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Abstract. In this paper, we introduce a new family of nonseparable orthogonal wavelets on the quincunx grid arising from Butterworth wavelets with an odd number of vanishing moments. Our wavelets are closely related to bireciprocal wave digital filters.

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Running head. Wavelets on the quincunx grid

1 Introduction

Wavelet techniques have attained attention during the last years. Many applications, such as image processing, make use of bivariate wavelet bases. Most of these bivariate wavelets are simply tensor products of univariate wavelets which have a number of drawbacks e.g. the „preferred directions effect" due to their separabilty. Only a few nonseparable bivariate bases have been constructed. See [10, 14] for the design of orthogonal, not differentiable wavelets, [3, 11] for a biorthogonal approach and the new paper [1] for the first construction of orthogonal nonseparable wavelets with compact support and arbitrary high smoothness.
In this paper, we propose a new family of nonseparable orthogonal wavelets on the quincunx grid. Note that the quincunx grid is a very popular choice since the corresponding multiresolution involves only one scaling function and one wavelet [10, 14, 3]. Our construction follows in a simple way from the design of univariate Butterworth wavelets of odd degree from the point of view of wave digital filters. It has the advantage that the approach can be extended to higher dimensions and to four–channel perfect reconstruction filter banks [7]. Furthermore, our design gives rise to an efficient implementation in the frequency domain by using the polyphase decomposition and to an efficient sampling of the continuous function at the starting point of the wavelet analysis [7]. The proof of the smoothness of our wavelets is still open.

2 Univariate Butterworth wavelets

We consider the scaling function $\varphi_N \in L_2(\mathbf{R})$ satisfying the refinement relation

$$\begin{aligned} \varphi_N(x) &= 2\sum_{k\in Z} h_k\,\varphi(2x-k)\,, \\ \hat{\varphi}_N(v) &= H_N(e^{-iv/2})\hat{\varphi}_N(\frac{v}{2})\,, \qquad H(z) = \sum_{k\in Z} h_k\, z^k\,, \end{aligned}$$

where H_N denotes the *N-th order Butterworth filter* [15] arising from

$$\begin{aligned} |H_N(z)|^2 &= \frac{(z+1)^{2N}}{(z+1)^{2N}+(-1)^N(z-1)^{2N}} \\ &= \frac{(\cos\frac{v}{2})^{2N}}{(\cos\frac{v}{2})^{2N}+(\sin\frac{v}{2})^{2N}} \qquad (z := e^{-iv})\,. \qquad (2.1) \end{aligned}$$

For odd N $(1 < N < \infty)$, the poles of (2.1) are given by 0 and $\pm i \cot \frac{k\pi}{2N}$ $(k = 1, ..., N-1)$. From (2.1) we can obtain different filters H_N. As an example, let us choose the filter having the poles $\pm i \cot \frac{k\pi}{2N}$ $(k = 1, ..., (N-1)/2)$. With $\alpha_k := (\cot \frac{k\pi}{2N})^2$ $(k = 1, ..., (N-1)/2)$ our filter can be written as

$$H_N(z) = C_N \frac{(z+1)^N}{(z^2+\alpha_1)...(z^2+\alpha_{\frac{N-1}{2}})} = C_N \frac{(z+1)^N}{U_N(z^2)}, \tag{2.2}$$

where the constant $C_N := (1+\alpha_1)...(1+\alpha_{\frac{N-1}{2}})/2^N$ is chosen such that $H(1) = 1$.
For $N = 1$, we obtain $H_1(e^{-iv}) = (1+e^{-iv})/2$ and φ_1 is the characteristic function on [0,1).
For $N = \infty$, we see that

$$H_\infty(e^{iv}) = \begin{cases} 1 & v \in [\frac{\pi}{2}, \frac{\pi}{2}), \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

is the ideal lowpass filter which corresponds to the sinc–function φ_∞.

Remark. The above choice of the filter is of particular interest for practical computations. It is easy to check that H_N (N odd) possesses the following *polyphase decomposition*

$$H_N(z) = \frac{1}{2}(E_N(z^2) + zF_N(z^2)) \tag{2.4}$$

with the *allpass filters* E_N, F_N given by

$$\begin{aligned} E_N(z^2) &= H_N(z) + H_N(-z) \\ &= \frac{(\alpha_1 z^2+1)(\alpha_3 z^2+1)...(\alpha_M z^2+1)}{(z^2+\alpha_1)(z^2+\alpha_3)...(z^2+\alpha_M)}, \end{aligned}$$

and

$$\begin{aligned} F_N(z^2) &= \frac{1}{z}(H_N(z) - H_N(-z)) \\ &= \frac{(\alpha_2 z^2+1)(\alpha_4 z^2+1)...(\alpha_L z^2+1)}{(z^2+\alpha_2)(z^2+\alpha_4)...(z^2+\alpha_L)} \end{aligned}$$

with

$$M := \begin{cases} (N-3)/2 & \text{for } N \equiv 1 \bmod 4 \\ (N-1)/2 & \text{for } N \equiv 3 \bmod 4 \end{cases}, L := \begin{cases} (N-1)/2 & \text{for } N \equiv 1 \bmod 4 \\ (N-3)/2 & \text{for } N \equiv 3 \bmod 4 \end{cases}.$$

Symbols H_N with polyphase decomposition (2.4), where E and F are allpass filters can be obtained in a more general way from bireciprocal wave digital filters (WDF). For a consideration of H_N from the point of view of WDF, we refer to [5, 6, 7].

Scaling functions and wavelets related to Butterworth filters were firstly considered in [13]. Since H_N has a finite number of poles, the coefficients h_k and the scaling function φ_N are of *exponential decay.* Further, -1 is a N–fold zero of H_N such that φ_N satisfies a *Strang–Fix condition of order* N and polynomials of degree $< N$ can be locally represented as linear combinations of integer translates of φ_N. By (2.1), we see that H_N satisfies the *orthogonality relation*

$$|H_N(e^{iv})|^2 + |H_N(e^{i(v+\pi)})|^2 = 1 \qquad (v \in [-\pi,\pi))\,. \tag{2.5}$$

Since moreover $|H_N(e^{iv})| > 0$ for all $v \in (-\pi,\pi)$, Cohen's criterion [2], p.39 implies that the integer translates of φ_N are orthonormal scaling functions. By [4, 12], the scaling functions φ_N become arbitrary smooth with increasing N.

The usual way to construct corresponding orthogonal wavelets consists in the *CQF–setting*

$$\begin{aligned}
\psi_N(x) &= 2\sum_{k\in Z} g_k\, \varphi_N(2x-k),\\
\hat{\psi}(v) &= e^{iv/2} H_N(-e^{iv/2})\, \hat{\varphi}_N(\frac{v}{2}), \quad G(z) = z^{-1}H(-z^{-1})\,.
\end{aligned}$$

Clearly, other choices of the wavelet filter are possible. See [13] for the characterization of all orthogonal filters with perfect reconstruction property. For our purposes, we define the wavelet filter by the *WDF–setting*

$$G_N(z) := H_N(-z) \quad (z := e^{-iv})\,, \tag{2.6}$$

i.e.

$$\hat{\psi}_N(v) = H_N(-e^{-iv/2})\, \hat{\varphi}_N(\frac{v}{2})\,.$$

Indeed, by (2.2),

$$\begin{aligned}
H_N(z)G_N(z^{-1}) + H_N(-z)G_N(-z^{-1}) &= H_N(z)H_N(-z^{-1}) + H_N(-z)H_N(z^{-1})\\
&= \frac{C_N^2\left((z-z^{-1})^N + (z^{-1}-z)^N\right)}{U_N(z^2)U_N(z^{-2})}\\
&= 0\,,
\end{aligned} \tag{2.7}$$

such that $(\varphi_N, \psi_N(\cdot - k))_{L_2} = 0$ $(k \in \mathbf{Z})$ and ψ_N is an orthogonal wavelet. Note that (2.7) follows also immediately by (2.4) since $E(z^{-1}) = 1/E(z)$ and $F(z^{-1}) = 1/F(z)$. However, neither for Butterworth filters of even order nor for Daubechies filter, (2.6) determines an orthonormal wavelet.

3 Butterworth wavelets on the quincunx grid

Let $\mathbf{A}$ denote one of the matrices $\mathbf{S} := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ or $\mathbf{R} := \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ related to the quincunx grid $\mathbf{A}\mathbf{Z}^2$ and let $\mathbf{B} := \mathbf{A}^T$. Note that $\mathbf{S}^2 = 2\mathbf{I}$ and that $\mathbf{R}^4 = -4\mathbf{I}$ with the 2×2 identitiy matrix $\mathbf{I}$. Since $|\det\mathbf{A}| = 2$, the representation system of $\mathbf{A}\mathbf{Z}^2$ in $\mathbf{Z}^2$ consists only of two representatives, for example $\{(0,0)^T, (1,0)^T\}$ and we have as in the univariate case only one wavelet Ψ such that $\{|\det\mathbf{A}|^{j/2}\, \Psi(\mathbf{A}^j x - k) : \mathbf{k} \in \mathbf{Z}^2; j \in \mathbf{Z}\}$ is an orthonormal basis of $L_2(\mathbf{R}^2)$. Therefore, the quincunx grid is a popular choice for the construction of two–dimensional wavelets [1, 3, 10, 14, 16]. We consider scaling functions Φ defined by the refinement equation

$$\begin{aligned} \Phi(\mathbf{x}) &= 2 \sum_{k \in Z^2} h_k\, \Phi(\mathbf{A}x + k), \\ \hat{\Phi}(\omega) &= H(e^{-i\mathbf{B}^{-1}\omega})\Phi(\mathbf{B}^{-1}\omega), \qquad H(e^{-i\mathbf{B}^{-1}\omega}) := \sum_{k \in Z^2} h_k\, e^{-ik\mathbf{B}^{-1}\omega} \end{aligned}$$

with the symbol

$$\begin{aligned} H(e^{i\omega}) &:= H_N(e^{i\frac{\omega_1+\omega_2}{2}})H_N(e^{i\frac{\omega_1-\omega_2}{2}}) + G_N(e^{i\frac{\omega_1+\omega_2}{2}})G_N(e^{i\frac{\omega_1-\omega_2}{2}}) \qquad (3.1) \\ &= H_N(e^{i\frac{\omega_1+\omega_2}{2}})H_N(e^{i\frac{\omega_1-\omega_2}{2}}) + H_N(-e^{i\frac{\omega_1+\omega_2}{2}})H_N(-e^{i\frac{\omega_1-\omega_2}{2}}). \end{aligned}$$

In the following, we always denote by H the bivariate mask (3.1), by H_N ($N \in \mathbf{N}$ odd) the univariate Butterworth filter (2.2) and by Φ and φ_N the corresponding scaling functions, respectively. Clearly, we have by definition of H_N that $H(1) = 1$. Again, our scaling functions Φ have exponential decay.

The idea for the construction of (3.1) arises from the consideration of the sinc–wavelets, i.e. from the special case $N = \infty$ [8]. By (2.3) and since

$$H(e^{i\mathbf{B}\omega}) := H_N(e^{i\omega_1})H_N(e^{i\omega_2}) + H_N(-e^{i\omega_1})H_N(-e^{i\omega_2}), \qquad (3.2)$$

the filters $H(e^{-i\omega})$ and $H(e^{-i\mathbf{B}\omega})$ behave for $N=\infty$ as in Figure 1 and by

$$\hat{\Phi}(\omega) = \prod_{j=1}^{\infty} H(e^{-i\mathbf{B}^{-j}\omega}) \tag{3.3}$$

it is easy to check that $\hat{\Phi}(\omega) = \hat{\varphi}_\infty(\omega_1)\hat{\varphi}_\infty(\omega_2)$.

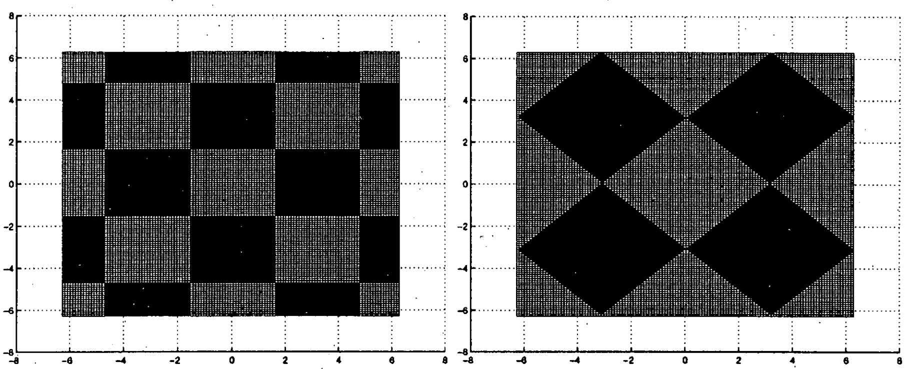


Figure 1. $|H(e^{i\mathbf{B}\omega})|$ (left) and $|H(e^{i\omega})|$ (right) for $N=\infty$ (view from above).

For the „Haar-case" $N=1$, definition (3.1) results in $H(e^{i\omega}) = H_1(e^{i\omega_1})$. „Induced" scaling functions of the above type were introduced in [3]. If $\mathbf{A} = \mathbf{S}$, then we obtain as corresponding scaling function $\Phi(x) = \varphi_1(x_2)\varphi_1(x_1-x_2)$. For $\mathbf{A}=\mathbf{R}$, the scaling function is the characteristic function of the „twin dragon".

Similarly, we define our wavelets by

$$\begin{aligned}\Psi(x) &:= 2\sum_{k\in Z^2} g_k\,\Phi(\mathbf{A}x-k)\,,\\ \hat{\Psi}(\omega) &= G(e^{-i\mathbf{B}^{-1}\omega})\,\Phi(\mathbf{B}^{-1}\omega)\end{aligned}$$

with

$$\begin{aligned}G(e^{i\omega}) &:= G_N(e^{i\frac{\omega_1+\omega_2}{2}})H_N(e^{i\frac{\omega_1-\omega_2}{2}}) + H_N(e^{i\frac{\omega_1+\omega_2}{2}})G_N(e^{i\frac{\omega_1-\omega_2}{2}}) \qquad (3.4)\\ &= H_N(-e^{i\frac{\omega_1+\omega_2}{2}})H_N(e^{i\frac{\omega_1-\omega_2}{2}}) + H_N(e^{i\frac{\omega_1+\omega_2}{2}})H_N(-e^{i\frac{\omega_1-\omega_2}{2}})\,.\end{aligned}$$

Theorem 3.1. For all $\omega \in [-\pi, \pi)$, the filters (3.1) and (3.4) satisfy the orthogonality relations

i) $|H(e^{i\omega})|^2 + |H(e^{i(\omega+(\pi,\pi)^T)})|^2 = 1$.

ii) $H(e^{i\omega})G(e^{-i\omega}) + H(e^{i(\omega+(\pi,\pi)^T)})G(e^{-i(\omega+(\pi,\pi)^T)}) = 0$.

Proof. 1. Since $\det \mathbf{B} \neq 0$, equation i) is equivalent to

$$|H(e^{i\mathbf{B}\omega})|^2 + |H(e^{i\mathbf{B}(\omega+(\pi,0)^T)})|^2 = 1.$$

By (3.2), the left–hand side can be rewritten as

$$\begin{aligned} |H(e^{i\mathbf{B}\omega})|^2 + |H(e^{i\mathbf{B}(\omega+(\pi,0)^T})|^2 &= (|H_N(e^{i\omega_1})|^2 + |H_N(-e^{i\omega_1})|^2) \\ &\quad (|H_N(e^{i\omega_2})|^2 + |H_N(-e^{i\omega_2})|^2) \\ &+ (H_N(e^{i\omega_1})H_N(-e^{-i\omega_1}) + H_N(-e^{i\omega_1})H_N(e^{-i\omega_1})) \\ &\quad (H_N(e^{i\omega_2})H_N(-e^{-i\omega_2}) + H_N(-e^{i\omega_2})H_N(e^{-i\omega_2})). \end{aligned}$$

Since by (2.5) and (2.7), the first product on the right–hand side is equal to 1 and the second product is equal to 0, we have proved assertion i).

2. Similarly, we obtain by the definitions (3.1) and (3.4) and by (2.7) that

$$\begin{aligned} H(e^{i\mathbf{B}\omega})G(e^{-i\mathbf{B}\omega}) &+ H(e^{i\mathbf{B}(\omega+(\pi,0)^T)})G(e^{-i\mathbf{B}(\omega+(\pi,0)^T)}) = \\ &\quad (|H_N(e^{i\omega_1})|^2 + |H_N(-e^{i\omega_1})|^2) \\ &\quad (H_N(e^{i\omega_2})H_N(-e^{-i\omega_2}) + H_N(-e^{i\omega_2})H_N(e^{-i\omega_2})) \\ &+ (|H_N(e^{i\omega_2})|^2 + |H_N(-e^{i\omega_2})|^2) \\ &\quad (H_N(e^{i\omega_1})H_N(-e^{-i\omega_1}) + H_N(-e^{i\omega_1})H_N(e^{-i\omega_1})) \\ &= 0. \quad \blacksquare \end{aligned}$$

Moreover, is easy to check the Cohen criterion (see [2], p.212 and p.213, Figure B.2) such that (3.3) converges in $L_2(\mathbf{R}^2)$. Therefore, we deal indeed with orthogonal scaling functions and corresponding orthogonal wavelets.

By definition of H and since -1 is an N–fold zero of H_N, we obtain immediately that

$$\frac{\partial^{n_1}}{\partial\omega_1^{n_1}} \frac{\partial^{n_2}}{\partial\omega_2^{n_2}} H(e^{i\omega})|_{\omega=(\pi,\pi)^T} = 0 \qquad \text{for } 0 \le n_1 + n_2 \le N-1.$$

Thus, bivariate polynomials of degree $< N$ can be locally represented as linear combinations of $\phi(\cdot - \mathbf{k})$ $(\mathbf{k} \in \mathbf{Z}^2)$ (cf. [9]).

Note that our construction can be extended to other bireciprocal WDF and to higher dimensional settings [7].

Except for $N = 1, \infty$, the smoothness of our wavelets is still an open question. It may be that the following form of the filters can be useful. By definition, we can write our filters in the form

$$\begin{aligned} |H(e^{i\mathbf{B}\omega})|^2 &= |H_N(e^{i\omega_1})|^2|H_N(e^{i\omega_2})|^2 + |H_N(-e^{i\omega_1})|^2|H_N(-e^{i\omega_2})|^2 \\ &+ H_N(e^{i\omega_1})H_N(e^{i\omega_2})H_N(-e^{-i\omega_1})H_N(-e^{-i\omega_2}) \\ &+ H_N(e^{-i\omega_1})H_N(e^{-i\omega_2})H_N(-e^{i\omega_1})H_N(-e^{i\omega_2}) \end{aligned}$$

and further by (2.7) and (2.1) as

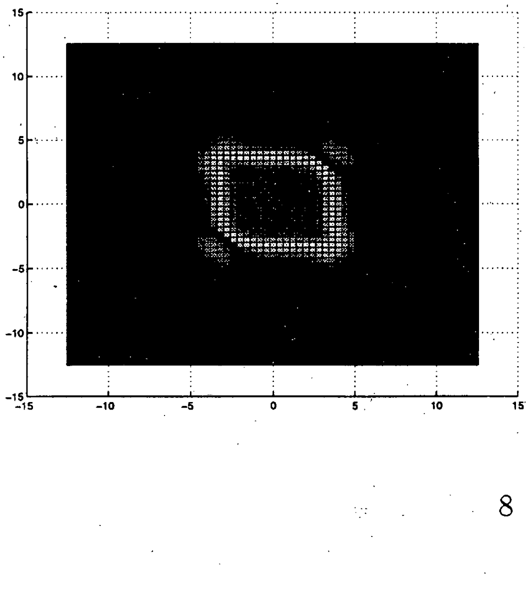
$$\begin{aligned} |H(e^{i\mathbf{B}\omega})|^2 &= \frac{(\cos\frac{\omega_1}{2}\cos\frac{\omega_2}{2})^{2N} + (\sin\frac{\omega_1}{2}\sin\frac{\omega_2}{2})^{2N}}{((\cos\frac{\omega_1}{2})^{2N} + (\sin\frac{\omega_1}{2})^{2N})((\cos\frac{\omega_2}{2})^{2N} + (\sin\frac{\omega_2}{2})^{2N})} \\ &+ 2H_N(e^{i\omega_1})H_N(-e^{-i\omega_1})H_N(e^{i\omega_2})H_N(-e^{-i\omega_2}). \end{aligned}$$

By (2.2), we verify that

$$H_N(z)H_N(-z^{-1}) = \frac{C_N^2(z-z^{-1})^N}{U_N(z^2)U_N(z^{-2})} = \frac{i^N(\sin\frac{v}{2}\cos\frac{v}{2})^N}{(\cos\frac{v}{2})^{2N} + (\sin\frac{v}{2})^{2N}} \quad (z := e^{iv}).$$

Consequently, we obtain the following simple form of our symbols

$$|H(e^{i\mathbf{B}\omega})|^2 = \frac{((\cos\frac{\omega_1}{2}\cos\frac{\omega_2}{2})^N - (\sin\frac{\omega_1}{2}\sin\frac{\omega_2}{2})^N)^2}{((\cos\frac{\omega_1}{2})^{2N} + (\sin\frac{\omega_1}{2})^{2N})((\cos\frac{\omega_2}{2})^{2N} + (\sin\frac{\omega_2}{2})^{2N})}.$$



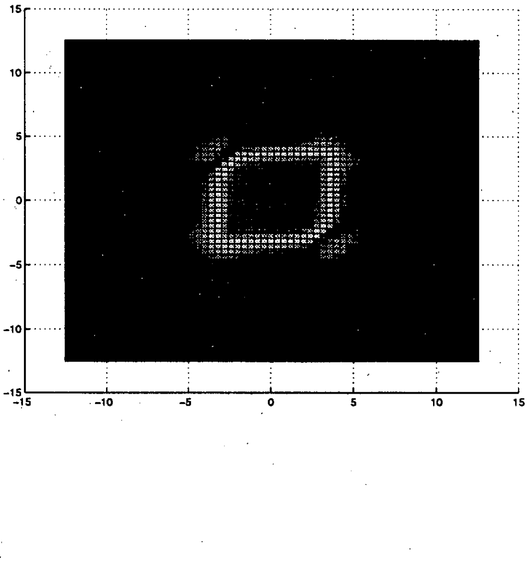


Figure 2. $|\hat{\Phi}(\omega)|$ with $\mathbf{A} = \mathbf{S}$ (left) and $\mathbf{A} = \mathbf{R}$ (right) for $N = 3$ (view from above).

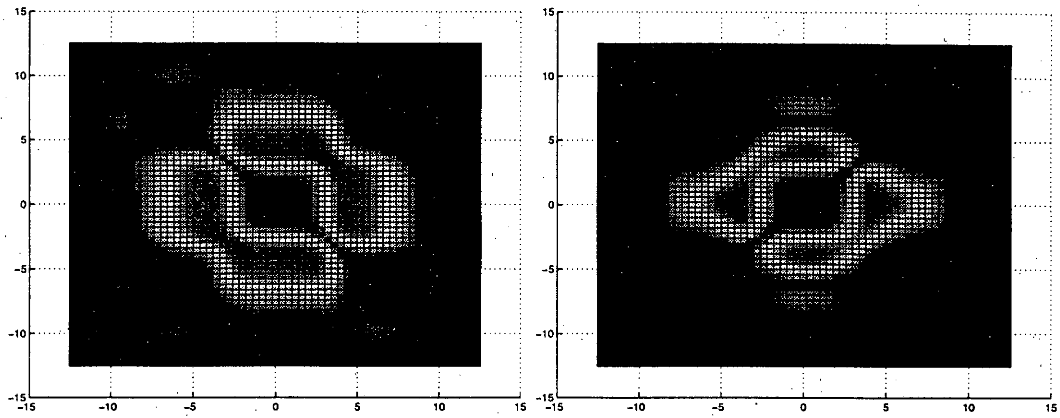


Figure 3. $|\hat{\Psi}(\omega)|$ with $\mathbf{A} = \mathbf{S}$ (left) and $\mathbf{A} = \mathbf{R}$ (right) for $N = 3$ (view from above).

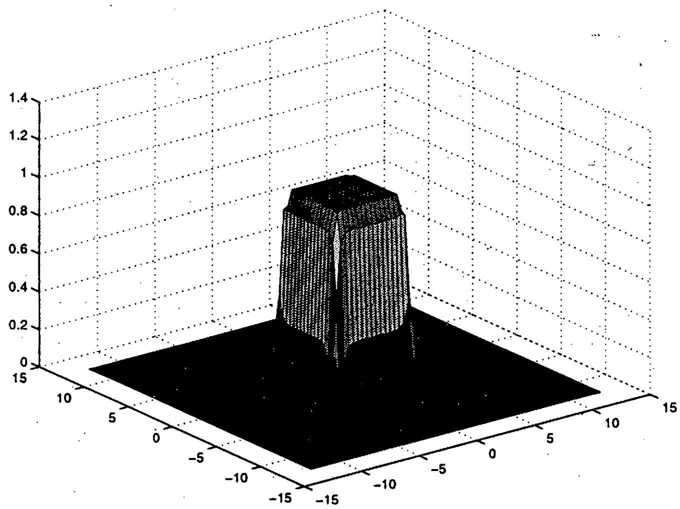


Figure 4. $|\hat{\Phi}(\omega)|$ for $N = 9$ with $\mathbf{A} = \mathbf{S}$.

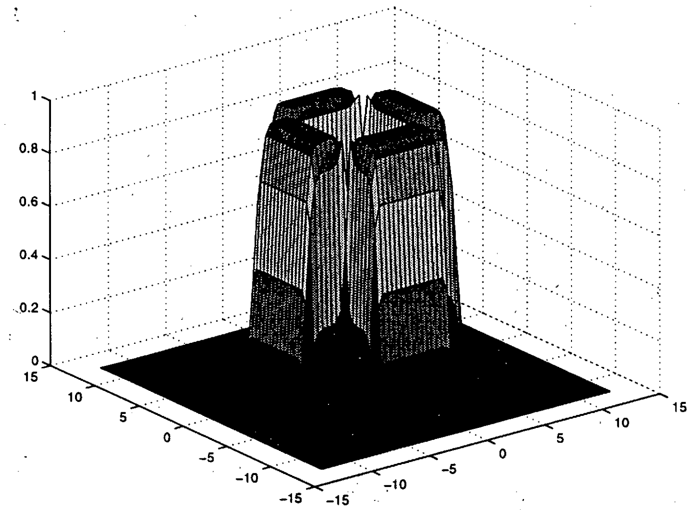


Figure 5. $|\hat{\Psi}(\omega)|$ for $N = 9$ with $\mathbf{A} = \mathbf{S}$.

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