Different Aspects of Nonlinear Stochastic Filtering Theory

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Dedication

This thesis is dedicate to my parents, my wife Aziza, and my daughter Sarah

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Abstract
This thesis studies different aspects of the linear and the nonlinear stochastic filtering problem. It consists of four chapters. In the first chapter we derive the Kalman and the extended Kalman filter algorithms and we study some of their qualitative properties. In the second chapter we present a unified general framework on particle filter methods. In particular, we show how the particle filter methods surmount the difficulties due to the Kalman approach to filtering and we compare different particle filter algorithms. In the third chapter we study a real life example, tracking the position and the speed of a car, then we compare the extended Kalman filter and the particle filter methods. Finally, in the fourth chapter we generalize the formulation of the filtering with the Zakai equation to the case of multidimensional systems with unbounded observation functions and an Ornstein-Uhlenbech type noise.

Zusammenfassung
Diese Dissertation studiert unterschiedliche Aspekte des linearen und nichtlinearen stochastischen filternproblems. Sie besteht aus vier Kapiteln. Im ersten Kapitel leiten wir den Kalman und den Extended Kalman Filteralgorithmen ab und wir studieren einige ihrer qualitativen Eigenschaften. Im zweiten Kapitel stellen wir einen vereinheitlichten allgemeinen Rahmen auf Partikelfiltermethoden dar. Insbesondere zeigen wir wie die Partikelfiltermethoden die Schwierigkeiten wegen Die Extended kalman Annäherung übersteigen und wir vergleichen unterschiedliche Partikelfilteralgorithmen. Im dritten Kapitel studieren wir ein reales Lebenbeispiel, schätzen die Position und die Geschwindigkeit eines Autos, dann vergleichen wir den Extended Kalman Filter und die Partikelfiltermethoden. Schließlich im vierten Kapitel generalisieren wir die Formulierung der Filternproblems mit der Zakai Gleichung zum Fall der mehrdimensionalen Systeme mit unbegrenzten Beobachtung Funktionen und einer Ornstein-Uhlenbech Art Geräusche.

keywords: Stochactic differential equation, Kalman filter, Monte Carlo methods, Zakai equation.

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i.i.d.	independent and identical distributed
a.s.	almost surely
a.e.	almost everywhere
pdf	probability density function
$\mathbb{R}$	The line of real numbers
$\mathbb{R}_+$	The set of non negative real numbers
$\mathbb{N}$	The set of non negative integer numbers
$\mathcal{M}_{n,p}(\mathbb{R})$	The set of $n \times p$-real matrices, $\mathcal{M}_{n,n}(\mathbb{R}) = \mathcal{M}_n(\mathbb{R})$
$GL_n(\mathbb{R})$	The set of invertible matrices of order n
$O_n(\mathbb{R})$	The orthogonal group of $\mathbb{R}^n$
$Ker(A)$, $\mathcal{R}(A)$	The kernel and the range of the matrix A
A^T	The matrix transpose of the matrix A
A^+	The matrix pseudo-inverse of the matrix A
A^{-1}	The matrix inverse of the matrix A
$\|x\|$	$= (x_1^2 + \cdots + x_n^2)^{1/2}$, for $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$
$\|A\|$	$= \sup\{\|Ax\|\,,\ x \in \mathbb{R}^p\,,\ , \|x\| = 1\}$, $A \in \mathcal{M}_{n,p}(\mathbb{R})$
$\mathbf{B}(\mathbb{R}^n)$	The Borel σ-algebra on $\mathbb{R}^n$
$\mathcal{B}(\mathbb{R}^n)$	The set of bounded Borel functions defined on $\mathbb{R}^n$
$\sigma(X_1, \ldots, X_n)$	The σ-algebra generated by the random variables $X_1, \ldots, X_n$
$\mathcal{A}^*$	The augmentation of the σ-algebra $\mathcal{A}$ by the null sets
$\mathcal{M}_F(\mathbb{R}^n)$	The set of the finite measures on $(\mathbb{R}^n, \mathbf{B}(\mathbb{R}^n))$
$\mathcal{P}(\mathbb{R}^n)$	The set of probability measures on $(\mathbb{R}^n, \mathbf{B}(\mathbb{R}^n))$
$(\mu, \varphi) \equiv \mu\varphi$	$= \int_{\mathbb{R}^n} \varphi(x) d\mu(x)$, for all $(\mu, \varphi) \in \mathcal{M}_F(\mathbb{R}^n) \times \mathcal{B}(\mathbb{R}^n)$
$\mathbf{I}_A$	The indicator function of the set A
$\mathcal{C}_b(I; J)$	The set of bounded, continuous functions defined on I with values in J
$\|\varphi\|$	$= \sup\{\|\varphi(x)\|\,,\ x \in I\}$, $\varphi \in \mathcal{C}_b(I; J)$
$\mathcal{C}^{n,k}(I \times J; K)$	The set of continuous functions $f(x, y)$ on $I \times J$ with values in K of $\mathcal{C}^n$ w.r.t. x and of $\mathcal{C}^k$ w.r.t. y
$\mathcal{C}_b^{n,k}(I \times J; K)$	$= \mathcal{C}^{n,k}(I \times J; K) \cap \mathcal{C}_b(I \times J; K)$
$L^p(X, \mathcal{A}, \lambda; Y)$	$= \{f : X \to Y, \mathcal{A}\text{-meas. and } \int_X \|f\|^p d\lambda < \infty\}$, $1 \leq p < \infty$
$L^\infty(X, \mathcal{A}, \lambda; Y)$	$= \{f : X \to Y, \mathcal{A}\text{-measurable and a.s. bounded}\}$.

Introduction

This thesis studies different aspects of the linear and the nonlinear stochastic filtering problem. It gives an introduction to the stochastic filtering theory together with new results, new proofs for classical results, and several examples and applications. The stochastic filtering problem arises in many different areas of science including tracking, automatic control, econometrics and others. It essentially deals with the estimation of the present state of a process $X = \{X_t\}_{t\in\mathbb{T}\subset\mathbb{R}_+}$, called the signal or the state process, based on measurements supplied by an associate process $Z = \{Z_t\}_{t\in\mathbb{T}}$, called the observation process.
In the Bayesian framework, a stochastic dynamical equation that provides the prior distribution of the state process and an observation equation that gives rise to the likelihood of the observations are available and all relevant information on the state X_t are included in the posterior distribution $P(X_t \in A\big|\{Z_s, s \in \mathbb{T}, s \leq t\})$. This is known as the Bayesian filtering problem called also the optimal filtering problem.
We will study the discrete time and the continuous time stochastic filtering problem ie. the cases $\mathbb{T} \subset \mathbb{N}$ and $\mathbb{T} = [a, b] \subset \mathbb{R}_+$, respectively.
The discrete filtering problem: Often the observations or measurements arrive sequentially in time, and one is interested in estimating recursively in time the evolving posterior distribution. In 1960, R.E. Kalman published his famous paper [43] on recursive minimum variance estimation in linear Gaussian dynamical systems. This paper introduces an algorithm known as the discrete Kalman filter. Since then, qualitative properties, including controllability, stability and others, of the Kalman filter were extensively studied [14, 39, 40, 45, 59].
However, in many realistic problems, dynamical systems involve elements of nonlinearity which exclude the use of the Kalman filter algorithm. One idea is to replace the original equations of the system by a family of linear Gaussian ones and to apply the Kalman algorithm, [40]. This method is known as the extended Kalman filter (EKF). The EKF algorithm has numerous drawbacks and its performance heavily depends on the system parameters behavior [8, 33, 52].
With the advance of computational power, sequential Monte Carlo methods, also called particle filter methods, have been developed to address the Bayesian filtering problem. Those methods are very flexible, often easy to implement. Moreover, these methods do not suppose linear, Gaussian or dimensional hypothesis on the models. Particle filter methods approximate the posterior distributions with empirical measures based on clouds of particles. The particles are sampled from appropriate distributions. The convergence is obtained when the number of sampled particles increases.

The continuous filtering problem: In the case of continuous time parameters, the filtering problem consists of estimating the statistics of a partially observed Markov process $X = \{X_t\}_{t\in[0,T]}$, called the signal process, based on the observation process $Z_t = \int_0^t h(X_s, s)ds + N_t$, $0 \leq t \leq T$, corrupted by a Gaussian noise process N.
The conditional expectation $\pi_t(\varphi) = E[\varphi(X_t, t)\big|\{Z_s, 0 \leq s \leq t\}]$, where φ is a bounded Borel function, gives the minimum variance estimation of the random variable $\varphi(X_t, t)$.
When the noise N is a Brownian motion, then under some regularity assumptions, see [7, 54, 57, 69, 77], the measure π_t satisfies a measure valued nonlinear stochastic partial differential equation (SPDE), called the Kushner-Stratonovitch equation. Moreover, the Kallianpur-Strieble formula states that

$$\pi_t(\varphi) = \frac{p_t(\varphi)}{p_t(1)} \ ,$$

where $p_t(\varphi)$ is the unnormalized conditional expectation. $p_t(\varphi)$ is characterized as the solution of a *linear* SPDE, called the Zakai equation.
Recently, interest has been developed when the noise is a Ornstein-Uhlenbeck or a general Gaussian process [9, 10, 56, 65].
Bhatt et al. [10] studied the case of Ornstein-Uhlenbeck noise. They considered 1-dimensional processes and bounded observation function h, they derived an analogous to the Zakai equation and they proved existence and uniqueness results.
The linearity of the Zakai equation makes its numerical approximations attractive. Crisan [16] and Crisan et al. [21] approximated the solution of the Zakai equation using branching particle methods. In [18], Crisan used a particle approximation method. A time dependent weight is attached to each particle. The particles move independently w.r.t. the law of the state process. He also gives a comparison with the branching particle methods.
Other methods using Galerkin approximation and Cameron-Martin version of the chaos decomposition can be found in [1, 38, 66].

Summary and main results of the thesis

Appendices We recall in a series of appendices various notions and tools that one will need. Specially, we present:

- The Euler-Maruyama discretization
- The Monte Carlo integration
- The Bayesian approach estimation
- The minimum variance estimation (MVE)
- The best affine minimum variance estimation (BAMVE)
- Errors and convergence criterions

Chapter 1 **First section:** We study linear systems that are not necessarily Gaussian. Instead of calculating the (nonlinear) minimum variance estimation (MVE), we derive a recursive algorithm for the best affine minimum variance estimation (BAMVE). When the system is Gaussian then BAMVE = MVE.
D.E. Catlin [14] gave similar results for the best linear minimum variance estimation (BLMVE) where its systems have no optional control inputs as ours.
We study then some qualitative properties of the Kalman filter. Firstly, we give lower and upper bounds on the error covariance matrix. Those bounds are based on stronger conditions than those of Jazwinski in [40] but much simpler to verify. Secondly, using the technical Lemma 1.1.7, we give in Proposition 1.1.6 a new and simple proof of the uniform asymptotic stability, see Definition 1.1.5, of the Kalman filter. We end this section by an illustrative example.
Second Section: We derive the extended Kalman filter (EKF) algorithm for nonlinear models with Gaussian noises. Our main result, Theorem 1.2.6, gives an upper bound on the error in the EKF algorithm for a class of nonlinear dynamical systems. The results are illustrated by numerical examples.

Chapter 2 Based on previous work including [3, 17, 19, 20, 22, 23, 24, 51, 62], we aim to present a unified general framework on particle filter methods.
We start with a general dynamical system and we simplify some *essentially known* approaches for proving the validity of those numerical methods. Some original developments are also presented. We first prove the almost sure or weak convergence of the particle filter algorithm through a generic model. Although the measures, defined by the particle filter algorithm, converge weakly to the right measure, one has no idea about the rate of convergence. Using the mean square convergence, we show that those measures converge with a rate proportional to $1/N_0$, N_0 is the initial number of particles injected to the particle filter algorithm.
The particle filter algorithm essentially consists of three stages: prediction, update and resampling. The importance sampling functions are tools to accomplish the prediction and the update stages. We give:

- Strategies of selection of the importance sampling functions
- A collection of importance sampling functions

In the resampling stage, we use branching mechanisms to renew the generations of the particles. We give:

- Strategies of selection of the branching mechanisms
- A collection of branching mechanisms

Several numerical and illustrative examples are provided.

Chapter 3 Tracking the position and the speed of an observed moving object has received much of interest in many scientific areas, see for example [36, 37, 48]. This Chapter deals with the estimation of the dynamics, position and speed, of a moving vehicle.

Scenario: A rolling car and a fixed observer who measures on-line his distance to the vehicle

We derive the car dynamical equation, state equation. The state equation is a 2-dimensional stochastic differential equation (SDE) with non Lipschitz and non linear growth bounded coefficients. We prove the existence and the uniqueness of solutions of a family of SDE's with non Lipschitz and non linear growth bounds conditions. We apply the result to the state equation.
We discretize the state equation and we prove that the Euler-Maruyama discretization converges strongly with order 1/2. A discretization to the observation equation is deduced. A comparison of the EKF and the particle filter algorithms shows that the latter one performs better.

Chapter 4 We present a generalization of both the work of Bensoussan in [7] and the work of Bhatt et al. in [10]:

- Multi-dimensional dynamical systems
- An Ornstein-Uhlenbeck type noise
- An unbounded observation function that admits a linear growth bound

We prove that the unnormalized measure satisfies the Zakai equation and under some additional assumptions:

1. We prove the existence and uniqueness of the solution of the Zakai equation
2. We construct a sequence of particle measures converging to the solution of the Zakai equation
3. We give a rate of convergence
4. We establish an implementation scheme

This will permit us to solve numerically the filtering problem via a Kallianpur-Striebel formula.
Computations are performed on a Pentium III, 730 MHz, with 256 Mb RAM under a Linux system (Suse 9.2) with Matlab and Scilab/plotlib.

Chapter 1

The Discrete Kalman and Extended Kalman Filter

Filtering is the problem of estimating the state of a system as a set of observations becomes available on-line. In 1960, R.E. Kalman published his famous paper [43] on recursive minimum variance estimation for linear Gaussian dynamical systems. The paper introduces an algorithm known as the discrete Kalman filter. The Kalman filter has been used in diverse areas including tracking, navigation, and guidance. Since Kalman's work, many papers and books appeared on the Kalman filter including continuous time systems [29, 39, 40, 47, 59].

In many cases, dynamical systems are nonlinear or non Gaussian and computing the minimum variance estimation, see appendix C.2, turns to be a difficult and sometimes an impossible exercise. One idea is to extend the Kalman work to the nonlinear and non Gaussian framework. This can be done by approximating the system equations by linear Gaussian ones and apply the Kalman filter algorithm. That gave birth to the extended Kalman Filter, [40, 67].

In this chapter we derive the Kalman and the extended Kalman filter algorithms, and we study some of their qualitative properties.

In subsection 1.1.1, we consider linear systems that are not necessarily Gaussian, then instead of calculating the minimum variance estimation (MVE), we give in Theorem 1.1.1 an analytical recursive algorithm for the best **affine** minimum variance estimation (BAMVE), see appendix C.2. If, in addition, the system is Gaussian then the BAMVE coincide with the MVE, Corollary 1.1.2. In the book of D.E. Catlin [14], one can find similar results for the best **linear** minimum variance estimation, see appendix C.2, and for systems that have no optional control inputs as ours.

In [40], Jazwinski studied some qualitative properties of the Kalman filter. He proved, under controllability and observability conditions [44], the uniform asymptotic stability of the filter and gave bounds on the error covariance matrix. In subsection 1.1.2, we consider stronger conditions, Assumptions 1.1-1.4, than those of Jazwinski, but they are much simpler to verify and allow us to give a lower and upper bounds on the error covariance matrix, Proposition 1.1.4.

We give in Proposition 1.1.6 a simple proof of the uniform asymptotic stability of the Kalman filter. In Corollary 1.1.8 we derive a result on the asymptotic behavior

of the filter with respect to its initialization. We end the subsection with an illustrative example.
In subsection 1.2.1, we derive the extended Kalman filter (EKF) algorithm for nonlinear models with Gaussian noises. In subsection 1.2.2, we exploit the parameters of the system to give an upper bound on the error in the EKF algorithm for a class of nonlinear dynamical systems. Finally, we give two numerical examples: in the first the errors remains bounded and in the second the errors are unbounded and the EKF fails to give good results.

1.1 The discrete Kalman filter

1.1.1 Kalman filter algorithm

n-dimensional state equation:

$$\mathbf{x}_{k+1} = \phi(k)\ \mathbf{x}_k + \Psi(k)\ \mathbf{a}_k + \Lambda(k)\ \mathbf{u}_k\,,\ k \geq 0 \tag{1.1}$$

- $\phi(k)$, $\Psi(k)$ and $\Lambda(k)$ are respectively $n \times n$, $n \times l$ and $n \times r$ known real matrices
- The optional control input $\mathbf{a}_k$ is a known l-dimensional discrete static process
- The process $\mathbf{u}_k$ is an r-dimensional white, called the state noise process
- The initial state $\mathbf{x}(0)$ is supposed to be a square integrable random variable
- For all $0 \leq j \leq k$,
$$E[\mathbf{u}_k] = 0 \text{ and } E[\mathbf{u}_k \mathbf{x}_j^T] = 0 \tag{1.2}$$
- The matrix $Q(k) = E[\mathbf{u}_k \mathbf{u}_k^T]$ is known for all $k \geq 0$

m-dimensional measurement equation:

$$\mathbf{z}_k = H(k)\ \mathbf{x}_k + \Pi(k)\ \mathbf{w}_k\,,\ k \geq 0 \tag{1.3}$$

- $H(k)$ is a known $m \times n$-real matrix, called the measurement matrix
- $\Pi(k)$ is a known $m \times p$-real matrix
- $\mathbf{w}_k$ is a p-dimensional white process, called measurement noise
- For all $0 \leq j \leq k$
$$E[\mathbf{w}_k] = 0\,,\ E[\mathbf{w}_k \mathbf{u}_j^T] = 0 \text{ and } E[\mathbf{w}_k \mathbf{x}_j^T] = 0 \tag{1.4}$$
- The matrix $R(k) = E[\mathbf{w}_k \mathbf{w}_k^T]$ is known for all $k \geq 0$

Denoting by $\hat{\mathbf{x}}(k|j)$ the BAMVE, see Appendix C.2, of $\mathbf{x}_k$ based on

$$y_j = \begin{bmatrix} \mathbf{z}_0 \\ \vdots \\ \mathbf{z}_j \end{bmatrix}. \tag{1.5}$$

Thus, $\hat{\mathbf{x}}(k|j)$ is the orthogonal projection onto the n-fold product of M_j with it self, M_j being the affine span of the components of the vector y_j.

If $j = k$, then $\hat{\mathbf{x}}(k|k)$ is the filtered estimate.

If $j < k$, then $\hat{\mathbf{x}}(k|j)$ is the predicted estimate.

If $j > k$, then $\hat{\mathbf{x}}(k|j)$ is the smoothed estimate.

We define the $n \times n$-matrix

$$P(k|j) = E[(\hat{\mathbf{x}}(k|j) - \mathbf{x}_k)(\hat{\mathbf{x}}(k|j) - \mathbf{x}_k)^T]\,. \tag{1.6}$$

$P(k|j)$ is the error covariance matrix of the estimate $\hat{\mathbf{x}}(k|j)$ to $\mathbf{x}_k$.

Theorem 1.1.1 (The Kalman Filter Algorithm)**.** *The BAMVE $\hat{\mathbf{x}}(k|k)$ may be generated recursively according to the following two stages*

1. ***Prediction***

$$\begin{aligned} \hat{\mathbf{x}}(k+1|k) &= \phi(k)\hat{\mathbf{x}}(k|k) + \Psi(k)\mathbf{a}_k\,, & (1.7)\\ P(k+1|k) &= \phi(k)P(k|k)\phi(k)^T + \Lambda(k)Q(k)\Lambda(k)^T\,. & (1.8) \end{aligned}$$

2. ***Update***

$$\begin{aligned} K(k+1) &= P(k+1|k)H^T\big[HP(k+1|k)H^T \\ &\quad +\Pi(k+1)R(k+1)\Pi(k+1)^T\big]^+\,, \\ \hat{\mathbf{x}}(k+1|k+1) &= \hat{\mathbf{x}}(k+1|k) + K(k+1)[\mathbf{z}_{k+1} - H\hat{\mathbf{x}}(k+1|k)]\,, & (1.9)\\ P(k+1|k+1) &= [I_n - K(k+1)H]P(k+1|k)\,. & (1.10) \end{aligned}$$

Where we have denoted simply $H(k+1)$ by H.
The matrix $K(k+1)$ is called the Kalman gain matrix.

Proof. If $\mu_y(j) = E[y_j]$, $\Gamma_y(j) = cov(y_j, y_j)$, $\mu_x(k) = E[\mathbf{x}_k]$ and $\Gamma_{x,y}(k,j) = cov(\mathbf{x}_k, y_j)$, for every $k, j \geq 0$, then the BAMVE of $\mathbf{x}_k$ and $\mathbf{x}_{k+1}$ based on y_k, see Theorem C.2.4, satisfies

$$\begin{aligned} \hat{\mathbf{x}}(k|k) &= \Gamma_{x,y}(k,k)\Gamma_y(k)^+(y_k - \mu_y(k)) + \mu_x(k)\,, \\ \hat{\mathbf{x}}(k+1|k) &= \Gamma_{x,y}(k+1,k)\Gamma_y(k)^+(y_k - \mu_y(k)) + \mu_x(k+1)\,. \end{aligned} \tag{1.11}$$

Firstly, we prove the equality (1.7). From the properties (1.2) and (1.4) of the noises we get

$$\begin{aligned} \mu_x(k+1) &= \Phi(k)\mu_x(k) + \Psi(k)\mathbf{a}_k \\ y_k - \mu_y(k) &= H(k)(\mathbf{x}_k - \mu_x(k)) + \Pi(k)\mathbf{w}_k \end{aligned}$$

This implies that

$$\begin{aligned}\Gamma_{x,y}(k+1,k) &= E[(\mathbf{x}_{k+1}-\mu_x(k+1))(y_k-\mu_y(k))^T] \\ &= E[(\phi(k)(\mathbf{x}_k-\mu_x(k)+\Lambda(k)\mathbf{u}_k)(y_k-\mu_y(k))^T] \\ &= \phi(k)\Gamma_{x,y}(k,k)\,. \end{aligned} \tag{1.12}$$

Now, using together the equalities (1.11) and (1.12) we obtain the equality (1.7). Secondly, we prove the equality (1.8). The equality (1.7) implies that

$$\hat{\mathbf{x}}(k+1|k)-\mathbf{x}_{k+1} = \phi(k)(\hat{\mathbf{x}}(k|k)-\mathbf{x}_k)-\Lambda(k)\mathbf{u}_k\,.$$

Since $\hat{\mathbf{x}}(k|k)-\mathbf{x}_k$ is orthogonal to $y_k-\mu_y(k)$, and $\mathbf{u}_k$ is uncorrelated with $\hat{\mathbf{x}}(k|k)-\mathbf{x}_k$ and $y_k-\mu_y(k)$, we deduce easily the result using the definition of $P(k+1|k)$ in (1.6). Finally, an application of Theorem C.2.6, 1., gives the equalities (1.9) and (1.10). □

- The process $\hat{\mathbf{z}}(k+1|k)=H(k+1)\hat{\mathbf{x}}(k+1|k)$ is called the predicted measure
- The process $\nu(k+1)=\mathbf{z}_{k+1}-\hat{\mathbf{z}}(k+1|k)$ is called the innovation
- If $\hat{\mathbf{x}}(k|k)$ and y_{k+1} are Gaussian, then $\hat{\mathbf{x}}(k+1|k+1)$ is Gaussian too

Corollary 1.1.2. *If the processes $\mathbf{u}_k$ and $\mathbf{w}_k$ are Gaussian and the process $\mathbf{x}_0$ is either deterministic or Gaussian, then, see Theorem C.2.8, 2., the MVE is Gaussian and satisfies the same relations in Theorem 1.1.1. Moreover,*

$$\mathbf{x}_k \,\big|\, y_k \sim \mathcal{N}(\hat{\mathbf{x}}(k|k),P(k|k)) \tag{1.13}$$

$$\mathbf{x}_{k+1} \,\big|\, y_k \sim \mathcal{N}(\hat{\mathbf{x}}(k+1|k),P(k+1|k)) \tag{1.14}$$

Equations (1.8) and (1.10) of Theorem 1.1.1 taken together constitute a recursively solvable matrix difference equation known as the discrete Riccati equation. These matrices may be computed in advance. In particular, $P(k|j)$ may be defined as the conditional covariance matrix

$$P(k|j)=E[(\hat{\mathbf{x}}(k|j)-\mathbf{x}_k)(\hat{\mathbf{x}}(k|j)-\mathbf{x}_k)^T\big|y_j]\,.$$

The relations in Theorem 1.1.1 still true and $P(k|j)=E[P(k|j)]$, see [40].

Remark 1.1.3. If a matrix A is symmetric positive definite we write $A>0$ and if A is symmetric positive semidefinite we write $A\geq 0$. Also, we write that $A\leq B$ if $B-A\geq 0$ and that $A<B$ if $B-A>0$.

For the remainder of this Chapter we suppose that the conditions of Corollary 1.1.2 hold and that $\mathbf{a}_k=0$, $\Lambda(k)=I_n$ and $\Pi(k)=I_m$, for all $k\geq 0$.
Suppose that $P(0|0)\geq 0$ and for all $k\geq 1$, $Q(k)$ and $R(k)$ are positive definite, then $[H(k)P(k|k-1)H(k)^T+R(k)]^{-1}$ exists, $P(k|k)$ is positive definite and

$$P(k|k)^{-1} = P(k|k-1)^{-1}+H(k)^TR(k)^{-1}H(k)\,, \tag{1.15}$$

$$K(k) = P(k|k)H(k)^TR(k)^{-1}\,. \tag{1.16}$$

1.1.2 Bounds and stability of the Kalman filter

This subsection deals with some qualitative properties of the Kalman filter, namely bounds, stability, convergence w.r.t. the initialization, and the propagation of errors. Suppose that the coefficients of the system satisfy the following assumptions:

Assumption 1.1. The covariances matrices $Q(k)$ and $R(k)$ are symmetric definite positive.

Assumption 1.2. The transition matrix $\Phi(k)$ is invertible.

Assumption 1.3. $P(0|0) > 0$.

Let us define, for every integer $k \geq 1$, the following symmetric positive matrix

$$\begin{aligned} I(k) &= \Phi(k)^{-T} H(k)^T R(k)^{-1} H(k) \Phi(k)^{-1} \\ &\quad + H(k+1)^T R(k+1)^{-1} H(k+1)\,, \end{aligned} \tag{1.17}$$

where $\Phi(k)^{-T} = (\Phi(k)^{-1})^T$.

Assumption 1.4. The matrices $Q(k)$ and $I(k)$ are uniformly bounded. That is, there exists two real numbers $0 < \alpha \leq \beta$ such that

$$0 < \alpha\, I_n \leq Q(k), I(k) \leq \beta\, I_n\,, \quad \forall k \geq 0\,.$$

Proposition 1.1.4. *Under the Assumptions 1.1-1.4, the error covariance matrices* $\{P(k|k)\}_{k\geq 1}$ *are uniformly bounded. More precisely,*

$$\frac{\alpha}{1+\alpha\,\beta}\, I_n \leq P(k|k) \leq \frac{1+\alpha\,\beta}{\alpha}\, I_n\,, \quad \forall k \geq 1\,. \tag{1.18}$$

Proof. Let us prove first the second inequality, ie.

$$P(k|k) \leq \frac{1+\alpha\,\beta}{\alpha}\, I_n\,, \quad \forall k \geq 1\,. \tag{1.19}$$

If we combine the equalities (1.7) and (1.9) we get

$$\mathbf{z}_k = H(k)\Phi(k)^{-1}\mathbf{x}_{k+1} - H(k)\Phi(k)^{-1}\mathbf{u}_k + \mathbf{w}_k\,.$$

Let us define the process

$$\bar{\mathbf{x}}(k+1|k+1) = I(k)^{-1}\{\Phi(k)^{-T}H(k)^T R(k)^{-1}\mathbf{z}_k + H(k+1)^T R(k+1)^{-1}\mathbf{z}_{k+1}\}\,.$$

Then, using the expression of the matrix $I(k)$ in (1.17), we obtain

$$\begin{aligned} \bar{\mathbf{x}}(k+1|k+1) &= I(k)^{-1}\{I(k)\mathbf{x}_{k+1} + \Phi(k)^{-T}H(k)^T R(k)^{-1}\mathbf{w}_k \\ &\qquad + H(k+1)^T R(k+1)^{-1}\mathbf{w}_{k+1} \\ &\qquad - \Phi(k)^{-T}H(k)^T R(k)^{-1} H(k)\Phi(k)^{-1}\mathbf{u}_k\}\,. \end{aligned}$$

From which we deduce

$$\begin{aligned}\mathbf{x}_{k+1} - \bar{\mathbf{x}}(k+1|k+1) &= I(k)^{-1}\{\Phi(k)^{-T}H(k)^T R(k)^{-1}H(k)\Phi(k)^{-1}\mathbf{u}_k \\ &\quad -(\Phi(k)^{-T}H(k)^T R(k)^{-1}\mathbf{w}_k \\ &\quad +H(k+1)^T R(k+1)^{-1}\mathbf{w}_{k+1})\} \qquad (1.20)\end{aligned}$$

The process $\bar{\mathbf{x}}(k+1|k+1)$ is, apriori, not the MVE, then

$$P(k+1|k+1) \leq E[(\mathbf{x}_{k+1} - \bar{\mathbf{x}}(k+1|k+1))(\mathbf{x}_{k+1} - \bar{\mathbf{x}}(k+1|k+1))^T]\,. \qquad (1.21)$$

Let us denote $\bar{E}_{k+1} = E[(\mathbf{x}_{k+1} - \bar{\mathbf{x}}(k+1|k+1))(\mathbf{x}_{k+1} - \bar{\mathbf{x}}(k+1|k+1))^T]$, if we use that $\Phi(k)^{-T}H(k)^T R(k)^{-1}H(k)\Phi(k)^{-1} \leq I(k)$, see (1.17), then from the equality (1.20) we get

$$\begin{aligned}\bar{E}_{k+1} &= I(k)^{-1} + cov(I(k)^{-1}\Phi(k)^{-T}H(k)^T R(k)^{-1}H(k)\Phi(k)^{-1}\mathbf{u}_k) \\ &\leq I(k)^{-1} + Q(k) \\ &\leq (\frac{1}{\alpha} + \beta)\, I_n\,. \qquad (1.22)\end{aligned}$$

It is clear that from the inequalities (1.21) and (1.22) we get the inequality (1.19). Now, we prove the first inequality, ie.

$$\frac{\alpha}{1+\alpha\,\beta}\, I_n \leq P(k+1|k+1)\,. \qquad (1.23)$$

Consider the following system

$$\begin{aligned}\bar{\mathbf{x}}_{k+1} &= \bar{\Phi}(k)\bar{\mathbf{x}}_k + \bar{\mathbf{u}}_k\,, \\ \bar{\mathbf{z}}_k &= \bar{\mathbf{x}}_k + \bar{\mathbf{w}}_k\,,\end{aligned}$$

where $\bar{\Phi}(k) = \Phi(k)^{-T}$ and $\bar{\mathbf{u}}_k$ and $\bar{\mathbf{w}}_k$ are independent Gaussian noises with covariance matrices given by

$$\begin{array}{rcccc}\bar{Q}(k) &=& E[\bar{\mathbf{u}}_k\bar{\mathbf{u}}_k^T] &=& \Phi(k)^{-T}H(k)^T R(k)^{-1}H(k)\Phi(k)^{-1} \\ \bar{R}(k) &=& E[\bar{\mathbf{w}}_k\bar{\mathbf{w}}_k^T] &=& Q(k)^{-1}\end{array}$$

We initialize the Kalman filter for this system by taking

$$\bar{P}(0|0) = P(0|0)^{-1} - H(0)^T R(0)^{-1}H(0)\,.$$

Then, using the equalities (1.8) and (1.15) we get by induction:

$$\bar{P}(k|k) = P(k|k)^{-1} - H(k)^T R(k)^{-1}H(k)\,,\ \forall k \geq 1\,.$$

In particular,

$$\bar{I}(k) = \Phi(k)Q(k)\Phi(k)^T + Q(k+1)\,,\ \forall k \geq 0\,.$$

Reasoning as for the first inequality, we get

$$\bar{P}(k|k) \leq \bar{I}(k)^{-1} + \bar{Q}(k)\,.$$

The definition of the matrix I, see (1.17), implies

$$P(k|k)^{-1} \leq \bar{I}(k)^{-1} + I(k)$$

Finally, since $\alpha\, I_n \leq Q(k+1) \leq \bar{I}(k)$ and $I(k) \leq \beta\, I_n$, then we deduce the inequality (1.23). □

Consider the following r-dimensional linear system

$$Z(k+1) = \Upsilon(k)Z(k)\,, \quad k \geq 0. \tag{1.24}$$

For every $0 \leq k_1 \leq k_2$, let us define the matrix $\Upsilon(k_2, k_1) = \Upsilon(k_2)\ldots\Upsilon(k_1)$.

Definition 1.1.5. The system (1.24) is said to be *uniformly asymptotically stable* if there exists two real constants $C > 0$ and $\gamma > 0$ such that

$$\|\Upsilon(k_2, k_1)\| \leq C\exp\left(-\gamma(k_2 - k_1)\right), \quad \forall k_2 \geq k_1 \tag{1.25}$$

where $\|\cdot\| = \|\cdot\|_2$ is the spectral norm.

A sufficient condition to get (1.25), see [45, 46], is to find scalar functions $V_k : \mathbb{R}^r \to \mathbb{R}$ and $\lambda_i : \mathbb{R} \to \mathbb{R}$, $1 \leq i \leq 3$ satisfying the following conditions

1. λ_i is continuous and $\lambda_i(0) = 0$ for $1 \leq i \leq 3$, and λ_1 and λ_2 are nondecreasing
2. $\lambda_1(x) \to +\infty$ as $x \to +\infty$ and $\lambda_3(x) < 0$ for $x \neq 0$
3. $V_k(0) = 0$ for all $k \geq 0$, and for some integers $0 < N \leq M$ and for all $k \geq M$ such that $Z(k) \neq 0$,

$$0 < \lambda_1(\|Z(k)\|) \leq V_k(Z(k)) \leq \lambda_2(\|Z(k)\|)\,, \tag{1.26}$$
$$V_k(Z(k)) - V_{k-N}(Z(k-N)) \leq \lambda_3(\|Z(k)\|)\,. \tag{1.27}$$

The function V is called a Lyapunov function for the system (1.24).
To prove the stability of the Kalman filter, we use equations (1.7) and (1.9) to write

$$\hat{\mathbf{x}}(k+1|k+1) = \Upsilon(k)\hat{\mathbf{x}}(k|k) + U(k)$$

where $\Upsilon(k) = [I_n - K(k+1)H(k+1)]\Phi(k)$ and $U(k) = K(k+1)\mathbf{z}_{k+1}$. The matrix Υ is called the state transition matrix of the filter. using the equality (1.10), we get

$$\Upsilon(k) = P(k+1|k+1)P(k+1|k)^{-1}\Phi(k)\,.$$

Consider the linear system

$$\begin{aligned}\hat{Z}(k+1) &= \Upsilon(k)\hat{Z}(k) \\ &= P(k+1|k+1)P(k+1|k)^{-1}\Phi(k)\hat{Z}(k)\end{aligned} \tag{1.28}$$

Proposition 1.1.6. *The Kalman filter, ie. the linear system (1.28), is uniformly asymptotically stable.*

For the proof we need the following Lemma

Lemma 1.1.7. *Suppose that P and R are two $n \times n-$symmetric definite positive matrices and suppose that*

$$0 < \alpha\, I_n \leq P \leq \beta\, I_n \quad \textit{and} \quad 0 < \lambda\, I_n \leq R \leq \mu\, I_n\,.$$

Then, there exists a real number $c > 0$ such that

$$(P+R)^{-1} - P^{-1} \leq -c\, I_n\,.$$

Proof. For any $n \times n$-matrix M, see [40] p.262, we have

$$(P^{-1} + M^T R M)^{-1} = P - PM^T(MPM^T + R^{-1})^{-1}MP\,.$$

Taking $M = P^{-1}$ we get

$$\begin{aligned}(P+R)^{-1} - P^{-1} &= P^{-1}((P^{-1}+PRP)^{-1} - P)P^{-1} \\ &= P^{-1}((P^{-1}+P^{-1}RP^{-1})^{-1} - P)P^{-1} \\ &= -P^{-1}(P^{-1}+R^{-1})^{-1}P^{-1}\end{aligned}$$

Denoting by $\gamma = \inf\{\|P^{-1}x\|\,,\ \|x\| = 1\}$. Since P^{-1} is continuous and invertible and $0 \notin \{x \in \mathbb{R}^n\,,\ \|x\| = 1\}$ is a compact then $\gamma > 0$. We deduce that

$$P^{-1}(P^{-1}+R^{-1})^{-1}P^{-1} \geq \gamma^2 \frac{\lambda\alpha}{\lambda+\alpha}\, I_n\ .$$

Then,

$$(P+R)^{-1} - P^{-1} \leq -\gamma^2 \frac{\lambda\alpha}{\lambda+\alpha}\, I_n\ .$$

It is sufficient to take $c = \gamma^2 \frac{\lambda\alpha}{\lambda+\alpha}$ and the proof is complete. □

Proof. (Proposition 1.1.6.) It is sufficient to find the right Lyapunov function V for the system (1.28). Let

$$V_k(\hat{Z}(k)) = \hat{Z}(k)^T P(k|k)^{-1}\hat{Z}(k)\,.$$

From Proposition 1.1.4 we have $\alpha/(1+\alpha\beta)\, I_n \leq P(k|k)^{-1} \leq (1+\alpha\beta)/\alpha\, I_n$, then

$$\frac{\alpha}{1+\alpha\beta}\|\hat{Z}(k)\|^2 \leq V_k(\hat{Z}(k)) \leq \frac{1+\alpha\beta}{\alpha}\|\hat{Z}(k)\|^2\,,$$

which proves (1.26). Now, we prove (1.27) for $N = M = 1$. Using (1.28), we get

$$\begin{aligned}&V_{k+1}(\hat{Z}(k+1)) - V_k(\hat{Z}(k)) \\ &\quad= \hat{Z}(k+1)^T P(k+1|k+1)^{-1}\hat{Z}(k+1) - \hat{Z}(k)^T P(k|k)^{-1}\hat{Z}(k) \\ &\quad= \hat{Z}(k)^T[\Phi(k)^T P(k+1|k)^{-1}P(k+1|k+1)P(k+1|k)^{-1}\Phi(k) \\ &\qquad\quad - P(k|k)^{-1}]\hat{Z}(k) \qquad (1.29)\end{aligned}$$

It is sufficient to show that the matrix in (1.29) is negative definite, ie. there exists a real number $c > 0$ such that

$$\Phi(k)^T P(k+1|k)^{-1}P(k+1|k+1)P(k+1|k)^{-1}\Phi(k) - P(k|k)^{-1} \leq -c\ I_n$$

From the equalities (1.8) and (1.15), we can write

$$\begin{aligned}&\Phi(k)^T P(k+1|k)^{-1}P(k+1|k+1)P(k+1|k)^{-1}\Phi(k) \\ &\quad= \Phi(k)^T P(k+1|k)^{-1}\big(P(k+1|k)^{-1} \\ &\qquad + H(k+1)R(k+1)^{-1}H(k+1)^T\big)^{-1}P(k+1|k)^{-1}\Phi(k) \\ &\quad\leq \Phi(k)^T P(k+1|k)^{-1}\Phi(k) \qquad (1.30) \\ &\quad= \Phi(k)^T(\Phi(k)P(k|k)\Phi(k)^T + Q(k))^{-1}\Phi(k) \\ &\quad= (P(k|k) + \Phi(k)^{-1}Q(k)\Phi(k)^{-T})^{-1}\,.\end{aligned}$$

In the inequality (1.30) we have used the fact that

$$P(k+1|k)^{-1} \leq P(k+1|k)^{-1} + H(k+1)R(k+1)^{-1}H(k+1)^T .$$

Then,

$$\begin{aligned}\Phi(k)^T P(k+1|k)^{-1}P(k+1|k+1)P(k+1|k)^{-1}\Phi(k) - P(k|k)^{-1} \\ \leq (P(k|k) + \Phi(k)^{-1}Q(k)\Phi(k)^{-T})^{-1} - P(k|k)^{-1}\end{aligned}$$

Since the matrices $P(k|k)$ and $\Phi(k)^{-1}Q(k)\Phi(k)^{-T}$ satisfy the conditions of the Lemma 1.1.7, we deduce the result. □

Corollary 1.1.8. *If* $\left(\mathbf{x}^1(0|0), P^1(0|0)\right)$ *and* $\left(\mathbf{x}^2(0|0), P^2(0|0)\right)$ *are two different initializations of the Kalman filter algorithm such that*

$$P^i(0|0) > 0 <, , \quad i \in \{1, 2\} .$$

Then, there exists two constants $C > 0$ *and* $\gamma > 0$ *such that*

$$\|P^1(k|k) - P^2(k|k)\| \leq C \exp(-\gamma k) .$$

The effect of the initialization $P(0|0)$ is exponentially forgotten with time. This is important if $P(0|0)$ is poorly known.

Proof. Let $K^1(k)$ and $K^2(k)$ be the Kalman gain matrices corresponding to $P^1(k|k)$ and $P^2(k|k)$ respectively. From the equality (1.16) we obtain

$$P^1(k|k)H(k)^T K^2(k)^T = K^1(k)H(k)P^2(k|k) , \forall\, k .$$

Denoting $\Delta(k) = P^1(k|k) - P^2(k|k)$, then

$$\begin{aligned}\Delta(k+1) &= P^1(k+1|k+1)(I_n - K^2(k+1)H(k+1))^T \\ &\quad -(I_n - K^1(k+1)H(k+1))P^2(k+1|k+1)^T \\ &= \left(P^1(k+1|k+1) - (I_n - K^1(k+1)H(k+1))Q(k)\right) \\ &\quad \times\left(I_n - K^2(k+1)H(k+1)\right)^T - \left(I_n - K^1(k+1)H(k+1)\right) \\ &\quad \times\left(P^2(k+1|k+1) - Q(k)(I_n - K^2(k+1)H(k+1))^T\right) \\ &= (I_n - K^1(k+1)H(k+1))\Phi(k) \\ &\quad \times\Delta(k)\Phi(k)^T(I_n - K^2(k+1)H(k+1))^T . \qquad (1.31)\end{aligned}$$

The matrices $\Upsilon^i(k) = (I_n - K^i(k+1)H(k+1))\Phi(k)$, $i = 1, 2$, are uniformly asymptotically stable by Proposition 1.1.6.
If $\Upsilon^i(k_2, k_1) = \Upsilon^i(k_2)\ldots\Upsilon^i(k_1)$ for $0 \leq k_1 \leq k_2$ and $i = 1, 2$, then

$$\begin{aligned}\Delta(k+1) &= \Upsilon^1(k)\Delta(k)\Upsilon^2(k)^T \\ &= \Upsilon^1(k, 0)\Delta(0)\Upsilon^2(k, 0)^T .\end{aligned}$$

Finally, from the inequality (1.25) we get the result. □

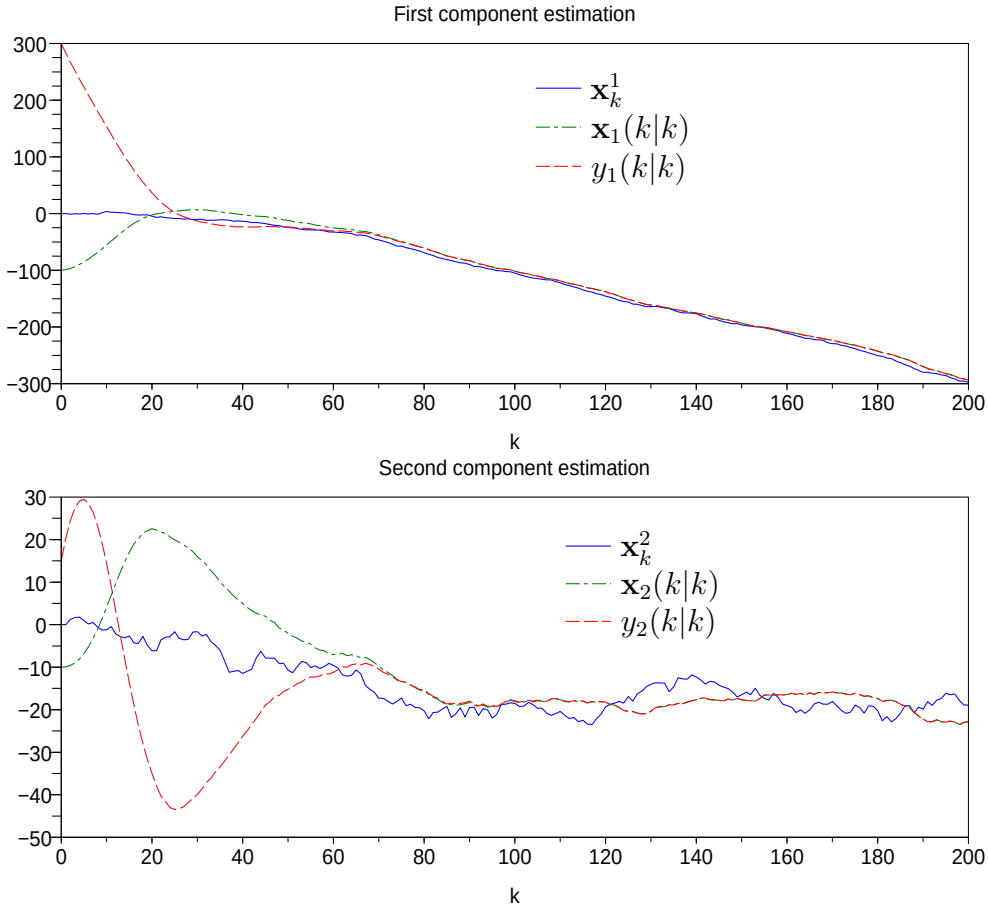

Figure 1.1: Two Kalman filter approximations of a path of the state $\mathbf{x}$ with two different initializations $\mathbf{x}(0|0)$ and $y(0|0)$.

Example 1.1.9 (Kalman filter). Consider the following linear dynamic system

2-dim. state equation $$\mathbf{x}_{k+1} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \mathbf{x}_k + \mathbf{u}_k$$

1-dim. measurement equation $$\mathbf{z}_k = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}_k + \mathbf{w}_k$$

where $\mathbf{x}_0 \in \mathbb{R}^2$ is an arbitrary fixed point, $T = 0.1$, and the processes $(\mathbf{u}_k)_k$ and $(\mathbf{w}_k)_k$ are independent and Gaussian with covariance matrices given respectively by $Q = Q(k) = I_2$ and $R = R(k) = 100$.
The matrix $I(k)$ defined in (1.17) is given by

$$I(k) = 10^{-2} \begin{bmatrix} 2 & -T \\ -T & T^2 \end{bmatrix} > 0\,.$$

In particular, $\lambda_1 I_2 \leq I(k) \leq \lambda_2 I_2$ where $\lambda_1 = 0.5(2 + T^2 - \sqrt{T^4 + 4}) > 0$ and $\lambda_2 = 0.5(2 + T^2 + \sqrt{T^4 + 4}) > 0$.
The Assumptions 1.1-1.4 hold and if the initialization of the Kalman algorithm is

such that $P(0|0) > 0$, then the Proposition 1.1.4, the Proposition 1.1.6 and the Corollary 1.1.8 say that

- The error covariance matrices $\{P(k|k)\}_k$ are uniformly bounded
- The filter is uniformly asymptotically stable
- The filter forgot exponentially its initialization

Suppose that

$$\mathbf{x}_0 = \begin{bmatrix} x_0^1 \\ x_0^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We apply the Kalman algorithm for two different initializations. Denoting $\mathbf{x}(k|k)$

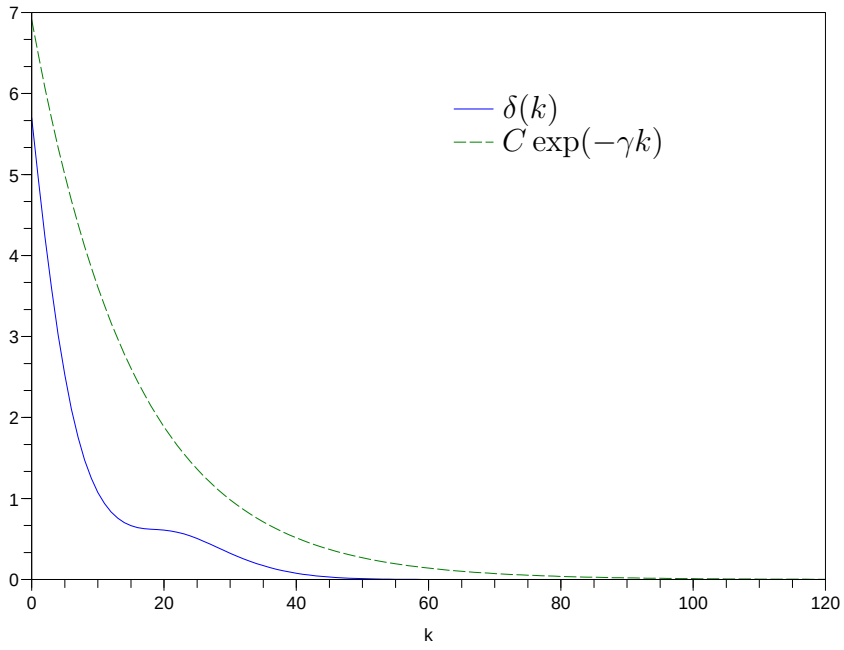

Figure 1.2: Kalman error covariance behavior: $\delta(k) = \|P_1(k|k) - P_2(k|k)\| \leq C\exp(-\gamma k)$.

and $y(k|k)$ the two update estimates of the state $\mathbf{x}_k$ relative to the following initializations:

1. $\mathbf{x}(0|0) = \begin{bmatrix} \mathbf{x}_1(0|0) \\ \mathbf{x}_2(0|0) \end{bmatrix} = \begin{bmatrix} -100 \\ -10 \end{bmatrix}$ and $P_1(0|0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} > 0$

2. $y(0|0) = \begin{bmatrix} y_1(0|0) \\ y_2(0|0) \end{bmatrix} = \begin{bmatrix} 300 \\ 15 \end{bmatrix}$ and $P_2(0|0) = \begin{bmatrix} 6 & -2 \\ -2 & 1 \end{bmatrix} > 0$

Figure 1.1: $k \in \{0, 1, \ldots, 200\}$

Up plot: A path of the first component $\mathbf{x}_k^1$ of the state and the two Kalman estimates $\mathbf{x}_1(k|k)$ and $y_1(k|k)$

Down plot: A path of the first component $\mathbf{x}_k^2$ of the state and the two Kalman estimates $\mathbf{x}_2(k|k)$ and $y_2(k|k)$

Let $\delta(k) = \|P_1(k|k) - P_2(k|k)\|$, there exists two constants C and $\gamma > 0$ such that

$$\delta(k) \leq C \exp(-\gamma\, k)\,.$$

Figure 1.2: We plot $(k, \delta(k))$, $0 \leq k \leq 120$, against the function $C\exp(-\gamma\, k)$, where $C = 6.7050$ and $\gamma = 0.7741744$.
The constants C and γ are obtained by a least squares fit of $\log(C)$ and γ.

1.2 The discrete extended Kalman filter

1.2.1 The extended Kalman filter algorithm

n-dimensional state equation $\mathbf{x}(k+1) = f_k(\mathbf{x}(k), \mathbf{u}(k))$, $k \geq 0$

m-dimensional measurement equation $\mathbf{z}(k) = h_k(\mathbf{x}(k), \mathbf{w}(k))$, $k \geq 0$

where

- $\mathbf{u}(k)$ is an $r-$dimensional Gaussian white process satisfying conditions (1.2)
- $\mathbf{w}(k)$ is an $p-$dimensional Gaussian white process, satisfying conditions (1.4)
- $\mathbf{x}(0)$ is either Gaussian or deterministic with mean μ_0 and covariance Γ_0

The filtering problem consists of calculating the minimum variance estimate (MVE) of the state given measurements up to the time of interest. That is, if for every $j \geq 0$,

$$y_j = \begin{bmatrix} \mathbf{z}(0) \\ \vdots \\ \mathbf{z}(j) \end{bmatrix}. \tag{1.32}$$

The MVE of state $\mathbf{x}(k)$ given y_j is given by $\hat{\mathbf{x}}(k|j) = E[\,\mathbf{x}(k)\,\big|\,y_j\,]$.
The estimate $\hat{\mathbf{x}}(k|k)$ is called the filtered estimate of $\mathbf{x}(k)$. The estimate $\hat{\mathbf{x}}(k|j)$ is called the predicted estimate, when $j < k$, and the smoothed estimate, when $j > k$.
The error covariance matrix of the estimate $\hat{\mathbf{x}}(k|j)$ to $\mathbf{x}(k)$ is

$$P(k|j) = E[(\hat{\mathbf{x}}(k|j) - \mathbf{x}(k))(\hat{\mathbf{x}}(k|j) - \mathbf{x}(k))^T \big| y_j]\,. \tag{1.33}$$

In many cases, dynamical systems are nonlinear or non Gaussian, and computing the MVE turns to be a difficult and sometimes an impossible exercise. One idea is to approximate the optimal solution (MVE) of the filter problem by approximating the system's dynamics. This can be done by replacing the original equations by a family of linear Gaussian equations obtained by a first order Taylor expansions around a

reference trajectory and apply the Kalman algorithm to the obtained linear system.
The validity of such approximations: How are good these linearizations, and which reference trajectory should be chosen?
We start with a **nominal reference trajectory:**

- $\mathbf{x}^*(0)$ fixed in $\mathbb{R}^n$
- $\mathbf{x}^*(k+1) = f_k(\mathbf{x}^*(k), 0)$, for all $k \geq 0$

Let us define the following two processes

$$\begin{aligned} \delta(k) &= \mathbf{x}(k) - \mathbf{x}^*(k)\,, & (1.34)\\ \nu(k) &= \mathbf{z}(k) - h_k(\mathbf{x}^*(k), 0)\,. & (1.35) \end{aligned}$$

A first order of Taylor's series expansion gives

$$\begin{aligned} f_k(\mathbf{x}(k), \mathbf{u}(k)) &\approx f_k(\mathbf{x}^*(k), 0) + D_x f_k(\mathbf{x}^*(k), 0)(\mathbf{x}(k) - \mathbf{x}^*(k)) \\ &\quad + D_u f_k(\mathbf{x}^*(k), 0)\mathbf{u}(k) & (1.36)\\ h_k(\mathbf{x}(k), \mathbf{w}(k)) &\approx h_k(\mathbf{x}^*(k), 0) + D_x h_k(\mathbf{x}^*(k), 0)(\mathbf{x}(k) - \mathbf{x}^*(k)) \\ &\quad + D_w h_k(\mathbf{x}^*(k), 0)\mathbf{w}(k) & (1.37) \end{aligned}$$

The functions f and h are supposed sufficiently regular and one can drop all higher than first order terms in the Taylor expansions. We denote for all $k \geq 0$,

$$\Phi(k) = D_x f_k(\mathbf{x}^*(k), 0)\,, \quad \Lambda(k) = D_u f(\mathbf{x}^*(k), 0)\,,$$

$$H(k) = D_x h_k(\mathbf{x}^*(k), 0)\,, \quad \Pi(k) = D_w h_k(\mathbf{x}^*(k), 0)\,.$$

Then,

$$\begin{aligned} \delta(k+1) &= \mathbf{x}(k+1) - \mathbf{x}^*(k+1) \\ &= f_k(\mathbf{x}(k), \mathbf{u}(k)) - f_k(\mathbf{x}^*(k), 0) \\ &\approx \Phi(k)\delta(k) + \Lambda(k)\mathbf{u}(k) & (1.38) \end{aligned}$$

Similarly,

$$\nu(k) \approx H(k)\delta(k) + \Pi(k)\mathbf{w}(k)\,. \quad (1.39)$$

We apply the Kalman filter algorithm to the *approximate* linear model (1.38) and (1.39). We denote the state estimate by $\hat{\delta}(j|k)$ and the error covariance matrix by $P(j|k)$. A reasonable choice of estimates to $\mathbf{x}(k+1)$ are

$$\begin{aligned} \hat{\mathbf{x}}(k+1|k) &= \mathbf{x}^*(k+1) + \hat{\delta}(k+1|k) \\ \hat{\mathbf{x}}(k+1|k+1) &= \mathbf{x}^*(k+1) + \hat{\delta}(k+1|k+1) \end{aligned}$$

We come now to the choice of the reference trajectory $(\mathbf{x}^*(k))_{k\geq 0}$. In the extended Kalman filter (EKF) setting, the reference trajectory $\{\mathbf{x}^*(k)\}_k$ is chosen on-line. That is, the first order Taylor's expansion (linearization) is made about the last predicted or filtered estimate as the algorithm proceeds.

The Extended Kalman Filter Algorithm

Initialization

$$\hat{\mathbf{x}}(0|0) = \mu_0\,, \quad P(0|0) = \Gamma_0\,.$$

Prediction

$$\Phi(k) = D_x f_k(\hat{\mathbf{x}}(k|k), 0)\,, \quad \Lambda(k) = D_u f_k(\hat{\mathbf{x}}(k|k), 0)\,,$$
$$\hat{\mathbf{x}}(k+1|k) = f_k(\hat{\mathbf{x}}(k|k), 0)\,,$$
$$P(k+1|k) = \Phi(k)P(k|k)\Phi(k)^T + \Lambda(k)Q(k)\Lambda(k)^T\,.$$

Update

$$H(k+1) = D_x h_k(\hat{\mathbf{x}}(k+1|k), 0)\,,\ \Pi(k+1) = D_w h_k(\hat{\mathbf{x}}(k+1|k), 0)\,,$$
$$K(k+1) = P(k+1|k)H(k+1)^T[H(k+1)P(k+1|k)H(k+1)^T$$
$$+\Pi(k+1)R(k+1)\Pi(k+1)^T]^+\,,$$
$$\hat{\mathbf{x}}(k+1|k+1) = \hat{\mathbf{x}}(k+1|k) + K(k+1)[\mathbf{z}(k+1) - h_{k+1}(\hat{\mathbf{x}}(k+1|k), 0)]\,,$$
$$P(k+1|k+1) = P(k+1|k) - K(k+1)H(k+1)P(k+1|k)\,.$$

Remark 1.2.1 (Drawbacks of the EKF).

1. The derivation of the Jacobian matrices are non trivial in most applications and often lead to significant implementation difficulties
2. Smoothness of the coefficients of a dynamical system may be not required and even they are smooth, higher than the first order terms of the Taylor expansions can not be neglected
3. The extended Kalman filter approximates the MVE's by Gaussian estimates. If they are non Gaussian, *e.g.* bimodal, then a Gaussian ones can never approximated it well.

1.2.2 On the convergence of the EKF

In general, the convergence of the EKF algorithm to the solution of the filtering problem may not be obtained. It depends on how good the linearized system approximates the true one, see Remark 1.2.1. The performance of the EKF depends on

- The regularity and the behavior of the functions f and h and their derivatives
- The noises and their covariances matrices Q and R
- The initialization of the algorithm

Krener [52], and Guo and Zhu [33] gave convergence results for some classes of deterministic nonlinear systems. They considered an EKF as an observer on the system, ie. Q and R are artificial inputs to control the quality of the convergence. Bertsekas [8] provides a non-stochastic analysis of the convergence and he interpret

the EKF as an incremental Gauss-Newton method and uses least squares methods. Here, we exploit directly the EKF algorithm and we give a suitable upper bound on the error for a class of nonlinear dynamical systems. This upper bound depends heavily on the coefficients of the system and on the covariance matrices of the noises. Let us consider $\mathcal{Z}(k) = \sigma\{\mathbf{z}(i)\,,\ 0 \leq i \leq k\}$. It is clear from the algorithm that $\hat{\mathbf{x}}(k|k)$ and $\hat{\mathbf{x}}(k+1|k)$ are $\mathcal{Z}(k)$-measurable. We define the following two processes:

$$\begin{aligned} V(k) &= \mathbf{x}(k) - \hat{\mathbf{x}}(k|k) \\ \tilde{V}(k) &= E[V(k)|\mathcal{Z}(k)] = E[\mathbf{x}(k)|\mathcal{Z}(k)] - \hat{\mathbf{x}}(k|k) \end{aligned}$$

We measure the performance of the EKF with the quantity $\epsilon_k = E[\tilde{V}(k)^T\tilde{V}(k)]^{1/2}$. In particular,

$$\epsilon_k^2 = E[\tilde{V}(k)^T\tilde{V}(k)] \leq E[V(k)^TV(k)]\,. \tag{1.40}$$

For a class of dynamical systems, we give an upper bound for ϵ_k, see Theorem 1.2.6. This upper bound depends on the initialization, on the regularity of the functions of the system, and on the noises and their covariances.
Suppose that the functions f and h sufficiently smooth, we make for simplicity the following notations:

- For all $\mathbf{X} = (X_1, \ldots, X_n) \in (\mathbb{R}^n)^n$ and $u \in \mathbb{R}^r$,

$$\hat{\Phi}^{\mathbf{X},u}(k) = \begin{bmatrix} D_x f_k^1(X_1, u) \\ \vdots \\ D_x f_k^n(X_n, u) \end{bmatrix}$$

- For all $x \in \mathbb{R}^n$ and $\mathbf{U} = (U_1, \ldots, U_n) \in (\mathbb{R}^r)^n$,

$$\hat{\Lambda}^{x,\mathbf{U}}(k) = \begin{bmatrix} D_u f_k^1(x, U_1) \\ \vdots \\ D_u f_k^n(x, U_n) \end{bmatrix}$$

- For all $\mathbf{X} = (X_1, \ldots, X_m) \in (\mathbb{R}^n)^m$ and $w \in \mathbb{R}^p$,

$$\hat{H}^{\mathbf{X},w}(k+1) = \begin{bmatrix} D_x h_k^1(X_1, w) \\ \vdots \\ D_x h_k^m(X_m, w) \end{bmatrix}$$

- For all $x \in \mathbb{R}^n$ and $\mathbf{W} = (X_1, \ldots, X_m) \in (\mathbb{R}^p)^m$,

$$\hat{\Pi}^{x,\mathbf{W}}(k+1) = \begin{bmatrix} D_w h_k^1(x, W_1) \\ \vdots \\ D_w h_k^m(x, W_m) \end{bmatrix}$$

- For any differentiable function $F : (x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to F(x, y) \in \mathbb{R}^{n_3}$ we define

$$|||D_x F||| = \sup_{(x_1,\ldots,x_{n_3},y)\in(\mathbb{R}^{n_1})^{n_3}\times\mathbb{R}^{n_2}} \| \begin{bmatrix} D_x F^1(x_1, y) \\ \vdots \\ D_x F^{n_3}(x_{n_3}, y) \end{bmatrix} \|$$

  In particular, $\|D_x F(x, y)\| \leq |||D_x F|||$. we define similarly $|||D_y F|||$

For every $k \geq 0$,

i. There exists $\mathbf{X}_{f,k} = (X^1_{f,k}, \ldots, X^n_{f,k}) \in [\mathbf{x}(k), \hat{\mathbf{x}}(k|k)]^n \subset (\mathbb{R}^n)^n$ and $\mathbf{U}_{f,k} = (U^1_{f,k}, \ldots, U^n_{f,k}) \in [0, \mathbf{u}(k)]^n \subset (\mathbb{R}^r)^n$ such that

$$f_k(\mathbf{x}(k), \mathbf{u}(k)) - f_k(\hat{\mathbf{x}}(k|k), 0) = \hat{\Phi}^{\mathbf{X}_{f,k},\mathbf{u}(k)}(k)(\mathbf{x}(k) - \hat{\mathbf{x}}(k|k)) + \hat{\Lambda}^{\hat{\mathbf{x}}(k|k),\mathbf{U}_{f,k}}(k)\mathbf{u}(k)$$

ii. There exists $\mathbf{X}_{h,k+1} = (X^1_{h,k+1}, \ldots, X^m_{h,k+1}) \in [\mathbf{x}(k+1), \hat{\mathbf{x}}(k+1|k)]^m$ and $\mathbf{W}_{h,k+1} = (W^1_{h,k+1}, \ldots, W^m_{h,k+1}) \in [0, \mathbf{w}(k+1)]^m$ such that if

$$\Delta h_{k+1} = h_{k+1}(\mathbf{x}(k+1), \mathbf{w}(k+1)) - h_{k+1}(\hat{\mathbf{x}}(k+1|k), 0)\,,$$

then

$$\begin{aligned} \Delta h_{k+1} &= \hat{H}^{\mathbf{X}_{h,k+1},\mathbf{w}(k+1)}(k+1)(\mathbf{x}(k+1) - \hat{\mathbf{x}}(k+1|k)) \\ &\quad +\hat{\Pi}^{\hat{\mathbf{x}}(k+1|k),\mathbf{W}_{h,k+1}}(k+1)\mathbf{w}(k+1) \\ &= \hat{H}^{\mathbf{X}_{h,k+1},\mathbf{w}(k+1)}(k+1)\hat{\Phi}^{\mathbf{X}_{f,k},\mathbf{u}(k)}(k)(\mathbf{x}(k) - \hat{\mathbf{x}}(k|k)) \\ &\quad +\hat{H}^{\mathbf{X}_{h,k+1},\mathbf{w}(k+1)}(k+1)\hat{\Lambda}^{\hat{\mathbf{x}}(k|k),\mathbf{U}_{f,k}}(k)\mathbf{u}(k) \\ &\quad +\hat{\Pi}^{\hat{\mathbf{x}}(k+1|k),\mathbf{W}_{h,k+1}}(k+1)\mathbf{w}(k+1) \end{aligned}$$

It follows that

$$\begin{aligned} V(k+1) &= [I_n - K(k+1)\hat{H}(k+1)]\hat{\Phi}(k)V(k) \\ &\quad +[I_n - K(k+1)\hat{H}(k+1)]\hat{\Lambda}(k)\mathbf{u}(k) \\ &\quad -K(k+1)\hat{\Pi}(k+1)\mathbf{w}(k+1)\,. \end{aligned} \tag{1.41}$$

In particular,

$$\begin{aligned} E[|V(k+1)|^2]^{1/2} &\leq E[|\{I_n - K(k+1)\hat{H}(k+1)\}\hat{\Phi}(k)\,V(k)|^2]^{1/2} \\ &\quad +E[|\{I_n - K(k+1)\hat{H}(k+1)\}\hat{\Lambda}(k)\mathbf{u}(k)|^2]^{1/2} \\ &\quad +E[|K(k+1)\hat{\Pi}(k+1)\mathbf{w}(k+1)|^2]^{1/2}\,. \end{aligned} \tag{1.42}$$

From now, we suppose that the following assumptions hold

Assumption 1.5. For all $k \geq 0$, $x \in \mathbb{R}^n$, $\mathbf{u} \in (\mathbb{R}^r)^m$ and $\mathbf{w} \in (\mathbb{R}^p)^m$,

- $\hat{\Lambda}^{x,\mathbf{u}}(k)Q(k)\hat{\Lambda}^{x,\mathbf{u}}(k)^T$ and $\hat{\Pi}^{x,\mathbf{w}}(k+1)R(k)\hat{\Pi}^{x,\mathbf{w}}(k+1)^T$ are invertible
- There exists positive constants $\eta_i\,,\ i = 1, 2$ and γ such that

- $\|Q(k)\|$, $\|\hat{\Lambda}^{x,\mathbf{u}}(k)Q(k)\hat{\Lambda}^{x,\mathbf{u}}(k)^T\| \leq \gamma$
- $\|R(k)\|$, $\|\hat{\Pi}^{x,\mathbf{w}}(k+1)R(k)\hat{\Pi}^{x,\mathbf{w}}(k+1)^T\| \leq \eta_1$
- $\|\big(\hat{\Pi}^{x,\mathbf{w}}(k+1)R(k)\hat{\Pi}^{x,\mathbf{w}}(k+1)^T\big)^{-1}\| \leq \eta_2$

Assumption 1.6. There exists positive constants $\alpha_i\,,\ \beta_i\,,\ i=1,2$ and γ such that

$$|||D_x f_k||| \leq \alpha_1\,, \quad |||D_u f_k||| \leq \alpha_2\,,$$
$$|||D_x h_k||| \leq \beta_1\,, \quad |||D_w h_k||| \leq \beta_2\,.$$

Assumption 1.7. $\tau = \beta_1^2\eta_2\gamma < 1$ and $\alpha_1 < 1-\sqrt{\tau}$.

Assumption 1.8. $P(0|0) \geq 0$.

Under the Assumption 1.5-1.8, we obtain

$$\begin{array}{rcl} P(k+1|k+1)^{-1} & = & P(k+1|k)^{-1} + H(k+1)^T\tilde{R}(k+1)^{-1}H(k+1)\,, \\ K(k+1) & = & P(k+1|k+1)H(k+1)^T\tilde{R}(k+1)^{-1}\,. \end{array} \tag{1.43}$$

Lemma 1.2.2. *There exists a real number $\theta > 1$ such that*

$$\theta^2\tau < 1 \quad \textit{and} \quad \alpha_1^2 < \frac{\theta-1}{\theta}(1-\theta\tau)\,.$$

Proof. For every $\theta \in (1, \sqrt{1/\tau})$ the first inequality of the Lemma is satisfied. For the second inequality we consider the function g defined on the interval $(0, \sqrt{1/\tau})$ by

$$g(\theta) = \frac{\theta-1}{\theta}(1-\theta\tau)\,, \quad \forall\,\theta \in (0, \frac{1}{\sqrt{\tau}})\,.$$

The function$g$ is $\mathcal{C}^1$-class on $(0, \sqrt{1/\tau})$ and for all $\theta \in (0, \frac{1}{\sqrt{\tau}})$ we have

$$g'(\theta) = \frac{1-\tau\theta^2}{\theta^2} \ > \ 0\,.$$

Then, The function g is a strictly increasing and from the Assumption 1.7, $\alpha_1^2 < (1-\sqrt{\tau})^2 = g(\frac{1}{\sqrt{\tau}})$. The two inequalities are satisfied for θ in $\big(\max\{1, g^{-1}(\alpha^2)\}\,,\ \sqrt{1/\tau}\big)$. □

Proposition 1.2.3. *Let us choose a real number $\theta > 1$ satisfying the Lemma 1.2.2. If $\|P(0|0)\| \leq \frac{\theta-1}{\alpha_1^2}\gamma$, then*

$$\|P(k|k)\| \leq \frac{\theta-1}{\alpha_1^2}\gamma\,, \quad \forall k \geq 0\,.$$

Remark 1.2.4.

- Since $\alpha_1^2 < \frac{\theta-1}{\theta}(1-\theta\tau)$, we choose α_1 such that

$$\big(\frac{\theta-1}{\theta}\big)^2 < \alpha_1^2 < \frac{\theta-1}{\theta}(1-\theta\beta_1^2\eta_2\gamma) \tag{1.44}$$

- For simplicity, we denote by
  - $\tilde{Q}(k) = \hat{\Lambda}(k)Q(k)\hat{\Lambda}(k)^T$
  - $\tilde{R}(k+1) = \hat{\Pi}(k+1)R(k)\hat{\Pi}(k+1)^T$

Proof. Since $\|P(0|0)\| \leq \frac{\theta-1}{\alpha_1^2}\gamma$, we prove the result by induction. Let $k \geq 0$ and suppose that $\|P(k|k)\| \leq \frac{\theta-1}{\alpha_1^2}\gamma$, then

$$P(k+1|k) = \Phi(k)P(k|k)\Phi(k)^T + \tilde{Q}(k)\,.$$

Then,

$$\begin{aligned}\|P(k+1|k)\| &\leq \|\Phi(k)\|^2\|P(k|k)\| + \|\tilde{Q}(k)\| \\ &= \alpha_1^2\frac{(\theta-1)}{\alpha_1^2}\gamma + \gamma = \theta\gamma\end{aligned}$$

We denote by $H = H(k+1)$. From the equalities (1.43) we get

$$\begin{aligned}P(k+1|k+1) &= \big(I_n - K(k+1)H\big)P(k+1|k) \\ &= \big(I_n - P(k+1|k+1)H^T\tilde{R}(k+1)^{-1}H\big)P(k+1|k)\,.\end{aligned}$$

This leads to

$$\|P(k+1|k+1)\|(1-\theta\beta_1^2\eta_2\gamma) \leq \theta\gamma$$

From (1.44) we have $1-\theta\beta_1^2\eta_2\gamma > 0$ and $\frac{\theta\gamma}{1-\theta\beta_1^2\eta_2\gamma} \leq \frac{\theta-1}{\alpha_1^2}\gamma$. Then,

$$\|P(k+1|k+1)\| \leq \frac{\theta-1}{\alpha_1^2}\gamma\,.$$

□

Corollary 1.2.5. *If $\lambda = (\theta-1)/(\alpha_1\theta) < 1$ and $\Theta = \theta\beta_1\eta_2\gamma$, then for all $k \geq 0$ we have*

1. $\|\{I_n - K(k+1)\hat{H}(k+1)\}\hat{\Phi}^{\mathbf{X}_{f,k},\mathbf{u}(k)}(k)\| \leq \lambda$
2. $\|I_n - K(k+1)\hat{H}^{\mathbf{X}_{h,k+1},\mathbf{w}(k+1)}(k+1)\| \leq \lambda \times 1/\alpha_1$
3. $\|K(k+1)\| \leq \lambda \times \Theta/\alpha_1$

From the inequality (1.42) we conclude that

$$E[|V(k+1)|^2]^{1/2} \leq \lambda E[|V(k)|^2]^{1/2} + \frac{\lambda\alpha_2}{\alpha_1}E[|\mathbf{u}(k)|^2]^{1/2} + \frac{\lambda\Theta\beta_2}{\alpha_1}E[|\mathbf{w}(k+1)|^2]^{1/2}\,.$$

The matrices $Q(k)$ and $R(k)$ are real symmetric and positive, then

$$\begin{array}{ccccc}\|Q(k)\| & \leq & E[|\mathbf{u}(k)|^2] & \leq & n\|Q(k)\| \\ \|R(k+1)\| & \leq & E[|\mathbf{w}(k+1)|^2] & \leq & m\|R(k+1)\|\end{array}$$

It follows that

$$E[|V(k+1)|^2]^{1/2} \leq \lambda E[|V(k)|^2]^{1/2} + \lambda(\sqrt{\gamma n}\ \frac{\alpha_2}{\alpha_1} + \sqrt{\eta_1 m}\ \frac{\beta_2 \Theta}{\alpha_1}) \,. \tag{1.45}$$

Iterating this inequality from 0 to k we get

$$\begin{aligned} E[|V(k)|^2]^{1/2} &\leq \lambda^k E[|V(0)|^2]^{1/2} + C\frac{\lambda(1-\lambda^k)}{1-\lambda} \\ &\leq \lambda^k E[|V(0)|^2]^{1/2} + C\frac{\lambda}{1-\lambda} \end{aligned} \tag{1.46}$$

where

$$C = \sqrt{\gamma n}\ \frac{\alpha_2}{\alpha_1} + \sqrt{\eta_1 m}\ \frac{\beta_2 \Theta}{\alpha_1} \ .$$

The constants C and λ depend heavily on the functions f, h and their Jacobian matrices, and on the covariance noises matrices Q and R. The algorithm forgot exponentially its initialization since $\lambda^k E[|V(0)|^2]^{1/2} \to 0$ as $k \to \infty$. The expression of the constant C shows that this upper bound is sensitive to the dimensions of the state and the observation spaces. Finally, smaller are the numbers λ, C and $E[|V(0)|^2]$ better are the approximations.
To summarize we have proved the following Theorem

Theorem 1.2.6 (On the convergence of the EKF)**.**
The EKF error is $\epsilon_k = E[(\mathbf{x}(k) - \hat{\mathbf{x}}(k|k))(\mathbf{x}(k) - \hat{\mathbf{x}}(k|k))^T]^{1/2}$.
Suppose that the conditions of Proposition 1.2.3 hold. Let us consider

$$\lambda = \frac{\theta - 1}{\alpha_1 \theta} \quad (<1)\,,\ \Theta = \theta\beta_1\eta_2\gamma \ \textit{and}\ \ C = \sqrt{\gamma n}\ \frac{\alpha_2}{\alpha_1} + \sqrt{\eta_1 m}\ \frac{\beta_2 \Theta}{\alpha_1} \ .$$

If $0 \leq \|P(0|0)\| \leq \frac{(\theta-1)\gamma}{\alpha_1^2}$, *then*

$$\epsilon_k \leq \lambda^k \epsilon_0 + \frac{\lambda}{1-\lambda} C \,.$$

Example 1.2.7 (Convergence of the EKF). Consider the following model:

2-dim. state equation $\mathbf{x}(k+1) = f(\mathbf{x}(k)) + \mathbf{u}(k)$

1-dim. measurement equation $\mathbf{z}(k) = h(\mathbf{x}(k)) + \mathbf{w}(k)$

where

- $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- for all $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$,
  - $f(x) = \begin{bmatrix} x_1/2 + \cos(x_2/(2\pi)) \\ x_2/2 + \sin(x_1/(2\pi)) \end{bmatrix}$

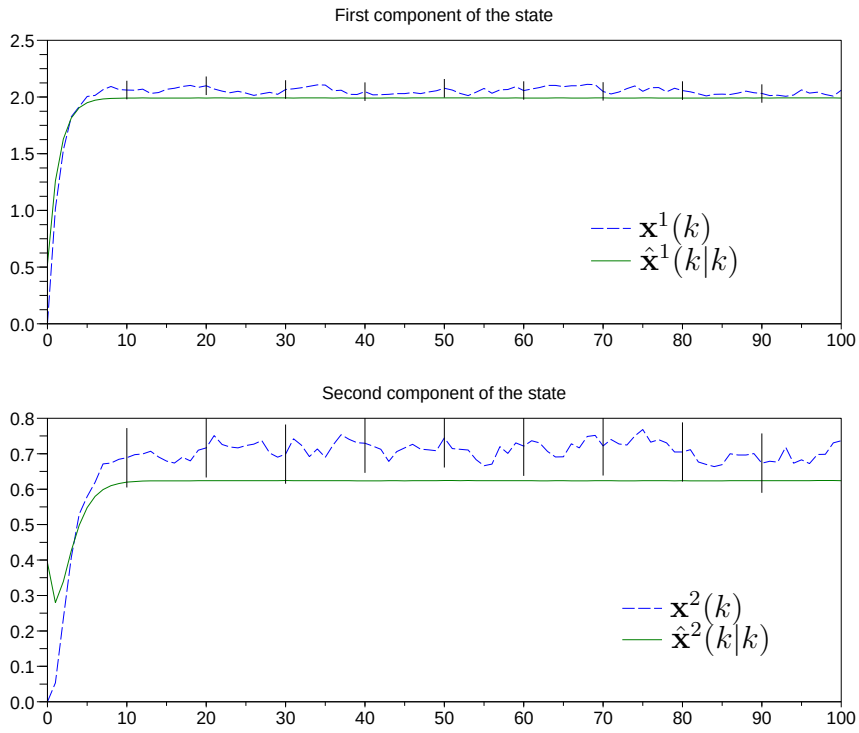

Figure 1.3: The EKF estimate of a path of the state process $\mathbf{x}$.

– $h(x) = 2(1 + x_1^2 + x_2^2)^{1/2}$

The Gaussian noises $\{\mathbf{u}(k)\}_k$ and $\{\mathbf{w}(k)\}_k$ are independent with covariance matrices given by

$$Q(k) = Q = \begin{bmatrix} 5 \times 10^{-3} & 0 \\ 0 & 5 \times 10^{-3} \end{bmatrix} \text{ and } R(k) = R = 100\,.$$

The Jacobian matrices of the functions f and h are given by

$$\begin{aligned} D_x f(x) &= \frac{1}{2\pi} \begin{bmatrix} \pi & -\sin(x_2/(2\pi)) \\ \cos(x_1/(2\pi)) & \pi \end{bmatrix}, \\ D_x h(x) &= 2 \begin{bmatrix} x_1/h(x) & x_2/h(x) \end{bmatrix}. \end{aligned}$$

In particular,

1. $\|D_x f\| \leq \alpha_1 = 0.7421$, $\|D_x h\| \leq \beta_1 = 2$ and $\alpha_2 = \beta_2 = 1$
2. $\|Q\| \leq \gamma = 5 \times 10^{-3}$, $\|R\| = 100 \leq \eta_1 = 100$ and $\|R^{-1}\| = 10^{-2} \leq \eta_2 = 10^{-2}$

Then, $\beta_1^2 \eta_2 \gamma < 1$. The conditions of Proposition 1.2.3 hold for $\theta = 2.6$. In particular,

$$\lambda = \frac{\theta - 1}{\alpha_1 \theta} = 0.8292\,, \quad C = 0.08571 \quad \text{and} \quad \frac{\lambda}{1-\lambda} \times C = 0.416124\,.$$

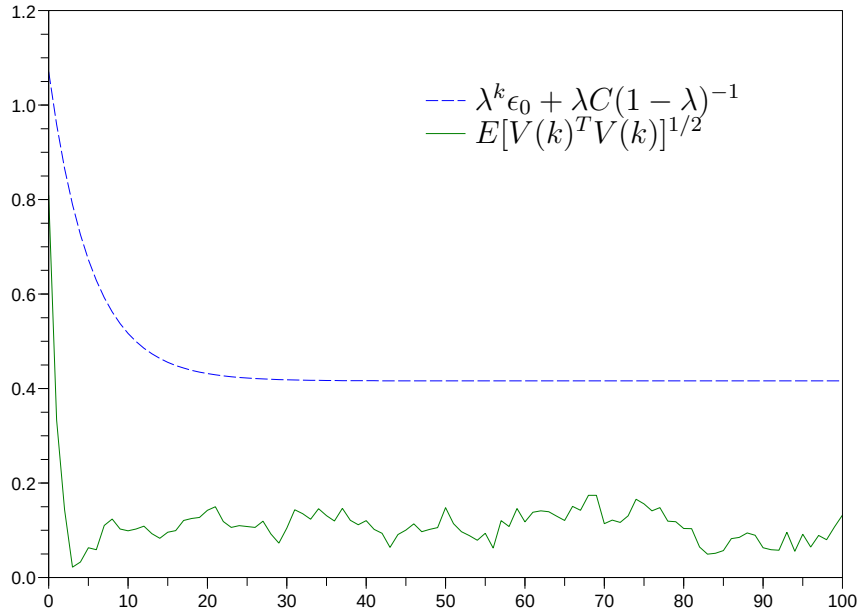

Figure 1.4: EKF Upbound error: $E[V(k)^T V(k)]^{1/2} \leq \lambda^k \epsilon_0 + \lambda C(1-\lambda)^{-1}$.

We initialize the EKF algorithm with

$$\begin{aligned} \hat{\mathbf{x}}(0|0) &= \begin{bmatrix} \pi/6 \\ \pi/8 \end{bmatrix}, \\ P(0|0) &= 10^{-2} \begin{bmatrix} 1.4 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Figure 1.3: The EKF estimate of the first and the second component of the state vector $\mathbf{x}$. The bars "$\Big|$" in the two plots represent the EKF errors, ie. $\sqrt{P(k|k)_{1,1}}$ and $\sqrt{P(k|k)_{2,2}}$.
Figure 1.4: Since $\|P(0|0)\| \leq \gamma(\theta-1)/\alpha_1^2 = 0.0145266$, then Theorem 1.2.6 says that

$$\epsilon_k = E[|\mathbf{x}(k) - \hat{\mathbf{x}}(k|k)|^2]^{1/2} \leq \lambda^k \epsilon_0 + C\lambda/(1-\lambda)$$

Example 1.2.8 (non convergent EKF algorithm). One example of non linear system for which an EKF algorithm may fail to converge is

State equation: $\mathbf{x}(k+1) = \mathbf{x}(k)(1-\mathbf{x}(k)^2) + \mathbf{u}(k)$ and $\mathbf{x}(0) = 0.05$

Measurement equation: $\mathbf{z}(k) = \mathbf{x}(k)^2 - \mathbf{x}(k)/2 + \mathbf{w}(k)$

where

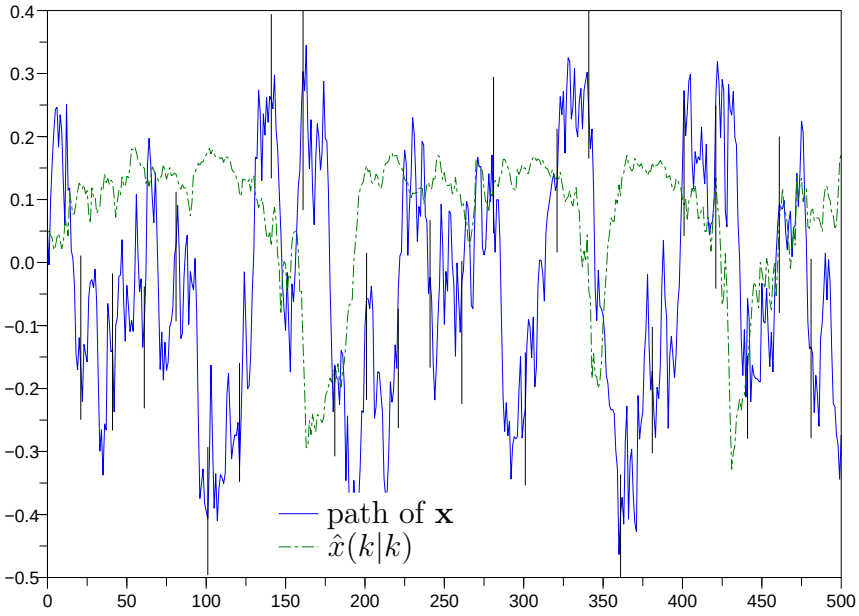

Figure 1.5: The EKF may fail.

- The processes $\mathbf{u}$ and $\mathbf{w}$ are Gaussian independent
- For all $k \geq 0$, $E[\mathbf{u}(k)^2] = Q = 0.005$ and $E[\mathbf{w}(k)^2] = 1$

This system is of high nonlinearity because of the presence of high degree polynomial terms. In particular, the hypothesis of Theorem 1.2.6 are not satisfied indeed the derivatives are non bounded.
We applied the EKF algorithm with the following initialization:

$$\hat{\mathbf{x}}(0|0) = 0.05 \text{ and } P(0|0) = 1\,.$$

Figure 1.5: Even a long time simulation ($N = 500$), the EKF path do not converge to the true one.
Figure 1.6:

- $\hat{\mathbf{x}}(k|k)$ approximates $E[\mathbf{x}(k)|\mathcal{Z}_k]$
- $P(k|k)$ approximates $E[(\mathbf{x}(k) - E[\mathbf{x}(k)|\mathcal{Z}_k])^2|\mathcal{Z}_k]$

Then, $E[(\mathbf{x}(k) - \hat{\mathbf{x}}(k|k))^2]$ and $E[P(k|k)]$ are both approximations to $E[(\mathbf{x}(k) - E[\mathbf{x}(k)|\mathcal{Z}_k])^2]$.
If the EKF gives good approximations, then $E[(\mathbf{x}(k) - \hat{\mathbf{x}}(k|k))^2] - E[P(k|k)]$ converges to zero or at least becomes small.
We simulate 100 paths of the EKF estimation to calculate $E[P(k|k)]$ and $E[(\mathbf{x}(k) - \hat{\mathbf{x}}(k|k))^2]$. The two trajectories fail to become close one to the other.

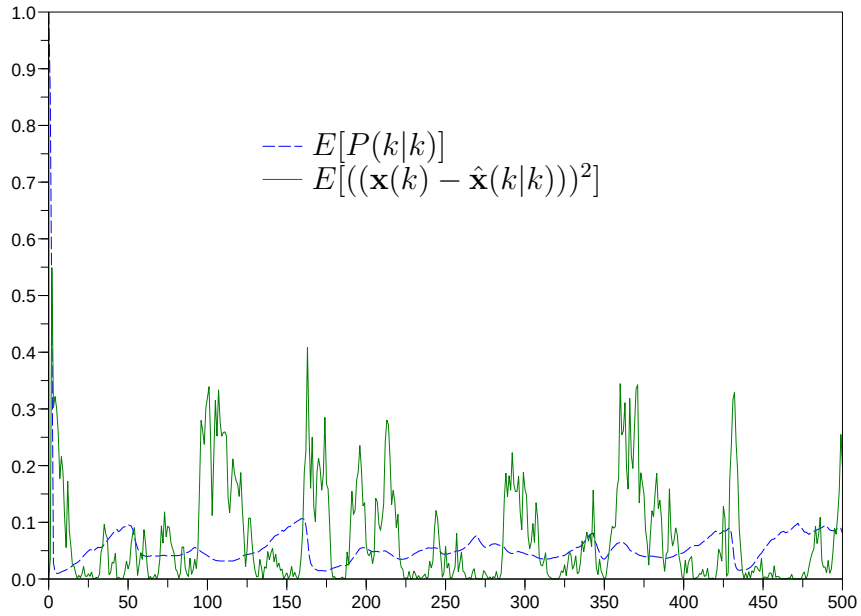

Figure 1.6: The two trajectories fail to become close one to the other.

Chapter 2

Particle Filter Methods

2.1 Introduction

The stochastic filtering problem deals with the estimation of the current state of a process X, called the signal or the state process, based on measurements supplied by an associate process Z, called the observation process. In the Bayesian framework, see Appendix C.1, a stochastic dynamical equation provides the prior distribution of the state process and an observation equation gives rise to the likelihood of the observation, and all relevant information on the state X_k are included in the posterior distribution $P(X_k \in A \big| Z_0, \ldots, Z_j)$. This problem is known as the Bayesian filtering also called the optimal filtering problem. Often the observations or measurements arrive sequentially in time, and one is interested in estimating recursively in time the evolving posterior distribution. The posterior distribution only admits an analytical expression for few special models including linear Gaussian models, in that case one derives an exact analytical recursion expression for the posterior called the Kalman filter, see Chapter 1. However, in many realistic problems real data involve elements of non linearity, non Gaussianity and high dimensionality which preclude analytical solutions. For over 3 decades several approximation strategies to the optimal filter were been proposed, for example:

- The extended Kalman filter, see Section 1.2
- The unscented filter [41, 42]
- Approximations by Gaussian sums [4, 74]
- Approximations using deterministic numerical integration methods [13]

These methods have numerous drawbacks, they depend on the dimension of the system, they are numerically expensive and they use mainly Gaussian distributions. Only in the past few years with the advance of computational power the Monte Carlo method, see Appendix C.3, has gained the full status of a numerical method capable of addressing many complex applications. The sequential Monte Carlo methods, also called particle filter methods, have been developed to address the Bayesian filtering problem. These methods are very flexible, often easy to implement, and have

the advantage of not being subject to any linearity, Gaussianity or dimensionality hypothesis on the models.
The aim of this chapter is to present a unified general framework on particle filter methods. We simplify some approaches for proving the validity of these numerical methods using some different techniques to those appeared in literature, this make them comprehensible and applicable. Also, some original developments and new results are presented. Our work is essentially based on the works of Doucet [22, 23], Liu and Chen [62], Crisan and Doucet [20], Doucet, Godsill and Andrieu [24], Crisan [17], Crisan, Del Moral and Lyons [19], Arulampalam, Maskell, Gordon and Clapp [3] and Kong, Liu and Wong [51].
The rest of the chapter is organized as follows. In Section 2.2 we apply a classical Monte Carlo method to resolve the filtering problem. This method is only idealistic since it requires the possibility of sampling from multivariate and non standard distributions. In Section 2.3, we consider a rather general model and we prove a recursion relation for the optimal filtering problem. This recursion represents the basis for the numerical algorithms and enable us to design iterative schemes, called particle filter methods or sequential Monte Carlo methods. In Section 2.4 we present the original particle filter algorithm and its drawbacks including the degeneracy phenomenon. The use of importance sampling functions and branching mechanisms allow us to surmount this degeneracy phenomenon and permit us to give a very general particle filter algorithm. Section 2.5 deals with the almost sure convergence of these algorithms. We prove convergence results for generic models that we apply to particle filter algorithms. In Section 2.6, we prove the mean square convergence and we give a rate of convergence. In Section 2.7 strategies of selection of importance sampling functions are discussed. In Section 2.8 we give a large family of correlated and independent branching mechanisms. Examples are studied through the Chapter supplied with comparison of different particle filter methods. In in the remainder of this Section, we recall some properties of Markov processes and their transition kernels.
The transition kernels of an $\mathbb{R}^n$-valued Markov process $X = \{X_k\}_k$ are the functions $(K_k(\cdot,\cdot))_k$ defined on $\mathbb{R}^n \times B(\mathbb{R}^n)$ by

$$K_k(x, A) := P(X_{k+1} \in A \big| X_k = x)\,.$$

We make the following assumption

Assumption 2.1. All probability distributions and all kernels in this Chapter are supposed to have a density function.

Then, for all $A \in B(\mathbb{R}^n)$,

$$\begin{aligned} K_k(x, A) &= P(X_{k+1} \in A \big| X_k = x) = \int_A p(x_{k+1}|x_k = x)dx_{k+1}\,, \\ K_k(x, dx_{k+1}) &= p(x_{k+1}|x_k = x)dx_{k+1}\,. \end{aligned}$$

If $X_{i:j} = (X_i, \ldots, X_j)$ and $P^x_{i:j}$ is the law of $X_{i:j}$ on $(\mathbb{R}^n)^{j-i+1}$, the Chapman-Kolmogorov equation gives

$$P^x_{0:k}(dx_{0:k}) = \pi_0(dx_0)\Pi_{j=1}^k K_j(x_{j-1}, dx_j)\,. \tag{2.1}$$

The Remark B.1.2 affirms that we can choose the kernels $K_k(\cdot,\cdot)$ to be regular. That is, for every $(k, x, A) \in \mathbb{N} \times \mathbb{R}^n \times B(\mathbb{R}^n)$

- $K_k(x,\cdot)$ is a probability measure on $\mathbb{R}^n$
- $K_k(\cdot, A)$ is bounded Borel function

For every $\mu \in \mathcal{M}_F(\mathbb{R}^n)$ and every $k \in \mathbb{N}$, the measure $\mu K_k(\cdot)$ is defined by

$$\mu K_k(A) = \int_{\mathbb{R}^n} K_k(x, A)\mu(dx)\,, \quad \forall A \in B(\mathbb{R}^n)$$

In particular, $\mu K_k(\cdot) \in \mathcal{M}_F(\mathbb{R}^n)$.
For every $\varphi \in \mathcal{B}(\mathbb{R}^n)$, let $K_k\varphi$ be the bounded Borel function defined on $\mathbb{R}^n$ by

$$K_k\varphi(x) = \int_{\mathbb{R}^n} \varphi(y)K_k(x, dy)\,, \quad \forall x \in \mathbb{R}^n\,.$$

Then,

$$\mu K_k\varphi = \int_{\mathbb{R}^n} \varphi(y)K_k(x, dy)\mu(dx)\,.$$

The Markov process X has a Feller transition kernel if for every $\varphi \in \mathcal{C}_b(\mathbb{R}^n)$,

$$K_k\varphi \in \mathcal{C}_b(\mathbb{R}^n)\,.$$

If $\mathcal{F}_k^x = \sigma\{X_j\,,\ 0 \le j \le k\}$, the Markov property implies that

$$E[\varphi(X_{k+1})|\mathcal{F}_k^x] = E[\varphi(X_{k+1})|X_k] = K_k\varphi(X_k)\,,\ \forall\varphi \in \mathcal{B}(\mathbb{R}^n)\,.$$

2.2 Perfect Monte Carlo sampling

Let us consider the following model:

1. The state $X = \{X_k\}_k$ is a Markov process of initial distribution π_0 and a transition kernel $K_k(x_{k-1}, dx_k) = p(x_k|x_{k-1})\,dx_k$
2. The observation process $Z = \{Z_k\}_k$ is conditionally independent given the process X, of marginal distribution with probability density function $p(z_k|x_k)$

Let us denote by $X_{0:k} = (X_0, \ldots, X_k)$ and by $Z_{0:k} = (Z_0, \ldots, Z_k)$.

Aim: Estimate recursively $p(x_{0:k}|z_{0:k})$ and essentially its associated feature $p(x_k|z_{0:k})$

Using Bayes' rule, we get the following recursive formula

$$p(x_{0:k+1}|z_{0:k+1}) = p(x_{0:k}|z_{0:k}) \times \frac{p(z_{k+1}|x_{k+1})p(x_{k+1}|x_k))}{p(z_{k+1}|z_{0:k})}\ .$$

This recursion is only academic in the sense that we cannot compute the densities $p(x_{0:k+1}|z_{0:k+1})$ and its marginal and we can not evaluate the associated high-dimensional integrals.

Suppose that we are able to simulate N i.i.d. random samples $\{X_{0:k}^{(i)},\ 1 \leq i \leq N\}$ according to $p(x_{0:k}|z_{0:k})$, then an empirical estimate of this distribution is given by

$$\pi_{0:k|k}(dx_{0:k}) = \frac{1}{N}\sum_{i=1}^{N} \delta_{\{X_{0:k}^{(i)}\}}(dx_{0:k})\,.$$

The rate of convergence of this method is of order greater or equal to $N^{1/2}$ and it is independent of the dimension $k \times n$, see appendix C.3. Unfortunately, it is usually impossible to sample efficiently from the posterior distribution $p(x_{0:k}|z_{0:k})$ at any time k, since it is multivariate, non standard and only known up to a proportional constant.

2.3 Bayesian model to the filtering problem

The state process: An $\mathbb{R}^n$-valued Markov process $\{X_k\,,\ k \in \mathbb{N}\}$ with a Feller transition kernel $K_k(x, dy)$

The observation process: An $\mathbb{R}^m$-valued stochastic process $\{Z_k\,,\ k \in \mathbb{N}\}$ such that

$$Z_k = h_k(X_k, W_k)\,, \qquad k \geq 1\,; \quad Z_0 = 0\,. \tag{2.2}$$

The function $h : \mathbb{N} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is a Borel measurable function such that:

1. For all $k \in \mathbb{N}$, the function $h_k(\cdot,\cdot)$ is continuous on $\mathbb{R}^n \times \mathbb{R}^m$.
2. For all $(k, x) \in \mathbb{N} \times \mathbb{R}^n$, $h_k(x,\cdot)$ is a $\mathcal{C}^1$-diffeomorphism of $\mathbb{R}^m$. We denote $h_k^{-1}(x,\cdot)$ its inverse.
3. For all $(k, x) \in \mathbb{N} \times \mathbb{R}^n$,

$$\Phi_k(x, z) = |\det(\frac{\partial}{\partial z} h_k^{-1}(x, z))| > 0\,, \quad \forall\, z \in \mathbb{R}^m\,.$$

4. For all $(k, z) \in \mathbb{N} \times \mathbb{R}^m$, $\Phi_k(\cdot, z)$ is bounded continuous.

The random vectors $W_k : \Omega \to \mathbb{R}^m$ are independent, independent of the process X with laws absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}^m$ and the densities $g_k(\cdot)$ of W_k are bounded and continuous.

Remark 2.3.1. If the noise in the observation process is simply additive, ie. $Z_k = h_k(X_k) + W_k$, then 2., 3. and 4. hold and $\Phi_k(x, z) = 1$ for all $z \in \mathbb{R}^m$.

Let $0 \leq i \leq j < \infty$, denoting by $X_{i:j} = (X_i, \ldots, X_j)$, $Z_{i:j} = (Z_i, \ldots, Z_j)$, $P_{i:j}^x$ the law of $X_{i:j}$ on $(\mathbb{R}^n)^{j-i+1}$ and $P_{i:j}^z$ the law of $Z_{i:j}$ on $(\mathbb{R}^m)^{j-i+1}$. The filtering problem consists of computing the conditional distribution of the state given the observations from time 0 to the current time k, denoted by $\pi_{k|k}$, that is

$$\begin{aligned} \pi_{k|k}(A) &= P(X_k \in A \big| Z_{0:k} = z_{0:k})\,, \quad \forall A \in B(\mathbb{R}^n) \\ \pi_{k|k}\varphi &= E[\varphi(X_k) \big| Z_{0:k} = z_{0:k}]\,, \quad \forall \varphi \in \mathcal{B}(\mathbb{R}^n) \end{aligned}$$

where $z_{0:k} = (z_0, \ldots, z_k) \in (\mathbb{R}^m)^{k+1}$ is a generic path in the space of paths of the observation process from time 0 to time k. The predicted conditional probability measure $\pi_{k|k-1}$ is

$$\begin{aligned} \pi_{k|k-1}(A) &= P(X_k \in A \big| Z_{0:k-1} = z_{0:k-1}) \,, \\ \pi_{k|k-1}\varphi &= E[\varphi(X_k)|Z_{0:k-1} = z_{0:k-1}] \,. \end{aligned}$$

The main result of this Section is a recursion relation satisfied by $\pi_{k|k}$ and $\pi_{k|k-1}$. It represents the basis for the optimal Bayesian solution of the filtering problem.

Theorem 2.3.2. *For a fixed path of the observation process from time* 0 *to time* k, $Z_{0:k} = z_{0:k} \in (\mathbb{R}^m)^{k+1}$, *we have*

Prediction

$$\pi_{k+1|k} = \pi_{k|k} K_k \tag{2.3}$$

Update

$$\frac{d\pi_{k+1|k+1}}{d\pi_{k+1|k}} = \frac{\tilde{g}_{k+1}}{\int_{\mathbb{R}^n} \tilde{g}_{k+1}(x)\pi_{k+1|k}(dx)} \tag{2.4}$$

where the function $\tilde{g}_k \in C_b(\mathbb{R}^n)$ *defined by* $\tilde{g}_k(\cdot) = g_k(h_k^{-1}(\cdot, z_k))\Phi_k(\cdot, z_k)$.

Remark 2.3.3. 1. Since $Z_0 = 0$ then $\pi_{0|0} = \pi_0$, where π_0 is the law of X_0.
2. It is easy to see that for any measurable function $\varphi : \mathbb{R}^n \to \mathbb{R}$, the recursion relations (2.3) and (2.4) imply

$$\begin{aligned} \big(\pi_{k+1|k}, \varphi\big) &= \big(\pi_{k|k}, K_k\varphi\big) \qquad \text{Prediction} \,. & (2.5) \\ \big(\pi_{k|k}, \varphi\big) &= \big(\pi_{k|k-1}, \tilde{g}_k\big)^{-1}\big(\pi_{k|k-1}, \tilde{g}_k\varphi\big) \qquad \text{Update} \,. & (2.6) \end{aligned}$$

For the proof of Theorem 2.3.2 we need the following Lemma. Let λ the Lebesgue measure on $(\mathbb{R}^m)^{j-i+1}$, then

Lemma 2.3.4. $P^z_{i:j}$ *is absolutely continuous w.r.t.* λ *and its Radon-Nikodym derivative is given by*

$$\frac{dP^z_{i:j}}{d\lambda}(z_{i:j}) = \int_{(\mathbb{R}^n)^{j-i+1}} \Pi_{k=i}^{j}\, g_k(h_k^{-1}(x_k, z_k))\Phi_k(x_k, z_k) P^x_{i:j}(dx_{i:j}) \,. \tag{2.7}$$

where $dx_{i:j} = dx_i \times \cdots \times dx_j$.

Proof. Let $C_{i:j} = C_i \times \cdots \times C_j$, where $C_i, \ldots, C_j$ are arbitrary in $B(\mathbb{R}^m)$. Using the property (B.1) of the conditional expectation, we get

$$\begin{aligned} P^z_{i:j}(C_{i:j}) &= P(\{Z_{i:j} \in C_{i:j}\}) \\ &= \int_{(\mathbb{R}^n)^{j-i+1}} P(Z_{i:j} \in C_{i:j} \big| X_{i:j} = x_{i:j}) P^x_{i:j}(dx_{i:j}) \,. \end{aligned} \tag{2.8}$$

But the processes X and W are independent and the W_k's are independent, then

$$\begin{aligned} P(Z_{i:j} \in C_{i:j} \big| X_{i:j} = x_{i:j}) &= E[\Pi_{k=i}^{j} \mathbf{I}_{C_k}(h_k(X_k, W_k)) \big| X_{i:j} = x_{i:j}] \\ &= E[\Pi_{k=i}^{j} \mathbf{I}_{C_k}(h_k(x_k, W_k))] \\ &= \Pi_{k=i}^{j} E[\mathbf{I}_{C_k}(h_k(x_k, W_k))] \\ &= \Pi_{k=i}^{j} \int_{C_k} g_k(h_k^{-1}(x_k, z_k)) \Phi_k(x_k, z_k) dz_k \,. \end{aligned} \tag{2.9}$$

where $\mathbf{I}_C$ is the characteristic function of the Borel set C. The equality (2.9) is obtained by a direct application of Theorem 2.2.7. in [40].
We substitute (2.9) and (2.8) and we apply Fubini's theorem we obtain (2.7). □

Proof. of Theorem 2.3.2
Since $X_{0:k+1}$ and $W_{0:k}$ are independent, then for all $\varphi \in \mathcal{B}(\mathbb{R}^n)$:

$$K_k \varphi(X_k) = E[\varphi(X_{k+1}) \big| \mathcal{F}_k^x] = E[\varphi(X_{k+1}) \big| \sigma(\mathcal{F}_k^x \,,\, \sigma(W_{0:k}))] \,.$$

But $Z_{0:k}$ is $\sigma(\mathcal{F}_k^x \,,\, \sigma(W_{0:k}))$-measurable, then

$$\begin{aligned} E[\varphi(X_{k+1}) \big| Z_{0:k}] &= E\big[E[\varphi(X_{k+1}) \big| \sigma(\mathcal{F}_k^x \,,\, \sigma(W_{0:k})] \big| Z_{0:k}\big] \\ &= E[K_k \varphi(X_k) \big| Z_{0:k}] \end{aligned}$$

which implies the first equality (2.3).
Using Lemma 2.3.4, we obtain the second equality (2.4) if we prove that for all $A \in B(\mathbb{R}^n)$

$$\pi_{k|k}(A) = \frac{\int_A \tilde{g}_k(x_k) \pi_{k|k-1}(dx_k)}{\int_{\mathbb{R}^n} \tilde{g}_k(x) \pi_{k|k-1}(dx)} \quad P_{0:k}^z\text{-a.s.} \tag{2.10}$$

Let $C_{0:k} = C_0 \times \cdots \times C_k$, where $C_0, \ldots, C_k$ are arbitrary in $B(\mathbb{R}^m)$. The property (B.1) of the conditional probability implies

$$\begin{aligned} \int_{C_{0:k}} \pi_{k|k}(A) P_{0:k}^z(dz_{0:k}) &= \int_{C_{0:k}} P\big(X_k \in A \big| Z_{0:k} = z_{0:k}\big) P_{0:k}^z(dz_{0:k}) \\ &= P\big(\{X_k \in A\} \cap \{Z_{0:k} \in C_{0:k}\}\big) \,. \end{aligned} \tag{2.11}$$

Then, it is sufficient to prove the following:

$$\int_{C_{0:k}} \frac{\int_A \tilde{g}_k(x_k) \pi_{k|k-1}(dx_k)}{\int_{\mathbb{R}^n} \tilde{g}_k(x) \pi_{k|k-1}(dx)} P_{0:k}^z(dz_{0:k}) = P(\{X_k \in A\} \cap \{Z_{0:k} \in C_{0:k}\}) \,. \tag{2.12}$$

Let us denote by $P_{k,0:k-1}^{x,z}(dx_k dz_{0:k-1})$ the joint pdf of X_k and $Z_{0:k-1}$. We claim the following identities:

a. $P\big(Z_k \in C_k \big| X_k = x_k,\, Z_{0:k-1} = z_{0:k-1}\big) = \int_{C_k} \tilde{g}_k(x_k) dz_k$

b. $P_{k,0:k-1}^{x,z}(dx_k dz_{0:k-1}) = \pi_{k|k-1}(dx_k) P_{0:k-1}^z(dz_{0:k-1})$

c. $P_{0:k}^z(dz_{0:k}) = \int_{\mathbb{R}^n} \tilde{g}_k(x_k) \pi_{k|k-1}(dx_k) dz_k P_{0:k-1}^z(dz_{0:k-1})$

These equalities will be proved here later on.
Let us denote

$$\Delta = \int_{C_{0:k}} \frac{\int_A \tilde{g}_k(x_k)\pi_{k|k-1}(dx_k)}{\int_{\mathbb{R}^n} \tilde{g}_k(x)\pi_{k|k-1}(dx)} P^z_{0:k}(dz_{0:k})\,.$$

If we use together the identities a., b. and c. we get

$$\begin{aligned}
\Delta &= \int_{C_{0:k}} \int_A \tilde{g}_k(x_k)\pi_{k|k-1}(dx_k)dz_k P^z_{0:k-1}(dz_{0:k-1}) \\
&= \int_{A\times C_{0:k-1}} \Big(\int_{C_k} \tilde{g}_k(x_k)dz_k\Big)\pi_{k|k-1}(dx_k)P^z_{0:k-1}(dz_{0:k-1}) \\
&= \int_{A\times C_{0:k-1}} P\big(Z_k \in C_k \big| X_k = x_k,\, Z_{0:k-1} = z_{0_k-1}\big) P^{x,z}_{k,0:k-1}(dx_k dz_{0:k-1}) \\
&= P(\{X_k \in A\} \cap \{Z_{0:k} \in C_{0:k}\})\,.
\end{aligned}$$

Finally, let us show the identities a., b. and c.
Since $\sigma(X_k, Z_{0:k-1} \subset \sigma(X_{0:k}, W_{0:k-1})$, we obtain

$$\begin{aligned}
P\big(Z_k \in C_k \big| X_k,\, Z_{0:k-1}\big) &= E[\mathbf{I}_{C_k}(Z_k)\big| X_k,\, Z_{0:k-1}] \\
&= E\big[P(Z_k \in C_k \big| X_{0:k},\, W_{0:k-1})\big| X_k,\, Z_{0:k-1}\big]\,. \qquad (2.13)
\end{aligned}$$

But Z_k and $W_{0:k-1}$ are independent, then

$$\begin{aligned}
P\big(Z_k \in C_k \big| X_{0:k},\, W_{0:k-1}\big) &= P\big(Z_k \in C_k \big| X_{0:k}\big) \\
&= P\big(Z_{0:k} \in (\mathbb{R}^m)^k \times C_k \big| X_{0:k}\big) \\
&= \int_{C_k} \tilde{g}_k(X_k)dz_k\,. \qquad (2.14)
\end{aligned}$$

From (2.13) and (2.14) we derive

$$P\big(Z_k \in C_k \big| X_k,\, Z_{0:k-1}\big) = \int_{C_k} \tilde{g}_k(X_k)dz_k\,.$$

This proves a.
For all $A \in B(\mathbb{R}^n)$, let us denote $\Psi(A) = P\big((X_k, Z_{0:k-1}) \in A \times C_{0:k-1}\big)$. Then,

$$\begin{aligned}
\Psi(A) &= \int_{C_{0:k-1}} P\big(X_k \in A \big| Z_{0:k-1} = z_{0:k-1}\big) P^z_{0:k-1}(dz_{0:k-1}) \\
&= \int_{A\times C_{0:k-1}} \pi_{k|k-1}(dx_k)P^z_{0:k-1}(dz_{0:k-1})\,,
\end{aligned}$$

This gives the equality b.
We use equality b. to prove the equality c. In fact, we have

$$\begin{aligned}
P^z_{0:k}(C_{0:k}) &= P(\{Z_k \in C_k\} \cap \{X_k \in \mathbb{R}^n\} \cap \{Z_{0:k-1} \in C_{0:k-1}\}) \\
&= \int P\big(Z_k \in C_k \big| X_k = x_k,\, Z_{0:k-1} = z_{0:k-1}\big) P^{x,z}_{k/0:k-1}(dx_k dz_{0:k-1}) \\
&= \int_{\mathbb{R}^n \times C_{0:k-1}} \int_{C_k} \tilde{g}_k(x_k)dz_k \pi_{k|k-1}(dx_k)P^z_{0:k-1}(dz_{0:k-1}) \\
&= \int_{C_{0:k}} \int_{\mathbb{R}^n} \tilde{g}_k(x_k)\pi_{k|k-1}(dx_k)dz_k P^z_{0:k-1}(dz_{0:k-1})\,. \qquad (2.15)
\end{aligned}$$

The equality c. is expressed by (2.15). □

Remark 2.3.5. Similar results and proofs was given in [1] in the case of additive noise in the observation equation.

2.4 Particle filter methods

Computation of the posterior density $\pi_{k|k}(dx_k)$ is in general an exercise in high dimension numerical integration and it seems better to design iterative schemes imitating the exact one, see Theorem 2.3.2. The particle filter methods proceed in this manner. These methods are very flexible, easy to implement, parallelizable and applicable in very general setting. More precisely, a particle filter method is a recursive algorithm based on the equations (2.3) and (2.4) that produces, at each time k, a cloud of particles. The empirical measure associated to these particles converges in some sense to the distribution $\pi_{k|k}$ as the number of particles growths. At time k, the algorithm generates N particles $\{X_k^{(i)}\}_{1\leq i\leq N}$ with an associated empirical measure

$$\pi_{k|k}^N(dx_k) = \frac{1}{N}\sum_{i=1}^N \delta_{\{X_k^{(i)}\}}(dx_k)$$

where $\delta_{\{x\}}(dx_k)$ denotes the delta-Dirac measure at the point x.
The algorithm is recursive in the sense that $\{X_k^{(i)}\}_{1\leq i\leq N}$ are produced using the observation at time k and the particles $\{X_{k-1}^{(i)}\}_{1\leq i\leq N}$ produced at time $k-1$. Recall that Theorem 2.3.2 states that

Prediction

$$\pi_{k|k-1} = \pi_{k-1|k-1}K_{k-1} \tag{2.16}$$

Update

$$\frac{d\pi_{k|k}}{d\pi_{k|k-1}} = \frac{\tilde{g}_k}{\int_{\mathbb{R}^n}\tilde{g}_k(x)\pi_{k|k-1}(dx)} \tag{2.17}$$

Then, intuitively we follow these recursions. Suppose that a set of particles $\{X_{k-1}^{(i)}\}_{1\leq i\leq N}$ distributed approximately according to $\pi_{k-1|k-1}(dx_{k-1})$ is given, then the associated empirical measure

$$\pi_{k-1|k-1}^N(dx_{k-1}) = \frac{1}{N}\sum_{i=1}^N \delta_{\{X_{k-1}^{(i)}\}}(dx_{k-1})\,.$$

is an approximation to $\pi_{k-1|k-1}$. In the prediction stage, we sample N particles:

$$\tilde{X}_k^{(i)} \;\sim\; K_{k-1}(X_{k-1}^{(i)}, dx_k)\,, \quad 1\leq i\leq N\,. \tag{2.18}$$

Remark 2.4.1. In the prediction stage, instead of sampling from $K_{k-1}(X_{k-1}^{(i)}, dx_k)$, Crisan and Doucet in [2] propose to sample from

$$K_{k-1}\pi_{k-1|k-1}^N(dx_k) = \frac{1}{N}\sum_{j=1}^{N} K_{k-1}(X_{k-1}^{(j)}, dx_k)\,.$$

The new particles $\{\tilde{X}_k^{(i)}\}_{1\leq i\leq N}$ approximate $\pi_{k|k-1}(dx_k)$ ie. the associated empirical distribution

$$\pi_{k|k-1}^N(dx_k) = \frac{1}{N}\sum_{i=1}^{N} \delta_{\{\tilde{X}_k^{(i)}\}}(dx_k) \tag{2.19}$$

is an approximation to $\pi_{k|k-1}$. Updating means that one plugs the measure (2.19) in the Update stage equation (2.17) to get $\tilde{\pi}_{k|k}^N(dx_k)$ the Monte Carlo approximation of $\pi_{k|k}(dx_{k-1})$, then

$$\begin{aligned}
\tilde{\pi}_{k|k}^N(dx_k) &= \frac{\tilde{g}_k(x_k)\pi_{k|k-1}^N(dx_k)}{\int_{\mathbb{R}^n}\tilde{g}_k(x_k)\pi_{k|k-1}^N(dx_k)} \\
&= \frac{\sum_{i=1}^N \tilde{g}_k(\tilde{X}_k^{(i)})\delta_{\{\tilde{X}_k^{(i)}\}}(dx_k)}{\sum_{i=1}^N \tilde{g}_k(\tilde{X}_k^{(i)})} \\
&= \sum_{i=1}^{N} w_k^{*(i)}\delta_{\{\tilde{X}_k^{(i)}\}}(dx_k)
\end{aligned}$$

where $w_k^{*(i)} = \frac{\tilde{g}_k(\tilde{X}_k^{(i)})}{\sum_{i=1}^N \tilde{g}_k(\tilde{X}_k^{(i)})} \propto \tilde{g}_k(\tilde{X}_k^{(i)})$ are the so-called importance weights. The measure $\tilde{\pi}_{k|k}^N$ is a weighted empirical measure approximation of $\pi_{k|k}$. The aim of a particle filter method is to obtain an unweighted empirical measure approximation of the form

$$\pi_{k|k}^N(dx_k) = \frac{1}{N}\sum_{i=1}^{N} \delta_{\{X_k^{(i)}\}}(dx_k)\,. \tag{2.20}$$

A resampling stage is used for duplicating the particles $\tilde{X}_k^{(i)}$ having high weights and discarding the others to focus on the zones of high posterior probabilities. This is can be achieved for example by sampling N times from the weighted empirical measure $\tilde{\pi}_{k|k}^N(dx_k)$. In fact, it generates $N_k^{(i)}$ copies of the ith particle and the $N_k^{(i)}$'s are distributed according to a multinomial distribution with parameters $(N; w_k^{*(1)}, \ldots, w_k^{*(N)})$. Consequently, the total number of particles alive during the system evolution don't change from a generation to another. Moreover,

$$E[N_k^{(i)}] = Nw_k^{*(i)} \quad \text{and} \quad var(N_k^{(i)}) = Nw_k^{*(i)}(1 - w_k^{*(i)})\,. \tag{2.21}$$

The variances $var(N_k^{(i)})$'s are referred to us as the Monte Carlo variations of the resampling stage. We summarize with the particle filter algorithm

Particle Filter Algorithm

Initialization stage: Sample N_0 particles $X_0^{(i)} \sim \pi_0 = \pi_{0|0}$.

$$\pi_{0|0}^N(dx_0) = \frac{1}{N_0}\sum_{i=1}^{N_0} \delta_{\{X_0^{(i)}\}}(dx_0)\,.$$

Prediction stage: At time $k-1$ ($k \geq 1$), Sample N particles $\tilde{X}_k^{(i)} \sim K_{k-1}(X_{k-1}^{(i)}, dx_k)$ to get an approximation to $\pi_{k|k-1}$

$$\tilde{\pi}_{k|k-1}^N(dx_k) = \frac{1}{N_{k-1}}\sum_{i=1}^{N_{k-1}} \delta_{\{\tilde{X}_k^{(i)}\}}(dx_k)\,.$$

Update stage: For $1 \leq i \leq N$ calculating the weights $w_k^{*(i)} \propto \tilde{g}_k(\tilde{X}_k^{(i)})$. The weighted empirical distribution approximation to $\pi_{k|k}$ is

$$\tilde{\pi}_{k|k}^N(dx_k) = \sum_{i=1}^{N} w_k^{*(i)} \delta_{\{\tilde{X}_k^{(i)}\}}(dx_k)\,.$$

Resampling stage: Sample N particles $X_k^{(i)} \sim \tilde{\pi}_{k|k}^N$. The unweighted empirical distribution approximation to $\pi_{k|k}$ is

$$\pi_{k|k}^N(dx_k) = \tfrac{1}{N_k}\textstyle\sum_{i=1}^{N_k} \delta_{\{X_k^{(i)}\}}(dx_k)\,.$$

Example 2.4.2. Let us consider the following nonlinear model [23, 49]:

State: $X_{k+1} = \frac{1}{2}X_k + 25X_k/(1+X_k^2) + 8\cos(1.2k) + U_k$

Observation: $Z_k = X_k^2/20 + W_k$

where $X_0 \sim N(0,10)$, $U_k \sim N(0,10)$ and $W_k \sim N(0,1)$.
Figure 2.1: We use the particle filter (PF) method with 60 particles to estimate a path of the process X. The PF estimate is better than the EKF estimate.
Figure 2.2: A particle filter method provides estimations of the posterior filtered distribution $p(X(k)|Z(0:k))$. This can be not given by the EKF method for such nonlinear system.

The particle filter algorithm is very intuitive and easy to implement, but unfortunately suffers from numerous drawbacks:

1. In the prediction stage: Some times it is difficult or impossible to sample efficiently from the density $K_k(x, dx)$ in (2.18)

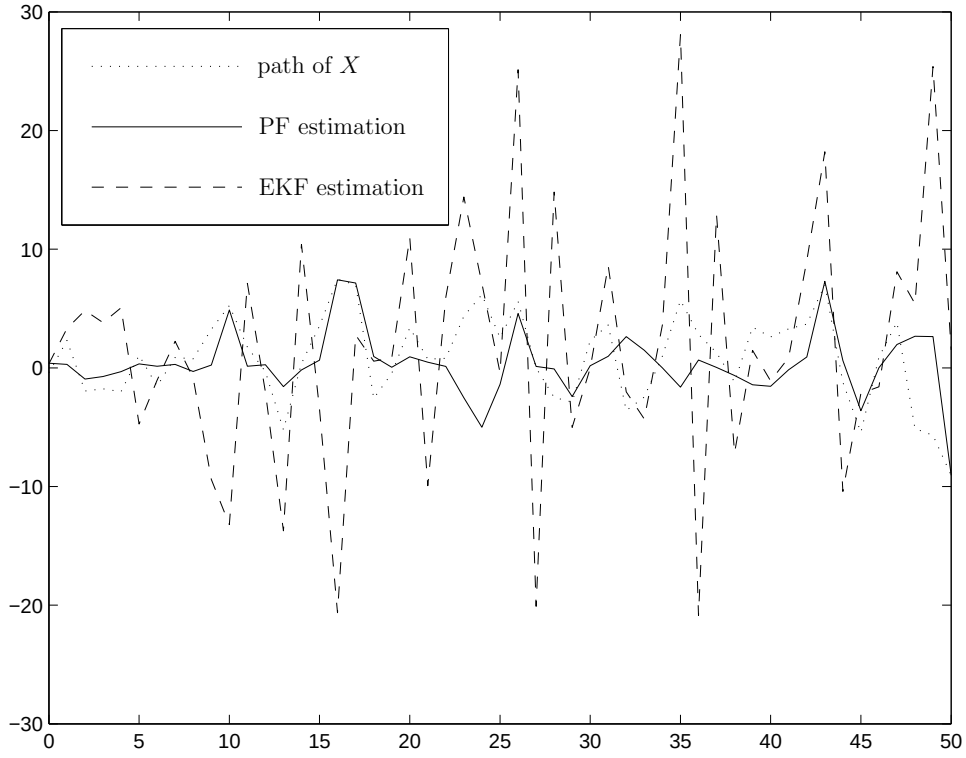

Figure 2.1: A comparison between the EKF and the PF methods in estimating a path of a process X.

2. In the update stage: The degeneracy phenomenon that comes up in practice. Since the weights $w_k^{*(i)} \propto \tilde{g}_k(\tilde{X}_k^{(i)})$, this phenomenon happen when the function $\tilde{g}_k$ generates all but few importance weights very close to zero, see figure 2.3. Then, essentially only few particles will be duplicated and the others discarded and we have no diversity in the next generations

3. In the resampling stage: A multinomial branching mechanism is used. This can introduce a large Monte Carlo variation, see (2.21)

To avoid the problem of sampling directly from $K_{k-1}(x_{k-1}, dx_k)$ in (2.18) and reduce the degeneracy phenomenon we sample the particles $\tilde{X}_k^{(i)}$ from a new kernel $\tilde{K}_{k-1}(X_{k-1}^{(i)}, dx_k)$ instead of $K_{k-1}(X_{k-1}^{(i)}, dx_k)$.
In the resampling stage, others branching mechanisms will be proposed. These mechanisms perform lowers variances, see (2.21), and reduce the degeneracy phenomenon. A branching mechanism depends on the weight of each particle and on the past trajectories of all the particles. This causes the total number of particles N to be time dependent, ie. $N = N_k$.

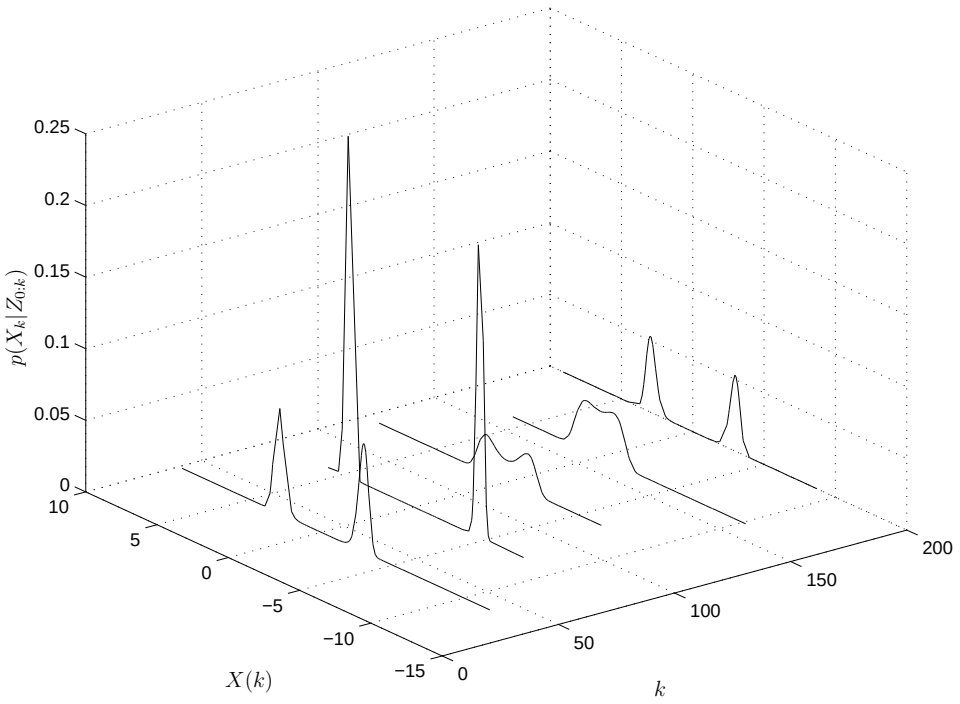

Figure 2.2: The Posterior distribution $p(X_k|Z_{0:k})$ using 160 particles

New prediction stage: Let us denote

$$\tilde{\mathcal{F}}_{k-1} = \sigma\{N_0\,, \ldots,\, N_{k-1}\,;\, \bigcup_{j=0}^{k-1}\{X_j^{(i)}\}_{1\le i\le N_j}\,;\, \bigcup_{j=1}^{k-1}\{\tilde{X}_j^{(i)}\}_{1\le i\le N_j}\}\,. \tag{2.22}$$

We sample $N = N_{k-1}$ independent particles $\{\tilde{X}_k^{(i)}\}_i$ given $\tilde{\mathcal{F}}_{k-1}$ as

$$\tilde{X}_k^{(i)} \;\sim\; \tilde{K}_{k-1}(X_{k-1}^{(i)}, dx_k)\,, \quad 1 \le i \le N\,. \tag{2.23}$$

In particular, for all $\varphi \in \mathcal{B}(\mathbb{R}^n)$,

$$E[\varphi(\tilde{X}_k^{(i)})\big|\tilde{\mathcal{F}}_{k-1}] = \int_{\mathbb{R}^n} \varphi(x_k)\tilde{K}_{k-1}(X_{k-1}^{(i)}, dx_k)\,. \tag{2.24}$$

The new transition kernels $(\tilde{K}_k)_k$ are chosen such that:

i. We can easily sample from $\tilde{K}_{k-1}(x_{k-1}, dx_k)$ for all $x_{k-1} \in \mathbb{R}^n$.
ii. The new weights produced by the new kernels have better properties.

Remark 2.4.3. Under general conditions, several choices on the kernels $(\tilde{K}_k)_k$ will be discussed in the Section 2.2.

We denote by $\tilde{\pi}^N_{k|k-1}$ the empirical measure associated with the particles $\{\tilde{X}_k^{(i)}\}$, ie.

$$\tilde{\pi}^N_{k|k-1}(dx_k) = \frac{1}{N}\sum_{i=1}^{N} \delta_{\{\tilde{X}_k^{(i)}\}}(dx_k) \tag{2.25}$$

$\tilde{\pi}^N_{k|k-1}$ is an approximation to $\pi_{k-1|k-1}\tilde{K}_{k-1}$.

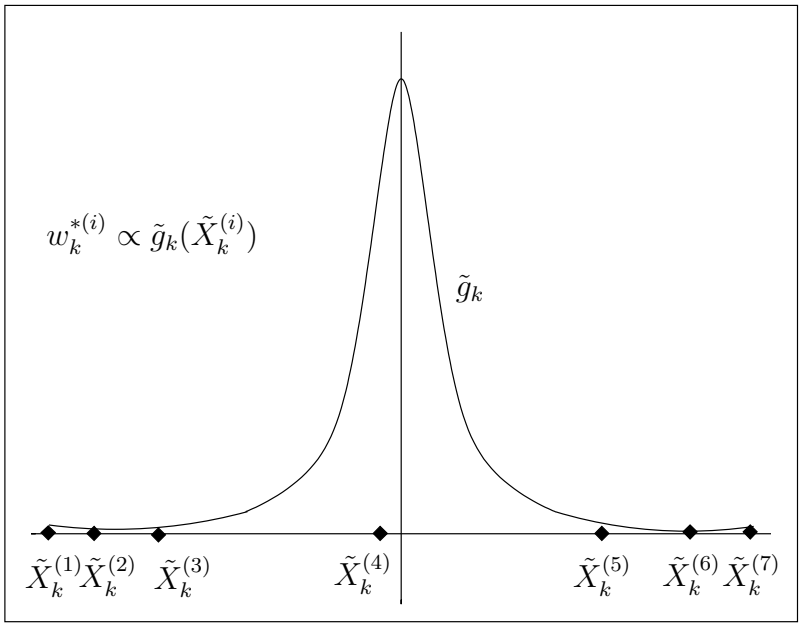

Figure 2.3: $\tilde{X}_k^{(4)}$ has a high weight while the other have weights close to zero.

New update stage: Let $\varphi \in \mathcal{B}(\mathbb{R}^n)$. The two equations (2.16) and (2.17) give us the following:
First,

$$\begin{aligned} \big(\pi_{k|k}, \varphi\big) &= \int_{\mathbb{R}^n} \varphi(x_t)\pi_{k|k}(dx_k) \\ &= \frac{\int_{\mathbb{R}^n} \varphi(x_t)\tilde{g}_k(x_k)\pi_{k|k-1}(dx_k)}{\int_{\mathbb{R}^n} \tilde{g}_k(x_k)\pi_{k|k-1}(dx_k)} \end{aligned} \tag{2.26}$$

Second, if we denote by $I = int_{\mathbb{R}^n}\varphi(x_t)\tilde{g}_k(x_k)\pi_{k|k-1}(dx_k)$, then

$$\begin{aligned} I &= \int_{\mathbb{R}^n} \varphi(x_t)\tilde{g}_k(x_k) \int_{\mathbb{R}^n} K_{k-1}(x_{k-1}, dx_k)\pi_{k-1|k-1}(dx_{k-1}) \\ &= \int_{\mathbb{R}^n} \varphi(x_t) \int_{\mathbb{R}^n} \frac{\tilde{g}_k(x_k)K_{k-1}(x_{k-1}, dx_k)}{\tilde{K}_{k-1}(x_{k-1}, dx_k)} \tilde{K}_{k-1}(x_{k-1}, dx_k)\pi_{k-1|k-1}(dx_{k-1}) \\ &= \int_{(\mathbb{R}^n)^2} \varphi(x_t)\tilde{w}_k(x_{k-1}, x_k)\tilde{K}_{k-1}(x_{k-1}, dx_k)\pi_{k-1|k-1}(dx_{k-1}) \end{aligned} \tag{2.27}$$

where

$$\tilde{w}_k(x_{k-1}, x_k) = \frac{\tilde{g}_k(x_k)K_{k-1}(x_{k-1}, dx_k)}{\tilde{K}_{k-1}(x_{k-1}, dx_k)} \ . \tag{2.28}$$

The kernel function $\tilde{K}_{k-1}$ may depend on the observations $Z_{0:k}$, but not on the Z_i's for $i > k$. Moreover, using Assumption 2.1,

$$\tilde{K}_{k-1}(x_{k-1}, dx_k) = \tilde{p}(x_k|x_{k-1}, z_{0:k})dx_k \ .$$

The functions $\{\tilde{p}(x_k|x_{k-1}, z_{0:k})\}_k$ are called importance sampling functions. Choosing a kernel $\tilde{K}_{k-1}$ is equivalent to choose an importance function $\tilde{p}(x_k|x_{k-1}, z_{0:k})$. In particular,

$$\tilde{w}_k(x_{k-1}, x_k) = \frac{\tilde{g}_k(x_k)p(x_k|x_{k-1})}{\tilde{p}(x_k|x_{k-1}, z_{0:k})}\,. \tag{2.29}$$

Plugging (2.27) in (2.26) where we take $\varphi \equiv 1$ for the inverse term in (2.26). Then, an approximation to $\big(\pi_{k|k}, \varphi\big)$ is given by

$$\big(\pi_{k|k}, \varphi\big) \approx \sum_{i=1}^{N} w_k^{(i)} \varphi(\tilde{X}_k^{(i)})$$

where, for all $1 \leq i \leq N$, $w_k^{(i)}$ is the normalized weight of the particle $\tilde{X}_k^{(i)}$ given by

$$w_k^{(i)} = \frac{\tilde{w}_k(X_{k-1}^{(i)}, \tilde{X}_k^{(i)})}{\sum_{j=1}^{N} \tilde{w}_k(X_{k-1}^{(j)}, \tilde{X}_k^{(j)})}\,. \tag{2.30}$$

An approximation to $\pi_{k|k}$ is given by

$$\tilde{\pi}_{k|k}^{N}(dx_k) = \sum_{i=1}^{N} w_k^{(i)} \delta_{\{\tilde{X}_k^{(i)}\}}(dx_k)\ . \tag{2.31}$$

New resampling stage: Let us denote by

$$\mathcal{F}_{k-1} = \sigma\{N_0\,, \ldots,\, N_{k-1}\,;\, \bigcup_{j=0}^{k-1} \{X_j^{(i)}\}_{1\leq i\leq N_j}\,;\, \bigcup_{j=1}^{k} \{\tilde{X}_j^{(i)}\}_{1\leq i\leq N_j}\} \tag{2.32}$$

If $N_k^{(i)}$ is the number of offsprings of the particle $\tilde{X}_k^{(i)}$, the branching mechanism is generally chosen such that for some constant $C > 0$ and for all $\varphi \in C_b(\mathbb{R}^n)$

$$E[N_k^{(i)}|\mathcal{F}_{k-1}] \;=\; w_k^{(i)} N_{k-1}\,, \tag{2.33}$$

$$E[\big|\sum_{i=1}^{N_{k-1}} N_k^{(i)} \varphi(\tilde{X}_k^{(i)}) - N_{k-1}\big(\tilde{\pi}_{k|k}^{N_{k-1}}, \varphi\big)\big|^2 \big|\mathcal{F}_{k-1}] \;\leq\; C N_{k-1} \|\varphi\|^2 \tag{2.34}$$

At the end of this stage, we obtain $N_k = \sum_{i=1}^{N_{k-1}} N_k^{(i)}$ particles $\{X_k^{(i)}\}$ indexed as follows

$$X_k^{(i)} = \tilde{X}_k^{(j)}\,, \quad 1 \leq j \leq N_{k-1}\,,\ 1 + \sum_{l=1}^{j-1} N_k^{(l)} \leq i \leq \sum_{l=1}^{j} N_k^{(l)}\,. \tag{2.35}$$

In Section 2.8, we will give several branching mechanisms.
Next, we give three properties of the evolution of the number of particles

Proposition 2.4.4. *For every $k \in \mathbb{N}$, there exist a constant $C = C(k) \geq 0$ such that*

1. $E[N_k] = N_0$
2. $P(N_k = 0) \leq C(k)N_0^{-1}$
3. $E[\,\left|\frac{N_k}{N_0} - 1\right|^2] \leq C(k)N_0^{-1}$

Proof. We have the following two properties:

i. N_{k-1} is $\mathcal{F}_{k-1}$-measurable.
ii. $N_k = \sum_{i=1}^{N_{k-1}} N_k^{(i)}$ on $\{N_{k-1} \neq 0\}$.

If $\mathbf{I}_{\{N_{k-1}\neq 0\}}$ the indicator function of the set $\{N_{k-1} \neq 0\}$, then

$$\begin{aligned} E[N_k|\mathcal{F}_{k-1}] &= \mathbf{I}_{\{N_{k-1}\neq 0\}} \sum_{i=1}^{N_{k-1}} E[N_k^{(i)}|\mathcal{F}_{k-1}] \\ &= \mathbf{I}_{\{N_{k-1}\neq 0\}} \sum_{i=1}^{N_{k-1}} w_k^{(i)} N_{k-1} \\ &= N_{k-1}\,. \end{aligned}$$

Then, N_k is an $\mathcal{F}_k$ -martingale and $E[N_k] = E[N_0] = N_0$. This proves 1.
Now, we suppose that 3. holds. For every $\epsilon > 0$, we have

$$\begin{aligned} P(N_k = 0) &\leq P(N_k \leq \epsilon N_0) \\ &\leq P(N_0 - N_k \geq (1-\epsilon)N_0) \\ &\leq \frac{1}{(1-\epsilon)^2} E[|\frac{N_k}{N_0} - 1|^2] \qquad (2.36) \\ &\leq \frac{C(k)N_0}{(1-\epsilon)^2 N_0^2} \qquad (2.37) \end{aligned}$$

where we have applied Doob's inequality in (2.36). This proves 2.
To prove 3., we take the expectation in both side in inequality (2.34) with $\varphi \equiv 1$, we get

$$E[|N_k - N_{k-1}|^2] \leq CE[N_{k-1}] = CN_0\,. \qquad (2.38)$$

In particular,

$$E[|N_1 - N_0|^2] \leq CN_0\,.$$

The property holds for $k = 1$. By induction, suppose that there exists a positive constant $C(k-1) > 0$ such that

$$E[|N_{k-1} - N_0|^2] \leq C(k-1)N_0\,.$$

This implies that

$$\begin{aligned} E[|N_k - N_0|^2] &\leq 2\{E[|N_k - N_{k-1}|^2] + E[|N_{k-1} - N_0|^2]\} \\ &\leq 2\{CE[N_{k-1}] + C(k-1)N_0\} \\ &\leq 2(C + C(k-1))N_0\,. \end{aligned}$$

It is sufficient to take $C(k) = 2(C + C(k-1))$. The proof is complete. □

To finish this section, we give in the next a generic and very general algorithm for the filtering problem.

General Particle Filter Algorithm

Initialization stage: Sample N_0 particles $X_0^{(i)} \sim \pi_0 = \pi_{0|0}$.

$$\pi_{0|0}^N(dx_0) = \frac{1}{N_0}\sum_{i=1}^{N_0}\delta_{\{X_0^{(i)}\}}(dx_0)\,.$$

Prediction stage: At time $k-1$ ($k \geq 1$), Sample N_{k-1} particles $\tilde{x}_k^{(i)} \sim \pi_{k-1|k-1}^N \tilde{K}_{k-1}(dx_k)$ according to (2.23) to get an approximation to $\pi_{k-1|k-1}\tilde{K}_{k-1}$

$$\tilde{\pi}_{k|k-1}^N(dx_k) = \frac{1}{N_{k-1}}\sum_{i=1}^{N_{k-1}}\delta_{\{\tilde{X}_k^{(i)}\}}(dx_k)\,.$$

Update stage For $1 \leq i \leq N_{k-1}$ calculating the weights $w_k^{(i)} \propto \tilde{w}(X_{k-1}^{(i)}, \tilde{X}_k^{(i)})$ w.r.t. (2.28). The weighted empirical distribution approximation to $\pi_{k|k}$ is

$$\tilde{\pi}_{k|k}^N(dx_k) = \sum_{i=1}^{N} w_k^{(i)}\delta_{\{\tilde{X}_k^{(i)}\}}(dx_k)\,.$$

Resampling stage: A fixed branching mechanism is applied to each particle $\tilde{X}_k^{(i)}$, which gives $N_k^{(i)}$ offsprings. If $N_k = \sum_{i=1}^{N_{k-1}} N_k^{(i)}$, one get by (2.35) a new set of particles $\{X_k^{(i)}\}_{1\leq i\leq N_k}$. The unweighted empirical distribution approximation to $\pi_{k|k}$ is

$$\pi_{k|k}^N(dx_k) = \tfrac{1}{N_k}\textstyle\sum_{i=1}^{N_k}\delta_{\{X_k^{(i)}\}}(dx_k)\,.$$

2.5 Generic model and convergence results

Denoting by $\mathcal{P}(\mathbb{R}^n)$ the space of probability measures on $\mathbb{R}^n$ and by $\mathcal{M}_F(\mathbb{R}^n)$ the space of finite measures on $\mathbb{R}^n$. It is clear that $\mathcal{P}(\mathbb{R}^n) \subset \mathcal{M}_F(\mathbb{R}^n)$.
The weak convergence on $\mathcal{M}_F(\mathbb{R}^n)$ is defined by

$$\{\mu_N \to \mu \quad \text{in } \mathcal{M}_F(\mathbb{R}^n)\} \Longleftrightarrow \{(\mu_N, \varphi) \to (\mu, \varphi)\,, \quad \forall\, \varphi \in C_b(\mathbb{R}^n)\}\,,$$

The weak convergence is defined similarly on $\mathcal{P}(\mathbb{R}^n)$.
Let us consider a countable family $\{\varphi_k\}_{k\geq 1} \subset C_b(\mathbb{R}^n)$ which is convergence determining, see [26], in $\mathcal{M}_F(\mathbb{R}^n)$. That is,

$$\{\mu_N \to \mu \quad \text{in } \mathcal{M}_F(\mathbb{R}^n)\} \Longleftrightarrow \{(\mu_N, \varphi_k) \to (\mu, \varphi_k)\,, \quad \forall\, k\}\,,$$

This allow us to define the distance $d(\cdot,\cdot)$ by

$$d(\nu,\mu) = \sum_{k=1}^{\infty} 2^{-k}\frac{|(\nu,\varphi_k) - (\mu,\varphi_k)|}{\|\varphi_k\|}\,,$$

The topology generated by this distance is called the weak topology. Its restriction on $\mathcal{P}(\mathbb{R}^n)$ gives the weak topology on $\mathcal{P}(\mathbb{R}^n)$. In particular, for any measure μ and sequence $\{\mu_N\}$ in $\mathcal{M}_F(\mathbb{R}^n)$ (or in $\mathcal{P}(\mathbb{R}^n)$)

$$\mu_N \to \mu \Longleftrightarrow d(\mu_N, \mu) \to 0\,.$$

A random measure is a random variable with values in $\mathcal{M}_F(\mathbb{R}^n)$. A sequence $\{\mu^{N,w}\}$ of random measures is said to be convergent to the random measure μ^ω in $\mathcal{M}_F(\mathbb{R}^n)$ a.s. if for almost every $w \in \Omega$

$$(\mu^{N,w}, \varphi) \to (\mu^\omega, \varphi) \quad \text{as} \quad N \to \infty\,, \quad \forall\, \varphi \in C_b(\mathbb{R}^n)$$

or, equivalently

$$d(\mu^{N,w}, \mu^\omega) \to 0 \quad \text{as} \quad N \to \infty \quad \text{a.s.}$$

2.5.1 Prediction-Update

Consider a kernel $K(\cdot,\cdot)$ on $\mathbb{R}^n \times B(\mathbb{R}^n)$ satisfying the following properties:

Properties 2.5.1. i. For all $x \in \mathbb{R}^n$, $K(x,\cdot) \in \mathcal{P}(\mathbb{R}^n)$

ii. For all $A \in B(\mathbb{R}^n)$, $K(\cdot, A)$ is a bounded Borel function

iii. For all $\varphi \in C_b(\mathbb{R}^n)$, $K\varphi \in C_b(\mathbb{R}^n)$
where $K\varphi(x) = \int_{\mathbb{R}^n} \varphi(y) K(x, dy)$

Let us fix a probability measure $\mu \in \mathcal{P}(\mathbb{R}^n)$. We define the probability ν on $\mathbb{R}^n$ by

$$(\nu, \varphi) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \varphi(x) K(y, dx) \mu(dy)\,, \quad \forall \varphi \in C_b(\mathbb{R}^n)\,.$$

Suppose that there exists a sequence of random vectors, not necessarily independents, $U_i : \Omega \to \mathbb{R}^n$, $i \geq 1$, such that the associated empirical measure $\mu^N(dx) = \frac{1}{N}\sum_{i=1}^N \delta_{\{U_i\}}(dx)$ converges to μ in $\mathcal{P}(\mathbb{R}^n)$ for almost every $w \in \Omega$. That is,

$$\mu^{N,w} = \frac{1}{N}\sum_{i=1}^N \delta_{\{U_i(w)\}} \longrightarrow \mu \quad \text{in } \mathcal{P}(\mathbb{R}^n) \quad \text{for almost all } \omega\,.$$

We fix a realization $\{u_i\}_{i\geq 1}$ of the sequence $\{U_i\}$, i.e. $u_i = U_i(w)$ for some $w \in \Omega$, such that

$$\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{\{u_i\}} \to \mu \quad \text{in } \mathcal{P}(\mathbb{R}^n)\,. \tag{2.39}$$

Let $V_i : \Omega \to \mathbb{R}^n\,,\ i \geq 1$ be a sequence of i.i.d. random vectors such that

$$V_i \ \sim \ K(u_i, dx)\,, \quad \text{for all } i \geq 1\,. \tag{2.40}$$

We denote by ν^N the empirical measure associated with the V_i's, ie.

$$\nu^N(dx) = \frac{1}{N}\sum_{i=1}^N \delta_{\{V_i\}}(dx)\,.$$

Remark 2.5.2. The measure μ will play the role of $\pi_{k-1|k-1}$, the kernel K the role of the kernel $\tilde{K}_{k-1}$ and ν of $\pi_{k-1|k-1}\tilde{K}_{k-1}$. The corresponding approximations μ^N and ν^N are successively associated to $\pi^N_{k-1|k-1}$ and $\tilde{\pi}^N_{k|k-1}$.

Let $W : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a continuous, bounded and strictly positive function. Define the probability measure λ on $\mathbb{R}^n$ by

$$(\lambda, \varphi) = \frac{\int_{\mathbb{R}^n \times \mathbb{R}^n} \varphi(x) W(y,x) K(y,dx)\mu(dy)}{\int_{\mathbb{R}^n \times \mathbb{R}^n} W(y,x) K(y,dx)\mu(dy)}, \quad \forall \varphi \in C_b(\mathbb{R}^n). \tag{2.41}$$

Also, we define the random weighted empirical measure $\lambda^{N,\cdot}$ on $\mathbb{R}^n$ by

$$\lambda^{N,\cdot}(dx) = \sum_{i=1}^{N} W_N^{(i)} \delta_{\{V_i\}}(dx),$$

where

$$W_N^{(i)} = \frac{W(u_i, V_i)}{\sum_{j=1}^N W(u_j, V_j)}.$$

Remark 2.5.3. The measure λ is associated to the measure $\pi_{k|k}$, t he function $W(\cdot,\cdot)$ is associated to the function $\tilde{w}(\cdot,\cdot)$, and the approximation λ^N to $\tilde{\pi}_{k|k}$.

Theorem 2.5.4. *For almost every $w \in \Omega$*

$$\lambda^{N,w} = \sum_{i=1}^{N} W_N^{(i)}(w) \delta_{\{V_i(w)\}} \longrightarrow \lambda \quad \textit{in} \quad \mathcal{P}(\mathbb{R}^n). \tag{2.42}$$

For the proof of this Theorem we need the following Lemma:

Lemma 2.5.5. *The empirical measure associated to the set $\{(V_i, u_i);\ 1 \le i \le N\}$ converges weakly almost surely to the probability measure defined on $\mathbb{R}^n \times \mathbb{R}^n$ by $K(y,dx)\mu(dy)$, ie. $\forall\ \psi(\cdot,\cdot) \in C_b(\mathbb{R}^n \times \mathbb{R}^n)$*

$$\frac{1}{N}\sum_{i=1}^{N} \psi(V_i(w), u_i) \longrightarrow \int_{(\mathbb{R}^n)^2} \psi(x,y) K(y,dx)\mu(dy) \quad \textit{as} \quad N \to \infty,$$

for almost all w in Ω.

Proof. Let $\psi(\cdot,\cdot) \in C_b(\mathbb{R}^n \times \mathbb{R}^n)$. If $\alpha(dxdy) = K(y,dx)\mu(dy)$ and $\alpha^{N,w}(dxdy) = \frac{1}{N}\sum_{i=1}^N \delta_{\{V_i(w),u_i\}}(dxdy)$, we need to prove that

$$(\alpha^{N,\omega}, \psi) - (\alpha, \psi) \to 0 \quad \text{as } N \to \infty.$$

We define the function $\varphi \in C_b(\mathbb{R}^n)$ by

$$\varphi(y) = \int_{\mathbb{R}^n} \psi(x,y) K(y,dx), \quad \forall y \in \mathbb{R}^n.$$

Then, $(\alpha, \psi) = (\mu, \varphi)$ and

$$(\alpha^{N,\omega}, \psi) - (\alpha, \psi) = (\alpha^{N,\omega}, \psi) - (\mu^N, \varphi) + (\mu^N, \varphi) - (\mu, \varphi)\,.$$

Since $(\mu^N, \varphi) \longrightarrow (\mu, \varphi) = (\alpha, \psi)$ as $N \to \infty$, it remains to prove that

$$(\alpha^{N,\omega}, \psi) - (\mu^N, \varphi) \longrightarrow 0 \text{ as } N \to \infty \; a.s.\,. \tag{2.43}$$

First, we have

$$(\mu^N, \varphi) = \frac{1}{N}\sum_{i=1}^{N} \varphi(u_i) = \frac{1}{N}\sum_{i=1}^{N} \int_{\mathbb{R}^n} \psi(x, u_i) K(u_i, dx)\,.$$

Second, from (2.40) we get

$$E[\psi(V_i, u_i)] = \int_{\mathbb{R}^n} \psi(x, u_i) K(u_i, dx)\,. \tag{2.44}$$

We conclude that

$$(\alpha^{N,\omega}, \psi) - (\mu^N, \varphi) = \frac{1}{N}\sum_{i=1}^{N} \Big(\psi(V_i, u_i) - \int_{\mathbb{R}^n} \psi(x, u_i) K(u_i, dx)\Big)$$

For simplicity we denote for all $1 \le i \le N$

$$\Delta_i\psi = \psi(V_i, u_i) - \int_{\mathbb{R}^n} \psi(x, u_i) K(u_i, dx)\,.$$

In particular, for all $1 \le i \le N$, $\big|\Delta_i\big| \le 2\|\psi\|$. Using the independency of the V_i's and the identity (2.44), we get

$$\begin{aligned}
E[\big((\mu^N, \varphi) - (\alpha^{N,\omega}, \psi)\big)^4] &= \frac{1}{N^4} E[\big(\sum_{i=1}^{N} \Delta_i\psi\big)^4] \\
&= \frac{1}{N^4}\sum_{i=1}^{N} E[\big(\Delta_i\psi\big)^4] + \frac{6}{N^4}\sum_{1\le i_1\ne i_2\le N} E[\big(\Delta_{i_1}\psi\big)^2\big(\Delta_{i_2}\psi\big)^2] \\
&\le \frac{1}{N^4} 2^4 (N + 6N(N-1))\|\psi\|^4 \\
&\le \frac{6\times 2^4}{N^2}\|\psi\|^4
\end{aligned}$$

This implies that

$$E[\sum_{N=1}^{\infty} \big((\mu^N, \varphi) - (\alpha^{N,\omega}, \psi)\big)^4] < \infty\,.$$

Then, $\sum_{N=1}^{\infty} \big((\mu^N, \varphi) - (\alpha^{N,\omega}, \psi)\big)^4 < \infty$ a.s. and for almost all ω

$$(\mu^N, \varphi) - (\alpha^{N,\omega}, \psi) \longrightarrow 0 \quad \text{as } N \to \infty\,.$$

which proves (2.43) and the proof is complete. □

Proof. (Theorem 2.5.4)
We apply Lemma 2.5.5 successively for $\psi(x,y)=\varphi(x)W(y,x)$ and $\psi(x,y)=W(y,x)$, we get almost surely as $N\to\infty$,

$$\frac{1}{N}\sum_{i=1}^{N}\varphi(V_i)W(u_i,V_i)\longrightarrow\int_{\mathbb{R}^n\times\mathbb{R}^n}\varphi(x)W(y,x)K(y,dx)\mu(dy)\,, \tag{2.45}$$

$$\frac{1}{N}\sum_{i=1}^{N}W(u_i,V_i)\longrightarrow\int_{\mathbb{R}^n\times\mathbb{R}^n}W(y,x)K(y,dx)\mu(dy)\,. \tag{2.46}$$

Since the function W is strictly positive, then for all $\varphi\in C_b(\mathbb{R}^n)$

$$\frac{\frac{1}{N}\sum_{i=1}^{N}\varphi(V_i)W(u_i,V_i)}{\frac{1}{N}\sum_{i=1}^{N}W(u_i,V_i)}\longrightarrow\frac{\int_{(\mathbb{R}^n)^2}\varphi(x)W(y,x)K(y,dx)\mu(dy)}{\int_{(\mathbb{R}^n)^2}W(y,x)K(y,dx)\mu(dy)}\,. \tag{2.47}$$

This complete the proof. □

2.5.2 Resampling

Definition 2.5.6. A probability measure $\nu\in\mathcal{P}(\mathbb{R}^n)$ satisfies the **branch property** if for any integer valued random variable $\tilde{N}$ satisfying

$$E[\tilde{N}]=N\geq 1\,,\ P(\tilde{N}=0)\leq CN^{-1}\ \text{ and }\ E[|\frac{\tilde{N}}{N}-1|^{\alpha}]\leq CN^{-1}\,, \tag{2.48}$$

where $C>0$, there exists a sequence of random variables $\xi_j:\Omega\longrightarrow\mathbb{R}^n$, $j\geq 1$, identically distributed w.r.t. ν, but not necessarily independent, such that for all $\varphi\in C_b(\mathbb{R}^n)$:

$$\begin{aligned}E[\textstyle\sum_{j=1}^{\tilde{N}}\varphi(\xi_j)]&=N\big(\nu,\varphi\big)\,,\\ E[\,\big|\textstyle\sum_{j=1}^{\tilde{N}}\varphi(\xi_j)-\tilde{N}\big(\nu,\varphi\big)\big|^2]&\leq CN\|\varphi\|^2\,.\end{aligned} \tag{2.49}$$

We correspond to the pair (ν,N) the random measure $\tilde{\nu}^{N,w}$ defined by

$$\begin{array}{rcll}\tilde{\nu}^{N,\omega}(dx) & = & \frac{1}{\tilde{N}(\omega)}\sum_{j=1}^{\tilde{N}(\omega)}\delta_{\{\xi_j(\omega)\}}(dx) & \text{if}\ \ \omega\notin\{\tilde{N}=0\}\,,\\ \tilde{\nu}^{N,w} & = & 0 & \text{if}\ \ \omega\in\{\tilde{N}=0\}\,.\end{array}$$

The random measure $\tilde{\nu}^{N,w}$ is almost surely a probability measure.
Let us fix a probability measure ν and consider a sequence $\{\nu_l\}_{l\geq 1}$ converging to ν in $\mathcal{P}(\mathbb{R}^n)$. Suppose that the both ν and $\{\nu_l\}_l$ satisfy the branch property uniformly, ie. for the same constant C.
We denote by $\tilde{\nu}_l^{N,\omega}$ the corresponding measure to the pair(ν_l,N).

Theorem 2.5.7. *Let $\{N_k\}_{k\geq 0}$ a sequence of strictly positive integers such that*

$$\sum_{k=1}^{\infty}\frac{1}{N_k}<\infty\,. \tag{2.50}$$

Then, for almost all $\omega\in\Omega$ and for all $\varphi\in C_b(\mathbb{R}^n)$,

$$(\tilde{\nu}_l^{N_k,\omega},\varphi)\longrightarrow(\nu,\varphi)\ \ as\ \ k,l\to\infty\,. \tag{2.51}$$

Proof. Since $\big(\nu_l, \varphi\big) \to \big(\nu, \varphi\big)$, it is sufficient to prove that for every $l \geq 1$

$$(\tilde{\nu}_l^{N_k,\omega}, \varphi) - (\nu_l, \varphi) \longrightarrow 0 \quad \text{as } k \to \infty\,.$$

We write $\Omega = \Omega_k \cup \bar{\Omega}_k$ where $\Omega_k = \{\tilde{N}_k \neq 0\}$. From (2.48) we get

$$P(\tilde{N}_k = 0) = P(\bar{\Omega}_k) \leq C_l \frac{1}{N_k} \quad \text{and} \quad P(\Omega_k) \geq 1 - C_l \frac{1}{N_k}\,.$$

Then,

$$\begin{aligned} E[\big|(\tilde{\nu}_l^{N_k,\cdot}, \varphi) - (\nu_l, \varphi)\big|^2] \;\leq\; & \int_{\Omega_k} \big|\frac{1}{\tilde{N}_k(\omega)} \sum_{j=1}^{\tilde{N}_k(\omega)} \varphi(\xi_{l,j}) - (\nu_l, \varphi)\big|^2 dP(\omega) \\ & + C\|\varphi\|^2 N_k^{-1}\,. \end{aligned} \tag{2.52}$$

We decompose the integral term in (2.52) as

$$\int_{\Omega_k} |\ldots|^2 dP = \int_{\Omega_k^1} |\ldots|^2 dP + \int_{\Omega_k^2} |\ldots|^2 dP = \Delta_1 + \Delta_2\,,$$

where $\Omega_k = \Omega_k^1 \cup \Omega_k^2$, $\Omega_k^1 = \{\omega \in \Omega_k\,,\ |\tilde{N}_k(\omega) - N_k| \leq \frac{N_k}{2}\}$ and $\Omega_k^2 = \Omega_k \setminus \Omega_k^1$. Since $E[(\frac{\tilde{N}_k}{N_k} - 1)^2] \leq C N_k^{-1}$, see (2.48), we obtain

$$\begin{aligned} P(\Omega_k^2) \;&\leq\; P(\{\omega\,,\ |\tilde{N}_k(\omega) - N| > \frac{N_k}{2}\}) \\ &\leq\; \frac{N_k^2}{(\frac{N_k}{2})^2} E[|\frac{\tilde{N}_k}{N_k} - 1|^2] \\ &\leq\; 2^2 C_l N_k^{-1} \end{aligned} \tag{2.53}$$

<u>The first term Δ_1:</u> For all $\omega \in \Omega_N^1$ we have $\frac{1}{N^i(\omega)} \leq \frac{2}{N}$, then

$$\begin{aligned} \Delta_1 \;&\leq\; (\frac{2}{N_k})^2 E[\big|\sum_{j=1}^{\tilde{N}_k} \varphi(\xi_{l,j}) - \tilde{N}_k\big(\nu, \varphi\big)\big|^2] \\ &\leq\; (\frac{2}{N_k})^2 \times C N_k \|\varphi\|^2 \\ &\leq\; 2^2 C_l \|\varphi\|^2 N_k^{-1}\,, \end{aligned} \tag{2.54}$$

<u>The second term Δ_2:</u> If we use (2.53), we get

$$\begin{aligned} \Delta_2 \;&=\; \int_{\Omega_k^2} |\frac{1}{\tilde{N}_k(\omega)} \sum_{j=1}^{\tilde{N}_k(\omega)} \varphi(\xi_{l,j}) - (\nu_l, \varphi)|^2 dP(\omega) \\ &\leq\; \int_{\Omega_k^2} \big(\frac{1}{\tilde{N}_k(\omega)} \sum_{j=1}^{\tilde{N}_k(\omega)} \|\varphi\| + \|\varphi\|)\big)^2 dP(\omega) \\ &\leq\; 2^4 C_l \|\varphi\|^2 N_k^{-1}\,, \end{aligned} \tag{2.55}$$

Taking together (2.52), (2.54) and (2.55) we obtain a constant $\Gamma = \Gamma(l, \varphi)$ such that

$$E[|(\tilde{\nu}_l^{N_k,\cdot}, \varphi) - (\nu_l, \varphi)|^2] \leq \Gamma N_k^{-1} . \tag{2.56}$$

It follows that

$$E[\sum_{k=1}^{\infty} |(\tilde{\nu}_l^{N_k,\cdot}, \varphi) - (\nu_l, \varphi)|^2] \leq \Gamma \sum_{k=1}^{\infty} N_k^{-1} < \infty .$$

and hence almost surely

$$\sum_{k=1}^{\infty} |(\tilde{\nu}_l^{N_k,\cdot}, \varphi) - (\nu_l, \varphi)|^2 < \infty .$$

which implies that $(\tilde{\nu}_l^{N_k,\cdot}, \varphi) - (\nu_l, \varphi) \to 0$ as $k \to \infty$. This complete the proof □

2.5.3 Application to the particle filter: almost sure convergence

We will apply Theorem 2.5.4 and Theorem 2.5.7 to obtain the almost sure convergence of the particle filter algorithm.
In the Prediction-Update stage we suppose that for all $k \geq 1$:

i. The new transition kernel $\tilde{K}_k(\cdot, \cdot)$ satisfies the conditions of properties 2.5.1

ii. The weight function $\tilde{w}_k(\cdot, \cdot)$, see (2.28), is bounded, continuous and strictly positive

Then, an application of theorem 2.5.4 gives

Proposition 2.5.8. *The sequence $\tilde{\pi}_{k|k}^N$ given in (2.31) converges to $\pi_{k|k}$ as the number of particles $N \to \infty$ almost surely.*

For every $k \geq 0$, $\tilde{\pi}_{k|k}^N$ is a weighted empirical measure. The resampling stage gives an unweighted empirical measure $\pi_{k|k}^N$. The resampling stage satisfies the conditions in Proposition 2.4.4, then the branch property holds. An application of Theorem 2.5.7 gives

Proposition 2.5.9. *Let us consider a sequence $\{N_0^j\}_{j\geq 1}$ of strictly positive integers such that*

$$\sum_{j=1}^{\infty} \frac{1}{N_0^j} < \infty .$$

If the branching mechanism satisfies the conditions (2.33) and (2.34), then for all $k \geq 0$

$$\pi_{k|k}^{N_k^j} \to \pi_{k|k} \quad as \ j \to \infty \quad a.s.$$

Remark 2.5.10. We will see in Section 2.8 that such branching mechanisms exist.

2.6 Mean square convergence of the particle filter

2.6.1 Convergence results

We still suppose that

i. The kernels $\{\tilde{K}_k(\cdot,\cdot)\}_k$ satisfy the conditions of Properties 2.5.1

ii. The weight functions $\tilde{w}_k(\cdot,\cdot)$ are bounded, continuous and strictly positive

iii. The branching mechanism satisfies the conditions (2.33) and (2.34)

The Proposition 2.5.8 and Proposition 2.5.9 show that the measure $\pi_{k|k}^{N_k}$ converges weakly to the measure $\pi_{k|k}$. Although the measures defined by the particle filter algorithm converge weakly to the right measure, we have no idea about the rate of the convergence. To this end we use the mean square convergence instead of the weak convergence.
A sequence of random measures $\{\mu^{N,\omega}\}_N$ converges in mean square to the random measure μ^ω in $\mathcal{M}_F(\mathbb{R}^n)$ (or in $\mathcal{P}(\mathbb{R}^n)$) if

$$\lim_{N\to\infty} E[\big((\mu^{N,\cdot},\varphi)-(\mu^{\cdot},\varphi)\big)^2]=0\,,\quad \forall\ \varphi\in\mathcal{B}(\mathbb{R}^n)\,.$$

For this mode of convergence we will show that, for every $k\geq 0$, the sequence of measures $\pi_{k|k}^{N_k}$ converges to the measure $\pi_{k|k}$ as the initial number of particles N_0 grows to infinity and we obtain a rate of convergence proportional to $1/N_0$. In particular, this convergence is independent of the state and the observation space dimensions.
The following two Propositions state that a mean square error proportional to $1/N_0$ is propagated while time evolves.

Proposition 2.6.1. *[Prediction-Update stage] Let us assume that there exist a constant $\Delta_{k-1}\geq 0$ such that for any $\varphi\in\mathcal{B}(\mathbb{R}^n)$*

$$E[\big((\pi_{k-1|k-1}^{N_{k-1}},\varphi)-(\pi_{k-1|k-1},\varphi)\big)^2]\leq \Delta_{k-1}\frac{\|\varphi\|^2}{N_0}\,. \tag{2.57}$$

Then, there exist a constant $\tilde{\Delta}_k\geq 0$ such that for any $\varphi\in\mathcal{B}(\mathbb{R}^n)$

$$E[\big((\tilde{\pi}_{k|k}^{\tilde{N}_k},\varphi)-(\pi_{k|k},\varphi)\big)^2]\leq \tilde{\Delta}_k\frac{\|\varphi\|^2}{N_0}\,.$$

During the Prediction-Update stage the number of particles don't change, ie. $\tilde{N}_k=N_{k-1}$.

Proposition 2.6.2. *[the Resampling stage] Let us assume that there exist a constant $\tilde{\Delta}_k\geq 0$ such that for any $\varphi\in\mathcal{B}(\mathbb{R}^n)$*

$$E[\big((\tilde{\pi}_{k|k}^{N_{k-1}},\varphi)-(\pi_{k|k},\varphi)\big)^2]\leq \tilde{\Delta}_k\frac{\|\varphi\|^2}{N_0}\,. \tag{2.58}$$

Then, there exist a constant $\Delta_k\geq 0$ such that for any $\varphi\in\mathcal{B}(\mathbb{R}^n)$

$$E[\big((\pi_{k|k}^{N_k},\varphi)-(\pi_{k|k},\varphi)\big)^2]\leq \Delta_k\frac{\|\varphi\|^2}{N_0}\,. \tag{2.59}$$

2.6.2 Proofs of Propositions 2.6.1 and 2.6.2

The Prediction-Update Stage: The number of particles $\tilde{N}_k = N_{k-1}$ don't change, for simplicity we denote it by N.
Let us consider the measures $\pi_{k-1:k|k}$ and $\pi^N_{k-1:k|k}$ on $(\mathbb{R}^n)^2$ such that for all $\phi \in \mathcal{B}(\mathbb{R}^n \times \mathbb{R}^n)$:

$$\begin{aligned}(\pi_{k-1:k|k}, \phi) &= \int_{\mathbb{R}^n \times \mathbb{R}^n} \phi(x_k, x_{k-1}) \tilde{K}_k(x_{k-1}, dx_k) \pi_{k-1|k-1}(dx_{k-1}) \\ (\pi^N_{k-1:k|k}, \phi) &= \begin{cases} \frac{1}{N}\sum_{i=1}^N \phi(\tilde{X}^{(i)}_k, X^{(i)}_{k-1}) & \text{on} \quad \{N > 0\} \\ 0 & \text{on} \quad \{N = 0\} \end{cases}\end{aligned}$$

In particular, we have

$$(\pi_{k|k}, \varphi) = \frac{(\pi_{k-1:k|k}, \varphi\tilde{w}_k)}{(\pi_{k-1:k|k}, \tilde{w}_k)} \tag{2.60}$$

$$(\tilde{\pi}^N_{k|k}, \varphi) = \frac{(\pi^N_{k-1:k|k}, \varphi\tilde{w}_k)}{(\pi^N_{k-1:k|k}, \tilde{w}_k)}, \quad \text{on} \quad \{N > 0\}. \tag{2.61}$$

We claim the following Lemma that we prove later on.

Lemma 2.6.3. *If the condition (2.57) holds, then there exists a constant $\hat{\Delta}_k \geq 0$ such that for all $\phi \in \mathcal{B}(\mathbb{R}^n \times \mathbb{R}^n)$*

$$E[\big((\pi_{k-1:k|k}, \phi) - (\pi^N_{k-1:k|k}, \phi)\big)^2] \leq \hat{\Delta}_k \frac{\|\phi\|^2}{N_0}.$$

proof of Proposition 2.6.1: For all $\varphi \in \mathcal{B}(\mathbb{R}^n)$, we have

$$\begin{aligned}(\pi_{k|k}, \varphi) - (\tilde{\pi}^N_{k|k}, \varphi) &= \frac{(\pi_{k-1:k|k}, \varphi\tilde{w}_k)}{(\pi_{k-1:k|k}, \tilde{w}_k)} - \frac{(\pi^N_{k-1:k|k}, \varphi\tilde{w}_k)}{(\pi^N_{k-1:k|k}, \tilde{w}_k)} \\ &= \frac{(\pi_{k-1:k|k}, \varphi\tilde{w}_k)}{(\pi_{k-1:k|k}, \tilde{w}_k)} - \frac{(\pi^N_{k-1:k|k}, \varphi\tilde{w}_k)}{(\pi_{k-1:k|k}, \tilde{w}_k)} + \frac{(\pi^N_{k-1:k|k}, \varphi\tilde{w}_k)}{(\pi_{k-1:k|k}, \tilde{w}_k)} \\ &\quad - \frac{(\pi^N_{k-1:k|k}, \varphi\tilde{w}_k)}{(\pi^N_{k-1:k|k}, \tilde{w}_k)} \\ &= \frac{1}{(\pi_{k-1:k|k}, \tilde{w}_k)}\big((\pi_{k-1:k|k}, \varphi\tilde{w}_k) - (\pi^N_{k-1:k|k}, \varphi\tilde{w}_k)\big) \\ &\quad + \frac{(\pi^N_{k-1:k|k}, \varphi\tilde{w}_k)}{(\pi_{k-1:k|k}, \tilde{w}_k)(\pi^N_{k-1:k|k}, \tilde{w}_k)}\big((\pi_{k-1:k|k}, \tilde{w}_k) - (\pi^N_{k-1:k|k}, \tilde{w}_k)\big).\end{aligned}$$

Then,

$$\begin{aligned}|(\pi_{k|k}, \varphi) - (\tilde{\pi}^N_{k|k}, \varphi)| &\leq \frac{1}{(\pi_{k-1:k|k}, \tilde{w}_k)}|(\pi_{k-1:k|k}, \varphi\tilde{w}_k) - (\pi^N_{k-1:k|k}, \varphi\tilde{w}_k)| \\ &\quad + \frac{\|\varphi\|}{(\pi_{k-1:k|k}, \tilde{w}_k)}|(\pi_{k-1:k|k}, \tilde{w}_k) - (\pi^N_{k-1:k|k}, \tilde{w}_k)|.\end{aligned}$$

If we denote by $\mathcal{E} = E[\big((\pi_{k|k},\varphi) - (\tilde{\pi}^N_{k|k},\varphi)\big)^2]^{1/2}$ and we use Lemma 2.6.3, we get

$$\begin{aligned}\mathcal{E} &\leq \frac{1}{(\pi_{k-1:k|k},\tilde{w}_k)} E[\big((\pi_{k-1:k|k},\varphi\tilde{w}_k) - (\pi^N_{k-1:k|k},\varphi\tilde{w}_k)\big)^2]^{1/2} \\ &\quad + \frac{\|\varphi\|}{(\pi_{k-1:k|k},\tilde{w}_k)} E[\big((\pi_{k-1:k|k},\tilde{w}_k) - (\pi^N_{k-1:k|k},\tilde{w}_k)\big)^2]^{1/2} \\ &\leq 2\frac{\sqrt{\hat{\Delta}_k}\|\varphi\| \times \|\tilde{w}_k\|}{(\pi_{k-1:k|k},\tilde{w}_k)\sqrt{N_0}} .\end{aligned}$$

The result follows by taking

$$\tilde{\Delta}_k = 4\frac{\hat{\Delta}_k\|\tilde{w}_k\|^2}{(\pi_{k-1:k|k},\tilde{w}_k)^2} .$$

□

The resampling stage:
The branching mechanism:

- Each particle $\tilde{X}^{(i)}_k$ branches to give birth to $N^{(i)}_k$ offsprings
- There exists a constant $C > 0$ such that for every $\varphi \in C_b(\mathbb{R}^n)$:

  i $E[N^{(i)}_k|\mathcal{F}_{k-1}] = w^{(i)}_k N_{k-1}$

  ii $E[\big|\sum_{i=1}^{N_{k-1}} N^{(i)}_k\varphi(\tilde{x}^{(i)}_k) - N_{k-1}\big(\tilde{\pi}^{N_{k-1}}_{k|k},\varphi\big)\big|^2|\mathcal{F}_{k-1}] \leq CN_{k-1}\|\varphi\|^2$

  where $\mathcal{F}_{k-1} = \sigma\{N_0,\ldots,N_{k-1}\,;\,\bigcup_{j=0}^{k-1}\{X^{(i)}_j\}_{1\leq i\leq N_j}\,;\,\bigcup_{j=1}^{k}\{\tilde{X}^{(i)}_j\}_{1\leq i\leq N_j}\}$

- At the end of this stage, we obtain the set $\{X^{(i)}_k\}$ consisting of $N_k = \sum_{i=1}^{N_{k-1}} N^{(i)}_k$ particles indexed as follows

$$X^{(i)}_k = \tilde{X}^{(j)}_k\,,\ 1\leq j\leq N_{k-1}\,,\ 1+\sum_{l=1}^{j-1} N^{(l)}_k \leq i \leq \sum_{l=1}^{j} N^{(l)}_k$$

Remark 2.6.4. From Proposition 2.4.4 we deduce that for every $k \geq 0$

$$E[\big|N_{k+1} - N_k\big|^2] \leq \Gamma_1(k+1)N_0 \ \text{ and } \ E[N_k^2] \leq \Gamma_2(k)N_0^2\,, \tag{2.62}$$

where $\Gamma_1(k+1) = 2^2(C(k+1)+C(k))$ and $\Gamma_2(k) = 2^2(C(k)+1)$.

proof of Proposition 2.6.2. On the event $\{N_k > 0\}$ we have

$$\pi^{N_k}_{k|k}(dx_k) = \frac{1}{N_k}\sum_{i=1}^{N_k}\delta_{x^{(i)}_k}(dx_k) = \frac{1}{N_k}\sum_{i=1}^{N_{k-1}} N^{(i)}_k\delta_{\tilde{x}^{(i)}_k}(dx_k)\,.$$

We write $\Omega = \bar{\Omega}_k \cup \Omega_k^1 \cup \Omega_k^2$ where $\bar{\Omega}_N = \Omega \setminus (\Omega_k^1 \cup \Omega_k^2) = \{N_k = 0\}$, $\Omega_k^1 = \{|N_k - N_0| \leq \frac{N_0}{2}\}$ and $\Omega_k^2 = \{|N_k - N_0| > \frac{N_0}{2}\}$. Then, from Proposition 2.4.4 and inequality (2.53) we get

$$P(\bar{\Omega}_k) \leq C(k)N_0^{-1} \quad \text{and} \quad P(\Omega_k^2) \leq 2^2 C(k)N_0^{-1}\,.$$

Since, by hypothesis, inequality (2.58) holds, it is sufficient to prove that there exist a constant Δ_k' such that

$$E[\big((\pi_{k|k}^{N_k}, \varphi) - (\tilde{\pi}_{k|k}^{N_{k-1}}, \varphi)\big)^2] \leq \frac{\Delta_k'}{N_0}\|\varphi\|^2\,. \tag{2.63}$$

In that case, we take $\Delta_k = 2(\Delta_k' + \tilde{\Delta}_k)$ to get the inequality (2.59).
Let us prove (2.63). For all $\varphi \in \mathcal{B}(\mathbb{R}^n)$ we denote

$$\mathcal{S}_{N_{k-1}}^{\varphi} = \sum_{i=1}^{N_{k-1}} N_k^{(i)} \varphi(\tilde{X}_k^{(i)}) - N_k(\tilde{\pi}_{k|k}^{N_{k-1}}, \varphi)\,.$$

Then,

$$\begin{aligned}
E[\big((\pi_{k|k}^{N_k}, \varphi) - (\tilde{\pi}_{k|k}^{N_{k-1}}, \varphi)\big)^2] &= \|\varphi\|^2 P(\bar{\Omega}_k) + E[\mathbf{I}_{\{N_k>0\}} \frac{1}{N_k^2}\big(\mathcal{S}_{N_{k-1}}^{\varphi})\big)^2] \\
&\leq \|\varphi\|^2 C(k)N_0^{-1} + E[\mathbf{I}_{\Omega_k^1 \cup \Omega_k^2} \frac{1}{N_k^2}\big(\mathcal{S}_{N_{k-1}}^{\varphi}\big)^2] \\
&\leq \|\varphi\|^2 C(k)N_0^{-1} + E[\mathbf{I}_{\Omega_k^1} \frac{1}{N_k^2}\big(\mathcal{S}_{N_{k-1}}^{\varphi}\big)^2] + 2^2\|\varphi\|^2 P(\Omega_k^2) \\
&\leq \frac{C(k)}{N_0}\|\varphi\|^2 + E[\mathbf{I}_{\Omega_k^1} \frac{1}{N_k^2}\big(\mathcal{S}_{N_{k-1}}^{\varphi}\big)^2] + \frac{2^4 C(k)}{N_0}\|\varphi\|^2 \qquad (2.64)
\end{aligned}$$

It remains the expectation term in (2.64). On Ω_k^1 we have $\frac{1}{N_k} \leq \frac{2}{N_0}$, then

$$\begin{aligned}
E[\mathbf{I}_{\Omega_k^1} \frac{1}{N_k^2}\big(\mathcal{S}_{N_{k-1}}^{\varphi}\big)^2] &\leq (\frac{2}{N_0})^2 E[\big(\mathcal{S}_{N_{k-1}}^{\varphi}\big)^2] \\
&\leq \frac{8}{N_0^2} E[\big(\mathcal{S}_{N_{k-1}}^{\varphi}\big)^2] + \frac{8}{N_0^2}\|\varphi\|^2 E[(N_k - N_{k-1})^2] \qquad (2.65)
\end{aligned}$$

The definition of the branching mechanism implies that

$$\begin{aligned}
E[\big(\mathcal{S}_{N_{k-1}}^{\varphi}\big)^2] &= E[E[\big(\mathcal{S}_{N_{k-1}}^{\varphi}\big)^2 | \mathcal{F}_{k-1}]] \\
&\leq E[C\|\varphi\|^2 N_{k-1}] = C\|\varphi\|^2 N_0\,. \qquad (2.66)
\end{aligned}$$

Also, from (2.62) we have

$$E[(N_k - N_{k-1})^2] \leq \Gamma_1(k)N_0\,. \tag{2.67}$$

Using (2.66) together with (2.67) in (2.65), we obtain the constant $\Gamma(k) = 8(\Gamma_1(k) + C)$ such that

$$E[\mathbf{I}_{\Omega_k^1} \frac{1}{N_k^2}\big(\sum_{i=1}^{N_{k-1}} N_k^{(i)} \varphi(\tilde{X}_k^{(i)}) - N_k(\tilde{\pi}_{k|k}^{N_{k-1}}, \varphi)\big)^2] \leq \frac{\Gamma(k)}{N_0}\|\varphi\|^2 \tag{2.68}$$

Finally, we substitute this inequality in (2.64) we get (2.63) for $\Delta_k' = (1 + 2^4)C(k) + \Gamma(k)$. The proof is complete. □

Proof of Lemma 2.6.3:

Proof. Let us denote

1. $\mathcal{E}^{\phi}_{k-1:k|k} = E[\big((\pi^N_{k-1:k|k}, \phi) - (\pi_{k-1:k|k}, \phi)\big)^2]$
2. $\Upsilon^k_i(\Phi) = \phi(\tilde{X}^{(i)}_k, X^{(i)}_{k-1}) - (\pi_{k-1:k|k}, \phi)$, for all $1 \leq i \leq N$

Then,

$$\begin{aligned}
\mathcal{E}^{\phi}_{k-1:k|k} &= E[\mathbf{I}_{\{N=0\}}\big((\pi_{k-1:k|k}, \phi)\big)^2] + E[\mathbf{I}_{\{N>0\}}\frac{1}{N^2}\big(\sum_{i=1}^N \Upsilon^k_i(\Phi)\big)^2] \quad (2.69)\\
&= E[\mathbf{I}_{\{N=0\}}\big((\pi_{k-1:k|k}, \phi)\big)^2] + E[\mathbf{I}_{\{N>0\}}\frac{1}{N^2}\sum_{i=1}^N \big(\Upsilon^k_i(\Phi)\big)^2]\\
&\quad + E[\mathbf{I}_{\{N>0\}}\frac{1}{N^2}\sum_{1\leq i\neq j\leq N} \Upsilon^k_i(\Phi)\Upsilon^k_j(\Phi)]\\
&= \mathcal{E}^{0,\phi}_{k-1:k|k} + \mathcal{E}^{1,\phi}_{k-1:k|k} + \mathcal{E}^{2,\phi}_{k-1:k|k} \quad (2.70)
\end{aligned}$$

We examine the three terms $\mathcal{E}^{i,\phi}_{k-1:k|k}$, $i \in \{0,1,2\}$.
The first term: $\mathcal{E}^{1,\phi}_{k-1:k|k} = E[\mathbf{I}_{\{N=0\}}\big((\pi_{k-1:k|k}, \phi)\big)^2]$.
The Proposition 2.4.4 implies that

$$\mathcal{E}^{0,\phi}_{k-1:k|k} \leq \|\phi\|^2 P(\{N=0\}) \leq \|\phi\|^2 \frac{C(k)}{N_0}\,. \quad (2.71)$$

The second term: $\mathcal{E}^{0,\phi}_{k-1:k|k} = E[\mathbf{I}_{\{N>0\}}\frac{1}{N^2}\sum_{i=1}^N \big(\Upsilon^k_i(\Phi)\big)^2]$.
In one hand, we have

$$E[\mathbf{I}_{\{N>0\}}\frac{1}{N^2}\sum_{i=1}^N \big(\Upsilon^k_i(\Phi)\big)^2] \leq 4\|\phi\|^2 E[\mathbf{I}_{\{N>0\}}\frac{1}{N}]$$

In the other hand, if we use (2.53) we get

$$\begin{aligned}
E[\mathbf{I}_{\{N>0\}}\frac{1}{N}] &= E[\mathbf{I}_{\{N>0\}}\mathbf{I}_{\{|N-N_0|\leq N_0/2\}}\frac{1}{N}] + E[\mathbf{I}_{\{N>0\}}\mathbf{I}_{\{|N-N_0|>N_0/2\}}\frac{1}{N}]\\
&\leq \frac{2}{N_0} + P(|N-N_0| > N_0/2)\\
&\leq \frac{2+4C(k)}{N_0}\,. \quad (2.72)
\end{aligned}$$

Then,

$$\mathcal{E}^{0,\phi}_{k-1:k|k} = E[\mathbf{I}_{\{N>0\}}\frac{1}{N^2}\sum_{i=1}^N \big(\Upsilon^k_i(\Phi)\big)^2] \leq \frac{8+2^4C(k)}{N_0}\|\phi\|^2\,. \quad (2.73)$$

The third term: $\mathcal{E}^{2,\phi}_{k-1:k|k} = E[\mathbf{I}_{\{N>0\}} \frac{1}{N^2} \sum_{1\leq i\neq j\leq N} \Upsilon^k_i(\Phi)\Upsilon^k_j(\Phi)]$.
Since $N = N_{k-1}$ is $\tilde{\mathcal{F}}_{k-1}$-measurable and the particles $\{\tilde{X}^{(i)}_k\}_i$ are independent given $\tilde{\mathcal{F}}_{k-1}$, see (2.22) and (2.23), then

$$\begin{aligned}\mathcal{E}^{2,\phi}_{k-1:k|k} &= E\big[\mathbf{I}_{\{N>0\}}\frac{1}{N^2}\sum_{1\leq i\neq j\leq N} E[\Upsilon^k_i(\Phi)\Upsilon^k_j(\Phi)|\tilde{\mathcal{F}}_{k-1}]\big] \\ &= E\big[\mathbf{I}_{\{N>0\}}\frac{1}{N^2}\sum_{1\leq i\neq j\leq N} E[\Upsilon^k_i(\Phi)|\tilde{\mathcal{F}}_{k-1}]\times E[\Upsilon^k_j(\Phi)|\tilde{\mathcal{F}}_{k-1}]\big]\,.\end{aligned}$$

Using (2.24), we get for all $1 \leq i \neq j \leq N$

$$\begin{aligned}E[\Upsilon^k_i(\Phi)|\tilde{\mathcal{F}}_{k-1}] &= E[\phi(\tilde{X}^{(i)}_k, X^{(i)}_{k-1})|\tilde{\mathcal{F}}_{k-1}] - (\pi_{k-1:k|k},\phi) \\ &= \int_{\mathbb{R}^n}\phi(x_k, X^{(i)}_{k-1})\tilde{K}_{k-1}(X^{(i)}_{k-1}, dx_k) - (\pi_{k-1:k|k},\phi)\end{aligned}$$

Let us denote $\varphi(x_{k-1}) = \int_{\mathbb{R}^n}\phi(x_{k-1},x_k)\tilde{K}_{k-1}(x_{k-1},dx_k)$. In particular, $\varphi \in \mathcal{B}(\mathbb{R}^n)$ and $(\pi_{k-1|k-1},\varphi) = (\pi_{k-1:k|k},\phi)$. Moreover,

i. $E[\Upsilon^k_i(\Phi)|\tilde{\mathcal{F}}_{k-1}] = \varphi(X^{(i)}_{k-1}) - (\pi_{k-1|k-1},\varphi)$
ii. $(\pi^N_{k-1|k-1},\varphi) = \frac{1}{N}\sum_{i=1}^N \varphi(X^{(i)}_{k-1})$

Then,

$$\begin{aligned}\frac{1}{N^2}\sum_{i\neq j} E[\Upsilon^k_i(\Phi)|\tilde{\mathcal{F}}_{k-1}]E[\Upsilon^k_j(\Phi)|\tilde{\mathcal{F}}_{k-1}] &= \frac{1}{N^2}\Big(\sum_{1\leq i\leq N}\big(\varphi(X^{(i)}_{k-1}) - (\pi_{k-1|k-1},\varphi)\big)\Big)^2 \\ &\quad -\frac{1}{N^2}\sum_{1\leq i\leq N}\big(\varphi(X^{(i)}_{k-1}) - (\pi_{k-1|k-1},\varphi)\big)^2 \\ &= \big((\pi^N_{k-1|k-1},\varphi) - (\pi^N_{k-1|k-1},\varphi)\big)^2 \\ &\quad -\frac{1}{N^2}\sum_{1\leq i\leq N}\big(\varphi(X^{(i)}_{k-1}) - (\pi_{k-1|k-1},\varphi)\big)^2\,.\end{aligned}$$

Using (2.57) and (2.72), we get

$$\begin{aligned}\mathcal{E}^{2,\phi}_{k-1:k|k} &\leq 4\|\Phi\|^2 E\big[\mathbf{I}_{\{N>0\}}\frac{1}{N}\big] + E[\big((\pi^N_{k-1|k-1},\varphi) - (\pi^N_{k-1|k-1},\varphi)\big)^2] \\ &\leq \frac{\Delta_{k-1}}{N_0}\|\Phi\|^2 + \frac{8+2^4C(k)}{N_0}\|\phi\|^2 \\ &\leq \frac{8+2^4C(k)+\Delta_{k-1}}{N_0}\|\phi\|^2\,. \qquad (2.74)\end{aligned}$$

Taking together (2.71), (2.73) and (2.74) we get a constant $\hat{\Delta}_k > 0$ such that

$$E[\big((\pi^N_{k-1:k|k},\phi) - (\pi_{k-1:k|k},\phi)\big)^2] \leq \frac{\hat{\Delta}_k}{N_0}\|\phi\|^2\,. \qquad (2.75)$$

The proof is complete. □

2.7 Selection of the importance function

The importance sampling strategy is used to limit and to reduce the degeneracy phenomenon, see figure 2.3. By considering a family of density functions $\{\tilde{p}(x_k|x_{k-1}, z_{0:k})\}_k$, called importance sample functions, the new weights of the particles are given by

$$w_k^{(i)} \propto \tilde{w}_k(X_{k-1}^{(i)}, \tilde{X}_k^{(i)}) = \frac{\tilde{g}_k(\tilde{X}_k^{(i)}) p(\tilde{X}_k^{(i)}|X_{k-1}^{(i)})}{\tilde{p}(\tilde{X}_k^{(i)}|X_{k-1}^{(i)}, z_{0:k})} \,. \tag{2.76}$$

The idea is to choose the importance function which minimizes the variance of the importance weights given the simulated trajectory and given the observations. Recall that

$$\tilde{\mathcal{F}}_{k-1} = \sigma\{N_0\,, \ldots,\, N_{k-1}\,;\, \bigcup_{j=0}^{k-1} \{X_j^{(i)}\}_{1\le i\le N_j}\,;\, \bigcup_{j=1}^{k-1} \{\tilde{X}_j^{(i)}\}_{1\le i\le N_j}\}$$

2.7.1 Optimal importance function

Optimal importance function

Proposition 2.7.1. *The conditional pdf $p(x_k|x_{k-1}, z_k)$ of the state X_k given the observation Z_k and the state X_{k-1} is the importance function that minimizes the variance of the importance weight $w_k^{(i)}$ given $\tilde{\mathcal{F}}_{k-1}$. This function is called the **optimal** importance function.*

Proof. We begin by the following two observations:

i. The Theorem 2.2.7. in [40] implies that

$$p(z_k|x_k) = \tilde{g}_k(x_k) \tag{2.77}$$

ii. Baye's rule allows us to write

$$\begin{aligned} p(x_k|x_{k-1}, z_k) &= \frac{p(x_k)p(x_{k-1}, z_k|x_k)}{p(x_{k-1}, z_k)} \\ &= \frac{p(x_k)p(z_k|x_{k-1}, x_k)p(x_{k-1}|x_k)}{p(z_k|x_{k-1})p(x_{k-1})} \\ &= \frac{p(x_k|x_{k-1})p(z_k|x_k)}{p(z_k|x_{k-1})} \end{aligned} \tag{2.78}$$

Using together (2.76), (2.77) and (2.78) we get

$$\begin{aligned} E_{p(x_k|x_{k-1},z_k)}[\tilde{w}_k(x_{k-1}, x_k)] &= \int \tilde{w}_k(x_{k-1}, x_k)p(x_k|x_{k-1}, z_k)dx_k \\ &= \int p(z_k|x_k)p(x_k|x_{k-1})dx_k \\ &= p(z_k|x_{k-1})\,. \end{aligned}$$

Using again (2.78) we get

$$\begin{aligned} var_{p(x_k|x_{k-1},z_k)}(\tilde{w}_k(x_{k-1},x_k)) &= E_{p(x_k|x_{k-1},z_k)}\big[\big(\tilde{w}_k(x_{k-1},x_k)-p(z_k|x_{k-1})\big)^2\big] \\ &= \int_{\mathbb{R}^n}(\tilde{w}_k(x_{k-1},x_k))^2 p(x_k|x_{k-1},z_k)dx_k - p^2(z_k|x_{k-1}) \\ &= \int \frac{p^2(z_k|x_k)p^2(x_k|x_{k-1})}{p(x_k|x_{k-1},z_k)}dx_k - p^2(z_k|x_{k-1}) \\ &= p(z_k|x_{k-1})\int p(z_k|x_k)p(x_k|x_{k-1})dx_k - p^2(z_k|x_{k-1}) \\ &= 0\,. \end{aligned}$$

□

The optimal importance function $p(x_k|x_{k-1},z_k)$ gives the weights

$$\begin{aligned} w_k^{(i)} \propto \tilde{w}_k(X_{k-1}^{(i)},\tilde{X}_k^{(i)}) &= \frac{p(z_k|\tilde{X}_k^{(i)})p(\tilde{X}_k^{(i)}|X_{k-1}^{(i)})}{p(\tilde{X}_k^{(i)}|X_{k-1}^{(i)},z_k)} \\ &= p(z_k|X_{k-1}^{(i)})\,. \end{aligned}$$

The weight $w_k^{(i)}$ do not depend on the $\tilde{X}_k^{(j)}$'s, this allows parallelization of the simulation of the $\tilde{X}_k^{(j)}$'s and the evaluation of the $w_k^{(j)}$'s.
To use the optimal importance function we have to be able to sample from $p(x_k|X_{k-1}^{(i)},z_k)$ and to evaluate, up to a proportional constant, the integral

$$p(z_k|X_{k-1}^{(i)}) = \int p(z_k|x_k)p(x_k|X_{k-1}^{(i)})dx_k\,. \tag{2.79}$$

This can be done for the following class of models.

Example 2.7.2 (Partial Gaussian state space models).
n-dimensional state equation: $X_{k+1} = f_k(X_k) + U_k$
m-dimensional observation equation: $Z_k = C_k X_k + W_k$

i. The processes X_0, U_k and W_k are mutually independent for all $k \geq 0$

ii. The processes $\{U_k\}_k$ and $\{W_k\}_k$ are Gaussian and

$$U_k \sim \mathcal{N}(0,\Sigma_u)\,,\ \Sigma_u > 0 \quad \text{and } W_k \sim \mathcal{N}(0,\Sigma_w)\,,\ \Sigma_w > 0\,. \tag{2.80}$$

iii. For all $k \geq 0$, C_k is an $m \times n$ real matrix and the function $f_k : \mathbb{R}^n \to \mathbb{R}^n$ is Borel measurable

We obtain

1. $X_k|X_{k-1},Z_k \sim \mathcal{N}(m_k,\Sigma_k)$
2. $p(z|X_{k-1}) \propto \exp\Big(-\frac{1}{2}\big(z-C_k f_k(X_{k-1})\big)^T\big(\Sigma_u + C_k\Sigma_w C_k^T\big)^{-1}\big(z-C_k f_k(X_{k-1})\big)\Big)$

where

$$\begin{aligned} \Sigma_k^{-1} &= \Sigma_u^{-1} + C_k^T \Sigma_w^{-1} C_k \\ m_k &= \Sigma_k (\Sigma_u^{-1} f_k(X_{k-1}) + C_k^T \Sigma_w^{-1} Z_k) \end{aligned}$$

The optimality of the Kalman filter for linear Gaussian systems allow us to compare the *optimal* particle filter algorithm to the original one.

Example 2.7.3. Let us consider the following 1-dimensional linear Gaussian system,

$$\begin{aligned} X_{k+1} &= X_k + U_k\,, \qquad X_0 \sim N(0,10)\,,\ U_k \sim N(0,10)\,, \\ Z_k &= X_k + W_k\,, \qquad\qquad W_k \sim N(0,1)\,. \end{aligned}$$

We apply the particle filter algorithms, optimal and original, using 60 particles and using a multinomial branching mechanism, to estimate the expectation and the variance. We compare these two algorithms relatively to the Kalman filter algorithm. The results are given in the following table:

	Kalman Alg.		Opt. Part. filter		Part. filter	
k	Expect..	Var.	Expect.	Var.	Expect.	Var.
10	- 6.66	0.91	- 6.53	0.86	- 6.84	0.83
25	- 5.89	0.91	- 6.14	0.92	- 5.43	1.29
50	- 33.57	0.91	- 33.30	0.58	- 31.02	0.04
100	- 0.46	0.91	- 0.40	0.98	- 0.50	1.11
150	0.481	0.91	0.59	0.65	0.21	0.65
200	- 21.08	0.91	- 21.23	0.91	- 21.11	0.76
250	- 9.00	0.91	- 8.98	0.87	- 8.92	1.36
300	- 21.64	0.91	- 21.71	0.62	- 22.09	0.77
400	- 31.66	0.91	- 31.52	1.00	- 31.97	0.29

The optimal importance function gives better results.

For many other models, such evaluations are impossible. One idea is to approximate the optimal importance function.

Approximation by local linearization

The idea is to linearize locally the observation equation to obtain an importance function that approximates the optimal one. Let us consider the system:

$$X_{k+1} = f_k(X_k) + U_k\,, \quad k \geq 0\,, \tag{2.81}$$

$$Z_k = h_k(X_k) + W_k\,, \quad k \geq 0\,. \tag{2.82}$$

i The processes U_k and W_k are Gaussian with non singular covariance matrices

ii The processes X_0, U_k and W_k are mutually independent

iii The function $f_k : \mathbb{R}^n \to \mathbb{R}^n$ is a Borel measurable function

iv The function $h_k : \mathbb{R}^n \to \mathbb{R}^m$ is supposed two-times differentiable

A Taylor expansion up to the first order of the observation equation (2.82) gives

$$\begin{aligned} Z_k &\approx h_k(f_k(X_{k-1})) + C_k(X_k - f_k(X_{k-1})) + W_k \\ &\approx C_k X_k + F_k(X_{k-1}) + W_k \,, \end{aligned} \tag{2.83}$$

where $C_k = \frac{\partial h_k(x)}{\partial x}\big|_{x=f(X_{k-1})}$ and $F_k(X_{k-1}) = h_k(f_k(X_{k-1})) - C_k f_k(X_{k-1})$.
Equations (2.81) and (2.83) define a new model. The observation is linear and Gaussian. Similar calculations to that in Example 2.7.2 suggest us to choose the importance function $\tilde{p}(x_k|x_{k-1}, z_k)$ to be the density of $\mathcal{N}(m_k, \Sigma_k)$, where

$$\begin{aligned} \Sigma_k^{-1} &= \Sigma_u^{-1} + C_k^T \Sigma_w^{-1} C_k \,, \\ m_k &= \Sigma_k \{\Sigma_u^{-1} f_k(X_{k-1}) + C_k^T \Sigma_w^{-1}(Z_k - F_k(X_{k-1}))\} \,. \end{aligned}$$

The associated weights are computing using (2.76).

Monte Carlo approximations

Assume that we can not evaluate analytically $p(z_k|X_{k-1}^{(i)})$ and (or) we can not sample from $p(x_k|X_{k-1}^{(i)}, z_k)$. Since the functions g_k are strictly positive and bounded and the kernels are Feller. Then, $p(z_k|x_k)$ and $p(x_k|x_{k-1})$ are bounded, see (2.5.1) and (2.77). In particular, from (2.78) we deduce that the ratio $p(x_k|X_{k-1}^{(i)}, z_k)/p(x_k|X_{k-1}^{(i)})$ is bounded say by M_k. It is possible then to sample from $p(x_k|X_{k-1}^{(i)}, z_k)$ using the **Accept-Reject** method, see [72].

1. Generate $y_k \sim p(x_k|X_{k-1}^{(i)})$, $u \sim \mathcal{U}_{[0,1]}$.
2. Accept $\tilde{X}_k^{(i)} = y_k$ if $u \leq p(y_k|X_{k-1}^{(i)}, z_k)/M_k p(y_k|X_{k-1}^{(i)})$.
3. Return to 1. otherwise.

For each $1 \leq i \leq N$, we use a Monte Carlo step to approximate $p(z_k|X_{k-1}^{(i)}) = \int p(z_k|x_k) p(x_k|X_{k-1}^{(i)}) dx_k$. This can be done by sampling N' i.i.d. random variables $\{X_k^{(i,j)}, 1 \leq j \leq N'\}$ according to $p(x_k|X_{k-1}^{(i)})$. Then,

$$p(z_k|X_{k-1}^{(i)}) \approx \bar{p}(z_k|X_{k-1}^{(i)}) = \frac{1}{N'} \sum_{j=1}^{N'} p(z_k|X_k^{(i,j)}) \,.$$

We use the particles $\{X_k^{(i,j)}\,,\ 1 \leq j \leq N'\}$ to approximate the measure $p(x_k|z_k, X_{k-1}^{(i)})\, dx_k$:

$$p(x_k|z_k, X_{k-1}^{(i)}) dx_k \approx \sum_{j=1}^{N'} \alpha_{i,j} \delta_{X_k^{(i,j)}}(dx_k) \,,$$

where

$$\alpha_{i,j} = \frac{\frac{1}{N'} p(z_k|X_k^{(i,j)})}{\bar{p}(z_k|X_{k-1}^{(i)})} .$$

Such approximations have numerous drawbacks. Using the Accept-Reject Method requires a random number of iterations and in an online framework this strategy is avoided. Also, the Monte Carlo approximations are valid when $N' \to \infty$ and this can be computationally expensive.

2.7.2 Prior importance function

A simple choice is to take as importance function the transition probability density function of the Markov process $\{X_k\}_k$, that is

$$\tilde{p}(x_k|z_k, x_{k-1}) = p(x_k|x_{k-1}) .$$

This yields to

$$w_k^{(i)} \propto p(z_k|\tilde{X}_k^{(i)}) .$$

This method is very sensitive to the observations. If non sufficient knowledge about the observations are available this method can be inefficient.

2.7.3 Fixed importance function

Another simple choice is to select an importance function independently from the simulated trajectories and from the observations. In such case,

$$\tilde{p}(x_k|z_k, x_{k-1}) = \tilde{p}(x_k) .$$

and the weights are given by

$$w_k^{(i)} \propto \frac{p(z_k|\tilde{X}_k^{(i)}) p(\tilde{X}_k^{(i)}|X_{k-1}^{(i)})}{\tilde{p}(\tilde{X}_k^{(i)})} .$$

This choice don't take in account the dynamic of the model and can leads to unbounded weights.

2.7.4 Rao-Blackwellisation

The Rao-Blackwellisation technique improves the accuracy of the particle filter, it reduces the variance of the weights, by analytically marginalizing some components of the state and only sampling from the remainders.
If we decompose the state process $X_k = (X_k^1, X_k^2) \in \mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, then

$$p(x_k)dx_k = p(x_k^1, x_k^2)dx_k^1 dx_k^2 = p(x_k^2|x_k^1)p(x_k^1)dx_k^1 dx_k^2 . \tag{2.84}$$

where $p(x_k^2|x_k^1)$ is the conditional pdf of X_k^2 given X_k^1. Moreover, Bayes rule implies

1. For any $\varphi \in C_b(\mathbb{R}^n)$,

$$(\pi_{k|k}, \varphi) = \int_{\mathbb{R}^n} \varphi(X_k) p(x_k|z_{0:k}) dx_k = \frac{\int_{\mathbb{R}^{n_1}} \tilde{\varphi}(x_k^1) p(x_k^1) dx_k^1}{\int_{\mathbb{R}^{n_1}} p(z_{0:k}|x_k^1) p(x_k^1) dx_k^1}$$

where

$$\tilde{\varphi}(x_k^1) = \int_{\mathbb{R}^{n_2}} \varphi(x_k^1, x_k^2) p(z_{0:k}|x_k^1, x_k^2) p(x_k^2|x_k^1) dx_k^2 \tag{2.85}$$

$$p(z_{0:k}|x_k^1) = \int_{\mathbb{R}^{n_2}} p(z_{0:k}|x_k^1, x_k^2) p(x_k^2|x_k^1) dx_k^2 \tag{2.86}$$

2. $p(z_{0:k}|x_k^1) p(x_k^1) = p(z_{0:k}) p(x_k^1|z_{0:k})$

If we denote by

$$\phi(x_k^1) = \frac{\tilde{\varphi}(x_k^1)}{p(z_{0:k}|x_k^1)} = \frac{\int_{\mathbb{R}^{n_2}} \varphi(x_k^1, x_k^2) p(z_{0:k}|x_k^1, x_k^2) p(x_k^2|x_k^1) dx_k^2}{\int_{\mathbb{R}^{n_2}} p(z_{0:k}|x_k^1, x_k^2) p(x_k^2|x_k^1) dx_k^2} \ .$$

Then,

$$(\pi_{k|k}, \varphi) = \frac{\int_{\mathbb{R}^{n_1}} \phi(x_k^1) p(x_k^1|z_{0:k}) dx_k^1}{\int_{\mathbb{R}^{n_1}} p(x_k^1|z_{0:k}) dx_k^1} \ . \tag{2.87}$$

The Rao-Blackwellisation technique is based on the following assumption

Assumption 2.2. Given a realization x_k^1 of X_k^1, we can evaluate analytically $\tilde{\varphi}(x_k^1)$ and $p(z_{0:k}|x_k^1)$ as a function of x_k^1.

Rao-Blackwellisation: we integrate out analytically x_k^2 in (2.85) and (2.86) and we use a particle filter method to estimate $\int_{\mathbb{R}^{n_1}} \phi(x_k^1) p(x_k^1|z_{0:k}) dx_k^1$ and $\int_{\mathbb{R}^{n_1}} p(x_k^1|z_{0:k}) dx_k^1$. We filter the state X_k^1 based on the observations $z_{0:k}$.
The importance sampling functions are $\{\tilde{p}(x_k^1|x_{k-1}, z_{0:k})\}_k$. In particular, $\tilde{p}(x_k^1|x_{k-1}, z_{0:k}) = \int_{\mathbb{R}^{n_2}} \tilde{p}(x_k^1, x_k^2|x_{k-1}, z_{0:k}) dx_k^2$ and the estimate to $(\pi_{k|k}, \varphi)$ is

$$(\pi_{k|k}^{*N}, \varphi) = \frac{\sum_{i=1}^N w_k^{*(i)} \phi(\tilde{X}_k^{1,(i)})}{\sum_{i=1}^N w_k^{*(i)}} \tag{2.88}$$

where

$$w_k^{*(i)} = \frac{p(\tilde{X}_k^{1,(i)}|z_{0:k})}{\tilde{p}(\tilde{X}_k^{1,(i)}|X_{k-1}^{(i)}, z_{0:k})} \quad \text{and} \quad \tilde{X}_k^{(i)} = (\tilde{X}_k^{1,(i)}, \tilde{X}_k^{2,(i)}) \, . \tag{2.89}$$

The Rao-Blackwellisation reduces the variance of the weights. In fact, if we denote by

$$w_k(x_k) = \frac{p(x_k|z_{0:k})}{\tilde{p}(x_k|x_{k-1}, z_{0:k})} = \frac{p(x_k^1, x_k^2|z_{0:k})}{\tilde{p}(x_k^1, x_k^2|x_{k-1}, z_{0:k})}$$

and by

$$w_k^*(X_k^1) = \frac{p(x_k^1|z_{0:k})}{\tilde{p}(x_k^1|x_{k-1}, z_{0:k})} \ .$$

Then,

Proposition 2.7.4.

$$var_{\tilde{p}(x_k^1|x_{k-1},z_{0:k})}\big(w_k^*(X_k^1)\big) \leq var_{\tilde{p}(x_k|x_{k-1},z_{0:k})}\big(w_k(X_k)\big)\,. \tag{2.90}$$

Proof. Let us consider three random vectors $\mathcal{X}^1$, $\mathcal{X}^2$ and $\mathcal{Y}$ and suppose that $\mathcal{X} = (\mathcal{X}^1, \mathcal{X}^2)$. The associated conditional probability density functions satisfy

$$p(x|y) = p(x^1, x^2|y) = p(x^2|x^1, y)p(x^1|y)\,.$$

If $w(x)$ is any Borel measurable function, then

$$\begin{aligned}
E_{p(x|y)}[w(\mathcal{X})] &= \int w(x)p(x|y)dx \\
&= \int w(x^1, x^2)p(x^2|x^1, y)p(x^1|y)dx^1dx^2 \\
&= \int \big(\int w(x^1, x^2)p(x^2|x^1, y)dx^2\big)p(x^1|y)dx^1 \\
&= E_{p(x^1|y)}\big[E_{p(x^2|x^1,y)}[w(\mathcal{X}^1, \mathcal{X}^2)\big|\mathcal{X}^1]\big]. \qquad (2.91)
\end{aligned}$$

Using the equality (2.91) for the functions w and w^2 we get the following decomposition of the conditional variance $var_{p(x|y)}(w(\mathcal{X}))$

$$\begin{aligned}
var_{p(x|y)}(w(\mathcal{X})) &= E_{p(x|y)}[w(\mathcal{X})^2] - (E_{p(x|y)}[w(\mathcal{X})])^2 \\
&= E_{p(x|y)}[w(\mathcal{X})^2] - \big(E_{p(x^1|y)}\big[E_{p(x^2|x^1,y)}[w(\mathcal{X}^1, \mathcal{X}^2)\big|\mathcal{X}^1]\big]\big)^2 \\
&= E_{p(x|y)}[w(\mathcal{X})^2] - E_{p(x^1|y)}\big[(E_{p(x^2|x^1,y)}[w(\mathcal{X}^1, \mathcal{X}^2)\big|\mathcal{X}^1])^2\big] \\
&\quad + E_{p(x^1|y)}\big[(E_{p(x^2|x^1,y)}[w(\mathcal{X}^1, \mathcal{X}^2)\big|\mathcal{X}^1])^2\big] \\
&\quad - (E_{p(x^1|y)}\big[E_{p(x^2|x^1,y)}[w(\mathcal{X}^1, \mathcal{X}^2)\big|\mathcal{X}^1]\big])^2 \\
&= E_{p(x^1|y)}[var_{p(x^2|x^1,y)}(w(\mathcal{X}^1, \mathcal{X}^2)\big|\mathcal{X}_1)] \\
&\quad + var_{p(x^1|y)}\big(E_{p(x^2|x^1,y)}[w(\mathcal{X}^1, \mathcal{X}^2)\big|\mathcal{X}^1]\big)\,. \qquad (2.92)
\end{aligned}$$

To apply the decomposition (2.92) to our case it is sufficient to see that

$$\begin{aligned}
E_{\tilde{p}(x_k^2|x_k^1,x_{k-1},z_{0:k})}[w_k(X_k^1, X_k^2)|X_k^1] &= \int \frac{p(x_k^1, x_k^2|z_{0:k})}{\tilde{p}(x_k^1, x_k^2|x_{k-1}, z_{0:k})\tilde{p}(x_k^2|x_k^1, x_{k-1}, z_{0:k})}dx_k^2 \\
&= \int \frac{p(x_k^1, x_k^2|z_{0:k})\tilde{p}(x_k^2|x_k^1, z_{0:k})}{\tilde{p}(x_k^2|x_k^1, x_{k-1}, z_{0:k})\tilde{p}(x_k^1|x_{k-1}, z_{0:k})}dx_k^2 \\
&= \int \frac{p(x_k^1, x_k^2|z_{0:k})}{\tilde{p}(x_k^1|x_{k-1}, z_{0:k})}dx_k^2 \\
&= w_k^*(X_k^1)\,. \qquad (2.93)
\end{aligned}$$

Then, taking $\mathcal{X} = X_k$, $\mathcal{Y} = (X_{k-1}, Z_{0:k})$ and $p(x|y) = \tilde{p}(x_k|x_{k-1}, z_{0:k})$ we get

$$\begin{aligned}
var_{\tilde{p}(x_k|x_{k-1},z_{0:k})}\big(w_k(X_k)\big) &= var_{\tilde{p}(x_k^1|x_{k-1},z_{0:k})}\big(w_k^*(X_k^1)\big) \\
&\quad + E_{\tilde{p}(x_k^1|x_{k-1},z_{0:k})}\big[var_{\tilde{p}(x_k^2|x_k^1,x_{k-1},z_{0:k})}\big(w_k(X_k)\big|X_k^1\big)\big]
\end{aligned}$$

The proof is complete. □

Remark 2.7.5. The optimal importance function for the Rao-Blackwellisation is the conditional pdf $\tilde{p}(x_k^1|x_{k-1}, z_{0:k}) = p(x_k^1|x_{k-1}^1, z_k)$ and the associated importance weights are $w_k^*(x_k^1) = p(z_k|z_{0:k-1}, x_{k-1}^1)$.

Example 2.7.6 (Conditionally Linear Gaussian State Space Model). Suppose that X_k^1 is a Markov process, denoting $p(x_k^1|x_{k-1}^1)$ its transition pdf. The process X_k^2 is supposed to be linear Gaussian conditionally upon X_k^1. More precisely, we suppose that

$$X_k^2 = A_k\big(X_k^1\big)X_{k-1}^2 + B_k\big(X_k^1\big)U_k\,, \tag{2.94}$$

where $A_k : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2\times n_2}$, $B_k : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2\times p}$ and $U_k \sim \mathcal{N}(0, I_p)$.
The observations are given by

$$Z_k = C_k\big(X_k^1\big)X_k^2 + D_k\big(X_k^1\big)W_k\,, \tag{2.95}$$

where $C_k : \mathbb{R}^{n_1} \to \mathbb{R}^{m\times n_2}$, $D_k : \mathbb{R}^{n_1} \to \mathbb{R}^{m\times r}$ and $W_k \sim \mathcal{N}(0, I_r)$. If one is interested in estimating $p(x_k^1|z_{0:k})$, $E[X_k^2|z_{0:k}]$ and $E[X_k^2(X_k^2)^T|z_{0:k}]$. The Rao-Blackwellisation method can be applied here, by using a particle filter method to approximate $p(x_k^1|z_{0:k})$, based on this estimate a Kalman filter is clearly optimal to integrating out x_k^2.

Remark 2.7.7. As applications to this example are the RSA (Random Sampling Algorithm) introduced by Akashi and Kumamoto in [2] and the algorithm for blind deconvolution introduced by Liu *et al.* in [61].

2.8 Selection of the branching mechanism

We initialize the particle filter algorithm by sampling $N_0 > 0$ independent particles according to π_0. At the end of each Update-Prediction stage we apply a branching mechanism.
At time $k-1$ the system consists of N_{k-1} particles. If we denote by

$$\mathcal{F}_{k-1} = \sigma\{N_0\,,\ldots,\,N_{k-1}\,;\ \bigcup_{j=0}^{k-1}\{X_j^{(i)}\}_{1\le i\le N_j}\,;\ \bigcup_{j=1}^{k}\{\tilde{X}_j^{(i)}\}_{1\le i\le N_j}\} \tag{2.96}$$

and by $N_k^{(i)}$ the number of offsprings of the particle $\tilde{X}_k^{(i)}$, then there exists a constant $C > 0$ such that for all $\varphi \in C_b(\mathbb{R}^n)$

$$E[N_k^{(i)}|\mathcal{F}_{k-1}] \;=\; w_k^{(i)}N_{k-1}\,, \tag{2.97}$$

$$E[\Big|\sum_{i=1}^{N_{k-1}} N_k^{(i)}\varphi(\tilde{X}_k^{(i)}) - N_{k-1}\big(\tilde{\pi}_{k|k}^{N_{k-1}},\varphi\big)\Big|^2|\mathcal{F}_{k-1}] \;\le\; CN_{k-1}\|\varphi\|^2\,. \tag{2.98}$$

At the end of this stage, we obtain the set $\{X_k^{(i)}\}$ consisting of $N_k = \sum_{i=1}^{N_{k-1}} N_k^{(i)}$ particles indexed as follows

$$X_k^{(i)} = \tilde{X}_k^{(j)}\,,\quad 1\le j\le N_{k-1}\,,\ 1+\sum_{l=1}^{j-1}N_k^{(l)} \le i \le \sum_{l=1}^{j}N_k^{(l)}\,.$$

The property (2.97) ensures that the estimator is unbiased, ie.

$$E[\pi_{k|k}|\mathcal{F}_{k-1}] = \tilde{\pi}_{k|k}\,.$$

This expresses that the averaged estimate coincides with the desired result.
The condition (2.98) is related to the deviation of the estimate. It ensures that after branching the obtained approximation is not far from the Updated-Predicted approximation. An unbiased branching is better if its deviation is smaller.
The Proposition 2.4.4 and its proof ensure that for all $k \geq 0$:

- The integer valued random variable N_k is an $\mathcal{F}_k$-martingale
- $E[N_k] = N_0$, $P(N_k = 0) \leq C(k)N_0^{\beta-\alpha}$ and $E[|\frac{N_k}{N_0} - 1|^\alpha] \leq C(k)N_0^{\beta-\alpha}$

The property $E[N_k] = N_0 > 0$ means that the system never dies.

Remark 2.8.1. If the numbers of offsprings $N_k^{(i)}$, $1 \leq i \leq N_{k-1}$ are conditionally independent given $\mathcal{F}_{k-1}$, then condition (2.98) is equivalent to

$$\sum_{i=1}^{N_{k-1}} E[\big(N_k^{(i)} - N_{k-1}w_k^{(i)}\big)^2|\mathcal{F}_{k-1}] \leq CN_{k-1}\,. \tag{2.99}$$

2.8.1 Independent branching numbers

Suppose that the integer valued random variables $\{N_k^{(i)}\,,\ 1 \leq i \leq N_{k-1}\}$ are conditionally independent given $\mathcal{F}_{k-1}$.

Bernoulli branching numbers

The Bernouilli branching numbers are defined by

$$P(N_k^{(i)} = j|\mathcal{F}_{k-1}) = \begin{cases} \epsilon(N_{k-1}w_k^{(i)}) & \text{if} \quad j = [N_{k-1}w_k^{(i)}] + 1\,, \\ 1 - \epsilon(N_{k-1}w_k^{(i)}) & \text{if} \quad j = [N_{k-1}w_k^{(i)}]\,, \end{cases}$$

where $[a]$ is the largest integer less than a and $\epsilon(a) = a - [a]$. In this case

i $E[N_k^{(i)}|\mathcal{F}_{k-1}] = N_{k-1}w_k^{(i)}$

ii $E\big[\big(N_k^{(i)} - N_{k-1}w_k^{(i)}\big)^2\big|\mathcal{F}_{k-1}\big] = \epsilon(N_{k-1}w_k^{(i)})(1 - \epsilon(N_{k-1}w_k^{(i)})) \in [0, 1/4]$

The condition (2.99) is satisfied for $C = 1/4$.

Binomial branching numbers

The Binomial branching numbers are defined by

$$P(N_k^{(i)} = j|\mathcal{F}_{k-1}) = \begin{pmatrix} j \\ N_{k-1} \end{pmatrix} (w_k^{(i)})^j (1 - w_k^{(i)})^{N_{k-1}-j}\,.$$

Then,

i $E[N_k^{(i)}|\mathcal{F}_{k-1}] = N_{k-1}w_k^{(i)}$

ii $E\big[\big(N_k^{(i)} - N_{k-1}w_k^{(i)}\big)^2\big|\mathcal{F}_{k-1}\big] = N_{k-1}w_k^{(i)}(1 - w_k^{(i)})$

Hence, (2.99) is satisfied for $C = 1$.

Negative binomial branching numbers

Suppose that for all $1 \leq i \leq N_{k-1}$,

$$p_i = \frac{w_k^{(i)}}{1+w_k^{(i)}}$$
$$P(N_k^{(i)} = j|\mathcal{F}_{k-1}) = \left(\begin{array}{c} N_{k-1}-1 \\ N_{k-1}+j-1 \end{array} \right) p_i^j (1-p_i)^{N_{k-1}} .$$

Then,

i $E[N_k^{(i)}|\mathcal{F}_{k-1}] = N_{k-1} w_k^{(i)}$

ii $E\big[\big(N_k^{(i)} - N_{k-1} w_k^{(i)}\big)^2 \big| \mathcal{F}_{k-1}\big] = N_{k-1} w_k^{(i)} (1 + w_k^{(i)})$

It is clear that (2.99) is satisfied for $C = 2$.

Poisson branching numbers

For all $k \geq 0$ and $1 \leq i \leq N_{k-1}$, the Poisson branching numbers is defined by

$$P(N_k^{(i)} = j|\mathcal{F}_{k-1}) = \exp(-N_{k-1} w_k^{(i)}) \frac{(N_{k-1} w_k^{(i)})^j}{j!} \tag{2.100}$$

In this case,

$$E[N_k^{(i)}|\mathcal{F}_{k-1}] = E\big[\big(N_k^{(i)} - N_{k-1} w_k^{(i)}\big)^2 \big| \mathcal{F}_{k-1}\big] = N_{k-1} w_k^{(i)} .$$

Hence (2.99) is satisfied for $C = 1$.

2.8.2 Negative correlated branching numbers

Multinomial branching mechanism

The multinomial branching mechanism, denoted by Multinomial$(N_{k-1}, w_k^{(1)}, \ldots, w_k^{(N_{k-1})})$, is defined by

$$P\big(N_k^{(i)} = \alpha_i; 1 \leq i \leq N_0\big) = \left\{ \begin{array}{cc} \frac{N_0!}{\alpha_1! \cdots \alpha_{N_0}!} \Pi_i w_k^{(i)} & \text{if } \sum_i \alpha_i = N_0 , \\ 0 & \text{otherwise.} \end{array} \right.$$

In this case the total number of particles remains unchanged.
For all $1 \leq i \neq j \leq N_0$,

i $E[N_k^{(i)}\big|\mathcal{F}_{k-1}] = N_0 w_k^{(i)}$

ii $E\big[\big(N_k^{(i)} - N_0 w_k^{(i)}\big)^2 \big| \mathcal{F}_{k-1}\big] = N_0 w_k^{(i)} (1 - w_k^{(i)})$

iii $E\big[\big(N_k^{(i)} - N_0 w_k^{(i)}\big)\big(N_k^{(j)} - N_0 w_k^{(j)}\big)\big|\mathcal{F}_{k-1}\big] = -N_0 w_k^{(i)} w_k^{(j)}$

Moreover, for any $\varphi \in C_b(\mathbb{R}^n)$,

$$E\big[\ \big|\sum_{i=1}^{N_0} N_k^{(i)}\varphi(\tilde{X}_k^{(i)}) - N_0\big(\tilde{\pi}_{k|k}^{N_0},\varphi\big)\big|^2\ \big|\mathcal{F}_{k-1}\big] = N_0\{(\tilde{\pi}_{k|k}^{N_0},\varphi^2) - \big((\tilde{\pi}_{k|k}^{N_0},\varphi)\big)^2\}$$
$$\leq N_0(\tilde{\pi}_{k|k}^{N_0},\varphi^2)$$
$$\leq N_0\|\varphi\|^2\,.$$

Hence, (2.98) is satisfied for $C = 1$.

Residual multinomial branching mechanism

Instead of using a multinomial mechanism with parameters $(N_{k-1}, w_k^{(1)}, \ldots, w_k^{(N_{k-1})})$, we define the numbers of offsprings $N_k^{(i)}$ for $1 \leq i \leq N_{k-1}$ by

$$N_k^{(i)} = [N_{k-1}w_k^{(i)}] + X_k^{(i)}\,.$$

The integer valued random variables $X_k^{(1)}, \ldots, X_k^{(N_{k-1})}$ are given by

$$(X_k^{(1)}, \ldots, X_k^{(N_{k-1})}) = \text{Multinomial}(M_k, q_k^{(1)}, \ldots, q_k^{(N_{k-1})})$$

where

$$M_k = N_{k-1} - \sum_{j=1}^{N_{k-1}} [N_{k-1}w_k^{(j)}] \text{ and } q_k^{(i)} = \frac{\epsilon(N_{k-1}w_k^{(i)})}{\sum_{j=1}^{N_{k-1}} \epsilon(N_{k-1}w_k^{(j)})}\ .$$

where $[a]$ is the largest integer less than a and $\epsilon(a) = a - [a]$. In this case one has also $N_k = N_{k-1} = N_0$ and the total number of particles remains unchanged. In addition, for all $1 \leq i \leq N_0$, one has

$$E[N_k^{(i)}\big|\mathcal{F}_{k-1}] = N_0 w_k^{(i)}\,.$$

For every $\varphi \in C_b(\mathbb{R}^n)$, one has

$$E\big[\ \big|\sum_{i=1}^{N_{k-1}} N_k^{(i)}\varphi(\tilde{X}_k^{(i)}) - N_0\big(\tilde{\pi}_{k|k}^{N_0},\varphi\big)\big|^2\ \big|\mathcal{F}_{k-1}\big] \leq M_k\|\varphi\|^2 \leq N_0\|\varphi\|^2\,.$$

This mechanism can perform better than the multinomial branching mechanism in the case when M_k is close to 0. In such case, we get a smaller conditional variance and smaller time computation !!.

Example 2.8.2. In this example, we compare several branching mechanisms. We take the 1-dimensional linear system of Example 2.7.3, that is

$$\begin{cases} X_{k+1} &= X_k + U_k\,, & X_0 \sim N(0,10)\,,\ U_k \sim N(0,10)\,,\\ Z_k &= X_k + W_k\,, & W_k \sim N(0,1)\,. \end{cases}$$

We apply the Kalman algorithm for a reference comparison and the particle filter algorithm with three different branching mechanisms, the multinomial, the residual multinomial and the Bernouilli schemes successively. For this model, see Figure 2.4, the Bernouilli performs better than the negative correlated branching mechanisms. It is also clear, that the residual multinomial scheme is at least better than the multinomial scheme.

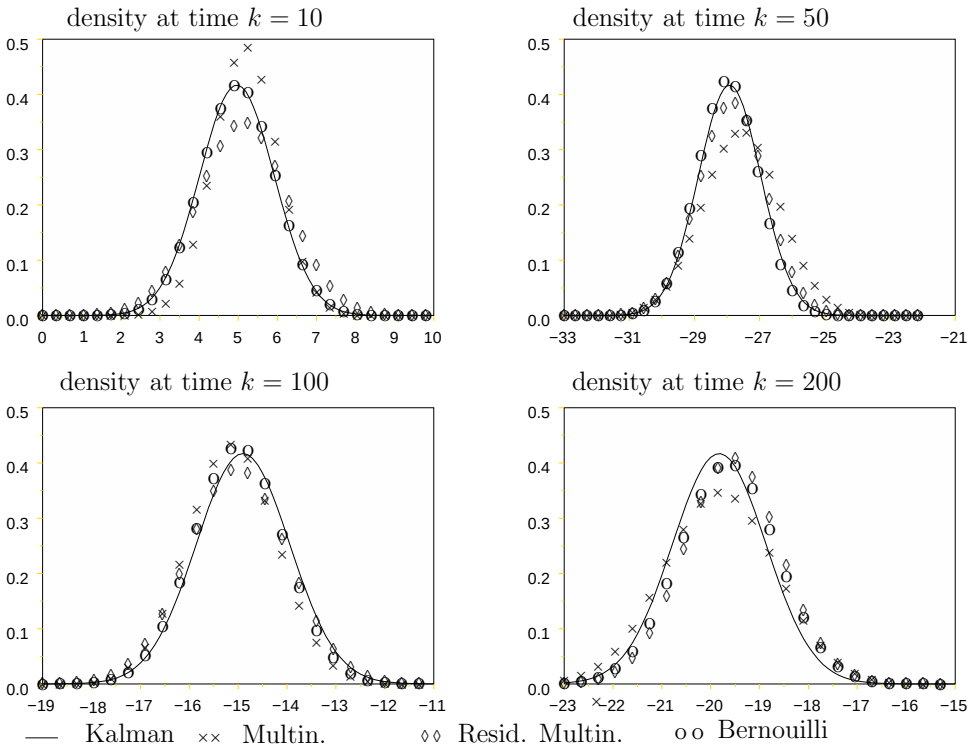

Figure 2.4: Posterior distribution estimations using 100 particles

Chapter 3

Vehicle Tracking Example

3.1 Vehicle dynamics equation

3.1.1 Introduction

In handling vehicle dynamics, see [63, 68, 75], two families of forces are considered. The first are the forces controlling the acceleration and the velocity of the car: tractive force, wheel force, braking force, rolling resistance, drag (= air resistance), etc. The second are those allowing the car to turn: friction of the wheels, angular moments, torque, etc.
Let us consider a car which descends a circular slope of radius R, see Figure 3.1. The angle between the weight vector F_g and the heading direction vector u of the car is denoted by θ. We take into account three of these forces mentioned above: the tractive force, the aerodynamic drag, the rolling resistance consisting of the friction rubber-road and the weight, and we add a noisy term = error in modeling + other friction forces. For simplicity, we suppose that all the forces act through the center of gravity of the car **CG**.

Tractive force: It is the force delivered by the engine via the rear wheels. The engine turns the wheels forward, the wheels push backwards on the road surface and, in reaction, the road surface pushes back in a forward direction. The engine only generates force and hence acceleration. The tractive force is

$$F_t = \frac{\Gamma G g}{r}\, u,$$

where

- u is the unit vector direction of car's heading
- Γ is the torque function of the engine in (Nm)
- $G = 3.545$ is the final drive ratio
- g is the gear ratio and $r = 0.3266\, m$ is the radius of tire

We suppose that during the tracking this force is of constant magnitude

$$\left|F_t\right| = 1200\ N\,. \tag{3.1}$$

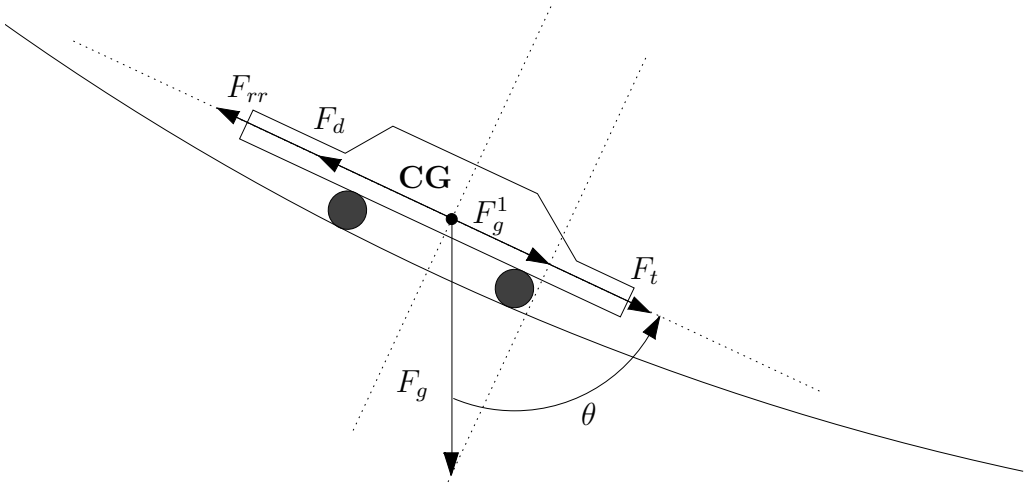

Figure 3.1: The car trajectory

Aerodynamic drag: The air resistance force acts on the front of the car when the car sifts through the air,

$$F_d = -C_{drag} \times |v| \times v, \tag{3.2}$$

where

- v is the velocity vector and $|v|$ the speed
- $C_{drag} = 0.5 \times \rho \times A \times C_d \approx 0.4257$ where
  - $C_d \approx 0.3$ is the coefficient of the friction
  - $\rho = 1.29\, kg/m^3$ is the air density
  - $A \approx 2.2\, m^2$ is the frontal area of the car

Rolling resistances:

1. Friction between the rubber and road surface as the wheels roll:

$$F_{rr} = -C_{rr} \times v\,, \tag{3.3}$$

   where $C_{rr} = 12.8\, Ns/m$ is the coefficient of friction

2. The weight $F_g = M\, g$ of the vehicle acts through its center of gravity and either pulls it back or forward, depending on the angle θ between the weight vector and the forward direction:

$$F_g^1 = M \cos(\theta) \left|g\right| u\,, \tag{3.4}$$

   where $M = 1200\, kg$ is the mass of the car and $|g| = 9.81 m/s^2$ is the gravitational acceleration

3.1.2 Vehicle dynamics equation

Newton's second law: The acceleration $\mathbf{a}$ of the car is proportional in magnitude and direction to the net force $F = F_t + F_d + F_{rr} + F_g^1$ and inversely proportional to its mass $M = 1200\,kg$. That is,

$$\mathbf{a} = \frac{F}{M}\ .$$

In addition, $dv = \mathbf{a}\,dt$ and $dp = v\,dt$ where p is the car position and dt is the time increment.
Parameterizing the position of the car by $s(t) = \int_0^t |v(u)|du$, we get

1. $\dot{s}(t) = \frac{ds}{dt}(t) = |v(t)|$ is the speed of the car at time t
2. $\ddot{s}(t) = \frac{d^2s}{dt^2}(t) = \frac{<v(t),\mathbf{a}(t)>}{|v(t)|}$ is the orthogonal projection of the acceleration on the axle directed by the vector u

Let us denote by $F_{\|}$ the orthogonal projection of the net force F on the axle directed by the vector u, then

$$\begin{aligned}\ddot{s} &= \frac{F_{\|}}{M}\\ &= \frac{1200 + M|g|\cos(\theta_0 + s/R) - C_{rr}\dot{s} - C_{drag}\dot{s}^2}{M}\\ &= 1 + 9.8\,\cos(\theta_0 + s/R) - 1.06\times 10^{-2}\,\dot{s} - 3.54\times 10^{-4}\,\dot{s}^2\,,\end{aligned}$$

where θ_0 is the angle between the weight vector and the initial direction heading of the car. Since the speed of the car takes only nonnegative values, then

$$\ddot{s}(t) = 1 + 9.8\cos(\theta_0 + s(t)/R) - 1.06\times 10^{-2}\dot{s}(t) - 3.54\times 10^{-4}\dot{s}^2(t)\mathbf{I}_{\{\dot{s}\geq 0\}}\,. \quad (3.5)$$

The indicator function $\mathbf{I}_{\{\dot{s}\geq 0\}}$ is introduced for technical considerations needed in next subsections.
This modeling supposes that the engine generates a constant force, gives constant values to the friction coefficients C_{drag} and C_{rr}, and doesn't take into account other frictions. We introduce, then, a noisy term increasing with the speed of the form $\big(\alpha + \beta\dot{s}(t)\big)W_2(t)$, where W_2 is a standard 1-dimensional white noise.
Aim: Estimating the position and the speed of the car.
If $x_1(t) = s(t)$ and $x_2(t) = \dot{s}(t)$, then the state vector is $x(t) = \begin{bmatrix} x_1(t)\\ x_2(t)\end{bmatrix}$
In addition,

$$\begin{cases} dx_1(t) &= x_2(t)\,dt\\ dx_2(t) &= \{a_0 + |g|\cos(\theta_0 + \frac{x_1(t)}{R}) - a_1x_2(t) - a_2x_2(t)^2\mathbf{I}_{\{x_2(t)\geq 0\}}\}dt\\ &\quad +\{\alpha + \beta x_2(t)\}du_2(t)\end{cases} \quad (3.6)$$

where

- $(a_0, a_1, a_2) = (1, 1.06\times 10^{-2}, 3.54\times 10^{-4})$
- $(\alpha, \beta) = (0, 2\times 10^{-2})$
- u_2 is a standard Brownian motion

The state equation: $dx(t) = b(x(t))dt + \sigma(x(t))du(t)\,,$ (3.7)

where for all $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$,

1. $b(x) = \begin{bmatrix} x_2 \\ a_0 + |g|\cos(\theta_0 + \frac{x_1}{R}) - a_1 x_2 - a_2 x_2^2 \mathbf{I}_{\{x_2 \geq 0\}} \end{bmatrix}$
2. $\sigma(x) = \begin{bmatrix} 0 & 0 \\ 0 & \beta x_2 \end{bmatrix}$
3. $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$

The random variable u_1 is a standard 1-dimensional Brownian motion independent with u_2.

3.2 Observations

We aim to estimate the position $x_1(t)$ and the speed $x_2(t)$ of the car. Suppose that we can only preform measurements on the position of the form $h(x(t)) = h(x_1(t))$, where h is a known function. These measurements suffers of errors due to such things as instrumental errors. We introduce informally a noise term of the form $\xi(x_1(t), x_2(t))W(t)$, where W is a standard 1-dimensional white noise and ξ is known function.
Since $W(t)$ is the derivative in distribution of a 1-dimensional Brownian motion $w(t)$ and since is equivalent to observe $\{Z(s)\,,\ 0 \leq s \leq t\}$ or $\{\int_0^s Z(u)du\,,\ 0 \leq s \leq t\}$, see [69], then
Observation equation: $z(t) = \int_0^t h(x(s))ds + \int_0^t \xi(x(s))dw(s)$,
where $w(t)$ is a 1-dimensional standard Brownian motion independent of the state noise $u(t)$. The observer is at the initial position $x_1(0)$ and he measures the distance $Z(t)$ from his position to the current position of the car $x_1(t)$, see Figure 3.2. This means that the function h is given by

$$h(x) = 2R\sin(\frac{x}{2R})\,, \quad \forall\ x \in \mathbb{R}\,.$$

Moreover, suppose that the function ξ is of the form

$$\xi(x_1, x_2) = \xi_0 + \xi_1 x_1 + \xi_2 x_2\,.$$

where $\xi_i\,,\ i \in \{1, 2, 3\}$ are some known non negative real numbers. Then,

$$z(t) = 2R\int_0^t \sin(\frac{x_1(s)}{2R})ds + \int_0^t \big(\xi_0 + \xi_1 x_1(s) + \xi_2 x_2(s)\big)dw(s)\,. \tag{3.8}$$

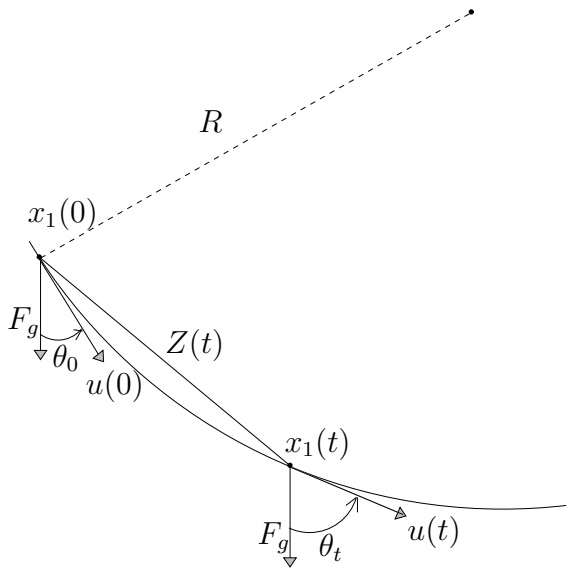

Figure 3.2: Observations

3.3 Discretization of the system

The state equation (3.7) is a 2-dimensional SDE which can be written as $dx_1(t)' = F(x_1(t), x_1(t)')dt + G(x_1(t), x_1(t))du_2(t)$, where $x_1(t)'$ denotes the sample path derivative of the process $x_1(t)$. Following the terminology in [12] and [31], a process satisfying such an equation is called a second order Ito process (SOIP).
The coefficients of the state equation (3.7) do not satisfy the classical conditions of the existence and uniqueness Theorem, since the drift term contains a second order polynomial term. In particular, we note that the conditions of the convergence of the Euler discretization in Appendix B.4 are not satisfied.
In this section we prove the existence and the uniqueness of the solution to the SDE (3.7) and we prove that the Euler discretization converges strongly to this solution with the same order of convergence, ie. $\gamma = 1/2$.
The discretization of the observation will be easily deduced since the coefficients of the observation equation (3.8) are Lipschitz.

3.3.1 The solution of the state equation (3.7)

Although the Lipschitz and the linear growth conditions are not satisfied for the SDE (3.6), the drift coefficient of the second component of contains a square order term $x_2^2\mathbf{I}_{\{x_2\geq 0\}}$, we prove existence and uniqueness of a strong solution.
In fact, we prove existence and uniqueness for a family of SDE's with non Lipschitz conditions and non linear growth bounds then deduce existence and uniqueness for the SDE (3.6).
Let us consider a second order Ito process $X_t = (X_t^1, X_t^2)$ such that for all $t \geq 0$,

$$\left\{ \begin{array}{rcl} dX_t^1 & = & X_t^2\, dt \\ dX_t^2 & = & \{\varphi(X_t^1, X_t^2) + \eta_1 X_t^2 - \eta_2 (X_t^2)^2 \mathbf{I}_{\{X_t^2\geq 0\}}\}dt + \{\sigma_1 + \sigma_2 X_t^2\}dU_t^2 \end{array} \right. \tag{3.9}$$

where

1. $X_0 = (a, b) \in \mathbb{R}^2$, where $a, b \geq 0$
2. $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is Lipschitz, continuous and admit linear growth bound
3. $(\eta_1, \eta_2) \in \mathbb{R}^2$ is such that $\eta_i > 0\,,\ i = 1, 2$ and $(\sigma_1, \sigma_2) \in \mathbb{R}^2$ is such that $\sigma_2 \neq 0$
4. $U_t = \left(U_t^i\right)_{i=1,2}$ is a 2-dimensional standard Brownian motion

The equation (3.9) can also be written in the following form:

$$dX_t = b(X_t)dt + \sigma(X_t)dU_t\ , \tag{3.10}$$

where $b(x) = \begin{bmatrix} x_2 \\ \varphi(x_1, x_2) + \eta_1 x_2 - \eta_2 x_2^2 \mathbf{I}_{\{x_2 \geq 0\}} \end{bmatrix}$ and $\sigma(x) = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_1 + \sigma_2 x_2 \end{bmatrix}$.

Remark 3.3.1. a. A reference filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is a complete filtration generated by a given Brownian motion on Ω

b. The pathwise uniqueness of solutions of (3.10) holds if whenever X and Y are two weak solutions defined on the same probability space and with the same reference family $\{\mathcal{F}_t\}_{t\geq 0}$ and the same $\mathcal{F}_t$-Brownian motion such that $X_0 = Y_0 = (a_1, a_2)$, then for all $t \geq 0$ we get $X_t = Y_t$ a.s

c. The strong uniqueness holds when given an initial condition $X_0 = (a_1, a_2)$ and an $\mathcal{F}_t$-Brownian motion $(\tilde{U}_t)$, there exists one and only one continuous $\mathcal{F}_t$-adapted process (X_t) satisfying (3.11)

The stochastic differential equation (3.11) is time homogeneous since the functions b and σ are time independent and the initial condition $X_0 = (a, b)$ is non random. In addition, the functions b and σ are continuous. Theorem 4.2.3 in [35] secures, up to an explosion time e, the existence of a weak solution. That is, given a reference filtration $\{\mathcal{F}_t\}_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, P)$ there exists an $\mathcal{F}_t$-Brownian motion $(\tilde{U}_t)$ and a continuous $\mathcal{F}_t$-adapted process (X_t) such that for almost all $\omega \in \Omega$ and all $t \in [0, e(\omega))$:

$$X_t(\omega) = X_0 + \int_0^t b(X_s(\omega))ds + \int_0^t \sigma(X_s(\omega))d\tilde{U}_s(\omega)\,. \tag{3.11}$$

However, the functions b and σ are continuous and locally Lipschitz, then by Theorem 4.3.1. in [35] the pathwise uniqueness holds and hence it has a unique strong solution by Theorem 4.1.1. in [35].
It remains to prove the global existence of the weak solution, i.e. $e = \infty$ a.s.
We apply the method of transformation of drift, see for example [35] pp. 190-197.
Denoting by for all $x \in \mathbb{R}^2$,

i. $\tilde{\varphi}(x) = \varphi(x) - \frac{\sigma_1\eta_2}{\sigma_2} x_2 \mathbf{I}_{\{x_2<0\}} - \frac{\sigma_1}{\sigma_2}(\eta_1 + \frac{\sigma_1\eta_2}{\sigma_2})$

ii. $\alpha_1 = \frac{1}{\sigma_2}(\eta_1 + \frac{\sigma_1\eta_2}{\sigma_2})$ and $\alpha_2 = -\frac{\eta_2}{\sigma_2}$

iii. $\gamma(x) = \begin{bmatrix} 0 \\ \alpha_1 + \alpha_2 x_2 \mathbf{I}_{\{x_2 \geq 0\}} \end{bmatrix}$

Then, for all $x = (x_1, x_2) \in \mathbb{R}^2$,

$$b(x) = \tilde{b}(x) + \sigma(x)\gamma(x)\,,$$

where $\tilde{b}(x) = \begin{bmatrix} x_2 \\ \tilde{\varphi}(x) \end{bmatrix}$. The functions $\tilde{\varphi}$ and $\tilde{b}$ are continuous and Lipschitz.
Let us consider an $\mathcal{F}_t$-Brownian motion $(U_t)_{t\geq 0}$ and the stochastic differential equation

$$dX_t = \tilde{b}(X_t)dt + \sigma(X_t)dU_t\,, \quad X_0 = (a,b) \in \mathbb{R}^2_+\,. \tag{3.12}$$

It is a classical result that the equation (3.12) has a unique strong solution, then there exists a continuous $\mathcal{F}_t$-adapted process X_t such that

$$X_t = X_0 + \int_0^t \tilde{b}(X_s)ds + \int_0^t \sigma(X_s)dU_s\,. \tag{3.13}$$

Moreover, $E[\sup_{0\leq s\leq t}|X_s|^{2p}] < \infty$ for every $t \geq 0$ and $p \geq 1$.
Let us define the following two processes:

$$\begin{aligned} M_t &= \exp\{-\frac{1}{2}\int_0^t |\gamma(X_s)|^2 ds + \int_0^t \gamma(X_s)dU_s\} \\ \tilde{U}_t &= U_t - \int_0^t \gamma(X_s)ds \end{aligned} \tag{3.14}$$

If we prove that the process M_t is a martingale, then by Theorem 4.4.1. and its corollary in [35] the process $(\tilde{U}_t)_{t\geq 0}$ is an $\mathcal{F}_t$-Brownian motion and the process $(X_t)_t$ is a solution of the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)d\tilde{U}_t$$

Lemma 3.3.2. *The process $(M_t)_{t\geq 0}$ is an $\mathcal{F}_t$-martingale.*

Proof. The process $Y_t = \int_0^t \gamma(X_s)dU_s$ is a square integrable $\mathcal{F}_t$-martingale, so by Theorem 3.5.2. in [35] the process M_t is an $\mathcal{F}_t$-supermartingale. It is martingale if its expectation is equal to one.
Let us prove that for all $T > 0$

$$E[M_t] = 1\,, \quad \forall\, t \in [0,T]\,.$$

We check first that there exists a constant $C < \infty$ such that

$$E[M_t|X_t|^2] < C\,, \quad \forall t \in [0,T]\,. \tag{3.15}$$

Ito's formula gives us the following equalities:

1. $d|X_t|^2 = 2X_t^T(\tilde{b}\,dt + \sigma\,dU_t) + tr(\sigma\sigma^T)dt)$

2. $dM_t|X_t|^2 = \big(M_t|X_t|^2\gamma^T + 2M_tX^T\sigma\big)dU_t + M_t\big(2X_t^T\tilde{b} + tr(\sigma\sigma^T) + X_t^T\sigma\gamma\big)dt$

3. For $\epsilon > 0$, we take $V(t,x) = V(x) = \frac{x}{1+\epsilon x}$, then

$$\begin{aligned} d\frac{M_t|X_t|^2}{1+\epsilon M_t|X_t|^2} &= \frac{1}{(1+\epsilon M_t|X_t|^2)^2}(M_t|X_t|^2\gamma^T + 2M_tX_t^T\sigma)dU_t \\ &\quad + \big[\frac{M_t}{(1+\epsilon M_t|X_t|^2)^2}(2X_t^T\tilde{b} + tr(\sigma\sigma^T) + X_t^T\sigma\gamma) \\ &\quad - \frac{\epsilon}{(1+\epsilon M_t|X_t|^2)^3}|2M_tX_t^T\sigma + M_t|X_t|^2\gamma^T|^2\big]dt \end{aligned}$$

Integrating from 0 to t and taking the expectation, we see that the first term is zero, then taking the derivative w.r.t. t we get

$$\begin{aligned} \frac{d}{dt}E[\frac{M_t|X_t|^2}{1+\epsilon M_t|X_t|^2}] &\leq E[\frac{M_t}{(1+\epsilon M_t|X_t|^2)^2}(2X_t^T\tilde{b} + tr(\sigma\sigma^T) + X_t^T\sigma\gamma)] \\ &\quad - E[\frac{\epsilon}{(1+\epsilon M_t|X_t|^2)^3}|2M_tX_t^T\sigma + M_t|X_t|^2\gamma^T|^2] \end{aligned}$$

The second term is negative, then

$$\frac{d}{dt}E[\frac{M_t|X_t|^2}{1+\epsilon M_t|X_t|^2}] \leq E[\frac{M_t}{(1+\epsilon M_t|X_t|^2)^2}(2X_t^T\tilde{b} + tr(\sigma\sigma^T) + X_t^T\sigma\gamma))]$$

Moreover,

$$\begin{aligned} X_t^T\sigma\gamma &= [X_t^1 \; X_t^2]\begin{bmatrix} 0 & 0 \\ 0 & \sigma_1 + \sigma_2X_t^2 \end{bmatrix}\begin{bmatrix} 0 \\ \alpha_1 + \alpha_2X_t^2\mathbf{I}_{\{X_t^2\geq 0\}} \end{bmatrix} \\ &= X_t^2(\sigma_1 + \sigma_2X_t^2)(\alpha_1 + \alpha_2X_t^2\mathbf{I}_{\{X_t^2\geq 0\}}) \end{aligned}$$

Since $\sigma_1\alpha_2 + \alpha_1\sigma_2 = \eta_1 > 0$ and $\sigma_2\alpha_2 < 0$, then $\sigma_2\alpha_2(X_t^2)^3\mathbf{I}_{\{X_t^2\geq 0\}} \leq 0$ and there exists a constant $K > 0$ such that

$$X_t^T\sigma\gamma \leq K(1 + (X_t^2)^2) \leq K(1 + |X_t|^2)\,.$$

The functions $\tilde{b}$ and σ admit linear growth bounds, then there exists a constant $K' > 0$

$$\begin{aligned} |X_t^T\tilde{b}(X_t)| &\leq K'(1 + |X_t|^2) \\ |tr(\sigma\sigma^T)| &\leq K'(1 + |X_t|^2) \end{aligned}$$

Let $\tilde{K} = 3K' + K$, $\tilde{K}$ is independent from ϵ, we deduce that

$$\begin{aligned} \frac{d}{dt}E[\frac{M_t|X_t|^2}{1+\epsilon M(t)|X_t|^2}] &\leq \tilde{K}E[\frac{M_t(1+|M_t|^2)}{(1+\epsilon M_t|X_t|^2)^2}] \\ &\leq \tilde{K}(E[\frac{M_t|X_t|^2}{1+\epsilon M_t|X_t|^2}] + E[\frac{M_t}{1+\epsilon M_t|X_t|^2}]) \\ &\leq \tilde{K}(E[\frac{M_t|X_t|^2}{1+\epsilon M_t|X_t|^2}] + 1) \end{aligned}$$

In the last inequality we use the fact that $E[M_t] \leq 1$ since M_t is a supermartingale and that $M_0 = 1$.
We apply Gronwall's inequality, see Appendix A.2, we obtain a constant $C > 0$ independent of ϵ such that

$$E[\frac{M_t|X_t|^2}{1+\epsilon M_t|X_t|^2}] \leq C$$

Taking $\epsilon \to 0$, Fatou's Lemma gives (3.15), ie $E[M_t|X_t|^2] < C$, for all $t \in [0,T]$.
We use Ito's formula to obtain

$$d\frac{M_t}{1+\epsilon M_t} = \frac{M_t\gamma(X_t)^T dU_t}{(1+\epsilon M_t)^2} - \frac{\epsilon M_t^2|\gamma(X_t)|^2 dt}{(1+\epsilon M_t)^3}\,.$$

Integrating from 0 to t and taking the expectation, we get

$$E[\frac{M_t}{1+\epsilon M_t}] = \frac{1}{1+\epsilon} - \epsilon E[\int_0^t \frac{M_s^2|\gamma(x_s)|^2 ds}{(1+\epsilon M_s)^3}]. \tag{3.16}$$

The integrand term $\frac{M_s^2|\gamma(x_s)|^2 ds}{(1+\epsilon M_s)^3}$ in (3.16) is bounded by $DM_t(|X_t|^2+1)$, for some constant D. This term is integrable by (3.15). Since $E[\frac{M_t}{1+\epsilon M_t}] \leq E[M_t] \leq 1$, Lebesgue's theorem gives that

$$E[\frac{M_t}{1+\epsilon M_t}] \to E[M_t]\,.$$

The result follows by taking ϵ to 0 in (3.16). □

To conclude, we have proved the following Theorem

Theorem 3.3.3. *For every $(a,b) \in \mathbb{R}_+^2$, the stochastic differential equation (3.10) admits one and only one strong solution $(X_t)_{t\geq 0}$ with $X_0 = (a,b)$. The solution is called a second order Ito process.*

Corollary 3.3.4. *For every $(a,b) \in \mathbb{R}_+^2$, the stochastic differential equation (3.7) or also the stochastic differential system (3.6) admits one and only one strong and pathwise solution x such that $x(0) = (a,b)$.*

3.3.2 Discretization of the state equation (3.7)

The speed of a car can not exceed a maximum speed that we denote M_c. We will prove the strong convergence of the Euler-Maruyama discretization of the state process $\{x(t)\}_t$ on a sufficient large subset Ω_{T,B_c} of Ω, the subset Ω_{T,B_c} depends on M_c and the time tracking duration T.
Suppose that $Y = (Y_t)_{t\geq 0}$ and $Z = (Z_t)_{t\geq 0}$ are two continuous $\mathcal{F}_t$-adapted processes with values in $\mathbb{R}^p$, for some $p \geq 1$, that are the unique pathwise solutions of the following SDE's

$$dY_t = F_1(Y_t)dt + G_1(Y_t)dW_t \tag{3.17}$$
$$dZ_t = F_2(Z_t)dt + G_2(Z_t)dW_t \tag{3.18}$$

where $(W_t)_t$ is a q-dimensional $\mathcal{F}_t$-Brownian motion and the vector function F_i and the matrix function G_i are locally Lipschitz for $i = 1, 2$.
Let $T > 0$ and $B \in B(\mathbb{R}^p)$ be closed and define $\Omega_{T,B} \subset \Omega$ by

$$\Omega_{T,B} = \{\omega \in \Omega\,,\ Y_t(\omega), Z_t(\omega) \in B\,,\ \forall\, t \in [0,T]\}\,.$$

Lemma 3.3.5. *Suppose that $Y_0 = Z_0 \in \mathbb{R}^p$, non random.*
If for almost all $\omega \in \Omega_{T,B}$ and for all $t \in [0,T]$

$$F_1(Y_t(\omega)) = F_2(Y_t(\omega)) \quad \textit{and} \quad G_1(Y_t(\omega)) = G_2(Y_t(\omega))\,,$$

then for almost all $\omega \in \Omega_{T,B}$ and for all $t \in [0,T]$

$$Y_t(\omega) = Z_t(\omega)\,.$$

Proof. Let $\{B_k\}_{k\geq 0}$ be an increasing sequence of compacts in $\mathbb{R}^p$ such that

$$B_k \nearrow B \quad \text{as} \quad k \to \infty\,.$$

Denoting by $\Omega_{T,B_k} = \{\omega \in \Omega\,,\ Y_t(\omega), Z_t(\omega) \in B_k\,,\ \forall\, t \in [0,T]\}$. Then,

i. $\Omega_{T,B_k} \nearrow \Omega_{T,B}$ as $k \to \infty$

ii. For almost all $\omega \in \Omega_{T,B_k}$ and for all $t \in [0,T]$,

$$F_1(Y_t(\omega)) = F_2(Y_t(\omega)) \quad \text{and} \quad G_1(Y_t(\omega)) = G_2(Y_t(\omega))$$

It is sufficient to prove the result on Ω_{T,B_k} for every $k \geq 0$.
Let $k \geq 0$ be arbitrary fixed, since B_k is a compact and the functions F_i, G_i, $i = 1, 2$, are locally Lipschitz then there exists a constant $C_k > 0$ such that for $i = 1, 2$ and for all $y, z \in B_k$

$$\begin{aligned} |F_i(y) - F_i(z)| &\leq C_k|y - z| \\ \|G_i(y) - G_i(z)\| &\leq C_k|y - z| \end{aligned}$$

In addition, almost surely on Ω_{T,B_k} we have

$$Y_t - Z_t = \int_0^t (F_2(Y_s) - F_2(Z_s))ds + \int_0^t (G_2(Y_s) - G_2(Z_s))dW_s$$

Then, using that $(a+b)^2 \leq 2(a^2+b^2)$ and Ito's isometry we get

$$\begin{aligned} E[\mathbf{I}_{\Omega_{T,B_k}}|Y_t - Z_t|^2] &\leq 2T \int_0^t E[\mathbf{I}_{\Omega_{T,B_k}}|F_2(Y_s) - F_2(Z_s)|^2]ds \\ &\quad +2 \int_0^t E[\mathbf{I}_{\Omega_{T,B_k}}\|G_2(Y_s) - G_2(Z_s)\|^2]ds \\ &\leq 2(T+1)C_k^2 \int_0^t E[\mathbf{I}_{\Omega_{T,B_k}}|Y_s - Z_s|^2]ds \end{aligned}$$

where $\mathbf{I}_{\Omega_{T,B_k}}$ is the indicator function of Ω_{T,B_k}.
Gronwall's inequality implies that $E[\mathbf{I}_{\Omega_{T,B_k}}|Y_t - Z_t|^2] = 0$, then $\mathbf{I}_{\Omega_{T,B_k}}|Y_t - Z_t|^2 = 0$ almost surely. □

Let $T > 0$ be the tracking duration and $M_c > 0$ be a high speed that can not be attained by a car. Let $\tilde{b} : \mathbb{R}^2 \to \mathbb{R}^2$ such that for all $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$,

$$\tilde{b}(x) = \begin{bmatrix} x_2 \\ a_0 + g\cos(\theta_0 + \frac{x_1}{R}) - a_1 x_2 - \Phi(x_2) \end{bmatrix}$$

where for all $y \in \mathbb{R}$,

$$\Phi(y) = \begin{cases} a_2 y^2 \mathbf{I}_{\{y \geq 0\}} & \text{if } y \leq M_c \\ a_2 M_c^2 & \text{if not} \end{cases} \tag{3.19}$$

It is clear that the function Φ is continuous, Lipschitz and admit a linear growth bound then it is also the case of the function $\tilde{b}$.
Let $(\tilde{x}(t))_t$ the unique pathwise solution of the following SDE

$$\begin{cases} \tilde{x}(0) = x(0) \in (\mathbb{R}_+)^2 \\ d\tilde{x}(t) = \tilde{b}(\tilde{x}(t))dt + \sigma(\tilde{x}(t))dU_t \end{cases} \tag{3.20}$$

We define the subspace Ω_{T,B_c} of Ω by

$$\Omega_{T,B_c} = \{x(t), \tilde{x}(t) \in B_c\,,\ \forall t \in [0,T]\}\,,$$

where $B_c = \mathbb{R} \times (-\infty, M_c]$.

Lemma 3.3.6. *For all $t \in [0,T]$,*

$$x(t) = \tilde{x}(t) \quad a.s. on \Omega_{T,B_c}\,.$$

Proof. It is clear that:

i. $b(x(t)) = \tilde{b}(x(t))$ almost surely in Ω_{T,B_c} and for all $t \in [0,T]$

ii. The diffusion coefficients of the two processes $x(t)$ and $\tilde{x}(t)$ are the same

The Lemma 3.3.5 implies the result. □

Let $N \geq 1$ be an integer such that $\frac{T}{N} < 1$ and $Y^\delta = (Y_t^\delta)_{0 \leq t \leq T}$ be the Euler-Maruyama time discretization process associated to the process $\{\tilde{x}(t)\}_{0 \leq t \leq T}$. Then,

$$\begin{cases} Y_0^\delta = x(0)\,, \\ Y_t^\delta = Y_k^\delta + (t - \tau_k)\tilde{b}(Y_k^\delta) + \sigma(Y_k^\delta)(U_t - U_k)\,, \quad \text{if } t \in [\tau_k, \tau_{k+1}]\,, \end{cases} \tag{3.21}$$

where

- $Y_k^\delta = Y_{\tau_k}^\delta$ and $U_k = U_{\tau_k}$
- $\tau_k = k\delta$ and $\delta = \frac{T}{N}$

In particular, if we denote $\Delta_k U = U_{k+1} - U_k$, then

$$Y_{k+1}^\delta = Y_k^\delta + \delta\tilde{b}(Y_k^\delta) + \sigma(Y_k^\delta)\Delta_k U\,,\ \text{for every } k \in \{0, 1, \ldots, N-1\}\,.$$

Proposition 3.3.7. *The Euler-Maruyama time discretization process Y^δ converges strongly with an order $1/2$to the process x on Ω_{T,B_c}. That is, for some constant $C > 0$ independent of δ*

$$E[\, \mathbf{I}_{\Omega_{T,B_c}} \left|x(T) - Y_T^\delta\right| \,] \leq C\, \delta^{1/2}\,.$$

Proof. The functions $\tilde{g}$ and σ are Lipschitz, admits growth bound and are time independent, then the functions $\tilde{g}$ and σ satisfy the conditions of the Appendix B.4. In addition, $Y_0^\delta = \tilde{x}(0) = x(0)$. This implies that the time discretization process Y^δ converges strongly to $(\tilde{x}(t))_t$ with an order $1/2$ ie. there exists a constant $C > 0$ independent of δ such that

$$E[\, \left|\tilde{x}(T) - Y_T^\delta\right| \,] \leq C\, \delta^{1/2}\,.$$

In the other hand, since $\mathbf{I}_{\Omega_{T,B}} \leq 1$ we get

$$\begin{aligned} E[\, \mathbf{I}_{\Omega_{T,B}} \left|x(T) - Y_T^\delta\right| \,] &= E[\, \mathbf{I}_{\Omega_{T,B}} \left|\tilde{x}(T) - Y_T^\delta\right| \,] \\ &\leq E[\, \left|\tilde{x}(T) - Y_T^\delta\right| \,] \\ &\leq C\, \delta^{1/2}\,. \end{aligned}$$

This proves the result. □

Remark 3.3.8. There exists a subsequence $\delta_k \to 0$ as $k \to \infty$ such that $Y_T^{\delta_k} \to \tilde{x}(T)$ almost surely. In particular, $Y_T^{\delta_k} \to x(T)$ almost surely on $\Omega_{T,B}$.

The Proposition 3.3.7 is almost useless if we have no knowledge about the constant C. We will give next an upper bound on C, this will permit us the control of the error of the convergence.
It is sufficient to find an explicit constant $C' = C'(T, \tilde{b}, \sigma, \epsilon)$ such that $E[\, \left|\tilde{x}(T) - Y_T^\delta\right| \,] \leq C'\, \delta^{1/2}$ for $\delta \leq \epsilon$ for some fixed $\epsilon \in (0, 1)$.
We need the following two Lemmas:

Lemma 3.3.9. *For all $(x, y) \in \mathbb{R}^2$, the functions $\tilde{b}$ and σ satisfy the following inequalities*

i. $|\sigma(x) - \sigma(y)| \leq \beta|x - y|$

ii. $< x - y\,,\, \tilde{b}(x) - \tilde{b}(y) > \leq\, \Delta_1|x - y|^2$

iii. $|\tilde{b}(x) - \tilde{b}(y)|^2 \leq \Delta_2|x - y|^2$

iv. $|\tilde{b}(x)| \leq A_1 + A_2|x|$

where $\Delta_1 = 1/2 + g/2R$, $\Delta_2 = 1 + g/R + a_1 + 2a_2M_c$, $A_1 = a_0 + g$ and $A_2 = \sqrt{1 + a_1^2} + a_2M_c$.

Proof. Based on the expressions of the functions $\tilde{b}$ and σ it is not difficult to verify the inequalities of the present Lemma. □

Lemma 3.3.10. *The process Y^δ given in (3.21) satisfies*

$$\sup_{0\le k\le N} E[\,|Y_k^\delta|^2\,] \le \frac{2A_1^2}{c}\big(\exp((1+c)T)-1\big) + \exp((1+c)T)|x(0)|^2\,.$$

where $c = \beta^2 + 2(\sqrt{1+a_1^2}+a_2M_c)^2$.

Proof. Let $k\in\{0,\dots,N-1\}$. Ito's formula gives us, for all $t\in[\tau_k,\tau_{k+1}]$,

$$d|Y_t^\delta|^2 = \{2 < Y_t^\delta\,,\,\tilde b(Y_k^\delta) > +|\sigma(Y_k^\delta)|^2\}\,dt + 2 < Y_t^\delta\,,\,\sigma(Y_k^\delta) > du_t\,.$$

We integrate $d|Y_t^\delta|^2$ from τ_k to t and we take the expectation we get

$$E[\,|Y_t^\delta|^2\,] = E[\,|Y_k^\delta|^2\,] + \delta E[\,|\sigma(Y_k^\delta)|^2\,] + 2\int_{\tau_k}^t E[< Y_s^\delta\,,\,\tilde b(Y_k^\delta) >]ds\,.$$

Using Lemma 3.3.9 and the fact that $2<x,y>\le |x|^2+|y|^2$ we get

$$E[\,|Y_t^\delta|^2\,] \le 2A_1^2\delta + \big(1+(\beta^2+2A_2^2)\delta\big)E[\,|Y_k^\delta|^2\,] + \int_{\tau_k}^t E[\,|Y_s^\delta|^2\,]\,ds\,.$$

If we denote by $c=\beta^2+2A_2^2$, then Gronwall inequality implies

$$E[\,|Y_t^\delta|^2\,] \le \exp(\delta)\big(2A_1^2\delta + (1+c\,\delta)E[\,|Y_k^\delta|^2\,]\big)\,,\ \ \forall t\in[\tau_k,\tau_{k+1}]\,.$$

In particular, for $t=\tau_{k+1}$ we have

$$E[\,|Y_{k+1}^\delta|^2\,] \le \exp(\delta)\big(2A_1^2\delta + (1+c\,\delta)E[\,|Y_k^\delta|^2\,]\big)\,.$$

Recall that $E[\,|Y_0^\delta|^2\,] = |x(0)|^2$. We deduce by induction that

$$E[\,|Y_{k+1}^\delta|^2\,] \le 2A_1^2\delta\{\sum_{j=1}^k (1+c\delta^{j-1})\exp(j\delta)\} + \exp(k\delta)(1+c\,\delta)^k|x(0)|^2\,. \qquad (3.22)$$

Finally, the following two observations give us the result:

1. Since $(1+c\delta)^k \le (1+c\frac{T}{N})^N \le \exp(cT)$ and $c\,\delta\exp(\delta)\le(1+c\,\delta)\exp(\delta)-1$,

$$\begin{aligned}\sum_{j=1}^k (1+c\,\delta)^{j-1}\exp(j\delta) &= \exp(\delta)\frac{(1+c\,\delta)^k\exp(k\delta)-1}{(1+c\,\delta)\exp(\delta)-1}\\ &\le \frac{1}{c\,\delta}(\exp((1+c)T)-1)\end{aligned}$$

2. $\exp(k\delta)(1+c\delta)^k|x(0)|^2 \le \exp((1+c)T)|x(0)|^2$

□

Let $\epsilon \in (0,1)$, we define Γ_ϵ by

$$\Gamma_\epsilon = (1+\Delta_2)\{2\epsilon A_1^2 T + B\} \tag{3.23}$$

where

- $B = \big(2\epsilon T A_2^2 + T\beta^2\big)\{\frac{2A_1^2}{c}(\exp((1+c)T) - 1) + \exp((1+c)T)|x(0)|^2\}$
- $c = \beta^2 + 2(\sqrt{1+a_1^2} + a_2 M_c)^2$

Theorem 3.3.11. *If $\delta \leq \epsilon$, then*

$$E[\,\big|\tilde{x}(T) - Y_T^\delta\big|^2\,] \leq \delta\,\Gamma_\epsilon \exp((1+2(\Delta_1+\beta^2))T)\,. \tag{3.24}$$

Before proving the Theorem, let us define the process $X^\delta = \{X_t^\delta\,,\ t \in [0,T]\}$ by

$$X_t^\delta = \sum_{k=0}^{N-1} Y_k^\delta \mathbf{I}_{[\tau_k,\tau_{k+1})}(t) + Y_T^\delta \mathbf{I}_{\{T\}}(t)\,. \tag{3.25}$$

In particular, for all $t \in [0,T]$

$$Y_t^\delta = x(0) + \int_0^t \tilde{b}(X_s^\delta)ds + \int_0^t \sigma(X_s^\delta)du_s\,. \tag{3.26}$$

Moreover, for all $t \in [0,T]$

$$E[|Y_s^\delta - X_s^\delta|^2] \leq 2\delta^2 A_1^2 + (2\delta^2 A_2^2 + \delta\beta^2) \sup_{0\leq k\leq N} E[\,|Y_k^\delta|^2\,] \tag{3.27}$$

In fact, for $k \in \{0,\ldots,N-1\}$ and for $s \in [\tau_k, \tau_{k+1})$ we have

$$\begin{aligned} |Y_s^\delta - X_s^\delta|^2 &= |(s-\tau_k)\tilde{b}(Y_k^\delta) + \sigma(Y_k^\delta)(u_s - u_k)|^2 \\ &= (s-\tau_k)^2|\tilde{b}(Y_k^\delta)|^2 + |\sigma(Y_k^\delta)(u_s-u_k)|^2 \\ &\quad +2(s-\tau_k) < \tilde{b}(Y_k^\delta)\,,\ \sigma(Y_k^\delta)(u_s-u_k) > \ . \end{aligned}$$

Since $u_s - u_k$ and Y_k^δ are independent, $E[u_s - u_k] = 0$, $E[|u_s-u_k|^2] \leq \delta$, then Fubini's Theorem, Lemma 3.3.9 and Lemma 3.3.10 gives the inequality (3.27).

Proof. (Theorem 3.3.11)
For all $t \in [0,T]$, we denote $e_t = \tilde{x}(t) - Y_t^\delta$, then

$$e_t = \int_0^t (\tilde{b}(\tilde{x}(s)) - \tilde{b}(X_s^\delta))ds + \int_0^t (\sigma(\tilde{x}(s)) - \sigma(X_s^\delta))du_s\,. \tag{3.28}$$

Applying the Ito formula to $|e_t|^2$ we get

$$\begin{aligned} d|e_t|^2 &= \{2 < e_t, \tilde{b}(\tilde{x}(t)) - \tilde{b}(X_t^\delta) > + |\sigma(\tilde{x}(t)) - \sigma(X_t^\delta)|^2\}dt \\ &\quad +2 < e_t, \sigma(\tilde{x}(t)) - \sigma(X_t^\delta) > du_t \end{aligned}$$

Integrating form 0 to t and taking the expectation we get

$$\begin{aligned} E[\ |e_t|^2\ ] &= 2E[\int_0^t < e_s, \tilde{b}(\tilde{x}(s)) - \tilde{b}(X_s^\delta) > ds] \\ &\quad + E[\int_0^t |\sigma(\tilde{x}(s)) - \sigma(X_s^\delta)|^2\} ds] \end{aligned} \tag{3.29}$$

Let us examine the two expectations of the right hand term in equality (3.29).
First term: Using Lemma 3.3.9 we get

$$\begin{aligned} 2E[\int_0^t < e_s, \tilde{b}(\tilde{x}(s)) - \tilde{b}(X_s^\delta) > ds] &= 2E[\int_0^t < e_s, \tilde{b}(\tilde{x}(s)) - \tilde{b}(Y_s^\delta) > ds] \\ &\quad + 2E[\int_0^t < e_s, \tilde{b}(Y_s^\delta) - \tilde{b}(X_s^\delta) > ds] \\ &\leq (2\Delta_1 + 1)E[\int_0^t |e_s|^2 ds] \\ &\quad + \Delta_2 E[\int_0^t |Y_s^\delta - X_s^\delta|^2 ds] \end{aligned} \tag{3.30}$$

where we use that $2 < x, y > \leq |x|^2 + |y|^2\ ,\ \forall x, y \in \mathbb{R}^2$.
Second term:

$$E[\int_0^t |\sigma(\tilde{x}(s)) - \sigma(X_s^\delta)|^2\} ds] \leq 2\beta^2 \{E[\int_0^t |e_s|^2\} ds] + E[\int_0^t |Y_s^\delta - X_s^\delta|^2\} ds]\} \tag{3.31}$$

Now, using (3.27), (3.30) and (3.31) in (3.29) we get

$$E[\ |e_t|^2\ ] \leq \Lambda + (1 + 2\Delta_1 + 2\beta^2) \int_0^t E[|e_s|^2] ds$$

where $\Lambda = (1 + \Delta_2)\{2\delta^2 A_1^2 T + (2\delta^2 T A_2^2 + T\delta\beta^2) \sup_{0 \leq k \leq N} E[|Y_k^\delta|^2]\}$.
The Gronwall inequality implies that

$$E[\ |e_t|^2\ ] \leq \Lambda \exp((1 + 2\Delta_1 + 2\beta^2)t)\,.$$

Finally, from Lemma 3.3.10 we get $\Lambda \leq \delta\Gamma_\epsilon$. □

Corollary 3.3.12. *If $C' = \sqrt{\Gamma_\epsilon} \exp((1/2 + \Delta_1 + \beta^2)T)$, then*

$$E[\ \mathbf{I}_{\Omega_{T,B}} \left|x(T) - Y_T^\delta\right|\ ] \leq C'\delta^{1/2}\,. \tag{3.32}$$

If $T = 1$, $R = 100$, $x(0) = 0$, $M_c = 150$ and $\epsilon = 0.01$, then $C' \approx 34.254$.

3.3.3 Discretization of the observation equation (3.8)

Let us first recall that the measurement equation is

$$z(t) = \int_0^t h(x(s)) ds + \int_0^t \xi(x(s)) dw(s)\,. \tag{3.33}$$

where $h(x) = 2R\sin(x_1/2R)$, $\xi(x) = \xi_0 + \xi_1 x_1 + \xi_2 x_2$ and ξ_i, $i \in \{1, 2, 3\}$, are known non negative real numbers.
We define the process Z^δ on $[0, T]$ by

i. $Z_0^\delta = 0$

ii. $Z_t^\delta = Z_k^\delta + (t-\tau_k)h(Y_k^\delta) + \xi(Y_k^\delta)(w(t)-w_k)$, for all $t \in [\tau_k, \tau_{k+1}]$

In particular, for all $t \in [0,T]$

$$Z_t^\delta = \int_0^t h(X_s^\delta)ds + \int_0^t \xi(X_s^\delta)dw_s \tag{3.34}$$

where the process X^δ was defined by the equation (3.25).

Lemma 3.3.13. *The process Z^δ converges strongly with order $1/2$ to the process $\tilde{z}$ given by*

$$\tilde{z}(t) = \int_0^t h(\tilde{x}(s))ds + \int_0^t \xi(\tilde{x}(s))dw(s)\,.$$

That is, there exists a constant $\tilde{C}$ independent of δ such that

$$E[\ \left|\tilde{z}(T) - Z_T^\delta\right|\ ] \leq \tilde{C}\delta^{1/2}\,.$$

Proof. The Theorem 3.3.11 and the inequality (3.27) state that there exists a constant C independent of δ such that

$$E[\ \left|\tilde{x}(t) - X_t^\delta\right|^2\ ] \leq C\delta\,, \quad \forall\ t \in [0,T]\,. \tag{3.35}$$

In addition, the functions h and ξ are Lipschitz:

$$\begin{aligned} |h(x)-h(y)| &\leq |x-y| \\ |\xi(x)-\xi(y)| &\leq (\xi_1+\xi_2)|x-y| \end{aligned}$$

Then, if we use successively that $(a+b)^2 \leq 2(a^2+b^2)$, the Cauchy-Schwartz inequality, the Ito isometry, the Fubini Theorem and the Lipschitz property of the functions h and ξ, we get

$$\begin{aligned} E[\ \left|\tilde{z}(T) - Z_T^\delta\right|^2\ ] &\leq 2TE[\int_0^t |h(\tilde{x}(s)) - h(X_s^\delta)|^2 ds] \\ &\quad +2E[\int_0^t |\xi(\tilde{x}(s)) - \xi(X_s^\delta)|^2 ds] \\ &\leq (2T+\xi_1+\xi_2)\int_0^t E[|\tilde{x}(s) - X_s^\delta|^2]ds \\ &\leq (2T+\xi_1+\xi_2)TC\delta\,. \end{aligned}$$

It is sufficient to take

$$\tilde{C} = \sqrt{(2T+\xi_1+\xi_2)TC}\ . \tag{3.36}$$

□

Remark 3.3.14. With the aid of Theorem 3.3.11, inequalities (3.27) and (3.36) it is not difficult to get an explicit expression for the constant $\tilde{C} = \tilde{C}(\epsilon, \tilde{b}, \sigma, T)$.

Theorem 3.3.15. *The time discretization process Z^δ converges strongly to the process $\{z(t)\}_{0\le t\le T}$ with an order $1/2$ on Ω_{T,B_c}. That is, for some constant $C>0$ independent of δ*

$$E[\ \mathbf{I}_{\Omega_{T,B_c}}\left|z(T)-Z_T^\delta\right|\ ]\le C\,\delta^{1/2}\,.$$

Proof. Recall that $\Omega_{T,B_c}=\{x(t),\tilde{x}(t)\in B_c=\mathbb{R}\times(-\infty,M_c]\,,\ \forall t\in[0,T]\}$.
For all $t\in[0,T]$:

i. $x(t)=\tilde{x}(t)$ a.s. on $\Omega_{T,B}$

ii. $\tilde{z}(t)=\int_0^t h(\tilde{x}(s))ds+\int_0^t \xi(\tilde{x}(s))dw(s)$ a.s. on $\Omega_{T,B}$

Then, for all $t\in[0,T]$

$$z(t)=\tilde{z}(t)\quad \text{a.s. on } \Omega_{T,B_c}\,.$$

We use Lemma 3.3.13 to conclude:

$$\begin{aligned} E[\ \mathbf{I}_{\Omega_{T,B}}\left|z(T)-Z_T^\delta\right|\ ] &= E[\ \mathbf{I}_{\Omega_{T,B}}\left|\tilde{z}(T)-Z_T^\delta\right|\ ]\\ &\le E[\ \left|\tilde{z}(T)-Z_T^\delta\right|\ ]\\ &\le \tilde{C}\delta^{1/2}\,. \end{aligned}$$

We get the result by taking $C=\tilde{C}$. □

The time discretization process Z^δ satisfies

$$\begin{aligned} Z_0^\delta &= 0\\ Z_{k+1}^\delta &= Z_k^\delta+\delta h(Y_k^\delta)+\xi(Y_k^\delta)\Delta_k w\,,\ \text{for all } k\in\{0,\dots,N-1\} \end{aligned}$$

where $\Delta_k w=w_{k+1}-w_k$. Let us define the discrete process $\{\zeta^\delta(k)\}_{0\le k\le N}$ by

$$\begin{aligned} \zeta^\delta(0) &= 0\\ \zeta^\delta(k+1) &= h(Y_k^\delta)+\xi(Y_k^\delta)w^\delta(k)\,,\ \text{for all } k\in\{0,\dots,N-1\} \end{aligned}$$

where $w^\delta(k)=\delta^{-1}\Delta_k w$. The process $\{w^\delta\}$ is a 1-dimensional standard white process and for all $k\in\{1,\dots,N\}$

$$Z_k^\delta=\delta\sum_{j=0}^{k}\zeta^\delta(j)\,. \tag{3.37}$$

In particular,

- For all $0\le k\le N$, $\sigma\{Z_j^\delta,\,0\le j\le k\}=\sigma\{\zeta^\delta(j),\,0\le j\le k\}$.
- To calculate Z_k^δ we have to store the $\zeta^\delta(j)$'s for $0\le j\le k-1$.

The filtering algorithms are recursive. That's means that we need only to keep the new information, represented here by $\zeta^\delta(k+1)$.
But at time $k+1$, the observation $\zeta^\delta(k+1)$ seems to be a measurement on the state at time k rather a measurement at time $k+1$ and just the noise $w^\delta(k)=\delta^{-1}(w_{k+1}-w_k)$ depends on the point time $k+1$.
It is more appropriate to take as observation the process z^δ defined by

$$z^\delta(k)=h(Y_k^\delta)+\xi(Y_k^\delta)w^\delta(k-1)\quad\forall k\in\{0,\dots,N\}\,. \tag{3.38}$$

Still the question of the validity of these observations.

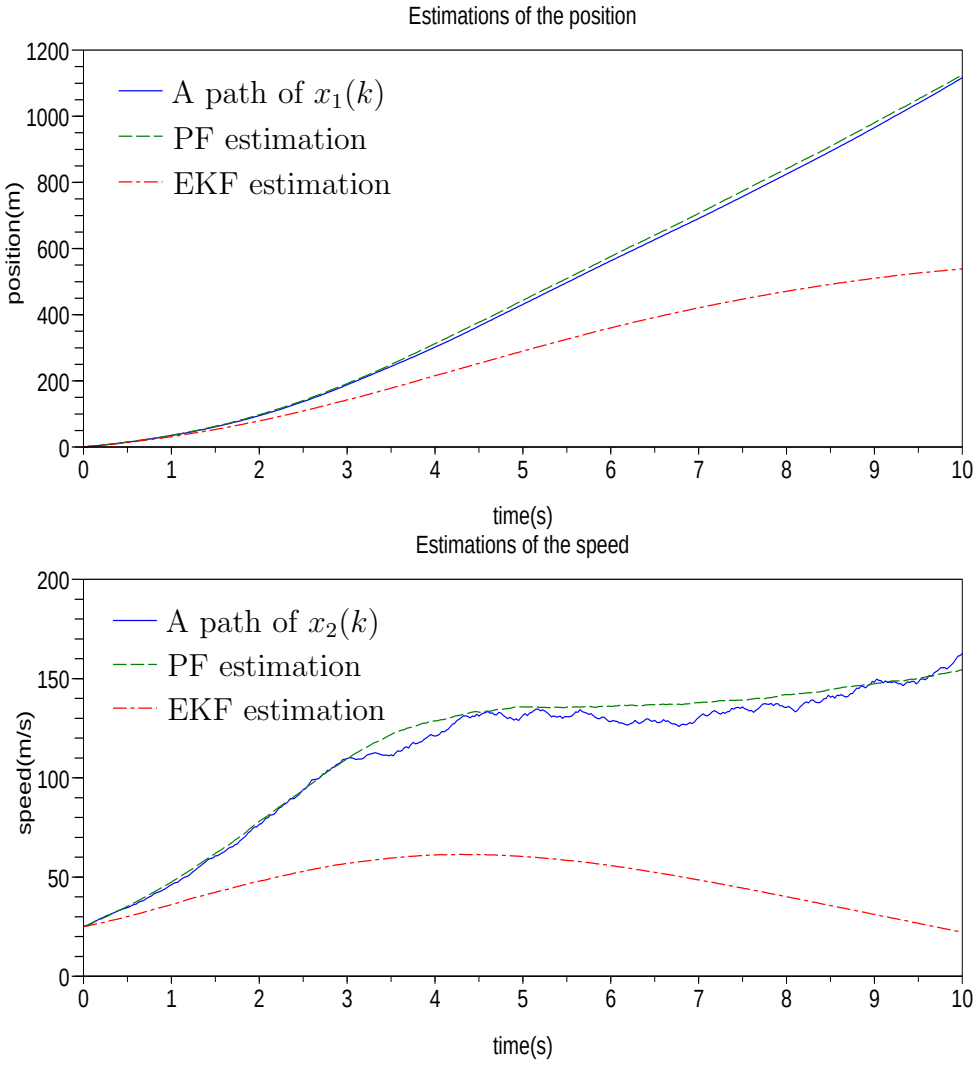

Figure 3.3: The particle filter and the EKF estimates of the position and the speed of the car.

Proposition 3.3.16.

1. *For all* $0 \leq k \leq N$, $\sigma\{\zeta^\delta(j)\,,\ 0 \leq j \leq k\} = \sigma\{z^\delta(j)\,,\ 0 \leq j \leq k\}$
2. *There exists a constant* Δ *independent of* δ *such that for all* $k \in \{0, \ldots, N\}$

$$E[\ |\zeta^\delta(k) - z^\delta(k)|\ ] \leq \Delta\,\delta\,.$$

Proof. Let $k \in \{0, \ldots, N-1\}$, then

$$\begin{aligned} |\zeta^\delta(k+1) - z^\delta(k+1)| &\leq |h(Y_k^\delta) - h(Y_{k+1}^\delta)| + |\xi(Y_k^\delta) - \xi(Y_{k+1}^\delta)|\dot{|}w^\delta(k)| \\ &\leq |Y_k^\delta - Y_{k+1}^\delta|\{1 + (\xi_1 + \xi_2)|w^\delta(k)|\}\,. \end{aligned}$$

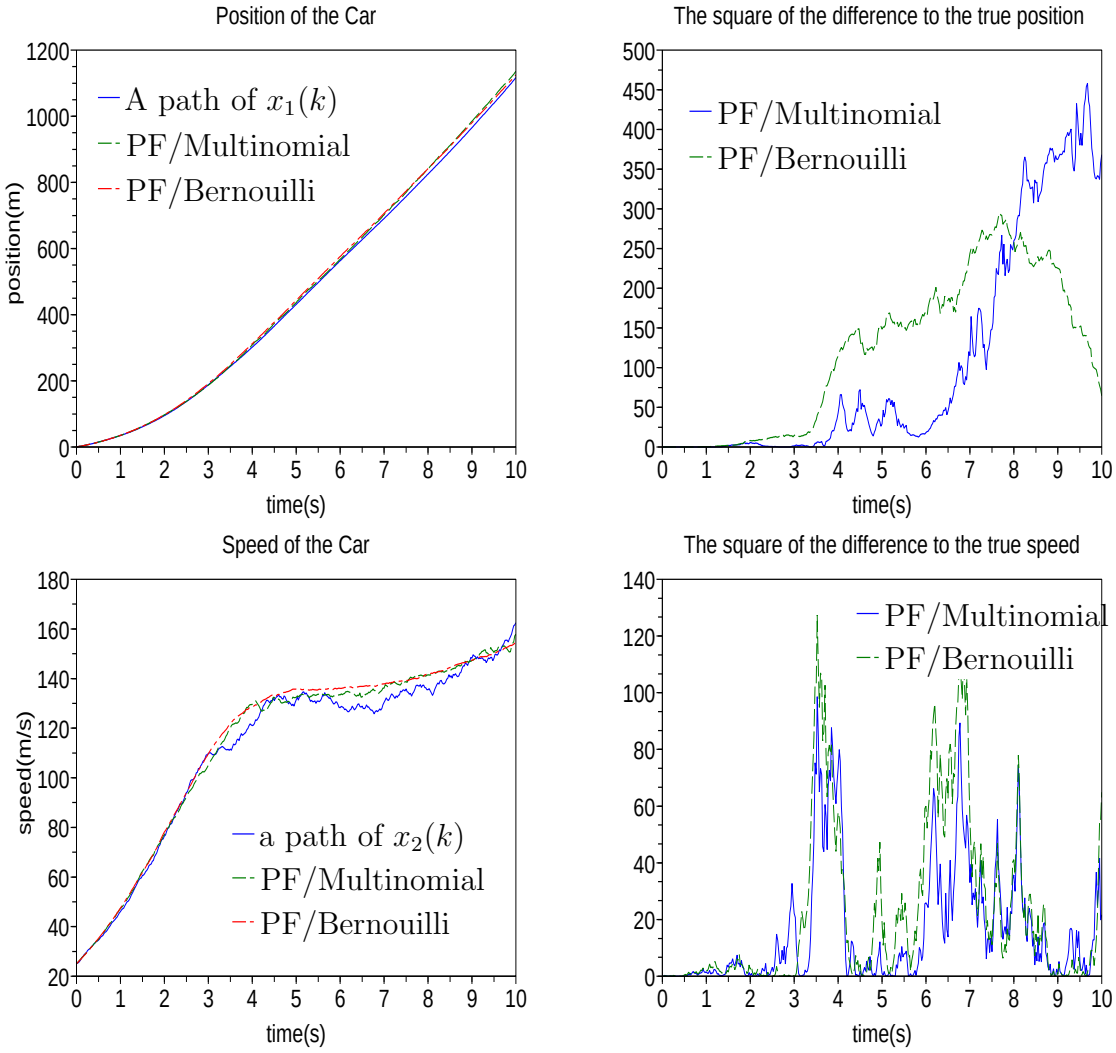

Figure 3.4: Branching mechanisms comparison: Multinomial/Bernouilli

Since the processes w^δ and Y^δ are independent and $E[|w^\delta(k)|] \leq \delta$ then using (3.27) and Lemma 3.3.10 we get

$$\begin{aligned} E[|\zeta^\delta(k+1) - z^\delta(k+1)|] &\leq \big(1 + (\xi_1+\xi_2)\delta\big) E[\,|Y_k^\delta - Y_{k+1}^\delta|^2\,]\,. \\ &\leq \delta\big(1+\delta(\xi_1+\xi_2)\big)\big(2\delta A_1^2 + (2\delta A_2^2 + \beta^2)\sup_k E[\,|Y_k^\delta|^2\,]\big)\,. \end{aligned}$$

It is sufficient to take $\Delta = (1+\xi_1+\xi_2)\{2A_1^2 + (2A_2^2+\beta^2)\sup_k E[\,|Y_k^\delta|^2\,]\}$. □

3.4 Estimations of the position and the speed

The discrete time model, equation (3.21) and equation (3.38), approximate the continuous time model, equation (3.7) and equation (3.8).
Let us fix δ, we denote $x(k) = Y_k^\delta$ and $z(k) = z^\delta(k)$, then

The state equation: $x(k+1) = f(x(k)) + \sigma(x(k))u(k)$

The observation equation: $z(k) = h(x(k)) + \xi(x(k))w(k)$

where

- The functions σ, h and ξ are defined previously
- For all $y \in \mathbb{R}$,
$$\Psi(y) = \begin{cases} a_2 y^2 \mathbf{I}_{\{y \geq 0\}} & \text{if } y \leq M_c\,, \\ a_2 M_c^2 & \text{if not}\,. \end{cases}$$
- For all $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$,
$$f(x) = \begin{bmatrix} x_1 + \delta x_2 \\ x_2 + \delta\{a_0 + g\cos(\theta_0 + \frac{x_1}{R}) - a_1 x_2 - \Psi(x_2)\} \end{bmatrix}$$
- $\{u(k)\}_k$ is a 2-dimensional standard white process independent with the 1-dimensional standard white process $\{w(k)\}_k$

Aim: Approximate the state vector $x(k)$ given the observations $\{z(0), \ldots, z(k)\}$ up to time k.
For simulations proposes we fix the constants of the system: $T = 10$, $N = 4000$, $M_c = 150$, $\theta_0 = \pi/6$, $R = 200$, $(\xi_0, \xi_1, \xi_2) = (1, 0.5, 0.05)$ and $x(0) = (0, 25)$.
We approximate the position and the speed of the car using successively the EKF algorithm and two particle filter methods using 160 particles. We use a multinomial and a Bernouilli branching mechanisms, respectively.
The initialization of the state for the different approximation methods coincide with the true initial state, ie.

$$\hat{x}(0|0) = \begin{bmatrix} 0 \\ 25 \end{bmatrix}.$$

Figure 3.3: We approximate the position and the speed of the car using the EKF and the particle filter with a Bernouilli mechanism, respectively. It is clear that the particle filter performs better and gives better approximations.
Figure 3.4: We compare the Bernouilli and the multinomial branching mechanisms and we plot the square of the difference to the true paths. A Bernouilli resampling scheme performs better.

Chapter 4

Zakai Equation with an Ornstein-Uhlenbeck Noise Type

4.1 Introduction

The filtering problem consists of estimating the statistics of a partially observed process X, called the signal process, based on observations or measurements corrupted by noises. The measurements are supplied by a process Z called the observation process:

$$Z_t = \int_0^t h(X_s, s)ds + N_t\,.$$

The conditional probability $\pi_t(\varphi) = E[\varphi(X_t, t)\big|\{Z_s, 0 \leq s \leq t\}]$, where φ is a bounded Borel function, gives the minimum variance estimate of $\varphi(X_t, t)$ based on the observations $\{Z_s, 0 \leq s \leq t\}$.
Many interests was developed to describe the nonlinear filtering with the help of stochastic partial differential equations. This connections make it possible to apply Monte Carlo and probabilistic methods for partial differential equations to filtering. The case when the noise N is a Brownian motion was extensively studied, see [7, 54, 57, 69]. Under some regularity conditions, π_t satisfies a measure valued nonlinear stochastic partial differential equation, called the Kushner-Stratonovitch equation.
In his famous paper [77], M. Zakai uses the Kallianpur-Striebel formula:

$$\pi_t(\varphi) = \frac{p_t(\varphi)}{p_t(1)}\,,$$

where $p_t(\varphi)$ is called the unnormalized conditional expectation, and he characterizes $p_t(\varphi)$ as the solution of a linear stochastic partial differential equation, called the Zakai equation. The linearity of the Zakai equation makes it simpler and more attractive for numerical methods than the Kushner-Stratonovitch equation.
Bensoussan [7] has established that p_t is the unique solution of the Zakai equation when the observation function h has linear growth bound.

Recently, interest has developed when the noise is an Ornstein-Uhlenbeck or a general Gaussian process [9, 10, 56, 65]. A. Bhatt et al. [10], study the case of Ornstein-Uhlenbeck noise. They derived an analogous to the Zakai equation and they prove existence and uniqueness results for 1-dimensional processes and bounded observation function h.
This chapter represents a generalization of the work of Bensoussan [7] and of Bhatt et al. [10]. We consider a multi-dimensional dynamical system such that

- The noise N is of Ornstein-Uhlenbeck type
- The observation function h has a linear growth bound

We prove that the unnormalized conditional expectation $p_t(\varphi)$ is a solution of the Zakai equation and we give uniqueness result. Then, we construct a sequence of particle measures that converges to the solution of the Zakai equation. We precise the rate of convergence and we establish an implementation scheme.
In the section 4.2 we describe in details the filtering problem under consideration. In section 4.3 we transform the observation process and the probability measure to obtain the (analogous) Zakai equation and under some additional assumptions we prove the existence and uniqueness of the solution. Finally, in section 4.4 we approximate the solution of the Zakai equation by a sequence of weighted empirical measures, we give a rate of convergence, an implementation scheme and an illustrative example.

4.2 The filtering problem

Let $(\Omega, \mathcal{A}, P)$ be a probability space equipped with a complete filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Let us consider

1. Two independent standard $\mathcal{F}_t$-Brownian motions $W = \{W_t\}_{t\geq 0}$ and $G = \{G_t\}_{t\geq 0}$ with values respectively in $\mathbb{R}^n$ and in $\mathbb{R}^m$
2. An $\mathcal{F}_0$-measurable $\mathbb{R}^n$-valued random variable ξ independent of W and G such that
$$E[|\xi|^2] = \int |x|^2 \pi_0(dx) < \infty\,. \tag{4.1}$$
where π_0 is the probability density of ξ
3. Two Borel functions $g(\cdot,\cdot) : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$ and $\sigma(\cdot,\cdot) : \mathbb{R}^n \times [0,\infty) \to \mathcal{M}_n(\mathbb{R})$ such that
Global linear growth bounds: There exists a constant $K > 0$ such that for all $t \geq 0$ and $x \in \mathbb{R}^n$ we have
$$\max(\,|g(x,t)|\,,\ ||\sigma(x,t)||\,) \leq K\,(1 + |x|) \tag{4.2}$$
Locally Lipschitz: For every $T > 0$, there exists a constant $K_T > 0$ such that for all $t \in [0,T]$ and $x, y \in \mathbb{R}^n$ we have
$$\max(\,|g(x,t) - g(y,t)|\,,\ ||\sigma(x,t) - \sigma(y,t)||\,) \leq K_T\,|x - y| \tag{4.3}$$

4. A jointly continuous Borel function $h(\cdot,\cdot) : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^m$ satisfying for all $t \geq 0$ and $x \in \mathbb{R}^n$
$$|h(x,t)| \leq K(1+|x|) \tag{4.4}$$

We consider the following model

The signal process:

$$dX_t = g(X_t,t)dt + \sigma(X_t,t)dW_t\,; \qquad X_0 = \xi \tag{4.5}$$

The observation process:

$$dZ_t = h(X_t,t)dt + dN_t\,; \qquad Z_0 = 0 \tag{4.6}$$

The noise process $N = \{N_t\}_{t\geq 0}$ is the unique continuous square integrable process satisfying the SDE

$$dN_t = \alpha(N_t,t)\,dt + \beta_t\,dG_t\,; \qquad N_0 = 0\,. \tag{4.7}$$

The functions α and β are non random and satisfy:

i. $(z,t) \in \mathbb{R}^m \times [0,\infty) \longrightarrow \alpha(z,t) = -A_t^{-1}A_t'z - A_t^{-1}B_t' \in \mathbb{R}^m$,
where

- The functions $A_\cdot \in \mathcal{C}^1\big([0,\infty), GL_m(\mathbb{R})\big)$ and $B_\cdot \in \mathcal{C}^1([0,\infty),\mathbb{R}^m)$
- $A_t' = dA_t/dt$ and $B_t' = dB_t/dt$

In particular, $A_\cdot^{-1} \in \mathcal{C}^1\big([0,\infty), GL_m(\mathbb{R})\big)$

ii. $t \in [0,\infty) \longrightarrow \beta_t \in GL_m(\mathbb{R})$ is continuous

For all $t \geq 0$, we define the bijective affine transformation L_t of $\mathbb{R}^m$ by

$$L_t(z) = A_t\,z + B_t\,, \quad \forall\, z \in \mathbb{R}^m\,. \tag{4.8}$$

Remark 4.2.1. 1. The conditions on the coefficients of the SDE (4.7) determine uniquely (strongly and path-wise) the process N. We take a continuous square integrable measurable version for the process N

2. We say that the process N is an Ornstein-Uhlenbeck type process because if $A_t = \exp(tA)$, A is a negative definite matrix of order m, $B_t = B$ a constant vector in $\mathbb{R}^m$ and $\beta_t = \Gamma$ a constant matrix of order m, the process N is an Ornstein-Uhlenbeck process

The conditions (4.2) and (4.3) implies that the SDE equation (4.5) admit a unique strong solution. Moreover,

1. For all $T > 0$, $X \in L^2(\Omega,\mathcal{A},P,\mathcal{C}(0,T;\mathbb{R}^n))$, i.e.
$$E[\sup_{0\leq t\leq T}|X_t|^2] < \infty\,, \quad \forall\, T > 0 \tag{4.9}$$

2. For every $k \geq 1$ such that $E[|\xi|^{2k}] < \infty$,

$$\begin{array}{rcl} E[|X_t|^{2k}] & \leq & (1 + E[|\xi|^{2n}]) \exp(Ct) \\ E[|X_t - \xi|^{2k}] & \leq & D(1 + E[|\xi|^{2k}])t^k \exp(Ct) \end{array} \tag{4.10}$$

where $0 \leq t \leq T < \infty$, $C = 2k(2k+1)K_T^2$ and D is a positive constant depending only on k, K_T and T

3. For every $k \geq 1$ such that $E[|\xi|^{2k}] < \infty$,

$$\begin{array}{rcl} E[\sup_{0\leq t\leq T} |X_t|^{2k}] & \leq & \tilde{D}\{E[|\xi|^{2n}] + T^k(1 + E[|\xi|^{2n}]) \exp(CT)\} \\ E[\sup_{0\leq t\leq T} |X_t - \xi|^{2k}] & \leq & \tilde{D}T^k(1 + E[|\xi|^{2k}]) \exp(CT) \end{array} \tag{4.11}$$

where $T < \infty$, and C and $\tilde{D}$ are positive constants depending only on k, K, K_T and T

Remark 4.2.2. In [50] one can find the proofs of the existence and the uniqueness of the solutions of the equations (4.5) and (4.7), their mentioned properties and the proofs of the inequalities (4.9), (4.10) and (4.11).

Given the observations $\{Z_s\,,\ 0 \leq s \leq t\}$, what is the best L^2-estimate $\hat{X}_t$ of the state X_t, based on these observations ?
Let us denote $\mathcal{Z}_t = \sigma(Z_s, 0 \leq s \leq t)^* \subset \mathcal{F}_t$, where $*$ denote the augmentation of a σ-algebra by the P-null sets.

a. The estimate $\hat{X}_t$ is based on the observations $\{Z_s\,,\ 0 \leq s \leq t\}$ if $\hat{X}_t$ is $\mathcal{Z}_t$-measurable

b. $\hat{X}_t$ is the best L^2-estimate if

$$\int_\Omega |X_t - \hat{X}_t|^2 dP = E[\,|X_t - \hat{X}_t|^2\,] = \inf\{E[\,|X_t - Y|^2\,]\,,\ Y \in \mathcal{K}_t\}$$

where $\mathcal{K}_t = \mathcal{K}(Z,t) = L^2(\Omega, \mathcal{Z}_t, P; \mathbb{R}^n) = \{Y \in= L^2(\Omega, P; \mathbb{R}^n)\,,\ Y \text{ is } \mathcal{Z}_t\text{-adapted}\}$

Lemma 4.2.3. *Let $\mathcal{H} \subset \mathcal{A}$ be a σ-algebra with $\mathcal{H}^* = \mathcal{H}$ and $X \in L^2(P)$ be $\mathcal{A}$-measurable. Put $N = L^2(\Omega, \mathcal{H}, P)$ and denote by $\mathcal{P}_N$ the orthogonal projection from the Hilbert space $L^2(P)$ into its closed subspace N. Then,*

$$\mathcal{P}_N(X) = E[\,X\big|\mathcal{H}\,]\,, \quad a.s..$$

Proof. $E[X\big|\mathcal{H}]$ is by definition the P-$a.s.$ unique function: $\Omega \to \mathbb{R}^n$ such that

- $E[X\big|\mathcal{H}]$ is $\mathcal{H}$-measurable
- For all $A \in \mathcal{H}$, $\int_A E[X\big|\mathcal{H}]dP = \int_A X dP$

However, $\mathcal{P}_N(X)$ is $\mathcal{H}$-measurable and for all $Y \in N$, $\int_\Omega Y(X - \mathcal{P}_N(X))dP = 0$
In particular, for all $A \in \mathcal{H}$ the choice $Y = 1_A$ gives

$$\int_A (X - \mathcal{P}_N(X))dP = 0\,.$$

□

Theorem 4.2.4. $\hat{X}_t = \mathcal{P}_{\mathcal{K}_t}(X_t) = E[\, X_t \big| \mathcal{Z}_t \,]$.

The filtering problem: For any arbitrary fixed positive number T, estimate the conditional expectation

$$\pi_t(\varphi) = E[\varphi(X_t, t)|\mathcal{Z}_t] \,, \ \forall t \in [0, T] \,,$$

where φ is a bounded Borel function on $\mathbb{R}^n \times [0, \infty)$. Clearly,

$$\pi_0(\varphi) = \int_{\mathbb{R}^n} \varphi(x)\pi_0(dx) \,.$$

4.3 The Zakai equation

Following the work of Bhatt et al. in [10], we apply an affine transformation to the observation equation to obtain a new observation equation. This will allow us, after a change of the probability measure, to obtain an evolution equation of the unnormalized conditional probability. We call this evolution equation the Zakai equation. Under some additional assumptions we will show that this unnormalized conditional probability is the unique solution of the Zakai equation.

4.3.1 Change of the probability measure

To derive the Zakai equation we will transform the observation equation (4.6) to obtain a new observation process Y. The process Y will be a continuous square integrable $\mathcal{F}_t$-martingale, independent of the signal process X in the probability space $(\Omega, \mathcal{A})$ endowed by a new probability measure $\tilde{P}$ obtained by

$$\frac{d\tilde{P}}{dP}\Big|_{\mathcal{F}_t} = \rho_t \,.$$

Let us define properly the processes Y and ρ.
First, let T be an arbitrary fixed positive real number. We define

1. The function H on $\mathcal{C}(0, T; \mathbb{R}^n) \times [0, T]$ by

$$(\mathbf{x}, t) \longrightarrow H(\mathbf{x}, t) = \frac{d}{dt}\big[A_t \int_0^t h(\mathbf{x}(s), s)ds\big] \in \mathbb{R}^m \,. \tag{4.12}$$

2. The process M by $M_t = L_t(N_t)$, then

$$M_t = \int_0^t A_s \, \beta_s \, dG_s \,, \quad \forall t \in [0, T] \,. \tag{4.13}$$

3. The process Y by $Y_t = L_t(Z_t)$, then

$$Y_t = \int_0^t H(X_\cdot, s) \, ds + M_t \,, \quad \forall \, t \in [0, T] \,. \tag{4.14}$$

4. The process $\rho = \{\rho_t\}_{0\le t\le T}$ such that

$$d\rho_t = -\rho_t H^T(X_\cdot, t) R_t^{-1} dM_t\,, \quad \rho_0 = 1\,.$$

where $R_s = (A_s\beta_s)(A_s\beta_s)^T$. The matrix R_s is symmetric positive definite

We have the following properties

i. The matrices A_t and A_t' are continuous on $[0,T]$, then there exists a constant $a_T > 0$ such that $\max\big(\|A_t\|, \|A_t'\|\big) \le a_T$ for all $t \in [0,T]$. This implies that for all $(\mathbf{x}, t) \in \mathcal{C}(0,T;\mathbb{R}^n) \times [0,T]$:

$$\begin{array}{rcl} |H(\mathbf{x},t)| & \le & a_T K(1 + |\mathbf{x}(t)| + \int_0^t (1 + |\mathbf{x}(s)|) ds) \\ |H(\mathbf{x},t)| & \le & \tilde{K}_T\big(1 + \sup_{0\le s\le t} |\mathbf{x}(s)|\big) \end{array} \tag{4.15}$$

where $\tilde{K}_T = a_T K(1+T)$

ii. The process $M = \{M_t\}_{0\le t\le T}$ is a centered continuous square integrable $\mathcal{F}_t^G$-martingale, independent of the Brownian motion W, where $\mathcal{F}_t^G = \sigma\{G_s\,,\ 0 \le s \le t\}^*$. Its quadratic variation is given by

$$\langle M\rangle_t = \int_0^t R_s\, ds\,, \quad \forall t \in [0,T]\,. \tag{4.16}$$

iii. Since L_t is a bijective deterministic affine transformation of $\mathbb{R}^m$ and $Y_t = L_t(Z_t)$, then for all $t \in [0,T]$ we have

$$\mathcal{Z}_t = \mathcal{Y}_t\,, \tag{4.17}$$

where $\mathcal{Y}_t = \sigma\{Y_s\,,\ 0 \le s \le t\}^*$. In particular, $E[\,\cdot\,|\mathcal{Z}_t] = E[\,\cdot\,|\mathcal{Y}_t]$

iv. The process ρ is explicitly given by

$$\rho_t = \exp\{-\int_0^t H^T(X_\cdot, s) R_s^{-1} dM_s - \frac{1}{2}\int_0^t H^T(X_\cdot, s) R_s^{-1} H(X_\cdot, s)\, ds\,\} \tag{4.18}$$

The measure $\tilde{P}$ is a probability if we show that the process ρ is a martingale.
Let us consider the process $M_t^X = -\int_0^t H^T(X_\cdot, s) R_s^{-1} dM_s$, $t \in [0,T]$. Since R_s^{-1} is nonrandom and continuous and $H(X_\cdot, t)$ is square integrable, see (4.9) and (4.15) and $\mathcal{F}_t$-adapted, then the Proposition 2.2.2. in [35] ensures that the process M_t^X is an $\mathcal{F}_t$-martingale. Its quadratic variation is $\langle M^X\rangle_t = \int_0^t H^T(X_\cdot, s) R_s^{-1} H(X_\cdot, s) ds$ for all $t \in [0,T]$. Moreover,

$$\rho_t = \exp(M_t^X - \frac{1}{2}\langle M^X\rangle_t)\,.$$

The Theorem 3.5.2 in [35] implies that the process ρ is an $\mathcal{F}_t$-super-martingale and $E[\rho_t] \le 1$. It is a martingale if $E[\rho_t] = 1$ for all $t \in [0,T]$.

Lemma 4.3.1. *For all $t \in [0,T]$, we have*

$$E[\rho_t] = 1\,.$$

Proof. We check first that there exists a constant $L_1 < \infty$ such that

$$E[\rho_t |X_t|^2] < L_1\,, \quad \forall t \in [0,T]\,. \tag{4.19}$$

Let us denote for simplicity $g(X_t,t)$, $\sigma(X_t,t)$ and $H(X_\cdot,t)$ by g, σ and H respectively. The Ito formula gives

1. $d|X_t|^2 = \{2X_t^T g + tr(\sigma\sigma^T)\}dt + 2X_t^T\sigma\, dW_t$
2. $d\rho_t|X_t|^2 = \rho_t\{2X_t^T g + tr(\sigma\sigma^T)\}dt - \rho_t|X_t|^2 H^T R^{-1} dM_t + 2\rho_t X_t^T\sigma dW_t$
3. Taking $U(t,x) = U(x) = x/(1+\epsilon x)$ for $\epsilon > 0$,

$$\begin{aligned} dU(t,X_t) &= d\frac{\rho_t|X_t|^2}{1+\epsilon\rho_t|X_t|^2} \\ &= \frac{\rho_t}{(1+\epsilon\rho_t|X_t|^2)^2}(-|X_t|^2 H^T R^{-1} dM_t + 2X_t^T\sigma dW_t) \\ &\quad +\{\frac{\rho_t}{(1+\epsilon\rho_t|X_t|^2)^2}(2X_t^T g + tr(\sigma\sigma^T)) \\ &\quad -\frac{\epsilon\rho_t^2}{(1+\epsilon\rho_t|X_t|^2)^3}(|X_t|^4 H^T R^{-1} H + 4X_t^T\sigma\sigma^T X_t)\}dt\,. \end{aligned}$$

We integrate $dU(t,X_t)$ from 0 to t, then we take the expectation, we see that the expectation of first term is zero. The derivative w.r.t. t gives

$$\begin{aligned} \frac{d}{dt}E[\frac{\rho(t)|X_t|^2}{1+\epsilon\rho_t|X_t|^2}] &\le E[\frac{\rho_t}{(1+\epsilon\rho_t|X_t|^2)^2}(2X_t^T g + tr(\sigma\sigma^T))] \\ &\quad -E[\frac{\epsilon\rho_t^2}{(1+\epsilon\rho_t|X_t|^2)^3}(|X_t|^4 H^T R^{-1} H + 4X_t^T\sigma\sigma^T X_t)] \end{aligned}$$

The last term of this last inequality is negative, then

$$\begin{aligned} \frac{d}{dt}E[\frac{\rho_t|X_t|^2}{1+\epsilon\rho_t|X_t|^2}] &\le E[\frac{\rho_t}{(1+\epsilon\rho_t|X_t|^2)^2}\{2X_t^T g + tr(\sigma\sigma^T)\}] \\ &\le E[\frac{\rho_t}{1+\epsilon\rho_t|X_t|^2}\{2X_t^T g + tr(\sigma\sigma^T)\}] \end{aligned}$$

The linear growth bound property (4.2) of g and σ, implies

$$\begin{array}{c} |x^T g(x,t)| \le K|x|(1+|x|) \le 2K(1+|x|^2) \\ |tr(\sigma(x,t)\sigma(x,t)^T)| \le 2nK^2(1+|x|^2) \end{array}\,, \quad \forall (x,t) \in \mathbb{R}^n \times [0,T]\,. \tag{4.20}$$

Then, using the fact that $\rho_t/(1+\epsilon\rho_t|X_t|^2) \le \rho_t$ and that $E[\rho_t] \le 1$, we get

$$\begin{aligned} \frac{d}{dt}E[\frac{\rho_t|X_t|^2}{1+\epsilon\rho_t|X_t|^2}] &\le \tilde{K}E[\frac{\rho_t(1+|X_t|^2)}{1+\epsilon\rho_t|X_t|^2}] \\ &\le \tilde{K}(1+E[\frac{\rho_t|X_t|^2}{1+\epsilon\rho_t|X_t|^2}]) \end{aligned}$$

where $\tilde{K} = 2(K + nK^2)$. The Gronwall inequality implies that

$$E[\frac{\rho_t|x_t|^2}{1+\epsilon\rho_t|x_t|^2}] \leq L_1$$

where $L_1 = \int_0^T \big(\tilde{K} + \tilde{K}^2 \int_0^t \exp(\tilde{K}(t-s))ds\big)dt$. The constant L_1 is independent from ϵ. If we apply Fatou's Lemma $\epsilon \to 0$ we get (4.19).
Next, we prove that there exists a constant $L_2 < \infty$ such that

$$E[\rho_t \int_0^t |X_s|^2 ds] < L_2\,, \quad \forall t \in [0,T]\,. \tag{4.21}$$

Using the Fubini Theorem, that the process ρ is an $\mathcal{F}_t$-super-martingale, ie. for all $0 \leq s \leq t$ $E[\rho_t\big|\mathcal{F}_s] \leq \rho_s$, and that the process$X_t$ is $\mathcal{F}_t$-measurable we get

$$\begin{aligned} E[\rho_t \int_0^t |X_s|^2 ds] &= \int_0^t E[\rho_t |X_s|^2]ds \\ &= \int_0^t E\big[E[\rho_t|X_s|^2\big|\mathcal{F}_s]\big]ds \\ &= \int_0^t E\big[E[\rho_t\big|\mathcal{F}_s]|X_s|^2\big]ds \\ &\leq \int_0^t E[\rho_s|X_s|^2]ds \leq L_2 = TL_1 \end{aligned}$$

which proves (4.21).
Finally, Ito's formula gives us

$$d\frac{\rho_t}{1+\epsilon\rho_t} = -\frac{\rho_t H^T R^{-1} dM_t}{(1+\epsilon\rho_t)^2} - \frac{\epsilon\rho_t^2 H^T R_t^{-1} H dt}{(1+\epsilon\rho_t)^3}$$

Then,

$$E[\frac{\rho_t}{1+\epsilon\rho_t}] = \frac{1}{1+\epsilon} - E[\int_0^t \frac{\epsilon\rho_s^2 H^T(X_\cdot,s)R_s^{-1}H(X_\cdot,s)ds}{(1+\epsilon\rho_s)^3}]. \tag{4.22}$$

The integrand part in the second hand term of equation (4.22) converges pointwise to 0 as $\epsilon \to 0$. In addition, the inequalities in (4.15) and the fact that $\epsilon\rho_t^2/(1+\epsilon\rho_t)^3 \leq \rho_t$ implies that there exists a constant L independent of ϵ such that

$$\frac{\epsilon\rho_s^2 H^T(X_\cdot,s)R_s^{-1}H(X_\cdot,s)ds}{(1+\epsilon\rho_s)^3} \leq L\rho_s(1+|X_s|^2 + \int_0^s |X_r|^2 dr)\,,$$

This quantity is integrable by (4.19) and (4.21), then the Lebesgue theorem implies that

$$E[\int_0^t \frac{\epsilon\rho_s^2 H^T(X_\cdot,s)R_s^{-1}H(X_\cdot,s)ds}{(1+\epsilon\rho_s)^3}] \Longrightarrow_{\epsilon\to 0} 0\,.$$

In addition, we have $E[\rho_t] \leq 1$, then

$$E[\frac{\rho_t}{1+\epsilon\rho_t}] \longrightarrow E[\rho_t]\,.$$

The claim follows by taking ϵ to 0 in (4.22). □

Lemma 4.3.1 shows that ρ_t is an integrable $\mathcal{F}_t$-martingale, then by Propositions 8.1.1. and 8.1.3. in [71] we can define a new probability measure $\tilde{P} \ll P$ such that the Radon-Nykodym derivative of $\tilde{P}$ with respect to P on $\mathcal{F}_t$ is

$$\frac{d\tilde{P}}{dP}\Big|_{\mathcal{F}_t} = \rho_t\,, \quad \forall t \in [0,T]\,.$$

If we denote by $\tilde{E}$ the expectation w.r.t $\tilde{P}$, then for all integrable and $\mathcal{F}_t$-measurable random variable $\zeta : \Omega \to \mathbb{R}^d$ we have

$$\begin{aligned} \tilde{E}[\zeta] &= E[\rho_t \zeta] && (4.23)\\ \tilde{E}[\zeta\big|\mathcal{F}_s] &= \rho_s^{-1} E[\rho_t\,\zeta\big|\mathcal{F}_s]\,, \quad \forall 0 \le s \le t && (4.24) \end{aligned}$$

Proposition 4.3.2. *On the probability space $(\Omega, \mathcal{A}, \tilde{P}, (\mathcal{F}_t)_{t\in[0,T]})$ we have:*

1. *The process Y is a continuous square integrable $\mathcal{F}_t$-martingale*
2. *The law of the process X remains unchanged under $\tilde{P}$ and the processes X and Y are independent*
3. *The law of the process Y under $\tilde{P}$ coincide with that of the process M under P. In particular, for all $t \in [0,T]$ we have*

$$\langle Y\rangle_t = \langle M\rangle_t$$

Proof. The process $M_t^X = -\int_0^t H(X_s,s)R_s^{-1}dM_s$ is a martingale and its quadratic variation is

$$\langle M^X\rangle_t = \int_0^t H^T(X_s,s)R_s^{-1}H(X_s,s)ds\,.$$

Then, the Theorem 4.4.1. in [35] implies that the process $\tilde{Y}_t = M_t - \langle M, M^X\rangle_t$ is a continuous square integrable local-martingale and for all $t \in [0,T]$,

$$\langle \tilde{Y}\rangle_t = \langle M\rangle_t\,.$$

from the equality (4.16) we get

$$\langle M, M^X\rangle_t = -\int_0^t H(X_s,s)R_s^{-1}d\langle M\rangle_s = -\int_0^t H(X_s,s)ds \qquad (4.25)$$

This implies that for all $t \in [0,T]$, $\tilde{Y}_t = Y_t$ almost surely, see (4.14).
The process Y is a martingale: It is sufficient to prove that

$$E[\sup_{0\le t\le T}|Y_t|\,] < \infty\,.$$

In fact, suppose that $E[\sup_{0\le t\le T}|Y_t|\,] < \infty$, then we can apply Ito's formula:

$$\begin{aligned} d(\rho_t Y_t) &= d\{\rho_t(M_t - \langle M, M^X\rangle_t\}\\ &= Y_t d\rho_t + \rho_t dM_t - \rho_t d\langle M, M^X\rangle_t + d\rho_t dM_t\\ &= Y_t d\rho_t + \rho_t dM_t \end{aligned}$$

Since $d\rho_t dM_t = -\rho_t H^T(X_t,t)dt = \rho_t d\langle M, M^X\rangle_t$, see (4.16), we get $\rho_t Y_t$ is an $\mathcal{F}_t$-martingale on $(\Omega\,,\,P)$ and for all $0 \le s \le t \le T$

$$\tilde{E}[Y_t\big|\mathcal{F}_s] = \rho_s^{-1} E[\rho_t Y_t\big|\mathcal{F}_s] = Y_s\,.$$

Let us show that $E[\sup_{0\le t\le T}|Y_t|\,] < \infty$. We have

i. $\sup_{0\le t\le T}|Y_t|^2 \le 4t^2\tilde{K}_T^2(1+\sup_{0\le s\le t}|X_s|^2) + 2t\sup_{0\le s\le t}|M_s|^2$

ii. $E[\sup_{0\le s\le t}|X_s|^2] < \infty$

iii. M is a square integrable martingale

iv. The Doob inequality:

$$E[\sup_{0\le s\le t}|M_s|^2] \le 4E[M_t|^2] < \infty$$

We deduce the result by using that $E[\sup_{0\le t\le T}|Y_t|]^2 \le E[\sup_{0\le t\le T}|Y_t|^2]$.
The law of the process X remains unchanged and the processes X and Y are independent: It is sufficient to prove that the law of (X,Y) under the probability $\tilde{P}$ is law of (X,M) under the probability P. We use characteristic functionals. Let $u \in L^\infty([0,T],\mathbb{R}^n)$ and $v \in L^\infty([0,T],\mathbb{R}^m)$, then

$$\begin{aligned}
\tilde{F}_{X,Y}(u,v) &= \tilde{E}[\exp(i\int_0^t u^T(s)dX_s + i\int_0^t v^T(s)dY_s)] \\
&= E[\rho_t \exp(i\int_0^t u^T(s)dX_s + i\int_0^t v^T(s)dY_s)] \qquad (4.26)\\
&= E[\exp(i\int_0^t u^T(s)dX_s)\exp(\int_0^t(-H^TR_s^{-1} + iv(s)^T)dM_s) \\
&\quad \exp(\int_0^t(-\frac{1}{2}H^TR_s^{-1}H + iv(s)^TH)ds] \qquad (4.27)
\end{aligned}$$

where we have replaced ρ_t by its expression in (4.26) and we have denoted $H(X_s,s)$ simply by H.
Let $\mathcal{F}_t^{\xi,W} = \sigma\{\xi; W_s, 0\le s\le t\}^*$. It is clear that X_t is $\mathcal{F}_t^{\xi,w}$-measurable. We see that given $\mathcal{F}_t^{\xi,W}$ the process $M_t^{\xi,W} = \int_0^t(-H^T(X_s,s)R_s^{-1} + iv(s)^TR_s)dM_s$ is a martingale with quadratic variation

$$\langle M^{\xi,W}\rangle_t = \int_0^t(-H^TR_s^{-1} + iv(s)^T)R_s(-H^TR_s^{-1} + iv(s)^T)^Tds \qquad (4.28)$$

We claim that

$$E[\exp(M_t^{\xi,W})|\mathcal{F}_t^{\xi,W}] = \exp(\frac{1}{2}\langle M^{\xi,W}\rangle_t)\,. \qquad (4.29)$$

We will show this equality later in the proof.
If we use the equalities (4.28) and (4.29) in the equation (4.27), we get

$$\begin{aligned}
\tilde{F}_{X,Y}(u,v) &= E\big[E[\exp(i\int_0^t u^T(s)dX_s)\times\exp(M_t^{\xi,W}) \\
&\qquad \exp(\int_0^t(-\frac{1}{2}H^TR_s^{-1}H(X_s,s)+iv(s)^TH)ds\big|\mathcal{F}_t^{\xi,W}]\big] \\
&= E\big[\exp(i\int_0^t u^T(s)dX_s)E[\exp(M_t^{\xi,W})|\mathcal{F}_t^{\xi,w}] \\
&\qquad \exp(\int_0^t(-\frac{1}{2}HR_s^{-1}H+iv(s)^TH)ds)\big] \\
&= E[\exp(i\int_0^t u^T(s)dX_s)\exp(-\frac{1}{2}\int_0^t v^T(s)R_sv(s))ds)] \qquad (4.30)
\end{aligned}$$

Given $\mathcal{F}_t^{\xi,W}$, the process $i\int_0^t v^T(s)dM_s$ is a martingale and its quadratic variation is

$$\langle i\int_0^t v^T(s)dM_s)\rangle_t = -\int_0^t v^T(s)R_sv(s)ds\,.$$

We deduce that

$$E[\exp(i\int_0^t v^T(s)dM_s)|\mathcal{F}_t^{\xi,w}] = \exp(-\frac{1}{2}\int_0^t v^T(s)R_sv(s)ds)\,. \qquad (4.31)$$

Finally, we use together (4.31) in (4.30) to get the result, that is

$$\tilde{F}_{X,Y}(u,v) = E[\exp(i\int_0^t u^T(s)dX_s + i\int_0^t v^T(s)dM_s)] = F_{X,M}(u,v)\,.$$

Then, the law of the martingale M under P is equal to the law of the martingale Y under $\tilde{P}$. In particular, M and Y have the same quadratic variation.
To finish the proof we prove the equality (4.29), that is

$$E[\exp(M_t^{\xi,W})|\mathcal{F}_t^{\xi,W}] = \exp(\frac{1}{2}\langle M^{\xi,W}\rangle_t)\,.$$

Recall that $\mathcal{F}_t^{\xi,W} = \sigma\{\xi; W_s, 0\le s\le t\}^*$ and X_t is $\mathcal{F}_t^{\xi,w}$-measurable. The equality (4.13) implies that $dM_t = A_s\beta_s dG_s$, G is a standard BM.
We write the process $M_t^{\xi,W}$ as a sum of two integrals

$$\begin{aligned}
M_t^{\xi,W} &= i\int_0^t v(s)^TQ_s^1dGs - \int_0^t H(X_s,s)^TQ_s^2dG_s \\
&= i\,I_1(t) \;-\; I_2(t)\,,
\end{aligned}$$

where $Q_s^1 = A_s\beta_s$ and $Q_s^2 = R_s^{-1}A_s\beta_s$.
Given $\mathcal{F}_t^{\xi,W}$, the two integrals I_1 and I_2 are jointly Gaussian.
Let us denote $I = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$ and $\Pi = E[\,I(t)I(t)^T\,|\mathcal{F}_t^{\xi,W}]$, then

a. $E[\, I(t) \,|\mathcal{F}_t^{\xi,W}] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

b. $\Pi = \begin{bmatrix} \int_0^t v^T(s) R_s v(s) ds & \int_0^t H^T v(s) ds \\ \int_0^t H^T v(s) ds & \int_0^t H^T R_s^{-1} H ds \end{bmatrix}$

Let us denote $\mathcal{E} = E[\exp(\int_0^t (-H^T R_s^{-1} + i v(s)^T) dM_s) | \mathcal{F}_t^{\xi,W}]$, then

$$\begin{aligned} \mathcal{E} &= E[\exp\Big(\begin{bmatrix} i \\ -1 \end{bmatrix}^T I(t) \Big) | \mathcal{F}_t^{\xi,W}] \\ &= \exp\big(\frac{1}{2} \begin{bmatrix} i \\ -1 \end{bmatrix}^T \Pi \begin{bmatrix} i \\ -1 \end{bmatrix}\big) \\ &= \exp\big(\frac{1}{2} \int_0^t H^T R_s^{-1} H ds - \frac{1}{2} \int_0^t v^T(s) R_s v(s) ds - i \int_0^t H^T v(s) ds\big) \\ &= \exp(\frac{1}{2} \langle M^{\xi,W} \rangle_t) \,. \end{aligned}$$

That is, $E[\exp(M^{\xi,W}) | \mathcal{F}_t^{\xi,W}] = \exp(\frac{1}{2} \langle M^{\xi,W} \rangle_t)$ and the proof is complete. □

Definition 4.3.3. For all bounded Borel function φ on $\mathbb{R}^n \times [0,T]$ and for all $t \in [0,T]$

1. $\eta_t = \rho_t^{-1} = \exp\{\int_0^t H^T(X_\cdot, s) R_s^{-1} dM_s + \frac{1}{2} \int_0^t H^T(X_\cdot, s) R_s^{-1} H(X_\cdot, s)\, ds\,\}$
2. $p_t(\varphi) = \tilde{E}[\varphi(X_t, t)\, \eta_t \big| \mathcal{Z}_t]$

Lemma 4.3.4 (The Kallianpur-Striebel formula)**.** *For any bounded Borel function φ on $\mathbb{R}^n \times [0,T]$ and for all $t \in [0,T]$ we have*

$$\pi_t(\varphi) = \frac{p_t(\varphi)}{p_t(1)} \,. \tag{4.32}$$

Proof. Let ζ be any $\mathcal{Z}_t$-measurable and bounded random variable. One has

$$E[\zeta\, \pi_t(\varphi)] = E[\zeta\, \varphi(X_t, t)] \,.$$

Then,

$$\tilde{E}[\zeta\, \pi_t(\varphi) \eta_t] = \tilde{E}[\zeta\, \varphi(X_t, t) \eta_t] \,.$$

This implies

$$\tilde{E}[\zeta\, \pi_t(\varphi) \tilde{E}[\eta_t | \mathcal{Z}_t]] = \tilde{E}\big[\zeta\, \tilde{E}[\varphi(X_t, t) \eta_t \big| \mathcal{Z}_t]\big] \,.$$

This gives the result. □

Before establishing the Zakai equation we give a Proposition that enables us to give a suitable space where the operator $p_\cdot(\cdot)$ will act and where we will prove the uniqueness of the solution of the Zakai equation.

Proposition 4.3.5. *Let $\psi : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ be a Borel function. Suppose that there exists a constant $\kappa > 0$ such that for all $\mathbf{x} \in \mathcal{C}([0,T],\mathbb{R}^n)$ we have*

$$|\psi(\mathbf{x}(t),t)| \leq \kappa(1 + |\mathbf{x}(t)|^2 + \int_0^t |\mathbf{x}(s)|^2 ds)\,. \tag{4.33}$$

Then, for almost all $t \in [0,T]$

i. $p_t(\psi) = \tilde{E}[\psi(X_t,t)\eta_t \big| \mathcal{Z}_t] \in L^1(\Omega, \mathcal{Z}_t, \tilde{P})$

ii. $\tilde{E}[\int_0^T p_t(\psi)dt] < \infty$

Proof. Since $E[\sup_{0\leq t\leq T} |X_t|^2] < \infty$, we get

$$\begin{aligned} \tilde{E}[|\psi(X_t,t)|\eta_t] &= E[|\psi(X_t,t)|] \\ &\leq \kappa E[(1 + |X_t|^2 + \int_0^t |X_s|^2 ds)] \\ &< \infty\,. \end{aligned}$$

Then, we can define $\tilde{E}[\psi(X_t,t)\eta_t \big| \mathcal{Z}_t]$ as the unique element of $L^1(\Omega, \mathcal{Z}_t, \tilde{P})$ satisfying

$$\tilde{E}[\gamma \tilde{E}[\psi(X_t,t)\eta_t \big| \mathcal{Z}_t]] = \tilde{E}[\gamma\psi(X_t,t)\eta_t] \quad \forall\ \gamma \in L^\infty(\Omega, \mathcal{Z}_t, \tilde{P})\,.$$

The Fubini Theorem implies

$$\begin{aligned} \tilde{E}[\int_0^T |\psi(X_t,t)|\eta_t dt] &= E[\int_0^T |\psi(X_t,t)|dt] \\ &\leq \kappa E[\int_0^T (1 + |X_t|^2 + \int_0^t |X_s|^2 ds)dt] \\ &< \infty\,. \end{aligned} \tag{4.34}$$

Since $\psi = \psi^+ - \psi^-$, we suppose that $\psi \geq 0$. Let us consider

$$\psi_t^k = \frac{k\psi(X_t,t)\eta_t}{k + \psi(X_t,t)} \wedge k \in L^2((0,T)\times\Omega; dt \otimes \tilde{P})\,.$$

If we denote $\hat{\psi}_t^k = \tilde{E}[\psi_t^k \big| \mathcal{Z}_t]$, then for almost all t, $\hat{\psi}_t^k \in L^2(\Omega, \mathcal{Z}_t, \tilde{P})$ and $\hat{\psi}_t^k \uparrow p_t(\psi)$ a.s.. In addition,

$$\begin{aligned} \tilde{E}[\int_0^T |\hat{\psi}_t^k|dt] &= \tilde{E}[\int_0^T \hat{\psi}_t^k sign(\hat{\psi}_t^k)dt] \\ &= \tilde{E}[\int_0^T \psi_t^k sign(\hat{\psi}_t^k)dt] \\ &\leq \tilde{E}[\int_0^T |\psi(X_t,t)|\eta_t dt] \end{aligned} \tag{4.35}$$

The inequality (4.35) and Lebesgue's Theorem imply the result. □

The Proposition 4.3.5 enables us to define the space where we will find solutions of the Zakai equation. Let us consider the space

$$\mathcal{B}_T = \{\psi : \mathbb{R}^n \times [0,T] \to \mathbb{R} \text{ Borel function satisfying (4.33)}\}. \tag{4.36}$$

We define the space of linear operators $\tilde{\mathcal{L}}^1_{\mathcal{Z}}(0,T)$ by

$$q_\cdot(\cdot) \in \tilde{\mathcal{L}}^1_{\mathcal{Z}}(0,T) \Leftrightarrow \begin{cases} q_\cdot(\psi) \in L^1((0,T)\times\Omega; dt\otimes\tilde{P}) \ \ \forall \psi \in \mathcal{B}_T \\ q_t(\psi) \in L^1(\Omega, \mathcal{Z}_t, \tilde{P}) \ \text{ for almost all } t \text{ in } [0,T] \end{cases} \tag{4.37}$$

In particular, we have

- $\mathcal{C}_b^{2,1}(\mathbb{R}^n \times [0,T];\mathbb{R}) \subset \mathcal{B}_T$
- The linear operator $p_\cdot(\cdot)$ belongs to the space $\tilde{\mathcal{L}}^1_{\mathcal{Z}}(0,T)$

4.3.2 The Zakai equation

We begin by a useful Lemma. For any bounded Borel function $\beta : [0,T] \to \mathbb{R}^m$, let us define

$$\theta_t = \exp(i\int_0^t \beta_s^T R_s^{-1} dY_s + \frac{1}{2}\int_0^t \beta_s^T R_s^{-1}\beta_s ds). \tag{4.38}$$

Then, $\theta_0 = 1$ and $d\theta_t = i\theta_t \beta_t^T R_t^{-1} dY_t$, $0 \le t \le T$. In addition, the process θ_t is a $\mathbb{C}$-valued $\mathcal{Z}_t$-martingale.

Lemma 4.3.6. *Let ζ be a $\mathcal{Z}_t$-measurable and $\tilde{P}$-integrable complex valued random variable. If $\tilde{E}[\zeta\,\theta_t] = 0$ for all bounded Borel function $\beta : [0,T] \to \mathbb{R}^m$, then*

$$\zeta = 0 \quad a.s.$$

Proof. The proof is similar to that of Lemma 4.1.4 in [7]. □

From now we suppose that

$$E[\sup_{0\le t\le T} |X_t|^3] < \infty. \tag{4.39}$$

A sufficient condition for which (4.39) holds is that $E[|\xi|^6] < \infty$, see (4.11).
Let us consider

- The matrix $a(x,t) = \frac{1}{2}\sigma(x,t)\sigma(x,t)^T = \big(a_{ij}(x,t)\big)_{1\le i,j\le n}$
- The second order differential operator $A = -\sum_{1\le i\le n} g_i \frac{\partial}{\partial x_i} - \sum_{1\le i,j\le n} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j}$

Theorem 4.3.7 (Zakai equation)**.** *For every $\varphi \in \mathcal{C}_b^{2,1}(\mathbb{R}^n \times [0,T];\mathbb{R})$ we have*

$$p_t(\varphi) = \pi_0(\varphi) + \int_0^t p_s\big((\frac{\partial}{\partial s} - A)\varphi\big)ds + \int_0^t p_s(\varphi H^T)R^{-1}dY_s. \tag{4.40}$$

Remark 4.3.8. The strategy of the following proofs is instead of proving properties for unbounded quantities we prove the same properties for bounded ones which depends on parameters like ϵ and λ and then taking the limit to zero to get the results for the unbounded cases. This permit us to give an elementary proof.

Proof. For every $\epsilon > 0$, we define

$$\eta_t^\epsilon = \frac{\eta_t}{1+\epsilon\eta_t}\ . \tag{4.41}$$

In particular, $\eta_t^\epsilon \longrightarrow \eta_t$ as $\epsilon \to 0$ a.s.
We will prove in Proposition 4.3.10 that for every $\varphi \in \mathcal{C}_b^2(\mathbb{R}^n;\mathbb{R})$

$$\begin{aligned}\tilde{E}[\eta_t^\epsilon\varphi(X_t,t)\big|\mathcal{Z}_t] &= \frac{\pi_0(\varphi)}{1+\epsilon} + \int_0^t \tilde{E}[\eta_s^\epsilon(\frac{\partial}{\partial s} - A)\varphi(X_s,s)\big|\mathcal{Z}_s]ds \\ &\quad - \int_0^t \tilde{E}[\frac{\epsilon\eta_s^2}{(1+\epsilon\eta_s)^3}\varphi(X_s,s)H^TR^{-1}H\big|\mathcal{Z}_s]ds \\ &\quad + \int_0^t \tilde{E}[\frac{\eta_s}{(1+\epsilon\eta_s)^2}\varphi(X_s,s)H^T\big|\mathcal{Z}_s]R^{-1}dY_s \end{aligned}\tag{4.42}$$

We obtain the Zakai equation if we can take the limit $\epsilon \to 0$ in the equality (4.42). It is clear that we can take the limit $\epsilon \to 0$ in the left-hand side of the equality (4.42), then

$$\tilde{E}[\eta_t^\epsilon\varphi(X_t,t)\big|\mathcal{Z}_t] \longrightarrow \tilde{E}[\eta_t\varphi(X_t,t)\big|\mathcal{Z}_t]\,.$$

It remain the three terms of the right-hand side of the equality (4.42):
First term: We have

- $\tilde{E}[\eta_t^\epsilon(\frac{\partial}{\partial t} - A)\varphi(X_t,t)\big|\mathcal{Z}_t] \longrightarrow \tilde{E}[\eta_t(\frac{\partial}{\partial t} - A)\varphi(X_t,t)\big|\mathcal{Z}_t]$ as $\epsilon \to 0$ a.s.
- $\tilde{E}[\eta_t(\frac{\partial}{\partial t} - A)\varphi(X_t,t)\big|\mathcal{Z}_t] \le \tilde{E}[\eta_t|(\frac{\partial}{\partial t} - A)\varphi(X_t,t)|\ \big|\mathcal{Z}_t]$
- Proposition 4.3.5 implies that $\tilde{E}[\eta_t|(\frac{\partial}{\partial t} - A)\varphi(X_t,t)|\ \big|\mathcal{Z}_t]$ is integrable on $[0,T]$

Then,

$$\int_0^t \tilde{E}[\eta_s^\epsilon(\frac{\partial}{\partial s} - A)\varphi(X_s,s)\big|\mathcal{Z}_s]ds \longrightarrow_{\epsilon\to 0} \int_0^t \tilde{E}[\eta_s(\frac{\partial}{\partial s} - A)\varphi(X_s,s)\big|\mathcal{Z}_s]ds\,.$$

Second term: The function φ is bounded and the function H satisfies the inequalities(4.15), then there exists a constant $L > 0$ such that

$$\Big|\int_0^t \tilde{E}[\frac{\epsilon\eta_s^2}{(1+\epsilon\eta_s)^3}\varphi(X_s,s)H^TR^{-1}H\big|\mathcal{Z}_s]ds\Big| \le L\int_0^t \tilde{E}[(1+|X_t|^2 + \int_0^s |X_u|^2du)\big|\mathcal{Z}_s]ds\,.$$

In addition,

- $\tilde{E}[\frac{\epsilon\eta_s^2}{(1+\epsilon\eta_s)^3}\varphi(X_s,s)H^TR^{-1}H\big|\mathcal{Z}_s] \longrightarrow 0$ as $\epsilon \to 0$
- $\int_0^t \tilde{E}[(1+|X_t|^2 + \int_0^s |X_u|^2du)\big|\mathcal{Z}_s]ds < \infty$ a.s.

Then,

$$\int_0^t \tilde{E}[\frac{\epsilon\eta_s^2}{(1+\epsilon\eta_s)^3}\varphi(X_s,s)H^T R^{-1}H\big|\mathcal{Z}_s]ds \longrightarrow 0 \quad \text{as } \epsilon \to 0 \quad \text{a.s.}$$

Third term: Let us denote

- $\alpha_t^\epsilon = \int_0^t \tilde{E}[\frac{\eta_s}{(1+\epsilon\eta_s)^2}\varphi(X_s,s)H^T\big|\mathcal{Z}_s]R^{-1}dY_s$
- $\alpha_t = \int_0^t \tilde{E}[\eta_s\varphi(X_s,s)H^T\big|\mathcal{Z}_s]R^{-1}dY_s$

In particular, $\alpha_t^\epsilon - \alpha_t = \int_0^t \tilde{E}[\frac{\epsilon\eta_s^2(2+\epsilon\eta_s)}{(1+\epsilon\eta_s)^2}\varphi(X_s,s)H^T\big|\mathcal{Z}_s]R^{-1}dY_s$.
We check first that

$$\alpha_t^\epsilon \longrightarrow \alpha_t \text{ as } \epsilon \to 0 \text{ in probability.} \tag{4.43}$$

From Theorem 2.3.4. in [55], we obtain (4.43) if we prove that

$$\int_0^t \big|\tilde{E}[\frac{\epsilon\eta_s^2(2+\epsilon\eta_s)}{(1+\epsilon\eta_s)^2}\varphi(X_s,s)H^T\big|\mathcal{Z}_s]\big|^2 ds \longrightarrow 0 \text{ as } \epsilon \to 0 \quad \text{a.s.} \tag{4.44}$$

The function φ is bounded and the function H satisfies the inequalities (4.15), then there exists a constant $L' > 0$ such that

$$\begin{aligned}&\big|\tilde{E}[\frac{\epsilon\eta_s^2(2+\epsilon\eta_s)}{(1+\epsilon\eta_s)^2}\varphi(X_s,s)H^T\big|\mathcal{Z}_s]\big|^2 \\ &\qquad \le L'\big|\tilde{E}[\frac{\epsilon\eta_s^2(2+\epsilon\eta_s)}{(1+\epsilon\eta_s)^2}\big((1+|X_s|^2)^{1/2}+\int_0^s |X_u|du\big)H^T\big|\mathcal{Z}_s]\big|^2 .\end{aligned}$$

Using the Fubini-Lebesgue Theorem (Theorem 1.8.5. in [64]), we get

$$\begin{aligned}&\tilde{E}\big[\int_0^T \tilde{E}[\frac{\epsilon\eta_s^2(2+\epsilon\eta_s)}{(1+\epsilon\eta_s)^2}\big((1+|X_s|^2)^{1/2}+\int_0^s |X_u|du\big)H^T\big|\mathcal{Z}_s]ds\big] \\ &\qquad = \tilde{E}[\int_0^T \frac{\epsilon\eta_s^2(2+\epsilon\eta_s)}{(1+\epsilon\eta_s)^2}\big((1+|X_s|^2)^{1/2}+\int_0^s |X_u|du\big)H^T ds] \longrightarrow_{\epsilon\to 0} 0\end{aligned}$$

Then, there exists a subsequence such that $dt\otimes\tilde{P}$-$a.s.$ we have

$$\tilde{E}[\frac{\epsilon\eta_s^2(2+\epsilon\eta_s)}{(1+\epsilon\eta_s)^2}\big((1+|X_s|^2)^{1/2}+\int_0^s |X_u|du\big)H^T\big|\mathcal{Z}_s] \to 0 .$$

Moreover, using that $\epsilon\eta_t^2(2+\epsilon\eta_t)(1+\epsilon\eta_t)^{-2} \le 2\eta_t$ we get

$$\big|\tilde{E}[\frac{\epsilon\eta_s^2(2+\epsilon\eta_s)}{(1+\epsilon\eta_s)^2}\varphi(X_s,s)H^T\big|\mathcal{Z}_s]\big|^2 \le 4L'\big|\tilde{E}[\eta_s\big((1+|X_s|^2)^{1/2}+\int_0^s |X_u|du\big)\big|\mathcal{Z}_s]\big|^2$$

In Lemma 4.3.14 and Lemma 4.3.15 we will prove that

$$\tilde{E}[\eta_s\big((1+|X_s|^2)^{1/2}+\int_0^s |X_u|du\big)\big|\mathcal{Z}_s] \in L^\infty(0,T) \tag{4.45}$$

We apply Lebesgue's Theorem to (4.44) we get (4.43) for the above subsequence. Now taking the "subsequence" limit in (4.42) we get (4.40) almost surely. □

Corollary 4.3.9. *For every θ as defined in (4.38) we have*

$$\tilde{E}[\theta_t p_t(\varphi)] = \pi_0(\varphi) + \tilde{E}\big[\int_0^t \theta_s p_s\big((\frac{\partial}{\partial s} - A)\varphi + i\varphi H^T R^{-1}\beta\big)ds\big] \tag{4.46}$$

Proof. We have

$$\big|\frac{\epsilon\eta_t^2\theta_t}{(1+\epsilon\eta_t)^3}\varphi H^T R^{-1} H\big| \leq C\eta_t(1 + \sup_{0\leq s\leq t}|X_s|^2)$$

Since $\tilde{E}[\eta_t(1+\sup_{0\leq s\leq t}|X_s|^2)] = E[1+\sup_{0\leq s\leq t}|X_s|^2] < \infty$, we take the limit $\epsilon \to 0$ in (4.47) and with the Fubini-Lebesgue Theorem, Theorem 1.8.5. in [64], we deduce the result. □

Let φ be a function in $\mathcal{C}_b^{2,1}(\mathbb{R}^n \times [0,T];\mathbb{R})$, then Ito's formula yields to

$$\begin{aligned} d(\theta_t\eta_t^\epsilon\varphi(X_t,t)) &= \big[\frac{i\eta_t\theta_t}{(1+\epsilon\eta_t)^2}\varphi H^T R^{-1}\beta + \theta_t\eta_t^\epsilon(\frac{\partial}{\partial t} - A)\varphi \\ &\quad - \frac{\epsilon\eta_t^2\theta_t}{(1+\epsilon\eta_t)^3}\varphi H^T R^{-1} H\big]dt \\ &\quad + [\frac{\eta_t\theta_t}{(1+\epsilon\eta_t)^2}\varphi H^t R^{-1} + i\theta_t\eta_t^\epsilon\varphi R^{-1}]dY_t \\ &\quad + \theta_t\eta_t^\epsilon D\varphi^T\sigma(X_t,t)dW_t \end{aligned}$$

If we integrate $d(\theta_t\eta_t^\epsilon\varphi(X_t,t))$ from 0 to t and we take the expectation w.r.t. $\tilde{P}$, we get

$$\begin{aligned} \tilde{E}[\theta_t\eta_t^\epsilon\varphi(X_t,t)] &= \frac{\pi_0(\varphi)}{1+\epsilon} + \tilde{E}\big[\int_0^t \theta_s\frac{i\eta_s}{(1+\epsilon\eta_s)^2}\varphi(X_s,s)H^T R^{-1}\beta ds\big] \\ &\quad + \tilde{E}\big[\int_0^t \theta_s\eta_s^\epsilon(\frac{\partial}{\partial s} - A)\varphi(X_s,s)ds\big] \\ &\quad - \tilde{E}\big[\int_0^t \theta_s\frac{\epsilon\eta_s^2}{(1+\epsilon\eta_s)^3}\varphi(X_s,s)H^T R^{-1}H ds\big] \end{aligned} \tag{4.47}$$

Proposition 4.3.10. *For every $\varphi \in \mathcal{C}_b^{2,1}(\mathbb{R}^n \times [0,T];\mathbb{R})$ and every $\epsilon > 0$ we have*

$$\begin{aligned} \tilde{E}[\eta_t^\epsilon\varphi(X_t,t)\big|\mathcal{Z}_t] &= \frac{\pi_0(\varphi)}{1+\epsilon} + \int_0^t \tilde{E}[\eta_s^\epsilon(\frac{\partial}{\partial s} - A)\varphi(X_s,s)\big|\mathcal{Z}_s]ds \\ &\quad - \int_0^t \tilde{E}[\frac{\epsilon\eta_s^2}{(1+\epsilon\eta_s)^3}\varphi(X_s,s)H^T R^{-1}H\big|\mathcal{Z}_s]ds \\ &\quad + \int_0^t \tilde{E}[\frac{\eta_s}{(1+\epsilon\eta_s)^2}\varphi(X_s,s)H^T\big|\mathcal{Z}_s]R^{-1}dY_s \end{aligned} \tag{4.48}$$

For the proof of Proposition 4.3.10 we need the following Lemma

Lemma 4.3.11. *i.* $\tilde{E}[\theta_t\eta_t^\epsilon\varphi(X_t,t)] = \tilde{E}[\theta_t\tilde{E}[\eta_t^\epsilon\varphi(X_t,t)\big|\mathcal{Z}_t]]$

ii. $\tilde{E}\big[\int_0^t \big(\theta_s\eta_s^\epsilon(\frac{\partial}{\partial s}-A)\varphi+\epsilon\theta_s\eta_s^2(1+\epsilon\eta_s)^{-3}\varphi H^TR^{-1}H\big)ds\big]$
$= \tilde{E}\big[\theta_t\int_0^t \tilde{E}[\eta_s^\epsilon(\frac{\partial}{\partial s}-A)\varphi+\epsilon\eta_s^2(1+\epsilon\eta_s)^{-3}\varphi H^TR^{-1}H\big|\mathcal{Z}_s]ds\big]$

iii. $\tilde{E}\big[\int_0^t i\theta_s\eta_s(1+\epsilon\eta_s)^{-2}\varphi(X_s,s)H^TR^{-1}\beta ds\big]$
$= \tilde{E}\big[\theta_t\int_0^t \tilde{E}[\eta_s(1+\epsilon\eta_s)^{-2}\varphi(X_s,s)H^T\big|\mathcal{Z}_s]R^{-1}dY_s\big]$

Proof. **The first equality:** Since θ_t is $\mathcal{Z}_t$-measurable, we obtain the equality.
The second equality: The inequalities (4.15) and (4.19) imply that

$$\theta_t\eta_t^\epsilon(\frac{\partial}{\partial t}-A)\varphi(X_t,t)+\frac{\theta_t\epsilon\eta_t^2\varphi(X_t,t)}{(1+\epsilon\eta_t)^3}H^TR^{-1}H\in L^1((0,T)\times\Omega;dt\otimes d\tilde{P})\,.$$

Then, The Fubini-Lebesgue Theorem (Theorem 1.8.5. in [64]) yields to

$$\tilde{E}\big[\int_0^t\big(\theta_s\eta_s^\epsilon(\frac{\partial}{\partial s}-A)\varphi(X_s,s)+\theta_s\frac{\epsilon\eta_s^2}{(1+\epsilon\eta_s)^3}\varphi(X_s,s)H^TR^{-1}H\big)ds\big]$$
$$=\int_0^t\tilde{E}\big[\theta_s\eta_s^\epsilon(\frac{\partial}{\partial s}-A)\varphi(X_s,s)+\theta_s\frac{\epsilon\eta_s^2}{(1+\epsilon\eta_s)^3}\varphi(X_s,s)H^TR^{-1}H\big]ds\,.$$

In the other hand, using that $\theta_s=\tilde{E}[\theta_t\big|\mathcal{Z}_s]$ we get

$$\tilde{E}\big[\theta_s\eta_s^\epsilon(\frac{\partial}{\partial t}-A)\varphi(X_s,s)+\theta_s\frac{\epsilon\eta_s^2}{(1+\epsilon\eta_s)^3}\varphi(X_s,s)H^TR^{-1}H\big]$$
$$=\tilde{E}\big[\theta_s\tilde{E}[\eta_s^\epsilon(\frac{\partial}{\partial t}-A)\varphi(X_s,s)+\frac{\epsilon\eta_s^2\varphi(X_s,s)}{(1+\epsilon\eta_s)^3}H^TR^{-1}H\big|\mathcal{Z}_s]\big]$$
$$=\tilde{E}\big[\theta_t\tilde{E}[\eta_s^\epsilon(\frac{\partial}{\partial t}-A)\varphi(X_s,s)+\frac{\epsilon\eta_s^2\varphi(X_s,s)}{(1+\epsilon\eta_s)^3}H^TR^{-1}H\big|\mathcal{Z}_s]\big]$$

The third equality: for similar arguments as for the second equality we get

$$\tilde{E}\big[\int_0^t\theta_s\frac{i\eta_s}{(1+\epsilon\eta_s)^2}\varphi(X_s,s)H^TR^{-1}\beta ds\big]=\tilde{E}\big[\int_0^t\theta_s\tilde{E}[\frac{i\eta_s}{(1+\epsilon\eta_s)^2}\varphi(X_s,s)H^T\big|\mathcal{Z}_s]R^{-1}\beta ds\big]$$

Let us define the process $\tilde{Y}_t=\int_0^t\tilde{E}[\frac{\eta_s}{(1+\epsilon\eta_s)^2}\varphi(X_s,s)H^T\big|\mathcal{Z}_s]R^{-1}dY_s$, the Ito formula gives

$$\begin{aligned}d(\theta_t\tilde{Y}_t) &= \{\theta_t\ \tilde{E}[\frac{\eta_t\varphi(X_s,s)}{(1+\epsilon\eta_s)^2}H^T\big|\mathcal{Z}_s]+i\ \tilde{Y}_t\theta_t\beta^T\}R^{-1}dY_t\\ &\quad+\theta_t\tilde{E}[\frac{i\eta_t}{(1+\epsilon\eta_t)^2}\varphi(X_t,t)H^T\big|\mathcal{Z}_t]R^{-1}\beta dt\,.\end{aligned}$$

We integrate $d(\theta_t\tilde{Y}_t)$ from 0 to t and we take the expectation, we get

$$\tilde{E}[\theta_t\,\tilde{Y}_t]=\tilde{E}[\int_0^t\theta_s\tilde{E}[\frac{i\eta_s}{(1+\epsilon\eta_s)^2}\varphi(X_s,s)H^T\big|\mathcal{Z}_s]R^{-1}\beta ds]\,.$$

Then,

$$\tilde{E}\big[\int_0^t \theta_s \frac{i\eta_s}{(1+\epsilon\eta_s)^2}\varphi(X_s,s)H^T R^{-1}\beta ds\big] = \tilde{E}[\theta_t \int_0^t \tilde{E}[\frac{\eta_s}{(1+\epsilon\eta_s)^2}\varphi(X_s,s)H^T\big|\mathcal{Z}_s]R^{-1}dY_s]\,.$$

□

Proof. (Proposition 4.3.10) The use of Lemma 4.3.11 in the equality (4.47) and of the density of the θ's in the way of Lemma 4.3.6 give the result. □

Let us define the following processes

i. $\tilde{X}_t = \int_0^t |X_s| ds$

ii. For all $\lambda > 0$, $\tilde{X}_t^\lambda = \tilde{X}_t(1+\lambda\tilde{X}_t)^{-1}$

Remark 4.3.12. If we apply Ito's formula to the product $\theta_t\eta_t^\epsilon\tilde{X}_t^\lambda$ then we integrate from 0 to t and finally we take the expectation w.r.t. $\tilde{P}$ we get

$$\begin{aligned}\tilde{E}[\theta_t\eta_t^\epsilon\tilde{X}_t^\lambda] &= \tilde{E}\big[\int_0^t \big(\frac{i\eta_s\theta_s}{(1+\epsilon\eta_s)^2}\tilde{X}_s^\lambda H^T R^{-1}\beta + \frac{\theta_s\eta_s^\epsilon|X_s|}{(1+\lambda\tilde{X}_s)^2} \\ &\quad -\frac{\epsilon\eta_s^2\theta_s}{(1+\epsilon\eta_s)^3}\tilde{X}_s^\lambda H^T R^{-1}H\big)ds\big]\,.\end{aligned}$$

Proposition 4.3.13.

$$\begin{aligned}\tilde{E}[\eta_t^\epsilon\tilde{X}_t^\lambda\big|\mathcal{Z}_t] &= \int_0^t \tilde{E}[\frac{\eta_s}{(1+\epsilon\eta_s)^2}\tilde{X}_s^\lambda H^T\big|\mathcal{Z}_s]R^{-1}dY_s \\ &\quad + \int_0^t \tilde{E}[\frac{\eta_s^\epsilon|X_s|}{(1+\lambda\tilde{X}_s)^2} - \frac{\epsilon\eta_s^2}{(1+\epsilon\eta_s)^3}\tilde{X}_s^\lambda H^T R^{-1}H\big|\mathcal{Z}_s]ds\,. \qquad (4.49)\end{aligned}$$

Proof. Like the proof of Proposition 4.3.10, we need to show that:

1. $\tilde{E}[\theta_t\eta_t^\epsilon\tilde{X}_t^\lambda] = \tilde{E}[\theta_t\tilde{E}[\eta_t^\epsilon\tilde{X}_t^\lambda)\big|\mathcal{Z}_t]]$
2. $\tilde{E}\big[\int_0^t \big(\frac{\theta_s\eta_s^\epsilon|X_s|}{(1+\lambda\tilde{X}_s)^2} - \frac{\epsilon\eta_s^2\theta_s}{(1+\epsilon\eta_s)^3}\tilde{X}_s^\lambda H^T R^{-1}H\big)ds\big] = \tilde{E}\big[\theta_t\int_0^t \tilde{E}[\frac{\eta_s^\epsilon|X_s|}{(1+\lambda\tilde{X}_s)^2} - \frac{\epsilon\eta_s^2}{(1+\epsilon\eta_s)^3}\tilde{X}_s^\lambda H^T R^{-1}H\big|\mathcal{Z}_s]ds\big]$
3. $\tilde{E}\big[\int_0^t \frac{i\eta_s\theta_s}{(1+\epsilon\eta_s)^2}\tilde{X}_s^\lambda H^T R^{-1}\beta ds\big] = \tilde{E}\big[\theta_t\int_0^t \tilde{E}[\frac{\eta_s}{(1+\epsilon\eta_s)^2}\tilde{X}_s^\lambda H^T\big|\mathcal{Z}_s]R^{-1}dY_s\big]$

The first equality: the process θ_t is $\mathcal{Z}_t$-measurable, then we obtain the equality.
The second equality: We have

$$\frac{\theta_t\eta_t^\epsilon|X_t|}{(1+\lambda\tilde{X}_t)^2} - \frac{\epsilon\eta_t^2\theta_s}{(1+\epsilon\eta_t)^3}\tilde{X}_t^\lambda H^T R^{-1}H \in L^1((0,T)\times\Omega; dt\otimes d\tilde{P})\,.$$

Then, we obtain easily the second equality.
The third equality: First, we have

$$\tilde{E}\big[\int_0^t \frac{i\eta_s\theta_s}{(1+\epsilon\eta_s)^2}\tilde{X}_s^\lambda H^T R^{-1}\beta ds\big] = \tilde{E}\big[\int_0^t \theta_s E[\frac{i\eta_s}{(1+\epsilon\eta_s)^2}\tilde{X}_s^\lambda H^T R^{-1}\beta\big|\mathcal{Z}_s]ds\big]$$

Then, if we consider the process $\tilde{Y}_t = \int_0^t \tilde{E}[\frac{\eta_s}{(1+\epsilon\eta_s)^2}\tilde{X}_s^\lambda H^T\big|\mathcal{Z}_s]R^{-1}dY_s$ and we write the Ito formula for the product $\theta_t\tilde{Y}_t$ then we integrate from 0 to t and we take the expectation $w.r.t\ \tilde{P}$ we get the result. □

The following two Lemmas proves (4.45). That is,

$$\tilde{E}[\eta_s\big((1+|X_s|^2)^{1/2} + \int_0^s |X_u|du\big)\big|\mathcal{Z}_s] \in L^\infty(0,T)$$

Let us denote, for all $x \in \mathbb{R}^n$ and $\lambda > 0$,

- $\Delta(x) = (1+|x|^2)^{1/2}$
- $\Delta_\lambda(x) = \big(\frac{1+|x|^2}{1+\lambda|x|^2}\big)^{1/2}$
- $\Delta_\lambda(x) \longrightarrow \Delta(x) = (1+|x|^2)^{1/2}$ as $\lambda \to 0$

Lemma 4.3.14.

$$t \longrightarrow \tilde{E}[\eta_t\Delta(X_t)\big|\mathcal{Z}_t](\omega) \ \in\ L^\infty(0,T) \quad \tilde{P}\text{-}a.s. \tag{4.50}$$

Proof. For any $\lambda > 0$, $\Delta_\lambda \in \mathcal{C}_b^2(\mathbb{R}^n;\mathbb{R})$. Then, the equality (4.48) applies to $\varphi = \Delta_\lambda$. In addition,

$$\big|\tilde{E}[\frac{\eta_s}{(1+\epsilon\eta_s)^2}\Delta_\lambda(X_s)H^T\big|\mathcal{Z}_s]\big| \le C_\epsilon E[1+\sup_{0\le s\le t}|X_s|^2] \le C'_{\epsilon,T}$$

Then,

i. $\tilde{E}[\eta_s(1+\epsilon\eta_s)^{-2}\Delta_\lambda(X_s)H^T\big|\mathcal{Z}_s] \longrightarrow \tilde{E}[\eta_s(1+\epsilon\eta_s)^{-2}\Delta(X_s)H^T\big|\mathcal{Z}_s]$ as $\lambda \to 0$

ii. $\tilde{E}[\eta_s(1+\epsilon\eta_s)^{-2}\Delta(X_s)H^T\big|\mathcal{Z}_s] \in L^2((0,T)\times\Omega; dt\otimes d\tilde{P})$

We conclude that the equality (4.48) apply also to Δ. In particular, the process

$$M_t^\epsilon = \int_0^t \tilde{E}[\frac{\eta_s}{(1+\epsilon\eta_s)^2}\Delta(X_s)H^T\big|\mathcal{Z}_s]R^{-1}dY_s$$

is a square integrable $\mathcal{Z}_t$-martingale and $\tilde{E}[M_t^\epsilon] = 0$. Moreover,

$$\begin{aligned} M_t^\epsilon &= \tilde{E}[\eta_t^\epsilon\Delta(X_t)\big|\mathcal{Z}_t] - \frac{\pi_0(\Delta)}{1+\epsilon} - \int_0^t \tilde{E}[\eta_s^\epsilon(\frac{\partial}{\partial s} - A)\Delta(X_s) \\ &\quad + \frac{\epsilon\eta_s^2}{(1+\epsilon\eta_s)^3}\Delta(X_t)H^T R^{-1}H\big|\mathcal{Z}_s]ds \end{aligned} \tag{4.51}$$

In other hand, for all $(x,t) \in \mathbb{R}^n \times [0,T]$

$$A\Delta(x) = \frac{x^T a(x,t) x}{(1+|x|^2)^{3/2}} - \frac{x^T g(x,t) + tr(a(x,t))}{(1+|x|^2)^{1/2}} \ .$$

Then, there exists a constant $D_T > 0$ such that

$$\big|A\Delta(x)\big| \leq D_T \Delta(x) \quad \forall\, (x,t) \in \mathbb{R}^n \times [0,T]\,.$$

In addition, using the inequalities (4.15) and the fact that $\epsilon\eta_t^2/(1+\epsilon\eta_t)^3 \leq \eta_t$ we can find a constant $D'_T > 0$ such that

$$\frac{\epsilon\eta_t^2}{(1+\epsilon\eta_t)^3}\Delta(X_t) H^T R^{-1} H \leq D'_T(1 + \sup_{0\leq s\leq t}|X_s|^3)$$

Then, Lebesgue's Theorem implies that M_t^ϵ converges as $\epsilon \to 0$ to a process M_t. In addition,

- The process M_t is a $\mathcal{Z}_t$-martingale
- $\tilde{E}[M_t] = 0$
- $M_t = \tilde{E}[\eta_t \Delta(X_t)\big|\mathcal{Z}_t] - \pi_0(\Delta) - \int_0^t \tilde{E}[\eta_s(\frac{\partial}{\partial s} - A)\Delta(X_s)\big|\mathcal{Z}_s]ds$

Since $\tilde{E}[M_t] = 0$, then the martingale M_t has a modification such that its sample paths are right continuous with left hand limits a.s., see Theorem 1.2.2 in [55]. We can assume that M_t has a.s. right continuous sample paths with left hand limits which implies that the paths are bounded on the compact $[0,T]$. This proves the Lemma. □

Lemma 4.3.15.

$$t \longrightarrow \tilde{E}[\eta_t \int_0^t |X_s| ds\big|\mathcal{Z}_t](\omega) \ \in \ L^\infty(0,T) \quad \tilde{P}\text{-a.s.} \tag{4.52}$$

Proof. Recall first that for all $\lambda > 0$, $\tilde{X}_t^\lambda = \tilde{X}_t(1+\lambda\tilde{X}_t)^{-1}$. We have

- For all $\mathbf{x} \in \mathcal{C}([0,T],\mathbb{R}^n)$ and $t \in [0,T]$

$$\frac{\int_0^t |\mathbf{x}(s)|ds}{1+\lambda\int_0^t |\mathbf{x}(s)|ds} \longrightarrow \int_0^t |\mathbf{x}(s)|ds \quad \text{as } \lambda \to 0\,.$$

- $\big|\tilde{E}[\eta_s(1+\epsilon\eta_s)^{-2}\tilde{X}_s^\lambda H^T\big|\mathcal{Z}_s]\big| \leq C_\epsilon E[1+\sup_{0\leq s\leq t}|X_s|^2] \leq C'_{\epsilon,T}$

Then, we have the following convergence in $L^2((0,T)\times\Omega; dt\otimes d\tilde{P})$:

$$\tilde{E}[\eta_s(1+\epsilon\eta_s)^{-2}\tilde{X}_s^\lambda H^T\big|\mathcal{Z}_s] \longrightarrow \tilde{E}[\eta_s(1+\epsilon\eta_s)^{-2}\tilde{X}_t H^T\big|\mathcal{Z}_s] \quad \text{as } \lambda \to 0\,.$$

We conclude that we can replace $\tilde{X}_t^\lambda$ by $\tilde{X}_t$ in equation (4.49). That is,

$$\begin{aligned}\tilde{E}[\eta_t^\epsilon \tilde{X}_t \big| \mathcal{Z}_t] &= \int_0^t \tilde{E}[\frac{\eta_s}{(1+\epsilon\eta_s)^2}\tilde{X}_s H^T \big| \mathcal{Z}_s] R^{-1} dY_s \\ &\quad + \int_0^t \tilde{E}[\ \eta_s^\epsilon |X_s| - \frac{\epsilon\eta_s^2}{(1+\epsilon\eta_s)^3}\tilde{X}_s H^T R^{-1} H \big| \mathcal{Z}_s] ds\,.\end{aligned}$$

Moreover, let us denote the process $M_t^\epsilon = \int_0^t \tilde{E}[\frac{\eta_s}{(1+\epsilon\eta_s)^2}\tilde{X}_s H^T \big| \mathcal{Z}_s] R^{-1} dY_s$, then

- M_t^ϵ is a square integrable martingale
- $\tilde{E}[M_t^\epsilon] = 0$
- $M_t^\epsilon = \tilde{E}[\eta_t^\epsilon \tilde{X}_t \big| \mathcal{Z}_t] - \int_0^t \tilde{E}[\eta_s^\epsilon |X_s| - \frac{\epsilon\eta_s^2}{(1+\epsilon\eta_s)^3}\tilde{X}_s H^T R^{-1} H \big| \mathcal{Z}_s] ds$

Since $\eta_t^\epsilon |X_t| \leq \eta_t |X_t|$ and $\frac{\epsilon\eta_t^2}{(1+\epsilon\eta_t)^3}\tilde{X}_t H^T R^{-1} H \leq C_T(1+\sup_{0\leq s\leq T}|X_t|^3)$, then Lebesgue's Theorem permit us to deduce that M_t^ϵ converges as $\epsilon \to 0$ and the limit M_t is a $\mathcal{Z}_t$-martingale with $\tilde{E}[M_t] = 0$. Moreover,

$$M_t = \tilde{E}\big[\eta_t \int_0^t |X_s| ds \big| \mathcal{Z}_t\big] - \int_0^t \tilde{E}[\ \eta_s^\epsilon |X_s|\ \big| \mathcal{Z}_s] ds\,.$$

Since $\tilde{E}[M_t] = 0$ then the martingale M_t has a modification such that its sample paths are right continuous with left hand limits a.s., see Theorem 1.2.2 in [55]. We can assume that M_t has a.s. right continuous sample paths with left hand limits which implies that the paths are bounded on the compact $[0,T]$. This proves the Lemma. □

4.3.3 Uniqueness of the solution to the Zakai equation

From now we suppose that the following assumptions hold:

Assumption 4.1. The initial condition of the state is deterministic, ie. there exist $x \in \mathbb{R}^n$ such that $X_0 = \xi = x$.

Assumption 4.2. The functions g and σ are $\mathcal{C}^3$-class in x and $\mathcal{C}^1$-class in t.

Assumption 4.3. The function $h(x,t)$ is $\mathcal{C}^1$-class in x and t.

We denote the solution of the state equation (4.5) by $X^x = \{X_t^x\}_t$. The solution $\Phi_t(x) = X_t^x$ has a modification that is almost surely a $\mathcal{C}^2$-class diffeomorphism in the variable x, see Theorem 2.3. in [54]. That is, for all $t \in [0,T]$ we have almost surely

$$\Phi_t : \mathbb{R}^n \to \mathbb{R}^n \quad \text{is a } \mathcal{C}^2\text{-class diffeomorphism}\,. \tag{4.53}$$

Denoting $f(x,t,\omega) = H(X_\cdot^x(\omega), t)$ then for almost all ω in Ω we have

$$H(X_\cdot^x(\omega), t) = \tilde{H}(X_t^x(\omega), t, \omega)\,, \tag{4.54}$$

where for all $y \in \mathbb{R}^n$, $\tilde{H}(y,t,\omega) = f(\Phi_t^{-1}(y), t, \omega)$.

i. For almost all ω, $\tilde{H}(\cdot,\cdot,\omega)$ is a jointly measurable function on $\mathbb{R}^n \times [0,T]$

ii. For all $x \in \mathbb{R}^n$, the process $\{\tilde{H}(x,t,\cdot)\}_t$ is $\mathcal{W}_t = \sigma\{W_s\,,\ 0 \le s \le t\}^*$ adapted

iii. For almost all ω and for all $t \in [0,T]$ the function $x \to \tilde{H}(x,t,\omega)$ is $\mathcal{C}^2$-class

Let us consider a subspace Ω' in Ω such that $\tilde{P}(\Omega') = 1$ and on which both (4.53) and (4.54) hold for all $\omega \in \Omega'$.
We fix ω in Ω' and t in $(0,T]$, and we simply write $\tilde{H}(x,s)$ for $\tilde{H}(x,s,\omega)$. We consider the following partial differential equation:

$$\left\{ \begin{array}{rcl} (\frac{\partial}{\partial s} - A)\psi(x,s) + i\psi(x,s)\tilde{H}(x,s)^T R_s^{-1}\beta_s &=& 0\,, \quad 0 \le s \le t \\ \psi(x,t) &=& \varphi(x,t) \end{array} \right. \tag{4.55}$$

Assumption 4.4. For all $\omega \in \Omega'$, the partial differential equation (4.55) has at least one solution $\Psi = \Psi_\omega \in \mathcal{C}^{2,1}(\mathbb{R}^n \times [0,T];\mathbb{C})$ such that its first derivative w.r.t. x has at most a polynomial growth.

We consider a standard $\mathbb{R}^n$-valued $\mathcal{F}_t$-Brownian motion $\tilde{W}$ independent with the process Y on the space $(\Omega, \tilde{P})$ and we denote by $\xi^x_{s,r}$ the solution of the SDE (4.5) on $[s,t]$driven by the noise $\tilde{W}$ in the place of the noise W such that $\xi^x_{s,s} = x$. That is, for all $r \in [s,t]$

$$\xi^x_{s,r} = x + \int_s^r g(\xi^x_{s,u},u)du + \int_s^r \sigma(\xi^x_{s,u},u)d\tilde{W}_u\,.$$

Let us define for all $s \in [0,t]$

$$\Upsilon_\omega(x,s) = \Upsilon(x,s) = \tilde{E}[\varphi(\xi^x_{s,t},t)\,\mathcal{E}^x_{s,t}]\,, \tag{4.56}$$

where $\mathcal{E}^x_{s,t} = \exp\left(i\int_s^t \tilde{H}(\xi^x_{s,r},r)^T R_r^{-1}\beta_r dr\right)$. Then,

- $\Upsilon \in \mathcal{C}^{2,1}(\mathbb{R}^n \times [0,t];\mathbb{C})$ and $\sup_{x,s}\left|\Upsilon(x,s)\right| \le \|\varphi\|$
- $d\mathcal{E}^x_{s,t} = i\mathcal{E}^x_{s,t}\tilde{H}(\xi^x_{s,t},t)^T R_t^{-1}\beta_t dt$ and $\mathcal{E}^x_{s,s} = 1$

Proposition 4.3.16. *Under the Assumption 4.4, the partial differential equation (4.55) has only one solution. This solution is bounded and coincide with Υ defined in (4.56).*

Proof. Let $\Psi \in \mathcal{C}^{2,1}(\mathbb{R}^n \times [0,t];\mathbb{C})$ a solution of (4.55), in particular $\Psi(x,t) = \varphi(x,t)$. We apply the Ito formula to the product $\mathcal{G}_{s,r}(x) = \Psi(\xi^x_{s,r},r)\mathcal{E}^x_{s,r}$ we get

$$\begin{array}{rcl} d\mathcal{G}_{s,r}(x) &=& \mathcal{E}^x_{s,r}\{(\dfrac{\partial}{\partial r} - A)\Psi(\xi^x_{s,r},r) + i\tilde{H}(\xi^x_{s,r},r)^T R_r^{-1}\beta_r\}dr \\ && +\mathcal{E}^x_{s,r}D\Psi(\xi^x_{s,r},r)^T\sigma(\xi^x_{s,r},r)d\tilde{W}_r\,. \end{array}$$

Integrating from s to t and taking the expectation w.r.t. $\tilde{P}$ we get

$$\tilde{E}[\varphi(\xi^x_{s,t},t)\mathcal{E}^x_{s,t}] - \Psi(x,s) = 0\,.$$

This proves the Proposition. □

Definition 4.3.17. For every $t \in [0,T]$, we define the family $\mathcal{P}_t$ as the set of stochastic processes $\{\Phi(x,s) = \Phi(x,s,\omega)\}_{0\le s\le t}$, $x \in \mathbb{R}^n$, satisfying

i. For each $s \in [0,t]$, the function $\Phi(\cdot, s)$ is a bounded $\mathcal{C}^2$-class function on $\mathbb{R}^n$ a.s.

ii. For each $x \in \mathbb{R}^n$, the process $\Phi(x,\cdot)$ is $\mathcal{F}_s$ adapted and has $\mathcal{C}^1$-class paths on $[0,t]$ a.s.

iii. For each $(x,s) \in \mathbb{R}^n \times [0,t]$,

$$\Phi(x,s) = \Phi(x,0) + \int_0^s \{A\Phi(x,r) - i\Psi(x,r)\tilde{H}(x,r)^T R_r^{-1}\beta_r\}dr \quad \text{a.s.} \tag{4.57}$$

Remark 4.3.18. The solution Ψ given in (4.56) of the partial differential equation (4.55) belongs to the space $\mathcal{P}_t$.

The Ito formula for the composition of processes, see Theorem 1.1. in [53], applies to the processes in $\mathcal{P}_t$ and for all $\Phi \in \mathcal{P}_t$

$$d\Phi(X_s^x, s) = (\frac{\partial}{\partial s} - A)\Phi(X_s^x, s)ds + D\Phi(X_s^x, s)^T\sigma(X_s^x, s)dW_s \quad \text{a.s.}$$

We prove next that elements of $\mathcal{P}_t$ satisfies the Zakai equation

Proposition 4.3.19. *For every $\Phi \in \mathcal{P}_t$ and $s \in [0,t]$ we have*

$$p_s(\Phi) = \pi_0(\Phi) + \int_0^s p_r\big((\frac{\partial}{\partial r} - A)\Phi\big)dr + \int_0^t p_r\big(\Phi H^T\big)dY_r\,. \tag{4.58}$$

Proof. The proof is similar to that of Theorem 4.3.7, the only difference is that the function Φ is bounded but not necessarily its first derivatives w.r.t. t and x and its second derivatives w.r.t x.
The process Φ satisfies the equation (4.57) $\tilde{P}$-a.s. and P-a.s., then

$$(\frac{\partial}{\partial s} - A)\Phi(X_s^x, s) = \Phi(X_s^x, s)H(X_\cdot^x, s)^T R_s^{-1}\beta_s\,, \quad \text{a.s.} \tag{4.59}$$

In addition, we have

- $E[\sup_{0\le t\le T}|X_t|^2] < \infty$
- $|H(\mathbf{x},t)| \le a_T K(1 + |\mathbf{x}(t)| + \int_0^t (1 + |\mathbf{x}(s)|)ds)$
- $|H(\mathbf{x},t)| \le \tilde{K}_T\big(1 + \sup_{0\le s\le t}|\mathbf{x}(s)|\big)$, where $\tilde{K}_T = a_T K(1+T)$

Together with Proposition 4.3.5, we get

1. $(\frac{\partial}{\partial s} - A)\Phi(X_s^x, s) \in L^1((0,t)\times\Omega; ds \otimes d\tilde{P})$
2. $\int_0^t \tilde{E}[\eta_s|(\frac{\partial}{\partial s} - A)\Phi(X_s^x, s)|\big|\mathcal{Z}_s]ds < \infty$

Then, in one hand the process Φ satisfies the equation (4.48) and in the other hand we can take the limit $\epsilon \to 0$ in this identity to get the Zakai equation. □

Remark 4.3.20. The equality (4.46) is valid for the function Φ and for every θ as defined in (4.38) and every $s \in [0,t]$

$$\begin{aligned}\tilde{E}[\theta_s p_s(\Phi)] &= \pi_0(\Phi) + \tilde{E}\big[\int_0^s \theta_r p_r\big((\frac{\partial}{\partial r} - A)\Phi + i\Phi H^T R^{-1}\beta\big)dr\big] \\ &= \pi_0(\Phi)\,.\end{aligned}$$

Now we are ready to prove the uniqueness of the solution to the Zakai equation. Let us first note that the Zakai equation (4.40) and the equality (4.46) still hold for complex values functions $\varphi \in \mathcal{C}_b^{2,1}(\mathbb{R}^n \times [0,T];\mathbb{C})$. Then, we define $\mathcal{B}_T^c$

$$\mathcal{B}_T^c = \{\psi : \mathbb{R}^n \times [0,T] \to \mathbb{C} \text{ Borel function satisfying (4.33)}\}\,.$$

In particular, $\mathcal{C}_b^{2,1}(\mathbb{R}^n \times [0,T];\mathbb{C}) \subset \mathcal{B}_T^c$.
The linear operator $p_\cdot(\cdot)$ extends naturally to belong to the space $\tilde{\mathcal{L}}^1_{\mathcal{Z}}(0,T;\mathbb{C})$ defined by

$$q_\cdot(\cdot) \in \tilde{\mathcal{L}}^1_{\mathcal{Z}}(0,T;\mathbb{C}) \Longleftrightarrow \left\{\begin{array}{c} q_\cdot(\psi) \in L^1((0,T)\times\Omega; dt\otimes\tilde{P})\,,\ \forall \psi \in \mathcal{B}_T^c \\ q_t(\psi) \in L^1(\Omega, \mathcal{Z}_t, \tilde{P})\,,\ \text{for almost all } t \text{ in } [0,T]\end{array}\right.$$

Theorem 4.3.21 (Uniqueness)**.** *Let $q_\cdot(\cdot) \in \tilde{\mathcal{L}}^1_{\mathcal{Z}}(0,T;\mathbb{C})$ such that for all $t \in [0,T]$ and for all $\Phi \in \mathcal{C}_b^{2,1}(\mathbb{R}^n \times [0,T];\mathbb{C}) \bigcup \mathcal{P}_t$ we have*

$$q_s(\Phi) = \pi_0(\Phi) + \int_0^s q_r\big((\frac{\partial}{\partial r} - A)\Phi\big)dr + \int_0^t q_r\big(\Phi H^T\big)dY_r\,,\ \text{a.s.} \tag{4.60}$$

Then, for all $t \in [0,T]$ and for every real valued bounded Borel function φ on $\mathbb{R}^n \times [0,T]$ we have

$$q_t(\varphi) = \tilde{E}[\eta_t \varphi(X_t^x, t)\big|\mathcal{Z}_t]\,.$$

Remark 4.3.22. $\pi_0(\Phi) = \Phi(x,0)$.

Proof. Let $t \in [0,T]$ and $\varphi \in \mathcal{C}_b^{2,1}(\mathbb{R}^n \times [0,T];\mathbb{R})$. We denote by Ψ the solution of the associated partial differential equation (4.55) with the terminal condition $\Psi(x,t) = \varphi(x,t)$ a.s.
The function (process) Ψ is an element of $\mathcal{P}_t$, then from Remark 4.3.20 we have

$$\tilde{E}[\theta_s p_s(\Psi)] = \pi_0(\Psi) = \tilde{E}[\theta_s q_s(\Psi)]\,,\quad \forall\, s \in [0,t]\,.$$

In particular, for $s = t$ we have for every θ as in (4.38)

$$\tilde{E}[\theta_t p_t(\varphi)] = \tilde{E}[\theta_t q_s(\varphi)]\,.$$

Using Lemma 4.3.6 we get, for all $t \in [0,T]$ and for all $\varphi \in \mathcal{C}_b^{2,1}(\mathbb{R}^n \times [0,T];\mathbb{R})$, that

$$p_t(\varphi) = q_s(\varphi)\,.$$

Since every bounded Borel function on $\mathbb{R}^n \times [0,T]$ can be approximated with elements of $\mathcal{C}_b^{2,1}(\mathbb{R}^n \times [0,T];\mathbb{R})$, we get the result. □

4.4 Particle approximation method

In our formulation, the filtering problem consists of computing

$$\pi_t(\varphi) = E[\varphi(X_t, t)|\mathcal{Z}_t], \quad \forall t \in [0, T]$$

where φ is a bounded Borel function on $\mathbb{R}^n \times [0, T]$ and $\mathcal{Z}_t = \sigma\{Z_s, 0 \leq s \leq t\}^* \subset \mathcal{F}_t$. The process $X = \{X_t\}_{0 \leq t \leq T}$ is the signal, given by (4.5). The process $Z = \{Z_t\}_{0 \leq t \leq T}$ is the observation, given by (4.6).
After an affine transformation of the observation equation ($Z \to Y$) and a change of the probability measure ($P \to \tilde{P}$), the process Y becomes a martingale independent of the process X whose law remains unchanged under the new probability measure. In addition,

- The Kallianpur-Striebel formula: $\pi_t(\varphi) = p_t(1)^{-1} p_t(\varphi)$
- $p_t(\varphi) = \tilde{E}[\varphi(X_t, t)\eta_t|\mathcal{Z}_t]$
- $\eta_t = \exp\left(\int_0^t H^T(X_\cdot, s)R_s^{-1}dY_s - \frac{1}{2}\int_0^t H^T(X_\cdot, s)R_s^{-1}H(X_\cdot, s)ds\right)$

If we suppose that the functions $\sigma, g \in C^{3,1}(\mathbb{R}^n \times [0, T])$ and $h \in C^{1,1}(\mathbb{R}^n \times [0, T])$, and that the initial state $X_0 = x \in \mathbb{R}^n$, then $p_\cdot(\cdot)$ is the unique solution of the Zakai equation. That is, if

- $a(x, t) = \frac{1}{2}\sigma(x, t)\sigma^T(x, t)$
- $A = -\sum_i g_i \frac{\partial}{\partial x_i} - \sum_{i,j} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$

Then, for all $t \in [0, T]$ and all $\varphi \in C_b^{2,1}(\mathbb{R}^n \times [0, T])$

$$p_t(\varphi) = \varphi(x, 0) + \int_0^t p_s(\frac{\partial \varphi}{\partial s} - A\varphi)ds + \int_0^t p_s(\varphi H^T)R_s^{-1}dY_s \quad \text{a.s.} \tag{4.61}$$

Definition 4.4.1. A set of weighted particles is a system of the form

$$\mathcal{P}_N = \{(X_i, w_i)\,,\ 1 \leq i \leq N\}\,,$$

where $N \geq 1$, $X_i \in \mathbb{R}^n$ and $w_i \geq 0$.
The set of particles $\mathcal{P}_N$ approximates the finite measure $\lambda \in \mathcal{M}_F(\mathbb{R}^n)$ if the weighted empirical measure

$$\lambda^N(dx) = \sum_{i=1}^N w_i \delta_{\{X_i\}}(dx)$$

converges to λ as $N \longrightarrow \infty$. The system $\mathcal{P}_N$ is unweighted if $w_i = 1/N$ for all i.

Because of the linearity of the Zakai equation, much interest was given to characterize the solution of the filtering problem as the solution of the Zakai equation via the Kallianpur-Striebel formula, [7, 9, 54, 56, 57, 65, 69]. Crisan [16, 18] and Crisan et al. [21] approximate the solution of the Zakai equation using *branching*

particle methods. This method offers an algorithm that begins with N particles then produces N generations. Each particle has a weight equal to $1/N$, lives a time length equal to $1/N$ and gives a number of offsprings depending on its trajectory. The number of particles may change from one generation to another. In [18], Crisan uses a particle approximation method. He attaches a time dependant weight to each of the N particles. The particles moves independently w.r.t. the law of the state process. He gives also a comparison with the branching particle methods. Other methods based on the Galerkin approximation or based on the Cameron-Martin version of the chaos decomposition can be found in [1, 38, 66].
We approximate the solution to the Zakai equation using a particle approximation method. We avoid the use of a branching particle method because of the presence of the integral $H(X_\cdot, s)$ in the expression of the martingale μ_t. It is not evident to express the process μ_t as a product of terms that are dependent on pairwise disjoint portions of the trajectory of the process X.
We will construct a sequence of weighted empirical measures U_N of the form

$$U_N(t) = \frac{1}{N}\sum_{i=1}^{N} \mu_t^i \delta_{\{V_t^i\}}$$

weakly convergent to the solution of the Zakai equation and we give the rate of convergence (Theorem 4.4.6). This will enable us to numerically solve the filtering problem via the Kallianpur-Strieble formula.
We first make two assumptions. The first assumption will allow us to consider many independent processes, all of law of the process X. We will need the second assumption to prove the main result of the next Section, (Theorem 4.4.6).

Assumption 4.5. We suppose that on the probability space $(\Omega, \tilde{P})$ we can consider many infinite independent $\mathbb{R}^n$-valued Brownian motions all independent of the process Y.

Assumption 4.6.

$$\Delta_H^T = \tilde{E}\big[\exp\big(2\int_0^T H^T(X_\cdot^x, t)R_t^{-1}H(X_\cdot^x, t)dt\big)\big] < \infty\,. \tag{4.62}$$

In particular, if the function h is bounded then Assumption 4.6 holds.

Remark 4.4.2. From now we work under the probability measure $\tilde{P}$.

4.4.1 Particle approximation

Let us consider, for any integer $N \geq 2$, $V_t^1, V_t^2, \ldots, V_t^N$ be N independent realizations of the signal X^x that are in addition independent of the process Y. Those random vectors are called particles. For each particle V_t^i we attach a weight μ_t^i given by

$$\mu_t^i = \exp\big(\int_0^t H(V_\cdot^i, s)^T R_s^{-1} dY_s - \frac{1}{2}\int_0^t H(V_\cdot^i, s)^T R_s^{-1} H(V_\cdot^i, s)ds\big)\,. \tag{4.63}$$

Let us consider the associated weighted empirical measure

$$U_N(t) = \frac{1}{N}\sum_{i=1}^{N} \mu_t^i \delta_{\{V_t^i\}}\,.$$

We fix an integer number $N \geq 2$, then there exists N independent Brownian motions $\{W_t^i\,;\ i = 1, \ldots, N\}$ on $(\Omega, \tilde{P})$ such that

- For all $1 \leq i \leq N$ and $t \in [0, T]$,

$$\begin{aligned} V_t^i &= x + \int_0^t g(V_s^i, s)ds + \int_0^t \sigma(V_s^i, s)dW_s^i \\ \mu_t^i &= 1 + \int_0^t \mu_s^i H(V_\cdot^i, s)^T R_s^{-1} dY_s \end{aligned} \tag{4.64}$$

- For all $\varphi \in \mathcal{C}^{2,1}(\mathbb{R}^n \times [0, T]; \mathbb{R})$,

$$\begin{aligned} d(\mu_t^i \varphi(V_t^i, t)) &= \mu_t^i(\frac{\partial}{\partial t} - A)\varphi(V_t^i, t)dt + \mu_t^i \varphi(V_t^i, t)H(V_\cdot^i, t)^T R_t^{-1} dY_t \\ &\quad + \mu_t^i D\varphi(V_t^i, t)^T \sigma(V_t^i, t)dW_t^i \end{aligned} \tag{4.65}$$

Remark 4.4.3. The law of each particle coincide with the law of the process X^x under $\tilde{P}$, since the law of the process X^x remains unchanged after the change of probability: $P \longrightarrow \tilde{P}$.

Lemma 4.4.4. *For all* $1 \leq i \leq N$, $t \in [0, T]$ *and* $\varphi \in \mathcal{B}(\mathbb{R}^n \times [0, T]; \mathbb{R})$

$$\begin{aligned} p_t(\varphi) &= \tilde{E}[\mu_t^i \varphi(V_t^i, t)\big|\mathcal{Z}_t]\,, \\ \tilde{E}\big[\big(\mu_t^i \varphi(V_t^i, t)\big)^2\big] &= \tilde{E}\big[\varphi(V_t^i, t)^2 \exp(\int_0^t H^T R_s^{-1} H(V_\cdot^i, s)ds)\big]\,. \end{aligned}$$

Proof. **The first equality:** The equality follows from the fact that each V^i has the law of the process X.
The second equality: We have

$$(\mu_t^i)^2 = \nu_t^i \times \exp(\int_0^t H^T R_s^{-1} H^T(V_\cdot^i, s)ds)\,,$$

where $\nu_t^i = \exp(\int_0^t 2H(V_\cdot^i, s)^T R_s^{-1} dY_s - \frac{1}{2}\int_0^t 4H^T R_s^{-1} H(V_\cdot^i, s)ds)$.
From Assumption 4.6 we deduce that the Novikov condition is satisfied, see Theorem 3.5.3 in [35], then the process ν_t^i is an $\mathcal{Y}_t = \sigma\{Y_s\,;\ 0 \leq s \leq t\}$-martingale. Moreover, $\tilde{E}[\nu_t^i\big|\mathcal{Y}_0] = 1$. In addition, the processes Y and V^i are independent, then

$$\begin{aligned} \tilde{E}\big[\big(\mu_t^i \varphi(V_t^i, t)\big)^2\big] &= \tilde{E}\big[\tilde{E}[\big(\mu_t^i \varphi(V_t^i, t)\big)^2\big|\mathcal{Y}_0]\big] \\ &= \tilde{E}\big[\varphi(V_t^i, t)^2 \exp(\int_0^t H^T R_s^{-1} H(V_\cdot^i, s)ds)\tilde{E}[\nu_t^i\big|\mathcal{Y}_0]\big] \\ &= \tilde{E}\big[\varphi(V_t^i, t)^2 \exp(\int_0^t H^T R_s^{-1} H(V_\cdot^i, s)ds)\big] \end{aligned}$$

□

Remark 4.4.5. If the function $\varphi \equiv 1$, then

$$\tilde{E}\big[(\mu_t^i)^2\big] = \tilde{E}\big[\exp(\int_0^t H^T R_s^{-1} H(V_\cdot^i, s)ds)\big] \leq \sqrt{\Delta_H^T}\,.$$

Denoting by $\mathcal{M}_F(\mathbb{R}^n)$ the space of finite measures on $\mathbb{R}^n$ endowed with the weak convergence topology. That is,

$$\{\mu_N \to \mu \text{ in } \mathcal{M}_F(\mathbb{R}^n)\} \Longleftrightarrow \{(\mu_N, \varphi) \to (\mu, \varphi)\,, \quad \forall\ \varphi \in C_b(\mathbb{R}^n \times [0,T])\}\,.$$

The weak topology on this space is metrizable. The distance $d(\cdot,\cdot)$ given by

$$d(\nu,\mu) = \sum_{k=1}^{\infty} 2^{-k} \frac{|(\nu,\varphi_k) - (\mu,\varphi_k)|}{\|\varphi_k\|}\,,$$

where the sequence $\{\varphi_k\}_{k\geq 1} \subset \mathcal{C}_b(\mathbb{R}^n \times [0,T])$ is convergent determining, generates the weak topology. Then,

$$\mu_N \to \mu \Longleftrightarrow d(\mu_N, \mu) \to 0\,.$$

A sequence $\{\mu^{N,w}\}$ of random measures converges to the random measure μ^ω a.s. if for all $\varphi \in \mathcal{C}_b(\mathbb{R}^n \times [0,T])$ and for almost every $w \in \Omega$ we have

$$d(\mu^{N,\cdot}, \mu^\cdot) \to 0 \ \text{ as } N \to \infty \ \text{ a.s.}$$

A sequence $\{\mu^{N,w}\}$ of random measures on $(\Omega, \tilde{P})$ is weakly convergent to μ^ω in $\mathcal{M}_F(\mathbb{R}^n \times [0,T])$ if

$$\tilde{E}\big[d(\mu^{N,\cdot}, \mu^\cdot)\big] \to 0 \ \text{ as } N \to \infty\,.$$

In the next Theorem we prove that the sequence U_N converges weakly to the measure p_t with a rate of convergence proportional to $N^{-1/2}$.

Theorem 4.4.6. *1. For all $t \in [0,T]$ and $\varphi \in \mathcal{B}(\mathbb{R}^n \times [0,T]; \mathbb{R})$ we have*

$$\tilde{E}\big[\big((U_N(t),\varphi) - p_t(\varphi)\big)^2\big] \leq \frac{\sqrt{\Delta_H^T}}{N}\,\|\varphi\|^2\,.$$

2. For all $t \in [0,T]$

$$\tilde{E}\big[d(U_N(t), p_t)\big] \leq \frac{2(\Delta_H^T)^{1/4}}{\sqrt{N}}\ .$$

Proof. 1. The particles are independent; the equalities of Lemma 4.4.4) and Assumption 4.6 imply that for all $t \in [0,T]$ and all $\varphi \in \mathcal{B}(\mathbb{R}^n \times [0,T]; \mathbb{R})$ we have

$$\begin{aligned}
\tilde{E}\big[\big((U_N(t),\varphi) - p_t(\varphi)\big)^2\big] &= \tilde{E}\big[\big(\frac{1}{N}\sum_{i=1}^N \mu_t^i \varphi(V_t^i, t) - p_t(\varphi)\big)^2\big] \\
&= \frac{1}{N^2}\sum_{i=1}^N \tilde{E}\big[\big(\mu_t^i \varphi(V_t^i,t) - p_t(\varphi)\big)^2\big] \\
&\leq \frac{1}{N^2}\sum_{i=1}^N \tilde{E}\big[\big(\mu_t^i \varphi(V_t^i,t)\big)^2\big] \\
&\leq \frac{\|\varphi\|^2\sqrt{\Delta_H^T}}{N}\ .
\end{aligned}$$

2. For all $k \geq 0$,

$$\begin{aligned}\tilde{E}[\frac{|(U_N(t),\varphi_k)-(p_t,\varphi_k)|}{2^k||\varphi_k||}] &\leq \frac{\tilde{E}[\left|(U_N(t),\varphi_k)-(p_t,\varphi_k)\right|^2]^{1/2}}{2^k||\varphi_k||} \\ &\leq \frac{(\Delta_H^T)^{1/4}}{N^{1/2}2^k}\end{aligned}$$

Since $\sum_k 2^{-k} < \infty$, then Fubini's Theorem implies

$$\begin{aligned}\tilde{E}[d(U_N(t),p_t)] &= \tilde{E}[\sum_{k=0}^{\infty}\frac{|(U_N(t),\varphi_k)-(p_t,\varphi_k)|}{2^k||\varphi_k||}] \\ &= \sum_{k=0}^{\infty}\tilde{E}[\frac{|(U_N(t),\varphi_k)-(p_t,\varphi_k)|}{2^k||\varphi_k||}] \\ &\leq \sum_{k=0}^{\infty}\frac{\tilde{E}[|(U_N(t),\varphi_k)-(p_t,\varphi_k)|^2]^{1/2}}{2^k||\varphi_k||} \\ &\leq \frac{2(\Delta_H^T)^{1/4}}{\sqrt{N}}\,.\end{aligned}$$

This proves the Theorem. □

4.4.2 Implementation

The numerical implementation of the empirical measure $U_N(t)$ needs the simulation of the particles $(V_t^i)_i$ and the evaluation (approximation) of the weights $(\mu_t^i)_i$.
We approximate the true trajectory V_t of each particle on the interval $[0,T]$ with a trajectory $\tilde{V}_t$ based on a discretization of the SDE (4.64). Depending on the smoothness of the functions g and σ, we can use one of the two strong scheme time discretizations: the Euler-Maruyama scheme or the Milstein scheme. The Euler-Maruayma scheme is implementable in our setting if, in addition to the linear growth bounds on g and σ see (4.2), we replace the local Lipschitz condition on g and σ w.r.t. x, see (4.3), by a global Lipschitz condition w.r.t. x and t, see Appendix B.4.

Assumption 4.7. For all $x, y \in \mathbb{R}^n$ and $s, t \in [0,T]$,

$$\max\{|g(x,s)-g(y,t)|, \|\sigma(x,s)-\sigma(y,t)\|\} \leq K(|x-y|+|s-t|)\,. \tag{4.66}$$

Let V_t be the trajectory of a generic particle. Then, there exists an $\mathbb{R}^n$-Brownian motion $\bar{W}$ such that for all $t \in [0,T]$,

$$V_t = x + \int_0^t g(V_s,s)ds + \int_0^t \sigma(V_s,s)d\bar{W}_s\,.$$

Let M be a positive integer sufficiently large such that

$$\delta = T/M < 1\,.$$

The equidistant time discretization trajectory $\tilde{V}_t$ of the generic trajectory V_t is:

- $\tilde{V}_0 = x$
- For all $k \in \{0, \dots, M-1\}$ and $t \in [\tau_k, \tau_{k+1}]$, where $\tau_k = k\delta$, we have

$$\tilde{V}_t = \tilde{V}_k + (t - \tau_k) g(\tilde{V}_k, \tau_k) + \sigma(\tilde{V}_k, \tau_k)(\bar{W}_t - \bar{W}_{\tau_k})$$

Let us denote $\Delta_k \bar{W} = \bar{W}_{\tau_{k+1}} - \bar{W}_{\tau_k}$, then

- $\tilde{V}_0 = x$
- For all $k \in \{0, \dots, M-1\}$,

$$\tilde{V}_{k+1} = \tilde{V}_k + \delta g(\tilde{V}_k, \tau_k) + \sigma(\tilde{V}_k, \tau_k)\Delta_k \bar{W}$$

The process $\tilde{V}_t$ converges strongly to the process V_t with order $1/2$ and there exists a constant C independent of M, see [73], such that

$$\tilde{E}[\sup_{0 \le t \le T} |V_t - \tilde{V}_t|^2] \le C\delta\,. \tag{4.67}$$

Moreover, even if it means to take a greater C in (4.67) we have

$$\tilde{E}[\sup_{0 \le t \le T} |\tilde{V}_t|^2] \le C_1 = Ce^C(1 + |x|^2)\,. \tag{4.68}$$

To approximate the weights μ_t^i we approximate the integrals in (4.63), we define the following process

- $\tilde{I}_0 = 0$
- For all $k \in \{1, \dots, M\}$,

$$\tilde{I}_k = \sum_{j=0}^{k-1} H_j(\tilde{V}_\cdot, \tau_j)^T R_{\tau_j}^{-1} \Delta_j Y - \frac{\delta}{2} \sum_{j=0}^{k-1} H_j^T R_{\tau_j}^{-1} H_j(\tilde{V}_\cdot, \tau_j) \tag{4.69}$$

- For all $k \in \{0, \dots, M-1\}$ and all $t \in [\tau_k, \tau_{k+1})$

$$\tilde{I}_t = \tilde{I}_k$$

  where

  - $\Delta_j Y = Y_{\tau_{j+1}} - Y_{\tau_j}$
  - For all $k \in \{0, \dots, M\}$,

$$H_k(\tilde{V}_\cdot, \tau_k) = A_{\tau_k} h(\tilde{V}_k, \tau_k) + \delta A'_{\tau_k}\big(\sum_{j=0}^{k-1} h(\tilde{V}_j, \tau_j)\big)\,. \tag{4.70}$$

Then, $e^{\tilde{I}_k}$ approximate the generic weight μ_t.
The total number of time increments M is the same for both approximations. Both H_k and I_k can be evaluated on-line since

- $H_k(\tilde{V}_\cdot, \tau_k) = A_{\tau_k} h(\tilde{V}_k, \tau_k) + \delta A'_{\tau_k} S_k$ where $S_0 = 0$ and $S_{k+1} = S_k + h(\tilde{V}_k, \tau_k)$
- $\tilde{I}_{k+1} = \tilde{I}_k + H_k(\tilde{V}_\cdot, \tau_k)^T R^{-1}_{\tau_k} \Delta_k Y - \frac{\delta}{2} H_k^T R^{-1}_{\tau_k} H_j(\tilde{V}_\cdot, \tau_k)$

Still the question of the validity of the approximation $\tilde{I}_t$ in (4.69) to the integral

$$I_t = \int_0^t H(V^i_\cdot, s)^T R_s^{-1} dY_s - \frac{1}{2} \int_0^t H(V^i_\cdot, s)^T R_s^{-1} H(V^i_\cdot, s) ds \,.$$

We need to make two additional assumptions, one on the function h and one on the matrix valued application $s \to R_s$.

Assumption 4.8. For all $x, y \in \mathbb{R}^n$ and $s, t \in [0, T]$

$$|h(x, s) - h(y, t)| \leq K(|x - y| + |s - t|) \,.$$

Assumption 4.9. For every $T > 0$, there exists a constant $K_T > 0$ such that for all $s, t \in [0, T]$,

$$\begin{aligned} \sup_{t \in [0,T]} \{\|R_t\| \,,\ \|R_t^{-1}\|\} &\leq K_T \\ \max\{\|R_s - R_t\| \,,\ \|R_s^{-1} - R_t^{-1}\|\} &\leq K_T |s - t|^{1/2} \end{aligned}$$

Since the process I_t involves the function H then the following Lemma is of utility

Lemma 4.4.7. *There exists a positive constant $\bar{K}$ such that for all $\mathbf{x}, \mathbf{y} \in \mathcal{C}([0, T]; \mathbb{R}^n)$ and $s, t \in [0, T]$ we have*

$$|H(\mathbf{x}_\cdot, s) - H(\mathbf{y}_\cdot, t)| \leq \bar{K}\{\|\mathbf{x} - \mathbf{y}\| + (1 + \|\mathbf{x}\| + \|\mathbf{y}\|)|s - t|\} \,. \tag{4.71}$$

Proof. The inequalities (4.4) and (4.15) together with the Assumption 4.8 give easily the result. □

Proposition 4.4.8. *There exists a constant $\tilde{C} > 0$ independent of M such that*

$$\tilde{E}[\ \left|I_t - \tilde{I}_t\right|\ ] \leq \tilde{C}\delta^{1/2} \quad \textit{for all } t \in [0, T] \,.$$

Proof. We fix M sufficiently large and we define the two processes N and $\tilde{N}$ such that for all $k \in \{0, \ldots, M - 1\}$ and $t \in [\tau_k, \tau_{k-1})$

$$\begin{aligned} N_t &= \int_0^t H(V_\cdot, s)^T R_s^{-1} dY_s \\ \tilde{N}_t &= \sum_{j=0}^{k-1} H_j(\tilde{V}_\cdot, \tau_j)^T R^{-1}_{\tau_j} \Delta_j Y \end{aligned}$$

The process N_t is clearly a square integrable martingale. The process $\tilde{N}_t$ is also a square integrable martingale since the increments of the martingale Y are independent and the function h and the process V satisfies successively the inequalities (4.4) and (4.9). In particular, their quadratic variations are

$$\begin{aligned}\langle N\rangle_t &= \int_0^t H(V_.,s)^T R_s^{-1} H(V_.^i,s)ds\,,\\ \langle \tilde{N}\rangle_t &= \sum_{j=0}^{k-1} H_j(\tilde{V}_.,\tau_j)^T R_{\tau_j}^{-1}\Big(\int_{\tau_j}^{\tau_{j+1}} R_s ds\Big) R_{\tau_j}^{-1} H_j(\tilde{V}_.,\tau_j)\,.\end{aligned}$$

Then, we can write the process $I_t - \tilde{I}_t$ as a sum of three terms:

$$I_t - \tilde{I}_t = T_1 + T_2 + T_3\,, \tag{4.72}$$

where

- $T_1 = N_t - \tilde{N}_t$
- $T_2 = \frac{1}{2}\big(\langle N\rangle_t - \langle \tilde{N}\rangle_t\big)$
- $T_3 = \frac{1}{2}\Big(\langle \tilde{N}\rangle_t - \delta \sum_{j=0}^{k-1} H_j^T R_{\tau_j}^{-1} H_j(\tilde{V}_.,\tau_j)\Big)$

We prove in three steps that

$$\tilde{E}[\big|T_i\big|] \le O(\delta^{1/2})\,, \quad i=1,2,3\,.$$

Step 1 We prove that for some constant $\Delta > 0$

$$\tilde{E}[\big|N_t - \tilde{N}_t\big|^2] \le \Delta\,\delta\,, \quad \text{for all } t\in[0,T]\,. \tag{4.73}$$

One has

$$\begin{aligned}N_t - \tilde{N}_t &= \int_0^t \{H(V_.,s) - H(\tilde{V}_.,s)\}^T R_s^{-1} dY_s\\ &\quad + \Big(\int_0^t H(\tilde{V}_.,s)^T R_s^{-1} dY_s - \sum_{j=0}^{k-1} H(\tilde{V}_.,\tau_j)^T R_{\tau_j}^{-1}\Delta_j Y\Big)\\ &\quad + \sum_{j=0}^{k-1}\{H(\tilde{V}_.,\tau_j) - H_j(\tilde{V}_.,\tau_j)\}^T R_{\tau_j}^{-1}\Delta_j Y\end{aligned} \tag{4.74}$$

Denoting successively by S_1, S_2 and S_3 the three terms in the right-hand side in equality (4.74). Then, $\tilde{E}[|N_t - \tilde{N}_t|^2] \le 3\sum_{i=1}^3 \tilde{E}[|S_i|^2]$. The idea is to found an upper for each $\tilde{E}[|S_i|^2]$.

First term

$$\begin{aligned}\tilde{E}[|S_1|^2] &= \tilde{E}[\int_0^t \{H(V_.,s) - H(\tilde{V}_.,s)\}^T R_s^{-1}\{H(V_.,s) - H(\tilde{V}_.,s)\}ds]\\ &\le K_T \tilde{E}[\int_0^t \big|H(V_.,s) - H(\tilde{V}_.,s)\big|^2 ds]\\ &\le K_T T \bar{K}^2 \tilde{E}[\sup_{0\le t\le T}|V_t - \tilde{V}_t|^2]\\ &\le K_T T \bar{K}^2 C\delta = \Delta_1\delta\,.\end{aligned} \tag{4.75}$$

Second term

$$\tilde{E}[|S_2|^2] = \tilde{E}[\Big|\sum_{j=0}^{k-1}\int_{\tau_j}^{\tau_{j+1}}\{H(\tilde{V}_.,s)^T R_s^{-1} - H(\tilde{V}_.,\tau_j)^T R_{\tau_j}^{-1}\}dY_s\Big|^2].$$

We have

$$\begin{aligned} H(\tilde{V}_.,s)^T R_s^{-1} - H(\tilde{V}_.,\tau_j)^T R_{\tau_j}^{-1} &= \{H(\tilde{V}_.,s) - H(\tilde{V}_.,\tau_j)\}^T R_s^{-1} \\ &\quad + H(\tilde{V}_.,\tau_j)^T\{R_s^{-1} - R_{\tau_j}^{-1}\}, \end{aligned}$$

Then,

$$\begin{aligned} \tilde{E}[|S_2|^2] &\le 2K_T\tilde{E}[\Big|\sum_{j=0}^{k-1}\int_{\tau_j}^{\tau_{j+1}}\big|H(\tilde{V}_.,s) - H(\tilde{V}_.,\tau_j)\big|^2 ds] \\ &\quad + 2\tilde{E}[\sum_{j=0}^{k-1}\int_{\tau_j}^{\tau_{j+1}} H(\tilde{V}_.,\tau_j)^T\{R_s^{-1} - R_{\tau_j}^{-1}\}R_s\{R_s^{-1} - R_{\tau_j}^{-1}\}H(\tilde{V}_.,\tau_j)ds\Big|^2] \\ &= B_1 + B_2 \end{aligned}$$

In one hand,

$$\begin{aligned} B_1 &\le 2K_T\bar{K}^2\tilde{E}[T(1 + 2\sup_{0\le t\le T}|\tilde{V}_t|)^2\delta^2] \\ &\le 2K_T\bar{K}^2\big(2T + 4\tilde{E}[\sup_{0\le t\le T}|\tilde{V}_t|^2]\big)\delta^2 \\ &\le \Delta_2\delta^2 . \end{aligned} \tag{4.76}$$

where, $\Delta_2 = 2K_T\bar{K}^2(2T + 4C_1)$, see the inequality (4.68).
In the other hand, from the Assumption 4.9 we get

$$\begin{aligned} B_2 &\le 2K_T\tilde{E}[\sum_{j=0}^{k-1}\int_{\tau_j}^{\tau_{j+1}}|H(\tilde{V}_.,\tau_j)|^2\|R_s^{-1} - R_{\tau_j}^{-1}\|^2 ds] \\ &\le 2(K_T')^3\tilde{E}[\sum_{j=0}^{k-1}\int_{\tau_j}^{\tau_{j+1}}|H(\tilde{V}_.,\tau_j)|^2 ds]\delta \\ &\le 4K_T^3T(1 + \tilde{E}[\sup_{0\le t\le T}|\tilde{V}_t|^2])\delta \\ &\le \Delta_3\delta \end{aligned} \tag{4.77}$$

where $\Delta_3 = 4(K_T')^3T(1 + C_1)$.
Third term

$$\begin{aligned} \tilde{E}[|S_3|^2] &= \tilde{E}[\sum_{j=0}^{k-1}\int_{\tau_j}^{\tau_{j+1}}(H - H_j)(\tilde{V}_.,\tau_j)^T R_{\tau_j}^{-1}R_s R_{\tau_j}^{-1}(H - H_j)(\tilde{V}_.,\tau_j)ds] \\ &\le (K_T')^3\tilde{E}[\sum_{j=0}^{k-1}\int_{\tau_j}^{\tau_{j+1}}\big|H(\tilde{V}_.,\tau_j) - H_j(\tilde{V}_.,\tau_j)\big|^2 ds] \\ &\le TK_T^3\max_j\tilde{E}[\big|H(\tilde{V}_.,\tau_j) - H_j(\tilde{V}_.,\tau_j)\big|^2] \end{aligned} \tag{4.78}$$

From the Assumption 4.8, we have

$$\begin{aligned}
|H(\tilde{V}_\cdot,\tau_k) - H_j(\tilde{V}_\cdot,\tau_k)| &= \big|A'_{\tau_j}\{\int_0^{\tau_k} h(\tilde{V}_s,s)ds - \delta\sum_{j=0}^{k-1} h(\tilde{V}_j,\tau_j)\}\big| \\
&= \big|A'_{\tau_j}\sum_{j=0}^{k-1}\int_{\tau_j}^{\tau_{j+1}} \big(h(\tilde{V}_s,s) - h(\tilde{V}_j,\tau_j)\big)ds\big| \\
&\leq a_T K \sum_{j=0}^{k-1}\int_{\tau_j}^{\tau_{j+1}} \{|\tilde{V}_s - \tilde{V}_j| + |s-\tau_j|\}ds \\
&\leq a_T K T\delta + a_T K T \max_j \sup_{s\in[\tau_j,\tau_{j+1}]} |\tilde{V}_s - \tilde{V}_j| \qquad (4.79)
\end{aligned}$$

We need to found an upper bound on $\tilde{E}[\sup_{s\in[\tau_j,\tau_{j+1}]}|\tilde{V}_s - \tilde{V}_j|^2]$.
From the definition of the process $\tilde{V}$, we have for all $s\in[\tau_j,\tau_{j+1}]$

$$\begin{aligned}
|\tilde{V}_s - \tilde{V}_j|^2 &\leq 2\delta^2|g(\tilde{V}_j,\tau_j)|^2 + 2\|\sigma(\tilde{V}_j,\tau_j)\|^2|W_s - W_{\tau_j}|^2 \\
&\leq 4K^2\delta^2(1+|\tilde{V}_j|^2) + 4K^2(1+|\tilde{V}_j|^2)|W_s - W_{\tau_j}|^2 .
\end{aligned}$$

Then,

$$\sup_{s\in[\tau_j,\tau_{j+1}]}|\tilde{V}_s - \tilde{V}_j|^2 \leq 4K^2\delta^2(1+|\tilde{V}_j|^2) + 4K^2(1+|\tilde{V}_j|^2)\sup_{s\in[\tau_j,\tau_{j+1}]}|W_s - W_{\tau_j}|^2 .$$

The processes $W_s - W_{\tau_j}$ and $\tilde{V}_j$ are independent, then the Doob inequality and the inequality (4.68) imply that

$$\begin{aligned}
\tilde{E}[\sup_{s\in[\tau_j,\tau_{j+1}]}|\tilde{V}_s - \tilde{V}_j|^2] &\leq 4K^2\tilde{E}[1+|\tilde{V}_j|^2]\tilde{E}[\sup_{s\in[\tau_j,\tau_{j+1}]}|W_s - W_{\tau_j}|^2] \\
&\quad +4K^2\delta^2\tilde{E}[1+|\tilde{V}_j|^2] \\
&\leq 16K^2\tilde{E}[1+|\tilde{V}_j|^2]\tilde{E}[|W_{\tau_{j+1}} - W_{\tau_j}|^2] \\
&\quad +4K^2\delta^2\tilde{E}[1+|\tilde{V}_j|^2] \\
&= 4K^2\delta^2(1+C_1) + 16K^2(1+C_1)\delta \\
&\leq 20K^2(1+C_1)\delta .
\end{aligned}$$

If we denote $\Delta_4 = a_T^2T^3K_T^3K^2(2+40K^2(1+C_1))$, then

$$\tilde{E}[|S_3|^2] \leq \Delta_4\delta , \qquad (4.80)$$

Conclusion: If $\Delta = 3\sum_{1\leq i\leq 4}\Delta_i$, then

$$\tilde{E}[\,\big|N_t - \tilde{N}_t\big|^2\,] \leq \Delta\,\delta .$$

Step 2 We show that

$$\tilde{E}[\,\big|\langle N\rangle_t - \langle\tilde{N}\rangle_t\big|\,] = O(\delta^{1/2}) , \quad \text{for all } t\in[0,T] .$$

Since $\langle N,\tilde{N}\rangle_t \le \langle N\rangle_t^{1/2}\langle\tilde{N}\rangle_t^{1/2}$, see Theorem 2.2.13 in [55], then $(\langle N\rangle_t^{1/2}-\langle\tilde{N}\rangle_t^{1/2})^2 \le \langle N-\tilde{N}\rangle_t$. Using Schwartz's inequality we get

$$\begin{aligned}\tilde{E}[\,|\langle N\rangle_t-\langle\tilde{N}\rangle_t|\,] &= \tilde{E}[\,|\langle N\rangle_t^{1/2}-\langle\tilde{N}\rangle_t^{1/2}||\langle N\rangle_t^{1/2}+\langle\tilde{N}\rangle_t^{1/2}|\,]\\ &\le \tilde{E}[\,|\langle N\rangle_t^{1/2}-\langle\tilde{N}\rangle_t^{1/2}|^2]^{1/2}\tilde{E}[\,|\langle N\rangle_t^{1/2}+\langle\tilde{N}\rangle_t^{1/2}|^2\,]^{1/2}\\ &\le \sqrt{2}\tilde{E}[\,\langle N-\tilde{N}\rangle_t]^{1/2}\tilde{E}[\,\langle N\rangle_t+\langle\tilde{N}\rangle_t]^{1/2}\\ &= \sqrt{2}\tilde{E}[\,|N_t-\tilde{N}_t|^2]^{1/2}\tilde{E}[\,|N_t|^2+|\tilde{N}_t|^2]^{1/2}\,.\end{aligned}$$

We denote by $\Gamma_H^T=\tilde{E}[\langle N\rangle_T|^2]=\tilde{E}[|N_T|^2]<\infty$, then

- $\tilde{E}[|N_t|^2]\le\Gamma_H^T$ for all $t\in[0,T]$
- $\tilde{E}[|\tilde{N}_t|^2]\le 2\Gamma_H^T+2\tilde{E}[|N_t-\tilde{N}_t|^2]$ for all $t\in[0,T]$

We deduce that

$$\begin{aligned}\tilde{E}[\,|\langle N\rangle_t-\langle\tilde{N}\rangle_t|\,] &\le \sqrt{2}\tilde{E}[\,|N_t-\tilde{N}_t|^2]^{1/2}(3\Gamma_H^T+2\tilde{E}[|N_t-\tilde{N}_t|^2])^{1/2}\\ &\le \{\sqrt{2}\sqrt{\Delta}(3\Gamma_H^T+2\Delta)^{1/2}\}\delta^{1/2}\,.\end{aligned}$$

Step 3 Finally, we show that

$$\tilde{E}[\,|\langle\tilde{N}\rangle_t-\delta\sum_{j=0}^{k-1}H_j(\tilde{V}_\cdot,\tau_j)^TR_{\tau_j}^{-1}H_j(\tilde{V}_\cdot,\tau_j)|\,]=O(\delta^{1/2})\,.$$

Let us denote $\lambda_t=\delta\sum_{j=0}^{k-1}H_j(\tilde{V}_\cdot,\tau_j)^TR_{\tau_j}^{-1}H_j(\tilde{V}_\cdot,\tau_j)$, then

$$\langle\tilde{N}\rangle_t-\lambda_t=\sum_{j=0}^{k-1}H_j(\tilde{V}_\cdot,\tau_j)^TR_{\tau_j}^{-1}\Big(\int_{\tau_j}^{\tau_{j+1}}(R_s-R_{\tau_j})ds\Big)R_{\tau_j}^{-1}H_j(\tilde{V}_\cdot,\tau_j)\,.$$

From Assumption 4.9 we get

$$\tilde{E}[\,|\langle\tilde{N}\rangle_t-\lambda_t|\,]\le\sum_{j=0}^{k-1}K_T^3\delta^{3/2}\tilde{E}[\,|H_j(\tilde{V}_\cdot,\tau_j)|^2]$$

In addition, from the definition of the process H_j, see (4.70), we get

$$\begin{aligned}|H_j(\tilde{V}_\cdot,\tau_j)| &\le a_TK(1+|\tilde{V}_j|)+a_TK\delta\sum_{i=0}^{j-1}(1+|\tilde{V}_j|)\\ &\le a_TK(1+\sup_t|\tilde{V}_t|)+a_TK\delta\sum_{i=0}^{N-1}(1+\sup_t|\tilde{V}_t|)\\ &\le a_TK(1+T)(1+\sup_t|\tilde{V}_t|)\end{aligned}$$

This implies that there exist a constant Γ independent of M such that

$$\tilde{E}[\,|H_j(\tilde{V}_\cdot,\tau_j)|^2]\le\Gamma\,.$$

Then, we deduce that

$$\tilde{E}[\,|\langle\tilde{N}\rangle_t-\lambda_t|\,]\le K_T^3T\Gamma\delta\,.$$

The proof is complete. □

4.4.3 Numerical example

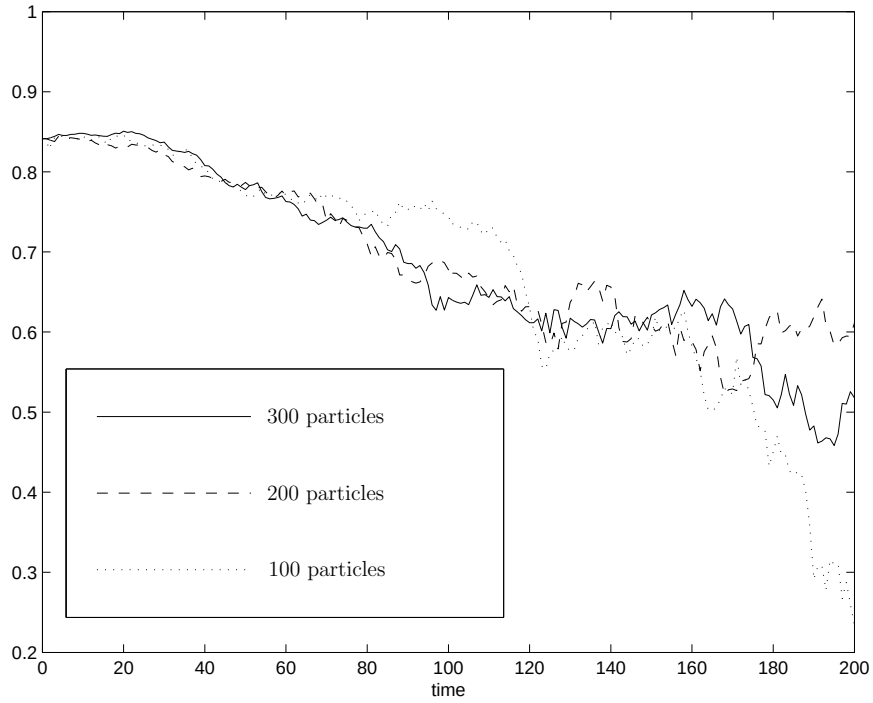

Figure 4.1: Conditional mean approximations

The signal process X:

- $X_0 = 1$
- $dX_t = g(X_t, t)dt + \sigma(X_t, t)dW_t$ for all $t \geq 0$

where

i. $W = \{W_t\}_t$ is a 1-dimensional standard Brownian motion

ii. $g(x, t) = (1 + \cos(t))x$

iii. $\sigma(x, t) = \sqrt{2}x$

The functions g and σ are $\mathcal{C}^\infty$-class, global Lipschitz w.r.t. (x, t) and admits a global linear growth bounds w.r.t x. Moreover, for all $\lambda > 0$ we have

$$A_t^\lambda = E[(X_t)^\lambda] = \exp\left(\lambda^2 t + \lambda \sin(t)\right).$$

For all $t \geq 0$, let us denote $B_t^\lambda = \exp\left(\lambda^2 t\right)$ and $\beta_t^\lambda = \exp\left(-\lambda^2 t\right)$, then

$$R_t^\lambda = (A_t^\lambda \beta_t^\lambda)^2 = \exp(2\lambda \sin(t)).$$

The observation process: $dZ_t = e^{-t/2}\cos(\frac{\pi}{2} - X_t)dt + dN_t^\lambda$, $t \geq 0$
where the process $N^\lambda = \{N_t^\lambda\}_t$ is given by

- $N_0^\lambda = 0$
- For all $t \geq 0$

$$\begin{aligned} dN_t^\lambda &= \{-(A_t^\lambda)^{-1}(A_t^\lambda)' N_t^\lambda - (A_t^\lambda)^{-1}(B_t^\lambda)'\}dt + \beta_t^\lambda dG_t \\ &= \{(\lambda^2 + \lambda\cos(t))N_t^\lambda + \lambda^2 e^{-\lambda\sin(t)}\}dt + e^{-\lambda^2 t}dG_t \end{aligned}$$

where $G = \{G_t\}_t$ is a 1-dimensional Brownian motion independent of W

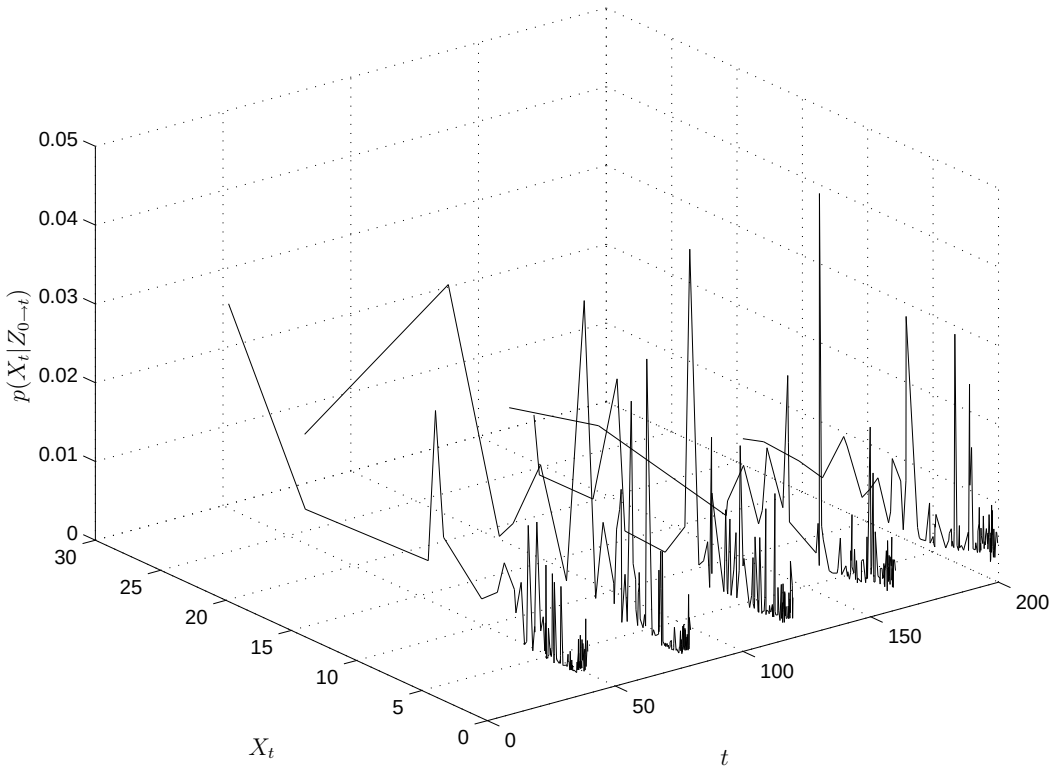

Figure 4.2: Conditional densities time evolution

We take $\varphi(x) = sin(x)$. In figure 4.1, we plot the normalized expected mean of $\varphi(X_t)$ using the particle approximation method and the Kallianpur-Striebel formula. We use 100, 200 and 300 particles successively. In figure 4.2, we plot the normalized conditional densities of the signal at various times, using 100 particles.

Appendices

Appendix A

A.1 Pseudo-inverse of a matrix

Let A be a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^m$; $n, m \geq 1$. Each of $\mathbb{R}^n$ and $\mathbb{R}^m$ is equipped by its canonical inner product. The mapping W obtained by restricting A to $(ker(A))^\perp$ with range $\mathcal{R}(A)$ is one to one and onto, hence

$$W^{-1} : \mathcal{R}(A) \to (ker(A))^\perp \text{ is linear}\,.$$

Denoting by A' the orthogonal projection in $\mathbb{R}^m$ onto $\mathcal{R}(A)$.

Definition A.1.1. The pseudo-inverse A^+ of A is defined by

$$A^+ : \mathbb{R}^m \to \mathbb{R}^n\,; \quad A^+ = W^{-1} \circ A'$$

Properties A.1.2. If A^T is the transpose matrix of A then

1. $A \circ A^+ = A'\,,\ A^+ \circ A' = A^+$
2. $A \circ A^+ \circ A = A\,,\ A^+ \circ A \circ A^+ = A^+\,,\ (A^+)^T = (A^T)^+$
3. If A^{-1} exists, then $A^+ = A^{-1}$
4. $A^+ = A^T \circ (A \circ A^T)^+ = (A^T \circ A)^+ \circ A^T$
5. If $b \in \mathbb{R}^m$ and $x_0 = A^+ b \in \mathbb{R}^n$ then

$$||Ax_0 - b|| \leq ||Ax - b||\,,\ \forall x \in \mathbb{R}^n$$

Moreover, if $b \in \mathcal{R}(A)$ then $Ax_0 = b$

Lemma A.1.3. *Let D be the following diagonal matrix*

$$D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. Then, if $\lambda_p > 0$ and $\lambda_{p+1} = 0$, we have

$$D' = diag(\underbrace{1, \ldots, 1}_{p}, 0, \ldots, 0) \quad \textit{and} \quad D^+ = diag(\lambda_1^{-1}, \ldots, \lambda_p^{-1}, 0 \ldots, 0)\,.$$

Theorem A.1.4. *[14] Let A be an $n \times n$ symmetric matrix and let $D = U^T AU$ be its spectral decomposition, in particular U is unitary. Then,*

$$A^+ = UD^+U^T \,.$$

Remark A.1.5. Using together Properties A.1.2, Theorem A.1.4 and Lemma A.1.3 we get the pseudo-inverse of any real $n \times n$ matrix.

One application of the pseudo-inverse calculus is the following

Theorem A.1.6. *Let $\{z_1, z_2, \ldots, z_k\}$ be any family of vectors in $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, consider $\hat{x} = \sum_{i=1}^k \alpha_i z_i$ its orthogonal projection onto span$\{z_1, z_2, \ldots, z_k\}$. Then,*

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} < z_1, z_1 > & \ldots & < z_1, z_k > \\ \vdots & & \vdots \\ < z_k, z_1 > & \ldots & < z_k, z_k > \end{bmatrix}^+ \begin{bmatrix} < z_1, x > \\ \vdots \\ < z_n, x > \end{bmatrix} .$$

A.2 Fatou Lemma and Gronwall Inequality

Fatou Lemma [11] If $f_1, f_2, \ldots$ is a sequence of nonnegative measurable functions in a measure space X, then

$$\int_X \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int_X f_n \,.$$

Gronwall Inequality [15] Suppose α and β are Lebesgue integrable in $[0, T]$ for some $T \in (0, \infty)$ and there exists a constant $L > 0$ such that

$$\alpha(t) \leq \beta(t) + L \int_0^t \alpha(s)ds \,, \quad \text{for all} \quad t \in [0, T] \,.$$

Then

$$\alpha(t) \leq \beta(t) + L \int_0^t \exp\left(L(t-s)\right)\beta(s)ds \,, \quad \text{for all} \quad t \in [0, T] \,.$$

Appendix B

B.1 Conditional probability

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{A}$. The conditional expectation $E[X|\mathcal{G}]$ of an integrable random vector X given (or based on) $\mathcal{G}$ is the integrable $\mathcal{G}$-measurable random vector, which exists uniquely P-a.s., such that

$$\int_A X dP = \int_A E[X|\mathcal{G}] dP\,, \qquad \forall A \in \mathcal{G}\,.$$

Properties of the conditional expectation, [76]

a) If $\alpha_1, \alpha_2 \in \mathbb{R}$ and X_1, X_2 are $\mathcal{A}$-measurable, then $E[\alpha_1 X_1 + \alpha_2 X_2|\mathcal{G}] = \alpha_1 E[X_1|\mathcal{G}] + \alpha_2 E[X_2|\mathcal{G}]$, P-a.s.

b) If X is a positive random variable then $E[X|\mathcal{G}] \geq 0$, P-a.s.

c) If $\mathcal{H}$ is a sub-σ-algebra of $\mathcal{G}$, then $E[E[X|\mathcal{G}]|\mathcal{H}] = E[X|\mathcal{H}]$

d) If $0 \leq X_n \nearrow X$, then $E[X_n|\mathcal{G}] \nearrow E[X|\mathcal{G}]$ P-a.s.

e) If X is $\mathcal{G}$-measurable, then $E[XY|\mathcal{G}] = XE[Y|\mathcal{G}]$ P-a.s.

f) If $\mathcal{H}$ is independent from $\sigma(\sigma(X), \mathcal{G})$, then $E[X|\sigma(\mathcal{H}, \mathcal{G})] = E[X|\mathcal{G}]$ P-a.s.

The conditional probability of $A \in \mathcal{A}$ given $\mathcal{G}$ is the random variable $P(A|\mathcal{G}) = E[I_A|\mathcal{G}]$, where I_A is the indicator function of A. In particular we have

$$P(A \cap B) = \int_B P(A|\mathcal{G}) dP\,, \quad \forall B \in \mathcal{G}\,.$$

If $Y_1, \ldots, Y_k$ are $\mathcal{A}$-measurable random variables, define the random vector Y by

$$Y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_k \end{bmatrix}.$$

The conditional expectation $E[X|Y]$ of X given Y is given by

$$E[X|Y] = E[X|\sigma(Y_1, \ldots, Y_k)]\,.$$

Definition B.1.1. The conditional distribution of X given $Y$is said to be regular if there exists a function $Q(\cdot,\cdot)$ defined on $\Omega \times \mathcal{B}(\mathbb{R}^n)$ such that

1. $\forall\, \omega \in \Omega$, $Q(\omega,\cdot)$ is a probability measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$
2. $\forall\, B \in \mathcal{B}(\mathbb{R}^n)$, $Q(\cdot, B)$ is $\sigma(Y_1,\ldots,Y_k)$ -measurable and $Q(\cdot,B) = P(X \in B|Y)(\cdot)$ P-a.s.

Remark B.1.2. All the random elements in this thesis take values in Borel spaces of the form $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, then all conditional distributions/probabilities are chosen to be regular (see [70], pp. 146-150).

The conditional expectation $E[X|Y]$ is $\sigma(Y_1,\ldots,Y_k)$-measurable, then there exists a Borel measurable function $m : \mathbb{R}^k \to \mathbb{R}^n$ such that $E[X|Y] = m(Y)$. We denote, see [14], for every $y \in \mathbb{R}^k$, $m(y)$ by $E[X|Y=y]$ the conditional expectation of X given the event $\{Y = y\}$. If P_Y is the probability distribution of Y, then by the formula of change of variable we get

$$\int_{Y\in A} X dP = \int_A m(y) P_Y(dy) = \int_A E[X|Y=y] P_Y(dy)\,, \quad \forall A \in \mathcal{B}(\mathbb{R}^k)\,.$$

Suppose that X and Y are independent, then for any Borel measurable function g and any measurable subset A of $\mathcal{A}$

$$\begin{aligned} E[g(X,Y)|Y=y] &= E[g(X,y)] \quad P_Y\text{-a.s.} \\ P(A\cap\{Y\in B\}) &= \int_B P(A|Y=y) P_Y(dy), \quad \forall B \in \mathcal{B}(\mathbb{R}^k)\,. \end{aligned} \tag{B.1}$$

The random vector X is said continuously distributed if $P_X \ll \lambda_n$, where λ_n is the Lebesgue measure on $\mathbb{R}^n$. By the Radom-Nikodym theorem, there exists a Borel measurable function p_X, called the probability density function (pdf) of X, s.t.

$$P_X(B) = P(X \in B) = \int_B p_X(x) d\lambda_n(x)\,, \quad \forall B \in \mathcal{B}(\mathbb{R}^n)\,.$$

$d\lambda_n(x)$ is shortly denoted dx. In particular, for any Borel measurable function g we have

$$\int_A g dP_X = \int_A g p_X d\lambda_n\,, \quad \forall A \in \mathcal{B}(\mathbb{R}^n)\,.$$

If p_X is continuous a.e., then at every point $(x_1,\ldots,x_n)$ of continuity we have

$$p_X(x_1,\ldots,x_n) = \frac{\partial^n F_X}{\partial x_1 \ldots \partial x_n}$$

where F_X is the distribution function of X.
If the $\mathbb{R}^p \times \mathbb{R}^q$-valued random vector $X = (X_1, X_2)$ is continuously distributed, then so are X_1 and X_2. In particular

$$p_{X_1}(x_1) = \int p_X(x_1, x_2) dx_2\,,$$

The random vectors X_1 and X_2 are independent if and only if $p_X(x_1, x_2) = p_{X_1}(x_1)p_{X_2}(x_2)$. Finally, we define the conditional density function $p_{X_1|X_2}(x_1|x_2)$ of X_1 given the event $\{X_2 = x_2\}$ for all x_1, x_2 by

$$\begin{array}{lll} \text{If } p_{X_2}(x_2) > 0\,, & \text{then} & p_{X_1|X_2}(x_1|x_2) = p_X(x_1, x_2)/p_{X_2}(x_2)\,, \\ \text{If } p_{X_2}(x_2) = 0\,, & \text{then} & p_{X_1|X_2}(x_1|x_2) = 0\,. \end{array}$$

In particular, we have the Baye's rule, that is

$$p_{X_1|X_2}(x_1|x_2) = \frac{p_{X_2|X_1}(x_2|x_1)p_{X_1}(x_1)}{p_{X_2}(x_2)}\,.$$

The conditional density function satisfies

$$E[g(X)|Y = y] = \int g(x)p_{X|Y}(x, y)dx\,, \quad \forall\ g \in \mathcal{B}(\mathbb{R}^n).$$

In particular,

$$\begin{aligned} E[X|Y = y] &= \int x p_{X|Y}(x, y)dx \\ P(X|Y) &= E[(X - E[X|Y])(X - E[X|Y])^T|Y] \end{aligned}$$

$E[X|Y = y]$ is conditional mean and $P(X|Y)$ is the conditional covariance matrix. One can see [40], pp. 36-42 for proofs.

B.2 The multivariate normal distribution

On the probability space $(\Omega, \mathcal{A}, P)$, consider the following two random vectors

$$X = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \qquad \text{and} \qquad Y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_m \end{bmatrix}\,, \quad n, m \geq 1\,.$$

If $\mu_X = E[X]$ and $\mu_Y = E[Y]$, then the covariance matrix of X and Y is given by

$$cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)^T] = (cov(X_i, Y_j))_{i,j}$$

If A and B are two constant matrices, then $cov(AX, BY) = A\, cov(X, Y)B^T$.
The matrix $cov(X, X)$ is symmetric, positive and semi-definite.
The family $\{X_1 - \mu_{X_1}, \ldots, X_n - \mu_{X_n}\}$ is linearly independent in $\mathcal{L}^2(P)$ if and only if the matrix $cov(X, X)$ is symmetric, positive and definite.

Definition B.2.1. A random variable X is normally distributed with mean μ and variance σ^2, we write $X \sim \mathcal{N}(\mu, \sigma^2)$, if X is continuously distributed w.r.t. the Lebesgue measure on $\mathbb{R}$ and its probability density function p_X is given by

$$p_X(x) = (\sqrt{2\pi}\sigma)^{-1} \exp\big(-\frac{(x-\mu)^2}{2\sigma^2}\big)\,, \quad \forall\ x \in \mathbb{R}\,.$$

We have the following properties, [14, 34]:

1. If $X \sim N(0,1)$, then $E[\exp(itX)] = \exp(-\frac{1}{2}t^2)\,,\ \forall t \in \mathbb{R}$
2. For all $k \geq 0$,
$$E[X^{2k+1}] = 0 \quad \text{and} \quad E[X^{2k}] = \frac{(2k)!}{2^k k!} \tag{B.2}$$
3. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\forall a, b \in \mathbb{R}$, $aX + b \sim \mathcal{N}(a\mu + b, (a\sigma)^2)$
4. If X_1, X_2 are independent and $X_i \sim \mathcal{N}(\mu_i, \sigma_i)\,,\ 1 \leq i, j \leq 2$, then
$$\alpha_1 X_1 + \alpha_2 X_2 \sim \mathcal{N}(\alpha_1\mu_1 + \alpha_2\mu_2, (\alpha_1\sigma_2)^2 + (\alpha_2\sigma_2)^2)$$
5. If X_1, X_2 are two independent random variables such that $S = X_1 + X_2$ is normally distributed, then X_1 and X_2 are normally distributed.

Theorem B.2.2. *[14] Let $\Gamma = cov(Y,Y)$ and $\mu = E[Y]$. Suppose that $det(\Gamma) \neq 0$ and that $Y_i \sim \mathcal{N}(\mu_i, \sigma_i)$, $1 \leq i \leq m$. Then,*

1. *The function*
$$f(z) = \frac{1}{(2\pi)^{\frac{m}{2}}\sqrt{\det\Gamma}} \exp\left[- < \Gamma^{-1}(z-\mu), (z-\mu) > /2\right]\,,\ \forall\ z \in \mathbb{R}^m \tag{B.3}$$
is a pdf for Y and Y is multivariate normal distributed: $Y \sim \mathcal{N}(\mu, \Gamma)$
2. *There exists a matrix $U \in O_m(\mathbb{R})$ such that $Z = UY$ has marginals that are pairwise independent and normally distributed*

Definition B.2.3. Let $X = \{X_\lambda\}$ be a system of random variables. We say that X is a Gaussian system if for any $n \geq 1$ and any $X_1, \ldots, X_n \in X$, we have $X_1, \ldots, X_n$ are jointly distributed and their joint distribution is normal.

Proposition B.2.4. *[34] Let $X = \{X_\lambda\}$ be a system of random variables. Then, X is a Gaussian system if and only if any finite affine combination of elements of X is normally distributed. In that case, the linear subspace and the closed linear subspace spanned by X in $\mathcal{L}^2(P)$ are Gaussian systems.*

Remark B.2.5. A random vector Y is said Gaussian, if the system formed by its components is Gaussian. In particular, for every $k \times m$-matrix A and k-dimensional vector b the vector $AY + b$ is Gaussian.

Theorem B.2.6. *[34] Let $X = \{X_\lambda\}_{\lambda\in\Lambda}$ be a finite Gaussian system. If $X' = \{X_{\lambda'}\}_{\lambda'\in\Lambda'}$ is a subsystem of X and $B' = \sigma\{X_{\lambda'},\ \lambda' \in \Lambda'\}$. Then, for all $\lambda \in \Lambda$, $E[X_\lambda|B']$ is the orthogonal projection of X_λ onto span$\{X_{\lambda'},\ \lambda' \in \Lambda'\}$ in $\mathcal{L}^2(P)$. In particular, $E[X_\lambda|B']$ is normally distributed.*

We finish this subsection by the following result.

Theorem B.2.7. *[40] If X and Y are two multivariate normal distributed random vectors such that $X \sim \mathcal{N}(m_X, \Gamma_X)$ and $Y \sim \mathcal{N}(m_Y, \Gamma_Y)$. Then, if we denote $\Gamma_{X,Y} = cov(X,Y)$, we get*

$$\begin{aligned} E[X|Y] &= m_X + \Gamma_{X,Y}\Gamma^{-1}(Y - m_Y)\,, \\ X|Y &\sim \mathcal{N}(m_X + \Gamma_{X,Y}\Gamma_Y^{-1}(Y - m_Y), \Gamma_X - \Gamma_{X,Y}\Gamma_Y^{-1}\Gamma_{X,Y}^T)\,. \end{aligned}$$

B.3 Limit and asymptotic convergence theorems

Convergence criterions Two classes of convergence criterions are commonly used. The first class contains three strong criterions of convergence and the second one contains two weak criterions.
Suppose that $\{X_k\}_{k\geq 1}$ and X are random vectors on a probability space $(\Omega, \mathcal{A}, P)$ with values in $\mathbb{R}^n$. For all $k \geq 0$, denoting by F_X and F_{X_k} the respective distribution functions of X and X_k.
Strong criterions of convergence

1. Convergence with probability one (w.p.1), called also almost sure convergence:

$$P(\{\omega \in \Omega\,, \quad \lim_{k\to\infty} |X_k(\omega) - X(\omega)| = 0\}) = 1\,.$$

   We write $X_k \to X$, P-a.s.

2. Mean square convergence: If $E[|X|^2] < \infty$ and $E[|X_k|^2] < \infty$, $\forall\ k$.

$$\lim_{k\to\infty} E[|X_k - X|^2] = 0$$

3. Convergence in probability, called also stochastic convergence:

$$\lim_{k\to\infty} P(\{\omega\,,\ |X_k(\omega) - X(\omega)| > \epsilon\}) = 0\,, \quad \forall\ \epsilon > 0\,.$$

   equivalent to

$$\lim_{k\to\infty} E[\frac{|X_k - X|}{1 + |X_k - X|}] = 0$$

In particular, 1. $\Rightarrow$ 3. and 2. $\Rightarrow$ 3.
Also, 3. implies that X_k has a subsequence satisfying 1.
Weak criterions of convergence

1. Convergence in distribution, called also convergence in law:
   For all continuity point x of F_X, one has

$$\lim_{k\to\infty} F_{X_k}(x) = F_X(x)\,.$$

   we write $X_k \to_{\mathcal{D}} X$ or $X_k \Rightarrow X$

2. Weak convergence: For all $f \in C_b(\mathbb{R}^n)$, one has

$$\lim_{k\to\infty} E[f(X_k)] = E[f(X)]\,.$$

   we write $X_k \xrightarrow{w} X$

Convergence in probability implies the convergence in distribution. In the case that X is non random, they are equivalent.

Strong laws of large numbers (SLLN) Let X be a $\mathbb{R}^n$-valued random vector. If $E[|X|^2] < \infty$, we set $\sigma^2(X) = E[|X - E[X]|^2]$. Suppose that $\{X_k\}_{k\geq 1}$ is a sequence of $\mathbb{R}^n$-valued random variables.

Theorem B.3.1 (Strong law of large numbers)**.** *Suppose that the X_k's are independent with finite expectations $E[X_k] = \mu_k$. Set for all $k \geq 1$, $\bar{X}_k = \frac{1}{k}\sum_{i=1}^k X_i$ and $\bar{\mu}_k = \frac{1}{k}\sum_{i=1}^k \mu_i$. Then,*

$$P(\{c_k|\bar{X}_k - \bar{\mu}_k| \to 0\}) = 1$$

for every sequence $\{c_k\}_k$ in $\mathbb{R}_+$ such that $c_k/k \to 0$ and $\sum_{k=1}^\infty k^{-2}c_k^2\sigma^2(X_k) < \infty$.

Corollary B.3.2. *Suppose in addition the $\sigma^2(X_k)$'s are uniformly bounded by $c \in \mathbb{R}_+$. Then,*

$$P(k^\alpha|\bar{X}_k - \bar{\mu}_k| \to 0) = 1\,, \quad \text{for every } \alpha < 1/2.$$

The next result removes the condition on the variance, but supposes that the X_k's are identically distributed.

Theorem B.3.3 (Kolmogorov law of large number)**.** *Let $X_1, X_2, \dots$ are independent identically distributed random vectors. If $\mu_1 = E[X_1] \in \mathbb{R}^n$, then*

$$P(\lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^k X_i = \mu_1) = 1\,.$$

Central limit theorem (CLT)

Theorem B.3.4 (Central Limit Theorem)**.** *Let $\{X_1, X_2, \dots\}$ be a family of independent identically distributed random vectors and denote $\Gamma = cov(X_1)$. If $E[|X_1|^2] < \infty$, then*

$$\sqrt{k}(\bar{X}_k - \mu_1) \to_{\mathcal{D}} U$$

where $\bar{X}_k = 1/k\sum_{i=1}^k X_i$ and $U \sim \mathcal{N}(0, \Gamma)$.

We finish this subsection by a result on the asymptotic behavior of functions of random vectors. Let $\{Z_k\}_k$ be a sequence of n-dimensional random vectors such that

$$b_k(Z_k - c) \to_{\mathcal{D}} U\,.$$

where

1. $U \sim \mathcal{N}(0, \Gamma)$, where Γ is a $n \times n$-real matrix
2. $\{b_k\}_k$ a sequence of real numbers such that $b_k \to \infty$
3. $c \in \mathbb{R}^n$

Consider a Borel measurable transformation $H : \mathbb{R}^n \to \mathbb{R}^m$, we have

Theorem B.3.5. *If H has a differential D at the point c, then*

$$b_k(H(Z_k) - H(c)) \to_{\mathcal{D}} DU\,.$$

Remark B.3.6. For proofs see, for example, [27].

B.4 The Euler-Maruyama discretization

Modeling a time varying physical quantity is of central importance in many scientific areas. In many problems, this quantity satisfies a continuous time stochastic differential equation (SDE) and its solution gives information on this quantity. Generally, solutions of a SDE are not analytically available. Even when such a solution can be found, it may be to complicated to visualize or to evaluate numerically. Necessity has thus lead to the development of methods for calculating numerical approximations to the solutions of a SDE.
The most widely applicable and commonly used of these are the time discretization, also called time discrete approximation or difference methods, in which the continuous time SDE is replaced by a discrete time stochastic difference equation which evolves recursively in discrete time. One hopes that sufficiently small time increments lead to more accuracy.
In the filtering problem context, strong convergence criterions called strong Taylor approximations are used to discretize the equations of the system. Strong Taylor approximations are based on the Ito-Taylor expansions and on the Stratonovich-Taylor expansions (Theorem 5.5.1 and Theorem 5.6.1 in [50]). We will present the Euler-Maruyama scheme. This method attains an order of strong convergence $\gamma = 0.5$. More accurate methods, like the Milstein scheme, can be found in [50].
First, let us consider the following $\mathbb{R}^d$-valued SDE

$$X_t = X_{t_0} + \int_{t_0}^{t} a(X_s, s)\, ds + \sum_{j=1}^{m} \int_{t_0}^{t} b^j(X_s, s)\, dW_s^j\,, \tag{B.4}$$

where $t \in [t_0, T]$, $W = (W_t)_{t\in[t_0,T]}$ is an $\mathbb{R}^m$-valued standard BM and

$$a = (a^k)_{1\le k\le d} : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^d\,,$$
$$b = (b^{k,j})_{1\le k\le d,\, 1\le j\le m} : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^{d\times m}\,.$$

For each $t \in [t_0, T]$, we set $\mathcal{A}_t = \sigma\{X_{t_0}\,;\ W_s\,,\ t_0 \le s \le t\}^*$. If $\delta_0 > 0$ and $\delta \in (0, \delta_0)$, then $(\tau)_\delta = \{\tau_n\,,\ n = 0, 1, 2, \dots\}$ is a time discretization of $[t_0, T]$ if :

i. $t_0 \le \tau_0 < \tau_1 < \cdots < \tau_n < \cdots \le T$

ii. $\sup(\tau_{n+1} - \tau_n) \le \delta$

iii. $n_t = \max(n \ge 0\,,\ \tau_n \le t) < \infty,\ \forall\ t \in [t_0, T)$

iv. τ_{n+1} is $\mathcal{A}_{\tau_n}$-measurable

An $\mathbb{R}^d$-valued continuous process $Y = (Y(t))_{t\in[t_0,T]}$ is a time discretization approximation of maximum step size δ based on $(\tau)_\delta$ of the process X if

1. $Y(\tau_n)$ is $\mathcal{A}_{\tau_n}$-measurable
2. $Y(\tau_{n+1})$ can be expressed as a function of $Y(\tau_0), Y(\tau_1), \dots, Y(\tau_n)$, $\tau_0, \dots, \tau_n, \tau_{n+1}$ and a finite number of $\mathcal{A}_{\tau_{n+1}}$-measurable random vectors which generate mainly the noises

In such a case the process Y is denoted by Y^δ.
The process Y^δ converges strongly with order $\gamma > 0$ at time T if there exists a constant C, independent of δ, such that

$$\epsilon(\delta) = E[\,|X_T - Y^\delta(T)|\,] \le C\delta^\gamma\,,\ \text{for all } \delta \in (0, \delta_0)\,. \tag{B.5}$$

Let $(\tau)_\delta = \{t_0 = \tau_0 < \tau_1 < \cdots < \tau_N = T\}$ be a time discretization of $[t_0, T]$. The nth time increment is $\Delta_n := \tau_{n+1} - \tau_n$, the maximum step is $\delta = \max_n \Delta_n$.
We consider only equidistant time discretization, that is

$$\tau_n = t_0 + n\delta\,, \quad \text{with} \quad \delta = \Delta_n = \Delta = (T - t_0)/N < 1\,.$$

The Euler-Maruyama scheme is the simplest strong Taylor discretization. For every $k \in \{1, \ldots, d\}$, the kth component of the Euler-Maruyama scheme is

- $Y^k_{n+1} = Y^k_n + a^k(Y^k_n, \tau_n)\Delta + \sum_{j=1}^m b^{k,j}(Y^k_n, \tau_n)\Delta_n W^j$
- for all $t \in [\tau_n, \tau_{n+1})$,

$$Y^k(t) = Y^k_n + a^k(Y^k_n, \tau_n)(t - \tau_n) + \sum_{j=1}^m b^{k,j}(Y^k_n, \tau_n)(W^j_t - W^j_{\tau_n}) \tag{B.6}$$

where $Y^k_n = Y^k(\tau_n)$ and for all $n \in \{0, \ldots, N-1\}$, $\Delta_n W^j = W^j_{\tau_{n+1}} - W^j_{\tau_n}$.
For all $n \in \{0, \ldots, N-1\}$, $\Delta_n W^1, \ldots, \Delta_n W^m$ are independent Gaussian and for all $1 \le j \le m$

$$E[\Delta_n W^j] = 0 \text{ and } E[(\Delta_n W^j)^2] = \Delta\,.$$

Suppose that there exist constants $C_i\,,\ i \in \{1, 2, 3, 4\}$ independent of δ such that for all $s,\, t \in [t_0, T]$ and $x,\, y \in \mathbb{R}^d$,

- $E(|X_0|^2) < \infty$
- $E(|X_0 - Y_0|^2)^{1/2} \le C_1\delta^{1/2}$
- $|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| \le C_2|x - y|$
- $|a(t,x)| + |b(t,y)| \le C_3(1 + |x|)$
- $|a(s,x) - a(t,x)| + |b(s,x) - b(t,x)| \le C_4(1 + |x|)|s - t|^{1/2}$

Then, the Euler scheme converges strongly with an order $\gamma = 1/2$, *i.e.* for a constant C independent of δ

$$E(|X_T - Y(T)|) \le C\delta^{1/2}\,.$$

Appendix C

C.1 Bayesian Approach Estimation

In a purely statistical setup, computational difficulties occurs at the level of statistical inference on a given probabilistic model (estimation, prediction, tests, variable selection, etc.).
Some of the most applied statistical techniques are maximum likelihood and Bayesian methods and the inferences that can be drawn from their use.
The maximum likelihood is associated with maximization problems, based on an implicit definition of the estimators as solutions of maximization problems.
The Bayesian methods, in which we are concerned here, deals with integration problems and proceed to give an explicit representation of estimators as an integral.
Suppose that a density $f(x, \theta)$ involves all statistical information about some real world situations, where the unknown parameter vector θ belongs to a Borel measurable set $\Theta \subset \mathbb{R}^n$.
The aim is to estimate the true unknown parameter vector θ_0 .
Suppose that prior information about $\theta \in \Theta$ are available, this permit us to treat θ as a random variable, usually square integrable. We can then represent all prior information by a density function $\tau(\theta)$, called prior density on θ .
Furthermore, suppose that we are able to sample from $f(x|\theta)$: the conditional density function given θ. Let $X = (X_1, \ldots, X_k)$ be a sequence of samples from $f(x|\theta)$.
The function $f(X|\theta)$ is the conditional density of $X = (X_1, \ldots, X_k)$ given θ.
Bayesian inference combine in some *optimal* way the prior information given by the prior density with the data in order to improve the inference about θ.
Let us make some notations and definitions before conducting such inference about θ in an *optimal* way.

- $f(X|\theta)$ is the conditional density of $X = (X_1, \ldots, X_k)$ given θ.
- $\tau(\theta)$ is the prior density on θ.
- $g(X)$ is the marginal density of X, ie. $g(X) = \int_\Theta f(X|\theta)\tau(\theta)d\theta$.
- $h(\theta|X)$ is the posterior density of θ given $X = (X_1, \ldots, X_k)$.
- $f(X, \theta)$ is the joint density of θ and $X = (X_1, \ldots, X_k)$.

In particular,

$$f(X, \theta) = f(X|\theta)\tau(\theta) = h(\theta|X)g(X)\,.$$

Then,

$$h(\theta|X) = \frac{f(X|\theta)\tau(\theta)}{g(X)} \propto f(X|\theta)\tau(\theta) \tag{C.1}$$

Let $T : (\mathbb{R}^m)^k \to \Theta$ be a Borel measurable function. The function T is called the estimator and $T(X) = T(X_1 \ldots, X_k)$ is called an estimate of θ. For any realization $x = (x_1, \ldots, x_k)$ of $X = (X_1 \ldots, X_k)$, we call $\theta^* = T(x_1, \ldots, x_k)$ a point estimation of θ.
A *loss* function $L(\theta, \theta^*)$ is defined to be any non-negative function equal to zero if the parameter and its estimator coincide, it measures the deficiency (loss) incurred when taking θ^* as a point estimation of θ.
We restrict our selves to the quadratic loss, called also the square error loss, that is

$$L(\theta, \theta^*) = \|\theta - \theta^*\|^2_{\mathbb{R}^n} = \|\theta - \theta^*\|^2 .$$

The *Bayesian risk* $r(\tau, T)$ of the estimator T with respect to the prior τ is

$$\begin{aligned} r(\tau, T) &= \int_\Theta \int_{(\mathbb{R}^m)^k} L(\theta, T(x)) f(x|\theta)\tau(\theta) dx d\theta \\ &= \int_\Theta \int_{(\mathbb{R}^m)^k} \|\theta - T(x)\|^2 f(x|\theta)\tau(\theta) dx d\theta , \end{aligned}$$

where $dx = dx_1 \ldots dx_k$. Loosely speaking $r(\tau, T)$ measures the average over θ of the average loss when we risk T as an estimator of a fixed θ. Finally, the *Bayesian estimator* T_τ is defined by

$$r(\tau, T_\tau) = \inf_T r(\tau, T) . \tag{C.2}$$

This criterion gives the optimal estimator in the Bayesian framework:

Theorem C.1.1. *For the square error loss $L(\theta, \theta^*) = \|\theta - \theta^*\|^2$, the Bayesian estimator is given by*

$$T_\tau(X) = E[\theta|X] = \int_\Theta \theta h(\theta|X) d\theta : \textit{ The posterior expectation.}$$

Proof. see Theorem 4.1.1. and its corollaries in [58]. □

An important class of numerical problems that arise in statistical inference is integration problems, which is generally associated with the Bayesian approach. With the advanced of computers in the last years, simulations methods, as Monte Carlo Methods, has proved a powerful performance and one can apply probabilistic results as the Laws of Large Numbers or the Central Limit Theorem to obtain an assessment of the convergence. These methods are essentially based on the possibility to generate random variables (usually i.i.d.) from distributions, not necessarily explicitly known, with the computer, see Chapter 2 in [72].

C.2 Minimum Variance Estimation (MVE)

Let us consider the $n+1$ random variable $X, Y_1, \ldots, Y_n$ in $\mathcal{L}^2(\Omega, P)$ and Y be random vector given by

$$Y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}.$$

An estimator $\hat{X}$ of X given Y is called minimum variance estimator (MVE) if for all Borel function h such that $||h \circ Y - X||^2 = \int_\Omega (h \circ Y - X)^2 dP < \infty$ we have

$$||\hat{X} - X|| \leq ||h \circ Y - X||. \tag{C.3}$$

More generally, if $\mathcal{A}$ is a subspace of Borel measurable functions on $\mathbb{R}^n$ such that

i. $||f \circ Y|| < \infty$ for all $f \in \mathcal{A}$

ii. $\mathcal{M}(\mathcal{A}, Y) = \{f \circ Y\,;\ \ f \in \mathcal{A}\}$ is a closed subspace of $\mathcal{L}^2(P)$

we define the best $\mathcal{A}$-MVE of X based on Y to be the orthogonal projection $\hat{X}$ of X onto $\mathcal{M}(\mathcal{A}, Y)$. The subspace $\mathcal{M}(\mathcal{A}, Y)$ is the set of $\mathcal{A}$ estimators based on Y.

1. If $\mathcal{A} = \{g : \mathbb{R}^n \to \mathbb{R}\,,\ g \text{ is a linear}\}$, then $\mathcal{M}(\mathcal{A}, Y)$ is a finite dimensional subspace of $\mathcal{L}^2(\Omega, P)$ called set of linear estimators based on Y and $\hat{X}$ is the best linear minimum variance estimator (BLMVE) of X based on Y

2. If $\mathcal{A}$ is the set of affine transformations from $\mathbb{R}^n$ to $\mathbb{R}$, then $\mathcal{M}(\mathcal{A}, Y)$ is a finite dimensional subspace of $\mathcal{L}^2(\Omega, P)$ and $\hat{X}$ is the best affine minimum variance estimator (BAMVE) of X based on Y

Remark C.2.1. These definitions extend to the case where X is a random vector.

Theorem C.2.2. *The MVE $\hat{X}$ of X based on Y is $\hat{X} = E[X|Y]$.*

Proof. Let h be any Borel function such that $||h \circ Y - X||^2 = \int_\Omega (h \circ Y - X)^2 dP < \infty$, we get the result using the orthogonality of $X - E[X|Y]$ and $E[X|Y] - h \circ Y$. That is,

$$\|X - h \circ Y\|^2 = \|X - E[X|Y]\|^2 + \|E[X|Y] - h \circ Y\|^2.$$

□

Remark C.2.3. The MVE $\hat{X}$ of X based on Y can be seen as the Bayesian estimator of X based on the observations $Y_1, \cdots, Y_n$.

Let us now characterizes the BLMVE and BAMVE. The following Theorem shows that's in general such estimators don't coincide and they are also different of the MVE.

Theorem C.2.4. *Consider two random vectors X and Y. Denote $\mu_x = E[X]$, $\mu_y = E[Y]$, $\Gamma_{11} = cov(X, X)$, $\Gamma_{12} = \Gamma_{21}^T = cov(X, Y)$ and $\Gamma_{22} = cov(Y, Y)$. Then,*

1. *The BLMVE of X based on Y is given by*

$$\hat{X} = KY\,, \quad \textit{where } K = E[XY^T]E[YY^T]^+\,.$$

Moreover, $E[(\hat{X}-X)(\hat{X}-X)^T] = E[XX^T] - KE[YX^T]$.

2. *The BAMVE of X based on Y is given by*

$$\hat{X} = HY + b\,, \quad \textit{where } H = \Gamma_{12}\Gamma_{22}^+ \quad \textit{and } b = \mu_x - H\mu_y\,.$$

Moreover, $E[(\hat{X}-X)(\hat{X}-X)^T] = \Gamma_{11} - H\Gamma_{21}$.

Remark C.2.5. The BLMVE of $X-\mu_x$ based on $Y-\mu_y$ is $\hat{X}-\mu_x$, where $\hat{X}$ is the BAMVE of X based on Y.

The following Theorem gives a recursive linear Bayesian estimation. This result has its application in the theory of Kalman filter.

Theorem C.2.6 (Static Updating Theorem-Bayesian Estimation)**.**
Let X, Y_1, Y_2 and W be random vectors such that $E[XW^T]=0$, $E[Y_1W^T]=0$ and $Y_2 = HX+W$, where $H \in \mathcal{M}_{m,n}(\mathbb{R})$. We denote $R = E[WW^T]$.
Suppose that $\hat{X}_1$ is the BLMVE of X based on Y_1, where $\hat{X}_1$ and $P_1 = E[(X-\hat{X}_1)(X-\hat{X}_1)^T]$ are known. If

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}\,.$$

Then, the BLMVE $\hat{X}$ of X based on Y and its error covariance P satisfy

1. $\hat{X} = \hat{X}_1 + P_1H^T[HP_1H^t + R]^+[Y_2 - H\hat{X}_1]$
2. $P = E[(X-\hat{X})(X-\hat{X})^T] = P_1 - P_1H^T[HP_1H^t+R]^+HP_1$

Corollary C.2.7. *If $\mu_w = E[W] = 0$, then the Theorem still hold if we replace BLMVE by BAMVE.*

We end this section by a result that makes the relation between the MVE and the BAMVE.

Theorem C.2.8. *If $(X,\,Y)$ is multivariate normal distributed then the MVE of X based on Y coincide with the BAMVE.*

C.3 Monte Carlo Integration

Looking at the generic problem of evaluating the finite integral

$$E_f[h(X)] = \int_{\mathbb{R}^k} h(x)f(x)dx\,. \tag{C.4}$$

For any function g, such that $supp(f) \subset supp(g)$, we have

$$E_f[h(X)] = \int_{\mathbb{R}^k} \frac{h(x)f(x)}{g(x)}g(x)dx = E_g[\frac{h(X)f(X)}{g(X)}]\,. \tag{C.5}$$

If $(X_1, \ldots, X_N)$ are i.i.d. sampled from the function g, then the integral (C.5) is approximated by

$$H_g^N = \frac{1}{N} \sum_{j=1}^{N} \frac{f(X_j)}{g(X_j)} h(X_j) \,. \tag{C.6}$$

Indeed, by the strong law of large numbers H_g^N converges almost surely to $E_f[h(X)]$. In addition, if

$$\Delta_g = var_g(\frac{h(X)f(X)}{g(X)}) < \infty \,, \tag{C.7}$$

or equivalently

$$E_g[h^2(X)\frac{f^2(X)}{g^2(X)}] = E_f[h^2(X)\frac{f(X)}{g(X)}] = \int_{\mathbb{R}^k} h^2(x)\frac{f^2(x)}{g(x)}\, dx < \infty \,, \tag{C.8}$$

the variance

$$var_g(H_g^N) = E_g[(H_g^N - E_g[H_g^N])^2] = \frac{1}{N}\Delta_g \tag{C.9}$$

can be estimated from the sample $(X_1, \ldots, X_N)$ through

$$\Delta_g^N = \frac{1}{N^2} \sum_{j=1}^{N} (\frac{f(X_j)}{g(X_j)} h(X_j) - H_g^N)^2 \,.$$

The speed of convergence of H_g^N can be assessed. In fact, from the central limit theorem,

$$\frac{H_g^N - E_f[h(X)]}{\sqrt{\Delta_g^N}}$$

is approximately distributed as a $\mathcal{N}(0, I_k)$ variable for large N. This leads to the construction of a convergence test and confidence bounds on the approximation of $E_f[h(X)]$.

If $f \equiv g$, this method is called the classical or perfect Monte Carlo Method. The instrumental function g, called the *impotance function*, is used to avoid a direct simulation from f and to gain in the speed of the convergence. When taking any importance function g such that $f \neq g$ and $supp(f) \subset supp(g)$, we talk about Importance Sampling Method. This method is of considerable interest since it puts little restrictions on the choice of the importance function g. The function g can be chosen from distributions that are easy to simulate. Moreover, the same sample generated from g can be used repeatedly, not only for different functions h but also for different densities f.

Although the distribution g can be almost any density, there are some choices better than others. It is natural to compare different distributions g for the evaluation of (C.4).

While (C.6) converges almost surely to (C.4), we have no idea about the computation time and the speed of the convergence and this can be determinant on the choice of the importance function. It is natural to choose g among the distributions leading to

finite variance of the estimator (C.6) or equivalently to one of the conditions (C.7) and (C.8). Importance functions g with unbounded ratio f/g are not appropriate for importance sampling because they leads to an infinite variance for many functions h. In addition, if the ratio f/g is unbounded, the weights $f(x_j)/g(x_j)$ will vary widely, giving too much importance to a few values x_j.
In [30], Geweke gives two types of sufficient conditions:

i. $f(x)/g(x) < M \ \forall \ x \in \mathbb{R}^k$ and $var_f(h) < \infty$

ii. The support of the integrand functions h of interest are include in some fixed compact $K \subset \mathbb{R}^k$, and $f(x) < F$ and $g(x) > \epsilon \ \forall \ x \in K$

Under one of those conditions and in view of (C.6), the distribution g leading to smaller variance of the estimators performs better. That is, a reduction of the variance accelerate the convergence.
Nevertheless, it is possible to exhibit the optimal distribution g corresponding to a given function h and a fixed distribution f:

Theorem C.3.1. *The choice of g that minimizes the variance of the estimator (C.6) is*

$$g^*(x) = \frac{|h(x)|f(x)}{\int_{\mathbb{R}^k} |h(z)|f(z)dz} \ .$$

Proof. see [72]. □

This optimality is formal since, when $h(x) > 0$, the optimal choice $g^*(x)$ requires the knowledge of $\int h(x)f(x)dx$: the integral of interest!.

Remark C.3.2. From a practical point of view, Theorem C.3.1 suggests looking for distributions g for which $|h|f/g$ is almost constant with finite variance.

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