Mai 1998

Mannheimer Manuskripte 233/98
Fakultät für Mathematik und Informatik

A generalization of Banach's fixed point theorem applied to non-linear stochastic evolution equations

T. Deck

A GENERALIZATION OF BANACH'S FIXED POINT THEOREM APPLIED TO NON–LINEAR STOCHASTIC EVOLUTION EQUATIONS

T. Deck

Department of Mathematics and Computer Science,
University of Mannheim, D-68131 Mannheim, Germany

Abstract

Banach's fixed point theorem for contraction operators on Banach spaces is generalized to inductive limits of Banach spaces. Within the framework of white noise analysis such spaces and (generalized) contraction operators arise naturally in the context of non-linear stochastic integral equations. In order to apply the fixed point theorem we establish topological isomorphisms between spaces of continuous mappings with values in generalized random variables, and those with values in U-functionals. As an application we prove that the Cauchy problem for a class of non-linear stochastic heat equations is well-posed. The same method also applies to stochastic Volterra equations, stochastic reaction-diffusion equations and anticipating stochastic differential equations.

1. Introduction

The present work is motivated by [BDP], where Cauchy problems for non–linear stochastic equations such as heat equations, Volterra equations and others were investigated. A solution of such a Cauchy problem is a random field $\Phi(t,x)$ with time parameter $t \in [0,T]$ and space parameter $x \in \mathbb{R}^d$. This random field defines the time evolution $t \mapsto \Phi(t,\cdot)$ which starts at some initial state $\Phi_0 = \Phi(0,\cdot)$. All examples discussed in [BDP] have the common feature that the solution Φ satisfies a fixed point equation of the following type:

$$\Phi = \Psi_0 + K(\Phi). \tag{1.1}$$

Here, K denotes a non-linear integral operator and Ψ_0 is a function of the initial state Φ_0. K and Ψ_0 depend on the specific problem, for the case of heat equations, e.g., see (5.7) and (5.8) in Section 5.

There are Cauchy problems satisfying (1.1) for which K is a contraction operator on a Banach space (cf. the monograph [DZ]). In this case it is not hard to prove that (1.1) has exactly one solution Φ, and that Φ depends continuously on the initial data Φ_0, i.e. that the mapping

$$\Phi_0 \mapsto \Phi \tag{1.2}$$

is continuous. In other cases, e.g. when space–derivatives of Φ are multiplied to noise terms or when one allows for anticipating objects, the methods described in [DZ] do not apply. In that case it is not clear if classical solutions (by this we mean L^p–valued random fields defined on $D_T := [0,T] \times \mathbb{R}^d$) exist at all. There are examples (even for linear heat equations, cf. [BDPS]) where no classical solution exists, but generalized ones do. In such situations white noise analysis (WNA) can provide a more appropriate framework because one works - from the beginning - with classical as well as with generalized random objects, such as Hida distributions $(\mathcal{S})^*$ or Kondratiev distributions $(\mathcal{S})^{-\beta}$, $\beta \in (0,1]$, cf. [KS], [KLS]. The solution Φ of (1.1) obtained by the methods of WNA is then typically an $(\mathcal{S})^{-\beta}$–valued random field on D_T. Of course, one finally likes to find conditions such that these solutions are classical random fields, e.g. L^2–valued ones. This difficult regularity question is excluded in the present paper. (Some results on this question – for linear heat equations – are given in [DP].) Instead, we focus on the continuity of the mapping (1.2).

Although stochastic evolution equations have been studied extensively by the techniques of WNA, cf. [HØUZ], the continuous dependence on initial data has not yet been studied in detail. The main obstacles are that $(\mathcal{S})^{-\beta}$ carries an inductive limit topology (which is not a metric topology), and that one cannot directly control the dependence of Φ on Φ_0. In particular, the inductive limit structure seems not to combine well with continuity questions for non–linear mappings (1.2). It is the main objective of the present paper to develop a general method by which these obstacles can be overcome. As a concrete example we consider a class of non-linear stochastic heat equations, and we interpret the associated integral equations (1.1) as fixed point problems on the space of bounded continuous functions from D_T into $(\mathcal{S})^{-\beta}$, denoted $C_b(D_T,(\mathcal{S})^{-\beta})$. We prove that the mapping (1.2) from $C_b(\mathbb{R}^d,(\mathcal{S})^{-\beta})$ to $C_b(D_T,(\mathcal{S})^{-\beta})$ is continuous with respect to (natural) inductive limit topologies on these spaces (see Theorem 5.3). Thereby we extend the existence and uniqueness results for (1.1) obtained in [BDP], and our final conclusion is that the Cauchy problems discussed in [BDP] actually are well–posed.

Our proof is based on two key observations: firstly, there is a generalization of Banachs fixed point theorem for the case of inductive limits of Banach spaces, and this in turn implies well-posedness on an abstract level. Secondly, the S–transformation induces a topological isomorphism from $C_b(D_T,(\mathcal{S})^{-\beta})$ onto the space $C_b(D_T,\mathcal{U}^\beta)$ consisting of bounded continuous functions with values in the U–functionals $\mathcal{U}^\beta$. On this latter space the analysis of (1.1) is tractable, and we can apply the generalized fixed point theorem there.

The paper is organized as follows: In Section 2 we generalize Banach's fixed point theorem. This part is independent of the white noise context, but is of course motivated by it. Because of its generality, this part might also be useful in other contexts. Section 3 provides the necessary facts about $(\mathcal{S})^{-\beta}$ and $\mathcal{U}^\beta$ for later reference. In Section 4 we introduce spaces of bounded functions with values in $(\mathcal{S})^{-\beta}$ or $\mathcal{U}^\beta$ and we prove that the S–transformation induces a topological isomorphism. Finally, we consider the non–linear heat equation, introduced in [BDP], as an example for a well–posed Cauchy problem. It is quite obvious that well–posedness also holds for the other examples discussed in [BDP], because they are all based on Banach's fixed point theorem in an appropriately chosen space of U–functionals.

Acknowledgement. It is a pleasure to thank J. Potthoff for valuable discussions on this paper.

2. A generalization of Banach's fixed point theorem

In the context of WNA various inductive limits of Banach spaces $(E_\nu, \|\cdot\|_\nu)$ arise, which have a special common feature: The index set J associated to $E = \cup_{\nu \in J} E_\nu$ is a *directed system*, i.e. there is a partial ordering $\prec$ on J such that for any $\alpha, \beta \in J$ there exists $\gamma \in J$ satisfying $\alpha \prec \gamma$ and $\beta \prec \gamma$. Moreover, if $\alpha \prec \beta$ then $E_\alpha \subset E_\beta$ and $\|u\|_\alpha \geq \|u\|_\beta$ for all $u \in E_\alpha$. In this paper we will consider exclusively inductive limits E of this type, and we denote them simply by $\cup_{\nu \in J} E_\nu$, without mentioning the ordering properties explicitly. Typical examples for index sets are given by $J = \mathbb{N}, \mathbb{N} \times \mathbb{Z}$ etc., with the usual (partial) ordering, cf. Section 5. For general properties of inductive limits we refer to [Ro].

Definition. *Let $E = \cup_{\nu \in J} E_\nu$ be an inductive limit of Banach spaces. A map $K : E \to E$ is called a strict contraction on E, if there exists $\alpha \in J$ such that for all $\nu \succ \alpha$ the following conditions are satisfied:*

(C1) *K maps E_ν into E_ν.*

(C2) *There exists $c_\nu \in [0, 1)$ such that for all $u, v \in E_\nu$*

$$\|K(u) - K(v)\|_\nu \leq c_\nu \|u - v\|_\nu.$$

Banach's fixed point theorem can now be generalized as follows:

Theorem 2.1. *Let K be a strict contraction on the inductive limit $E = \cup_{\nu \in J} E_\nu$. Then, for each $v \in E$, there exists a unique solution u_v of the fixed point problem*

$$u = v + K(u).$$

Moreover, the mapping $h : v \mapsto u_v$ from E into E is continuous.

Proof: We start with existence and uniqueness. Let $\alpha \in J$ be such that K is contractive on E_ν, for all $\nu \succ \alpha$. We shall distinguish between the restriction $K|_{E_\nu} : E_\nu \to E$ and the operator $K_\nu : E_\nu \to E_\nu$ obtained from $K|_{E_\nu}$ by restricting in addition the image space.

For fixed $v \in E$ choose $\beta \succ \alpha$ such that $v \in E_\beta$. Then the operator $K^v(u) := v + K(u)$, $u \in E$, yields a contraction K^v_β on the subspace E_β. By Banach's fixed point theorem there exists a unique $u \in E_\beta$ which solves $u = K^v_\beta(u)$. Now let $\tilde{u} \in E$ be another solution of $u = K^v(u)$. Then $\tilde{u} \in E_{\tilde{\beta}}$ for some $\tilde{\beta} \in J$. Choose $\gamma \succ \beta, \tilde{\beta}$. Then $u, \tilde{u} \in E_\gamma$ are fixed points for the contraction operator K^v_γ on the Banach space E_γ and thus $u = \tilde{u}$.

It remains to show that $h : v \mapsto u_v$ is continuous from E into E. A mapping h from E into a topological space X is continuous if and only if $h|_{E_\nu}$ is continuous for each $\nu \in J$. Fix $\nu \in J$. If $\nu \succ \alpha$ we have $u_v \in E_\nu$ and we can estimate

$$\begin{aligned} \|u_v - u_{\tilde{v}}\|_\nu &= \|v + K(u_v) - \tilde{v} - K(u_{\tilde{v}})\|_\nu \\ &\leq \|v - \tilde{v}\|_\nu + \|K(u_v) - K(u_{\tilde{v}})\|_\nu \\ &\leq \|v - \tilde{v}\|_\nu + c_\nu \|u_v - u_{\tilde{v}}\|_\nu, \end{aligned}$$

where $c_\nu \in [0, 1)$. With $k_\nu := 1 - c_\nu > 0$ we obtain

$$\|u_v - u_{\tilde{v}}\|_\nu \leq k_\nu^{-1} \|v - \tilde{v}\|_\nu.$$

This implies that $h_\nu : E_\nu \to E_\nu$ is continuous. Since the embedding $i_\nu : E_\nu \hookrightarrow E$ is continuous too, the same holds for the composition $h|_{E_\nu} = i_\nu \circ h_\nu$.

If $\nu \not\succ \alpha$ choose $\tilde{\nu} \succ \nu, \alpha$. The continuity of $h|_{E_\nu} = h|_{E_{\tilde{\nu}}} \circ i_{\nu,\tilde{\nu}}$ follows from the one of $h|_{E_{\tilde{\nu}}}$ and the continuous embedding $i_{\nu,\tilde{\nu}} : E_\nu \hookrightarrow E_{\tilde{\nu}}$. ■

The following elementary lemma is stated for later reference. It will be useful when we study the relation between spaces of white noise distributions and U–functionals.

Lemma 2.2. *Let $S : \cup_{\nu\in J} E_\nu \to \cup_{\nu'\in L} H_{\nu'}$ be a linear mapping between inductive limits of Banach spaces E, H and let $\alpha \in J$ be fixed. Assume that for every $\nu \succ \alpha$ there exists $\nu' \in L$ and $k_\nu \geq 0$ such that*

(I1) *S maps E_ν into $H_{\nu'}$ and*

(I2) *$\|Su\|_{\nu'} \leq k_\nu \|u\|_\nu$ for all $u \in E_\nu$.*

Then S is continuous. In particular, if S is a linear isomorphism and also S^{-1} satisfies properties analogous to (I1) *and* (I2)*, then S is a topological isomorphism.*

Proof: S is continuous if and only if every restriction $S|_{E_\nu}$ is continuous. For $\nu \succ \alpha$ this holds in view of (I1), (I2) and the continuity of $H_{\nu'} \hookrightarrow H$. For $\nu \not\succ \alpha$ choose $\tilde{\nu} \succ \nu, \alpha$. Then $S|_{E_\nu} = S|_{E_{\tilde{\nu}}} \circ i_{\nu,\tilde{\nu}}$ is continuous. ■

An application of this lemma when J and L do not coincide is given in Section 5. We remark that the S–transformation from white noise analysis is an isomorphism of the type described in Lemma 2.2, cf. [BT]. Also all other isomorphisms between inductive limits of Banach spaces considered in this paper will have the property, that E_ν is "shifted" into $H_{\nu'}$. For convenience we will therefore denote an isomorphism with the properties (I1,I2) stated in Lemma 2.2 as a *shift isomorphism*.

3. Topological aspects of white noise analysis

In this section we fix notations and recall some facts from WNA. We also discuss some more recent results from the literature, partly in slightly modified form, so that we obtain a uniform representation. Since the modifications are straightforward and inessential, we do not give proofs.

Let $\mathcal{S}(\mathbb{R}^k)$ denote the Schwartz space of real, rapidly decreasing C^∞–functions and $\mathcal{S}^*(\mathbb{R}^k)$ its topological dual space. The harmonic oscillator $H : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is the bijective operator $H = -d^2/dx^2 + x^2 + 1$. A family $\{|\cdot|_{2,p} : p \in \mathbb{N}_0\}$ of norms on $\mathcal{S}(\mathbb{R}^k)$ is defined by $|\xi|_{2,p} := |\tilde{H}^p \xi|_2$, where $\tilde{H} := H^{\otimes k}$ and $|\cdot|_2$ is the norm in $L^2(\mathbb{R}^k)$. For all $p, n \in \mathbb{N}_0$ these norms satisfy

$$|\xi|_{2,p} \leq \frac{1}{2^n} |\xi|_{2,p+n}. \tag{3.1}$$

We denote the complexified spaces by $\mathcal{S}_\mathbb{C}(\mathbb{R}^k)$, $\mathcal{S}^*_\mathbb{C}(\mathbb{R}^k)$, and we keep the same notation for the corresponding norms. In this paper we work with the white noise probability space $(\mathcal{S}^*(\mathbb{R}^k), \mathcal{B}, \mu)$. The σ-algebra $\mathcal{B}$ is generated by the functions X_ξ, $\xi \in \mathcal{S}(\mathbb{R}^k)$, which are defined by the dual pairing with $\omega \in \mathcal{S}^*(\mathbb{R}^k)$, i.e. by $X_\xi(\omega) := \langle \omega, \xi \rangle$. The white noise probability measure μ is such that the X_ξ are Gaussian random variables, and such that the algebra $\mathcal{P}$ of polynomials generated by the set $\{X_\xi | \xi \in \mathcal{S}(\mathbb{R}^k)\}$ is dense in $L^2_{\mathbb{C}}(\mu) =: (L^2)$. For more details and proofs see [HKPS], pp. 1-7.

The notion of generalized random variables is based on the Wiener-Itô decomposition of (L^2) into a (Hilbert) sum of orthogonal subspaces $\mathcal{H}_n$,

$$(L^2) = \oplus_{n=0}^{\infty} \mathcal{H}_n.$$

This decomposition allows to associate with each monomial $X_{\xi_1} \cdots X_{\xi_n}$ its orthogonal projection $: X_{\xi_1} \cdots X_{\xi_n} :$ into $\mathcal{H}_n$. If Q_0 denotes the space of constants in (L^2) and

$$Q_n := \text{span}[: X_{\xi_1} \cdots X_{\xi_n} : | \ \xi_1, \ldots, \xi_n \in \mathcal{S}(\mathbb{R}^k)], \quad n \in \mathbb{N},$$

then the (algebraic) orthogonal decomposition $\mathcal{P} = \oplus_{n=0}^{\infty} Q_n$ holds. One can now define a linear, bijective operator $\Gamma_{\tilde{H}} : \mathcal{P} \to \mathcal{P}$ by its action on monomials,

$$\Gamma_{\tilde{H}} : X_{\xi_1} \cdots X_{\xi_n} : = : X_{\tilde{H}\xi_1} \cdots X_{\tilde{H}\xi_n} : ,$$

and a system of inner product norms on $\mathcal{P}$, as follows: Let $\Phi = \sum_{n=0}^{\infty} \Phi_n$ be the (finite) decomposition of $\Phi \in \mathcal{P}$ into components $\Phi_n \in Q_n$, and define

$$\|\Phi\|^2_{\beta,p} := \sum_{n=0}^{\infty} n!^{\beta} \|\Gamma^p_{\tilde{H}} \Phi_n\|_2^2, \qquad p \in \mathbb{Z}, \ \beta \in [-1, 1].$$

It is then straightforward to show that $\|\cdot\|_{\beta,p} \geq \|\cdot\|_{\beta,p+1}$ for all $p \in \mathbb{Z}$. We denote the $\|\cdot\|_{\beta,p}$–completion of $\mathcal{P}$ by $(\mathcal{S})^{\beta}_p$. In case $p, \beta \geq 0$ this completion can be identified with a subspace of (L^2). Moreover, the embedding operators $i_{p+1,p} : (\mathcal{S})^{\beta}_{p+1} \hookrightarrow (\mathcal{S})^{\beta}_p$ are of Hilbert–Schmidt type. Since the dual space of $(\mathcal{S})^{\beta}_p$ is canonically isomorphic to $(\mathcal{S})^{-\beta}_{-p}$, one obtains for $\beta \in [0, 1]$:

$$(\mathcal{S})^{\beta} \subset \cdots \subset (\mathcal{S})^{\beta}_1 \subset (\mathcal{S})^{\beta}_0 \subset (L^2) \subset (\mathcal{S})^{-\beta}_0 \subset (\mathcal{S})^{-\beta}_{-1} \cdots \subset (\mathcal{S})^{-\beta}, \tag{3.2}$$

where we abbreviated $(\mathcal{S})^{\beta} := \bigcap_{p \in \mathbb{N}} (\mathcal{S})^{\beta}_p$ and $(\mathcal{S})^{-\beta} := \bigcup_{p \in \mathbb{N}} (\mathcal{S})^{-\beta}_{-p}$. If one equips $(\mathcal{S})^{\beta}$ with the projective limit topology then its dual $((\mathcal{S})^{\beta})^*$ is canonically isomorphic to $(\mathcal{S})^{-\beta}$. Henceforth we equip $(\mathcal{S})^{-\beta}$ with the inductive limit topology τ_{ind}. It is well-known that τ_{ind} and the strong topology τ_{strong} on $(\mathcal{S})^{-\beta}$ coincide, see Appendix 5 in [HKPS]. Finally we mention that for $\beta \in [0, 1]$ the operator $\Gamma_{\tilde{H}}$ extends by continuity to a topological isomorphism

$$\Gamma_{\tilde{H}} : \ (\mathcal{S})^{\beta} \to (\mathcal{S})^{\beta}.$$

Example: Let $\beta \in [0, 1)$, $\xi = \xi_1 + i\xi_2 \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^k)$ and $X_\xi := X_{\xi_1} + iX_{\xi_2}$. Then the series $\sum_{n=0}^{\infty} : X^n_\xi : /n!$ converges in $(\mathcal{S})^{\beta}$ to an element denoted $: e^{X_\xi} :$. The continuity of $\Gamma_{\tilde{H}}$ implies that

$$\Gamma_{\tilde{H}} : e^{X_\xi} : = : e^{X_{\tilde{H}\xi}} : .$$

We remark that $:e^{X_\xi}: = e^{X_\xi - \frac{1}{2}|\xi|_2^2}$, and that $\mathcal{E} := \text{span}[:e^{X_\xi}: \mid \xi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^k)]$ is dense in (L^2). For $\beta = 1$ the series $:e^{X_\xi}:$ converges in $(\mathcal{S})_p^1$ if and only if $|\xi|_{2,p} < 1$. This implies that $:e^{X_\xi}:$ is not in $(\mathcal{S})^1$, unless when $\xi = 0$.

Remarks. 1. The space $(\mathcal{S})^{-0}$ is the well-known space of Hida distributions, cf. [HKPS], and the elements in $(\mathcal{S})^{-\beta}$ for $\beta \in (0,1]$ are called Kondratiev distributions. The latter were introduced in [KS] and [KLS]. The construction given in [KLS] applies for general nuclear spaces $\mathcal{N}$ and therefore avoids the use of $\tilde{H}$. However, for the white noise case $\mathcal{N} = \mathcal{S}(\mathbb{R}^k)$ this construction simplifies in the obvious manner, as described above. 2. Usually the operator $\hat{H} = -\sum_{i=1}^k (\partial^2/\partial x_i^2 + x_i^2) + 1$, instead of $\tilde{H} = H^{\otimes k}$, is used to define the norms $|\cdot|_{2,p}$ and $\|\cdot\|_{\beta,p}$ on $\mathcal{S}(\mathbb{R}^k)$ and $\mathcal{P}$, respectively. Using the Hermite basis in $\mathcal{S}(\mathbb{R}^k)$ one can show that the estimates $|\hat{H}^p\xi|_2 \le |\tilde{H}^p\xi|_2 \le |\hat{H}^{(k+1)p}\xi|_2$ hold for all $\xi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^k)$. Similarly,

$$\|\Gamma_{\hat{H}}^p \Phi_n\|_2 \le \|\Gamma_{\tilde{H}}^p \Phi_n\|_2 \le \|\Gamma_{\hat{H}}^{(k+1)p} \Phi_n\|_2,$$

for all $\Phi \in \mathcal{P}$. Therefore the spaces $(\mathcal{S})^\beta$ obtained by the two choices of operators coincide as topological spaces, and the same holds for their duals $(\mathcal{S})^{-\beta}$.

A basic tool in white noise analysis is the S-transformation. For $\beta \in [0,1)$ it is defined via the dual pairing $\langle \cdot, \cdot \rangle$ between $(\mathcal{S})^{-\beta}$ and $(\mathcal{S})^\beta$,

$$S: \ (\mathcal{S})^{-\beta} \to \mathcal{U}^\beta, \quad S\Phi(\xi) := \langle \Phi, :e^{X_\xi}: \rangle. \tag{3.3}$$

We recall the definition of $\mathcal{U}^\beta$ in (3.3). For $\beta \in [0,1)$ the space $\mathcal{U}^\beta$ consists of all functions $u : \mathcal{S}_{\mathbb{C}}(\mathbb{R}^k) \to \mathbb{C}$ which satisfy:
(U1) For all $\xi, \eta \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^k)$, the mapping $z \mapsto u(\xi + z\eta)$ is entire.
(U2) There exist $K_1, K_2 > 0$ and $p \in \mathbb{N}_0$ such that for all $\xi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^k)$,

$$|u(\xi)| \le K_1 \exp\{K_2 |\xi|_{2,p}^{2/(1-\beta)}\}.$$

The space $\mathcal{U}^\beta$ can be topologized as follows. For $p \in \mathbb{N}_0$ define

$$\mathcal{U}_p^\beta := \{u \in \mathcal{U}^\beta : |u|_{\beta,p} < \infty\},$$

where

$$|u|_{\beta,p} := \sup_{\xi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^k)} |u(\xi)| \exp\{-|\xi|_{2,p}^{2/(1-\beta)}\}.$$

Then $(\mathcal{U}_p^\beta, |\cdot|_{\beta,p})$ is a Banach space. (This follows from [BDP, Prop.3] as a simple special case.) In view of (3.1) it is clear that $\mathcal{U}_p^\beta \subset \mathcal{U}_{p+1}^\beta$, $|u|_{\beta,p} \ge |u|_{\beta,p+1}$ for all $u \in \mathcal{U}_p^\beta$, and that the constant K_2 in (U2) can be absorbed into an appropriate norm: if $n \in \mathbb{N}_0$ is such that $2^{2n} \ge K_2^{1-\beta}$ one finds $\exp\{K_2|\xi|_{2,p}^{2/(1-\beta)}\} \le \exp\{|\xi|_{2,p+n}^{2/(1-\beta)}\}$. From this we obtain

$$\mathcal{U}_0^\beta \subset \mathcal{U}_1^\beta \subset \cdots \subset \bigcup_{p \in \mathbb{N}_0} \mathcal{U}_p^\beta = \mathcal{U}^\beta. \tag{3.4}$$

We equip $\mathcal{U}^\beta$ with the inductive limit topology τ_{ind} of the Banach spaces $\mathcal{U}_p^\beta$.

The S-transformation (3.3) is also defined for $\beta = 1$, but only for those $\xi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^k)$ which satisfy $|\xi|_{2,p} < 1$, where $p \in \mathbb{N}_0$ is the smallest index such that $\Phi \in (\mathcal{S})^{-1}_{-p}$. The pairing $\langle \cdot, \cdot \rangle$ in (3.3) is then the one between $(\mathcal{S})^{-1}_{-p}$ and $(\mathcal{S})^{1}_{p}$, and $\mathcal{U}^1$ is defined as follows:

Let $\mathcal{U}^1 := \mathrm{Hol}_0(\mathcal{S}_{\mathbb{C}}(\mathbb{R}^k))$ be the algebra of germs of functions holomorphic at $0 \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^k)$. (For the notion of holomorphy in locally convex spaces we refer to [Di].) Each germ $\hat{u} \in \mathcal{U}^1$ can be represented by a holomorphic function u defined on an open neighborhood $\mathcal{V} \subset \mathcal{S}_{\mathbb{C}}(\mathbb{R}^k)$ of zero. In particular, there exists $p \in \mathbb{N}_0$ and $\delta > 0$ such that u is defined on

$$\mathcal{V}_p(\delta) := \{\xi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^k) : |\xi|_{2,p} < \delta\}, \tag{3.5}$$

and moreover that u is bounded on $\mathcal{V}_p(\delta)$. Notice that when $n \in \mathbb{N}_0$ is such that $2^{-n} \leq \delta$ it follows from (3.1) that

$$\mathcal{V}_{p+n}(1) \subset \mathcal{V}_p(\delta). \tag{3.6}$$

This implies that for each $\hat{u} \in \mathcal{U}^1$ there exists a representative u defined on $\mathcal{V}_p := \mathcal{V}_p(1)$, if p is chosen large enough. These observations motivate the following definition: let $\mathcal{U}^1_p$ be the space of $\hat{u} \in \mathcal{U}^1$, for which there exists a representative u defined on $\mathcal{V}_p$. From $\mathcal{V}_{p+1} \subset \mathcal{V}_p$ and the remarks given above we conclude that (3.4) also holds for $\beta = 1$. The uniqueness theorem for holomorphic functions implies that there exists exactly one representative u of $\hat{u} \in \mathcal{U}^1_p$ with $\mathcal{V}_p$ as domain of definition. This fact combined with boundednes of u allows us to define the following norm on $\mathcal{U}^1_p$:

$$|\hat{u}|_{1,p} := \sup_{\xi \in \mathcal{V}_p} |u(\xi)|.$$

From $\mathcal{V}_{p+1} \subset \mathcal{V}_p$ we immediately obtain $|\hat{u}|_{1,p} \geq |\hat{u}|_{1,p+1}$, for all $\hat{u} \in \mathcal{U}^1_p$. Henceforth we equip $\mathcal{U}^1$ with the inductive limit topology of the Banach spaces $\mathcal{U}^1_p$, $p \in \mathbb{N}_0$. The 1:1–correspondence of $\hat{u} \in \mathcal{U}^1_p$ with a bounded holomorphic function u on $\mathcal{V}_p$ allows us to simply identify these objects. For the remainder of the paper we make this identification.

Summarizing, we can write for all $\beta \in [0, 1]$

$$|u|_{\beta,p} = \sup_{\xi \in \mathcal{V}^\beta_p} \{|u(\xi)| \cdot w_{\beta,p}(\xi)\},$$

if we introduce the weight functions $w_{\beta,p}$ and the domains $\mathcal{V}^\beta_p$:

$$w_{\beta,p}(\xi) := e^{-|\xi|^{2/(1-\beta)}_{2,p}}, \quad \mathcal{V}^\beta_p := \mathcal{S}_{\mathbb{C}}(\mathbb{R}^k), \quad \text{for } \beta \in [0, 1), \tag{3.7a}$$

and

$$w_{1,p}(\xi) := 1, \quad \mathcal{V}^1_p := \mathcal{V}_p. \tag{3.7b}$$

With these preparations we can now state a fundamental result in WNA.

Theorem 3.1. *For each $\beta \in [0, 1]$ the S–transformation* (3.3) *is a shift isomorphism from $(\mathcal{S})^{-\beta}$ onto $\mathcal{U}^\beta$.*

Remark. The main part of the proof of Theorem 3.1 is to show that S is a linear isomorphism, while the continuity of S and S^{-1} essentially follows as a by–product of this proof, cf. [PS], [KLPSW],[BT] and [KSWY]. In the proof one shows norm estimates of type (I2), as stated in Lemma 2.2. We calculated the constants in these estimates explicitly for our present needs. The result reads as follows:

Let $\beta \in [0,1]$, $p \in \mathbb{N}_0$ and $\Phi \in (\mathcal{S})^{-\beta}_{-p}$. Then $S\Phi \in \mathcal{U}^{\beta}_{p+1}$, and

$$|S\Phi|_{\beta,p+1} \leq 2\|\Phi\|_{-\beta,-p}. \tag{3.8a}$$

Let $\beta \in [0,1]$, $p \in \mathbb{N}_0$ and $u \in \mathcal{U}^{\beta}_{p}$. Then $S^{-1}u \in (\mathcal{S})^{-\beta}_{-p'}$, and

$$\|S^{-1}u\|_{-\beta,-p'} \leq 2|u|_{\beta,p}. \tag{3.8b}$$

In case $\beta \in [0,1)$ we have $p' = p+4$, and in case $\beta = 1$ we have $p' = p+k+1$. (Recall that k comes from $\mathcal{S}(\mathbb{R}^k)$.)

An immediate consequence of Theorem 3.1 is the following. Since $u_1, u_2 \in \mathcal{U}^{\beta}$ implies that the pointwise defined product $u_1 \cdot u_2$ is also in $\mathcal{U}^{\beta}$ one can define the *Wick product* for $\Phi, \Psi \in (\mathcal{S})^{-\beta}$ as follows:

$$\Phi \diamond \Psi := S^{-1}(S\Phi \cdot S\Psi).$$

4. Isomorphisms between spaces of bounded continuous functions

This section contains the main technical part of the present paper. It provides some insight in the spaces of bounded continuous functions with values in $(\mathcal{S})^{-\beta}$ and in $\mathcal{U}^{\beta}$. For the remainder of this paper we make the convention that a statement holds for all $\beta \in [0,1]$ if the domain for β is not explicitly given.

The isomorphism property of the S–transformation implies that a set $B \subset (\mathcal{S})^{-\beta}$ is bounded in $(\mathcal{S})^{-\beta}$ if and only if $S(B)$ is bounded in $\mathcal{U}^{\beta}$. In what follows it will be useful that bounded sets in $(\mathcal{S})^{-\beta}$ and $\mathcal{U}^{\beta}$ can be characterized in a simple way:

Lemma 4.1. *Let E stand for $(\mathcal{S})^{-\beta}$ or $\mathcal{U}^{\beta}$, and E_p for $(\mathcal{S})^{-\beta}_{-p}$ or $\mathcal{U}^{\beta}_{p}$. Then $B \subset E$ is strongly bounded if and only if there exists $p \in \mathbb{N}_0$ such that B is a bounded subset of E_p.*

Proof. It is well–known that strongly bounded sets in the dual of a countably Banach space are characterized as stated by the lemma, see e.g. [Co, Theorem 1.12]. It thus remains to prove the claim for $E = \mathcal{U}^{\beta}$ and $E_p = \mathcal{U}^{\beta}_{p}$.

Let $B \subset \mathcal{U}^{\beta}$ be bounded. Then $S^{-1}(B) \subset (\mathcal{S})^{-\beta}$ is bounded, i.e. there exists $q \in \mathbb{N}_0$ such that $\|S^{-1}u\|_{-\beta,-q} \leq c$ for all $u \in B$. From (3.8a) we obtain

$$|u|_{\beta,q+1} \leq 2\|S^{-1}u\|_{-\beta,-q} \leq 2c, \quad \text{for all } u \in B.$$

Thus B is bounded in $\mathcal{U}^{\beta}_{p}$, with $p = q+1$.

Conversely, let $B \subset \mathcal{U}^{\beta}_{p}$ be bounded. Then $B \subset \mathcal{U}^{\beta}$ is bounded too, because the embedding $i_p : \mathcal{U}^{\beta}_{p} \hookrightarrow \mathcal{U}^{\beta}$ is linear and continuous. ■

Notice that the argument in the proof used for $(\mathcal{S})^{-\beta}$ does not apply for $\mathcal{U}^\beta$, because $\mathcal{U}^\beta$ is not a dual space. Instead, we had to use the S-transformation. In what follows we denote by $B(M, E)$ *the space of bounded mappings* $f : M \to E$, i.e. those f for which $f(M)$ is a bounded subset of E.

Corollary 4.2. *For* $\beta \in [0,1)$ *and any set* M *the following holds:*

(a) $\Phi \in B(M, (\mathcal{S})^{-\beta}) \iff$ *There exists* $p \in \mathbb{N}_0$ *and* $K_1, K_2 \geq 0$ *such that*

$$|S\Phi(x)(\xi)| \leq K_1 \exp\{K_2 |\xi|_{2,p}^{2/(1-\beta)}\}, \qquad \textit{for all } x \in M. \tag{4.1}$$

(b) *Let* $\Phi, \Psi \in B(M, (\mathcal{S})^{-\beta})$ *and define* $\Phi \diamond \Psi(x) := \Phi(x) \diamond \Psi(x)$, *for all* $x \in M$. *Then* $\Phi \diamond \Psi \in B(M, (\mathcal{S})^{-\beta})$.

Proof. (a) $\Phi \in B(M, (\mathcal{S})^{-\beta})$ implies that $S\Phi(M)$ is bounded in $\mathcal{U}^\beta$. Thus there exists $c > 0$ such that

$$\sup_{\xi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^k)} |S\Phi(x)(\xi)| \exp\{-|\xi|_{2,p}^{2/(1-\beta)}\} \leq c, \qquad \text{for all } x \in M,$$

and some $p \in \mathbb{N}_0$. Now (4.1) follows with $K_1 = c$, $K_2 = 1$. Conversely assume (4.1). From (3.1) we obtain

$$|S\Phi(x)(\xi)| \leq K_1 \exp\{|\xi|_{2,p+n}^{2/(1-\beta)}\}, \qquad \text{for all } x \in M,$$

and some $n \in \mathbb{N}_0$. Therefore $S\Phi(M)$ is bounded in $\mathcal{U}_{p+n}^\beta$ which implies $\Phi \in B(M, (\mathcal{S})^{-\beta})$. To prove (b) observe that (4.1) holds for Φ, Ψ with constants p, K_1 and K_2 indexed by Φ and Ψ. With $p = \max\{p_\Phi, p_\Psi\}$ we can estimate

$$\begin{aligned} |S(\Phi(x) \diamond \Psi(x))(\xi)| &= |S\Phi(x)(\xi) \cdot S\Psi(x)(\xi)| \\ &\leq K_1^\Phi K_1^\Psi \exp\{(K_2^\Phi + K_2^\Psi)|\xi|_{2,p}^{2/(1-\beta)}\}. \end{aligned}$$

■

Remarks. 1. The estimate (4.1) arises frequently as a technical condition in the context of integration or differentiation w.r.t. $x \in M$, when $M = \mathbb{R}^n$. It also arises in the context of limits of sequences $(\Phi_n)_{n\in\mathbb{N}}$ in $(\mathcal{S})^{-\beta}$, i.e. when $M = \mathbb{N}$, cf. [PS], [KLS], [DPVW]. We find it worth mentioning that this somewhat clumsy condition on Φ just states that Φ is a strongly bounded function. 2. Part (b) of the corollary is sometimes useful when one considers stochastic differential equations, because the noise terms (and maybe also other terms) are typically multiplied in the Wick sense.

For the application in Section 5 we have to consider spaces $C_b(M, E)$ of bounded continuous functions on $M \subset \mathbb{R}^n$ with values in some topological vector space E. When $(E, \|\cdot\|)$ is a Banach space and (M, d) a metric space we define the norm

$$\|f\|_\infty := \sup_{x \in M} \|f(x)\|, \quad f \in C_b(M, E).$$

Then $(C_b(M,E), \|\cdot\|_\infty)$ is a Banach space, too. In particular we can topologize the space $C_b(M,(\mathcal{S})^{-\beta}_{-p})$ with the norm $\|\cdot\|_{-\beta,-p,\infty}$, and $C_b(M,\mathcal{U}^\beta_p)$ with the norm $|\cdot|_{\beta,p,\infty}$.

We denote by $C_b^w(M,(\mathcal{S})^{-\beta})$ and $C_b^s(M,(\mathcal{S})^{-\beta})$ the spaces of weakly, respectively strongly continuous bounded functions on M. In view of $\tau_{\text{weak}} \neq \tau_{\text{strong}}$ one would expect $C_b^w(M,(\mathcal{S})^{-\beta}) \neq C_b^s(M,(\mathcal{S})^{-\beta})$. The following lemma states however that these spaces actually coincide. Because of the continuous embedding $(\mathcal{S})^{-\beta}_{-p} \hookrightarrow (\mathcal{S})^{-\beta}$ we will subsequently identify the space $C_b(M,(\mathcal{S})^{-\beta}_{-p})$ with a subspace of $C_b(M,(\mathcal{S})^{-\beta})$.

Lemma 4.3. *Let M be a metric space. Then*

$$C_b^w(M,(\mathcal{S})^{-\beta}) = C_b^s(M,(\mathcal{S})^{-\beta}) = \bigcup_{p\in\mathbb{N}_0} C_b(M,(\mathcal{S})^{-\beta}_{-p}).$$

Proof. Let $\Phi \in \cup_{p\in\mathbb{N}_0} C_b(M,(\mathcal{S})^{-\beta}_{-p})$. Then the mapping $x \mapsto \Phi(x)$ from M into $(\mathcal{S})^{-\beta}_{-p}$ is continuous and bounded for an appropriate p. Since the embedding $(\mathcal{S})^{-\beta}_{-p} \hookrightarrow (\mathcal{S})^{-\beta}$ is continuous with respect to τ_{strong} we obtain $\Phi \in C_b^s(M,(\mathcal{S})^{-\beta})$. From $\tau_{\text{weak}} \subset \tau_{\text{strong}}$ we find $C_b^s(M,(\mathcal{S})^{-\beta}) \subset C^w(M,(\mathcal{S})^{-\beta})$. Since the strongly bounded sets coincide with the weakly bounded sets, see [Co, Theorem 1.14], we finally obtain

$$\bigcup_{p\in\mathbb{N}_0} C_b(M,(\mathcal{S})^{-\beta}_{-p}) \subset C_b^s(M,(\mathcal{S})^{-\beta}) \subset C_b^w(M,(\mathcal{S})^{-\beta}). \tag{4.2}$$

Assume conversely $\Phi \in C_b^w(M,(\mathcal{S})^{-\beta})$. Then $\Phi(M)$ is strongly bounded. In view of Lemma 4.1 there exists $p \in \mathbb{N}_0$ such that $\Phi(M) \subset (\mathcal{S})^{-\beta}_{-p}$ is bounded. Let $\varphi \in (\mathcal{S})^\beta$ and consider

$$(\Phi(x),\varphi)_{-\beta,-p} = (\Gamma_{\tilde{H}}^{-p}\Phi(x), \Gamma_{\tilde{H}}^{-p}\varphi)_2 = \langle\overline{\Phi(x)}, \Gamma_{\tilde{H}}^{-2p}\varphi\rangle.$$

From $\Gamma_{\tilde{H}}^{-2p}\varphi \in (\mathcal{S})^\beta$ and $\Phi \in C_b^w(M,(\mathcal{S})^{-\beta})$ we now obtain that

$$(\Phi(x_n),\varphi)_{-\beta,-p} \to (\Phi(x),\varphi)_{-\beta,-p}, \quad \text{if} \quad x_n \to x. \tag{4.3}$$

Since $(\mathcal{S})^\beta$ is dense in $(\mathcal{S})^{-\beta}_{-p}$ and $\Phi(M)$ is bounded in $(\mathcal{S})^{-\beta}_{-p}$ it follows that (4.3) extends to all $\varphi \in (\mathcal{S})^{-\beta}_{-p}$. This shows that $\Phi(x_n)$ is a weakly convergent sequence in $(\mathcal{S})^{-\beta}_{-p}$. But weakly convergent sequences are mapped to strongly convergent sequences by compact operators (see [RS]). Since the embedding $(\mathcal{S})^{-\beta}_{-p} \hookrightarrow (\mathcal{S})^{-\beta}_{-p-1}$ is Hilbert–Schmitt, it is in particular compact and thus $\Phi(x_n)$ converges to $\Phi(x)$ with respect to $\|\cdot\|_{-\beta,-p-1}$–norm. This shows that $\Phi \in C_b(M,(\mathcal{S})^{-\beta}_{-p-1})$ and thus

$$C_b^w(M,(\mathcal{S})^{-\beta}) \subset \bigcup_{p\in\mathbb{N}_0} C_b(M,(\mathcal{S})^{-\beta}_{-p}).$$

In view of (4.2) this concludes the proof. ■

In the following we simply write $C_b(M, (\mathcal{S})^{-\beta})$ for both topologies on $(\mathcal{S})^{-\beta}$. Because of $C_b(M, (\mathcal{S})^{-\beta}_{-p}) \subset C_b(M, (\mathcal{S})^{-\beta}_{-p-1})$, and $C_b(M, (\mathcal{S})^{-\beta}) = \cup_{p\in\mathbb{N}_0} C_b(M, (\mathcal{S})^{-\beta}_{-p})$ we equip $C_b(M, (\mathcal{S})^{-\beta})$ with the inductive limit topology of the spaces $C_b(M, (\mathcal{S})^{-\beta}_{-p})$.

We next investigate the space $C_b(M, \mathcal{U}^\beta)$ in more detail. First consider $u \in B(M, \mathcal{U}^\beta_p)$. This function u is by definition continuous in $x \in M$ iff for each $\epsilon > 0$ there exists $\delta > 0$ such that $d(x, \bar{x}) \leq \delta$ implies

$$|u(x) - u(\bar{x})|_{\beta,p} = \sup_{\xi\in\mathcal{V}^\beta_p} \{|u(x)(\xi) - u(\bar{x})(\xi)| w_{\beta,p}(\xi)\} \leq \epsilon.$$

Thus u is continuous if and only if the family of functions $u_\xi(x) := u(x)(\xi) w_{\beta,p}(\xi)$ (with family parameter ξ) is equicontinuous. For convenience we denote this by saying that u is *p–equicontinuous.* Of course, p-equicontinuity is a property which is harder to verify than continuity for every *fixed* ξ. This motivates the definition of the following spaces:

For $p \in \mathbb{N}_0$ let $\mathcal{U}^\beta_p(M)$ be the space of all mappings $u : M \to \mathcal{U}^\beta_p$ which satisfy the following boundednes and (pointwise) continuity condition:

(B) $|u|_{\beta,p,\infty} := \sup_{x\in M} |u(x)|_{\beta,p} < \infty$.

(C) For each $\xi \in \mathcal{V}^\beta_p$ the map $x \mapsto u(x)(\xi)$ is continuous on M.

Notice that the definition of $|\cdot|_{\beta,p,\infty}$ on the space $\mathcal{U}^\beta_p(M)$ coincides with the one for $|\cdot|_{\beta,p,\infty}$ on the space $C_b(M, \mathcal{U}^\beta_p)$.

Remark. For $\beta \in [0, 1)$ one can identify the space of mappings $\mathcal{U}^\beta_p(M)$ with a space of functions $u : M \times \mathcal{S}_{\mathbb{C}}(\mathbb{R}^k) \to \mathbb{C}$ via

$$u(x, \xi) \equiv u(x)(\xi), \tag{4.4}$$

where u must satisfy (B), (C) and (U1). For $\beta = 1$ we remark that a bounded function $u : \mathcal{V}_p \to \mathbb{C}$ is holomorphic if and only if the following holds (see [Di, Section 2.2]):

(U1') For all $\xi, \eta \in \mathcal{V}_p$ there exists an open set $V_{\xi,\eta}$ around zero in $\mathbb{C}$ such that the function $f(z) := u(\xi + z\eta)$ is holomorphic in $V_{\xi,\eta}$.

This characterization implies that the domain of holomorphy $V_{\xi,\eta}$ of f can always be extended to the natural domain of f, i.e. to $\mathcal{O}_{\xi,\eta} := \{z \in \mathbb{C} : |\xi + z\eta|_{2,p} < 1\}$. Therefore one can identify $\mathcal{U}^1_p(M)$ via (4.4) with the space of functions $u : M \times \mathcal{V}^1_p \to \mathbb{C}$ which satisfy (B), (C) and (U1'). This is the point of view adopted in [BDP]. (To be precise, in [BDP] the spaces $\mathcal{U}^\beta_{p,l}(M)$ instead of $\mathcal{U}^\beta_p(M)$ are considered; these spaces are discussed in Section 5.) By a trivial modification of the proof given for Prop.3 in [BDP] it follows that $(\mathcal{U}^\beta_p(M), |\cdot|_{\beta,p,\infty})$ is a Banach space.

In view of (3.4) it is clear that $\mathcal{U}^\beta_p(M) \subset \mathcal{U}^\beta_{p+1}(M)$ for all $p \in \mathbb{N}_0$, and that $|u|_{\beta,p,\infty} \geq |u|_{\beta,p+1,\infty}$ for all $u \in \mathcal{U}^\beta_p(M)$. We put the inductive limit topology τ_{ind} on the space

$$\mathcal{U}^\beta(M) := \bigcup_{p\in\mathbb{N}_0} \mathcal{U}^\beta_p(M).$$

The following theorem is basic for our applications.

Theorem 4.4. *Let M be a metric space. Then the composition $\hat{S} : \Phi \mapsto S \circ \Phi$ is a shift isomorphism from $C_b(M, (\mathcal{S})^{-\beta})$ onto $\mathcal{U}^\beta(M)$.*

Proof. Let $\Phi \in C_b(M; (\mathcal{S})^{-\beta})$. Since $S : (\mathcal{S})^{-\beta} \to \mathcal{U}^\beta$ is linear and continuous we have $S \circ \Phi \in C_b(M, \mathcal{U}^\beta)$. Moreover,

$$C_b(M, \mathcal{U}^\beta) \subset \mathcal{U}^\beta(M), \tag{4.5}$$

because p-equicontinuity implies continuity for fixed ξ. Thus $S \circ \Phi \in \mathcal{U}^\beta(M)$. Lemma 4.3 shows that $\Phi \in C_b(M, (\mathcal{S})^{-\beta}_{-p})$ for some $p \in \mathbb{N}_0$. The estimate (3.8a) yields

$$\sup_{x \in M} |S\Phi(x)|_{\beta,p+1} \leq 2 \sup_{x \in M} \|\Phi(x)\|_{-\beta,-p} < \infty.$$

This shows $|\hat{S}\Phi|_{\beta,p+1,\infty} \leq 2\|\Phi\|_{-\beta,-p,\infty}$, and we obtain with Lemma 2.2 that $\Phi \mapsto S \circ \Phi$ defines a continuous mapping from $C_b(M, (\mathcal{S})^{-\beta})$ into $\mathcal{U}^\beta(M)$. Injectivity of $\hat{S}$ follows from the injectivity of S.

Now let $u \in \mathcal{U}^\beta(M)$. Then we have $|u(x)|_{\beta,p} \leq c < \infty$ for all $x \in M$ and some $c \geq 0$, $p \in \mathbb{N}_0$. Using the estimate (3.8b) we obtain

$$\|\hat{S}^{-1}u\|_{-\beta,-p',\infty} \leq 2|u|_{\beta,p,\infty} \leq 2c, \tag{4.6}$$

where p' is chosen appropriately. This shows that $S^{-1}(u(M))$ is bounded in $(\mathcal{S})^{-\beta}_{-p'}$. We also have by assumption that $x \mapsto u(x)(\xi)$ is continuous for each ξ. As in the proof of Lemma 4.3 we conclude from

$$u(x)(\xi) = \langle S^{-1}u(x), :e^{X_\xi}:\rangle = (\overline{S^{-1}u(x)}, :e^{X_{H^{2p'}\xi}}:)_{-\beta,-p'}$$

that $x \mapsto S^{-1}u(x)$ is strongly continuous from M into $(\mathcal{S})^{-\beta}_{-p'-1}$. (This time we use that span$[:e^{X_\xi}:, \xi \in \mathcal{V}^\beta_{p'}]$ is dense in $(\mathcal{S})^{-\beta}_{-p'}$.) Thus $\hat{S}^{-1}u \in C_b(M, (\mathcal{S})^{-\beta})$. Again, injectivity of $\hat{S}^{-1}$ follows from the one of S^{-1}. The estimate (4.6) and Lemma 2.2 imply that $\hat{S}^{-1}$ is continuous. ■

Corollary 4.5. *Let M be a metric space, then*

$$\bigcup_{p \in \mathbb{N}_0} \mathcal{U}^\beta_p(M) = \bigcup_{p \in \mathbb{N}_0} C_b(M, \mathcal{U}^\beta_p) = C_b(M, \mathcal{U}^\beta).$$

The topology τ_{ind} defined by the spaces $\mathcal{U}^\beta_p(M)$ coincides with τ_{ind} defined by the spaces $C_b(M, \mathcal{U}^\beta_p)$. In particular, $\hat{S}$ is a shift isomorphism from $C_b(M, (\mathcal{S})^{-\beta})$ onto $C_b(M, \mathcal{U}^\beta)$.

Proof. We first show the equality

$$\bigcup_{p \in \mathbb{N}_0} \mathcal{U}^\beta_p(M) = \bigcup_{p \in \mathbb{N}_0} C_b(M, \mathcal{U}^\beta_p). \tag{4.7}$$

" $\subset$ ": Let $u \in \mathcal{U}_p^\beta(M)$. From the proof of Theorem 4.4 we find that $\Phi := \hat{S}^{-1}u \in C_b(M, (\mathcal{S})_{-p'-1}^{-\beta})$, and (3.8a) yields

$$|u(x)|_{\beta,p'+2} = |S\Phi(x)|_{\beta,p'+2} \le 2\|\Phi(x)\|_{-\beta,-(p'+1)} \le c, \text{ for all } x \in M.$$

Similarly, we obtain

$$|u(\bar{x}) - u(x)|_{\beta,p'+2} \le 2\|\Phi(\bar{x}) - \Phi(x)\|_{-\beta,-(p'+1)}, \text{ for all } \bar{x}, x \in M.$$

This implies $u \in C_b(M, \mathcal{U}_{p'+2}^\beta)$.
" $\supset$ ": Let $u \in C_b(M, \mathcal{U}_p^\beta)$. Then the defining condition (B) for $\mathcal{U}_p^\beta(M)$ is obviously satisfied, and (C) follows from (4.5), thus $u \in \mathcal{U}_p^\beta(M)$.

We next show that the inductive limit topologies on $\bigcup \mathcal{U}_p^\beta(M)$ and on $\bigcup C_b(M, \mathcal{U}_p^\beta)$ coincide: The embedding $i : \mathcal{U}_p^\beta(M) \hookrightarrow \cup_{p\in\mathbb{N}_0} C_b(M, \mathcal{U}_p^\beta)$ is continuous, because of

$$|iu|_{\beta,p'+2,\infty} = |u|_{\beta,p'+2,\infty} \le |u|_{\beta,p,\infty}.$$

The embedding $i : C_b(M, \mathcal{U}_p^\beta) \hookrightarrow \mathcal{U}^\beta(M)$ is continuous, because

$$|iu|_{\beta,p,\infty} = |u|_{\beta,p,\infty}.$$

Lemma 2.2 now implies the equality of the inductive limit topologies.

Finally, we show the non–obvious part of the equality

$$\bigcup_{p\in\mathbb{N}_0} C_b(M, \mathcal{U}_p^\beta) = C_b(M, \mathcal{U}^\beta).$$

For $u \in C_b(M, \mathcal{U}^\beta)$ the set $u(M)$ is bounded, thus $S^{-1}(u(M))$ is bounded. Moreover, $S^{-1} \circ u : M \to (\mathcal{S})^{-\beta}$ is continuous, thus $S^{-1} \circ u \in C_b(M, (\mathcal{S})^{-\beta})$. With Theorem 4.4 we obtain $u \in \mathcal{U}^\beta(M)$, and (4.7) concludes the proof. ∎

Remark. The equality (4.7) is somewhat unexpected. It implies that every bounded function $f : M \to \mathcal{U}^\beta$ which is continuous for fixed ξ is automatically p-equicontinuous, for an appropriate p. (More precisely, the proof of Corollary 4.5 shows that $\mathcal{U}_p^\beta(M) \subset C_b(M, \mathcal{U}_{p'+2}^\beta)$, where p' is given after (3.8b).) We did not find a proof of this last statement which avoids the use of the S–transformation. The crucial point is the observation made in the proof of Lemma 4.3: A weakly continuous function with values in $(\mathcal{S})^{-\beta}$ is automatically strongly continuous, because the embeddings $(\mathcal{S})_{-p}^{-\beta} \hookrightarrow (\mathcal{S})_{-p-1}^{-\beta}$ are compact operators. This type of argument is not available for U-functionals. Instead, we had to go back and forth with the S-transformation to prove (4.7).

5. Application to non–linear stochastic heat equations

In order to solve the fixed point equation $x = y + Kx$ in a Banach space $(E, \|\cdot\|)$ it is sometimes possible to introduce a "weighted" norm $\|\cdot\|_w$ on E such that K becomes a contraction operator with respect to this new norm. The most well–known example are ordinary differential equations (transformed to integral equations) which can be solved by Picard iteration on the Banach space $(C([0,T]), \|\cdot\|_\infty)$. In this case an appropriate weighted norm reads

$$\|f\|_w = \sup_{t\in[0,T]} |f(t)e^{-tc}|, \tag{5.1}$$

where $c > 0$ is some properly chosen constant, cf. [Mo,Wa]. In [BDP] a similar method has been worked out for so–called Banach spaces of U–functionals. These spaces are closely related to the $\mathcal{U}_p^\beta(M)$ from the previous section, for the choice $M = D_T = [0,T] \times \mathbb{R}^d$. The spaces $\mathcal{U}_{p,l}^\beta(D_T)$, $l \in \mathbb{N}$, introduced in [BDP] differ from the $\mathcal{U}_p^\beta(D_T)$ essentially by a time dependent weight factor similar to (5.1), cf. (3.7):

$$w_{\beta,p,l}(t,\xi) := e^{-l\{t+(1+t)|\xi|_{2,p}^{2/(1-\beta)}\}}, \quad \mathcal{V}_p^\beta(\delta) := \mathcal{S}_\mathbb{C}(\mathbb{R}^k), \quad \text{for } \beta \in [0,1)$$

and

$$w_{1,p,l}(t,\xi) := e^{-tl}, \quad \mathcal{V}_p^1(\delta) := \mathcal{V}_p(\delta).$$

With these notations the spaces $\mathcal{U}_{p,l}^\beta(D_T)$ can be defined as follows. Let $\mathcal{U}_{p,l}^\beta(D_T)$ be the space of all mappings $u : D_T \to \mathcal{U}^\beta$ which satisfy the following boundednes and continuity conditions:

(B') $u(t,x)$ is defined on $\mathcal{V}_p^\beta(\delta)$ for all $(t,x) \in D_T$, and

$$|u|_{\beta,p,l} := \sup_{D_T \times \mathcal{V}_p^\beta(\delta)} |u(t,x)(\xi) w_{\beta,p,l}(t,\xi)| < \infty.$$

(C') For each $\xi \in \mathcal{V}_p^\beta(\delta)$ the map $(t,x) \mapsto u(t,x)(\xi)$ is continuous on D_T.

Remarks. 1. $\mathcal{U}_{p,l}^\beta(D_T)$ was defined in [BDP] as a space of complex valued functions on $D_T \times \mathcal{V}_p^\beta(\delta)$. With the identification $u(t,x)(\xi) \equiv u(t,x,\xi)$ the spaces obtained by these different definitions coincide. See also the remark in Section 4 which follows the definition of $\mathcal{U}_p^\beta(M)$. 2. The parameters $l \in \mathbb{N}$ and $\delta \in (0,1]$ were introduced in [BDP] in order to obtain contraction operators by an appropriate choice of l and δ. Since we do not use different values of δ for the definition of τ_{ind} (as in the following lemma) we suppressed δ in the notation of $\mathcal{U}_{p,l}^\beta(D_T)$.

Notice that $\mathbb{N}_0 \times \mathbb{N}$ is a directed system with respect to the usual partial ordering

$$(p,l) \prec (p',l') \iff p \le p',\ l \le l',$$

and that $\mathcal{U}_{p,l}^\beta(D_T) \subset \mathcal{U}_{p',l'}^\beta(D_T)$ and $|\cdot|_{\beta,p,l} \ge |\cdot|_{\beta,p',l'}$.

Lemma 5.1. *Let $\beta \in [0,1]$, then*

$$\bigcup_{p\in\mathbb{N}_0} \mathcal{U}_p^\beta(D_T) \;=\; \bigcup_{p\in\mathbb{N}_0, l\in\mathbb{N}} \mathcal{U}_{p,l}^\beta(D_T). \tag{5.2}$$

The topology τ_{ind} on $\mathcal{U}^\beta(D_T)$, defined by $\{\mathcal{U}_p^\beta(D_T) : p \in \mathbb{N}_0\}$, coincides with the topology τ_{ind}^w, defined by $\{\mathcal{U}_{p,l}^\beta(D_T) : p \in \mathbb{N}_0, l \in \mathbb{N}\}$.

Proof. We first show that the defining conditions (B,C) and (B',C') follow from each other for appropriate parameter values.

Case $\beta \in [0,1)$: Since the conditions (C) and (C') coincide in this case it suffices to consider (B) and (B'). Let $p \in \mathbb{N}_0, l \in \mathbb{N}$ and $u \in \mathcal{U}_p^\beta(D_T)$. A simple calculation gives

$$|u|_{\beta,p,l} \leq |u|_{\beta,p}, \tag{5.3}$$

from which we obtain (B'). Now let $p \in \mathbb{N}_0, l \in \mathbb{N}$ and $u \in \mathcal{U}_{p,l}^\beta(D_T)$. Then the estimate

$$|u|_{\beta,p'} \leq e^{lT} |u|_{\beta,p,l}, \tag{5.4}$$

is easily verified with $p' = p + n$, where $n \in I\!N$ is such that $l(1+T)|\xi|_{2,p}^{2/(1-\beta)} \leq |\xi|_{2,p+n}^{2/(1-\beta)}$. Thus (B) is satisfied (with p replaced by p').

Case $\beta = 1$: Let $p \in \mathbb{N}_0, l \in \mathbb{N}$ and $u \in \mathcal{U}_p^1(D_T)$. Since $\mathcal{V}_p(\delta) \subset \mathcal{V}_p(1)$ the estimate (5.3) also holds for $\beta = 1$, which implies (B'). (C') is an immediate consequence of (C) and of $\mathcal{V}_p(\delta) \subset \mathcal{V}_p(1)$. Now let $p \in \mathbb{N}_0, l \in \mathbb{N}$ and $u \in \mathcal{U}_{p,l}^1(D_T)$. From (3.6) and (B') we find that (5.4) holds, with $p' = p + n$, where $n \in \mathbb{N}$ is such that $\mathcal{V}_{p+n}(1) \subset \mathcal{V}_p(\delta)$. Thus (B') is satisfied (with p replaced by p').

From (5.3) we find $\mathcal{U}_p^\beta(D_T) \subset \mathcal{U}_{p,l}^\beta(D_T)$ for all $p \in \mathbb{N}_0, l \in \mathbb{N}$, and (5.4) implies $\mathcal{U}_{p,l}^\beta(D_T) \subset \mathcal{U}_{p'}^\beta(D_T)$. Since the spaces $\mathcal{U}_p^\beta(D_T)$ and $\mathcal{U}_{p,l}^\beta(D_T)$ form increasing chains w.r.t. their partial orderings we arrive at (5.2). In view of Lemma 2.2 the estimates (5.3) and (5.4) imply that the identity mapping on $\mathcal{U}^\beta(D_T)$ is continuous with respect to both inductive limit topologies. ■

Remark. Although $\tau_{\text{ind}} = \tau_{\text{ind}}^w$ we will keep the notation τ_{ind}^w in the following. The point is that the notion of a contraction operator depends on the defining norms of the inductive limit topology (the index w denotes the weighted norms). Notice that the situation for (5.1) is similar: the norms $\|\cdot\|_\infty$ and $\|\cdot\|_w$ generate the same topology on $C([0,1])$, but only w.r.t. $\|\cdot\|_w$ one obtains a contraction operator (for the Picard iteration).

We are now prepared to consider non-linear stochastic heat equations. Let A be a uniformly elliptic, second order differential operator on D_T. In [BDP] the following stochastic Cauchy problem was studied:

$$\begin{aligned} \frac{\partial\Phi}{\partial t} - A\Phi &= F(\Phi) + \nabla G(\Phi) \diamond N \\ \Phi|_{t=0} &= \Phi_0. \end{aligned} \tag{5.5}$$

We will not restate here all the conditions on A, on the non–linearities F, G and on the noise $N = (N_1, \ldots, N_d)$, for details we refer to [BDP, Sec.2.2]. The only property which is essential for the present paper is the fact that a solution of (5.5) must satisfy the following integral equation:

$$\Phi = \Psi_0 + K(\Phi). \tag{5.6}$$

Here Ψ_0 is expressed by a (weak) integral over Φ_0 and the heat kernel q of A,

$$\begin{aligned}\Psi_0(t,x) &:= \int_{\mathbb{R}^d} \Phi_0(y) q(t,x;0,y)dy, \quad t > 0,\\ \Psi_0(0,x) &:= \Phi_0(x).\end{aligned} \tag{5.7}$$

The non–linear integral operator K reads

$$\begin{aligned}K(\Phi)(t,x) = &\int_0^t \int_{\mathbb{R}^d} q(t,x;s,y)\{F(\Phi) - G(\Phi) \diamond \nabla N\}(s,y)dyds\\ &- \int_0^t \int_{\mathbb{R}^d} \nabla_y q(t,x;s,y) \cdot N(s,y) \diamond G(\Phi)(s,y)dyds.\end{aligned} \tag{5.8}$$

Remark. (5.6) is an immediate consequence of the concept of a *mild solution*: Instead of (5.5) one requires the weaker condition that an associated integral equation is satisfied. An integration by parts in this integral equation then leads to (5.6), see [BDP].

Our aim is to prove that (5.5) is well–posed in the mild sense. (The precise meaning of this is given in Theorem 5.3). We first consider Ψ_0 as a (linear) function of Φ_0.

Lemma 5.2. *Let $\Phi_0 \in C_b(\mathbb{R}^d, (\mathcal{S})^{-\beta})$ and Ψ_0 be given by (5.7). Then $(t,x) \mapsto \Psi_0(t,x)$ is an element in $C_b(D_T, (\mathcal{S})^{-\beta})$. Moreover, $\Phi_0 \mapsto \Psi_0$ is continuous from $C_b(\mathbb{R}^d, (\mathcal{S})^{-\beta})$ to $C_b(D_T, (\mathcal{S})^{-\beta})$.*

Proof: From Lemma 4.3 we obtain that there exists $p \in \mathbb{N}_0$ such that $\Phi_0 \in C_b(\mathbb{R}^d, (\mathcal{S})^{-\beta}_{-p})$. In particular there exists $c > 0$ such that

$$\|\Phi_0(y)\|_{-\beta,-p} \le c, \quad \text{for all } y \in \mathbb{R}^d.$$

Since $\int_{\mathbb{R}^d} q(t,x;0,y)dy = 1$ this estimate and (5.7) implies

$$\|\Psi_0(t,x)\|_{-\beta,-p} \le \sup_{y\in\mathbb{R}^d} \|\Phi_0(y)\|_{-\beta,-p} \le c, \quad \text{for all } (t,x) \in D_T. \tag{5.9}$$

By definition we have $S\Psi_0(0,x)(\xi) = S\Phi_0(x)(\xi)$, and

$$S\Psi_0(t,x)(\xi) = \int_{\mathbb{R}^d} S\Phi_0(y)(\xi) q(t,x;0,y)dy, \quad t > 0. \tag{5.10}$$

Since $\hat{S}\Phi_0 \in \mathcal{U}^\beta(\mathbb{R}^d)$ it follows from (5.10) and from standard fact about the heat kearnel, that $(t,x) \mapsto S\Psi_0(t,x)(\xi)$ is continuous on D_T for each ξ. The boundednes of Ψ_0, expressed by (5.9), then implies that $\hat{S}\Psi_0 \in \mathcal{U}^\beta(D_T)$. Theorem 4.4 now shows that $\Psi_0 \in C_b(D_T, (\mathcal{S})^{-\beta})$. Finally (5.9) gives $\|\Psi_0\|_{-\beta,-p,\infty} \leq \|\Phi_0\|_{-\beta,-p,\infty}$, so Lemma 2.2 yields the continuity of the map $\Phi_0 \mapsto \Psi_0$. ■

We next show that K maps $C_b(D_T, (\mathcal{S})^{-\beta})$ to itself. In [BDP] it has been proved that the functions

$$\begin{aligned} f(u)(t,x,\xi) &:= S(F(t,x,S^{-1}u(t,x,\cdot))(\xi), \\ g(u)(t,x,\xi) &:= S(G(t,x,S^{-1}u(t,x,\cdot))(\xi),, \\ n(t,x,\xi) &:= SN(t,x)(\xi) \end{aligned}$$

are such that $\hat{K} : \mathcal{U}^\beta(M) \to \mathcal{U}^\beta(M)$ defined by

$$\begin{aligned} (\hat{K}u)(t,x,\xi) := &\int_0^t \int_{\mathbb{R}^d} q(t,x;s,y)\{f(u) - g(u)\cdot\nabla n\}(s,y,\xi)dyds \\ &- \int_0^t \int_{\mathbb{R}^d} \nabla_y q(t,x;s,y) g(u)(s,y,\xi)dyds \end{aligned} \tag{5.11}$$

is a contraction operator on $\mathcal{U}^\beta_{p,l}(D_T)$, for all $p \geq p_0$, $l \geq l_0$, and for a fixed value of δ. It follows from Theorem 4.4 that the mapping $\hat{S}^{-1}(\hat{K}u) : (t,x) \mapsto S^{-1}\{(\hat{K}u)(t,x,\cdot)\}$ is an element in $C_b(D_T, (\mathcal{S})^{-\beta})$. Moreover, since the S^{-1}–transformation (for (t,x) fixed) commutes with the integrals and with ∇ in (5.11) it follows that $\hat{S}^{-1}(\hat{K}u)$ coincides with the r.h.s. of (5.8). We conclude that K maps $C_b(D_T, (\mathcal{S})^{-\beta})$ into itself and that

$$\hat{K} = \hat{S} \circ K \circ \hat{S}^{-1}. \tag{5.12}$$

Theorem 5.3. *The Cauchy problem* (5.5) *is well-posed in the mild sense, i.e. for every* $\Phi_0 \in C_b(\mathbb{R}^d, (\mathcal{S})^{-\beta})$ *a unique solution* $\Phi \in C_b(D_T, (\mathcal{S})^{-\beta})$ *of* (5.6) *exists, and the mapping* $\Phi_0 \mapsto \Phi$ *is continuous.*

Proof: Let $v \in \mathcal{U}^\beta(D_T)$ and consider the equation

$$u = v + \hat{K}(u). \tag{5.13}$$

Since $\hat{K}$ is a contraction operator on $(\mathcal{U}^\beta(D_T), \tau^w_{\text{ind}})$ we obtain from Theorem 2.1 that (5.13) has a unique solution which depends continuously on v. (5.12) and (5.13) imply

$$\hat{S}^{-1}u = \hat{S}^{-1}v + K(\hat{S}^{-1}u).$$

Since $\hat{S}^{-1}$ is a topological isomorphism from $\mathcal{U}^\beta(D_T)$ onto $C_b(D_T, \mathcal{U}^\beta)$ we find that for each $\Psi \in C_b(D_T, (\mathcal{S})^{-\beta})$ the equation

$$\Phi = \Psi + K(\Phi)$$

has a unique solution $\Phi_\Psi \in C_b(D_T, (\mathcal{S})^{-\beta})$. Moreover, $\Psi \mapsto \Phi_\Psi$ is a continuous map on $C_b(D_T, (\mathcal{S})^{-\beta})$. Since Ψ_0, given by (5.7), depends continuously on Φ_0 (Lemma 5.2) it follows that (5.6) has a unique solution which depends continuously on Φ_0. ■

Remark. The Cauchy problems and integral equations (e.g. the Volterra equation) discussed in [BDP] have all been treated by Banach's fixed point theorem. The major difference to (5.5) on the technical level is that one has to modify the definition of the spaces $\mathcal{U}^{\beta}_{p,l}(D_T)$ in an inessential way (essentially one has to replace D_T by other subsets of $\mathbb{R}^n$). It is therefore straightforward to adapt the results of the present paper to the needs of these examples. In particular Theorem 5.3 will hold – in slightly modified form – for these applications.

References

[BDP] F.E. Benth, T. Deck and J. Potthoff, *A white noise approach to a class of nonlinear stochastic heat equations*, J. Funct. Anal. 146 (1997), 382-415.

[BDPS] F.E. Benth, T. Deck, J. Potthoff and L. Streit, *Nonlinear evolution equations with gradient coupled noise*, Lett. Math. Phys. 43 (1998), 267-278.

[BT] F. E. Benth and M. Timpel, *Topological Aspects of the Characterization of Hida Distributions – A Remark*, Stochastics and Stochastics Reports 51 (1994), 293-299.

[Co] F. Constantinescu, *Distributionen und ihre Anwendungen in der Physik*, Teubner 1974.

[DP] T. Deck and J. Potthoff, *On a class of stochastic partial differential equations related to turbulent transport*, to appear in Prob. Theorie and Rel. Fields.

[DPVW] T. Deck, J. Potthoff, G. Vage and H. Watanabe, *Stability of solutions of PDE'S with random drift and viscosity limit.* to appear in Appl. Math. Optim.

[Di] S. Dineen, *Complex Analysis in Locally Convex Spaces*, Mathematical Studies 57, North Holland, Amsterdam, 1981.

[DZ] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Encyclopedia of Mathematics and its Applications 44, Cambridge Univ. Press 1992.

[HKPS] T. Hida, H.–H. Kuo, J. Potthoff and L. Streit, *White Noise – An Infinite Dimensional Calculus.* Dordrecht, Kluwer (1993).

[HØUZ] H. Holden, B. Øksendal, J. Uboe and T. Zhang, *Stochastic Partial differential Equations – A Modeling, White Noise Functional Approach*, Birkhäuser, 1996.

[KLPSW] Y. Kondratiev, P. Leukert, J. Potthoff, L. Streit and W. Westerkamp, *Generalized functionals in Gaussian spaces: The characterization theorem revisited*, J. Funct. Anal. 141 (1996), 301–318.

[KSWY] Y. Kondratiev, L. Streit, W. Westerkamp and J. Yan, *Generalized Functions in Infinite Dimensional Analysis*, IIAS Reports No. 1995-002, Kyoto, 1995.

[KLS] Y. Kondratiev, P. Leukert and L. Streit, *Wick Calculus in Gaussian Analysis*, Acta Appl. Math. 44, 269-294 (1996).

[KS] Y. Kondratiev and L. Streit, *Spaces of white noise distributions: Constructions, Descriptions, Applications*, I. Rep. Math. Phys. 33 (1993), 341-366.

[Mo] D. Morgenstern, *Beiträge zur nichtlinearen Funktionalanalysis,* Dissertation, Technische Universität Berlin, 1952.

[PS] J. Potthoff and L. Streit: *A Characterization of Hida Distributions*, J. Funct. Anal. 101, pp. 212–229 (1991).

[Ro] A. and W. Robertson, *Topological Vector Spaces,* Cambridge Univ. Press 1973.

[RS] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I, Functional analysis*, Academic Press 1980.

[Wa] W. Walter, *Gewöhnliche Differentialgleichungen*, Springer 1972.