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Parabolic differential equations with Hölder continuous and unbounded coefficients

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Abstract. We consider one-dimensional linear parabolic differential equations with Hölder continuous, unbounded drifts. We first extend the classical parametrix method of E. Levi to prove existence of fundamental solutions. From this we derive existence and uniqueness of solutions for the corresponding Cauchy problem. Our results extend to equations in higher dimensions with additional unbounded potential.

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1. Introduction

Let $T > 0$ be fixed and let u be a function on $[0,T] \times \mathbb{R}$. We are going to study the differential equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + b(t,x)\frac{\partial u}{\partial x} \tag{1.1a}$$

subject to the continuous initial condition

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}. \tag{1.1b}$$

We assume that the drift b in (1.1a) is locally Hölder continuous of exponent $\alpha \in (0,1]$, and at most Hölder growing of exponent $\beta \in (0,1)$, while u_0 is assumed to be of quadratic exponential growth (the precise conditions are given in Section 2). Under these conditions we show that (1.1a) has a fundamental solution (Theorem 2.3), and this gives rise to a unique solution of (1.1) in the class of functions of quadratic exponential growth (Theorem 2.4). The interesting limiting case $\beta = 1$ (Lipschitz growth) cannot be treated by our approach, for details see the remark that follows Lemma 3.2.

Remark. The connection of (1.1) with stochastic ordinary differential equations [F2] indicates that these results had to be expected. Indeed these results are known under additional differentiability conditions on b, see e.g. [Be],[E1],[E2]. But these conditions rule out a number of basic examples which arise naturally in the context of stochastic partial differential equations (see the example given in Section 4). The case with (two-sided) unbounded, only Hölder continuous drifts b has apparently not been treated in the literature.

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Let us now recall the basic idea of E. Levi's parametrix method [Le], which we employ in the following. We do this in *informal way,* because we only want to motivate equation (1.6) which is the starting point for us later. Assume that u solves (1.1a). We write this as

$$\frac{\partial u}{\partial t} = Au + \Phi, \tag{1.2}$$

with $A = \frac{1}{2}\partial_x^2$ and $\Phi := b\partial_x u$. Under suitable conditions on Φ and on $u(s) := u(s, \cdot)$ the theory of strongly continuous semigroups shows that for $t > s \geq 0$ we have

$$u(t) = e^{(t-s)A}u(s) + \int_s^t e^{(t-r)A}\Phi(r)dr, \tag{1.3}$$

where e^{tA} operates on functions by convolution with the "parametrix"

$$Z_t(x) := (2\pi t)^{-\frac{1}{2}} e^{-\frac{x^2}{2t}}. \tag{1.4}$$

Putting informally $u(s, x) = \delta_y(x)$ – the Dirac delta function at fixed $y \in \mathbb{R}$ – the representation (1.3) shows that the fundamental solution $p(t, x, s, y)$ of (1.2) should satisfy

$$p(t, x, s, y) = Z_{t-s}(x - y) + \int_s^t \int_{\mathbb{R}} Z_{t-r}(x - z)\Phi(r, z)dzdr. \tag{1.5}$$

Denoting $L = \frac{1}{2}\partial_x^2 + b\partial_x$ and applying $L - \partial_t$ to this equation shows that Φ (which also depends on s and y) should satisfy the Volterra type equation

$$\Phi(t, x, s, y) = \Phi_1(t, x, s, y) + \int_s^t \int_{\mathbb{R}} \Phi_1(t, x, r, z)\Phi(r, z, s, y)dzdr, \tag{1.6}$$

with singular integral kernel

$$\Phi_1(t, x, s, y) := b(t, x)\partial_x Z_{t-s}(x - y). \tag{1.7}$$

Subsequently we will show that (1.6) has a solution Φ which defines a fundamental solution by (1.5), and this in turn gives rise to the unique solution of (1.1).

Remark. For bounded coefficients the construction of a fundamental solution based on the parametrix method relies heavily on this boundedness, see e.g. [F1], [KO], [E2]. We will see that basically the same construction works for Hölder growing drifts. In this sense our approach is elementary. But there are some non obvious modifications in the proof (they are discussed in an informal way at the beginning of Section 3). We stress that these modifications, with only minor changes, also go through in the d-dimensional case, when we add a Hölder growing potential, and when the second order term in (1.1a) is replaced by a uniformly elliptic operator with bounded, Hölder continuous coefficients, cf. Eq. (4.1) and its discussion in Section 4. (The details of these generalizations will be given in [Kr].) We restrict our discussion here to Eq. (1.1) which exhibits all the basic difficulties. This makes the arguments more transparent, and keeps the technicalities at a minimum.

The paper is organized as follows. In Section 2 we give the basic definitions and state the main results. Proofs and some more technical properties are postponed to Section 3. In the final Section 4 we make some remarks on the extension of our results, and we discuss their relation to stochastic parabolic differential equations.

2. Statement of results

Throughout this paper we fix $\alpha \in (0,1]$, $\beta \in (0,1)$ and $T > 0$. For technical reasons it will sometimes be necessary to consider time points $t_0 \in [0,T)$. A function f on $[t_0,T] \times \mathbb{R}$ is called *locally Hölder continuous in x with exponent α, uniformly with respect to $t \in [t_0,T]$*, if for each bounded set $K \subset \mathbb{R}$ there is a constant c_K such that

$$|f(t,x) - f(t,y)| \leq c_K|x-y|^\alpha, \quad \forall t \in [t_0,T],\ \forall x,y \in K. \tag{2.1a}$$

We say that f is *globally Hölder bounded with exponent β* if

$$|f(t,x)| \leq c_f(|x|^\beta + 1), \quad \forall t \in [t_0,T],\ \forall x \in \mathbb{R}, \tag{2.1b}$$

with a suitable constant c_f. For the rest of this paper we assume the following on (1.1):

(A) The drift b is a continuous function in $[0,T]\times\mathbb{R}$, which is locally Hölder continuous in x with exponent α, uniformly with respect to $t \in [0,T]$, and globally Hölder bounded with exponent β.

(B) u_0 is a continuous function of *quadratic exponential growth* (with parameter $h > 0$), i.e. for some $c_0 \geq 0$ the estimate $|u_0(x)| \leq c_0 e^{hx^2}$ holds for all $x \in \mathbb{R}$.

Our first result concerns Equation (1.6). Define the convolution type multiplication

$$f * g(t,x,s,y) := \int_s^t \int_{\mathbb{R}} f(t,x,r,z) g(r,z,s,y) dz dr$$

for functions f,g on $\Delta_T := \{(t,x,s,y) \in \mathbb{R}^4 : t,s \in [0,T], t > s\}$ which are such that the (Lebesgue) integrals exist. Then we can write (1.6) as $\Phi = \Phi_1 + \Phi_1 * \Phi$, and m iterations of this equation give $\Phi = \Phi_1 + \Phi_1^{*2} + \cdots + \Phi_1^{*m} + \Phi_1^{*m} * \Phi$. The following key proposition states that a solution Φ to (1.6) can indeed be obtained by taking m to infinity.

Proposition 2.1. *Assume the drift b satisfies* (A). *Then $\Phi_m := \Phi_1^{*m}$ is well-defined for all $m \geq 2$, and a pointwise solution of* (1.6) *is given by the Neumann series*

$$\Phi(t,x,s,y) := \sum_{m=1}^{\infty} \Phi_m(t,x,s,y), \tag{2.2}$$

which converges uniformly on compact sets $K \subset \Delta_T$. Φ has the following properties:

(a) *Fix $(s,y) \in [0,T) \times \mathbb{R}$ and $s < t_0 < T$. Then $\Phi(t,x,s,y)$ is locally Hölder continuous in x for every exponent $\hat{\alpha} \in (0,\alpha)$, uniformly with respect to $t \in [t_0,T]$.*

(b) *Fix $(s,y) \in [0,T) \times \mathbb{R}$. Then $\Phi(t,x,s,y)$ is continuous in $(t,x) \in (s,T] \times \mathbb{R}$.*

(c) *For any $\lambda^* \in (0,1)$ and any $h > 0$ there exists a constant $C = C_{\lambda^*,h} > 0$ such that*

$$|\Phi(t,x,s,y)| \le C\frac{e^{-\frac{\lambda^*(x-y)2}{2(t-s)}}}{t-s}e^{hy^2}. \tag{2.3}$$

In (2.3) we tacitly assume that this estimate holds on the full domain Δ_T. Subsequently we will suppress this domain whenever there is no danger of confusion. We remark that the properties (a,b,c) imply that the function p defined in (1.5) is sufficiently smooth, and that one can interchange derivatives with the integral in (1.5). It is then not hard to conclude that p is indeed a fundamental solution to (1.1a). In the following we say that a function $(t,x) \mapsto g(t,x)$ is $C^{1,2}$ if g has continuous partial derivatives $\partial_x^2 g$, $\partial_x g$, $\partial_t g$ on its domain of definition.

Definition 2.2. Let $L = \frac{1}{2}\partial_x^2 + b\partial_x$. A *fundamental solution of $\partial_t u = Lu$* is a function p on Δ_T with the following two properties:

(F1) Fix $(s,y) \in [0,T) \times \mathbb{R}$. Then $p(\cdot,\cdot,s,y)$ is $C^{1,2}$ and satisfies $(L-\partial_t)p(t,x,s,y) = 0$ for all $(t,x) \in (s,T] \times \mathbb{R}$.

(F2) For every continuous function f of quadratic exponential growth and all $x \in \mathbb{R}$

$$\lim_{t\downarrow s}\int_{\mathbb{R}} p(t,x,s,y)f(y)dy = f(x). \tag{2.4}$$

We remark that property (2.4) is often required only for bounded functions f, but we need this property for our unbounded initial conditions u_0.

Theorem 2.3. *Let Z and Φ be given by* (1.4) *and* (2.2), *and suppose that b satisfies* (A). *Then Equation* (1.5) *defines a fundamental solution p of $\partial_t u = Lu$.*

As expected the fundamental solution p gives rise to the solution of the corresponding Cauchy problem. Recall that a continuous function u on $[0,T] \times \mathbb{R}$ is called a *solution to the Cauchy problem* (1.1) if u is $C^{1,2}$ on $(0,T] \times \mathbb{R}$ and satisfies (1.1a) on this domain, and moreover (1.1b) holds.

Theorem 2.4. *Assume that b satisfies* (A) *and that u_0 satisfies* (B) *with $h < \frac{1}{2T}$. Then*

$$u(t,x) := \int_{\mathbb{R}} p(t,x,0,y)u_0(y)dy, \quad (t,x) \in (0,T] \times \mathbb{R}, \tag{2.5}$$

and $u(0,x) := u_0(x)$, $x \in \mathbb{R}$, is well-defined, and u is a solution to the Cauchy problem (1.1). *This solution is unique in the class of functions which are of some quadratic exponential growth in x, uniformly with respect to $t \in [0,T]$.*

Observe that the unbounded drift b in (1.1a) does not decrease the natural maximal existence interval $[0,T]$ for the solution u: For $b = 0$ the special solution $\int_{\mathbb{R}} Z_t(x-y)e^{hy^2}dy$ to (1.1a) does not explode at $t = T$ only if $h < \frac{1}{2T}$, and under this condition also a solution to (1.1a) - with unbounded b - exists for all $t \in [0,T]$.

3. Proofs

The basic problem with the construction of a fundamental solution via $\Phi = \sum \Phi_m$ is the convergence of the series $\sum \Phi_m$ given in (2.2). This is because the Φ_m contain products of the unbounded drift b, so the usual sup-norm estimates on these terms do not work. The key idea to prove the convergence of $\sum \Phi_m$ is to estimate the terms $\Phi_m = \Phi_1 * \Phi_{m-1}$ successively as follows. First fix $\lambda \in (0,1)$ and recall the well-known estimates

$$|\partial_x^n Z_{t-s}(x-y)| \le c_{\lambda,n} \frac{e^{-\frac{\lambda(x-y)^2}{2(t-s)}}}{(t-s)^{\frac{n+1}{2}}}, \quad n \in \mathbb{N}_0, \tag{3.1}$$

which hold for suitable constants $c_{\lambda,n}$. Then fix $\varepsilon \in (0,1)$ and notice that (3.1) and our assumption $|b(t,x)| \le c_b(|x|^\beta + 1)$ imply that $\Phi_1(t,x,s,y) = b(t,x)\partial_x Z_{t-s}(x-y)$ satisfies

$$|\Phi_1(t,x,s,y)| \le c_{\lambda,1} c_b (|x|^\beta + 1) e^{-\varepsilon \frac{\lambda(x-y)^2}{2(t-x)}} \frac{e^{-(1-\varepsilon)\frac{\lambda(x-y)^2}{2(t-s)}}}{t-s}. \tag{3.2}$$

Clearly the product of the first two functions is bounded in x. More precisely we have the following estimate which plays a key role in the present paper,

$$|x|^\beta e^{-\varepsilon\frac{\lambda(x-y)^2}{2(t-s)}} \le \left(|y| + (\frac{2\beta(t-s)}{\lambda\varepsilon})^{1/2} \right)^\beta \quad \text{on } \Delta_T, \tag{3.3}$$

and which follows by a simple extreme value consideration. Hence (3.2) yields

$$|\Phi_1(t,x,s,y)| \le c_1[(|y| + k(\varepsilon\lambda))^\beta + 1] \frac{e^{-\frac{(1-\varepsilon)\lambda(x-y)^2}{2(t-s)}}}{t-s}, \tag{3.4}$$

where $c_1 = c_{\lambda,1}c_b$ depends on λ but not on ε, and $k(\varepsilon\lambda) := (\frac{2\beta T}{\varepsilon\lambda})^{1/2}$. Observe that we replaced the growth estimate (3.2) for Φ_1 in x by *the same type of estimate* (3.4) in y. The point is now that this replacement successively applies to $|\Phi_m| = |\Phi_1 * \Phi_{m-1}|$, but this comes at the cost of a factor ε in the exponential. We will see in the proof of the following lemma that such factors indeed show up successively, so instead of using a fixed ε, we better choose a sequence $\varepsilon_m > 0$ such that $\prod(1-\varepsilon_m) > 0$ (which holds iff $\sum \varepsilon_m < \infty$). Since the bound (3.3) blows up as ε_m goes to zero it is not obvious that $\sum \Phi_m$ converges, but indeed it does, as we show in Lemma 3.2. We will frequently use the abbreviation

$$F_{\varepsilon\lambda}(y) := (|y| + k(\varepsilon\lambda))^\beta + 1.$$

Lemma 3.1. *Let $\varepsilon, \lambda \in (0,1)$, define $\varepsilon_1 := \varepsilon$ and choose $\varepsilon_m > 0$ such that $\sum_{m=1}^\infty \varepsilon_m < 1$, so $\lambda^* := \lambda \prod_{m=1}^\infty (1-\varepsilon_m) > 0$. Suppose the conditions of Proposition 2.1. Then the functions $\Phi_m = \Phi_1^{*m}$ are well-defined on Δ_T, and there is a constant $H = H(\varepsilon, \lambda) > 0$ such that*

$$|\Phi_m(t,x,s,y)| \le \frac{H^m}{\Gamma(\frac{m}{2})} F^m_{\varepsilon_m\lambda^*}(y) \frac{e^{-\frac{\lambda^*(x-y)^2}{2(t-s)}}}{(t-s)^{\frac{3-m}{2}}}, \quad m \in \mathbb{N}. \tag{3.5}$$

Proof: We first show by induction, that Φ_m is well-defined and that there is a constant H such that, with $\lambda_m := \lambda \prod_{j=1}^m (1-\varepsilon_j)$, we have

$$|\Phi_m(t,x,s,y)| \leq \frac{H^m}{\Gamma(\frac{m}{2})} \frac{e^{-\frac{\lambda_m (x-y)^2}{2(t-s)}}}{(t-s)^{\frac{3-m}{2}}} \prod_{i=1}^m (\frac{2\pi}{\lambda_i})^{1/2} F_{\varepsilon_i \lambda_i}(y). \tag{3.6}$$

For $m=1$ this estimate follows from (3.4), with $H = c_1(\lambda_1/2)^{1/2}$. Now suppose (3.6) is true for $m \geq 1$, then this estimate and (3.4) imply

$$\begin{aligned} |\Phi_{m+1}| &\leq \int_s^t \int_{\mathbb{R}} |\Phi_1(t,x,r,z)||\Phi_m(r,z,s,y)| dz dr \\ &\leq \frac{c_1 H^m}{\Gamma(\frac{m}{2})} \int_s^t \int_{\mathbb{R}} F_{\varepsilon\lambda}(z) \frac{e^{-\frac{\lambda_1 (x-z)^2}{2(t-r)}}}{t-r} \frac{e^{-\frac{\lambda_m (z-y)^2}{2(r-s)}}}{(r-s)^{\frac{3-m}{2}}} dz dr \prod_{i=1}^m (\frac{2\pi}{\lambda_i})^{1/2} F_{\varepsilon_i \lambda_i}(y). \end{aligned} \tag{3.7}$$

Now $F_{\varepsilon\lambda}(z) \leq |z|^\beta + k(\varepsilon\lambda)^\beta + 1 \leq c_2(|z|^\beta + 1)$, with $c_2 := k(\varepsilon\lambda)^\beta + 1$. This estimate and (3.3), with ε replaced by ε_{m+1}, applied to (3.7) gives

$$|\Phi_{m+1}| \leq \frac{c_1 c_2 H^m}{\Gamma(m/2)} F_{\varepsilon_{m+1}\lambda_m}(y) \int_s^t \int_{\mathbb{R}} \frac{e^{-\frac{\lambda_1 (x-z)^2}{2(t-r)}}}{t-r} \frac{e^{-\frac{\lambda_{m+1} (z-y)^2}{2(r-s)}}}{(r-s)^{\frac{3-m}{2}}} dz dr \prod_{i=1}^m (\frac{2\pi}{\lambda_i})^{1/2} F_{\varepsilon_i \lambda_i}(y).$$

Now we use the elementary fact that for $a, b \in (-\infty, \frac{3}{2})$ and $\gamma > 0$ we have

$$\int_s^t \int_{\mathbb{R}} \frac{e^{-\frac{\gamma (x-z)^2}{2(t-r)}}}{(t-r)^a} \frac{e^{-\frac{\gamma (z-y)^2}{2(r-s)}}}{(r-s)^b} dz dr = (\frac{2\pi}{\gamma})^{1/2} \frac{\Gamma(\frac{3}{2}-a)\Gamma(\frac{3}{2}-b)}{\Gamma(3-a-b)} \frac{e^{-\frac{\gamma (x-y)^2}{2(t-s)}}}{(t-s)^{a+b-3/2}}. \tag{3.8}$$

With $a=1$, $b=\frac{3-m}{2}$, $\gamma = \lambda_{m+1}$ and $k(\varepsilon_{m+1}\lambda_m) < k(\varepsilon_{m+1}\lambda_{m+1})$ we conclude

$$|\Phi_{m+1}| \leq \frac{c_1 c_2 H^m \Gamma(1/2)}{\Gamma((m+1)/2)} \frac{e^{-\frac{\lambda_{m+1} (x-y)^2}{2(t-s)}}}{(t-s)^{\frac{3-(m+1)}{2}}} \prod_{i=1}^{m+1} (\frac{2\pi}{\lambda_i})^{1/2} F_{\varepsilon_i \lambda_i}.$$

In particular Φ_m is well–defined. This proves (3.6) with $H := c_1 \max\{(\lambda_1/2)^{1/2}, c_2\Gamma(1/2)\}$. With $k(\varepsilon_i\lambda_i) < k(\varepsilon_m\lambda^*)$, $i \leq m$, and (3.6) we finally derive our assertion (3.5). ■

Lemma 3.2. *Assume the drift b satisfies* (A)*. Then the series* (2.2) *converges uniformly in compact subsets of Δ_T, the limit Φ satisfies the estimate* (2.3)*, and $\Phi = \Phi_1 + \Phi_1 * \Phi$.*

Proof: Consider the m-dependent factors in (3.5) and use $(|a|+|b|)^m \leq 2^m(|a|^m + |b|^m)$ and $(|a|+|b|)^\beta \leq |a|^\beta + |b|^\beta$ to estimate

$$\sum_{m=1}^\infty \frac{H^m}{\Gamma(\frac{m}{2})} \frac{F^m_{\varepsilon_m\lambda^*}(y)}{(t-s)^{\frac{1-m}{2}}} \leq \sum_{m=1}^\infty \frac{C_1 H_1^m}{\Gamma(\frac{m}{2})} |y|^{m\beta} + \sum_{m=1}^\infty \frac{C_1 H_1^m}{\Gamma(\frac{m}{2})} [k(\varepsilon_m\lambda^*)^\beta + 1]^m, \tag{3.9}$$

where $C_1 := T^{-1/2}$ and $H_1 := 2T^{1/2}H$. We estimate the two series separately, and we denote by C_1, C_2 etc. suitable constants. $\Gamma(m/2) \sim (m!)^{1/2}(2\pi)^{1/4}m^{-1/4}(\frac{1}{2})^{(m-1)/2}$ gives

$$\begin{aligned}
\sum_{m=1}^{\infty} \frac{C_1 H_1^m}{\Gamma(\frac{m}{2})}|y|^{m\beta} &\leq \sum_{m=1}^{\infty} \frac{C_2(2H_1)^m}{(m!)^{1/2}}|y|^{m\beta}m^{1/4} \\
&\leq C_2 \left(\sum_{m=1}^{\infty}\left(\frac{(4H_1)^m}{(m!)^{1/2}}|y|^{m\beta}\right)^2\right)^{1/2} \left(\sum_{m=1}^{\infty}\left(\frac{m^{1/4}}{2^m}\right)^2\right)^{1/2} \\
&= C_3 e^{H_2|y|^{2\beta}} \leq c(h)e^{h|y|^2}.
\end{aligned}$$

The last estimate holds for every $h > 0$ and a suitable constant $c(h)$ because $\beta < 1$. We estimate the second term in (3.9) for the special sequence $\varepsilon_m := \varepsilon \cdot m^{\delta - \frac{1}{\beta}}$, where $\delta > 0$ is chosen such that $\delta - 1/\beta < -1$ (so $\sum \varepsilon_m < \infty$). Then

$$\begin{aligned}
\sum_{m=1}^{\infty} \frac{C_1 H_1^m}{\Gamma(\frac{m}{2})}[k(\varepsilon_m\lambda^*)^\beta + 1]^m &\leq \sum_{m=1}^{\infty} \frac{C_2(2H_1)^m}{(m!)^{1/2}}\left(\left[\left(\frac{4\beta T}{\lambda^*}\right)^{1/2} m^{\frac{1}{2}(\frac{1}{\beta}-\delta)}\right]^\beta + 1\right)^m m^{1/4} \\
&\leq \sum_{m=1}^{\infty} \frac{C_3 H_3^m}{(m!)^{1/2}} m^{\frac{m}{2}(1-\delta\beta)} m^{1/4} < \infty,
\end{aligned}$$

with $H_3 := 4H_1[(4\beta T/\lambda^*)^{\beta/2} + 1]$. Combining the inequalities for the two series in (3.9), we eventually get, with a suitable constant $C(h)$:

$$|\Phi(t,x,s,y)| \leq \sum_{m=1}^{\infty} |\Phi_m(t,x,s,y)| \leq C(h)\frac{e^{-\frac{\lambda^*(x-y)^2}{2(t-s)}}}{t-s}e^{hy^2}. \tag{3.10}$$

To summarize: Φ is well-defined on Δ_T and (2.3) holds. Now let $K \subset \Delta_T$ be compact. Then $\inf\{t-s|(t,x,s,y) \in K\} > 0$, so (3.9) implies the uniform convergence of $\sum \Phi_m$ on K. Finally notice that $\Phi_1 * \Phi$ is well-defined because by (3.10) we can estimate $\Phi_1 * \Phi$ in the same way as we estimated $\Phi_1 * \Phi_1$ in the proof of Lemma 3.1. Moreover, (3.10) shows that Lebesgue's dominated convergence theorem applies, so we can interchange summation and integration (i.e. convolution) to obtain

$$\Phi_1 + \Phi_1 * \Phi = \Phi_1 + \Phi_1 * \sum_{m=1}^{\infty} \Phi_m = \Phi_1 + \sum_{m=1}^{\infty} \Phi_1 * \Phi_m = \Phi. \qquad \blacksquare$$

Recall from the proof that $\lambda^* = \lambda \prod_{m=1}^{\infty}(1-\varepsilon_m)$ with $\varepsilon_m = \varepsilon \cdot m^{\delta-\frac{1}{\beta}}$. This shows that $\lambda^* < \lambda$ can be chosen arbitrary close to λ, by the choice of a suitably small $\varepsilon > 0$. Since we can choose $\lambda \in (0,1)$ as we like this implies that $\lambda^* < 1$ can get arbitrary close to 1, as claimed in part (c) of Proposition 2.1.

Remark. The estimate (3.9) is also valid if we put $\beta = 1$, and it is sharp for $y = 0$. But in the limit $\beta \nearrow 1$ our choice of ε_m yields $\varepsilon_m = \frac{\varepsilon}{m}$, and this violates the condition $\sum \varepsilon_m < \infty$, i.e. we have $\prod(1-\varepsilon_m) = 0$. Therefore we have to choose another (summable) sequence

$\varepsilon_m > 0$ to investigate whether (3.9) converges. We write this ε_m as $\frac{\delta_m}{m}$ and observe that $\sum \frac{\delta_m}{m} < \infty$ implies $\liminf \delta_m = 0$. The root test applied to the second sum in (3.9) shows that this sum must diverge. Therefore our method of estimation based on (3.3) does not allow us to conclude that $\sum |\Phi_m|$ converges in case $\beta = 1$. The situation does not change if in (3.9) we use the slightly smaller bound (3.6) instead of (3.5).

Having established that Φ is a well–defined solution of (1.6) the next step is to show that p defined in (1.5) is sufficiently smooth. This smoothness will follows essentially from two continuity properties of Φ. We begin with the first one, the local Hölder continuity, as stated in part (a) of Proposition 2.1. In fact we will need a slightly stronger result than this continuity for the Cauchy problem, in order to handle the unbounded domain of integration in the y-variable:

Lemma 3.3. *The function $x \mapsto \Phi(t,x,s,y)$ is locally Hölder continuous as follows: For $K, h > 0$, $\hat{\alpha} \in (0, \alpha)$, $\lambda^{**} \in (0, \lambda^*)$, and $\gamma := \alpha - \hat{\alpha}$ there is a constant $C > 0$ such that*

$$|\Phi(t,x,s,y) - \Phi(t,x',s,y)| \le C|x-x'|^{\hat{\alpha}} \frac{e^{-\frac{\lambda^{**}(x-y)^2}{2(t-s)}} + e^{-\frac{\lambda^{**}(x'-y)^2}{2(t-s)}}}{(t-s)^{(3-\gamma)/2}} e^{hy^2} \tag{3.11}$$

for all $y \in \mathbb{R}$, $t > s \ge 0$ and $|x|, |x'| \le K$.

Proof: It suffices to prove (3.11) for each term of $\Phi = \Phi_1 + \Phi_1 * \Phi$ separately.

Hölder continuity for Φ_1: Abbreviate $\Phi_1(x) := \Phi_1(t,x,s,y)$. We distinguish two cases, the first case is $|x-x'|^2 > t-s$. This gives $(t-s)^{(1-\gamma)/2} \le |x-x'|^{1-\gamma} = |x-x'|^{\hat{\alpha}}|x-x'|^{1-\alpha}$ with $1-\alpha > 0$. So (3.4) yields

$$|\Phi_1(x)| \le \frac{C_1(t-s)^{\frac{1-\gamma}{2}} F_{\varepsilon\lambda^*}(y)}{(t-s)^{(1-\gamma)/2}} \frac{e^{-\frac{\lambda^*(x-y)^2}{2(t-s)}}}{t-s} \le C_2|x-x'|^{\hat{\alpha}} F_{\varepsilon\lambda^*}(y) \frac{e^{-\frac{\lambda^*(x-y)^2}{2(t-s)}}}{(t-s)^{\frac{3-\gamma}{2}}}.$$

Interchanging x with x' and adding the two estimates yields

$$|\Phi_1(x) - \Phi_1(x')| \le C_3|x-x'|^{\hat{\alpha}} F_{\varepsilon\lambda^*}(y) \frac{e^{-\frac{\lambda^*(x-y)^2}{2(t-s)}} + e^{-\frac{\lambda^*(x'-y)^2}{2(t-s)}}}{(t-s)^{\frac{3-\gamma}{2}}}. \tag{3.12}$$

Now $F_{\varepsilon\lambda^*}(y) \le c(h)e^{hy^2}$ (for any $h > 0$) gives (3.11). Next we consider the second case, $|x-x'|^2 \le t-s$. Here we write $\Phi_1(x) - \Phi_1(x')$ as the sum of the two terms

$$\Phi_{1,1}(x,x') := [b(t,x) - b(t,x')]\partial_x Z_{t-s}(x-y)$$
$$\Phi_{1,2}(x,x') := b(t,x')[\partial_x Z_{t-s}(x-y) - \partial_{x'} Z_{t-s}(x'-y)],$$

and estimate them separately. The Hölder continuity of b, (3.1) and $|x-x'|^2 \le t-s$ give an estimate for $\Phi_{1,1}$:

$$|\Phi_{1,1}(x,x')| \le C_4|x-x'|^{\alpha} \frac{e^{-\frac{\lambda(x-y)^2}{2(t-s)}}}{t-s} \le C_4|x-x'|^{\hat{\alpha}} \frac{e^{-\frac{\lambda(x-y)^2}{2(t-s)}}}{(t-s)^{\frac{2-\gamma}{2}}}. \tag{3.13}$$

To estimate $\Phi_{1,2}$ observe that $-(z-y)^2 \leq -\frac{\lambda^*}{\lambda}(x'-y)^2 + \frac{\lambda^*}{\lambda-\lambda^*}(x'-z)^2$ holds for all real x, x', y, z. For $z \in (x, x')$ and $|x-x'|^2 \leq t-s$ this implies $\exp\{-\lambda\frac{(z-y)^2}{2(t-s)}\} \leq C_5 \exp\{-\lambda^* \frac{(x'-y)^2}{2(t-s)}\}$. Using this, the mean value theorem and (3.1) (for $n=2$) we find

$$|\partial_x Z_{t-s}(x-y) - \partial_{x'} Z_{t-s}(x'-y)| \leq C_6 |x-x'| \frac{e^{-\frac{\lambda^*(x'-y)^2}{2(t-s)}}}{(t-s)^{3/2}}.$$

$b(t,x')$ is bounded because $|x'| \leq K$, and with $|x-x'|^2 \leq t-s$ the former estimate implies

$$|\Phi_{1,2}(x,x')| \leq C_6 |b(t,x')||x-x'| \frac{e^{-\frac{\lambda^*(x'-y)^2}{2(t-s)}}}{(t-s)^{3/2}} \leq C_7 |x-x'|^{\hat{\alpha}} \frac{e^{-\frac{\lambda^*(x'-y)^2}{2(t-s)}}}{(t-s)^{\frac{3-\gamma}{2}}}. \tag{3.14}$$

Combining (3.13) and (3.14) implies (3.11), so the two cases show that Φ_1 satisfies (3.11).

Hölder continuity for $\Phi_1 * \Phi$: Replacing h by $h/2$ in (3.10) and combining this with (3.12) yields

$$\begin{aligned} |\Phi_1 * \Phi(x) - \Phi_1 * \Phi(x')| &\leq \int_s^t \int_{\mathbb{R}} |\Phi_1(t,x,r,z) - \Phi_1(t,x',r,z)||\Phi(r,z,s,y)| dz dr \\ &\leq C_8 |x-x'|^{\hat{\alpha}} e^{hy^2/2} [(\Psi(t,x) + \Psi(t,x')], \end{aligned} \tag{3.15}$$

with the abbreviation

$$\Psi(t,x) := \int_s^t \int_{\mathbb{R}} F_{\varepsilon\lambda^*}(z) \frac{e^{-\frac{\lambda^*(x-z)^2}{2(t-r)}}}{(t-r)^{\frac{3-\gamma}{2}}} \frac{e^{-\frac{\lambda^*(z-y)^2}{2(r-s)}}}{r-s} dz dr.$$

We apply (3.3) with an $\varepsilon^* \in (0,1)$ such that $(1-\varepsilon^*)\lambda^* = \lambda^{**}$ and use (3.8) to obtain

$$\begin{aligned} |\Psi(t,x)| &\leq C_9 F_{\varepsilon^*\lambda^*}(y) \int_s^t \int_{\mathbb{R}} \frac{e^{-\frac{\lambda^*(x-z)^2}{2(t-s)}}}{(t-r)^{\frac{3-\gamma}{2}}} \frac{e^{-\frac{(1-\varepsilon^*)\lambda^*(z-y)^2}{2(r-s)}}}{r-s} dz dr \\ &\leq C_{10} F_{\varepsilon^*\lambda^*}(y) \frac{e^{-\frac{\lambda^{**}(x-y)^2}{2(t-s)}}}{(t-s)^{\frac{3-\gamma}{2}}}, \end{aligned}$$

where in the last estimate we made us of $(t-s)^{1/2} \leq T^{1/2}$. Since we can replace x by x' in this estimate (3.15) yields

$$|\Phi_1 * \Phi(x) - \Phi_1 * \Phi(x')| \leq C_{11} |x-x'|^{\hat{\alpha}} e^{hy^2/2} F_{\varepsilon\lambda^*}(y) \frac{e^{-\frac{\lambda^{**}(x-y)^2}{2(t-s)}} + e^{-\frac{\lambda^{**}(x'-y)^2}{2(t-s)}}}{(t-s)^{(3-\gamma)/2}},$$

and this finally gives (3.11). ■

Lemma 3.4. *Fix $(s,y) \in [0,T) \times \mathbb{R}$. Then $\Phi(t,x,s,y)$ is continuous in $(t,x) \in (s,T] \times \mathbb{R}$.*

Proof: Since $\Phi = \sum \Phi_m$ is uniformly convergent on compacts of Δ_T it suffices to prove the continuity of the functions $\Phi_m(\cdot,\cdot,s,y)$. Clearly Φ_1 is continuous on Δ_T. We use this to prove that $\Phi_{m+1} = \Phi_1 * \Phi_m$ is continuous in the fixed point (t,x). Let $0 < \delta < (t-s)/4$. For $|t-t'| < \delta$ and $x, x' \in \mathbb{R}$ we then have

$$\begin{aligned} |\Phi_{m+1}(t,x,s,y) - \Phi_{m+1}(t',x',s,y)| = |&\int_s^{t-\delta}\int_{\mathbb{R}} \Phi_1(t,x,r,z)\Phi_m(r,z,s,y)dzdr \\ &+ \int_{t-\delta}^{t}\int_{\mathbb{R}} \Phi_1(t,x,r,z)\Phi_m(r,z,s,y)dzdr \\ &- \int_s^{t'-\delta}\int_{\mathbb{R}} \Phi_1(t',x',r,z)\Phi_m(r,z,s,y)dzdr \\ &- \int_{t'-\delta}^{t'}\int_{\mathbb{R}} \Phi_1(t',x',r,z)\Phi_m(r,z,s,y)dzdr|. \end{aligned} \tag{3.16}$$

We are going to estimate the four terms, denoted $|A_1 + A_2 - A_3 - A_4|$, on the right hand side separately. The same arguments leading to the estimate that follows (3.7) show that

$$\begin{aligned} |A_2| &\le C_1 F^{m+1}_{\varepsilon_{m+1}\lambda^*}(y) \int_{t-\delta}^{t}\int_{\mathbb{R}} \frac{e^{-\frac{\lambda^*(x-z)^2}{2(t-r)}}}{t-r} \frac{e^{-\frac{\lambda^*(z-y)^2}{2(r-s)}}}{(r-s)^{\frac{3-m}{2}}} dzdr \\ &\le \frac{C_1 F^{m+1}_{\varepsilon_{m+1}\lambda^*}(y)}{(\frac{t-s}{2})^{\frac{3-m}{2}}} \int_{t-\delta}^{t}\int_{\mathbb{R}} \frac{e^{-\frac{\lambda^*(x-z)^2}{2(t-r)}}}{t-r} dzdr = C_2(y)\sqrt{\delta}. \end{aligned}$$

(In the second line we used $t - \delta - s \ge \frac{t-s}{2}$.) A trivial modification of this estimation yields $|A_4| \le C_2(y)\sqrt{\delta}$. (Note that $C_2(y)$ only depends on y, but not on (t',x'), as long as $|t-t'| < \delta$.) So for any given $\varepsilon > 0$ we can choose $\delta \in (0, \frac{t-s}{4})$ such that $|A_2 - A_4| \le \varepsilon/2$. With this fixed δ we are now left to estimate the term $|A_1 - A_3|$ in (3.16). We assume $t' \le t$ (the case $t' \ge t$ is treated similarly):

$$\begin{aligned} |A_1 - A_3| \quad \le \quad & \int_s^{t'-\delta}\int_{\mathbb{R}} |[\Phi_1(t,x,r,z) - \Phi_1(t',x',r,z)]\Phi_m(r,z,s,y)|dzdr \\ &+ \int_{t'-\delta}^{t-\delta}\int_{\mathbb{R}} |\Phi_1(t,x,r,z)\Phi_m(r,z,s,y)|dzdr. \end{aligned} \tag{3.17}$$

Estimate the first integral in (3.17) by replacing $t'-\delta$ by $t-\delta$, and observe that the integrand goes to 0 as $(t',x') \to (t,x)$. Moreover, in view of (3.4) this integrand is uniformly dominated by $cF_{\varepsilon\lambda}(z)|\Phi_m(r,z,s,y)|$ whenever t' is such that $|t-t'| \le \delta/2$, and $x, x' \in \mathbb{R}$. Thus the dominated convergence theorem implies that the first term in (3.17) goes to 0, as $(t',x') \to (t,x)$, and the convergence of the second term to 0 is trivial. Thus $|A_1 - A_3| \le \varepsilon/2$ for all (t',x') sufficiently close to (t,x). In view of our former estimate $|A_2 - A_4| \le \varepsilon/2$ this proves the lemma. ∎

Observe that Proposition 2.1 follows from Lemma 3.1 to Lemma 3.4. We are now going to prove that the function p defined in (1.5) is a fundamental solution to (1.1a). We need the following well-known result which summarizes some facts about so-called volume potentials. For proofs we refer to [F1, pp. 7-13].

Lemma 3.5. *Let $0 \leq t_0 < T$ and $f \in C([t_0, T] \times \mathbb{R})$ be locally Hölder continuous in x, uniformly with respect to $t \in [t_0, T]$. Moreover assume $|f(t,x)| \leq ce^{hx^2}$ with $h < \frac{1}{2(T-t_0)}$. Then the volume potential of f,*

$$V_f(t,x) := \int_{t_0}^{t}\int_{\mathbb{R}} Z_{t-r}(x-z)f(r,z)dzdr$$

is continuous on $[t_0, T] \times \mathbb{R}$ and $C^{1,2}$ on $(t_0, T] \times \mathbb{R}$. The ∂_x and ∂_x^2 derivatives of V_f can be calculated by interchanging them with the above integral. Moreover,

$$\partial_t V_f(t,x) = f(t,x) + \int_{t_0}^{t}\int_{\mathbb{R}} \partial_t Z_{t-r}(x-z)f(r,z)dzdr.$$

Proof of Theorem 2.3: First we are going to show, that $p(t,x,s,y)$ given by (1.5) satisfies for fixed (s,y) the equation $(L-\partial_t)p = 0$. Choose $t_0 \in (s,t)$ and notice that the properties (a,b,c) stated in Proposition 2.1 are exactly such that the function $f(t,x) := \Phi(t,x,s,y)$, defined on $[t_0, T] \times \mathbb{R}$, satisfies the conditions of Lemma 3.5. Thus the volume potential

$$V_\Phi(t,x) := \int_{t_0}^{t}\int_{\mathbb{R}} Z_{t-r}(x-z)\Phi(r,z,s,y)dzdr$$

is $C^{1,2}$, and the assertion of Lemma 3.5 shows that

$$(L-\partial_t)V_\Phi(t,x) = \int_{t_0}^{t}\int_{\mathbb{R}} (L-\partial_t)Z_{t-r}(x-z)\Phi(r,z,s,y)dzdr - \Phi(t,x,s,y). \tag{3.18}$$

A simple application of Lebesgue's dominated convergence theorem shows that we also have

$$(L-\partial_t)\int_{s}^{t_0}\int_{\mathbb{R}} Z_{t-r}(x-z)\Phi(r,z,s,y)dzdr = \int_{s}^{t_0}\int_{\mathbb{R}} (L-\partial_t)Z_{t-r}(x-z)\Phi(r,z,s,y)dzdr,$$

and thus we can replace t_0 by s in (3.18). Now $(L-\partial_t)Z_{t-r}(x-z) = b(t,x)\partial_x Z_{t-r}(x-y) = \Phi_1(t,x,s,y)$, so with (1.5) and the r.h.s. of (3.18) (t_0 replaced by s) we find

$$(L-\partial_t)p = \Phi_1 + \Phi_1 * \Phi - \Phi = 0.$$

So property (F1) of a fundamental solution holds. To infer (F2) we use that $Z_{t-s}(x-y)$ is a fundamental solution to $\partial_t u = \frac{1}{2}\partial_x^2$. Hence

$$\begin{aligned}\lim_{t\downarrow s}\int_{\mathbb{R}} p(t,x,s,y)f(y)dy &= \lim_{t\downarrow s}\int_{\mathbb{R}} Z_{t-s}(x-y)f(y)dy \\ &\quad + \lim_{t\downarrow s}\int_{\mathbb{R}}\int_{s}^{t}\int_{\mathbb{R}} Z_{t-r}(x-z)\Phi(r,z,s,y)f(y)dzdrdy \\ &=: f(x) + \lim_{t\downarrow s} I(t,x,s)\end{aligned} \tag{3.19}$$

Recall that $|f(y)| \leq c\exp\{hy^2\}$ for a suitable $h > 0$. Using property (c) of Proposition 2.1 for this h and also (3.8) we find

$$\int_s^t \int_{\mathbb{R}} |Z_{t-r}(x-z)\Phi(r,z,s,y)f(y)|dzdr \leq C_1 \int_s^t \int_{\mathbb{R}} \frac{e^{-\frac{\lambda(x-z)^2}{2(t-r)}}}{(t-r)^{1/2}} \frac{e^{-\frac{\lambda^*(z-y)^2}{2(r-s)}}}{r-s} e^{2hy^2} dzdr$$
$$\leq C_2 e^{-\frac{\lambda^*(x-y)^2}{2(t-s)}} e^{2hy^2}. \tag{3.20}$$

This shows that in (3.19) we have $\lim_{t\downarrow s} I(t,x,s) = 0$, so (F2) is indeed satisfied. ■

Corollary 3.6. *Under the assumptions of Theorem 2.3 for every $h > 0$ there is a $c(h) > 0$ such that*

$$|p(t,x,s,y)| \leq c(h) \frac{e^{-\frac{\lambda^*(x-y)^2}{2(t-s)}}}{(t-s)^{1/2}} e^{hy^2}. \tag{3.21}$$

Proof: From the definition (1.5) of p we have

$$|p(t,x,s,y)| \leq |Z_{t-s}(x-y)| + \int_s^t \int_{\mathbb{R}} |Z_{t-r}(x-z)\Phi(r,z,s,y)|dzdr. \tag{3.22}$$

The second term can be estimated as in (3.20) with $f(y) = 1 \leq e^{hy^2}$ for all $h > 0$, i.e. we can choose $h > 0$ in (3.20) arbitrary small. By (2.3) and $\lambda > \lambda^*$ (3.22) implies (3.21). ■

Remark. The estimate (3.21) will be convenient for us in the following but it is not optimal. Recall that we derived the factor $\exp hy^2$ in (3.21) from a factor $\exp c|y|^{2\beta}$ in the proof of Lemma 3.2 with a suitable (but not arbitrary small) $c > 0$.

Proof of Theorem 2.4: By assumption $|u_0(y)| \leq ce^{\delta y^2}$ with $\delta < \frac{1}{2T}$. Write $\delta =: \frac{\bar{\lambda}}{2T}$ with $\bar{\lambda} < 1$, and choose $\lambda^* \in (\bar{\lambda}, 1)$. In view of (3.21) the function u in (2.5) is well-defined on $(0,T] \times \mathbb{R}$. By the definition of p in (1.5) and with (3.1) we can write it as

$$u(t,x) = \int_{\mathbb{R}} Z_t(x-y)u_0(y)dy + \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} Z_{t-r}(x-z)\Phi(r,z,0,y)u_0(y)dzdrdy.$$

In view of (3.10) also $\hat{u}(r,z) := \int_{\mathbb{R}} \Phi(r,z,0,y)u_0(y)dy$ is well-defined and it is simple to check that we can apply Fubini to obtain

$$u(t,x) = \int_{\mathbb{R}} Z_t(x-y)u_0(y)dy + \int_0^t \int_{\mathbb{R}} Z_{t-r}(x-z)\hat{u}(r,z)dzdr. \tag{3.23}$$

Using $|u_0(y)| \leq ce^{\delta y^2}$, and the estimate (2.3) we get

$$|\hat{u}(t,x)| \leq C_1 \int_{\mathbb{R}} \frac{e^{-\frac{\lambda^*(x-y)^2}{2t}}}{t} e^{\bar{\delta}y^2} dy \leq C_2 \frac{e^{\bar{\delta}x^2}}{t^{1/2}}, \tag{3.24}$$

with $\tilde{\delta} > \delta$ so close to δ that $\tilde{\delta} < \frac{\lambda^*}{2T}$. Clearly $\hat{u}(t,x)$ is continuous in $(0,T]\times\mathbb{R}$. Furthermore, the local Hölder continuity (3.11) of Φ implies

$$\begin{aligned}|\hat{u}(t,x) - \hat{u}(t,x')| &\leq \int_{\mathbb{R}} |\Phi(t,x,0,y) - \Phi(t,x',0,y)| \cdot |u_0(y)| dy \\ &\leq C_3 \int_{\mathbb{R}} |x-x'|^{\hat{\alpha}} \frac{e^{-\frac{\lambda^{**}(x-y)^2}{2t}} + e^{-\frac{\lambda^{**}(x'-y)^2}{2t}}}{t^{(3-\gamma)/2}} e^{\tilde{\delta}y^2} dy \\ &\leq C_4 |x-x'|^{\hat{\alpha}},\end{aligned}$$

for any $t \geq t_0 > 0$ and $|x|, |x'| \leq K$. So $\hat{u}(t,x)$ is also locally Hölder continuous in x, uniformly w.r.t. $t \in [t_0, T]$. With Lemma 3.5 we conclude

$$(L-\partial_t) \int_{t_0}^{t} \int_{\mathbb{R}} Z_{t-r}(x-z)\hat{u}(r,z)dzdr = \int_{t_0}^{t} \int_{\mathbb{R}} (L-\partial_t) Z_{t-r}(x-z)\hat{u}(r,z)dzdr - \hat{u}(t,x). \quad (3.25)$$

(3.1) combined with (3.24) implies that the functions $(r,z) \mapsto \partial_x^n Z_{t-r}(x-z)\hat{u}(r,z)$ (with $n = 0,1,2$) are integrable on $(0,t_0] \times \mathbb{R}$ (because $t > t_0$), and they are dominated by integrable functions, locally uniform w.r.t. (t,x). Thus Lebesgue's theorem implies that

$$(L-\partial_t) \int_{0}^{t_0} \int_{\mathbb{R}} Z_{t-r}(x-z)\hat{u}(r,z)dzdr = \int_{0}^{t_0} \int_{\mathbb{R}} (L-\partial_t) Z_{t-r}(x-z)\hat{u}(r,z)dzdr. \quad (3.26)$$

Combining (3.25) and (3.26) yields

$$\begin{aligned}(L-\partial_t)u(t,x) &= \int_{\mathbb{R}} (L-\partial_t) Z_t(x-y) u_0(y) dy \\ &\quad + \int_0^t \int_{\mathbb{R}} (L-\partial_t) Z_{t-r}(x-z)\hat{u}(r,z)dzdr - \hat{u}(t,x) \\ &= \int_{\mathbb{R}} [\Phi_1(t,x,0,y) + \Phi_1 * \Phi(t,x,0,y) - \Phi(t,x,0,y)] u_0(y) dy = 0.\end{aligned}$$

To summarize, u is $C^{1,2}$ on $(0,T] \times \mathbb{R}$ and $\partial_t u = Lu$. It is well-known that the first term on the r.h.s. of (3.23) together with the boundary values $u_0(x)$ defines a continuous function on $[0,T] \times \mathbb{R}$. Moreover, the estimate (3.20) implies (with I introduced in (3.19) and $f = u_0$) that $|I(t,x,0)| \leq C_5 t^{1/2} e^{hx^2}$ with a suitable $h > 0$. This shows that the second term in (3.23) goes to 0 as (t,x) goes to a boundary point $(0,x_0)$, which implies that u is indeed continuous on all of $[0,T] \times \mathbb{R}$. That u is of quadratic exponential growth for any parameter $\tilde{\delta} > \delta$ follows from (3.21) and condition (B):

$$|u(t,x)| \leq \int_{\mathbb{R}} |p(t,x,0,y)||u_0(y)| dy \leq C_1 \int_{\mathbb{R}} \frac{e^{-\frac{\lambda^*(x-y)^2}{2t}}}{t^{1/2}} e^{\delta y^2} dy \leq C_2 \, e^{\tilde{\delta}x^2}.$$

It therefore remains to show uniqueness of u, which can be done by the maximum principle. Indeed, the uniqueness assertion of Corollary 4.2 in Section 6 of [F2] applies without modification to our situation. ■

4. Remarks on extensions and on stochastic PDE's.

Informal discussion of extensions. Recall that our proofs are mainly based on two estimates. The first one is (3.1) for the parametrix, and the second one is (3.3) which implies the estimate (3.4) for the singular integral kernel Φ_1. The first estimate generalizes to d dimensions for the parametrix $Z(t,x,s,y)$ which is associated to the operator $A = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t,x)\partial_{x^i}\partial_{x^j}$, provided A is uniformly elliptic and has bounded, Hölder continuous coefficients a_{ij}, see e.g. [F1]. The second estimate (3.3) easily extends to d dimensions. Therefore it is quite clear that our proofs go through for these extensions. When we add a Hölder continuous, Hölder growing potential to our equation, i.e.

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t,x)\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b_i(t,x)\frac{\partial u}{\partial x^i} + c(t,x)u \tag{4.1}$$

we have to estimate the term $c(t,x)Z(t,x,s,y)$ similarly as we did for $b(t,x)\partial_x Z_{t-s}(x-y)$. Clearly this additional estimate is easier than the previous one because Z is less singular than $\partial_x Z$. This indicates that our results can be extended to equations (4.1) with coefficients having the before mentioned properties. The technical details will be given in [Kr].

Connection to stochastic partial differential equations. Our motivation for this work originated from Cauchy problems of type (1.1) where the drift b is not an ordinary function, but is a *continuous random field on* $[0,T]\times\mathbb{R}$. This means that the continuous function $(t,x)\mapsto b(t,x,\omega)$ depends on an additional "random parameter" ω. It is natural to consider (1.1) for each fixed ω separately, and this essentially reduces the stochastic problem to a non-stochastic one.

Example. Let $(B^1_s, B^2_s)_{s\geq 0}$ be a two-dimensional normal Brownian motion and define the time independent field $b(x,\omega) := B^1_x(\omega)$ for $x \geq 0$ and $b(x,\omega) := B^2_{-x}(\omega)$ for $x \leq 0$. The sample functions of this field are nowhere differentiable but they are locally Hölder continuous for every exponent $\alpha \in (0,\frac{1}{2})$. Moreover they are two-sided unbounded and for $\beta \geq \frac{1}{2}$ they satisfy the global Hölder type growth condition $|b(x,\omega)| \leq c(\omega)(|x|^\beta + 1)$. Notice that this example meets all the requirements that we made in this paper.

Equations of type (1.1a) (in 3-dim. space with random b) arise in the theory of advection-diffusion phenomena and in turbulent diffusion, see e.g. [Ba],[CC],[CF],[HM], and references given there. A related example comes from the filtering theory of diffusion processes: the so-called robust Zakai equation is a parabolic differential equation with random coefficients [Da]. The right side of (1.1a) defines a so-called random operator and such operators also appear in stochastic partial differential equations with additional singular noise terms. Since such equations are frequently investigated in the mild sense [DZ],[DP], the existence of a fundamental solution is of basic interest in that context. Two recent examples with random fundamental solutions are discussed in [LN] and [ALN].

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