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BEREZIN-TOEPLITZ QUANTIZATION AND BEREZIN SYMBOLS FOR ARBITRARY COMPACT KÄHLER MANIFOLDS

Martin Schlichenmaier

Department of Mathematics and Computer Science
University of Mannheim, D7, 27
D-68131 Mannheim, Germany
E-mail:schlichenmaier@math.uni-mannheim.de

Abstract

For phase-space manifolds which are compact Kähler manifolds relations between the Berezin-Toeplitz quantization and the quantization using Berezin's coherent states and symbols are studied. First, results on the Berezin-Toeplitz quantization of arbitrary quantizable compact Kähler manifolds due to Bordemann, Meinrenken and Schlichenmaier are recalled. It is shown that the covariant symbol map is adjoint to the Toeplitz map. The Berezin transform for compact Kähler manifolds is discussed.

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1. INTRODUCTION

For phase-space manifolds which are complex Kähler manifolds different quantization schemes of geometric origin (i.e. related to the complex resp. the Kähler structure) have been considered. Some of them are connected with the name of Berezin. Mainly phase-space manifolds which are either domains in $\mathbb{C}^n$ or certain homogeneous spaces were studied. Only recently results on the quantization of arbitrary compact Kähler manifolds were obtained. An incomplete list of names related to such results contains J. Rawnsley, S. Gutt, M. Cahen [15],[7],[8], M. Bordemann, E. Meinrenken, M. Schlichenmaier [4], S. Klimek, A. Lesniewski [13], D. Borthwick, A. Uribe [5]. For the quantization of additional structures (i.e. the category of vector bundles over a compact Kähler manifold) see also E. Hawkins [12].

Here I like to relate for the compact Kähler case the Berezin-Toeplitz quantization (sometimes just called Toeplitz quantization) with the quantization using Berezin's coherent states, resp. Berezin's symbols.

Firstly, I will recall results on the Berezin-Toeplitz quantization which were obtained in joint work with M. Bordemann and E. Meinrenken [4]. I presented them in more detail already at the Białowieża 1995 workshop [17]. Then I will recall Berezin's coherent states and symbols [2] in their reformulation and generalization due to Rawnsley [15],[7]. I will show for compact Kähler manifolds that the Toeplitz operator map and the covariant symbol map of Berezin are adjoint if one takes the Hilbert-Schmidt norm for the operators and the, by the epsilon function of Rawnsley, deformed Lebesgue measure for the functions. I will close with introducing the Berezin transform for compact Kähler manifolds and discussing certain asymptotic properties of it.

In this way I extend results known for the bounded symmetric domains in $\mathbb{C}^n$ to arbitrary compact Kähler manifolds. The study of the Berezin transform for such domains in $\mathbb{C}^n$ (hence in some sense at the opposite edge of the set of manifolds) goes back to F. Berezin [2] and later A. Unterberger and H. Upmeier [19], M. Engliš [9],[10] and J. Peetre [11].

2. BEREZIN- TOEPLITZ QUANTIZATION

Let (M,ω) be a quantizable Kähler manifold. The complex manifold M is the phase-space and the Kähler form ω (a nondegenerate closed 2-form of type $(1,1)$) is taken as symplectic form. Let (L,h,∇) be an associated quantum line bundle. Here L is a holomorphic line bundle, h a hermitian metric (conjugate linear in the first argument), and ∇ the unique connection in L which is compatible with the complex structure and the metric. With respect to a holomorphic frame of the bundle the metric h can be given by a local function $\hat{h}$ and then the connection will be fixed as

$$\nabla_{|} = \partial + (\partial \log \hat{h}) + \overline{\partial}\ . \tag{2.1}$$

The quantization condition is the requirement that the curvature form F of the line bundle coincides with the Kähler form of the manifold up to a factor $(-\mathrm{i}\ )$, i.e.

$$F(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} = -\mathrm{i}\ \omega(X,Y), \tag{2.2}$$

for arbitrary vector fields X and Y. Again, with respect to a local holomorphic frame this reads as

$$\omega = \mathrm{i}\ \overline{\partial}\partial \log \hat{h}\ . \tag{2.3}$$

In the following we will restrict our situation to the compact quantizable Kähler case. But note that after some necessary modifications a lot of the constructions can be extended to the non-compact case as well. See [17] for more details and for examples.

The first important **observation** is that by the quantization condition the line bundle L is a positive line bundle, resp. in the language of algebraic geometry an ample line bundle. This says that a positive tensor power $L^{\otimes m_0}$ of L is very ample, i.e. M can be embedded as algebraic submanifold into projective space $\mathbb{P}^N(\mathbb{C})$ with the help of a basis $\{s_0, s_1, \ldots, s_N\}$ of the global holomorphic sections of $L^{\otimes m_0}$. In particular, every quantizable compact Kähler manifold is a projective algebraic manifold. Vice versa, every projective manifold is a quantizable Kähler manifold with Kähler form given by the restriction of the Kähler form of $\mathbb{P}^N(\mathbb{C})$, (the Fubini-Study form) and quantum line bundle given by the restriction of the hyperplane bundle. Again, see [17] for details.

Let us assume for the following that L is already very ample. If not, then we replace L by $L^{\otimes m_0}$ and the Kähler form ω by $m_0\,\omega$ without changing the complex manifold structure on M. Take $\Omega = \frac{1}{n!}\omega^{\wedge n}$ ($n = \dim_{\mathbb{C}} M$) as volume form on M. Let $\Gamma_\infty(M, L)$ be the space of differentiable global sections of L with scalar product and norm

$$\langle \varphi, \psi \rangle := \int_M h(\varphi, \psi)\,\Omega\,, \qquad ||\varphi|| := \sqrt{\langle \varphi, \varphi \rangle}\,. \tag{2.4}$$

Denote by $\mathrm{L}^2(M, L)$ the L^2-completion of $\Gamma_\infty(M, L)$ and by $\Gamma_{hol}(M, L)$ the finite-dimensional closed subspace of global holomorphic sections. Let

$$\Pi : \mathrm{L}^2(M, L) \to \Gamma_{hol}(M, L), \tag{2.5}$$

be the projection on this subspace. As usual let $C^\infty(M)$ be the algebra of complex-valued C^∞-functions on M. Recall that $(C^\infty(M), \cdot, \{.,.\})$ is via the symplectic form ω a Poisson algebra. Its Lie structure $\{.,.\}$ is defined as

$$\{f, g\} := \omega(X_f, X_g), \quad \text{with} \quad \omega(X_f, .) = df(.)\,. \tag{2.6}$$

Definition 2.1. (a) For $f \in C^\infty(M)$ the *Toeplitz operator* T_f is defined as

$$T_f := \Pi \circ (f \cdot) : \quad \Gamma_{hol}(M, L) \to \Gamma_{hol}(M, L), \qquad s \mapsto \Pi(f \cdot s)\,. \tag{2.7}$$

(b) The map

$$T : C^\infty(M) \to \mathrm{End}(\Gamma_{hol}(M, L)), \qquad f \mapsto T_f, \tag{2.8}$$

is called the *Berezin-Toeplitz quantization map.*

In words: The Toeplitz operator T_f associated to the differentiable function f multiplies the holomorphic section with this function and projects the obtained differentiable section back to a holomorphic one.

Clearly the map T is linear but it is neither a homomorphism of associative algebras (in general $T_f \cdot T_g \neq T_{fg}$), nor a Lie algebra homomorphism (in general $T_{\{f,g\}} \neq [T_f, T_g]$). By the quantization process a lot of classical informations get lost. The algebra $\mathrm{End}(\Gamma_{hol}(M, L))$ is finite-dimensional in contrast to the infinite-dimensional algebra $C^\infty(M)$. The right thing to do is to consider instead of only L all its tensor powers $L^m := L^{\otimes m}$ for $m \in \mathbb{N}$. With respect to the induced metric $h^{(m)}$ on L^m we obtain now the corresponding scalar products on the space of global sections of L^m, the projection operator

$$\Pi^{(m)} : \Gamma_\infty(M, L^m) \to \Gamma_{hol}(M, L^m), \tag{2.9}$$

the Toeplitz operators

$$T_f^{(m)} : \Gamma_{hol}(M, L^m) \to \Gamma_{hol}(M, L^m), \tag{2.10}$$

and the Berezin-Toeplitz quantization map

$$T^{(m)} : C^\infty(M) \to \mathrm{End}(\Gamma_{hol}(M, L^{(m)})) \ , \tag{2.11}$$

for every $m \in \mathbb{N}$. These maps have the correct semi-classical behavior for $m \to \infty$ (resp. for $\hbar = \frac{1}{m} \to 0$) as shown by Bordemann, Meinrenken and Schlichenmaier [4]:

Theorem 2.1. *For $f, g \in C^\infty(M)$ we have*

$$\text{(a)} \qquad ||T_f^{(m)}|| \to ||f||_\infty, \qquad m \to \infty, \tag{2.12}$$

$$\text{(b)} \qquad ||m \,\mathrm{i}\, [T_f^{(m)}, T_g^{(m)}] - T_{\{f,g\}}^{(m)}|| \to 0 \ , \qquad m \to \infty, \tag{2.13}$$

$$\text{(c)} \qquad ||T_f^{(m)} T_g^{(m)} - T_{f \cdot g}^{(m)}|| \to 0 \ , \qquad m \to \infty, \tag{2.14}$$

where for the operators the operator norm and for the functions the sup-norm have been chosen.

For more detailed results on the asymptotics, see [4],[17]. The proofs employ the theory of generalized Toeplitz operators due to Boutet de Monvel and Guillemin [6]. It is possible to construct a star product (i.e. a deformation quantization) with these techniques [17],[18].

Using similar techniques Borthwick and Uribe [5] were able to prove the same kind of semi-classical behavior in the setting of compact symplectic manifolds with almost-complex structure.

Theorem 2.1 together with Proposition 3.2 implies that this quantization is a strict quantization. See for example the recent book by N.P. Landsman [14] for its definition.

3. BEREZIN COHERENT STATES AND SYMBOLS

We use the definition of Berezin's coherent states in its coordinate independent version and extension due to Rawnsley [15],[7], see also [3],[18]. Let the situation be as above. In particular, we assume that L is already very ample. Later we will consider again any m^{th} power of L. Let $\pi : L \to M$ be the bundle projection and L_0 the total space of L with the zero section $0(M)$ removed. For $q \in L_0$ fixed and $s \in \Gamma_{hol}(M, L)$ arbitrary we obtain via

$$s(\pi(q)) = \hat{q}(s) \cdot q \tag{3.1}$$

a linear form

$$\hat{q} \ : \ \Gamma_{hol}(M, L) \ \to \ \mathbb{C} \ , \qquad s \mapsto \hat{q}(s) \ . \tag{3.2}$$

Hence, with the scalar product there exists exactly one holomorphic section e_q with

$$\langle e_q, s\rangle = \hat{q}(s), \qquad \text{for all} \quad s \in \Gamma_{hol}(M, L)\ . \tag{3.3}$$

One calculates

$$e_{cq} = \bar{c}^{-1} \cdot e_q, \qquad c \in \mathbb{C}^*\ . \tag{3.4}$$

The element e_q is called *coherent vector (associated to q)*. Note that $e_q \equiv 0$ would imply $\hat{q} = 0$. This yields via (3.1) that all sections will vanish at the point $x = \pi(q)$. But this is a contradiction to the very-ampleness of L. Hence, $e_q \not\equiv 0$ and due to (3.4) the element $[e_q] := \{s \in \Gamma_{hol}(M, L) \mid \exists c \in \mathbb{C}^* : s = c \cdot e_q\}$ with $\pi(q) = x$ is a well-defined element of the projective space $\mathbb{P}(\Gamma_{hol}(M, L))$ only depending on $x \in M$. It is called the *coherent state (associated to $x \in M$)*.

The *coherent state embedding* is the antiholomorphic embedding

$$M \quad \to \quad \mathbb{P}(\Gamma_{hol}(M, L)) \;\cong\; \mathbb{P}^N(\mathbb{C}), \qquad x \mapsto [e_{\pi^{-1}(x)}]. \tag{3.5}$$

In abuse of notation we will understand in this context under $\pi^{-1}(x)$ always a non-zero element of the fiber over x. The coherent state embedding is up to conjugation the Kodaira embedding with respect to an orthonormal basis of the sections. See [1] for further considerations of the geometry involved.

We need also the *coherent projectors* used by Rawnsley

$$P_{\pi(q)} = \frac{|e_q\rangle\langle e_q|}{\langle e_q, e_q\rangle}\ . \tag{3.6}$$

Here we used the convenient bra-ket notation. From (3.4) it follows that the projector is well-defined on M. Let us relate the projectors to the metric in the bundle with the help of Rawnsley's epsilon function as defined in [15]

$$\epsilon(\pi(q)) := |q|^2 \langle e_q, e_q\rangle, \quad \text{with} \quad |q|^2 := h(\pi(q))(q, q). \tag{3.7}$$

Take two sections s_1 and s_2. At a fixed point $x = \pi(q)$ we can write $s_1(x) = \hat{q}(s_1)q$ and $s_2(x) = \hat{q}(s_2)q$ and hence for the metric (using (3.3))

$$h(s_1, s_2)(x) = \overline{\hat{q}(s_1)} \cdot \hat{q}(s_2) \cdot |q|^2 = \langle s_1, e_q\rangle\langle e_q, s_2\rangle |q|^2 = \langle s_1, P_x s_2\rangle \cdot \epsilon(x)\ . \tag{3.8}$$

The *covariant Berezin symbol* $\sigma(A)$ of an operator $A \in \mathrm{End}(\Gamma_{hol}(M, L))$ is defined as

$$\sigma(A) : M \to \mathbb{C}, \qquad x \mapsto \sigma(A)(x) := Tr(AP_x) = \frac{\langle e_q, Ae_q\rangle}{\langle e_q, e_q\rangle}, \quad q \in \pi^{-1}(x), q \neq 0\ . \tag{3.9}$$

The symbol $\sigma(A)$ is real-analytic and obeys

$$\sigma(A^*) = \overline{\sigma(A)}\ . \tag{3.10}$$

A closer inspection shows that the linear symbol map

$$\sigma : \mathrm{End}(\Gamma_{hol}(M, L)) \to C^\infty(M) \tag{3.11}$$

is injective, see [7], or [4, Prop. 4.1].

Let us now introduce the modified measure (note that $\epsilon(x) > 0$, for all $x \in M$)

$$\Omega_\epsilon := \epsilon(x)\Omega(x) \tag{3.12}$$

on M. The corresponding scalar product on $C^\infty(M)$ is denoted by $\langle .,. \rangle_\epsilon$. The *contravariant Berezin symbol* $\check{\sigma}(A) \in C^\infty(M)$ of an operator is defined by the representation of the operator A as integral

$$A = \int_M \check{\sigma}(A)(x) P_x \, \Omega_\epsilon(x), \tag{3.13}$$

if such a representation exists.

Proposition 3.1. *The Toeplitz operator T_f admits a representation (3.13) with*

$$\check{\sigma}(T_f) = f \ , \tag{3.14}$$

i.e. the function f is the contravariant symbol of the Toeplitz operator T_f.

Proof. Set

$$A := \int_M f(x) P_x \, \Omega_\epsilon(x) \tag{3.15}$$

then $\check{\sigma}(A) = f$. For arbitrary $s_1, s_2 \in \Gamma_{hol}(M, L)$ we calculate (using (3.8))

$$\begin{aligned} \langle s_1, A s_2 \rangle &= \int_M f(x) \langle s_1, P_x s_2 \rangle \Omega_\epsilon(x) = \int_M f(x) h(s_1, s_2)(x) \Omega(x) \\ &= \int_M h(s_1, f s_2)(x) \Omega(x) = \langle s_1, f s_2 \rangle = \langle s_1, T_f s_2 \rangle \ . \end{aligned} \tag{3.16}$$

Hence $T_f = A$. □

We introduce on $\mathrm{End}(\Gamma_{hol}(M, L))$ the Hilbert-Schmidt norm

$$\langle A, C \rangle_{HS} = Tr(A^* \cdot C) \ . \tag{3.17}$$

Theorem 3.1. *The Toeplitz map $f \to T_f$ and the covariant symbol map $A \to \sigma(A)$ are adjoint:*

$$\langle A, T_f \rangle_{HS} = \langle \sigma(A), f \rangle_\epsilon \ . \tag{3.18}$$

Proof.

$$\langle A, T_f \rangle = Tr(A^* \cdot T_f) = Tr(A^* \int_M f(x) P_x \, \Omega_\epsilon(x)) = \int_M f(x) Tr(A^* \cdot P_x) \Omega_\epsilon(x). \quad (3.19)$$

Now applying the definition (3.9) and equation (3.10)

$$\langle A, T_f \rangle = \int_M f(x) \sigma(A^*) \Omega_\epsilon(x) = \int_M \overline{\sigma(A)}(x) f(x) \Omega_\epsilon(x) = \langle \sigma(A), f(x) \rangle_\epsilon \, . \quad (3.20)$$

□

The same is valid for every operator C which admits a contravariant symbol $\check{\sigma}(C)$

$$\langle A, C \rangle_{HS} = \langle \sigma(A), \check{\sigma}(C) \rangle_\epsilon \, . \quad (3.21)$$

In the compact Kähler case this will not be an additional result due to the following proposition.

Proposition 3.2. *The Toeplitz map $f \to T_f$ is surjective. In particular, every operator in* $\mathrm{End}(\Gamma_{hol}(M, L))$ *has a contravariant symbol.*

Proof. Choose A an operator orthogonal to $\mathrm{Im}\, T$, i.e. $\langle A, T_f \rangle = 0$ for all $f \in C^\infty(M)$. Hence, Theorem 3.1 implies $\langle \sigma(A), f \rangle_\epsilon = 0$ for all $f \in C^\infty(M)$, i.e. $\sigma(A) = 0$. By the injectivity of the symbol map this implies $A = 0$. Hence the T_f span the whole $\mathrm{End}(\Gamma_{hol}(M, L))$. □

Now the question arises when the measure Ω_ϵ will be the standard measure (up to a scalar). For $M = \mathbb{P}^N(\mathbb{C})$ from the homogeneity of the bundle, and all the other data it follows $\epsilon \equiv const$. By a result of Rawnsley [15], resp. Cahen, Gutt and Rawnsley [7], $\epsilon \equiv const$ if and only if the quantization is projectively induced. This means that using the conjugate of the coherent state embedding, the Kähler form ω of M coincides with the pull-back of the Fubini-Study form. Note that not every quantization is projectively induced, see the discussion in [17].

Appendix. To compare the global description with Berezin's original approach we have to choose a section $s_0 \in \Gamma_{hol}(M, L)$, $s_0 \not\equiv 0$. On the open set $V := \{x \in M \mid s_0(x) \neq 0\}$. the section s_0 is a holomorphic frame for the bundle L. Hence, every holomorphic (resp. differentiable) section s can be described as $s(x) = \hat{s}(x) s_0(x)$ with a holomorphic (resp. differentiable) function on V. The map $s \mapsto \hat{s}$ gives an isometry of $\Gamma_{hol}(M, L)$, resp. $\Gamma_\infty(M, L)$ with the L^2 space of holomorphic, resp. differentiable functions on V with respect to the measure

$$\mu_{s_0}(x) = h(s_0, s_0) \Omega(x) \, . \quad (3.22)$$

The scalar product can be given as

$$\langle s, t \rangle = \int_V \overline{\hat{s}} \cdot \hat{t} \cdot h(s_0, s_0) \Omega(x) = \int_V \overline{\hat{s}} \cdot \hat{t} \cdot \exp(-K(x)) \Omega(x) \, . \quad (3.23)$$

Here K is a local Kähler potential with respect to the frame s_0. It is given by

$$K(x) = -\log h(s_0, s_0)(x) \, . \quad (3.24)$$

Note that by the quantization condition (2.3) we have indeed $\omega = \mathrm{i}\, \partial\overline{\partial} K$.

4. THE BEREZIN TRANSFORM

Starting from $f \in C^\infty(M)$ we can assign to f its Toeplitz operator $T_f \in \mathrm{End}(\Gamma_{hol}(M,L))$ and then assign to T_f the covariant symbol $\sigma(T_f)$ which is again an element of $C^\infty(M)$. Altogether we obtain a map $f \mapsto B(f) = \sigma(T_f)$. This map is called *Berezin transform.*

Recall that f is the contravariant symbol of the Toeplitz operator T_f. Hence B gives a correspondence between contravariant symbols and covariant symbols of operators. The Berezin transform was introduced and studied by Berezin [2] for certain classical symmetric domains in $\mathbb{C}^n$. These results where extended by Unterberger and Upmeier [19], see also Engliš [9],[10] and Engliš and Peetre [11]. As seen above the Berezin transform makes sense also in the compact Kähler case which we consider here. Let me study it in some more detail.

For s, t holomorphic sections of L and $f \in C^\infty(M)$ we have $\langle t, T_f s\rangle = \langle t, fs\rangle$. Hence,

$$B(f)(x) = \sigma(T_f)(x) = \frac{\langle e_q, f e_q\rangle}{\langle e_q, e_q\rangle}, \quad x = \pi(q)\ , q \neq 0\ . \tag{4.1}$$

It is possible to find another useful description. From Proposition 3.1 follows

$$T_f = \int_M f(y) \frac{|e_{q'}\rangle\langle e_{q'}|}{\langle e_{q'}, e_{q'}\rangle}\,\Omega_\epsilon(y), \quad y = \pi(q')\ , q' \neq 0\ , \tag{4.2}$$

and we obtain with $q \in \pi^{-1}(x), q \neq 0$

$$B(f)(x) = \frac{\langle e_q, T_f e_q\rangle}{\langle e_q, e_q\rangle} = \int_M f(y) \frac{\langle e_{q'}, e_q\rangle}{\langle e_{q'}, e_{q'}\rangle} \frac{\langle e_q, e_{q'}\rangle}{\langle e_q, e_q\rangle}\,\Omega_\epsilon(y)\ . \tag{4.3}$$

We set

$$K(x,y) := \frac{|\langle e_{q'}, e_q\rangle|^2}{\langle e_{q'}, e_{q'}\rangle\langle e_q, e_q\rangle}\ . \tag{4.4}$$

Clearly, $0 \le K(x,y) \le 1$ for every $x, y \in M$. Note that K is one of the two-point functions introduced in [8]. Now the Berezin transform can be written with the help of an integral kernel as

$$B(f)(x) = \int_M f(y) K(x,y) \Omega_\epsilon(y)\ . \tag{4.5}$$

Everything can be done for any positive power m of the line bundle L.

Definition 4.1. Let $m \in \mathbb{N}$. The map

$$B : C^\infty(M) \to C^\infty(M), \qquad f \mapsto B^{(m)}(f) = \sigma^{(m)}(T_f^{(m)}) \tag{4.6}$$

is called the *Berezin transform of level m.*

Note that (nearly) everything depends on m, the scalar product, the coherent states $e_q^{(m)}$, the symbol maps, the epsilon function, the integral kernel $K^{(m)}(x,y)$, etc. The asymptotic behavior of $B^{(m)}$, resp. of $B^{(m)}(f)$ as $m \to \infty$ contains interesting information. In the classical bounded symmetric domain case in $\mathbb{C}^n$ considered by Berezin [2] the asymptotics contains a lot of information about the quantization. There the first two terms in the asymptotics are corresponding to the fact, that the quantization has the correct semi-classical behavior. See also Engliš [10] for results on some more general domains in $\mathbb{C}^n$.

In the compact Kähler case we have

Theorem 4.1. *Let $f \in C^\infty(M)$ then*

$$|B^{(m)}(f)|_\infty = |\sigma^{(m)}(T_f^{(m)})|_\infty \quad \leq \quad ||T_f^{(m)}|| \quad \leq \quad |\check{\sigma}(T_f^{(m)})|_\infty = |f|_\infty \ . \tag{4.7}$$

Proof. The two out-most equalities are by definition. To simplify the notation let us drop the super-script (m) in the proof. Using the Cauchy-Schwarz inequality we calculate $(x = \pi(q))$

$$|\sigma(T_f)(x)|^2 = \frac{|\langle e_q, T_f e_q\rangle|^2}{\langle e_q, e_q\rangle^2} \leq \frac{\langle T_f e_q, T_f e_q\rangle}{\langle e_q, e_q\rangle} \leq ||T_f||^2 \ . \tag{4.8}$$

Here the last step follows from the definition of the operator norm

$$||T_f|| = \sup_{\substack{s \in \Gamma_{hol}(M,L) \\ s \neq 0}} \frac{||T_f s||}{||s||} \ . \tag{4.9}$$

This shows the first inequality in (4.7). For the second inequality introduce the multiplication operator M_f on $\Gamma_\infty(M,L)$. Then $||T_f|| = ||\Pi\, M_f\, \Pi|| \leq ||M_f||$ and for $\varphi \in \Gamma_\infty(M,L)$, $\varphi \neq 0$

$$\frac{||M_f\varphi||^2}{||\varphi||^2} = \frac{\int_M h(f\varphi, f\varphi)\Omega}{\int_M h(\varphi,\varphi)\Omega} = \frac{\int_M \overline{f(z)}f(z)h(\varphi,\varphi)\Omega}{\int_M h(\varphi,\varphi)\Omega} \leq |f|_\infty^2 \ . \tag{4.10}$$

Hence,

$$||T_f|| \leq ||M_f|| = \sup_{\varphi \neq 0} \frac{||M_f\varphi||}{||\varphi||} \leq |f|_\infty. \tag{4.11}$$

□

Corollary 4.1. *Let A be an operator of $\Gamma_{hol}(M, L^m)$ then*

$$|\sigma^{(m)}(A)|_\infty \quad \leq \quad ||A|| \quad \leq \quad |\check{\sigma}^{(m)}(A)|_\infty \ . \tag{4.12}$$

Proof. By Proposition 3.2 every operator is a Toeplitz operator. Hence we can apply Theorem 4.1. □

Warning: The statement of Proposition 3.2 should not be misinterpreted in the way that given a (natural) family of operators $A^{(m)}$ we will have $A^{(m)} = T_f^{(m)}$. It only states that for a fixed m there is a function $f^{(m)}$ such that $A^{(m)} = T_{f^{(m)}}^{(m)}$. An important example is given by the operator $Q^{(m)}$ of geometric quantization (with Kähler polarisation). By a result of Tuynman we have $Q^{(m)} = T^{(m)}_{\mathrm{i}\,(f-\frac{1}{2m}\Delta f)}$ [16], see also [3].

By the above results we see that $B^{(m)}$ is norm contracting. In particular $B^{(m)}(f)$ is bounded by $|f|_\infty$. If we consider the asymptotic expansion

$$B^{(m)}(f) = A_0(f) + A_1(f)\frac{1}{m} + A_2(f)\frac{1}{m^2} + \cdots , \tag{4.13}$$

then in the non-compact Kähler case of bounded symmetric domains [19] and for the planar domains of hyperbolic type in $\mathbb{C}$ [9] it is shown that $A_0 = id$ and that the A_i are polynomials in the invariant differential operators (generalized Laplacians), resp. in the Laplace-Beltrami operator only depending on the geometry of M. In particular note, that the fundamental regions for compact Riemann surfaces of genus $g \geq 2$ are planar domains of hyperbolic type. Similar results are expected also for the general compact Kähler case. Let me quote one result. For $\epsilon \equiv const$, i.e. the projectively induced quantization case, all $\epsilon^{(m)}$ will (individually) be constant. From (4.1) it follows with the help of the stationary phase theorem in a way similar to the proof of part (a) of Theorem 2.1 (see [4], [18, p.83], [17])

$$(B^{(m)}(f))(x) = f(x) + O(\frac{1}{m}) \ . \tag{4.14}$$

Clearly, from an asymptotics given by (4.14) part (a) of Theorem 2.1 follows with the help of Theorem 4.1. Hence, as expected the zero order part of the asymptotics of $B^{(m)}$ is related to the first condition of the correct semi-classical behavior. Details and further results have to be postponed to a forthcoming paper. Let me close with the remark that the asymptotic expansion (4.13) with $A_0 = id$ gives a relation between the star product obtained via Berezin-Toeplitz quantization and the star product obtained via Berezins's recipe [2],[8]. See also [8] for results partly related to the asymptotics.

Note added in proof: In the meantime for general compact Kähler manifolds the equation (4.14) has been proven in joint work with Alexander Karabegov [20].

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