Mannheimer Mathematische Manuskripte
185/94

YANG-MILLS AND DIRAC FIELDS IN A BAG, CONSTRAINTS AND REDUCTION

Jędrzej Śniatycki*, Günter Schwarz⋆ and Larry Bates†.

185/94

Abstract

The structure of the constraint set in the Yang-Mills-Dirac theory in a contractible bounded domain is analysed under the bag boundary conditions. The gauge symmetry group is identified, and it is proved that its action on the phase space is proper and admits slices. The reduced phase space is shown to be the union of symplectic manifolds, each of which corresponds to a definite mode of symmetry breaking.

---

* Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta. Canada. Research partially supported by NSERC Research Grant SAP0008091.

⋆ Department of Mathematics and Computer Science, University of Mannheim, 6800 Mannheim, Germany. Research partially supported by DFG Grant Schw. 485/2-1.

† Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. Research partially supported by NSERC Research Grant OGP0042416.

1. Introduction.

In a previous paper we have proved the existence and uniqueness theorems for minimally interacting Yang-Mills and Dirac fields in a bounded contractible domain $M \subset \mathbb{R}^3$, [1]. The aim of this paper is to study the structure of the space of solutions.

Our results were obtained for Cauchy data $A \in H^2(M)$, $E \in H^1(M)$, and $\Psi \in H^2(M)$, where and $H^k(M)$ is the Sobolev space of fields on M which are square integrable together with their derivatives up to the order k, satisfying the boundary conditions

$$nE = 0, \quad tB = 0, \quad in_j\gamma^j\Psi|_{\partial M} = \Psi|_{\partial M}, \tag{1.1a}$$

$$nA = 0, \; in_j\gamma^j\{\gamma^0(\gamma^k\partial_k + im)\Psi\}|_{\partial M} = \gamma^0(\gamma^k\partial_k + im)\Psi|_{\partial M}\,. \tag{1.1b}$$

Here we use the notation established in [1]. In particular, nE denotes the normal component of the "electric" part, tB the tangential component of the "magnetic" part of the field strength on the boundary ∂M of M. Thus, the extended phase space of the theory under consideration is

$$P = \{(A,E,\Psi) \in H^2(M) \times H^1(M) \times H^2(M) | \text{satisfying } (1.1a,b)\}. \tag{1.2}$$

The variational principle underlying the theory gives rise to a (weak) symplectic structure on P. Let θ be a 1-form on P such that, for every $p = (A,E,\Psi) \in P$ and $a\frac{\delta}{\delta A} + e\frac{\delta}{\delta E} + \psi\frac{\delta}{\delta\Psi} \in T_pP$,

$$\langle\theta(A,E,\Psi)|a\frac{\delta}{\delta A} + e\frac{\delta}{\delta E} + \psi\frac{\delta}{\delta\Psi}\rangle = \int_M (E\cdot a + \Psi^\dagger\psi)d_3x\;, \tag{1.3}$$

The symplectic form ω of P is the exterior differential of θ,

$$\omega = d\theta\;. \tag{1.4}$$

Let G be the structure group of the theory, presented as a matrix group, and g be the Lie algebra of G. We assume that G is compact, and that g admits an ad-invariant metric. The group $GS(P)$ of gauge symmetries consists of maps $\phi : M \longrightarrow G$ such that their action on the variables (A,E,Ψ), given by

$$A \mapsto \phi A \phi^{-1} + \phi\, grad \phi^{-1}, \quad E \mapsto \phi E \phi^{-1}, \quad \Psi = \phi\Psi, \tag{1.5}$$

leaves the extended phase space P invariant. The infinitesimal action of an element ξ of the Lie algebra $gs(P)$ of $GS(P)$ is given by

$$A \mapsto A - D_A\xi, \quad E \mapsto E - [E,\xi], \quad \Psi = \Psi + \xi\Psi, \tag{1.6}$$

where

$$D_A\xi = grad\xi + [A,\xi] \tag{1.7}$$

is the covariant derivative of ξ with respect to the connection defined by A. It gives rise to a vector field ξ_P on P such that

$$\xi_P(A,E,\Psi) = -(D_A\xi)\frac{\delta}{\delta A} - [E,\xi]\frac{\delta}{\delta E} + \xi\Psi\frac{\delta}{\delta\Psi}. \tag{1.8}$$

The action of $GS(P)$ preserves the 1-form θ. Hence, it is Hamiltonian with the equivariant momentum map $J : P \to gs(P)^*$ such that

$$\langle J(A,E,\Psi) \mid \xi\rangle = \langle\theta \mid \xi_P(A,E,\Psi)\rangle = \int_M \{-E\cdot D_A\xi + \Psi^\dagger\xi\Psi\} d_3x\,. \tag{1.9}$$

Here $gs(P)^*$ denotes the L^2 dual of $gs(P)$, that is the space of square integrable maps from M to the dual g^* of the Lie algebra g of the structure group G. For each $\xi \in gs(P)$, the function $J_\xi : P \to \mathbb{R}$ given by

$$J_\xi(A,E,\Psi) = \langle J(A,E,\Psi) \mid \xi\rangle \tag{1.10}$$

is called the momentum associated to ξ. The vector field ξ_P is the Hamiltonian vector field of J_ξ, i.e.

$$\xi_P \lrcorner\, \omega = dJ_\xi. \tag{1.11}$$

Integrating by parts on the right hand side of Eq. (1.9), and taking into account the boundary condition $nE = 0$, we obtain

$$\langle J(A,E,\Psi) \mid \xi\rangle = \int_M \{(divE + [A;E])\xi + \Psi^\dagger\xi\Psi\} d_3x. \tag{1.12}$$

For every $\xi \in gs(P)$,

$$\Psi^\dagger\xi\Psi = -j\cdot\xi, \tag{1.13}$$

where j is the source term in the Yang-Mills-Dirac theory. Hence, the constraint equation of the theory

$$divE + [A;E] = j \,, \tag{1.14}$$

is equivalent to the vanishing of the momentum map J.

The presentation of the constraint set as the zero level $J^{-1}(0)$ of the momentum map J, enables one to study its structure in terms of the action of the group of gauge symmetries. It was first done by Arms [2], who discussed the structure of the constraint set for pure Yang-Mills fields in compact spaces (no boundary) in general terms, without specifying the topology of the function spaces under consideration. The structure of the zero level of the momentum map, corresponding to a Hamiltonian action of a Hilbert-Lie group on a Hilbert manifold was studied, under additional technical assumptions, by Arms, Marsden and Moncrief, [3]. Special cases were considered by Mitter and Vialet [4], Atiyah and Bott [5], Kondracki and Rogulski [6] and Huebschmann [7,8].

Functional analytic assumptions made in this paper are consequences of the results of [1]. They fail to satisfy two basic assumptions made in [3]: (i) neither the differential of J nor its adjoint are elliptic, (ii) the extended phase space **P** is not invariant under the interchange of **A** and E. Hence, we cannot use the results of Arms, Marsden and Moncrief, [3]. Instead, we follow the main idea of their paper, and prove the necessary intermediate steps. In particular, we prove the properness of the action of *GS*(**P**) and of the existence of slices for this action. From this we show that the reduced phase space is the union of symplectic manifolds labelled by the conjugacy classes of compact subalgebras of *gs*(**P**). Each of these symplectic manifolds consists of the fields (**A**,**E**,**Ψ**) with a definite mode of symmetry breaking.

In the finite dimensional case the partition of the reduced phase space into symplectic manifolds can be refined using conjugacy classes of compact subgroups of *GS*(**P**) rather than compact subalgebras. In this case one obtains a stratification, with strata which can described algebraically in terms of the Poisson algebra, c.f. [9,10].

Similar results for central Yang-Mills connections on surfaces has been obtained in [8]. An adaptation of their approach to our phase space will be studied elsewhere.

The paper is organized as follows. In Section 2 we discuss, in a proper functional analytic framework, the gauge symmetry group and its action. The structure of the zero level of the momentum map is analysed in Section 3. A stratification of the reduced phase space is studied in Section 4. Section 5 contains discussion of symmetry breaking corresponding to each stratum. The almost complex structure in the L^2 completion of **P** is discussed in Appendix A. The properness of the action of *GS*(**P**) is proved in Appendix B. The slice theorem is proved in Appendix C.

2. Gauge Symmetries and the Momentum Map.

The requirement that (1.6) gives an action of $\xi \in gs(\mathbf{P})$ in the space **P**, defined by (1.2), implies that $grad\xi \in H^2(M)$. Since M is bounded, it follows that $\xi \in H^3(M)$. Moreover, the action of ξ has to preserve the boundary conditions. The conditions (1.1a) are the usual bag boundary conditions and are gauge invariant. The conditions (1.1b) are satisfied if and only if $n \cdot grad\xi = 0$. Hence,

$$gs(\mathbf{P}) = \{\xi : M \longrightarrow \mathfrak{g} \,|\, \xi \in H^3(M) \text{ and } n \cdot grad\xi = 0\} \ . \tag{2.1}$$

The L^2 dual $gs(\mathbf{P})^*$ of $gs(\mathbf{P})$, considered here, is the space of square integrable maps from M to the dual $\mathfrak{g}^*$ of $\mathfrak{g}$, that is

$$gs(\mathbf{P})^* = \{\nu : M \longrightarrow \mathfrak{g}^* \,|\, \nu \in L^2(M)\}. \tag{2.2}$$

The evaluation of $\nu \in gs(\mathbf{P})^*$ on $\xi \in gs(\mathbf{P})$ is given by pointwise evaluation and integration

$$\langle \nu | \xi \rangle = \int_M \nu \cdot \xi \; d_3x. \tag{2.3}$$

The momentum map J defined in Eq. (1.9) is a continuous map from **P** to $gs(\mathbf{P})^*$.

GS(**P**) has a manifold structure with the tangent bundle space spanned by *gs*(**P**). The presentation of the structure group G as a matrix group, and boundedness.

of M, enable us to present $GS(\mathbf{P})$ as a group of maps ϕ from M to G of Sobolev class $H^3(M)$. Moreover, the boundary conditions (1.1) require that $n \cdot grad\phi = 0$. Hence,

$$GS(\mathbf{P}) = \{\phi : M \rightarrow G \,|\, \phi \in H^3(M) \text{ and } n \cdot grad\phi = 0\}. \tag{2.4}$$

Since M is contractible and G is connected, $GS(\mathbf{P})$ is connected. However, it need not be simply connected.

Proposition 2.1.

The exponential mapping $exp : gs(\mathbf{P}) \rightarrow GS(\mathbf{P})$ is a diffeomorphism of a neighbourhood of $0 \in gs(\mathbf{P})$ onto a neighbourhood of the identity in $GS(\mathbf{P})$.

Proof. Let U be a neighbourhood of $0 \in \mathfrak{g}$ and V a neighbourhood of the identity $e \in G$, such that the exponential mapping $exp : \mathfrak{g} \rightarrow G$ is a diffeomorphism of U onto V, and let $ln : V \rightarrow U$ be the inverse of this diffeomorphism. Since, by the Sobolev embedding theorem, each $\phi \in GS(\mathbf{P})$ is a continuous map from M to G, the sets

$$\mathbf{V} = \{\phi \in GS(\mathbf{P}) \,|\, range\ \phi \subseteq V\}$$

is open in $GS(\mathbf{P})$. Similarly, the set

$$\mathbf{U} = \{\xi \in gs(\mathbf{P}) \,|\, range\ \xi \subseteq U\}$$

is open in $gs(\mathbf{P})$. For every $\phi \in \mathbf{V}$, $ln \circ \phi$ is in $gs(\mathbf{P})$, and its range is in U. Hence, $ln \circ \phi \in \mathbf{U}$. Let $\boldsymbol{exp} : gs(\mathbf{P}) \rightarrow GS(\mathbf{P})$ denote the exponential for the gauge algebra. For every $\xi \in gs(\mathbf{P})$, $\boldsymbol{exp}(\xi) = exp \circ \xi$. Hence, for every $\phi \in \mathbf{V}$, $\boldsymbol{exp}(ln \circ \phi) = exp \circ ln \circ \phi = \phi$, which implies that $\boldsymbol{exp}(\mathbf{U}) = \mathbf{V}$.

Q.E.D.

The main property of the action of $GS(\mathbf{P})$ in $\mathbf{P}$ used in this paper is its properness.

THEOREM 1. The action of $GS(\mathbf{P})$ in $\mathbf{P}$ is proper.

That is, for every sequence p_n converging to q in $\mathbf{P}$ and every sequence ϕ_n in $GS(\mathbf{P})$ such that $\phi_n p_n$ converges to p, the sequence ϕ_n has a convergent subsequence with limit ϕ, and $\phi q = p$.

PROOF is given in Appendix B.

For each $\mathbf{p} \in \mathbf{P}$, we denote by $O_{\mathbf{p}}$ the orbit of $GS(\mathbf{P})$ through $\mathbf{p}$,

$$O_{\mathbf{p}} = \{\phi\mathbf{p} \mid \phi \in GS(\mathbf{P})\}. \tag{2.5}$$

All orbits $O_{\mathbf{p}}$ of $GS(\mathbf{P})$ are closed since, if $\phi_n\mathbf{p}$ is a convergent sequence of points in $O_{\mathbf{p}}$ with limit $\mathbf{q}$, then the sequence ϕ_n has a convergent subsequence with limit ϕ and $\mathbf{q} = \phi\mathbf{p}$, which implies that $\mathbf{q} \in O_{\mathbf{p}}$.

For every subspace $\mathbf{V}$ of $T_{\mathbf{p}}\mathbf{P}$, we denote by $\mathbf{V}^{\omega}$ the symplectic annihilator of $\mathbf{V}$, that is

$$\mathbf{V}^{\omega} = \{\mathbf{w} \in T_{\mathbf{p}}\mathbf{P} \mid \omega(\mathbf{v},\mathbf{w}) = 0 \quad \forall \quad \mathbf{v} \in \mathbf{V}\}. \tag{2.6}$$

Note that that $\mathbf{V}^{\omega}$ is closed, and if $\mathbf{V}$ is closed, then $(\mathbf{V}^{\omega})^{\omega} = \mathbf{V}$.

PROPOSITION 2.2.

For each $\mathbf{p} \in \mathbf{P}$,

$$T_{\mathbf{p}}O_{\mathbf{p}} = (\ker\, dJ_{\mathbf{p}})^{\omega}. \tag{2.7}$$

PROOF. If $\xi_{\mathbf{P}}$ is the Hamiltonian vector field of J_{ξ}, c.f. Eq. (1.11), then for every $\mathbf{v} \in T_{\mathbf{p}}\mathbf{P}$,

$$\omega(\xi_{\mathbf{P}}(\mathbf{p}),\mathbf{v}) = \langle dJ_{\mathbf{p}}(\mathbf{v}) \mid \xi\rangle. \tag{2.8}$$

Since $T_{\mathbf{p}}O_{\mathbf{p}} = \{\xi_{\mathbf{P}}(\mathbf{p}) \mid \xi \in gs(\mathbf{P})\}$ it follows that $\mathbf{v} \in (T_{\mathbf{p}}O_{\mathbf{p}})^{\omega}$ if and only if

$v \in ker\, dJ_p$. Hence, $(T_p O_p)^{\omega} = ker\, dJ_p$, and therefore $T_p O_p = (ker\, dJ_p)^{\omega}$, since *ker* dJ_p is closed.

Q.E.D.

PROPOSITION 2.3

For every $p \in P$, *range* dJ_p is a closed subspace of $gs(P)^*$ with finite codimension.

PROOF. For $p = (A,E,\Psi)$ and $(a,e,\psi) \in T_pP$, Eq. (1.12) implies that

$$\langle dJ_p(a,e,\psi) | \xi \rangle = \int_M \{-\, (div(e) + [A,e] + [E,a])\xi + \psi^{\dagger}\xi\Psi + \Psi^{\dagger}\xi\psi\} d_3x \,.$$

Hence, $dJ_p = T + S : T_pP \rightarrow L^2(M,g)$, where

$$T(a,e,\psi) = -\, div(e) \quad \text{and} \quad S(a,e,\psi) = -\,[A,e] - [E,a] + \psi^{\dagger}\otimes\Psi + \Psi^{\dagger}\otimes\psi \,.$$

The Hodge decomposition, cf. [11], applied to square integrable zero forms on M, implies that $L^2(M,g) = \mathcal{C} \oplus \mathcal{K}$, where $\mathcal{K}$ is the space of constant g-valued functions and $\mathcal{C} = \{div(v) | v \in H^1(M,g),\ nv = 0\}$. Both $\mathcal{C}$ and $\mathcal{K}$ are closed subspaces of $L^2(M,g)$. Since *range* $T = \mathcal{C}$, it follows that the range of T is closed. Moreover, *cokernel* $T = L^2(M,g)/range\ T \simeq \mathcal{K}$ has finite dimension, since $dim\ \mathcal{K} = dim$ g. Hence, T is semi-Fredholm.

Further, if $v_n = (a_n,e_n,\psi_n)$ is a bounded sequence in T_pP, then the sequence

$$\{Sv_n\} = \{-\,[A,e_n] - [E,a_n] + \psi_n^{\dagger}\otimes\Psi + \Psi^{\dagger}\otimes\psi_n\}$$

is bounded in $H^1(M,g) \subset L^2(M,g)$. Since the embedding of $H^1(M,g)$ into $L^2(M,g)$ is compact, it follows that the sequence $\{Sv_n\}$ has a convergent subsequence. That is, the operator S is compact. This implies that $dJ_p = T + S$ is semi-Fredholm, that is it has closed range and finite codimension, c.f. [12].

Q.E.D.

For each $p \in P$ we denote by gs_p the gauge symmetry (isotropy) algebra of p, that is

$$gs_p = \{\xi \in gs(P) \mid \xi_P(p) = 0\}, \tag{2.9}$$

and by GS_p gauge symmetry (isotropy) group of p,

$$GS_p = \{\phi \in GS(P) \mid \phi p = p\}. \tag{2.10}$$

By properness of the action of $GS(P)$ in P, each sequence $\{\phi_n\}$ in GS_p has a convergent subsequence, which implies that GS_p is compact. Consequently, the Lie algebra gs_p is finite dimensional. It is isomorphic to a subalgebra of the structure algebra $\mathfrak{g}$; a construction of such an ismomorphism is given in Section 5.

The annihilator of the subalgebra $\mathfrak{h} \subseteq gs(P)$ is the subspace $\mathfrak{h}^a \subseteq gs(P)^*$ defined by

$$\mathfrak{h}^a = \{\nu \in gs(P)^* \mid \langle \nu | \xi \rangle = 0 \ \forall \ \xi \in \mathfrak{h}\}. \tag{2.11}$$

PROPOSITION 2.4.

The range of the map $dJ_p : T_pP \to gs(P)^*$ is given by the annihilator of the symmetry algebra of p, that is

$$\mathit{range}\ dJ_p = (gs_p)^a . \tag{2.12}$$

PROOF. By (1.11), for each $\xi \in gs(P)$, and $p \in P$,

$$\langle dJ_p(\cdot) | \xi \rangle = \xi_P(p) \lrcorner \, \omega . \tag{2.13}$$

Since ω is non-degenerate, it follows from (2.9) that

$$gs_p = \{\xi \in gs(P) \mid \langle dJ_p(v) | \xi \rangle = 0 \ \forall \ v \in T_pP\} = (\mathit{range}\ dJ_p)^a. \tag{2.14}$$

Since $\mathit{range}\ dJ_p$ is closed, taking annihilators of both sides we obtain

$$(gs_p)^a = (\mathit{range}\ dJ_p)^{aa} = \mathit{range}\ dJ_p ,$$

provided that $(\mathit{range}\ dJ_p)^{aa}$ is the closure of $\mathit{range}\ dJ_p$.

In order to prove the last assertion, denote by R_p the closure of $\mathit{range}\ dJ_p$ in the topological dual $gs(P)'$ of $gs(P)$. The polar of R_p is

$$(R_p)^0 = \{\xi \in gs(P) \mid \langle v \mid \xi\rangle = 0 \ \forall\ v \in R_p\},$$

and the bi-polar

$$(R_p)^{00} = \{v \in gs(P)' \mid \langle v \mid \xi\rangle = 0 \ \forall\ \xi \in (R_p)^0\}$$

is the closure of R_p in $gs(P)'$, c.f. [13]. By definition R_p is closed so that $R_p = (R_p)^{00}$. Since *range* dJ_p is dense in R_p, it follows that

$$(R_p)^0 = (\mathit{range}\ dJ_p)^a .$$

Hence,

$$(\mathit{range}\ dJ_p)^{aa} = (R_p)^{00} \cap gs(P)^* = R_p \cap gs(P)^* ,$$

which implies that $(\mathit{range}\ dJ_p)^{aa}$ is the closure of *range* dJ_p in $gs(P)^*$.

Q.E.D.

We conclude from Proposition 2.4 that p is a regular point of the momentum map J if and only if p has no infinitesimal symmetries, i.e. $gs_p = \{0\}$. In this case $J^{-1}(J(p))$ is a manifold in a neighbourhood of p with the tangent space

$$T_p J^{-1}(J(p)) = \mathit{ker}\ dJ_p .$$

Singular points of the momentum map have non trivial algebras of infinitesimal symmetries. Let $\mathfrak{h}$ be the Lie algebra of a connected compact subgroup H of $GS(P)$. We denote by $P_{\mathfrak{h}}$ the set of points p in P such that $gs_p = \mathfrak{h}$, that is

$$P_{\mathfrak{h}} = \{p \in P \mid gs_p = \mathfrak{h}\}. \tag{2.15}$$

It follows from Proposition 2.4 that $p \in P_{\mathfrak{h}}$ if and only if for all $v \in T_pP$

$$\langle dJ_p(v) \mid \xi\rangle = 0 \quad \forall \quad \xi \in \mathfrak{h} \quad \text{and} \quad \langle dJ_p(v) \mid \zeta\rangle \neq 0 \quad \forall \quad \zeta \notin \mathfrak{h}. \tag{2.16}$$

Let $\bar{P}_{\mathfrak{h}}$ be the set of points in P such that their symmetry algebra gs_p contains $\mathfrak{h}$, i.e.

$$\bar{P}_{\mathfrak{h}} = \{p \in P \mid \mathfrak{h} \subseteq gs_p\}. \tag{2.17}$$

It is clear that $\bar{P}_{\mathfrak{h}} = \bigcup_{\mathfrak{k} \supseteq \mathfrak{h}} P_{\mathfrak{k}}$.

Proposition 2.5.

(1) $\bar{P}_{\mathfrak{h}}$ is a closed affine subspace of **P**.

(2) $P_{\mathfrak{h}}$ is a submanifold of **P** with the tangent space

$$T_pP_{\mathfrak{h}} = \{v \in T_pP \,|\, \langle D^2J_p(u,v)|\xi\rangle = 0 \quad \forall \quad u \in T_pP_{\mathfrak{h}},\ \xi \in \mathfrak{h}\}. \tag{2.18}$$

(3) The restrictions of the symplectic form ω to $\bar{P}_{\mathfrak{h}}$ and $P_{\mathfrak{h}}$ are symplectic.

Proof.

(1) It follows from the fact that the action of *GS*(**P**) in **P** is continuous and affine.

(2) Let **p** be a point in $P_{\mathfrak{h}}$. Then, for every $\zeta \notin \mathfrak{h}$, the linear map $\langle dJ_p(.)|\zeta\rangle$: $T_pP \longrightarrow \mathbb{R}$ does not vanish identically. Since dJ is continuous, there exists a neighbourhood U of **p** in **P** such that, for all $q \in U$, $\langle dJ_q(.)|\zeta\rangle \neq 0$ for every ζ not in $\mathfrak{h}$. This implies that $U \cap \bar{P}_{\mathfrak{h}} \subseteq P_{\mathfrak{h}}$. Hence, $P_{\mathfrak{h}}$ is an open submanifold of $\bar{P}_{\mathfrak{h}}$ and a submanifold of **P**. The expression (2.18) for the tangent space of $P_{\mathfrak{h}}$ is obtained by differentiating $\langle dJ_p(.)|\xi\rangle = 0$ for all $\xi \in \mathfrak{h}$ and all $p \in P_{\mathfrak{h}}$.

(3) Here we have to use the almost complex structure described in Appendix A. Let $\tilde{P}$ be the L^2-closure of **P**. Then $\tilde{P}_{\mathfrak{h}} = \{p \in \tilde{P} \,|\, \mathfrak{h} \subseteq gs_p\}$ is the L^2-closure of $\bar{P}_{\mathfrak{h}}$. Since $\mathcal{J}$ is *GS*(**P**) invariant, it follows that $\mathcal{J}\tilde{P}_{\mathfrak{h}} = \tilde{P}_{\mathfrak{h}}$. Suppose that $u \in T\bar{P}_{\mathfrak{h}}$ is such that **u** is in the kernel of $\omega|\bar{P}_{\mathfrak{h}}$. Since $T\bar{P}_{\mathfrak{h}} \subset T\tilde{P}_{\mathfrak{h}}$,

$$\langle \mathcal{J}u|v\rangle_{L^2} = \tilde{\omega}(u,v) = \omega(u,v) = 0 \tag{2.19}$$

for every $v \in T\bar{P}_{\mathfrak{h}}$. By construction $T\bar{P}_{\mathfrak{h}}$ is dense in $T\tilde{P}_{\mathfrak{h}}$, hence $\mathcal{J}u = 0$, and $u = 0$. Therefore, $\bar{P}_{\mathfrak{h}}$ is symplectic. Since $P_{\mathfrak{h}}$ is an open submanifold of $\bar{P}_{\mathfrak{h}}$ it is also symplectic.

Q.E.D.

The last essential property of the action of *GS*(**P**) in **P** needed here is the existence of slices. A slice through a point $p \in P$ for the action of *GS*(**P**) is a submanifold S_p of **P** containing **p**, and such that

(1) S_p is transverse and complementary to the orbit O_p at p, that is

$$T_pS_p \oplus T_pO_p = T_pP. \tag{2.20}$$

(2) S_p is transverse to all $GS(\mathbf{P})$ orbits, that is, for each $q \in S_p$,

$$T_qS_p + T_qO_q = T_qP. \tag{2.21}$$

(3) S_p is invariant under the action of the gauge symmetry group GS_p of p.

(4) For $\mathbf{q} \in S_p$ and $\phi \in GS(\mathbf{P})$, if $\phi\mathbf{q} \in S_p$ then $\phi \in GS_p$.

THEOREM 2 (Slice Theorem).

For each $\mathbf{p} \in \mathbf{P}$ there exists a slice S_p through p for the action of $GS(\mathbf{P})$ such that T_pS_p is L^2 orthogonal to T_pO_p.

PROOF is given in Appendix C.

3. Structure of the constraint set.

The constraint set is the zero level of the momentum map J. It follows from Proposition 2.4 that $J^{-1}(0)$ need not be a manifold in neighbourhoods of points admitting infinitesimal symmetries. We shall show that it is foliated by submanifolds labelled by the Lie algebras $\mathfrak{h}$ of compact subgroups of $GS(\mathbf{P})$. Let $\mathbf{M}_{\mathfrak{h}}$ be the intersection of $J^{-1}(0)$ with the submanifold $\mathbf{P}_{\mathfrak{h}}$,

$$\mathbf{M}_{\mathfrak{h}} = J^{-1}(0) \cap \mathbf{P}_{\mathfrak{h}}. \tag{3.1}$$

We denote by $\mu_{\mathfrak{h}}$ the pull-back of ω by the inclusion map $j_{\mathfrak{h}} : \mathbf{M}_{\mathfrak{h}} \to \mathbf{P}$,

$$\mu_{\mathfrak{h}} = j_{\mathfrak{h}}^{*}\omega . \tag{3.2}$$

By $\mathfrak{n}(\mathfrak{h})$ we denote the normaliser of $\mathfrak{h}$ in $gs(\mathbf{P})$,

$$\mathfrak{n}(\mathfrak{h}) = \{\xi \in gs(\mathbf{P}) \,|\, [\xi,\zeta] \in \mathfrak{h} \;\forall\; \zeta \in \mathfrak{h}\} . \tag{3.3}$$

THEOREM 3.

For every compact connected subgroup H of $GS(\mathbf{P})$, with Lie algebra $\mathfrak{h}$, $(\mathbf{M}_{\mathfrak{h}},\mu_{\mathfrak{h}})$ is a co-isotropic submanifold of $(\mathbf{P},\omega)$. The null distribution of $\mu_{\mathfrak{h}}$ is spanned by the vector fields $\xi_{\mathbf{P}}$, for $\xi \in \mathfrak{n}(\mathfrak{h})$.

The proof of this theorem will be given in a series of propositions. First, we observe that the annihilator $\mathfrak{h}^a$ is closed, and hence

$$gs(\mathbf{P})^* = \mathfrak{h}^a \oplus (\mathfrak{h}^a)^\perp . \tag{3.4}$$

We denote by $\pi_{\mathfrak{h}} : gs(\mathbf{P})^* \longrightarrow \mathfrak{h}^a$ the projections on the first component, and by K the composition of J with $\pi_{\mathfrak{h}}$,

$$K = \pi_{\mathfrak{h}} \circ J : \mathbf{P} \longrightarrow \mathfrak{h}^a. \tag{3.5}$$

It should be noted that the map K depends on the choice of the infinitesimal symmetry algebra $\mathfrak{h}$, but we shall not label it by the subscript $\mathfrak{h}$ in order to simplify the notation. Moreover, $J^{-1}(0) \subseteq K^{-1}(0)$ for every $\mathfrak{h}$, so that

$$\mathbf{M}_{\mathfrak{h}} = J^{-1}(0) \cap \mathbf{P}_{\mathfrak{h}} \subseteq K^{-1}(0) \cap \mathbf{P}_{\mathfrak{h}} . \tag{3.6}$$

PROPOSITION 3.1.

For every $p \in K^{-1}(0) \cap \mathbf{P}_{\mathfrak{h}}$, $K^{-1}(0)$ is a submanifold of $\mathbf{P}$ in a neighbourhood of p and

$$T_p K^{-1}(0) = \ker\, dJ_p. \tag{3.7}$$

PROOF. Since $p \in \mathbf{P}_{\mathfrak{h}} \cap K^{-1}(0)$, it follows from Prop. 2.4 that

$$\mathit{range}\ dK_p = \pi_{\mathfrak{h}}(\mathit{range}\ dJ_p) = \pi_{\mathfrak{h}}(\mathfrak{h}^a) = \mathfrak{h}^a,$$

which implies that K is a submersion. Hence, $K^{-1}(0)$ is locally a submanifold of $\mathbf{P}$. Moreover, $T_p K^{-1}(0) = \ker\, dK_p = \ker\, d(\pi_{\mathfrak{h}} \circ J)_p = \ker\, dJ_p$.

Q.E.D.

PROPOSITION 3.2.

$K^{-1}(0) \cap P_{\mathfrak{h}}$ is a submanifold of $P_{\mathfrak{h}}$. For every $p \in K^{-1}(0) \cap P_{\mathfrak{h}}$,

$$T_p(K^{-1}(0) \cap P_{\mathfrak{h}}) = \{v \in \mathit{ker}\ dJ_p \,|\, \langle D^2J_p(v,w)|\xi\rangle = 0\ \forall\ \xi \in \mathfrak{h},\ w \in T_pP\} \qquad (3.8).$$

PROOF. For every $p \in K^{-1}(0) \cap P_{\mathfrak{h}}$, K is a submersion in a neighbourhood of p. Hence, for every $\nu \in \mathfrak{h}^a$, there exists a vector field X_ν in P such that $\nu = dK(X_\nu) = \pi_{\mathfrak{h}}(dJ(X_\nu))$ in a neighbourhood of p. Also, $\nu \in \mathfrak{h}^a$ implies that $\langle dJ(X_\nu)|\xi\rangle = 0$ for all $\xi \in \mathfrak{h}$. Differentiating this equation in direction of $w \in T_pP$ we obtain

$$\langle D^2J_p(w,X_\nu(p))|\xi\rangle + \langle dJ_p(DX_\nu(w))|\xi\rangle = 0\ .$$

However, $\langle dJ_p(DX_\nu(w))|\xi\rangle = 0$ since $\langle dJ_p(.)|\xi\rangle = 0$ for all $\xi \in \mathfrak{h}$. Hence, $\langle D^2J_p(w,X_\nu(p))|\xi\rangle = 0$ for all $\xi \in \mathfrak{h}$, which implies by Eq. (2.18) that $X_\nu(p) \in T_pP_{\mathfrak{h}}$. Therefore, the restriction of K to $P_{\mathfrak{h}}$ is a submersion at p, and $K^{-1}(0) \cap P_{\mathfrak{h}} = (K|P_{\mathfrak{h}})^{-1}(0)$ is a submanifold of $P_{\mathfrak{h}}$. Eq. (3.8) follows from (3.7).

Q.E.D.

Most of the following analysis is local. Therefore, we fix a point p_0 in $J^{-1}(0) \cap P_{\mathfrak{h}}$ and discuss the structure of $M_{\mathfrak{h}}$ in its neighbourhood.

Let S_{p_0} be a slice for the GS(P) action such that $T_{p_0}S_{p_0}$ is the L^2 orthogonal complement of $T_{p_0}\mathcal{O}_{p_0}$,

$$T_{p_0}S_{p_0} = (T_{p_0}\mathcal{O}_{p_0})^{\perp}. \qquad (3.9)$$

PROPOSITION 3.3.

$P_{\mathfrak{h}} \cap S_{p_0}$ is a manifold in a neighbourhood of p_0 with tangent space

$$T_{p_0}(P_{\mathfrak{h}} \cap S_{p_0}) = \{v \in (T_{p_0}\mathcal{O}_{p_0})^{\perp} \,|\, \langle D^2J(v,w)|\xi\rangle = 0\ \forall\ \xi \in \mathfrak{h},\ w \in T_{p_0}P\}. \qquad (3.10)$$

PROOF. Let $P_{(\mathfrak{h})}$ the set of points on the GS(P) orbits through $P_{\mathfrak{h}}$, that is

$$P_{(\mathfrak{h})} = \{\phi q \,|\, \phi \in GS(P),\ q \in P_{\mathfrak{h}}\}\ . \qquad (3.11)$$

If $\psi q = p$, and $q \in P_{\mathfrak{h}}$, then $\psi\phi\psi^{-1}p = p \quad \forall \quad \phi \in H$. Thus, for every $q \in P_{(\mathfrak{h})}$, the symmetry group GS_q is conjugate to H. Moreover, property (4) of a slice yields

$$P_{(\mathfrak{h})} \cap S_p = \{\phi q \in S_p | \phi \in GS(P),\ q \in P_{\mathfrak{h}}\} =$$
$$= \{\phi q \in S_p | \phi \in H,\ q \in P_{\mathfrak{h}}\} = P_{\mathfrak{h}} \cap S_p .$$

By definition of $P_{(\mathfrak{h})}$ and property 2/ of the slice, $P_{(\mathfrak{h})}$ and S_{p_0} are transverse at p_0. It follows that $P_{\mathfrak{h}} \cap S_{p_0}$ is a submanifold of S_{p_0} in a neighbourhood of p_0. Further, Eq. (2.16), implies that

$$P_{\mathfrak{h}} \cap S_{p_0} = \{p \in S_{p_0} | \langle dJ_p(.)|\xi\rangle = 0 \ \forall\ \xi \in \mathfrak{h}\}. \tag{3.12}$$

Differentiating this condition at p_0 and taking into account Eq. (3.9) we obtain (3.10).

Q.E.D.

PROPOSITION 3.4.

$M_{\mathfrak{h}}$ is an open subset of $P_{\mathfrak{h}} \cap K^{-1}(0)$, and

$$M_{\mathfrak{h}} \cap S_{p_0} = P_{\mathfrak{h}} \cap S_{p_0} \cap K^{-1}(0) \tag{3.13}$$

PROOF. By Eq. (3.12), $\langle dJ_p(.)|\xi\rangle = 0$ for all $p \in P_{\mathfrak{h}} \cap S_{p_0} \cap K^{-1}(0)$ and all $\xi \in \mathfrak{h}$. Therefore, the function $\langle J|\xi\rangle$ on P is constant on connected components of $P_{\mathfrak{h}} \cap S_{p_0} \cap K^{-1}(0)$. By hypothesis $J(p_0) = 0$. Therefore $\langle J(p)|\xi\rangle = 0$ for all p in the connected component of $P_{\mathfrak{h}} \cap S_{p_0} \cap K^{-1}(0)$ containing p_0. Hence, $J|P_{\mathfrak{h}} \cap S_{p_0} \cap K^{-1}(0)$ has range in $\mathfrak{h}^a$. By definition $K = \pi_{\mathfrak{h}} \circ J : P \longrightarrow \mathfrak{h}^a$, so that $J(P_{\mathfrak{h}} \cap S_{p_0} \cap K^{-1}(0)) = K(P_{\mathfrak{h}} \cap S_{p_0} \cap K^{-1}(0)) = 0$. Since $M_{\mathfrak{h}} = P_{\mathfrak{h}} \cap J^{-1}(0)$ and $J^{-1}(0) \subseteq K^{-1}(0)$, it follows that

$$M_{\mathfrak{h}} \cap S_{p_0} = P_{\mathfrak{h}} \cap S_{p_0} \cap K^{-1}(0).$$

The preceding local analysis is valid for every $p_0 \in \in P_{\mathfrak{h}} \cap J^{-1}(0)$. In order to show that $M_{\mathfrak{h}}$ is open in $P_{\mathfrak{h}} \cap K^{-1}(0)$, for each $p_0 \in P_{\mathfrak{h}} \cap J^{-1}(0)$, we choose a slice S_{p_0} through p_0 satisfying (3.9), and a neighbourhood V_{p_0} of p_0 in O_{p_0} such that

$S_{p_0} \times V_{p_0}$ is an open neighbourhood of p_0 in P. The collection

$$\{(S_{p_0} \times V_{p_0}) \cap P_{\mathfrak{h}} \cap K^{-1}(0) \,|\, p_0 \in M_{\mathfrak{h}}\} \qquad (3.14)$$

is an open covering of $M_{\mathfrak{h}}$ in $P_{\mathfrak{h}} \cap K^{-1}(0)$. If $p \in P_{\mathfrak{h}} \cap K^{-1}(0)$ is contained in $(S_{p_0} \times V_{p_0})$, for some $p_0 \in M_{\mathfrak{h}}$, then there exists $\phi \in GS(P)$ such that p is contained in the slice $S'_{\phi p_0} = \phi(S_{p_0})$ through ϕp_0, satisfying (3.9), which need not to belong to the collection of slices chosen in (3.14). Hence,

$$p \in P_{\mathfrak{h}} \cap S'_{\phi p_0} \cap K^{-1}(0) = P_{\mathfrak{h}} \cap S'_{\phi p_0} \cap J^{-1}(0) \subseteq P_{\mathfrak{h}} \cap J^{-1}(0) = M_{\mathfrak{h}}.$$

This implies that the union of sets in (3.14) is contained in $M_{\mathfrak{h}}$. Hence, $M_{\mathfrak{h}}$ is open in $P_{\mathfrak{h}} \cap K^{-1}(0)$.

Q.E.D.

PROPOSITION 3.5

$M_{\mathfrak{h}} \cap S_{p_0}$ is a symplectic manifold in a neighbourhood of p_0 with tangent space

$$T_{p_0}(M_{\mathfrak{h}} \cap S_{p_0}) = \qquad (3.15)$$
$$= \{v \in (T_{p_0}O_{p_0})^{\perp} \cap \mathit{ker}\, dJ_{p_0} \,|\, \langle D^2J(v,w)|\xi\rangle = 0 \;\forall\; \xi \in \mathfrak{h},\, w \in T_{p_0}P\}\,.$$

PROOF. Since $M_{\mathfrak{h}} \cap S_{p_0} = P_{\mathfrak{h}} \cap S_{p_0} \cap K^{-1}(0)$, we prove the statement for $P_{\mathfrak{h}} \cap S_{p_0} \cap K^{-1}(0)$. Consider the restriction $K|P_{\mathfrak{h}} \cap S_{p_0}$. For each $v \in T_{p_0}P$, $dK_{p_0}(v) = \pi_{\mathfrak{h}}(dJ_{p_0}(v))$. Hence, using Eq. (3.10),

$$\mathit{range}\; d(K|(P_{\mathfrak{h}} \cap S_{p_0}))_{p_0} =$$
$$= \{\pi_{\mathfrak{h}}(dJ_{p_0}(v)) \,|\, v \in (T_{p_0}O_{p_0})^{\perp} \text{ and } \langle D^2J(v,v)|\xi\rangle = 0 \;\forall\; \xi \in \mathfrak{h},\, w \in T_{p_0}P\}.$$

Since $T_{p_0}O_{p_0} \subseteq \mathit{ker}\; dK_{p_0}$, the condition $v \in (T_{p_0}O_{p_0})^{\perp}$ can be omitted and, using Proposition 2.5, we obtain

$$\mathit{range}\; d(K|(P_{\mathfrak{h}} \cap S_{p_0}))_{p_0} =$$
$$\{\pi_{\mathfrak{h}}(dJ_{p_0}(v)) \,|\, v \in T_{p_0}P_{\mathfrak{h}} \text{ and } \langle D^2J(v,w)|\xi\rangle = 0 \;\forall\; \xi \in \mathfrak{h},\, w \in T_{p_0}P\} =$$
$$= \{\pi_{\mathfrak{h}}(dJ_{p_0}v) \,|\, v \in T_{p_0}P_{\mathfrak{h}}\} = \mathit{range}\; d(K|P_{\mathfrak{h}})_{p_0}.$$

In the proof of Prop. 3.2 we have shown that $K|P_{\mathfrak{h}}$ is a submersion at p_0. Hence,

$K|(P_{\mathfrak{h}} \cap S_{p_0})$ is a submersion at p_0, which implies that $P_{\mathfrak{h}} \cap S_{p_0} \cap K^{-1}(0) = (K|(P_{\mathfrak{h}} \cap S_{p_0}))^{-1}(0)$ is locally a submanifold of $P_{\mathfrak{h}} \cap S_{p_0}$. The expression (3.15) for the tangent space follows from (3.7) and (3.10).

Let $v \in T_{p_0}(P_{\mathfrak{h}} \cap S_{p_0} \cap J^{-1}(0))$. Since $v \in T_{p_0}(P_{\mathfrak{h}} \cap S_{p_0})$, Eqs. (3.10), (2.7) and (A.7), imply that $\mathcal{J}v \in \mathcal{J}((T_{p_0}O_{p_0})^{\perp}) = ker\ dJ_{p_0}$, where $\mathcal{J}$ is the almost complex structure discussed in Appendix A. Also, $v \in ker\ dJ_{p_0}$, so that $\mathcal{J}v \in \mathcal{J}(ker\ dJ_{p_0}) = (T_{p_0}O_{p_0})^{\perp}$. Moreover, Lemma A.1 yields $\langle D^2J_{p_0}(\mathcal{J}v,\mathcal{J}w)|\xi\rangle = \langle D^2J_{p_0}(v,w)|\xi\rangle = 0$ for all $\xi \in \mathfrak{h}$, $w \in T_{p_0}P$. Hence, $\mathcal{J}v \in T_{p_0}(P_{\mathfrak{h}} \cap S_{p_0} \cap K^{-1}(0))$, cf. Eq. (3.15).

Let $v \in T_{p_0}(P_{\mathfrak{h}} \cap S_{p_0} \cap K^{-1}(0))$ be such that $\omega(v,w) = 0$ for all $w \in T_{p_0}(P_{\mathfrak{h}} \cap S_{p_0} \cap K^{-1}(0))$. Then $\tilde{\omega}(v,\tilde{w}) = 0$ for all $\tilde{w}$ in the L^2 closure of $T_{p_0}(P_{\mathfrak{h}} \cap S_{p_0} \cap K^{-1}(0))$. Taking, $\tilde{w} = \mathcal{J}v$ we obtain $\|v\|^2_{L^2} = \langle v|v\rangle_{L^2} = \tilde{\omega}(\mathcal{J}v,v) = 0$. Hence, $v = 0$, which implies that $T_{p_0}(P_{\mathfrak{h}} \cap S_{p_0} \cap K^{-1}(0))$ is symplectic. Hence, $P_{\mathfrak{h}} \cap S_{p_0} \cap K^{-1}(0)$ is symplectic in a neighbourhood of p_0.

Q.E.D.

We have shown in Proposition 3.4 that, $M_{\mathfrak{h}}$ is an open submanifold of $P_{\mathfrak{h}} \cap K^{-1}(0)$, the manifold structure of which has been established in Proposition 3.2. Moreover, $M_{\mathfrak{h}} \cap S_{p_0}$ is symplectic by Proposition 3.6. It remains to show that the pull-back $\mu_{\mathfrak{h}}$ of ω to $M_{\mathfrak{h}}$ is co-isotropic with the null spanned by the vector fields ξ_P, for ξ in the normaliser $\mathfrak{n}(\mathfrak{h})$ of $\mathfrak{h}$, given by (3.3).

Let N(H) denote the normaliser of H in $GS(P)$, that is

$$N(H) = \{\phi \in GS(P) | \phi^{-1}\chi\phi \in H\ \forall\ \chi \in H\}\ , \tag{3.16}$$

and $N_0(H)$ be the connected component of the identity in N(H). It is a subgroup of $GS(P)$ with the Lie algebra $\mathfrak{n}(\mathfrak{h})$ given by Eq. (3.3). The connected component of the intersection of $P_{\mathfrak{h}}$ with O_{p_0} containing p_0 can be described in terms of $N_0(H)$ as follows,

$$component\ (P_{\mathfrak{h}} \cap O_{p_0}) = \{\phi p_0 | \phi \in N_0(H)\}. \tag{3.17}$$

The tangent space to $P_{\mathfrak{h}} \cap O_{p_0}$ at p_0 is given by

$$T_{p_0}(P_{\mathfrak{h}} \cap O_{p_0}) = \{\xi_P(p_0) \mid \xi \in \mathfrak{n}(\mathfrak{h})\}. \tag{3.18}$$

For every $\xi \in \mathfrak{n}(\mathfrak{h})$ and every $w \in T_{p_0}M_{\mathfrak{h}} = T_{p_0}(P_{\mathfrak{h}} \cap J^{-1}(0))$,

$$\mu_{\mathfrak{h}}(\xi_P(p_0), w) = \omega(\xi_P(p_0), w) = \langle dJ_{p_0}(w) \mid \xi\rangle = 0 .$$

This implies that $T_{p_0}(P_{\mathfrak{h}} \cap O_{p_0})$ is contained in the null space of $\mu_{\mathfrak{h}}$ at p_0.

Let S_{p_0} be a slice through p_0 satisfying Eq. (3.8). Then, by the Slice Theorem,

$$T_{p_0}M_{\mathfrak{h}} = T_{p_0}(M_{\mathfrak{h}} \cap O_{p_0}) \oplus T_{p_0}(M_{\mathfrak{h}} \cap S_{p_0}) = T_{p_0}(P_{\mathfrak{h}} \cap O_{p_0}) \oplus T_{p_0}(M_{\mathfrak{h}} \cap S_{p_0}),$$

since $O_{p_0} \subseteq J^{-1}(0)$. By Proposition 3.5, $T_{p_0}(M_{\mathfrak{h}} \cap S_{p_0})$ is symplectic. Hence, $\{\zeta_P(p_0) \mid \zeta \in \mathfrak{n}(\mathfrak{h})\} = T_{p_0}(P_{\mathfrak{h}} \cap O_{p_0})$ is the null space of $\mu_{\mathfrak{h}}$ at p_0. This completes the proof of Theorem 3.

4. Reduction.

The reduced phase space $\check{P}$ of the system is defined as the space of $GS(P)$ orbits in the constraint set $J^{-1}(0)$,

$$\check{P} = J^{-1}(0)/GS(P). \tag{4.1}$$

We denote by $\rho : J^{-1}(0) \longrightarrow \check{P}$ the natural projection, assigning to each $p \in J^{-1}(0)$ the orbit $O_p \in \check{P}$,

$$\rho(p) = O_p. \tag{4.2}$$

Since the action of $GS(P)$ in P is proper, the quotient topology in $\check{P}$ is Hausdorff. This can be seen as follows. If $p, q \in J^{-1}(0)$ are such that $\rho(p)$ and $\rho(q)$ cannot be separated by open sets, then there exists a sequence p_n in $J^{-1}(0)$ such that $\rho(p_n)$ converges both to $\rho(p)$ and $\rho(q)$. Let S_p and S_q be slices through p and q, respectively. For sufficiently large n, there exist $\phi_n, \psi_n \in GS(P)$ such that $\phi_n p_n \in S_p$ and $\psi_n p_n \in S_q$. Hence, $\phi_n p_n \to p$ and $\psi_n p_n \to q$ as $n \to \infty$. Thus, $\phi_n \psi_n^{-1}(\psi_n p_n) \to p$, while $\psi_n p_n \to q$, which implies that $\phi_n \psi_n^{-1}$ has a convergent subsequence with limit χ and $\chi q = p$. Hence, $p \in O_q$ and $\rho(p) = \rho(q)$.

If H_1 and H_2 are conjugate compact subgroups of $GS(\mathbf{P})$, with Lie algebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$, respectively, then the $GS(\mathbf{P})$-orbits of $\mathbf{P}_{\mathfrak{h}_1}$ and $\mathbf{P}_{\mathfrak{h}_2}$ coincide, that is $\mathbf{P}_{(\mathfrak{h}_1)} = \mathbf{P}_{(\mathfrak{h}_2)}$, c.f. Eq. (3.11), and $\rho(\mathbf{P}_{\mathfrak{h}_1}) = \rho(\mathbf{P}_{\mathfrak{h}_2})$. For every Lie algebra $\mathfrak{h}$ of a compact subgroup H of $GS(\mathbf{P})$, we denote by $\check{\mathbf{P}}_{(\mathfrak{h})}$ the projection of $J^{-1}(0) \cap \mathbf{P}_{(\mathfrak{h})}$ to $\check{\mathbf{P}}$,

$$\check{\mathbf{P}}_{(\mathfrak{h})} = \rho(J^{-1}(0) \cap \mathbf{P}_{(\mathfrak{h})}), \tag{4.3}$$

and by $\rho_{\mathfrak{h}} : \mathbf{M}_{\mathfrak{h}} \longrightarrow \check{\mathbf{P}}_{(\mathfrak{h})}$ the restriction of ρ to $\mathbf{M}_{\mathfrak{h}} = J^{-1}(0) \cap \mathbf{P}_{\mathfrak{h}}$, considered as a map to $\check{\mathbf{P}}_{(\mathfrak{h})}$.

THEOREM 4.

$\check{\mathbf{P}}_{(\mathfrak{h})}$ is a weakly symplectic manifold with an exact symplectic form

$$\check{\omega}_{(\mathfrak{h})} = d\check{\theta}_{(\mathfrak{h})}. \tag{4.4}$$

For every $\mathfrak{h}$ in the conjugacy class $(\mathfrak{h})$, $\rho_{\mathfrak{h}} : M_{\mathfrak{h}} \longrightarrow \check{\mathbf{P}}_{(\mathfrak{h})}$ is a submersion, and

$$\rho_{\mathfrak{h}}^{*}\check{\theta}_{(\mathfrak{h})} = \theta | M_{\mathfrak{h}} \quad \text{and} \quad \rho_{\mathfrak{h}}^{*}\check{\omega}_{(\mathfrak{h})} = \mu_{\mathfrak{h}}. \tag{4.5}$$

The proof of this theorem will be split into several propositions.

PROPOSITION 4.1.

Each connected component of $\check{\mathbf{P}}_{(\mathfrak{h})}$ is a smooth manifold. For each $\mathfrak{h}$ in the conjugacy class $(\mathfrak{h})$, the map $\rho_{\mathfrak{h}} : \mathbf{M}_{\mathfrak{h}} \longrightarrow \check{\mathbf{P}}_{(\mathfrak{h})}$ is a submersion.

PROOF. Since H is normal in N(H), the quotient N(H)/H is a group. We want to define an action of N(H)/H in $\mathbf{M}_{\mathfrak{h}}$. Let $\mathbf{p}$ and $\mathbf{q} = \phi\mathbf{p}$ be two points on the same orbit of N(H) in $\mathbf{M}_{\mathfrak{h}}$. Since, $\chi\mathbf{p} = \mathbf{p}$ for every $\chi \in H$, it follows that q is uniquely determined by $\mathbf{p}$ and the equivalence class $[\phi]$ of ϕ in N(H)/H. Hence, we can set $[\phi]\mathbf{p} = \phi\mathbf{p}$. This defines a left action of N(H)/H in $\mathbf{M}_{\mathfrak{h}}$. Moreover, $\check{\mathbf{P}}_{(\mathfrak{h})} = \rho_{\mathfrak{h}}(\mathbf{M}_{\mathfrak{h}})$ coincides with the space of N(H)/H orbits in $\mathbf{M}_{\mathfrak{h}}$,

$$\check{\mathbf{P}}_{(\mathfrak{h})} = \mathbf{M}_{\mathfrak{h}}/(N(H)/H) \ . \tag{4.6}$$

The $(N(H)/H)$ orbits in $\mathbf{M}_{\mathfrak{h}}$ are unions of connected components. Each $\phi \in (N(H)/H)$, which is not in $N_0(H)/H$ acts as a diffeomorphism between connected components. Hence, connected components of $\check{\mathbf{P}}_{(\mathfrak{h})}$ are diffeomorphic to the quotients of connected components of $\mathbf{M}_{\mathfrak{h}}$ by $N_0(H)/H$.

The map $\rho_{\mathfrak{h}} : \mathbf{M}_{\mathfrak{h}} \longrightarrow \check{\mathbf{P}}_{(\mathfrak{h})}$ is onto. Furthermore, it follows from the Slice Theorem that, for each $\mathbf{p} \in \mathbf{M}_{\mathfrak{h}}$, there exists a slice $\mathbf{S}_{\mathbf{p}}$ through $\mathbf{p}$ for the action of $GS(\mathbf{P})$ in $\mathbf{P}$. By Proposition 3.5, $\mathbf{M}_{\mathfrak{h}} \cap \mathbf{S}_{\mathbf{p}}$ is a smooth manifold. It is a slice for the action of $N_0(H)/H$ in $\mathbf{M}_{\mathfrak{h}}$. Then, by Eq. (4.6), there exists an open neighbourhood $\mathbf{U}_{\mathbf{p}}$ of $\mathbf{p}$ in $\mathbf{M}_{\mathfrak{h}} \cap \mathbf{S}_{\mathbf{p}}$ such that $\rho_{\mathfrak{h}} | \mathbf{U}_{\mathbf{p}}$ is a homeomorphism of $\mathbf{U}_{\mathbf{p}}$ onto $\rho_{\mathfrak{h}}(\mathbf{U}_{\mathbf{p}}) \subset \check{\mathbf{P}}_{(\mathfrak{h})}$. The family of sets $\{\rho_{\mathfrak{h}}(\mathbf{U}_{\mathbf{p}}) | \mathbf{p} \in \mathbf{M}_{\mathfrak{h}}\}$ is an open cover of $\check{\mathbf{P}}_{(\mathfrak{h})}$. Suppose $\mathbf{p}_1$ and $\mathbf{p}_2$ be in the same connected component of $\mathbf{M}_{\mathfrak{h}}$ and $\check{\mathbf{p}}_0 \in \rho_{\mathfrak{h}}(\mathbf{U}_{\mathbf{p}_1}) \cap \rho_{\mathfrak{h}}(\mathbf{U}_{\mathbf{p}_2})$. For any $\mathbf{p}_0 \in \rho_{\mathfrak{h}}^{-1}(\check{\mathbf{p}}_0)$ in the same connected component as $\mathbf{p}_1$ and $\mathbf{p}_2$, let $\mathbf{U}_{\mathbf{p}_0}$ be the chosen neighbourhood of $\mathbf{p}_0$ in $\mathbf{M}_{\mathfrak{h}} \cap \mathbf{S}_{\mathbf{p}_0}$. There exist $\phi_1, \phi_2 \in N_0(H)$ such that $\phi_1\mathbf{p}_0 \in \mathbf{S}_{\mathbf{p}_1}$ and $\phi_2\mathbf{p}_0 \in \mathbf{S}_{\mathbf{p}_2}$. Moreover, there is a map Φ from a neighbourhood $\mathbf{V}$ of $\phi_1\mathbf{p}_0$ in $\mathbf{M}_{\mathfrak{h}} \cap \mathbf{S}_{\mathbf{p}_1}$ to $N_0(H)/H$ such that, for each $\mathbf{p} \in \mathbf{V}$, $\phi_2\Phi(\mathbf{p})\phi_1^{-1}\mathbf{p} \in \mathbf{S}_{\mathbf{p}_2}$, and $\Phi(\phi_1\mathbf{p}_0) = identity$. Since the action of $N_0(H)/H$ in $\mathbf{M}_{\mathfrak{h}}$ is locally free, for sufficiently small $\mathbf{V}$ the map $\Phi : \mathbf{V} \longrightarrow N_0(H)/H$ satisfying the above conditions is unique and smooth. Hence, the map

$$\mathbf{V} \longrightarrow \mathbf{M}_{\mathfrak{h}} \cap \mathbf{S}_{\mathbf{p}_2} : \mathbf{p} \longmapsto \phi_2\Phi(\mathbf{p})\phi_1^{-1}\mathbf{p}$$

is a diffeomorphism onto its image. This ensures that the family maps $\{(\rho_{\mathfrak{h}} | \mathbf{U}_{\mathbf{p}})^{-1} | \mathbf{p} \in \mathbf{M}_{\mathfrak{h}}\}$ defines a smooth atlases in connected components of $\check{\mathbf{P}}_{(\mathfrak{h})}$. In the differentiable structure defined in this way the map $\rho_{\mathfrak{h}} : \mathbf{M}_{\mathfrak{h}} \longrightarrow \check{\mathbf{P}}_{(\mathfrak{h})}$ is clearly a submersion.

It remains to show that the obtained differentiable structure in $\check{\mathbf{P}}_{(\mathfrak{h})}$ is independent of the choice of $\mathfrak{h}$ in the conjugacy class $(\mathfrak{h})$. If $\mathfrak{h}_1$ is another representative of $(\mathfrak{h})$, and H_1 is the group generated by it, then there exists $\phi \in GS(\mathbf{P})$ such that $\phi H \phi^{-1} = H_1$. Moreover, by Eq. (2.11), $\phi(\mathbf{P}_{\mathfrak{h}}) = \mathbf{P}_{\mathfrak{h}_1}$, and $\phi | \mathbf{P}_{\mathfrak{h}}$ is a diffeomorphism of $\mathbf{P}_{\mathfrak{h}}$ onto $\mathbf{P}_{\mathfrak{h}_1}$.

Q.E.D.

PROPOSITION 4.2.

For every $\mathfrak{h}$ in $(\mathfrak{h})$, the pull-back of the 1-form θ to $\mathbf{M}_{\mathfrak{h}}$, denoted $\theta_{\mathfrak{h}} | \mathbf{M}_{\mathfrak{h}}$, is N(H) invariant, and it vanishes on the vectors $\xi_{\mathbf{P}}(p)$, with $p \in \mathbf{M}_{\mathfrak{h}}$ and $\xi \in \mathfrak{n}(\mathfrak{h})$. It pushes forward to a unique 1-form $\check{\theta}_{(\mathfrak{h})}$ on $\check{\mathbf{P}}_{(\mathfrak{h})}$ such that

$$\overset{*}{\rho}_{\mathfrak{h}} \check{\theta}_{(\mathfrak{h})} = \theta_{\mathfrak{h}} | \mathbf{M}_{\mathfrak{h}}. \tag{4.7}$$

The exterior differential $\check{\omega}_{(\mathfrak{h})} = d\check{\theta}_{(\mathfrak{h})}$ is a weakly symplectic form on $\check{\mathbf{P}}_{(\mathfrak{h})}$, and

$$\overset{*}{\rho}_{\mathfrak{h}} \check{\omega}_{(\mathfrak{h})} = \mu_{\mathfrak{h}}. \tag{4.8}$$

The forms $\check{\theta}_{(\mathfrak{h})}$ and $\check{\omega}_{(\mathfrak{h})}$ are independent of the choice of $\mathfrak{h}$ in $(\mathfrak{h})$.

PROOF. $\langle \theta | \xi_{\mathbf{P}}(p) \rangle = J_{\xi}(p) = 0$ for all $p \in \mathbf{M}_{\mathfrak{h}} = \mathbf{P}_{\mathfrak{h}} \cap J^{-1}(0)$ and all $\xi \in gs(\mathbf{P})$. This implies that $\theta_{\mathfrak{h}} | \mathbf{M}_{\mathfrak{h}}$ vanishes on $\xi_{\mathbf{P}}(p)$ for all $p \in \mathbf{M}_{\mathfrak{h}}$ and all $\xi \in \mathfrak{n}(\mathfrak{h})$. Since θ is $GS(\mathbf{P})$ invariant, $\theta_{\mathfrak{h}} | \mathbf{M}_{\mathfrak{h}}$ is N(H) invariant. Hence, $\theta_{\mathfrak{h}} | \mathbf{M}_{\mathfrak{h}}$ pushes forward to a 1-form $\check{\theta}_{(\mathfrak{h})}$, satisfying (4.7), which does not depend on the choice of $\mathfrak{h}$ in $(\mathfrak{h})$.

It follows from (4.7) that

$$\overset{*}{\rho}_{\mathfrak{h}} \check{\omega}_{(\mathfrak{h})} = \overset{*}{\rho}_{\mathfrak{h}} d\check{\theta}_{(\mathfrak{h})} = d\overset{*}{\rho}_{\mathfrak{h}} \check{\theta}_{(\mathfrak{h})} = d(\theta_{\mathfrak{h}} | \mathbf{M}_{\mathfrak{h}}) = d\theta_{\mathfrak{h}} | \mathbf{M}_{\mathfrak{h}} = \omega_{\mathfrak{h}} | \mathbf{M}_{\mathfrak{h}} = \mu_{\mathfrak{h}},$$

which proves (4.8). The independence of $\check{\omega}_{(\mathfrak{h})}$ from the choice of $\mathfrak{h}$ in $(\mathfrak{h})$ follows from the independence of $\check{\theta}_{(\mathfrak{h})}$.

Q.E.D.

This completes the proof of Theorem 4.

5. Symmetry breaking.

Yang-Mills potentials represent connections in a right principal bundle Q over M with structure group G. Since M is contractible, the bundle Q is trivial,

$$Q = M \times G \tag{5.1}$$

and the action of G in Q is given by

$$Q \times G \rightarrow Q : ((x,g),h) \mapsto ((x,g)\cdot h) = (x,gh). \tag{5.2}$$

The associated bundle Q[G] of Q with typical fibre G and the adjoint action of G on itself is called the group bundle of Q. Sections of Q[G] correspond to automorphisms of Q covering the identity transformation in M. In this context, the group *GS*(**P**) of gauge symmetries of **P** can be identifies with the group of sections of Q[G], of class $H^3(M)$, which satisfy the boundary condition (2.4).

Sections of associated bundles correspond to equivariant maps from the principal bundle to the typical fibre. Thus, each element $\phi \in GS(\mathbf{P})$ corresponds to a map $\phi^{\#} : Q \rightarrow G$ such that, for every $(x,g) \in Q$,

$$\phi^{\#}((x,g)) = g^{-1}\phi(x)g \ . \tag{5.3}$$

The adjoint bundle of Q is the associated bundle Q[$\mathfrak{g}$] with typical fibre $\mathfrak{g}$ and the adjoint action of G on $\mathfrak{g}$. The space of sections of Q[$\mathfrak{g}$] is the Lie algebra of the group of sections of the group bundle Q[G]. The Lie algebra *gs*(**P**) consists of sections of the adjoint bundle, which are of Sobolev class $H^3(M)$ and satisfy the boundary condition (2.1). Each $\xi : M \rightarrow \mathfrak{g}$ in *gs*(**P**) corresponds to an equivariant map $\xi^{\#} : P \rightarrow \mathfrak{g}$ such that

$$\xi^{\#}(x,e) = \xi(x) \ . \tag{5.4}$$

Let $\mathfrak{h} \subset gs(\mathbf{P})$ be the symmetry algebra of $\mathbf{p} = (A,E,\Psi) \in \mathbf{P}$, and H the connected subgroup of *GS*(**P**) with Lie algebra $\mathfrak{h}$. We have shown in Sec. 2 that H is compact. Let x_0 be a fixed point in M, then

$$H_0 = \{h(x_0) \,|\, h \in H\} \tag{5.5}$$

is a closed subgroup of G isomorphic to H. Consider the set

$$Q_0 = \{(x,g) \in P \,|\, h^{\#}(x,g) = h(x_0)\ \forall\ h \in H\}. \tag{5.6}$$

The right action of $k \in G$ leaves Q_0 invariant if and only if k is in the centralizer $Z[H_0]$ of H_0, defined by

$$Z[H_0] = \{k \in G \,|\, kg = gk\ \forall\ g \in H_0\}. \tag{5.7}$$

Actually, Q_0 is a right principal bundle over M with structure group $Z[H_0]$. Furthermore, for each $g \in G$, the set

$$H_{(x_0,g)} = \{h^{\#}(x_0,g) \,|\, h \in H\} = \{g^{-1}h(x_0)g \,|\, h \in H\}$$

is a group conjugate to H_0 with the centralizer conjugate to $Z[H_0]$. Hence, for a fixed $H \subset GS(P)$, the principal bundle Q is foliated by principal sub-bundles with conjugate structure groups.

The Yang-Mills potential **A** gives a local description of a connection in Q relative to the trivialization given by the product structure (5.1). The corresponding connection form α on Q is given by

$$\alpha = \gamma^{-1}A\gamma + \gamma^{-1}d\gamma\ , \tag{5.8}$$

where γ is the embedding of G into the matrix group $gl(\mathbb{R}^k)$. The horizontal distribution *hor*TQ on Q is the kernel of the connection form. The connection in Q is said to reduce to a connection in a sub-bundle Q_0 if $horTQ|Q_0 \subset horTQ_0$.

Proposition 5.1.

Let $\mathfrak{h} \subset gs(\mathbf{P})$ be the stability algebra of $\mathbf{p} = (A,E,\Psi) \in \mathbf{P}$, and H the connected subgroup of $GS(\mathbf{P})$ with Lie algebra $\mathfrak{h}$. Then, connection in Q defined by **A** reduces to a connection in the sub-bundle Q_0 with structure group $Z[H_0]$.

Proof. Let q(t) be a horizontal curve in Q. For every $h \in H$, $\frac{d}{dt} h^{\#}(q(t))$ is the equivariant function on Q describing the covariant derivative of the section h of Q[G] along q(t). However, $\mathring{H} \subseteq GS_{\mathbf{p}}$ implies that every $h \in H$ is covariantly constant.

Hence, every horizontal curve through Q_0 is contained in Q_0, which means that the connection in Q defined by **A** reduces to a connection in Q_0.

Q.E.D.

The electric component **E** of the field strength is a $\mathfrak{g}$-valued 1-form on M. It can be interpreted as a section of the bundle $T^*M \otimes Q[\mathfrak{g}]$ over M. Let $\mathbf{E}^{\#}$ be the corresponding equivariant form on Q with values in $\mathfrak{g}$. If $\mathfrak{h}$ is the symmetry algebra of **E**, then (1.6) yields

$$[\mathbf{E},\xi] = 0 \quad \forall \quad \xi \in \mathfrak{h}. \tag{5.9}$$

This implies that

$$[\mathbf{E}^{\#},\xi^{\#}] = 0 \quad \forall \quad \xi \in \mathfrak{h}. \tag{5.10}$$

At points of Q_0, given by (5.6), we have $\xi^{\#}|Q_0 = \xi(x_0)$. Hence, for each point $(x,g) \in Q_0$,

$$[\mathbf{E}^{\#}(x,g),\xi(x_0)] = 0 \quad \forall \quad \xi \in \mathfrak{h}, \tag{5.11}$$

which implies that, for all $(x,g) \in Q_0$, $\mathbf{E}^{\#}(x,g)$ is in the Lie algebra $\mathfrak{z}[H_0]$ of the centralizer $Z[H_0]$ of H_0. Hence, the electric component **E** of the field strength reduces to a section $\mathbf{E}_0$ of the bundle $T^*Q_0 \otimes Q[\mathfrak{z}[H_0]]$ over M.

The matter field Ψ is a section of the associated bundle of Q, with typical fibre $\mathbb{R}^n \otimes \mathbb{R}^4$, where $\mathbb{R}^n$ is the space of the fundamental representation of (the matrix group) G, and the factor $\mathbb{R}^4$ describes the spin degrees of freedom. It follows from (1.6) that $(\mathbf{A},\mathbf{E},\Psi) \in \mathbf{P}_{\mathfrak{h}}$ implies that

$$\xi\Psi = 0 \quad \forall \quad \xi \in \mathfrak{h}. \tag{5.12}$$

Let $\Psi^{\#}$ be the equivariant function from Q to $\mathbb{R}^n \otimes \mathbb{R}^4$ corresponding to Ψ. Then, (5.12) yields

$$\xi^{\#}\Psi^{\#} = 0 \quad \forall \quad \xi \in \mathfrak{h}. \tag{5.13}$$

For each point $(x,g) \in Q_0$,

$$\xi(x_0)\Psi^{\#}(x,g) = 0 \quad \forall \quad \xi \in \mathfrak{h}, \tag{5.14}$$

which implies that $\Psi^{\#}(x,g)$ is in the subspace

$$V_0 = \{z \in \mathbb{R}^n \otimes \mathbb{R}^4 \,|\, \xi(x_0)z = 0 \;\forall\; \xi \in \mathfrak{h}\} \tag{5.15}$$

of $\mathbb{R}^n \otimes \mathbb{R}^4$ annihilated by Lie algebra $\mathfrak{h}_0$ of H_0. The action of the centralizer $Z[H_0]$ preserves V_0. Hence, the matter field Ψ reduces to a section Ψ_0 of the associated bundle of Q_0, with typical fibre $V_0 \subseteq \mathbb{R}^n \otimes \mathbb{R}^4$.

Thus, we have proved

THEOREM 5.

For every $(\mathbf{A},\mathbf{E},\Psi) \in \mathbf{P}_{\mathfrak{h}}$, the Cauchy data $(\mathbf{A},\mathbf{E})$ for the Yang-Mills theory with the structure (internal symmetry) group G reduce to Cauchy data for a Yang-Mills theory with principal bundle

$$Q_0 = \{(x,g) \in P \,|\, h^{\#}(x,g) = h(x_0) \;\forall\; h \in H\}$$

and structure (internal symmetry) group

$$Z[H_0] = \{g \in G \,|\, gh(x_0) = h(x_0)g \;\forall\; h \in H\},$$

where x_0 is and arbitrary fixed point of M, and the matter field Ψ reduces to a section of the associated bundle of Q_0 with typical fibre

$$V_0 = \{z \in \mathbb{R}^n \otimes \mathbb{R}^4 \,|\, \xi(x_0)z = 0 \;\forall\; \xi \in \mathfrak{h}\}.$$

The change of the base point x_0 correspond to passing from Q_0 to another principal sub-bundle of Q with conjugate structure group.

It follows from Theorem 5 that each symplectic manifold $\check{\mathbf{P}}_{(\mathfrak{h})}$ in $\check{\mathbf{P}}$ corresponds to symmetry breaking from the original internal symmetry group G to conjugacy classes of subgroups $Z[H_0]$ centralizing H_0. It should be noted that the symmetry breaking encountered here is completely intrinsic, it does not require additional Higgs fields. On the other hand, it does not lead to vector bosons.

Appendix A. Completion and almost complex structure.

One of the technical assumptions in [3] is the existence of an appropriate almost complex structure, which in Yang-Mills theory acts by interchanging A and E. However, in our phase space **P** the variables A and E appear asymmetrically, and we do not have existence and uniqueness theorems in spaces symmetric under the interchange of A and E.

Let $\tilde{\mathbf{P}}$ denote the completion of **P** in the L^2 norm. The weak symplectic form ω in **P** induces a strong symplectic form $\tilde{\omega}$ in $\tilde{\mathbf{P}}$. The L^2 scalar product $\langle . | . \rangle_{L^2}$ defines a Riemannian metric in $\tilde{\mathbf{P}}$. Let $\mathcal{J} : T\tilde{\mathbf{P}} \longrightarrow T\tilde{\mathbf{P}}$ be defined by

$$\mathcal{J}(\delta A, \delta E, \delta\Psi) = (-\delta E, \delta A, i\Psi) \tag{A.1}$$

for every $(\delta A, \delta E, \delta\Psi) \in T\tilde{\mathbf{P}}$. Then, $\mathcal{J}^2 = -1$, and

$$\tilde{\omega}(\mathcal{J}u, \mathcal{J}v) = \tilde{\omega}(u,v) = \langle \mathcal{J}u | v \rangle_{L^2} = -\langle u | \mathcal{J}v \rangle_{L^2} \tag{A.2}$$

for all $u,v \in T\tilde{\mathbf{P}}$. Thus, $\mathcal{J}$ is an almost complex structure on $\tilde{\mathbf{P}}$. The action of $GS(\mathbf{P})$ in **P** extends to an action in $\tilde{\mathbf{P}}$ preserving its symplectic form, the Riemannian metric and the almost complex structure.

Let **V** be a closed subspace of $T_p\mathbf{P}$ and let $\tilde{\mathbf{V}}$ be its closure in $T_p\tilde{\mathbf{P}}$. The symplectic annihilator $\mathbf{V}^{\omega}$ of **V** is defined by

$$\mathbf{V}^{\omega} = \{\mathbf{u} \in T_p\mathbf{P} \,|\, \omega(\mathbf{u},\mathbf{v}) = 0 \ \forall \ \mathbf{v} \in \mathbf{V}\} . \tag{A.3}$$

Similarly, the symplectic annihilator of $\tilde{\mathbf{V}}$ in $T_p\tilde{\mathbf{P}}$ is

$$\tilde{\mathbf{V}}^{\tilde{\omega}} = \{\mathbf{u} \in T_p\tilde{\mathbf{P}} \,|\, \tilde{\omega}(\mathbf{u},\mathbf{v}) = 0 \ \forall \ \mathbf{v} \in \tilde{\mathbf{V}}\} \tag{A.4}$$

Since **V** is closed, we have

$$(\mathbf{V}^{\omega})^{\omega} = \mathbf{V} . \tag{A.5}$$

We denote by $\mathbf{V}^{\perp}$ the L^2-orthogonal complement of **V** in $T_p\mathbf{P}$, and $\tilde{\mathbf{V}}^{\perp}$ the L^2 orthogonal complement of its closure $\tilde{\mathbf{V}}$ in $T_p\tilde{\mathbf{P}}$. We have

$$(\mathbf{V}^{\perp})^{\omega} = (\tilde{\mathbf{V}}^{\perp})^{\tilde{\omega}} \cap T_p\mathbf{P} \tag{A.6}$$

Moreover, by Eq. (A.2)

$$(\tilde{V}^{\perp})^{\tilde{\omega}} = \{u \in T_p\tilde{P} \,|\, \tilde{\omega}(u,v) = 0 \;\forall\; v \in \tilde{V}^{\perp}\} = \{u \in T_p\tilde{P} \,|\, \mathcal{J}u \in (\tilde{V}^{\perp})^{\perp}\} = \mathcal{J}\tilde{V}.$$

Hence,

$$(V^{\perp})^{\omega} = \mathcal{J}\tilde{V} \cap T_pP. \qquad (A.7)$$

In the following we shall use the notation

$$\mathcal{J}V = \mathcal{J}\tilde{V} \cap T_pP. \qquad (A.8)$$

Since $\mathcal{J}^2 = -I$, we have

$$\mathcal{J}^2V = V, \text{ and } (\mathcal{J}V) \cap V = \{0\} . \qquad (A.9)$$

For each $p \in P$, the second derivative of the momentum map $J : P \longrightarrow gs(P)^*$ is a symmetric bilinear map $D^2J_p : T_pP \times T_pP \longrightarrow gs(P)^*$. It extends to a symmetric bilinear map $D^2\tilde{J}_p : T_p\tilde{P} \times T_p\tilde{P} \longrightarrow gs(P)^*$.

Lemma A.1.

For all $v,w \in T_pP$, and all $\xi \in gs(P)$,

$$\langle D^2\tilde{J}_p(\mathcal{J}v,\mathcal{J}w) \,|\, \xi\rangle = \langle D^2J_p(v,w) \,|\, \xi\rangle. \qquad (A.10)$$

Proof. For $\xi \in gs(P)$, and a constant vector field X in P, with $X(p) = w$, the equation $\tilde{\omega}(\xi_{\tilde{P}},X) = \langle dJ(X) \,|\, \xi\rangle$, differentiated at p in the direction $v \in T_pP$ yields

$$\tilde{\omega}(D\xi_{\tilde{P}}(p)v,w) = \langle D^2J_p(v,w) \,|\, \xi\rangle .$$

Since the action of $GS(P)$ in $\tilde{P}$ preserves the almost complex structure $\mathcal{J}$, for every $\xi \in gs(P)$, we have $\mathcal{J} \circ T\xi_{\tilde{P}} = T\xi_{\tilde{P}} \circ \mathcal{J}$. Hence,

$$\langle D^2J_p(\mathcal{J}v,\mathcal{J}w) \,|\, \xi\rangle = \tilde{\omega}(D\xi_{\tilde{P}}(p)\mathcal{J}v,\mathcal{J}w) = \tilde{\omega}(d\xi_{\tilde{P}}(p)v,w) = \langle D^2J_p(v,w) \,|\, \xi\rangle.$$

Q.E.D.

Appendix B. Properness of the action of the gauge symmetry group.

The gauge symmetry group $GS(P)$ consists of map $\phi : M \longrightarrow G$ in the Sobolev class $H^3(M)$ such that $n \cdot grad\phi = 0$, (2.4). Its action in P is given by (1.5). In order to prove that this action is proper, we need to show that, for every sequence $p_n = (A_n, E_n, \Psi_n)$ converging to $p_\infty = (A_\infty, E_\infty, \Psi_\infty) \in P$, and every sequence ϕ_n in $GS(P)$ such that $\phi_n p_n$ converges to $p = (A,E,\Psi)$, the sequence ϕ_n has a convergent subsequence with limit ϕ and $\phi p_\infty = p$.

The gauge transformations act on A, E, and Ψ independently. Hence, we may consider first the action of $GS(P)$ on the connections. For a sequence A_n converging to A_∞, and a sequence ϕ_n in $GS(P)$, let

$$C_n = \phi_n A_n \phi_n^{-1} + \phi_n d\phi_n^{-1} \tag{B.1}$$

denote A_n transformed by ϕ_n. This implies

$$d\phi_n = \phi_n A_n - C_n \phi_n. \tag{B.2}$$

By hypothesis, the sequences A_n and C_n converge in $H^2(M)$ to A_∞ and A, respectively. In particular, their $H^2(M)$ norms $\|A_n\|_{H^2}$ and $\|C_n\|_{H^2}$ are bounded. Furthermore, the $L^2(M)$ norms $\|\phi_n\|_{L^2}$ of ϕ_n are bounded since M and G are compact. Eq. (B.2) implies that also the $L^2(M)$ norms $\|d\phi_n\|_{L^2}$ of $d\phi_n$ are bounded. Hence, the $H^1(M)$ norms $\|\phi_n\|_{H^1}$ of ϕ_n are bounded. Repeating this argument twice, we conclude that the $H^3(M)$ norms of ϕ_n are bounded. By Rellich's Lemma the sequence ϕ_n has a subsequence convergent to ϕ in $H^2(M)$. Without loss of generality, we can restrict our argument to this subsequence, and assume that ϕ_n converges to ϕ in $H^2(M)$. Hence, the sequence $C_n = \phi_n A_n \phi_n^{-1} + \phi_n d\phi_n^{-1}$ converges to $\phi A_\infty \phi^{-1} + \phi d\phi^{-1}$ in $H^1(M)$,

$$\|\phi A_\infty \phi^{-1} + \phi d\phi^{-1} - C_n\|_{H^1} \to 0 \text{ as } n \to \infty. \tag{B.3}$$

By hypothesis, C_n converges to A in $H^2(M)$. Hence,

$$\|\phi A_\infty \phi^{-1} + \phi d\phi^{-1} - A\|_{H^1} \leq \|\phi A_\infty \phi^{-1} + \phi d\phi^{-1} - C_n\|_{H^1} + \|C_n - A\|_{H^1} \to 0 \text{ as } n \to \infty.$$

This implies that

$$A = \phi A_\infty \phi^{-1} + \phi d\phi^{-1} , \tag{B.4}$$

and hence,

$$d\phi = \phi A_\infty - A\phi . \tag{B.5}$$

Since the right hand side of (B.5) belongs to $H^2(M)$, it follows that $d\phi \in H^2(M)$, so that $\phi \in H^3(M)$.

Using (B.2) and (B.5), we observe that

$$\|d\phi_n - d\phi\|_{H2} = \|\phi_n A_n - C_n\phi_n - (\phi A_\infty - A\phi)\|_{H2} \leq$$
$$\|\phi_n A_n - \phi A_\infty\|_{H2} + \|C_n\phi_n - A\phi)\|_{H2}.$$

As $n \to \infty$ the right hand side tends to zero, because $\phi_n \to \phi$, $A_n \to A_\infty$, and $C_n \to A$ in $H^2(M)$. Hence, $\|d\phi_n - d\phi\|_{H2} \to 0$, which implies that $\phi_n \to \phi$ in $H^3(M)$. This proves the properness of the action of $GS(P)$ on the space of $H^2(M)$ connections satisfying the boundary conditions (1.1).

In remains to show that ϕ takes E_∞ to E and Ψ_∞ to Ψ. By hypothesis $E_n \to E_\infty$ and $\phi_n E_n \phi_n^{-1} \to E$ in $H^1(M)$. Since $\phi_n \to \phi$ in $H^3(M)$, and a pointwise multiplication of functions in $H^1(M)$ by functions in $H^3(M)$ is a continuous map from $H^1(M) \times H^3(M)$ to $H^1(M)$, we obtain

$$E = \lim_{H^1(M)} (\phi_n E_n \phi_n^{-1}) = (\lim_{H^3(M)} \phi_n)(\lim_{H^1(M)} E_n)(\lim_{H^3(M)} \phi_n^{-1}) = \phi E_\infty \phi^{-1}.$$

In a similar manner we obtain

$$\Psi = \lim_{H^2(M)} (\phi_n \Psi_n) = \lim_{H^3(M)} (\phi_n) \lim_{H^2(M)} (\Psi_n) = \phi\Psi_\infty.$$

This completes the proof of properness of the action of $GS(P)$ in P.

Appendix C. Proof of the slice theorem.

We establish here the slice theorem for infinite dimensional groups, c.f. [14]. Since the assumptions made here are more general than in the body of the paper, we use an independent notation following that of Appendix 2 of [15].

Let M be a Hilbert manifold, and G a Hilbert Lie group, with a continuous proper smooth left action $\Phi : G \times M \longrightarrow M$. In the following we use the notation $\Phi_g(m) = \Phi(g,m)$. Let $\mathfrak{g}$ be the Lie algebra of G. For each $m \in M$, we denote by G_m the isotropy group of m, by $\mathfrak{g}_m$ the Lie algebra of G_m, and by $\mathcal{O}_m = G\cdot m$ the orbit of G through m. Since the action is proper G_m is compact and the orbit $\mathcal{O}_m$ is closed. The tangent space $T_m\mathcal{O}_m$ can be presented as $\mathfrak{g}\cdot m = T\Phi(\mathfrak{g},0)(e,m)$, and $\mathfrak{g}_m\cdot m = 0$.

HYPOTHESES:

(a) The group G is a Lie group in the sense that the exponential map gives a diffeomorphism of a neighbourhood of $0 \in \mathfrak{g}$ onto a neighbourhood of $e \in G$.

(b) The action Φ is proper.

(c) Bochner Linearization Lemma, [16]. There is a G_m invariant neighbourhood U of $m \in M$ and a diffeomorphism $\psi : U \to T_mM$ such that:

$$\psi(m) = 0 \quad \text{and} \quad T_m\psi = \textit{identity} \tag{C.1}$$

and, for every $g \in G_m$ and $p \in U$

$$\psi(\Phi_g(p)) = T_m\Phi_g(\psi(p)) \ . \tag{C.2}$$

These assumptions are stronger than needed to get slices, but they allow us to control the topology of the space of orbits of the group action. They are satisfied by the gauge symmetry group $GS(\mathbf{P})$ considered in this paper. Proposition 2.1 guarantees assumption (a). Properness of the action of $GS(\mathbf{P})$ is proved in Appendix B. The Bochner Linearization Lemma follows from the fact that the action of $GS(\mathbf{P})$ is affine.

First we need a lemma.

LEMMA C.1.

Given $m \in M$, let L be a submanifold of G through e such that

$$\mathfrak{g} = \mathfrak{g}_m \oplus T_eL, \tag{C.3}$$

and let S be a submanifold of M through m such that

$$T_mM = T_m\mathcal{O}_m \oplus T_mS. \tag{C.4}$$

Then there is an open set $U \times V \subseteq L \times S$ such that $\Phi|(U \times V)$ is a diffeomorphism onto an open neighbourhood W of $m \in M$.

PROOF. Let $D\Phi : TG \times TM \longrightarrow TM$ denote the derivative of Φ, and $D_i\Phi$ be the restriction of $D\Phi$ to the i'th factor. Since $\Phi(e,m) = m$ for all $m \in M$, we have that $D_2\Phi_{(e,m)} = identity$, and so $D\Phi_{(e,m)}$ is surjective. Now *ker* $D_1\Phi_{(e,m)} = \mathfrak{g}_m$ by definition, and also, by definition *image* $D_1\Phi_{(e,m)} = T_m O_m$.

Choosing L and S so that we can make the identifications

$$T_e L \cong \mathfrak{g}/\mathfrak{g}_m \tag{C.5}$$

$$T_m S \cong T_m M / T_m O_m \tag{C.6}$$

we have that $D\Phi|(T_e L \times T_m S)$ is an isomorphism. Since M is a Hilbert manifold the Lemma now follows by the inverse function theorem.

Q.E.D.

COROLLARY C.2.

If $\Phi_g V \cap V \neq \emptyset$ for some $g \in U \subseteq L \subset G$, and $V \subseteq S$, then $g = e$.

PROOF. Let $m \in V$ be such that $\Phi(g,m) = \Phi(e,m')$ with $m' \in V$. Since Φ is a local diffeomorphism on $U \times V$ it follows that $(g,m) = (e,m')$, so that $g = e$.

Q.E.D.

LEMMA C.3.

For every neighbourhood $\tilde{U}$ on M containing m, there is a G_m invariant open set U containing m with $U \subseteq \tilde{U}$.

PROOF. Since M is a Hilbert manifold, it is first countable. Hence, there exists a sequence $\{U_n\}$ of neighbourhoods of m in M such that $U_n \subseteq U_{n-1}$, $\bigcap_{n=1}^{\infty} U_n = \{m\}$, and

$G_m \cdot U_n$ is not contained in $\tilde{U}$. Suppose now that the statement of the lemma is false. Then $G_m \cdot U_n$ is not contained in $\tilde{U}$ for all n. Hence, there exist sequences $m_n \in U_n$ and $g_n \in G_m$ such that $g_n m_n \notin \tilde{U}$. Since the action of G is proper, the isotropy group G_m is compact and the sequence g_n has a convergent subsequence. Without loss of generality we may assume that g_n converges to $g \in G_m$. The sequence m_n converges to m by construction. The continuity of the action of G in M implies that $g_n m_n$ converges to $g \cdot m = m$, which contradicts the statement that $g_n m_n \notin \tilde{U}$ for all n.

Q.E.D.

Slice Theorem.

For each $m \in M$, there exists a smooth submanifold S of M through m such that

(1) $$T_m M = T_m \mathcal{O}_m \oplus T_m S. \tag{C.7}$$

(2) $$T_p M = T_p \mathcal{O}_p + T_p S \quad \forall \quad p \in S. \tag{C.8}$$

(3) $$G_m \cdot S \subseteq S . \tag{C.9}$$

(4) For $p \in S$, and $g \in G$, if $\Phi_g(p) \in S$ then $g \in G_m$. (C.10)

Proof. We prove the existence of a slice by constructing a candidate S_ε and showing that properties (1) through (4) hold.

Observe that if $k \in G_m$, $kg \cdot m = kgk^{-1} \cdot m$, or

$$\Phi_k \circ \Phi_g(m) = \Phi_{kgk^{-1}}(m). \tag{C.11}$$

If $g = exp(t\xi)$, $\xi \in \mathfrak{g}$, then the 1-parameter groups $t \mapsto k[exp(t\xi)]k^{-1}$ and $t \mapsto exp(tAd_k\xi)$ have the same tangent vector $Ad_k\xi$ at $t = 0$. Hence, differentiating (C.11) with respect to t at t = 0 we get

$$T_m\Phi_k T_e\Phi_m \cdot \xi = T_e\Phi_m(Ad_k\xi) \tag{C.12}$$

which tells us that $T_m\Phi_k$ leaves $T_m\mathcal{O}_m$ invariant.

Since G_m is compact, there is a G_m invariant inner product on T_mM. So $(T_m\mathcal{O}_m)^{\perp}$ is a G_m invariant subspace. Using the local linearizing diffeomorphism ψ (from the Bochner Lemma) the submanifold

$$S_{\varepsilon} = \psi^{-1}((T_m\mathcal{O}_m)^{\perp} \cap B_{\varepsilon}), \tag{C.13}$$

where B_{ε} is a ball of radius ε in T_mM (with respect to the G_m invariant inner product) is G_m invariant. So S_{ε} has property (3). Moreover, $T_mS_{\varepsilon} = (T_m\mathcal{O}_m)^{\perp}$, since $T_m\psi$ = *identity*. Hence, property (1) holds as well.

We argue that Property (2) is an open condition in S_{ε} as follows. Observe that $\Phi|(G \times S_{\varepsilon}) : G \times S_{\varepsilon} \to M$ is a submersion at (e,m). Hence it is a submersion at (e,p), for all p in a neighbourhood of m in S_{ε}.

Now it remains to show that we can find $\varepsilon > 0$ so that (4) holds. Suppose that it does not hold for any $\varepsilon > 0$. This would imply that there is a sequence of points $\{m_n\}$ with $m_n \in S_{1/n}$, and a sequence $g_n \in G$, such that $g_n \notin G_m$, and $g_nm_n \in S_{1/n}$. Hence, $m_n \to m$ and $g_nm_n \to m$. Since the action of G in M is proper, it follows that there exists convergent subsequence of g_n. Without loss of generality, we may assume that $g_n \to g$. Moreover, $g_nm_n \to gm = m$, which implies that $g \in G_m$. Hence, $g^{-1}g_n \to e$, $g \in G_m$ and $g_n \notin G_m$.

G_m acts in G be multiplication on the left, and the orbit of this action through the identity in G coincides with G_m. Applying Lemma C.1 to the action of G_m in G, we conclude that there is a submanifold L of G transverse to G_m at e, and an open set $U \times V \subseteq G_m \times L$ such that the multiplication $(k,l) \mapsto kl$ is a diffeomorphism onto some open neighbourhood W of e in G. Thus, we may assume that $g^{-1}g_n = k_nl_n$, with $k_n \in G_m$ and $l_n \in L$. Since, g and k_n are in G_m and $g_n \notin G_m$, it follows that $l_n = k_n^{-1}g^{-1}g_n \notin G_m$ for all n.

We now apply Lemma C.1 to $U \times V \subseteq L \times S_{\varepsilon}$. For sufficiently large n, $g_nm_n = gk_nl_nm_n$ is in $V \subseteq S_{\varepsilon}$. It follows from Corollary C.2 that $gk_nl_n = e$ for n large enough. Hence, $l_n = k_n^{-1}g^{-1} \in G_m$, which contradicts the result above. This contradiction establishes (4).

Q.E.D.

We should remark that, for the case under consideration in this paper, that is for G = *GS*(**P**), there is a natural *GS*(**P**) invariant weak inner product on on the manifold M = **P** given by the L^2 scalar product. In this case, we can take $(T_m O_m)^\perp$ to be the L^2 orthogonal complement of $T_m O_m$. As long as the ball B_ε is defined with respect to the strong G_m invariant inner product on M, the manifold S_ε defined by (C.13) will satisfy properties (1) through (4). Hence, for the gauge symmetry group *GS*(**P**) one can always choose a slice S through m satisfying the condition (3.8), requiring that $T_m S$ is the L^2 orthogonal complement of $T_m O_m$.

References.

[1] G. Schwarz and J. Śniatycki, "Yang-Mills and Dirac fields in a bag, existence and uniqueness theorems", *Comm. Math. Phys.* (to appear)

[2] J. Arms, "The structure of the solution set for the Yang-Mills equations", *Math. Proc. Camb. Phil. Soc.*,**90** (1981), 361-372.

[3] J. Arms, J.E. Marsden and V. Moncrief, Symmetry and bifurcation of momentum maps", *Comm. Math. Phys.*, **78** (1981), 455-478.

[4] P. Mitter and C. Viallet, "On the bundle of connections and the gauge orbit manifold in Yang-Mills theory", *Comm. Math. Phys.*, 79 (1981), 457 - 472.

[5] M. Atiyah and R. Boot, "The Yang-Mills equations over Riemann surfaces", *Phil. Trans. R. Soc. London*, A **308** (1982) 523-615,

[6] W. Kondracki and J. Rogulski, "On the stratification of the orbit space for the action of automorphisms on connections", *Dissertationes Mathematicae*, CCL, Warsaw, 1986.

[7] J. Huebschmann, "The singularities of Yang-Mills connections over a surface. I. The local model", preprint, February 1992.

[8] J. Huebschmann, "The singularities of Yang-Mills connections over a surface. II. The stratification", preprint, February 1992.

[9] R. Sjamaar, "Singular orbit spaces in Riemannian and Symplectic geometry", Thesis, University of Utrecht, 1990.

[10] R. Sjamaar and E. Lerman, "Stratified symplectic spaces and reduction", *Ann. of Math.* 134 (1991), 375-422.

[11] C.B. Morrey, *Multiple Integrals in the Calculus of Variations*, Springer Verlag, Berlin, 1966.

[12] T. Kato, *Perturbation Theory for Linear Operators*, Springer Verlag, Berlin, 1966.

[13] A. Pietsch, *Nukleare Lokalkonvexe Raüme*, Akademie-Verlag, Berlin, 1965.

[14] R. Palais, "On the existence of slices for actions of noncompact Lie groups, *Ann. Math.*, 73 (1961), 295-323.

[15] R. Cushman and L. Bates, *Global Aspects of Classical Integrable Systems*, in preparation.

[16] S. Bochner, "Compact groups of differentiable transformations", *Ann. of Math.* 46 (1945) 372-381.