IDENTIFICATION OF BEREZIN-TOEPLITZ DEFORMATION QUANTIZATION

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ABSTRACT. We give a complete identification of the deformation quantization which was obtained from the Berezin-Toeplitz quantization on an arbitrary compact Kähler manifold. The deformation quantization with the opposite star-product proves to be a differential deformation quantization with separation of variables whose classifying form is explicitly calculated. Its characteristic class (which classifies star-products up to equivalence) is obtained. The proof is based on the microlocal description of the Szegö kernel of a strictly pseudoconvex domain given by Boutet de Monvel and Sjöstrand.

1. INTRODUCTION

In the seminal work [1] Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer drew the attention of both physical and mathematical communities to a well posed mathematical problem of describing and classifying up to some natural equivalence the formal associative differential deformations of the algebra of smooth functions on a manifold. The deformed associative product is traditionally denoted $\star$ and called star-product.

If the manifold carries a Poisson structure, or a symplectic structure (i.e. a non-degenerate Poisson structure) or even more specific if the manifold is a Kähler manifold with symplectic structure coming from the Kähler form one naturally asks for a deformation of the algebra of smooth functions in the "direction" of the given Poisson structure. According to [1] this deformation is treated as a quantization of the corresponding Poisson manifold.

Due to work of De Wilde and Lecomte [14], Fedosov [18], and Omori, Maeda and Yoshioka [32] it is known that every symplectic manifold admits a deformation quantization in this sense. The deformation quantizations for a fixed symplectic structure

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can be classified up to equivalence by formal power series with coefficients in two-dimensional cohomology of the underlying manifold, see [5], [15], [17], [31], [40]. Kontsevich [27] showed that every Poisson manifold admits a deformation quantization and that the equivalence classes of deformation quantizations on a Poisson manifold can be parametrized by the formal deformations of the Poisson structure.

Despite the general existence and classification theorems it is of importance to study deformation quantization for manifolds with additional geometric structure and ask for deformation quantizations respecting in a certain sense this additional structure. Examples of this additional structure are the structure of a complex manifold or symmetries of the manifold.

Another natural question in this context is how some naturally defined deformation quantizations fit into the classification of all deformation quantizations.

In this article we will deal with Kähler manifolds. Quantization of Kähler manifolds via symbol algebras was considered by Berezin in the framework of his quantization program developed in [3],[4]. In this program Berezin considered symbol algebras with the symbol product depending on a small parameter $\hbar$ which has a prescribed semi-classical behavior as $\hbar \to 0$. To this end he introduced the covariant and contravariant symbols on Kähler manifolds. However, in order to study quantization via symbol algebras on Kähler manifolds he, as well as most of his successors, was forced to consider Kähler manifolds which satisfy very restrictive analytic conditions. These conditions were shown to be met by certain classes of homogeneous Kähler manifolds, e.g., $\mathbb{C}^n$, generalized flag manifolds, Hermitian symmetric domains etc. The deformation quantization obtained from the asymptotic expansion in $\hbar$ as $\hbar \to 0$ of the product of Berezin's covariant symbols on these classes of Kähler manifolds was studied in a number of papers by Moreno, Ortega-Navarro ([29], [30]); Cahen, Gutt, Rawnsley ([11], [12], [13]); see also [25]. This deformation quantization is differential and respects the separation of variables into holomorphic and anti-holomorphic ones in the sense that left star-multiplication (i.e. the multiplication with respect to the deformed product) with local holomorphic functions is pointwise multiplication, and right star-multiplication with local anti-holomorphic functions is also point-wise multiplication, see Section 2 for the precise definition. It was shown in [22] that such deformation quantizations "with separation of variables" exist for every Kähler manifold. Moreover, a complete classification (not only up to equivalence) of all differential deformation quantizations with separation of variables was given. They are parameterized by formal closed forms of type $(1,1)$. The basic results are sketched in Section 2 below. Independently a similar existence theorem was proven by Bordemann and Waldmann [7] along the lines of Fedosov's construction. The corresponding classifying (1,1)-form was calculated in [26]. Yet another construction was given by Reshetikhin and Takhtajan in [34]. They directly derive it from Berezin's integral formulas which are treated formally, i.e., with the use of the formal method of stationary phase. The classifying form of deformation quantization from [34] can be easily obtained by the methods developed in this paper.

In [16] Engliš obtained asymptotic expansion of Berezin transform on a quite general class of complex domains which do not satisfy the conditions imposed by Berezin.

For general compact Kähler manifolds (M, ω_{-1}) which are quantizable, i.e. admit a quantum line bundle L it was shown by Bordemann, Meinrenken and Schlichenmaier [6] that the correspondence between the Berezin-Toeplitz operators and their contravariant symbols associated to L^m has the correct semi-classical behavior as $m \to \infty$. Moreover, it was shown in [35],[36], [38] that it is possible to define a deformation quantization via this correspondence. For this purpose one can not use the product of contravariant symbols since in general it can not be correctly defined.

The approach of [6] was based on the theory of generalized Toeplitz operators due to Boutet de Monvel and Guillemin [8], which was also used by Guillemin [19] in his proof of the existence of deformation quantizations on compact symplectic manifolds.

The deformation quantization obtained in [35],[36], which we call the Berezin-Toeplitz deformation quantization, is defined in a natural way related to the complex structure. It fulfils the condition to be 'null on constants' (i.e. $1 \star g = g \star 1 = g$), it is self-adjoint (i.e. $\overline{f \star g} = \overline{g} \star \overline{f}$), and admits a trace of certain type (see [38] for details).

As one of the results of this article we will show that the Berezin-Toeplitz deformation quantization is differential and has the property of separation of variables, though with the roles of holomorphic and antiholomorphic variables swapped. To comply with the conventions of [22] we consider the opposite to the Berezin-Toeplitz deformation quantization (i.e., the deformation quantization with the opposite star-product) which is a deformation quantization with separation of variables in the usual sense.

We will show how the Berezin-Toeplitz deformation quantization fits into the classification scheme of [22]. Namely, we will show that the classifying formal (1,1)-form of its opposite deformation quantization is

$$\tilde{\omega} = -\frac{1}{\nu}\omega_{-1} + \omega_{can}, \tag{1.1}$$

where ν is the formal parameter, ω_{-1} is the Kähler form we started with and ω_{can} is the closed curvature (1,1)-form of the canonical line bundle of M with the Hermitian fibre metric determined by the symplectic volume. Using [23] and (1.1) we will calculate the classifying cohomology class (classifying up to equivalence) of the Berezin-Toeplitz deformation quantization. This class was first calculated by E. Hawkins in [20] by K-theoretic methods with the use of the index theorem for deformation quantization ([17], [31]).

In deformation quantization with separation of variables an important role is played by the formal Berezin transform $f \mapsto I(f)$ (see [24]). In this paper we associate to a deformation quantization with separation of variables also a non-associative "formal twisted product" $(f, g) \mapsto Q(f, g)$. Here the images are always in the formal power series over the space $C^\infty(M)$. In the compact Kähler case by considering all tensor powers L^m of the line bundle L and with the help of Berezin-Rawnsley's coherent states [33], it is possible to introduce for every level m the Berezin transform $I^{(m)}$ and also some "twisted product" $Q^{(m)}$. The key result of this article is that the analytic asymptotic expansions of $I^{(m)}$, resp. of $Q^{(m)}$ define formal objects which coincide with I and Q for some deformation quantization with separation of variables whose classifying form ω is completely determined in terms of the form $\tilde{\omega}$ (Theorem 5.9). To prove this we use the integral representation of the Szegö kernel on a strictly pseudoconvex domain obtained by Boutet de Monvel and Sjöstrand in [9] and a theorem by Zelditch [41] based on [9]. We also use the method of stationary phase and introduce its formal counterpart which we call "formal integral".

Since the analytic Berezin transform $I^{(m)}$ has the asymptotics given by the formal Berezin transform it follows also that the former has the expansion

$$I^{(m)} = \mathrm{id} + \frac{1}{m}\Delta + O(\frac{1}{m^2}), \tag{1.2}$$

where Δ is the Laplace-Beltrami operator on M.

It is worth mentioning that the above formal form ω is the formal object corresponding to the asymptotic expansion of the pullback of the Fubini-Study form via Kodaira embedding of M into the projective space related to L^m as $m \to +\infty$. This asymptotic expansion was obtained by Zelditch in [41] as a generalization of a theorem by Tian [39].

The article is organized as follows. In Section 2 we recall the basic notions of deformation quantization and the construction of the deformation quantization with separation of variables given by a formal deformation of a (pseudo-)Kähler form.

In Section 3 formal integrals are introduced. Certain basic properties, like uniqueness are shown.

In Section 4 the covariant and contravariant symbols are introduced. Using Berezin-Toeplitz operators the transformation $I^{(m)}$ and the twisted product $Q^{(m)}$ are introduced. Integral formulas for them using 2-point, resp. cyclic 3-point functions defined via the scalar product of coherent states are given.

Section 5 contains the key result that $I^{(m)}$ and $Q^{(m)}$ admit a well-defined asymptotic expansion and that the formal objects corresponding to these expansions are given by I and Q respectively.

Finally in Section 6 the Berezin-Toeplitz star product is identified with the help of the results obtained in Section 5.

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2. Deformation quantizations with separation of variables

Given a vector space V, we call the elements of the space of formal Laurent series with a finite principal part $V[\nu^{-1},\nu]]$ formal vectors. In such a way we define formal functions, differential forms, differential operators, etc. However we shall often call these formal objects just functions, operators, and so on, omitting the word formal.

Now assume that V is a Hausdorff topological vector space and $v(m)$, $m \in \mathbb{R}$, is a family of vectors in V which admits an asymptotic expansion as $m \to \infty$, $v(m) \sim \sum_{r\geq r_0}(1/m^r)v_r$, where $r_0 \in \mathbb{Z}$. In order to associate to such asymptotic families the corresponding formal vectors we use the "formalizer" $\mathbb{F} : v(m) \mapsto \sum_{r\geq r_0} \nu^r v_r \in V[\nu^{-1},\nu]]$.

Let (M,ω_{-1}) be a real symplectic manifold of dimension $2n$. For any open subset $U \subset M$ denote by $\mathcal{F}(U) = C^\infty(U)[\nu^{-1},\nu]]$ the space of formal smooth complex-valued functions on U. Set $\mathcal{F} = \mathcal{F}(M)$. Denote by $\mathbb{K} = \mathbb{C}[\nu^{-1},\nu]]$ the field of formal numbers.

A deformation quantization on (M,ω_{-1}) is an associative $\mathbb{K}$-algebra structure on $\mathcal{F}$, with the product $\star$ (named star-product) given for $f = \sum \nu^j f_j$, $g = \sum \nu^k g_k \in \mathcal{F}$ by the following formula:

$$f \star g = \sum_r \nu^r \sum_{i+j+k=r} C_i(f_j, g_k). \tag{2.1}$$

In (2.1) C_r, $r = 0,1,\ldots$, is a sequence of bilinear mappings $C_r : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ where $C_0(\varphi,\psi) = \varphi\psi$ and $C_1(\varphi,\psi) - C_1(\psi,\varphi) = i\{\varphi,\psi\}$ for $\varphi,\psi \in C^\infty(M)$ and $\{\cdot,\cdot\}$ is the Poisson bracket corresponding to the form ω_{-1}.

Two deformation quantizations $(\mathcal{F},\star_1)$ and $(\mathcal{F},\star_2)$ on (M,ω_{-1}) are called equivalent if there exists an isomorphism of algebras $B : (\mathcal{F},\star_1) \to (\mathcal{F},\star_2)$ of the form $B = 1 + \nu B_1 + \nu^2 B_2 + \ldots$, where B_k are linear endomorphisms of $C^\infty(M)$.

We shall consider only those deformation quantizations for which the unit constant 1 is the unit in the algebra $(\mathcal{F},\star)$.

If all C_r, $r \geq 0$, are local, i.e., bidifferential operators, then the deformation quantization is called differential. The equivalence classes of differential deformation quantizations on (M,ω_{-1}) are bijectively parametrized by the formal cohomology classes from $(1/i\nu)[\omega_{-1}] + H^2(M,\mathbb{C}[[\nu]])$. The formal cohomology class parametrizing a star-product $\star$ is called the characteristic class of this star-product and denoted $cl(\star)$.

A differential deformation quantization can be localized on any open subset $U \subset M$. The corresponding star-product on $\mathcal{F}(U)$ will be denoted also $\star$.

For $f,g \in \mathcal{F}$ denote by L_f, R_g the operators of left and right multiplication by f,g respectively in the algebra $(\mathcal{F},\star)$, so that $L_f g = f \star g = R_g f$. The associativity of the star-product $\star$ is equivalent to the fact that L_f commutes with R_g for all $f,g \in \mathcal{F}$. If a deformation quantization is differential then L_f, R_g are formal differential operators.

Now let (M,ω_{-1}) be pseudo-Kähler, i.e., a complex manifold such that the form ω_{-1} is of type (1,1) with respect to the complex structure. We say that a differential deformation quantization $(\mathcal{F},\star)$ is a deformation quantization with separation of variables if for any open subset $U \subset M$ and any holomorphic function a and antiholomorphic function b on U the operators L_a and R_b are the operators of point-wise multiplication by a and b respectively, i.e., $L_a = a$ and $R_b = b$.

A formal form $\omega = (1/\nu)\omega_{-1} + \omega_0 + \nu\omega_1 + \ldots$ is called a formal deformation of the form $(1/\nu)\omega_{-1}$ if the forms ω_r, $r \geq 0$, are closed but not necessarily nondegenerate (1,1)-forms on M.

It was shown in [22] that all deformation quantizations with separation of variables on a pseudo-Kähler manifold (M, ω_{-1}) are bijectively parametrized by the formal deformations of the form $(1/\nu)\omega_{-1}$.

Recall how the star-product with separation of variables $\star$ on M corresponding to the formal form $\omega = (1/\nu)\omega_{-1} + \omega_0 + \nu\omega_1 + \ldots$ is constructed. For an arbitrary contractible coordinate chart $U \subset M$ with holomorphic coordinates $\{z^k\}$ let $\Phi = (1/\nu)\Phi_{-1} + \Phi_0 + \nu\Phi_1 + \ldots$ be a formal potential of the form ω on U, i.e., $\omega = -i\partial\bar{\partial}\Phi$ (notice that in [22] - [26] a potential Φ of a closed (1,1)-form ω is defined via the formula $\omega = i\partial\bar{\partial}\Phi$).

The star-product corresponding to the form ω is such that $L_{\partial\Phi/\partial z^k} = \partial\Phi/\partial z^k + \partial/\partial z^k$ and $R_{\partial\Phi/\partial\bar{z}^l} = \partial\Phi/\partial\bar{z}^l + \partial/\partial\bar{z}^l$ on U. The set $\mathcal{L}(U)$ of all left multiplication operators on U is completely described as the set of all formal differential operators commuting with the point-wise multiplication operators by antiholomorphic coordinates $R_{\bar{z}^l} = \bar{z}^l$ and the operators $R_{\partial\Phi/\partial\bar{z}^l} = \partial\Phi/\partial\bar{z}^l + \partial/\partial\bar{z}^l$. One can immediately reconstruct the star-product on U from the knowledge of $\mathcal{L}(U)$. The local star-products agree on the intersections of the charts and define the global star-product $\star$ on M.

One can express the characteristic class $cl(\star)$ of the star-product with separation of variables $\star$ parametrized by the formal form ω in terms of this form (see [23]). Unfortunately, there were wrong signs in the formula for $cl(\star)$ in [23] which should be read as follows:

$$cl(\star) = (1/i)([\omega] - \varepsilon/2), \tag{2.2}$$

where ε is the canonical class of the complex manifold M, i.e., the first Chern class of the canonical holomorphic line bundle on M.

Given a deformation quantization with separation of variables $(\mathcal{F}, \star)$ on the pseudo-Kähler manifold (M, ω_{-1}), one can introduce the *formal Berezin transform* I as the unique formal differential operator on M such that for any open subset $U \subset M$, holomorphic function a and antiholomorphic function b on U the relation $I(ab) = b \star a$ holds (see [24]). One can check that $I = 1 + \nu\Delta + \ldots$, where Δ is the Laplace-Beltrami operator corresponding to the pseudo-Kähler metric on M. The *dual* star-product $\tilde{\star}$ on M defined for $f, g \in \mathcal{F}$ by the formula $f \tilde{\star} g = I^{-1}(Ig \star If)$ is a star-product with separation of variables on the pseudo-Kähler manifold $(M, -\omega_{-1})$. For this deformation quantization the formal Berezin transform equals I^{-1}, and thus the dual to $\tilde{\star}$ is again $\star$.

Denote by $\tilde{\omega} = -(1/\nu)\omega_{-1} + \tilde{\omega}_0 + \nu\tilde{\omega}_1 + \ldots$ the formal form parametrizing the star-product $\tilde{\star}$. The opposite to the dual star-product, $\star' = \tilde{\star}^{op}$, given by the formula $f \star' g = I^{-1}(If \star Ig)$, also defines a deformation quantization with separation of variables on M but with the roles of holomorphic and antiholomorphic variables swapped. Differently said, $(\mathcal{F}, \star')$ is a deformation quantization with separation of variables on the pseudo-Kähler manifold $(\overline{M}, \omega_{-1})$ where $\overline{M}$ is the manifold M with the opposite complex structure. The formal Berezin transform I establishes an equivalence of deformation quantizations $(\mathcal{F}, \star)$ and $(\mathcal{F}, \star')$.

Introduce the following non-associative operation $Q(\cdot,\cdot)$ on $\mathcal{F}$. For $f, g \in \mathcal{F}$ set $Q(f,g) = If \star Ig = I(f \star' g) = I(g \tilde{\star} f)$. We shall call it formal twisted product. The importance of the formal twisted product will be revealed later.

A trace density of a deformation quantization $(\mathcal{F}, \star)$ on a symplectic manifold M is a formal volume form μ on M for which the functional $\kappa(f) = \int_M f\mu, f \in \mathcal{F}$, has the trace property, $\kappa(f \star g) = \kappa(g \star f)$ for all $f, g \in \mathcal{F}$ where at least one of the functions f, g has compact support. It was shown in [24] that on a local holomorphic chart $(U, \{z^k\})$ any formal trace density μ can be represented in the form $c(\nu)\exp(\Phi + \Psi) dz d\bar{z}$, where $c(\nu) \in \mathbb{K}$ is a formal constant, $dz d\bar{z} = dz^1 \dots dz^n d\bar{z}^1 \dots d\bar{z}^n$ is the standard volume on U and $\Phi = (1/\nu)\Phi_{-1} + \dots, \Psi = (1/\nu)\Psi_{-1} + \dots$ are formal potentials of the forms $\omega, \tilde{\omega}$ respectively such that the relations

$$\partial\Phi/\partial z^k = -I(\partial\Psi/\partial z^k),\ \partial\Phi/\partial\bar{z}^l = -I(\partial\Psi/\partial\bar{z}^l),\ \text{and}\ \Phi_{-1} + \Psi_{-1} = 0 \tag{2.3}$$

hold. Vice versa, any such form is a formal trace density.

3. Formal integrals, jets, and almost analytic functions

Let $\phi = (1/\nu)\phi_{-1} + \phi_0 + \nu\phi_1 + \dots$ and $\mu = \mu_0 + \nu\mu_1 + \dots$ be, respectively, a smooth complex-valued formal function and a smooth formal volume form on an open set $U \subset \mathbb{R}^n$. Assume that $x \in U$ is a nondegenerate critical point of the function ϕ_{-1} and μ_0 does not vanish at x. We call a $\mathbb{K}$-linear functional K on $\mathcal{F}(U)$ such that

(a) $K = K_0 + \nu K_1 + \dots$ is a formal distribution supported at the point x;
(b) $K_0 = \delta_x$ is the Dirac distribution at the point x;
(c) $K(1) = 1$ (normalization condition);
(d) for any vector field ξ on U and $f \in \mathcal{F}(U)$ $\quad K\big(\xi f + (\xi\phi + \operatorname{div}_\mu \xi) f\big) = 0$,

a (normalized) *formal integral at the point x associated to the pair (ϕ, μ)*.

It is clear from the definition that a formal integral at a point x is independent of a particular choice of the neighborhood U and is actually associated to the germs of (ϕ, μ) at x. Usually we shall consider a contractible neighborhood U such that μ_0 vanishes nowhere on U.

We shall prove that a formal integral at the point x associated to the pair (ϕ, μ) is uniquely determined. One can also show the existence of such a formal integral, but this fact will neither be used nor proved in what follows.

We call two pairs (ϕ, μ) and (ϕ', μ') equivalent if there exists a formal function $u = u_0 + \nu u_1 + \dots$ on U such that $\phi' = \phi - u,\ \mu' = e^u \mu$.

Since the expression $\xi\phi + \operatorname{div}_\mu \xi$ remains invariant if we replace the pair (ϕ, μ) by an equivalent one, a formal integral is actually associated to the equivalence class of the pair (ϕ, μ). This means that a formal integral actually depends on the product $e^{\phi}\mu$ which can be thought of as a part of the integrand of a "formal oscillatory integral". In the sequel it will be shown that one can directly produce formal integrals from the method of stationary phase.

Notice that if K is a formal integral associated to a pair (ϕ, μ) it is then associated to any pair $(\phi, c(\nu)\mu)$, where $c(\nu)$ is a nonzero formal constant.

It is easy to show that it is enough to check condition (d) for the coordinate vector fields $\partial/\partial x^k$ on U. Moreover, if U is contractible and such that μ_0 vanishes nowhere on it, one can choose an equivalent pair of the form (ϕ', dx), where $dx = dx^1 \dots dx^n$ is the standard volume form.

Proposition 3.1. *A formal integral $K = K_0 + \nu K_1 + \dots$ at a point x, associated to a pair $(\phi = (1/\nu)\phi_{-1} + \phi_0 + \nu\phi_1 + \dots, \mu)$ is uniquely determined.*

Proof. We assume that K is defined on a coordinate chart $(U, \{x^k\})$, $\mu = dx$, and take $f \in C^\infty(U)$. Since $\mathrm{div}_{dx}(\partial/\partial x^k) = 0$, the last condition of the definition of a formal integral takes the form

$$K\big(\partial f/\partial x^k + (\partial\phi/\partial x^k)f\big) = 0. \tag{3.1}$$

Equating to zero the coefficient at ν^r, $r \geq 0$, of the l.h.s. of (3.1) we get $K_r(\partial f/\partial x^k) + \sum_{s=0}^{r+1} K_s\big((\partial\phi_{r-s}/\partial x^k)f\big) = 0$, which can be rewritten as a recurrent equation

$$K_{r+1}\big((\partial\phi_{-1}/\partial x^k)f\big) = \text{r.h.s. depending on } K_j,\ j \leq r. \tag{3.2}$$

Since x is a nondegenerate critical point of ϕ_{-1}, the functions $\partial\phi_{-1}/\partial x^k$ generate the ideal of functions vanishing at x. Taking into account that $K_{r+1}(1) = 0$ for $r \geq 0$ we see from (3.2) that K_{r+1} is determined uniquely. Thus the proof proceeds by induction. □

Let V be an open subset of a complex manifold M and Z be a relatively closed subset of V. A function $f \in C^\infty(V)$ is called almost analytic at Z if $\bar\partial f$ vanishes to infinite order there.

Two functions $f_1, f_2 \in C^\infty(V)$ are called equivalent at Z if $f_1 - f_2$ vanishes to infinite order there.

Consider open subsets $U \subset \mathbb{R}^n$ and $\tilde U \subset \mathbb{C}^n$ such that $U = \tilde U \cap \mathbb{R}^n$, and a function $f \in C^\infty(U)$. A function $\tilde f \in C^\infty(\tilde U)$ is called an almost analytic extension of f if it is almost analytic at U and $\tilde f|_U = f$.

It is well known that every $f \in C^\infty(U)$ has an almost analytic extension uniquely determined up to equivalence.

Fix a formal deformation $\omega = (1/\nu)\omega_{-1} + \omega_0 + \nu\omega_1 + \dots$ of the form $(1/\nu)\omega_{-1}$ on a pseudo-Kähler manifold (M, ω_{-1}). Consider the corresponding star-product with separation of variables $\star$, the formal Berezin transform I and the formal twisted product Q on M. We are going to show that for any point $x \in M$ the functional $K_x^I(f) = (If)(x)$ on $\mathcal{F}$ and the functional K_x^Q on $\mathcal{F}(M \times M)$ such that $K_x^Q(f \otimes g) = Q(f, g)(x)$ can be represented as formal integrals.

Let $U \subset M$ be a contractible coordinate chart with holomorphic coordinates $\{z^k\}$. Given a smooth function $f(z,\bar{z})$ on U, where U is considered as the diagonal of $\tilde{U} = U \times \overline{U}$, one can choose its almost analytic extension $\tilde{f}(z_1,\bar{z}_1,z_2,\bar{z}_2)$ on $\tilde{U}$, so that $\tilde{f}(z,\bar{z},z,\bar{z}) = f(z,\bar{z})$. It is a substitute of the *holomorphic* function $f(z_1,\bar{z}_2)$ on $\tilde{U}$ which in general does not exist.

Let $\Phi = (1/\nu)\Phi_{-1} + \Phi_0 + \nu\Phi_1 + \dots$ be a formal potential of the form ω on U and $\tilde{\Phi}$ its almost analytic extension on $\tilde{U}$. In particular, $\tilde{\Phi}(x,x) = \Phi(x)$ for $x \in U$. Introduce an analogue of the Calabi diastatic function on $U \times U$ by the formula $D(x,y) = \tilde{\Phi}(x,y) + \tilde{\Phi}(y,x) - \Phi(x) - \Phi(y)$. We shall also use the notation $D_k(x,y) = \tilde{\Phi}_k(x,y) + \tilde{\Phi}_k(y,x) - \Phi_k(x) - \Phi_k(y)$ so that $D = (1/\nu)D_{-1} + D_0 + \nu D_1 + \dots$.

Let $\tilde{\omega}$ be the formal form corresponding to the dual star-product $\tilde{\star}$ of the star-product $\star$. Choose a formal potential Ψ of the form $\tilde{\omega}$ on U, satisfying equation (2.3), so that $\mu_{tr} = e^{\Phi+\Psi} dz d\bar{z}$ is a formal trace density of the star-product $\star$ on U.

Theorem 3.2. *For any point $x \in U$ the functional $K^I_x(f) = (If)(x)$ on $\mathcal{F}(U)$ is the formal integral at x associated to the pair (ϕ^x, μ_{tr}), where $\phi^x(y) = D(x,y)$.*

Remark. In the proof of the theorem we use the notion of jet of order N of a formal function $f = \sum \nu^r f_r$ at a given point. It is also a formal object, the formal series of jets of order N of the functions f_r.

Proof. The condition that x is a nondegenerate critical point of the function $\phi^x_{-1}(y) = D_{-1}(x,y)$ directly follows from the fact that Φ_{-1} is a potential of the non-degenerate (1,1)-form ω_{-1}. The conditions (a-c) of the definition of formal integral are trivially satisfied. It remains to check the condition (d). Replace the pair (ϕ^x, μ_{tr}) by the equivalent pair $(\phi^x + \Phi + \Psi, dz d\bar{z}) = (\tilde{\Phi}(x,y) + \tilde{\Phi}(y,x) - \Phi(x) + \Psi(y), dz d\bar{z})$. Put $x = (z_0,\bar{z}_0), y = (z,\bar{z})$. For $\xi = \partial/\partial z^k$ the condition (d) takes the form

$$I\Big(\partial f/\partial z^k + (\partial/\partial z^k)\big(\tilde{\Phi}(z_0,\bar{z}_0,z,\bar{z}) + \tilde{\Phi}(z,\bar{z},z_0,\bar{z}_0) + \Psi(z,\bar{z})\big) f\Big)(z_0,\bar{z}_0) = 0.$$

We shall check it by showing that

(i) $I\big(\partial f/\partial z^k + (\partial \Psi/\partial z^k) f\big) = -If \star (\partial\Phi/\partial z^k)$;

(ii) $I\Big(\big(\partial\tilde{\Phi}(z_0,\bar{z}_0,z,\bar{z})/\partial z^k\big) f\Big)(z_0,\bar{z}_0) = 0$;

(iii) $I\Big(\big(\partial\tilde{\Phi}(z,\bar{z},z_0,\bar{z}_0)/\partial z^k\big) f\Big)(z_0,\bar{z}_0) = \big(If \star \partial\Phi/\partial z^k\big)(z_0,\bar{z}_0)$.

First, $I\big(\partial f/\partial z^k + (\partial\Psi/\partial z^k) f\big) = I\big((\partial\Psi/\partial z^k)\tilde{\star} f\big) = If \star I(\partial\Psi/\partial z^k) = -If \star (\partial\Phi/\partial z^k)$, which proves (i).

The function $\psi(z,\bar{z}) = \tilde{\Phi}(z_0,\bar{z}_0,z,\bar{z})$ is almost antiholomorphic at the point $z = z_0$. Thus, the full jet of the function $\partial\psi/\partial z^k$ at the point $z = z_0$ is equal to zero, which proves (ii).

The function $\theta(z,\bar{z}) = \partial\tilde{\Phi}(z,\bar{z},z_0,\bar{z}_0)/\partial z^k$ is almost holomorphic at the point $z = z_0$. For a holomorphic function a we have $I(af) = I(a \tilde{\star} f) = If \star Ia = If \star a$. Since $I(\theta f)(z_0,\bar{z}_0)$ and $(If \star \theta)(z_0,\bar{z}_0)$ considered modulo ν^N depend on the jets of finite order of the functions θ and f at the point z_0 taken modulo $\nu^{N'}$ for sufficiently big

N', we can approximate θ by a formal holomorphic function a making sure that the jets of sufficiently high order of θ and a at the point z_0 coincide modulo $\nu^{N'}$. Then $I(\theta f)(z_0,\bar{z}_0) \equiv I(af)(z_0,\bar{z}_0) \equiv (If \star a)(z_0,\bar{z}_0) \equiv (If \star \theta)(z_0,\bar{z}_0) \pmod{\nu^N}$. Since N is arbitrary, $I(\theta f)(z_0,\bar{z}_0) = (If \star \theta)(z_0,\bar{z}_0)$ identically. The functions $\partial\Phi/\partial z^k$ and θ have identical holomorphic parts of jets at the point z_0, i.e., all the holomorphic partial derivatives (of any order) of these functions at the point z_0 coincide. Since a left star-multiplication operator of deformation quantization with separation of variables differentiates its argument only in holomorphic directions, we get that $(If \star \theta)(z_0,\bar{z}_0) = (If \star (\partial\Phi/\partial z^k))(z_0,\bar{z}_0)$. This proves (iii).

The check for $\xi = \partial/\partial \bar{z}^l$ is similar, which completes the proof of the theorem. □

The following lemma and theorem can be proved by the same methods as Theorem 3.2.

Lemma 3.3. *For any vector field ξ on U and $x \in U$ $\quad I\big(\xi_x \phi^x\big)(x) = 0$, where $\phi^x(y) = D(x,y)$.*

($\xi_x \phi^x$ denotes differentiation of ϕ^x w.r.t. the parameter x.)

Introduce a 3-point function T on $U \times U \times U$ by the formula $T(x,y,z) = \tilde{\Phi}(x,y) + \tilde{\Phi}(y,z) + \tilde{\Phi}(z,x) - \Phi(x) - \Phi(y) - \Phi(z)$.

Theorem 3.4. *For any point $x \in U$ the functional K_x^Q on $\mathcal{F}(U \times U)$ such that $K_x^Q(f \otimes g) = Q(f,g)(x)$ is the formal integral at the point $(x,x) \in U \times U$ associated to the pair $(\psi^x, \mu_{tr} \otimes \mu_{tr})$, where $\psi^x(y,z) = T(x,y,z)$.*

4. Covariant and contravariant symbols

In the rest of the paper let (M, ω_{-1}) be a compact Kähler manifold. Assume that there exists a quantum line bundle (L,h) on M, i.e., a holomorphic hermitian line bundle with fibre metric h such that the curvature of the canonical connection on L coincides with the Kähler form ω_{-1}.

Let m be a non-negative integer. The metric h induces the fibre metric h^m on the tensor power $L^m = L^{\otimes m}$. Denote by $L^2(L^m)$ the Hilbert space of square-integrable sections of L^m with respect to the norm $\|s\|^2 = \int h^m(s)\Omega$, where $\Omega = (1/n!)(\omega_{-1})^n$ is the symplectic volume form on M. The Bergman projector B_m is the orthogonal projector in $L^2(L^m)$ onto the space $H_m = \Gamma_{hol}(L^m)$ of holomorphic sections of L^m.

Denote by k the metric on the dual line bundle $\tau : L^* \to M$ induced by h. It is a well known fact that $D = \{\alpha \in L^* | k(\alpha) < 1\}$ is a strictly pseudoconvex domain in L^*. Its boundary $X = \{\alpha \in L^* | k(\alpha) = 1\}$ is a S^1-principal bundle.

The sections of L^m are identified with the m-homogeneous functions on L^* by means of the mapping $\gamma_m : s \mapsto \psi_s$, where $\psi_s(\alpha) = \langle \alpha^{\otimes m}, s(x)\rangle$ for $\alpha \in L^*_x$. Here $\langle \cdot, \cdot \rangle$ denotes the bilinear pairing between $(L^*)^m$ and L^m.

There exists a unique S^1-invariant volume form $\tilde{\Omega}$ on X such that for every $f \in C^\infty(M)$ the equality $\int_X (\tau^* f)\tilde{\Omega} = \int_M f\Omega$ holds.

The mapping γ_m maps $L^2(L^m)$ isometrically onto the weight subspace of $L^2(X, \tilde{\Omega})$ of weight m with respect to the S^1-action. The Hardy space $\mathcal{H} \subset L^2(X, \tilde{\Omega})$ of square integrable traces of holomorphic functions on L^* splits up into weight spaces, $\mathcal{H} = \oplus_{m=0}^{\infty} \mathcal{H}_m$, where $\mathcal{H}_m = \gamma_m(H_m)$.

Denote by S and $\hat{B}_m$ the Szegö and Bergman orthogonal projections in $L^2(X, \tilde{\Omega})$ onto $\mathcal{H}$ and $\mathcal{H}_m$ respectively. Thus $S = \sum_{m=0}^{\infty} \hat{B}_m$. The Bergman projection $\hat{B}_m$ has a smooth integral kernel $\mathcal{B}_m = \mathcal{B}_m(\alpha, \beta)$ on $X \times X$.

For each $\alpha \in L^* - 0$ ('-0' means the zero section removed) one can define a coherent state $e_\alpha^{(m)}$ as the unique holomorphic section of L^m such that for each $s \in H_m$ $\langle s, e_\alpha^{(m)} \rangle = \psi_s(\alpha)$ where $\langle \cdot, \cdot \rangle$ is the hermitian scalar product on $L^2(L^m)$ antilinear in the second argument.

Since the line bundle L is positive it is known that there exists a constant m_0 such that for $m > m_0$ $\dim H_m > 0$ and all $e_\alpha^{(m)}, \alpha \in L^* - 0$, are nonzero vectors. From now on we assume that $m > m_0$ unless otherwise specified.

The coherent state $e_\alpha^{(m)}$ is antiholomorphic in α and for a nonzero $c \in \mathbb{C}$ $e_{c\alpha}^{(m)} = \bar{c}^m e_\alpha^{(m)}$. Notice that in [10] coherent states are parametrized by the points of $L - 0$.

For $s \in L^2(L^m)$ $\langle s, e_\alpha^{(m)} \rangle = \langle s, B_m e_\alpha^{(m)} \rangle = \langle B_m s, e_\alpha^{(m)} \rangle = \psi_{B_m s}(\alpha)$. The mapping γ_m intertwines the Bergman projectors B_m and $\hat{B}_m$, for $s \in L^2(L^m)$ $\psi_{B_m s} = \hat{B}_m \psi_s$. Thus, on the one hand, $\langle s, e_\alpha^{(m)} \rangle = \hat{B}_m \psi_s(\alpha) = \int_X \mathcal{B}_m(\alpha, \beta) \psi_s(\beta) \tilde{\Omega}(\beta)$. On the other hand, $\langle s, e_\alpha^{(m)} \rangle = \langle \psi_s, \psi_{e_\alpha^{(m)}} \rangle = \int_X \psi_s(\beta) \overline{\psi_{e_\alpha^{(m)}}(\beta)} \tilde{\Omega}(\beta)$. Taking into account that $\langle e_\beta^{(m)}, e_\alpha^{(m)} \rangle = \psi_{e_\beta^{(m)}}(\alpha) = \overline{\psi_{e_\alpha^{(m)}}(\beta)}$ we finally get that $\langle e_\beta^{(m)}, e_\alpha^{(m)} \rangle = \psi_{e_\beta^{(m)}}(\alpha) = \mathcal{B}_m(\alpha, \beta)$. In particular, one can extend the kernel $\mathcal{B}_m(\alpha, \beta)$ from $X \times X$ to a holomorphic function on $(L^* - 0) \times (\overline{L^* - 0})$ such that for nonzero $c, d \in \mathbb{C}$

$$\mathcal{B}_m(c\alpha, d\beta) = (c\bar{d})^m \mathcal{B}_m(\alpha, \beta). \tag{4.1}$$

For $\alpha, \beta \in L^* - 0$ the following inequality holds.

$$|\mathcal{B}_m(\alpha, \beta)| = \left| \langle e_\alpha^{(m)}, e_\beta^{(m)} \rangle \right| \leq \|e_\alpha^{(m)}\| \|e_\beta^{(m)}\| = (\mathcal{B}_m(\alpha, \alpha) \mathcal{B}_m(\beta, \beta))^{\frac{1}{2}}. \tag{4.2}$$

The covariant symbol of an operator A in the space H_m is the function $\sigma(A)$ on M such that

$$\sigma(A)(x) = \frac{\langle A e_\alpha^{(m)}, e_\alpha^{(m)} \rangle}{\langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle}$$

for any $\alpha \in L_x^* - 0$.

Denote by M_f the multiplication operator by a function $f \in C^\infty(M)$ on sections of L^m. Define the Berezin-Toeplitz operator $T_f^{(m)} = B_m M_f B_m$ in H_m. If an operator in H_m is represented in the form $T_f^{(m)}$ for some function $f \in C^\infty(M)$ then the function f is called its contravariant symbol.

With these symbols we associate two important operations on $C^\infty(M)$, the Berezin transform $I^{(m)}$ and a non-associative binary operation $Q^{(m)}$ which we call twisted product, as follows. For $f, g \in C^\infty(M)$ $I^{(m)} f = \sigma(T_f^{(m)})$, $Q^{(m)}(f,g) = \sigma(T_f^{(m)} T_g^{(m)})$.

We are going to show in Section 5 that both $I^{(m)}$ and $Q^{(m)}$ have asymptotic expansions in $1/m$ as $m \to +\infty$, such that if the asymptotic parameter $1/m$ in these expansions is replaced by the formal parameter ν then we get the formal Berezin transform I and the formal twisted product Q corresponding to some deformation quantization with separation of variables on (M, ω_{-1}) which can be completely identified. We shall mainly be interested in the opposite to its dual deformation quantization. The goal of this paper is to show that it coincides with the Berezin-Toeplitz deformation quantization obtained in [36],[38].

In order to obtain the asymptotic expansions of $I^{(m)}$ and $Q^{(m)}$ we need their integral representations. To calculate them it is convenient to work on X rather than on M. We shall use the fact that for $f \in C^\infty(M)$, $s \in \Gamma(L^m), \psi_{M_f s} = (\tau^* f) \cdot \psi_s$. For $x \in M$ denote by X_x the fibre of the bundle X over x, $X_x = \tau^{-1}(x) \cap X$. For $x, y, z \in M$ choose $\alpha \in X_x$, $\beta \in X_y$, $\gamma \in X_z$ and set

$$\begin{aligned} (4.3) \quad u_m(x) = \mathcal{B}_m(\alpha,\alpha), \qquad & v_m(x,y) = \mathcal{B}_m(\alpha,\beta)\mathcal{B}_m(\beta,\alpha), \\ & w_m(x,y,z) = \mathcal{B}_m(\alpha,\beta)\mathcal{B}_m(\beta,\gamma)\mathcal{B}_m(\gamma,\alpha). \end{aligned}$$

It follows from (4.1) that $u_m(x), v_m(x,y), w_m(x,y,z)$ do not depend on the choice of α, β, γ and thus relations (4.3) correctly define functions u_m, v_m, w_m. The function w_m is the so called cyclic 3-point function studied in [2]. Notice that $u_m(x) = \mathcal{B}_m(\alpha,\alpha) = \|e_\alpha^{(m)}\|^2 > 0$, $v_m(x,y) = \mathcal{B}_m(\alpha,\beta)\mathcal{B}_m(\beta,\alpha) = |\mathcal{B}_m(\alpha,\beta)|^2 \geq 0$ and

$$(4.4) \qquad |w_m(x,y,z)|^2 = v_m(x,y) v_m(y,z) v_m(z,x).$$

It follows from (4.2) that

$$(4.5) \qquad v_m(x,y) \leq u_m(x) u_m(y).$$

For $\alpha \in X_x$ we have

$$\begin{aligned} (4.6) \quad (I^{(m)} f)(x) = \sigma(T_f^{(m)})(x) = \frac{\langle T_f^{(m)} e_\alpha^{(m)}, e_\alpha^{(m)} \rangle}{\langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle} = \frac{\langle B_m M_f B_m e_\alpha^{(m)}, e_\alpha^{(m)} \rangle}{\mathcal{B}_m(\alpha,\alpha)} = \\ \frac{\langle M_f e_\alpha^{(m)}, e_\alpha^{(m)} \rangle}{\mathcal{B}_m(\alpha,\alpha)} = \frac{\langle (\tau^* f) \psi_{e_\alpha^{(m)}}, \psi_{e_\alpha^{(m)}} \rangle}{\mathcal{B}_m(\alpha,\alpha)} = \frac{1}{\mathcal{B}_m(\alpha,\alpha)} \int_X (\tau^* f) \psi_{e_\alpha^{(m)}}(\beta) \overline{\psi_{e_\alpha^{(m)}}(\beta)} \tilde{\Omega}(\beta) = \\ \frac{1}{\mathcal{B}_m(\alpha,\alpha)} \int_X \mathcal{B}_m(\alpha,\beta)\mathcal{B}_m(\beta,\alpha)(\tau^* f)(\beta)\tilde{\Omega}(\beta) = \frac{1}{u_m(x)} \int_M v_m(x,y) f(y) \Omega(y). \end{aligned}$$

Similarly we obtain that

$$(4.7)\quad Q^{(m)}(f,g)(x) = \frac{1}{\mathcal{B}_m(\alpha,\alpha)} \int_{X\times X} \mathcal{B}_m(\alpha,\beta)\mathcal{B}_m(\beta,\gamma)\mathcal{B}_m(\gamma,\alpha)(\tau^* f)(\beta)(\tau^* g)(\gamma)\tilde{\Omega}(\beta)\tilde{\Omega}(\gamma) = \frac{1}{u_m(x)} \int_{M\times M} w_m(x,y,z)f(y)g(z)\Omega(y)\Omega(z).$$

5. Asymptotic expansion of the Berezin transform

In [9] a microlocal description of the integral kernel $\mathcal{S}$ of the Szegö projection S was given. The results in [9] were obtained for a strictly pseudoconvex domain with a smooth boundary in $\mathbb{C}^{n+1}$. However, according to the concluding remarks in [9], these results are still valid for the domain D in L^* (see also [6], [41]).

It was proved in [9] that the Szegö kernel $\mathcal{S}$ is a generalized function on $X \times X$ singular on the diagonal of $X \times X$ and smooth outside the diagonal. The Szegö kernel $\mathcal{S}$ can be expressed via the Bergman kernels $\mathcal{B}_m$ as follows, $\mathcal{S} = \sum_{m\geq 0} \mathcal{B}_m$, where the sum should be understood as a sum of generalized functions.

For $(\alpha, \beta) \in X \times X$ and $\theta \in \mathbb{R}$ set $r_\theta(\alpha,\beta) = (e^{i\theta}\alpha, \beta)$. Since each $\mathcal{H}_m$ is a weight space of the S^1-action in the Hardy space $\mathcal{H}$, one can recover $\mathcal{B}_m$ from the Szegö kernel,

$$(5.1)\qquad \mathcal{B}_m = \frac{1}{2\pi}\int_0^{2\pi} e^{-im\theta} r_\theta^* \mathcal{S}\, d\theta.$$

This equality should be understood in the weak sense.

Let E_1, E_2 be closed disjoint subsets of M. Set $F_i = \tau^{-1}(E_i) \cap X$, $i = 1, 2$. Thus F_1, F_2 are closed disjoint subsets of X or, equivalently, $F_1 \times F_2$ is a closed subset of $X \times X$ which does not intersect the diagonal. For $\mathcal{S}$ and $\mathcal{B}_m$ considered as smooth functions outside the diagonal of $X \times X$ equality (5.1) holds in the ordinary sense, from whence it follows immediately that

$$(5.2)\qquad \sup_{F_1\times F_2} |\mathcal{B}_m| = O\left(\frac{1}{m^N}\right)$$

for any $N \in \mathbb{N}$.

Now let E be a closed subset of M and $x \in M \setminus E$. Then (5.2) implies that

$$(5.3)\qquad \sup_{y\in E} v_m(x,y) = O\left(\frac{1}{m^N}\right)$$

for any $N \in \mathbb{N}$.

In [41] Zelditch proved that the function u_m on M expands in the asymptotic series $u_m \sim m^n \sum_{r\geq 0}(1/m^r) b_r$ as $m \to +\infty$, where $b_0 = 1$ $(n = (1/2)\dim_{\mathbb{R}} M)$. More precisely, he proved that for any $k, N \in \mathbb{N}$

$$\left| u_m - \sum_{r=0}^{N-1} m^{n-r} b_r \right|_{C^k} = O(m^{n-N}). \tag{5.4}$$

Therefore

$$\sup_M \frac{1}{u_m} = O\left(\frac{1}{m^n}\right). \tag{5.5}$$

Using (4.6),(5.3) and (5.5) it is easy to prove the following proposition.

Proposition 5.1. *Let $f \in C^\infty(M)$ be a function vanishing in a neighborhood of a point $x \in M$. Then $|(I^{(m)}f)(x)| = O(1/m^N)$ for any $N \in \mathbb{N}$, i.e., $(I^{(m)}f)(x)$ is rapidly decreasing as $m \to +\infty$.*

Thus for arbitrary $f \in C^\infty(M)$ and $x \in M$ the asymptotics of $(I^{(m)}f)(x)$ as $m \to +\infty$ depends only on the germ of the function f at the point x.

Let E be a closed subset of M. Fix a point $x \in M \setminus E$. The function $w_m(x,y,z)$ with $y \in E$ can be estimated using (4.4) and (4.5) as follows.

$$|w_m(x,y,z)|^2 \le v_m(x,y)u_m(x)u_m(y)\big(u_m(z)\big)^2. \tag{5.6}$$

Using (5.3), (5.4) and (5.6) we obtain that for any $N \in \mathbb{N}$

$$\sup_{y\in E, z\in M} |w_m(x,y,z)| = O\left(\frac{1}{m^N}\right). \tag{5.7}$$

Similarly,

$$\sup_{y\in M, z\in E} |w_m(x,y,z)| = O\left(\frac{1}{m^N}\right) \tag{5.8}$$

for any $N \in \mathbb{N}$.

Using (4.7), (5.5), (5.7) and (5.8) one can readily prove the following proposition.

Proposition 5.2. *For $x \in M$ and arbitrary functions $f, g \in C^\infty(M)$ such that f or g vanishes in a neighborhood of x $Q^{(m)}(f,g)(x)$ is rapidly decreasing as $m \to +\infty$.*

This statement can be reformulated as follows. For arbitrary $f, g \in C^\infty(M)$ and $x \in M$ the asymptotics of $Q^{(m)}(f,g)(x)$ as $m \to +\infty$ depends only on the germs of the functions f, g at the point x.

We are going to show how formal integrals can be obtained from the method of stationary phase.

Let ϕ be a smooth function on an open subset $U \subset M$ such that (i) $\operatorname{Re}\phi \le 0$; (ii) there is only one critical point $x_c \in U$ of the function ϕ, which is moreover a nondegenerate critical point; (iii) $\phi(x_c) = 0$.

Consider a classical symbol $\rho(x,m) \in S^0(U \times \mathbb{R})$ (see [21] for definition and notation) which has an asymptotic expansion $\rho \sim \sum_{r\ge 0}(1/m^r)\rho_r(x)$ such that $\rho_0(x_c) \neq 0$, and a smooth nonvanishing volume form dx on U. Set $\mu(m) = \rho(x,m)dx$.

We can apply the method of stationary phase with a complex phase function (see [21] and [28]) to the integral

$$S_m(f) = \int_U e^{m\phi} f \mu(m), \tag{5.9}$$

where $f \in C_0^\infty(U)$. Notice that the phase function in (5.9) is $(1/i)\phi$ so that the condition $\mathrm{Im}\,\big((1/i)\phi\big) \geq 0$ is satisfied.

Taking into account that $\dim_{\mathbb{R}} M = 2n$ and $\phi(x_c) = 0$ we obtain that $S_m(f)$ expands to an asymptotic series $S_m(f) \sim \sum_{r=0}^{\infty}(1/m^{n+r})\tilde{K}_r(f)$ as $m \to +\infty$. Here $\tilde{K}_r$, $r \geq 0$, are distributions supported at x_c and $\tilde{K}_0 = c_n \delta_{x_c}$, where c_n is a nonzero constant. Thus $\mathbb{F}(S_m(f)) = \nu^n \tilde{K}(f)$, where $\mathbb{F}$ is the "formalizer" introduced in Section 2 and $\tilde{K}$ is the functional defined by the formula $\tilde{K} = \sum_{r\geq 0} \nu^r \tilde{K}_r$. Consider the normalized functional $K(f) = \tilde{K}(f)/\tilde{K}(1)$, so that $K(1) = 1$. Then $\mathbb{F}(S_m(f)) = c(\nu)K(f)$, where $c(\nu) = \nu^n c_n + \dots$ is a formal constant.

Proposition 5.3. *For $f \in C_0^\infty(U)$ $S_m(f)$ given by (5.9) expands in an asymptotic series in $1/m$ as $m \to +\infty$. $\mathbb{F}\big(S_m(f)\big) = c(\nu)K(f)$, where K is the formal integral at the point x_c associated to the pair $((1/\nu)\phi, \mathbb{F}(\mu))$ and $c(\nu)$ is a nonzero formal constant.*

Proof. Conditions (a-c) of the definition of formal integral are satisfied. It remains to check condition (d). Let ξ be a vector field on U. Denote by $\mathbf{L}_\xi$ the corresponding Lie derivative. We have $0 = \int_U \mathbf{L}_\xi(e^{m\phi} f \mu(m)) = \int_U e^{m\phi}\big(\xi f + (m\xi\phi + \mathrm{div}_\mu \xi) f\big)\mu(m)$. Applying $\mathbb{F}$ we obtain that $0 = \mathbb{F}\Big(\int_U e^{m\phi}\big(\xi f + (m\xi\phi + \mathrm{div}_\mu\xi) f\big)\mu(m)\Big) = c(\nu) K\Big(\xi f + \big(\xi((1/\nu)\phi) + \mathrm{div}_{\mathbb{F}(\mu)}\xi\big) f\Big)$, which concludes the proof. □

Our next goal is to get an asymptotic expansion of the Bergman kernel $\mathcal{B}_m$ in a neighborhood of the diagonal of $X \times X$ as $m \to +\infty$. An asymptotic expansion of $\mathcal{B}_m$ on the diagonal of $X \times X$ was obtained in [41] (see (5.4)). As in [41], we use the integral representation of the Szegö kernel $\mathcal{S}$ given by the following theorem. We denote $n = \dim_{\mathbb{C}} M$.

Theorem 5.4. (L. Boutet de Monvel and J. Sjöstrand, [9], Theorem 1.5. and § 2.c) *Let $\mathcal{S}(\alpha,\beta)$ be the Szegö kernel of the boundary X of the bounded strictly pseudoconvex domain D in the complex manifold L^*. There exists a classical symbol $a \in S^n(X \times X \times \mathbb{R}^+)$ which has an asymptotic expansion*

$$a(\alpha,\beta,t) \sim \sum_{k=0}^{\infty} t^{n-k} a_k(\alpha,\beta)$$

so that

$$\mathcal{S}(\alpha,\beta) = \int_0^\infty e^{it\varphi(\alpha,\beta)} a(\alpha,\beta,t)\,dt, \tag{5.10}$$

where the phase $\varphi(\alpha,\beta) \in C^\infty(L^* \times L^*)$ is determined by the following properties:

- $\varphi(\alpha,\alpha) = (1/i)\big(k(\alpha)-1\big)$;
- $\bar\partial_\alpha\varphi$ *and* $\partial_\beta\varphi$ *vanish to infinite order along the diagonal;*
- $\varphi(\alpha,\beta) = \overline{-\varphi(\beta,\alpha)}$.

The phase function φ is thus almost analytic at the diagonal of $L^* \times \overline{L^*}$. It is determined up to equivalence at the diagonal.

Fix an arbitrary point $x_0 \in M$. Let s be a local holomorphic frame of L^* over a contractible open neighborhood $U \subset M$ of the point x_0 with local holomorphic coordinates $\{z^k\}$. Then $\alpha(x) = s(x)/\sqrt{k(s(x))}$ is a smooth section of X over U. Set $\Phi_{-1}(x) = \log k\big(s(x)\big)$, so that

$$\alpha(x) = e^{(-1/2)\Phi_{-1}(x)}s(x). \tag{5.11}$$

It follows from the fact that L is a quantum line bundle (i.e., that ω_{-1} is the curvature form of the Hermitian holomorphic line bundle L) that Φ_{-1} is a potential of the form ω_{-1} on U.

Let $\tilde\Phi_{-1}(x,y) \in C^\infty(U\times\overline{U})$ be an almost analytic extension of the potential Φ_{-1} from the diagonal of $U \times \overline{U}$. Denote $D_{-1}(x,y) := \tilde\Phi_{-1}(x,y) + \tilde\Phi_{-1}(y,x) - \Phi_{-1}(x) - \Phi_{-1}(y)$. Since $\tilde\Phi_{-1}(x,x) = \Phi_{-1}(x)$, we have $D_{-1}(x,x) = 0$. In local coordinates

$$D_{-1}(x,y) = -Q_{x_0}(x-y) + O(|x-y|^3), \tag{5.12}$$

where

$$Q_{x_0}(z) = \sum \frac{\partial^2\Phi_{-1}}{\partial z^k\partial\bar z^l}(x_0)z^k\bar z^l$$

is a positive definite quadratic form (since ω_{-1} is a Kähler form).

The following statement is an immediate consequence of (5.12).

Lemma 5.5. *There exists a neighborhood $U' \subset U$ of the point x_0 such that for any two different points $x,y \in U'$ one has* $\operatorname{Re} D_{-1}(x,y) < 0$.

Taking, if necessary, $(1/2)\big(\tilde\Phi_{-1}(x,y)+\overline{\tilde\Phi_{-1}(y,x)}\big)$ instead of $\tilde\Phi_{-1}(x,y)$ choose $\tilde\Phi_{-1}$ such that $\tilde\Phi_{-1}(y,x) = \overline{\tilde\Phi_{-1}(x,y)}$. Replace U by a smaller neighborhood (retaining for it the notation U) such that $\operatorname{Re} D_{-1}(x,y) < 0$ for any different x,y from this neighborhood.

For a point α in the restriction $L^*|_U$ of the line bundle L^* to U represented in the form $\alpha = vs(x)$ with $v \in \mathbb{C}, x \in U$ one has $k(\alpha) = |v|^2k\big(s(x)\big)$.

One can choose the phase function $\varphi(\alpha,\beta)$ in (5.10) of the form

$$\varphi(\alpha,\beta) = (1/i)\big(v\bar w e^{\tilde\Phi_{-1}(x,y)} - 1\big), \tag{5.13}$$

where $\alpha = vs(x), \beta = ws(y) \in L^*|_U$.

Denote $\chi(x,y) := \tilde\Phi_{-1}(x,y) - (1/2)\Phi_{-1}(x) - (1/2)\Phi_{-1}(y)$. Notice that $\chi(x,x) = 0$.

The following theorem is a slight generalization of Theorem 1 from [41].

Theorem 5.6. *There exists an asymptotic expansion of the Bergman kernel $\mathcal{B}_m(\alpha(x), \alpha(y))$ on $U \times U$ as $m \to +\infty$, of the form*

$$\mathcal{B}_m(\alpha(x), \alpha(y)) \sim m^n e^{m\chi(x,y)} \sum_{r \geq 0} (1/m^r) \tilde{b}_r(x, y) \tag{5.14}$$

such that (i) for any compact $E \subset U \times U$ and $N \in \mathbb{N}$

$$\sup_{(x,y)\in E} \left| \mathcal{B}_m(\alpha(x), \alpha(y)) - m^n e^{m\chi(x,y)} \sum_{r=0}^{N-1} (1/m^r) \tilde{b}_r(x, y) \right| = O(m^{n-N}); \tag{5.15}$$

(ii) $\tilde{b}_r(x, y)$ is an almost analytic extension of $b_r(x)$ from the diagonal of $U \times U$, where b_r, $r \geq 0$, are given by (5.4); in particular, $\tilde{b}_0(x, x) = 1$.

Proof. Using integral representations (5.1) and (5.10) one gets for $x, y \in U$

$$\mathcal{B}_m(\alpha(x), \alpha(y)) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{-im\theta} e^{it\varphi(r_\theta(\alpha(x),\alpha(y)))} a(r_\theta(\alpha(x), \alpha(y)), t) dt d\theta. \tag{5.16}$$

Changing variables $t \mapsto mt$ in (5.16) gives

$$\mathcal{B}_m(\alpha(x), \alpha(y)) = \frac{m}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{im(t\varphi(r_\theta(\alpha(x),\alpha(y)))-\theta)} a(r_\theta(\alpha(x), \alpha(y)), mt) dt d\theta. \tag{5.17}$$

In order to apply the method of stationary phase to the integral in (5.17) the following preparations should be made.

Using (5.13) and (5.11) express the phase function of the integral in (5.17) as follows:

$$Z(t, \theta; x, y) := t\varphi(r_\theta(\alpha(x), \alpha(y))) - \theta = (t/i)\big(e^{i\theta} e^{\chi(x,y)} - 1\big) - \theta. \tag{5.18}$$

In order to find the critical points of the phase Z (with respect to the variables (t, θ); the variables (x, y) are parameters) consider first the equation

$$\partial_t Z(t, \theta; x, y) = (1/i)\big(e^{i\theta} e^{\chi(x,y)} - 1\big) = 0. \tag{5.19}$$

It follows from $\tilde{\Phi}_{-1}(y, x) = \overline{\tilde{\Phi}_{-1}(x, y)}$ that $\operatorname{Re} \chi(x, y) = (1/2)D_{-1}(x, y)$. Since $D_{-1}(x, y) < 0$ for $x \neq y$ one has $|e^{\chi(x,y)}| = e^{\operatorname{Re} \chi(x,y)} < 1$ for $x \neq y$ whence it follows that (5.19) holds only if $x = y$ and thus Z has critical points only if $x = y$. Since $\chi(x, x) = 0$ one gets that $\partial_t Z(t, \theta; x, x) = (1/i)(e^{i\theta} - 1)$ and $\partial_\theta Z(t, \theta; x, x) = te^{i\theta} - 1$. As in the proof of Theorem 1 from [41], one shows that for each $x \in U$ the only critical point of the phase function $Z(t, \theta; x, x)$ is $(t = 1, \theta = 0)$. It does not depend on x and, moreover, is nondegenerate.

One has $\operatorname{Im} Z(t, \theta; x, y) = \operatorname{Im} \big((t/i)(e^{i\theta} e^{\chi(x,y)} - 1) - \theta\big) = t\big(1 - \operatorname{Re} (e^{i\theta} e^{\chi(x,y)})\big) \geq 0$ since $|e^{\chi(x,y)}| \leq 1$.

Finally, a simple calculation shows that the germs of the functions $Z(t, \theta; x, y)$ and $(1/i)\chi(x, y)$ at the point $(t = 1, \theta = 0, x = x_0, y = x_0)$ are equal modulo the ideal generated by $\partial_t Z$ and $\partial_\theta Z$.

Applying now the method of stationary phase to the integral in (5.17) one obtains the expansion (5.14) satisfying (5.15).

It follows from (5.4) and (4.3) that $\tilde{b}_r(x,x) = b_r(x)$ and $\tilde{b}_0(x,x) = b_0(x) = 1$. It remains to show that all $\tilde{b}_r, r \geq 0$, are almost analytic along the diagonal of $U \times \overline{U}$. One has

$$\mathcal{B}_m(\alpha(x), \alpha(y)) = e^{(-m/2)(\Phi_{-1}(x)+\Phi_{-1}(y))}\mathcal{B}_m(s(x), s(y)).$$

The function $\mathcal{B}_m(s(x), s(y))$ is holomorphic on $U \times \overline{U}$. Let ξ and η be arbitrary holomorphic and antiholomorphic vector fields on U, respectively. Then $\xi_y \mathcal{B}_m(s(x), s(y)) = 0$ and $\eta_x \mathcal{B}_m(s(x), s(y)) = 0$ (the subscripts x, y show in which variable the vector field acts). Thus

$$\big(\eta_x + \frac{m}{2}\eta_x \Phi_{-1}(x)\big)\mathcal{B}_m(\alpha(x), \alpha(y)) = e^{(-m/2)\Phi_{-1}(x)}\eta_x e^{(m/2)\Phi_{-1}(x)}\mathcal{B}_m(\alpha(x), \alpha(y)) = 0.$$

Analogously, $\big(\xi_y + (m/2)\xi_y\Phi_{-1}(x)\big)\mathcal{B}_m(\alpha(x), \alpha(y)) = 0$. Let A_N be a product of N derivations on $U \times U$. Then, using integral representation (5.17), expand $0 = A_N\big(\eta_x + (m/2)\eta_x\Phi_{-1}(x)\big)\mathcal{B}_m(\alpha(x), \alpha(y))$ to the asymptotic series

(5.20)
$$A_N\big(\eta_x + \frac{m}{2}\eta_x\Phi_{-1}(x)\big)\big(m^n e^{m\chi(x,y)}\sum_{r\geq 0}(1/m^r)\tilde{b}_r(x,y)\big) = e^{m\chi(x,y)}\sum_{r\geq r_0}(1/m^r)c_r(x,y)$$

for some $c_r \in C^\infty(U \times U)$ and $r_0 \in \mathbb{Z}$, and with the norm estimate of the partial sums in the r.h.s. term in (5.20) analogous to (5.15). Since $\chi(x,x) = 0$ one gets that all $c_r(x,x) = 0$. From this fact one can prove by induction over N that $\eta_x\tilde{b}_r$ vanishes to infinite order at the diagonal of $U \times U$. Similarly, $\xi_y\tilde{b}_r$ vanishes to infinite order at the diagonal. Thus $\tilde{b}_r$ is almost analytic along the diagonal. □

Choose a symbol $b(x,y,m) \in S^0((U \times U) \times \mathbb{R})$ such that it has the asymptotic expansion $b \sim \sum_{r=0}^{\infty}(1/m^r)\tilde{b}_r$. Then $\mathcal{B}_m(\alpha(x), \alpha(y))$ is asymptotically equivalent to $m^n e^{m\chi(x,y)}b(x,y,m)$ on $U \times U$. One has $\chi(x,y) + \chi(y,x) = \tilde{\Phi}_{-1}(x,y) + \tilde{\Phi}_{-1}(y,x) - \Phi_{-1}(x) - \Phi_{-1}(y) = D_{-1}(x,y)$ and $\chi(x,y) + \chi(y,z) + \chi(z,x) = \tilde{\Phi}_{-1}(x,y) + \tilde{\Phi}_{-1}(y,z) + \tilde{\Phi}_{-1}(z,x) - \Phi_{-1}(x) - \Phi_{-1}(y) - \Phi_{-1}(z) = T_{-1}(x,y,z)$ (the last equality is the definition of T_{-1}). Thus the functions

$$v_m(x,y) = \mathcal{B}_m(\alpha(x), \alpha(y))\mathcal{B}_m(\alpha(y), \alpha(x)) \text{ and}$$
$$w_m(x,y,z) = \mathcal{B}_m(\alpha(x), \alpha(y))\mathcal{B}_m(\alpha(y), \alpha(z))\mathcal{B}_m(\alpha(z), \alpha(x))$$

are asymptotically equivalent to

$$m^{2n}e^{mD_{-1}(x,y)}b(x,y,m)b(y,x,m) \text{ and } m^{3n}e^{mT_{-1}(x,y,z)}b(x,y,m)b(y,z,m)b(z,x,m)$$

respectively. It is easy to show that for the functions $\phi^x_{-1}(y) = D_{-1}(x,y)$ and $\psi^x_{-1}(y,z) = T_{-1}(x,y,z)$ the points $y = x$ and $(y,z) = (x,x)$ respectively are nondegenerate critical ones.

Since $\tilde{b}_0(x,x) = 1$ one can take a smaller contractible neighborhood $V \Subset U$ of x_0 such that $\tilde{b}_0(x,y)$ does not vanish on the closure of $V \times V$. One can choose V such that for any $x \in V$ the only critical points of the functions $\phi^x_{-1}(y)$ on V and $\psi^x_{-1}(y,z)$ on $V \times V$ are $y = x$ and $(y,z) = (x,x)$ respectively.

The identity $T_{-1}(x,y,z) = (1/2)(D_{-1}(x,y) + D_{-1}(y,z) + D_{-1}(z,x))$ implies that $\operatorname{Re} T_{-1}(x,y,z) \le 0$ for $x,y,z \in V$.

The symbol $b(x,y,m)$ does not vanish on $V \times V$ for sufficiently big values of m. It follows from (5.4) that $1/u_m(x)$ and $(m^n b(x,x,m))^{-1}$ are asymptotically equivalent for $x \in V$. Denote

(5.21)

$$\mu_x(m) = \frac{b(x,y,m)b(y,x,m)}{b(x,x,m)}\Omega(y), \ \tilde{\mu}_x(m) = \frac{b(x,y,m)b(y,z,m)b(z,x,m)}{b(x,x,m)}\Omega(y)\Omega(z).$$

Taking into account (4.6) we get for $f,g \in C_0^\infty(V)$ and $x \in V$ the following asymptotic equivalences,

(5.22)

$$\big(I^{(m)}f\big)(x) \sim m^n \int_V e^{m\phi^x_{-1}} f \mu_x(m) \text{ and } Q^{(m)}(f,g)(x) \sim m^{2n} \int_{V\times V} e^{m\psi^x_{-1}} (f \otimes g)\tilde{\mu}_x(m).$$

(In (5.22) $\big(f \otimes g\big)(y,z) = f(y)g(z)$.)

Applying Proposition 5.3 to the first integral in (5.22) we obtain that $\mathbb{F}\big(\big(I^{(m)}f\big)(x)\big) = c(\nu,x)L^I_x(f)$, where the functional L^I_x on $\mathcal{F}(V)$ is the formal integral at the point x associated to the pair $((1/\nu)\phi^x_{-1}, \mathbb{F}(\mu_x))$ and $c(\nu,x)$ is a formal function. It is easy to show that $c(\nu,x)$ is smooth.

Similarly we obtain from (5.22) that $\mathbb{F}\big(Q^{(m)}(f,g)(x)\big) = d(\nu,x)L^Q_x(f \otimes g)$ where the functional L^Q_x on $\mathcal{F}(V \times V)$ is the formal integral at the point (x,x) associated to the pair $((1/\nu)\psi^x_{-1}, \mathbb{F}(\tilde{\mu}_x))$ and $d(\nu,x)$ is a smooth formal function.

Since the unit constant 1 is a contravariant symbol of the unit operator $\mathbf{1}$, $T^{(m)}_1 = \mathbf{1}$, and $\sigma(\mathbf{1}) = 1$, we have $I^{(m)}1 = 1, Q^{(m)}(1,1) = 1$, and thus $\mathbb{F}(I^{(m)}1) = 1$ and $\mathbb{F}\big(Q^{(m)}(1,1)\big) = 1$. Taking the functions f,g in (5.22) to be equal to 1 in a neighborhood of x and applying Proposition 5.1 and Proposition 5.2 we get that $c(\nu,x) = 1$ and $d(\nu,x) = 1$.

Since $b_0(x,y)$ does not vanish on $V \times V$ we can find a formal function $\tilde{s}(x,y)$ on $V \times V$ such that $\mathbb{F}(b(x,y,m)) = e^{\tilde{s}(x,y)}$. Set $s(x) = \tilde{s}(x,x)$. In these notations

$$\begin{aligned}(5.23) \quad \mathbb{F}(\mu_x) &= \exp(\tilde{s}(x,y) + \tilde{s}(y,x) - s(x))\Omega(y) \text{ and} \\ \mathbb{F}(\tilde{\mu}_x) &= \exp(\tilde{s}(x,y) + \tilde{s}(y,z) + \tilde{s}(z,x) - s(x))\Omega(y)\Omega(z).\end{aligned}$$

It follows from Theorem 5.6 that $\tilde{s}$ is an almost analytic extension of the function s from the diagonal of $V \times V$. According to (5.4), $\mathbb{F}(u_m) = (1/\nu^n)e^s$.

Denote $\tilde{\Phi} = (1/\nu)\tilde{\Phi}_{-1} + \tilde{s}$, $\Phi = (1/\nu)\Phi_{-1} + s$, $D(x,y) = \tilde{\Phi}(x,y) + \tilde{\Phi}(y,x) - \Phi(x) - \Phi(y) = (1/\nu)D_{-1}(x,y) + (\tilde{s}(x,y) + \tilde{s}(y,x) - s(x) - s(y))$, $T(x,y,z) = \tilde{\Phi}(x,y) + \tilde{\Phi}(y,z) +$

$\tilde{\Phi}(z,x) - \Phi(x) - \Phi(y) - \Phi(z)$. The pair $((1/\nu)\phi^x_{-1}, \mathbb{F}(\mu_x)) = ((1/\nu)\phi^x_{-1}, \exp(\tilde{s}(x,y) + \tilde{s}(y,x) - s(x))\Omega(y))$ is then equivalent to the pair $(\phi^x, e^s\Omega)$, where $\phi^x(y) = D(x,y)$. Similarly, the pair $((1/\nu)\psi^x_{-1}, \mathbb{F}(\tilde{\mu}_x))$ is equivalent to the pair $(\psi^x, e^s\Omega \otimes e^s\Omega)$, where $\psi^x(y,z) = T(x,y,z)$.

Thus we arrive at the following proposition.

Proposition 5.7. *For $f,g \in C_0^\infty(V)$, $x \in V$, $\big(I^{(m)}f\big)(x)$ and $Q^{(m)}(f,g)(x)$ expand in asymptotic series in $1/m$ as $m \to +\infty$. $\mathbb{F}\big(\big(I^{(m)}f\big)(x)\big) = L^I_x(f)$ and $\mathbb{F}\big(Q^{(m)}(f,g)(x)\big) = L^Q_x(f \otimes g)$, where the functional L^I_x on $\mathcal{F}(V)$ is the formal integral at the point x associated to the pair $(\phi^x, e^s\Omega)$ and the functional L^Q_x on $\mathcal{F}(V \times V)$ is the formal integral at the point (x,x) associated to the pair $(\psi^x, e^s\Omega \otimes e^s\Omega)$.*

Now let $\star$ denote the star-product with separation of variables on (V, ω_{-1}) corresponding to the formal deformation $\omega = -i\partial\bar{\partial}\Phi$ of the form $(1/\nu)\omega_{-1}$, so that Φ is a formal potential of ω. Let I be the corresponding formal Berezin transform, $\tilde{\omega}$ the formal form parametrizing the dual star-product $\tilde{\star}$ and Ψ the solution of (2.3) so that $\mu_{tr} = e^{\Phi+\Psi} dz d\bar{z}$ is a formal trace density for the star-product $\star$.

Choose a classical symbol $\rho(x,m) \in S^0(V \times \mathbb{R})$ which has an asymptotic expansion $\rho \sim \sum_{r \geq 0}(1/m^r)\rho_r$ such that

$$\mathbb{F}(\rho)e^s\Omega = \mu_{tr}. \tag{5.24}$$

Clearly, (5.24) determines $\mathbb{F}(\rho)$ uniquely.

For $f \in C_0^\infty(V)$ and $x \in V$ consider the following integral

$$(P_m f)(x) = m^n \int_V e^{m\phi^x_{-1}} f \rho \mu_x, \tag{5.25}$$

where $\phi^x_{-1}(y) = D_{-1}(x,y)$ and μ_x is given by (5.21).

Proposition 5.8. *For $f \in C_0^\infty(V)$ and $x \in V$ $(P_m f)(x)$ has an asymptotic expansion in $1/m$ as $m \to +\infty$. $\mathbb{F}\big((P_m f)(x)\big) = c(\nu)(If)(x)$, where $c(\nu)$ is a nonzero formal constant.*

Proof. It was already shown that the phase function $(1/i)\phi^x_{-1}$ of integral (5.25) satisfies the conditions required in the method of stationary phase. Thus Proposition 5.3 can be applied to (5.25). We get that $\mathbb{F}\big((P_m f)(x)\big) = c(\nu,x)K_x(f)$, where K_x is a formal integral at the point x associated to the pair $((1/\nu)\phi^x_{-1}, \mathbb{F}(\rho\mu_x))$ and $c(\nu,x)$ is a nonvanishing formal function on V. It follows from (5.23) and (5.24) that $\mathbb{F}(\rho\mu_x) = \mathbb{F}(\rho)\mathbb{F}(\mu_x) = \mathbb{F}(\rho)\exp(\tilde{s}(x,y) + \tilde{s}(y,x) - s(x))\Omega(y) = \exp(\tilde{s}(x,y) + \tilde{s}(y,x) - s(x) - s(y))\mu_{tr} = \exp(D(x,y) - (1/\nu)D_{-1}(x,y))\mu_{tr} = \exp(\phi^x - (1/\nu)\phi^x_{-1})\mu_{tr}$, where $\phi^x(y) = D(x,y)$. The pair $((1/\nu)\phi^x_{-1}, \mathbb{F}(\rho\mu_x))$ is thus equivalent to the pair (ϕ^x, μ_{tr}). Applying Theorem 3.2 we get that

$$\mathbb{F}\big((P_m f)(x)\big) = c(\nu,x)\big(If\big)(x). \tag{5.26}$$

It remains to show that $c(\nu,x)$ is actually a formal constant. Let x_1 be an arbitrary point of V. Choose a function $\epsilon \in C_0^\infty(V)$ such that $\epsilon = 1$ in a neighborhood $W \subset V$

of x_1. Let ξ be a vector field on V. Then, using (5.23), we obtain

$$(5.27)\quad \frac{1}{\nu}\xi_x\phi^x_{-1}(y) + \mathbb{F}\left(\frac{\xi_x\mu_x}{\mu_x}(y)\right) = \\ \frac{1}{\nu}\xi_x D_{-1}(x,y) + \xi_x(\tilde{s}(x,y) + \tilde{s}(y,x) - s(x)) = \xi_x D(x,y) = \xi_x\phi^x.$$

On the one hand, taking into account (5.27) we get for $x \in W$ that

(5.28)
$$\mathbb{F}\big((\xi P_m\epsilon)(x)\big) = \mathbb{F}\left(m^n\xi\int_V e^{m\phi^x_{-1}}\epsilon\rho\mu_x\right) = \mathbb{F}\left(m^n\int_V e^{m\phi^x_{-1}}\left(m\xi_x\phi^x_{-1} + \frac{\xi_x\mu_x}{\mu_x}\right)\epsilon\rho\mu_x\right) = \\ c(\nu,x)I\left(\frac{1}{\nu}\xi_x\phi^x_{-1}(y) + \mathbb{F}\left(\frac{\xi_x\mu_x}{\mu_x}(y)\right)\right) = c(\nu,x)I(\xi_x\phi^x) = 0.$$

The last equality in (5.28) follows from Lemma 3.3. On the other hand, for $x \in W$ we have from (5.26) that $\mathbb{F}\big((P_m\epsilon)(x)\big) = c(\nu,x)$, from whence $\mathbb{F}\big((\xi P_m\epsilon)(x)\big) = \xi\mathbb{F}\big((P_m\epsilon)(x)\big) = \xi c(\nu,x)$. Thus we get from (5.28) that $\xi c(\nu,x) = 0$ on W for an arbitrary vector field ξ, from which the Proposition follows. □

It follows from (5.22) and (5.25) that for $f \in C_0^\infty(V)$ $\big(I^{(m)}(f\rho)\big)(x)$ is asymptotically equivalent to $(P_m f)(x)$. Passing to formal asymptotic series we get from Proposition 5.7 and Proposition 5.8 that $c(\nu)(If)(x) = \mathbb{F}\big((P_m f)(x)\big) = \mathbb{F}\big(\big(I^{(m)}(f\rho)\big)(x)\big) = L^I_x(f\mathbb{F}(\rho))$, where L^I_x is the formal integral at the point x associated to the pair $(\phi^x, e^s\Omega)$. Thus

$$(5.29)\qquad c(\nu)(If)(x) = L^I_x(f\mathbb{F}(\rho)).$$

The formal function $\mathbb{F}(\rho)$ is invertible (see (5.24)). Setting $f = 1/\mathbb{F}(\rho)$ in (5.29) we get $c(\nu)\big(I(1/\mathbb{F}(\rho))\big)(x) = L^I_x(1) = 1$ for all $x \in V$. Since the formal Berezin transform is invertible and $I(1) = 1$, we finally obtain that

$$(5.30)\qquad \mathbb{F}(\rho) = c(\nu).$$

Now (5.24) can be rewritten as follows,

$$(5.31)\qquad c(\nu)e^s\Omega = \mu_{tr} = e^{\Phi+\Psi}dzd\bar{z}.$$

In local holomorphic coordinates the symplectic volume Ω can be expressed as follows, $\Omega = e^\theta dzd\bar{z}$. The closed (1,1)-form $\omega_{can} = -i\partial\bar{\partial}\theta$ does not depend on the choice of local holomorphic coordinates and is defined globally on M. The form ω_{can} is the curvature form of the canonical connection of the canonical holomorphic line bundle on M equipped with the Hermitian fibre metric determined by the volume form Ω. Its de Rham class $\varepsilon = [\omega_{can}]$ is the first Chern class of the canonical holomorphic line bundle on M and thus depends only on the complex structure on M. The class ε is called the canonical class of the complex manifold M.

One can see from (5.31) that $c(\nu) = c_0 + \nu c_1 + \dots$, where $c_0 \neq 0$. Thus there exists a formal constant $d(\nu)$ such that $e^{d(\nu)} = c(\nu)$ and $d(\nu) + s + \theta = \Phi + \Psi$. Therefore the formal potential Ψ of the form $\tilde{\omega}$ is expressed explicitly, $\Psi = d(\nu) - (1/\nu)\Phi_{-1} + \theta$, from whence it follows that

$$\tilde{\omega} = -(1/\nu)\omega_{-1} + \omega_{can}. \tag{5.32}$$

Formula (5.32) defines $\tilde{\omega}$ globally on M. Thus the corresponding star-product $\tilde{\star}$ and therefore its dual star-product $\star$ are also globally defined.

Theorem 3.2, Theorem 3.4, Proposition 5.1, Proposition 5.2 Proposition 5.7, formulas (5.29), (5.30) and (5.31) imply the following theorem, which is the central technical result of the paper.

Theorem 5.9. *For any $f, g \in C^\infty(M)$ and $x \in M$ $(I^{(m)}f)(x)$ and $Q^{(m)}(f,g)(x)$ expand to asymptotic series in $1/m$ as $m \to +\infty$. $\mathbb{F}\Big(\big(I^{(m)}f\big)(x)\Big) = \big(If\big)(x)$ and $\mathbb{F}\big(Q^{(m)}(f,g)(x)\big) = Q(f,g)(x)$, where I and Q are the formal Berezin transform and the formal twisted product corresponding to the star-product with separation of variables $\star$ on (M, ω_{-1}) whose dual star-product $\tilde{\star}$ on $(M, -\omega_{-1})$ is parametrized by the formal form $\tilde{\omega} = -(1/\nu)\omega_{-1} + \omega_{can}$.*

Remark. As shown in [37] we have the following chain of inequalities

$$|I^{(m)}(f)|_\infty = |\sigma(T_f^{(m)})|_\infty \quad \le \quad ||T_f^{(m)}|| \quad \le \quad |f|_\infty\,. \tag{5.33}$$

Here $||..||$ denotes the operator norm with respect to the norm of the sections of L^m and $|..|_\infty$ the sup-norm on $C^\infty(M)$. Choose as $x_e \in M$ a point with $|f(x_e)| = |f|_\infty$. From Theorem 5.9 and the fact that the formal Berezin transform has as leading term the identity it follows that $|(I^{(m)}f)(x_e) - f(x_e)| \le A/m$ with a suitable constant A. This implies $\big||f(x_e)| - |(I^{(m)}f)(x_e)|\big| \le A/m$ and hence

$$|f|_\infty - \frac{A}{m} = |f(x_e)| - \frac{A}{m} \quad \le \quad |(I^{(m)}f)(x_e)| \quad \le \quad |(I^{(m)}f)|_\infty\,. \tag{5.34}$$

Putting (5.33) and (5.34) together we obtain

$$|f|_\infty - \frac{A}{m} \quad \le \quad ||T_f^{(m)}|| \quad \le \quad |f|_\infty\,. \tag{5.35}$$

This provides another proof of [6], Theorem 4.1.

6. The identification of the Berezin-Toeplitz star-product

In this section $\star$ will denote the star-product with separation of variables on (M, ω_{-1}) whose dual $\tilde{\star}$ is the star-product with separation of variables on $(M, -\omega_{-1})$ parametrized by the formal form $\tilde{\omega} = -(1/\nu)\omega_{-1} + \omega_{can}$.

Let $I = 1 + \nu I_1 + \nu^2 I_2 + \ldots$ and $Q = Q_0 + \nu Q_1 + \ldots$ denote the formal Berezin transform and the formal twisted product corresponding to $\star$. Theorem 5.9 asserts that for given $f, g \in C^\infty(M)$, $r \in \mathbf{N}$, $x \in M$ there exist constants A, B such that for sufficiently big values of m the following inequalities hold:

$$(6.1)\qquad \left|(I^{(m)}f)(x)-\sum_{i=0}^{r-1}\frac{1}{m^i}I_i(f)(x)\right|\le\frac{A}{m^r},$$

$$(6.2)\qquad \left|Q^{(m)}(f,g)(x)-\sum_{i=0}^{r-1}\frac{1}{m^i}Q_i(f,g)(x)\right|\le\frac{B}{m^r}.$$

It was proved in [36],[38] that Berezin-Toeplitz quantization on a compact Kähler manifold M gives rise to a star-product on M. This star-product $\star^{BT}$ is given by a sequence of bilinear operators $\{C_k\}$, $k\ge 0$, on $C^\infty(M)$ satisfying the following conditions. For $f,g\in C^\infty(M)$ and any $r\in\mathbf{N}$ there exists a constant C such that

$$(6.3)\qquad \left\|T_f^{(m)}T_g^{(m)}-T_{f\star_{[r]}g}^{(m)}\right\|\le C/m^r,$$

where $f\star_{[r]}g=\sum_{k=0}^{r-1}(1/m^k)C_k(f,g)$. The conditions (6.3) determine the star-product $\star^{BT}$ uniquely. We call $\star^{BT}$ the Berezin-Toeplitz star-product.

Recall that for $f,g\in C^\infty(M)$ $\sigma(T_f^{(m)})=I^{(m)}(f)$, $\sigma(T_f^{(m)}T_g^{(m)})=Q^{(m)}(f,g)$.

Passing from operators to their covariant symbols in (6.3) and using the inequality $|\sigma(A)|\le\|A\|$ we get that

$$(6.4)\qquad \left|Q^{(m)}(f,g)(x)-I^{(m)}(f\star_{[r]}g)(x)\right|\le C/m^r.$$

It follows from (6.1) that

$$(6.5)\qquad \left|\frac{1}{m^k}I^{(m)}\big(C_k(f,g)\big)(x)-\sum_{i=0}^{r-k-1}\frac{1}{m^{i+k}}I_i\big(C_k(f,g)\big)(x)\right|\le\frac{A_k}{m^r}.$$

Summing up inequalities (6.2) and (6.5) for $k=0,1,\dots,r-1$, we obtain that

$$(6.6)\qquad \Big|\Big(Q^{(m)}(f,g)(x)-I^{(m)}(f\star_{[r]}g)(x)\Big)-\sum_{i=0}^{r-1}\frac{1}{m^i}\Big(Q_i(f,g)(x)-\sum_{j+k=i}I_j\big(C_k(f,g)\big)(x)\Big)\Big|\le\frac{D}{m^r}.$$

for some constant D. It follows from (6.4) and (6.6) that

$$\left|\sum_{i=0}^{r-1}\frac{1}{m^i}\Big(Q_i(f,g)(x)-\sum_{j+k=i}I_j\big(C_k(f,g)\big)(x)\Big)\right|\le\frac{E}{m^r},$$

for some constant E, which infers that for $i=0,1,\dots$

$$(6.7)\qquad Q_i(f,g)=\sum_{j+k=i}I_j(C_k(f,g)).$$

Equalities (6.7) mean that $Q(f,g)=I(f\star^{BT}g)$. Since I is invertible we immediately obtain that the star-products $\star'$ and $\star^{BT}$ coincide. Thus the Berezin-Toeplitz deformation quantization is completely identified as the deformation quantization with separation of variables on $(\overline{M}, \omega_{-1})$ whose star-product $\star^{BT}$ is opposite to $\tilde{\star}$.

Using (2.2) we can calculate the characteristic class $cl(\star^{BT})$ of the Berezin-Toeplitz star-product $\star^{BT}$.

It follows from (2.2) and (5.32) that the characteristic class of the star-product $\tilde{\star}$ equals to $cl(\tilde{\star}) = (1/i)\big(-[(1/\nu)\omega_{-1}] + \varepsilon/2\big)$. It is easy to show that the characteristic class of the opposite star-product $\star'$ is equal to $-cl(\tilde{\star})$. Since $\star^{BT} = \star'$, we finally get that the characteristic class of the Berezin-Toeplitz deformation quantization is given by the formula $cl(\star^{BT}) = (1/i)\big([(1/\nu)\omega_{-1}] - \varepsilon/2\big)$.

The characteristic class of the Berezin-Toeplitz deformation quantization was first calculated by Eli Hawkins in [20] by K-theoretic methods.

As a concluding remark we would like to draw the readers attention to the fact that the classifying form ω of the star-product $\star$ is the formal object corresponding to the asymptotic expansion as $m \to +\infty$ of the pullback $\omega^{(m)}$ of the Fubini-Study form on the projective space $\mathbb{P}(H_m^*)$ via Kodaira embedding of M into $\mathbb{P}(H_m^*)$. Here H_m^* denotes the Hilbert space dual to $H_m = \Gamma_{hol}(L^m)$ (see Section 4). It was proved by Zelditch [41] that $\omega^{(m)}$ admits a complete asymptotic expansion in $1/m$ as $m \to +\infty$. As an easy consequence of the results obtained in this article one can show that $\mathbb{F}(\omega^{(m)}) = \omega$.

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