.97%. As the time horizon grows, the cost converges toward zero: The normalised guarantee cost is equal to 3.52, 1.93%, 1.08%, and 0.58% for 10, 20, 30, and 40 years maturity, respectively. Please note that if the guarantor would like to keep the guarantee cost at zero level, he or she should exclusively invest in low-risk securities. On the other hand low risk also means low return expectation. The individuals investing in the retirement plans, however, expect not only a high safety level but, high return perspectives too. An investment policy that is carried out too carefully could cause outflow of the clients to the competitors. So while making the risk management, guarantors have to take into account the business strategy as well.

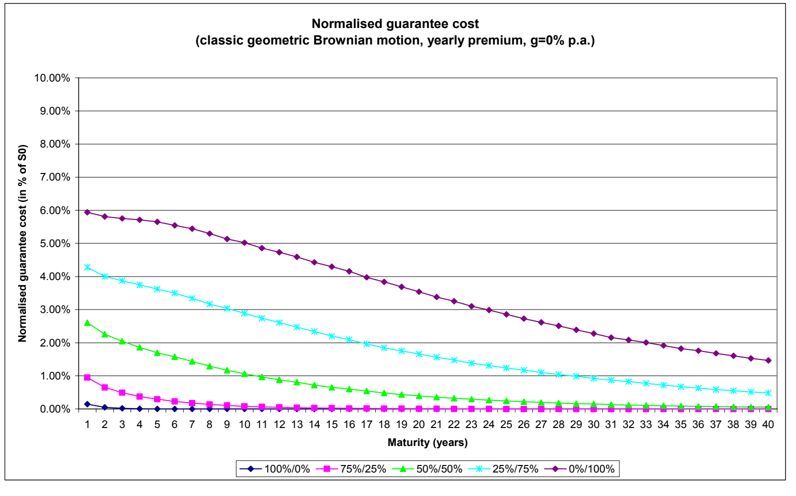
As mentioned above, the cost of the high-level guarantee (g = 4% p.a.) behaves differently, than in the case of the low-level guarantee. It grows up to a certain maximum and then decreases very slowly, so that it remains high even at long-term horizons (cf. graph 4.3). The normalised cost of the guarantee for the pure bond investment equals 1.28% for the one-year guarantee and grows to 2.61%, 2.66%, 2.51%, and 2.34% as the time horizon increases to 10, 20, 30, and 40 years, respectively. In the case of the pure stock investment, the cost is much higher and equals 5.60%, 9.63%, 10.93%, 11.16%, and 10.88% for one, 10, 20, 30, and 40 years maturity, respectively.

The cost of the guarantee within the Markov switching approach is higher for the investment with the low equity proportion, compared to the classic geometric Brownian motion. This has the opposite effect on investment with high stock ratio, e.g. the normalised cost of the low-level guarantee, with the pure equity investment strategy and one-year maturity is 1.97 per cent point larger for the geometric Brownian motion without Markov switching, compared to the case with regime switching (cf. graph 4.2 and 4.4). The difference decreases to 0.88 per cent point, if the time horizon increases to 40 years. In the case of the high-level guarantee, the spread between the prices simulated with both methods grows as the maturity time increases. The difference is equal to 2.38 per cent point for the one-year contract with the pure equity investment and increases to 8.07 per cent point as the maturity increases to 40 years.

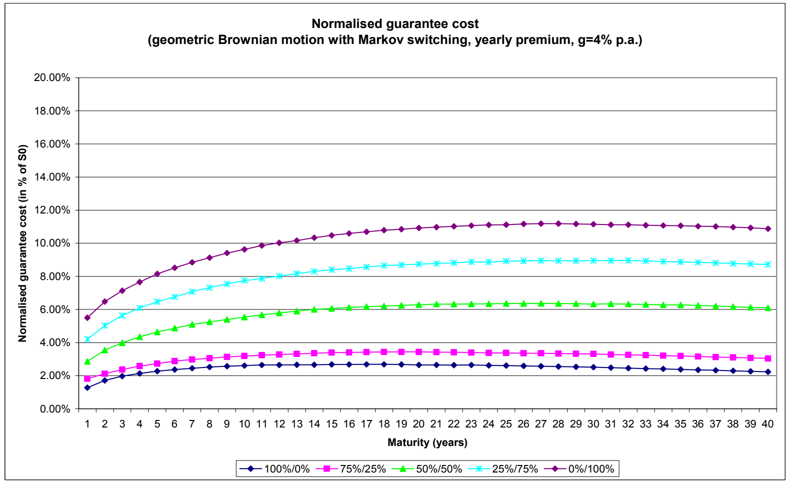
Graph 4.1 Normalised guarantee cost (*g* = 0% p.a., geometric Brownian motion with regime switching)



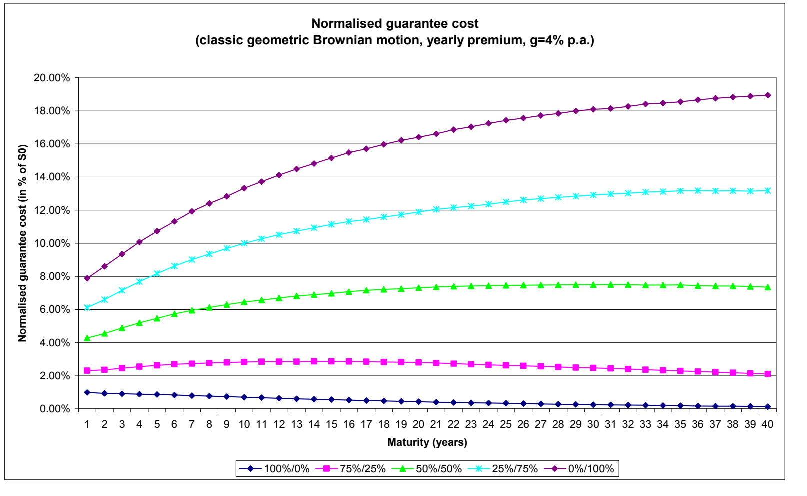
Graph 4.2 Normalised guarantee cost (*g* = 0% p.a., classic geometric Brownian motion)



Graph 4.3 Normalised guarantee cost (g = 4% p.a., geometric Brownian motion with Markov switching)



Graph 4.4 Normalised guarantee cost (*g* = 4% p.a., classic geometric Brownian motion)



5. Risk Analysis

5.1. Risk Quantification

Conventionally, the standard deviation is used to quantify financial risks. This risk measure includes both the "risk" to loose the invested capital and the "chance" of the positive return. In the case of the maturity guarantee, however, one needs other measures, because the positive deviance from the guaranteed value cannot be considered as a risk. This is the reason why, for the purpose of this study, other measures, so-called "shortfall risk measures", are more suitable: the shortfall probability, the shortfall expectation, and the mean expected loss (cf. ALBRECHT, MAURER and RUCKPAUL 2001). The shortfall probability $SP(G_T)$ is the probability that the risky asset has not performed better than the guaranteed value at the maturity

$$SP(G_T) = P(S_T < G_T), \tag{5.1}$$

the shortfall expectation $SE(G_T)$ is the expected loss

$$SE(G_T) = E[\max\{G_T - S_T, 0\}], \tag{5.2}$$

and the mean expected loss $MEL(G_T)$ is the expected loss in the case of shortfall

$$MEL(G_T) = E[G_T - S_T \mid S_T < G_T]. \tag{5.3}$$

The relationship of these three risk measures is given by

$$SE(G_T) = MEL(G_T) \cdot SP(G_T). \tag{5.4}$$

To make the comparison of different values for various maturities possible, the normalised shortfall expectation, $\overline{SE}(G_T)$, and the normalised MEL, $\overline{MEL}(G_T)$, are introduced. They are defined as "the relation of the discounted shortfall risk measure to the net present value of the paid contributions":

$$\overline{SE}(G_T) = \frac{e^{-rT} SE(G_T)}{\sum_{t=0}^{T-1} C_t e^{-rt}}, \tag{5.5}$$

$$\overline{MEL}(G_T) = \frac{e^{-rT} MEL(G_T)}{\sum_{t=0}^{T-1} C_t e^{-rt}}. \tag{5.6}$$

5.2. Results of the Simulation

5.2.1. Design of the Study

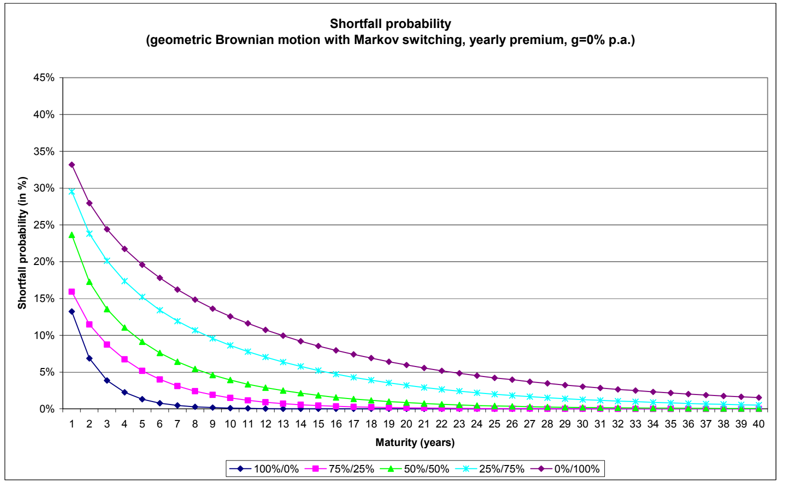
In this section the risk connected to the maturity guarantee will be analysed. As in section 4.1, the values simulated within the Markov switching approach will be discussed and briefly compared to the output of the simulation under the classic geometric Brownian motion. The assumptions made in section 4.1 will remain the same with two exceptions: The risk measures were determined under the real-world probability measure P and the number of simulation runs was increased to 750,000 due to a more precise estimation of the distribution tails.

5.2.2. Shortfall Probability

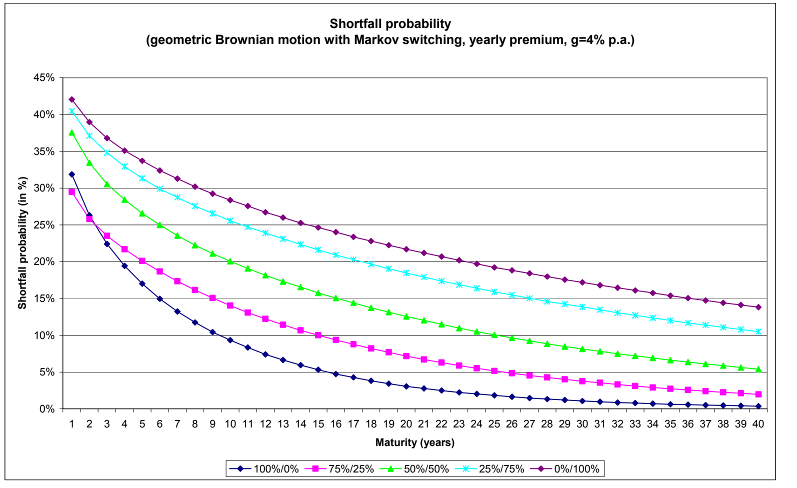
First, the shortfall probability for the low-level guarantee (g = 0% p.a.) in dependence on the investment strategy will be discussed (cf. graph 5.1). The probability of a shortfall is high or very high for the contracts with short maturity but it decreases as the maturity time grows. For the pure bond investment and one-year maturity the shortfall probability equals 13.24%, but two years later equals 3.87%. For the 10-year maturity the shortfall probability equals only 0.11%. The convergence of the shortfall probability toward zero is slower if the equity ratio in the portfolio grows. The probability of a shortfall in the case of the fifty-fifty bond/stock investment is equal to 23.81% for the one-year maturity guarantee, but only 3.94% for the 10-year guarantee, and close to zero after 40 years. For the 100%-equity investment the probability of shortfall equals 33.19%, 12.51%, 5.95%, 3.02%, and 1.54% for the one-, 10-, 20-, 30-, and 40-year guarantee, respective. Graph 5.3 shows the probabilities of shortfall for the high-level guarantee (g = 4% p.a.). As expected, the probabilities are higher than those for the low-level guarantee, due to the increase in risk. The convergence rate is much slower as at g = 0% p.a. Even the pure bond investment does not have the shortfall probability of 0% for any simulated maturity.

In comparison to the classic geometric Brownian motion, the regime switching approach results in lower shortfall probabilities. This is consistent with the observation that the normalised guarantee cost is lower for the Markov switching approach if compared to the classic geometric Brownian motion. The shortfall probabilities simulated with the classic

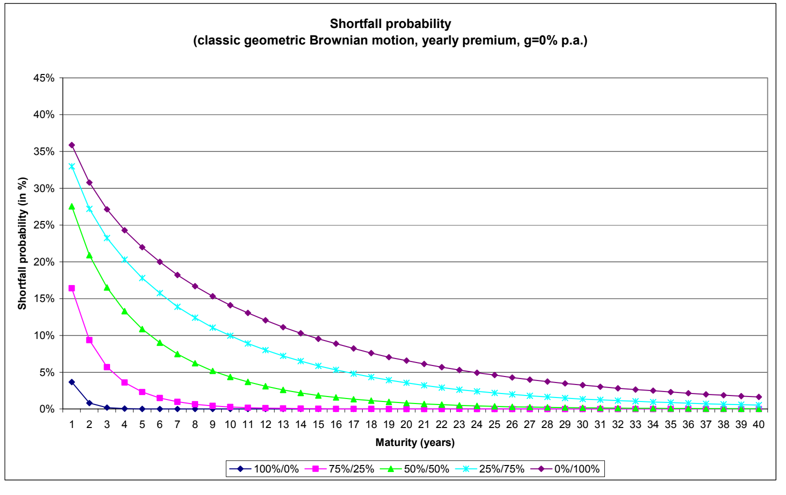
Graph 5.1 Shortfall probability (*g* = 0% p.a., geometric Brownian motion with Markov switching)



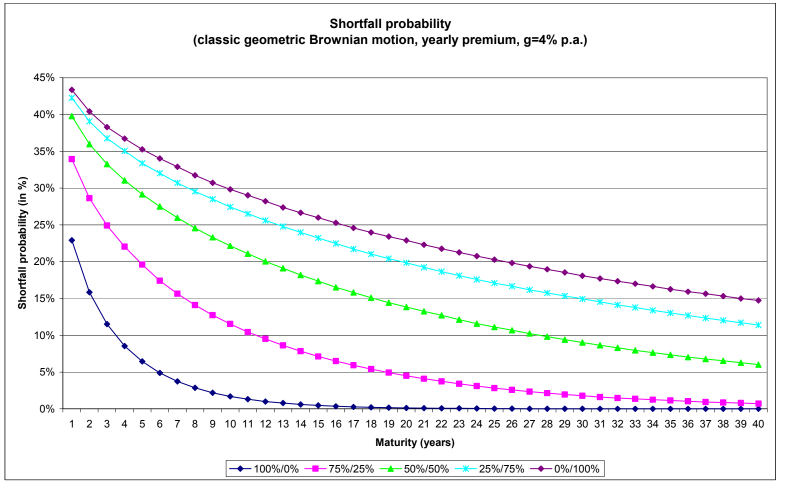
Graph 5.3 Shortfall probability (g = 4% p.a., geometric Brownian motion with Markov switching)



Graph 5.2 Shortfall probability (g = 0% p.a., classic geometric Brownian motion)



Graph 5.4 Shortfall probability (*g* = 4% p.a., classic geometric Brownian motion)



geometric Brownian motion method are, generally speaking, up to two times higher than the ones simulated with the Markov switching approach. This effect is much weaker for the high-level guarantee (cf. graph 5.2 and 5.4).

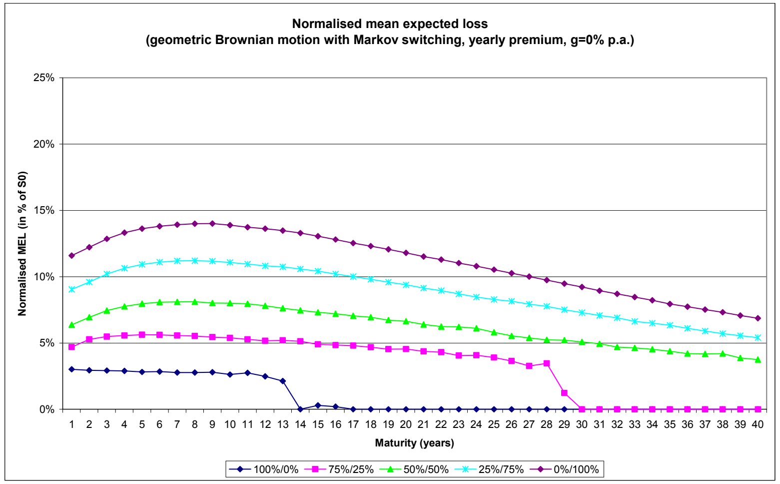
5.2.3 Mean Expected Loss

Another important risk measure is the mean expected loss (MEL), which quantifies the amount of money the guarantor will have to pay out, if the investment strategy fails to reach the guaranteed value. Graph 5.5 shows the normalised MEL for the low-level guarantee ($g = 0\%$ p.a.). In case of the contract with the pure bond investment strategy the normalised MEL equals 3.02% for the one-year maturity. The normalised MEL decreases slowly as the maturity time grows. Please note, that for maturity equal or greater than 17 the normalised MEL equals zero. It is due to the fact, that none of the simulation runs produced shortfall. The normalised MEL for the portfolio with the pure stock investment equals 11.59% for the one-year maturity contract. It reaches the maximum at the level of 14.01% with a maturity of nine years and starts to decline slowly to 6.86% after 40 years.

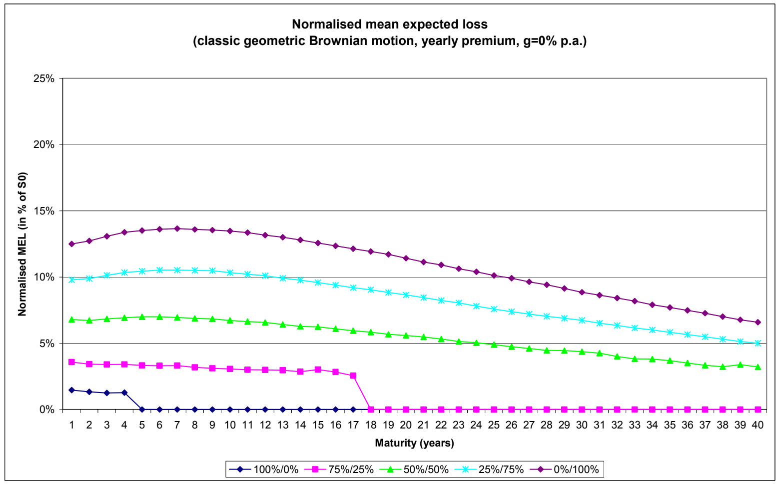
Graph 5.7 shows the normalised MEL for the high-level guarantee ($g = 4\%$ p.a.). The pure bond investment has a normalised MEL of 3.96% for the one-year maturity and it rises to 5.84% for the maturity of 40 years. The contract with the pure stock investment strategy has a normalised MEL between 12.93% (one-year maturity) and 23.88% (40 years maturity). This difference shows again, that it is very risky to guarantee the return on the gross premiums of 4% p.a., especially with high investment in stocks.

Graphs 5.6 and 5.8 show the normalised MEL for the high- and low-level guarantee, respectively, from the classic geometric Brownian motion simulation. Note that for contracts with the low stock investment, the risk quantified with this measure is lower - and for contracts with high stock participation higher - than in the Markov switching approach. It is conform to the observation in section 4.3.2 that if the bond ratio in the investment portfolio is high, the guarantee cost is higher for the Markov switching approach. If the equity proportion in the investment portfolio is high, the cost is higher under the classic geometric Brownian motion.

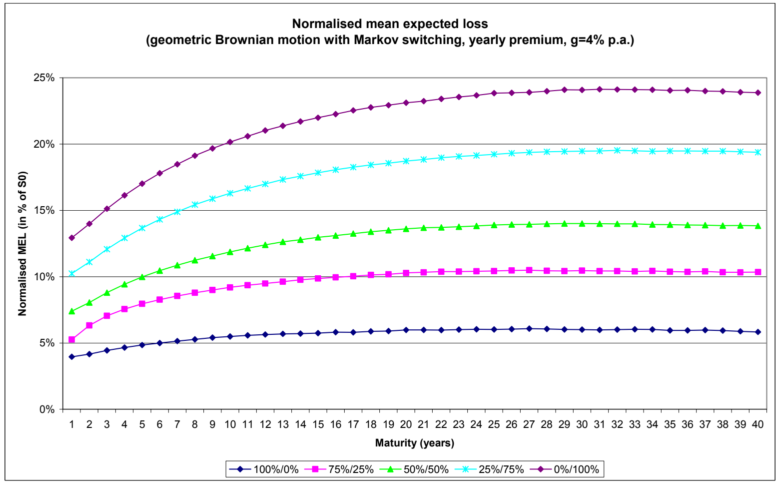
Graph 5.5 Normalised MEL (*g* = 0% p.a., geometric Brownian motion with Markov switching)



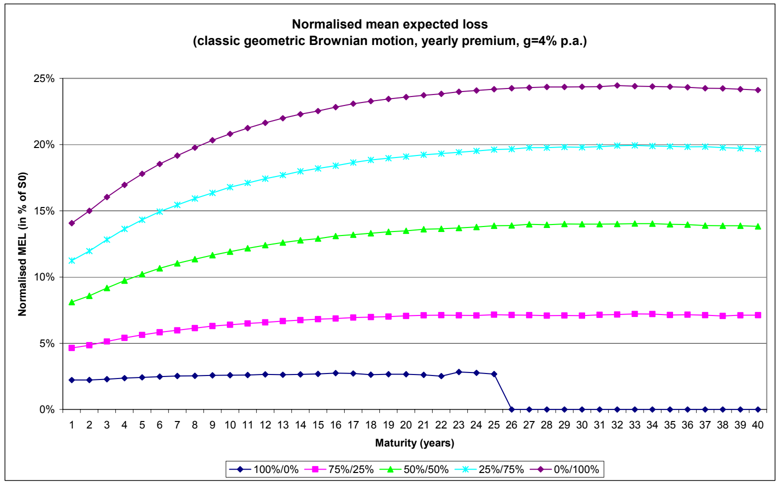
Graph 5.6 Normalised MEL (*g*=0% p.a., classic geometric Brownian motion)



Graph 5.7 Normalised MEL (*g* = 4% p.a., geometric Brownian motion with Markov switching)



Graph 5.8 Normalised MEL (*g* = 4% p.a., classic geometric Brownian motion)



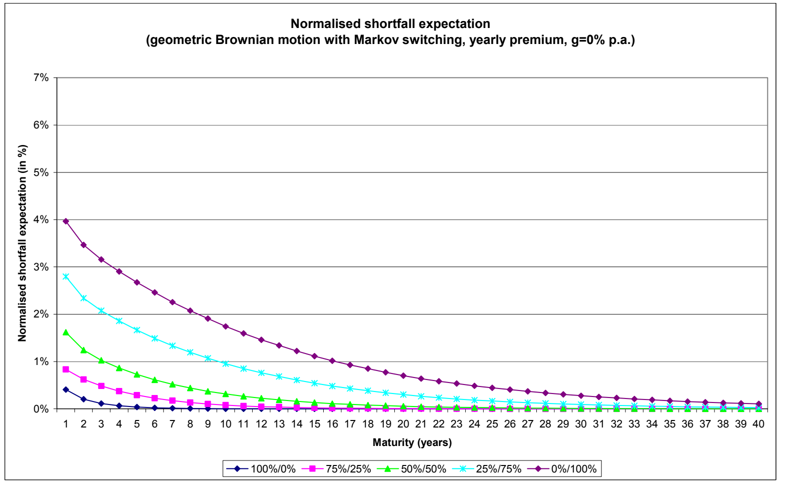
5.2.4. Shortfall Expectation

The mean expected loss should be interpreted in comprehension to the shortfall probability. This is easier with the knowledge of the shortfall expectation which is the product of both risk measures (cf. equation (5.4)). Please consider the following example of the pure equity investment with the low-level guarantee ($g = 0\%$). The normalised MEL equals 11.59% and 14.01% for the one- and nine-year maturity, respectively. The shortfall probability for the same maturity times equals 33.19% and 13.62%, respectively. It is difficult to say, if the shortfall risk decreases or increases as the maturity grows. The product of the both risk measures (also the shortfall expectation) gives the value of 3.9% and 1.9% for the one- and nine-year maturity. This shows, that the shortfall risk falls as the maturity increases.

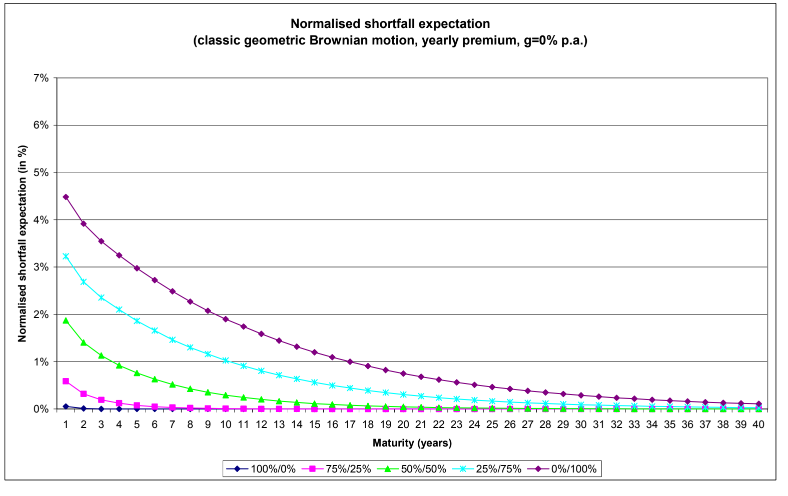
Graph 5.9 demonstrates the normalised shortfall expectation for the low-level guarantee ($g = 0\%$ p.a.). The normalised shortfall expectation is nearly zero for the pure bond investment, independent of the contract duration. This risk measure is higher for the pure equity investment case and equals 3.9% for the one year contract and decreases with time to 1.74%, 0.70%, 0.28% and 0.11% for 10, 20, 30, and 40 years maturity, respectively.

For the high-level guarantee ($g = 4\%$ p.a.), the risk measured with the shortfall expectation is higher and does not fall with the increase of the maturity as rapidly as in the low-level guarantee case (cf. graph 5.11). E.g. the pure bond strategy has the normalised shortfall expectation of 1.28% for the one-year contract and after 20 years is nearly zero (<0.2%). The fifty-fifty bond/equity investment strategy does not reach zero even after 40 years. The normalised shortfall expectations are equal to 2.84% and 0.75%, for a duration of one year and for 40 years respectively. For the contract with the pure equity investment strategy, these values equal 5.40% and 3.30%, respectively. Please be aware that for investment with the equity ratio over 50% and short maturities the normalised shortfall expectation does not decrease with time. It increases until it reaches a maximum and only then it begins to fall. Nevertheless, the maxima are only slightly higher than the values for the one-year contract. E.g. the maximum of the pure equity investment is reached at 5.78% after seven years which is only 0.28 per cent point higher than the normalised shortfall expectation for the maturity of one year.

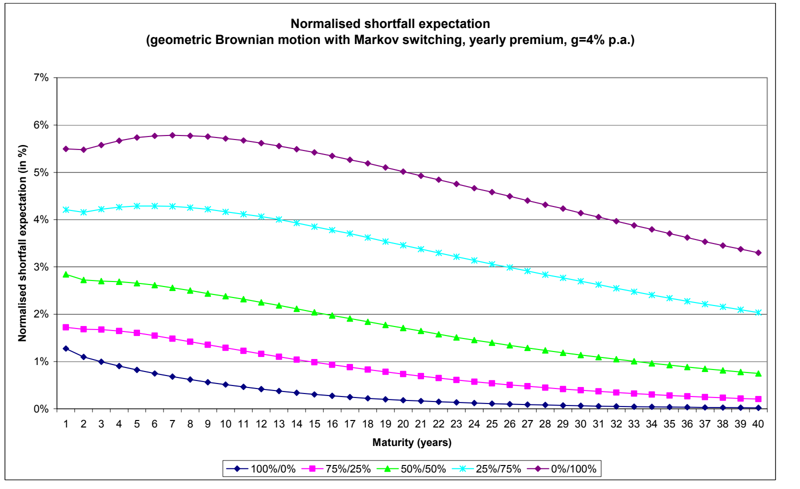
Graph 5.9 Normalised shortfall expectation (*g* = 0% p.a., geometric Brownian motion with Markov switching)



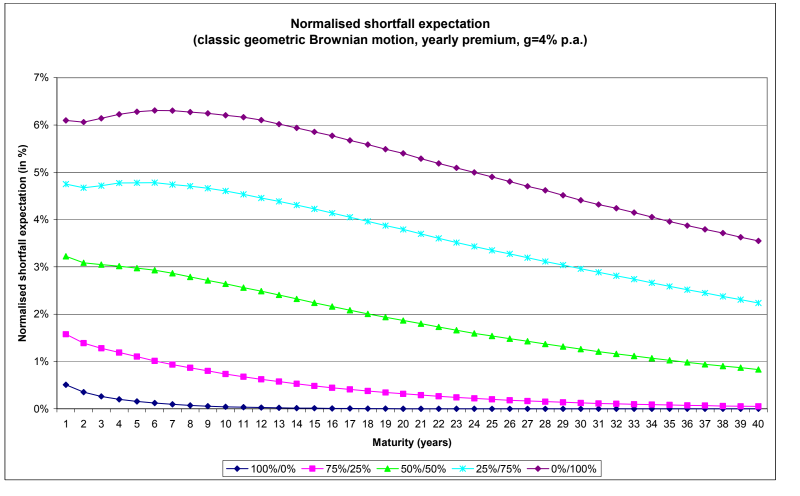
Graph 5.10 Normalised shortfall expectation (*g* = 0% p.a., classic geometric Brownian motion)



Graph 5.11 Normalised shortfall expectation (*g* = 4% p.a., geometric Brownian motion with Markov switching)



Graph 5.12 Normalised shortfall expectation (*g* = 4% p.a., classic geometric Brownian motion)



Graphs5.10 and 5.12 enable the comparison of the values discussed above with those simulated under the geometric Brownian motion without Markov switching. Generally speaking, the risk measured with the shortfall expectation when the distribution parameter are constant, is higher than in the case of stochastic volatility. This effect is highest for the high-level guarantee with an equity investment ratio over 50%.

6. Conclusion and Outlook

This study looks at maturity guarantees with deterministic rates of return embedded in the modern retirement saving plans as offered, among other countries, in Germany. The first part of the paper (section 2 and 3) presents the theoretic foundation of the study. First, the geometric Brownian motion with Markov switching is presented. As statistical tests have shown, this approach models the behaviour of the asset prices in a better way than the standard geometric Brownian motion approach (cf. PIASKOWSKI 2005, HARDY 2001a). It is for this reason that it was implemented in this work. At the end of the first part the WEBB's (2003) option pricing theory for Markov switching process is introduced.

The second part (section 4) applies Webb's option pricing method to determining the cost of the guarantee. At the same time it discusses how the guarantee cost depends on (a) the contract duration, on (b) the investment strategy, and on (c) the guarantee rate. Generally speaking, the cost of the guarantee rises as, *ceteris paribus*, the guarantee rate increases. The same relationship is true for the stock proportion in the investment portfolio. The impact of the maturity time is not so straightforward. For low guarantee rates, the cost decreases as, *ceteris paribus*, the time horizon grows. The opposite is true for contracts with high guarantee rates. The comparison with the result of the guarantee pricing under the classic geometric Brownian motion has shown that the relationships between the guarantee cost and its determinants are the same under both stochastic processes. The Markov switching approach leads to lower cost of the guarantee for almost all investment strategies (with the exception of those with a very low equity ratio). This is correct for both the low- and the high-level guarantees.

The third part (section 5) analyses the performance risk associated with providing the investment guarantees. To quantify this risk three shortfall risk measures were applied: the shortfall probability, the shortfall expectation, and the mean expected loss. The shortfall probability is high for the short-time horizon. It converges toward zero, but the convergence speed is dependent on the guarantee rate. The lower, *ceteris paribus*, the guarantee rate is, the faster is the convergence rate. For some guarantees with (a) a low guarantee rate, and (b) a long time horizon, and (c) a low equity investment rate, the shortfall probability is equal to zero.

The mean expected loss grows, *ceteris paribus*, with the increase of the guarantee rate. The same relationship holds true for the equity ratio in the investment portfolio. The dependence on the contract duration is not straightforward. The MEL increases up to a certain maximum and then decreases. In the case of the low-level guarantees, the maximum is reached after a few years and a cost decrease is noticeable. For the high-level guarantees, this effect is observable after 35 years, if at all, and is very weak.

The joint impact of the shortfall probability and the MEL on the shortfall risk is quantified with the shortfall expectation. Generally it decreases as, *ceteris paribus*, (a) the guarantee rate falls, or as (b) the equity ratio in the investment portfolio decreases, or as (c) the contract duration becomes longer. The analysis of the shortfall expectation shows as well, that if one of the two values, the shortfall probability and the MEL, grows and the other value declines, the probability of a shortfall has the stronger impact on the risk. The comparison with the values of the shortfall risk measures with the values computed under the classic geometric Brownian motion shows, that in both cases the above stated relationships are the same. Nevertheless, the shortfall risk under the classic method is higher than under the model with Markov switching, which is true for all three risk measures. The regular exceptions are contracts with low equity ratio in the investment portfolio and short maturity times. Then the opposite is the case.

This study shows that maturity guarantees should not be offered at no charge to the clients as it was the custom of insurance companies in the past. The reason is that the guarantees have an economic value which can be considerable and that they can be highly risky. Only contracts that at the same time have (a) a low rate of guaranteed return, and (b) a long maturity, and (c) a low equity proportion in the investment portfolio may be considered as "free of charge" and as "not risky".

It is possible to find an investment strategy that makes it possible to finance the cost of the guarantee. The financing should be carried out by a reliable charging and provision policy of the guarantor. The above discussed pricing method and risk measures are good tools to determine this policy. Please note that this paper only takes into account strategies with a constant proportion of equity in the investment portfolio. Therefore, it may be assumed that adequate active investment policies will decrease the economic cost of the guarantee as well as the associated performance risks. MAURER and SCHLAG (2003) showed that it is true for the investment risk, so it should hold true for the guarantee pricing as well.

Issues, such as periodical guarantees, remain open for further research. These guarantees are embedded in a large number of financial products, e.g. life insurance policies in Germany. Guarantees with a stochastic guarantee rate pose a very interesting topic for future study also, as they are provided in many public retirement systems (esp. in Latin America, in Central and Eastern Europe). A further interesting issue will be the incorporation of the stochastic risk-free rate, which would bring the results closer to the "true" guarantee cost. The issue is of particular importance for the long-term guarantees, which are embedded in financial retirement products, because in the real world the risk-free rate of return is, indeed, stochastic. As the sensitivity analysis (which I am delighted to send to you upon request) has shown, the change of the risk-free return has a significant impact on the cost and risk associated with the investment guarantees.

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