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The generalized Lichnerowicz formula and analysis of Dirac operators

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Abstract. We study Dirac operators acting on sections of a Clifford module \mathcal{E} over a Riemannian manifold M . We prove the intrinsic decomposition formula for their square, which is the generalisation of the well-known formula due to Lichnerowicz [L]. This formula enables us to distinguish Dirac operators of simple type. For each Dirac operator of this natural class the local Atiyah-Singer index theorem holds. Furthermore, if M is compact and $\dim M = 2n \geq 4$, we derive an expression for the Wodzicki function $W_{\mathcal{E}}$, which is defined via the non-commutative residue on the space of all Dirac operators $\mathcal{D}(\mathcal{E})$. We calculate this function for certain Dirac operators explicitly. From a physical point of view this provides a method to derive gravity, resp. combined gravity/Yang-Mills actions from the Dirac operators in question.

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1. Introduction

In 1928, when developing the quantum theory of the electron, Dirac introduced his famous first-order operator - the square-root of the so-called wave-operator (d'Alembertian operator). Generalisations of this operator, called 'Dirac operators', have come to play a fundamental role in the mathematics of our century, particularly in the interrelations between topology, geometry and analysis. To mention only a few applications, Dirac operators are the most important tools in proving the general Atiyah-Singer index theorem for pseudo-differential operators both in a topological [AS] and more analytical way [G]. Dirac operators also assume a significant place in Connes' non-commutative geometry [C] as the main ingredient of a K-cycle, where they encode the geometric structure of the underlying non-commutative 'quantum-spaces'. In mathematical physics, in contrast, Dirac operators had almost fallen into oblivion, except in modifications of Dirac's original application and Witten's proof of the positive mass conjecture in general relativity [W] where he used the Dirac operator on the spinor bundle S of a four-dimensional Lorentzian manifold. More recently, however, Connes and Lott derived the action of the standard model of elementary particles using a special K-cycle [CL]. Moreover, the Einstein-Hilbert gravity action can be reproduced via the non-commutative residue from a Dirac operator on a Clifford module \mathcal{E} over a four-dimensional Riemannian manifold as it was explicitly shown in [K] and [KW].

Nowadays a Dirac operator on a Clifford module $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ is understood as an odd-parity first order operator $D: \Gamma(\mathcal{E}^\pm) \rightarrow \Gamma(\mathcal{E}^\mp)$, whose square D^2 is a generalized laplacian. In view of this rather general definition, all Dirac operators of the examples mentioned above are in a sense structurally the most simple ones as they correspond to Clifford connections on the respective Clifford module \mathcal{E} . Hence, it may not be surprising that other types of Dirac operators have more or less failed to be studied in the literature.

Here our subject is a more thorough treatment of Dirac operators acting on sections of a Clifford module \mathcal{E} under two aspects: We prove the intrinsic decomposition formula (3.14) for their square, which is the generalisation of the well-known formula due to Lichnerowicz [L], and apply this formula to calculate explicitly the

the Wodzicki function $W_{\mathcal{E}}$ on the space of all Dirac operators $\mathcal{D}(\mathcal{E})$ in theorem 6.4. This complex-valued function $W_{\mathcal{E}}: \mathcal{D}(\mathcal{E}) \rightarrow \mathbb{C}$ is defined via the non-commutative residue, which has been extensively studied by Guillemin and Wodzicki (cf. [Gu], [W1], [W2]) and represents a link between Dirac operators and (gravity-) action functionals as already mentioned.

We now give a summary of the different sections: After fixing our notation we recall Quillen's statement (cf. [BGV]), that 'Dirac operators are a quantisation of the theory of connections' in the first section. More precisely, we show in lemma 2.1, that given any Dirac operator D , there exists a connection ∇ on the Clifford module \mathcal{E} such that $D = c\circ\nabla$. This is the essential property to prove the generalized Lichnerowicz formula (3.14) in section 3. Since there is a one-to-one correspondence between Clifford superconnections and Dirac operators (cf. [BGV]), we present in section 4 formula (3.14) in this context. In the following section, we classify Dirac operators 'of simple type' with respect to the decomposition of their square. For this kind of Dirac operators, the local Atiyah-Singer index theorem as proven by Getzler (cf. [G]) for Dirac operators corresponding to Clifford connections, also holds (theorem 5.6). In section 6 we turn our attention to the Wodzicki function $W_{\mathcal{E}}$ defined on the space of all Dirac operators $\mathcal{D}(\mathcal{E})$. By the help of our decomposition formula (3.14) we obtain an explicit expression for the Wodzicki function $W_{\mathcal{E}}(\tilde{D})$ of an arbitrary Dirac operator \tilde{D} in theorem 6.4. In the last section, we apply this result to calculate the Wodzicki function $W_{\mathcal{E}}$ for various Dirac operators. These examples are also inspired by physics. Note that by this method we are able to derive the combined Einstein-Hilbert/Yang-Mills action out of one (special) Dirac operator. From a physical point of view, this can be understood as unification of Einstein's gravity and Yang-Mills gauge theories (cf. [AT2]).

For the reader who is less familiar with the notions of Clifford module, Clifford connection and Dirac operator we recommend the recent book [BGV] which can also serve as an excellent introduction into the theory of superconnections and Clifford superconnections.

2. Dirac operators

Let M be an even-dimensional Riemannian manifold and $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ a \mathbb{Z}_2 -graded vector bundle over M . A Dirac operator acting on sections of \mathcal{E} is an odd-parity first order operator

$$D: \Gamma(\mathcal{E}^\pm) \longrightarrow \Gamma(\mathcal{E}^\mp) \quad (2.1)$$

such that D^2 is a generalized laplacian. Here we consider such bundles \mathcal{E} provided with a fixed \mathbb{Z}_2 -graded left action $c: C(M) \times \mathcal{E} \rightarrow \mathcal{E}$ of the Clifford bundle $C(M)$, i.e. Clifford modules. For convenience of the reader and to fix our conventions we recall that the Clifford bundle $C(M)$ is a vector bundle over M whose fibre at $x \in M$ consists of the Clifford algebra $C(T_x^*M)$ generated by T_x^*M with respect to the relations $v \star w + w \star v = -2g_x(v, w)$ for all $v, w \in T_x^*M$. It is well-known that for an even-dimensional spin manifold M any Clifford module \mathcal{E} is a twisted bundle $\mathcal{E} = S \otimes E$. Here S denotes the spinor bundle and E is a vector bundle with trivial Clifford action uniquely determined by \mathcal{E} .

We will regard only those Dirac operators D that are compatible with the given Clifford module structure of \mathcal{E} . This means that

$$[D, f] = c(df) \quad (2.2)$$

holds for all $f \in C^\infty(M)$. Property (2.2) fully characterizes these Dirac operators : If P is a differential operator $P: \Gamma(\mathcal{E}^\pm) \rightarrow \Gamma(\mathcal{E}^\mp)$ with $[P, f] = c(df)$ for all $f \in C^\infty(M)$, then P is a Dirac operator, cf. [BGV]. Given any connection $\nabla^\mathcal{E}: \Gamma(\mathcal{E}^\pm) \rightarrow \Gamma(T^*M \otimes \mathcal{E}^\pm)$ on \mathcal{E} which respects the grading, the first-order operator $D_{\nabla^\mathcal{E}}$ defined by the following composition

$$\Gamma(\mathcal{E}^\pm) \xrightarrow{\nabla^\mathcal{E}} \Gamma(T^*M \otimes \mathcal{E}^\pm) \hookrightarrow \Gamma(C(M) \otimes \mathcal{E}) \xrightarrow{c} \Gamma(\mathcal{E}^\mp) \quad (2.3)$$

obviously is a Dirac operator. Note that, in the case of $\mathcal{E} = S \otimes E$, the above construction yields a canonical Dirac operator with respect to any fixed connection ∇^E on E by taking the tensor product connection $\nabla^\mathcal{E} := \nabla^S \otimes \mathbb{1}_E + \mathbb{1}_S \otimes \nabla^E$. Here ∇^S denotes the spin connection on S uniquely determined by the metric structure on M .

For our purpose it is important to realize that every Dirac operator \tilde{D} can be constructed as in (2.3). This follows from

LEMMA 2.1. *Let \mathcal{E} be a Clifford module over an even-dimensional manifold M and $\tilde{D}: \Gamma(\mathcal{E}^\pm) \rightarrow \Gamma(\mathcal{E}^\mp)$ be an arbitrary Dirac operator compatible with the Clifford module structure. Then there exists a connection $\tilde{\nabla}^\mathcal{E}: \Gamma(T^*M \otimes \mathcal{E}^\pm) \rightarrow \Gamma(\mathcal{E}^\pm)$ on \mathcal{E} such that $\tilde{D} = D_{\tilde{\nabla}^\mathcal{E}} := c \circ \tilde{\nabla}^\mathcal{E}$.*

PROOF: Let $D_{\nabla^\mathcal{E}} := c \circ \nabla^\mathcal{E}$ be a Dirac operator on \mathcal{E} constructed as in (2.3). Since $D_{\nabla^\mathcal{E}}$ and \tilde{D} are compatible with the Clifford action on \mathcal{E} we obtain

$$[D_{\nabla^\mathcal{E}} - \tilde{D}, f] = c(df) - c(df) = 0 \quad (2.4)$$

for all $f \in C^\infty(M)$. Hence $D_{\nabla^\mathcal{E}} - \tilde{D}$ can be considered as a section $A \in \Gamma(\text{End}^-(\mathcal{E}))$. Now let the linear bundle map

$$\nu: C(M) \longrightarrow T^*M \otimes C(M) \quad (2.5)$$

be locally defined by $\nu(c(dx^{i_1}) \star \cdots \star c(dx^{i_k})) := dx^{i_1} \otimes c(dx^{i_2}) \star \cdots \star c(dx^{i_k})$ with $i_1 < i_2 < \cdots < i_k < 2n$. Furthermore, we denote by $\text{End}_{C(M)}\mathcal{E}$ the algebra bundle of bundle endomorphisms of \mathcal{E} supercommuting with the action of $C(M)$. Then the composition of this map ν with the canonical isomorphism $\text{End}\mathcal{E} \cong C(M) \otimes \text{End}_{C(M)}\mathcal{E}$ induces a map

$$\Gamma(\text{End}^-(\mathcal{E})) \xrightarrow{\nu} \Gamma(T^*M \otimes \text{End}^+(\mathcal{E})) = \Omega^1(M, \text{End}^+(\mathcal{E})) \quad (2.6)$$

which we denote with the same symbol for convinience. Let $\omega \in \Omega^1(M, \text{End}^+(\mathcal{E}))$ be the image of $A \in \Gamma(\text{End}^-(\mathcal{E}))$ under this map ν . Then $\tilde{\nabla}^\mathcal{E} := \nabla^\mathcal{E} + \omega$ obviously defines a new covariant derivative on \mathcal{E} which respects the \mathbb{Z}_2 -grading, and the Dirac operators $D_{\tilde{\nabla}^\mathcal{E}}$ and \tilde{D} coincide.

□

We will now turn our attention to a specific class of connections on a Clifford module \mathcal{E} , namely the Clifford connections. Let us recall that a connection $\nabla^\mathcal{E}: \Gamma(\mathcal{E}^\pm) \rightarrow \Gamma(T^*M \otimes \mathcal{E}^\pm)$ is called a Clifford connection, if for any $a \in \Gamma(C(M))$ and $X \in \Gamma(TM)$ we have

$$[\nabla_X^\mathcal{E}, c(a)] = c(\nabla_X a). \quad (2.7)$$

In this formula, ∇ denotes the Levi-Civita connection extended to the Clifford

bundle $C(M)$. In the case of $\mathcal{E} = S \otimes E$, the above mentioned tensor product connections $\nabla^S \otimes \mathbb{I}_E + \mathbb{I} \otimes \nabla^E$ are Clifford connections. Furthermore, applying a partition of unity argument, one is able to construct a Clifford connection on every Clifford module \mathcal{E} . In lemma 2.1 we can therefore chose $\nabla^\mathcal{E}$ to be a Clifford connection, i.e.

$$\tilde{\nabla}^\mathcal{E} = \nabla^\mathcal{E} + \omega. \quad (2.8)$$

This is the only property we need in order to prove our generalisation of the Lichnerowicz formula.

For later use we introduce the following notion: We call a graduation on the bundle $\text{End}_{C(M)}(\mathcal{E})$ with the property

$$\text{End}^+(\mathcal{E}) \cong (C(M)^+ \hat{\otimes} \text{End}_{C(M)}^+(\mathcal{E})) \oplus (C(M)^- \hat{\otimes} \text{End}_{C(M)}^-(\mathcal{E})) \quad (2.9)$$

$$\text{End}^-(\mathcal{E}) \cong (C(M)^+ \hat{\otimes} \text{End}_{C(M)}^-(\mathcal{E})) \oplus (C(M)^- \hat{\otimes} \text{End}_{C(M)}^+(\mathcal{E})) \quad (2.10)$$

a twisting graduation of \mathcal{E} . Here $\hat{\otimes}$ denotes the \mathbb{Z}_2 -graded tensor product. In the case of a twisted spin bundle $\mathcal{E} = S \otimes E$, obviously any twisting graduation $\text{End}_{C(M)}(\mathcal{E}) = \text{End}_{C(M)}^+(\mathcal{E}) \oplus \text{End}_{C(M)}^-(\mathcal{E})$ induces a graduation on the twisting bundle E and conversely. Since any Clifford module \mathcal{E} may be decomposed as $S \otimes E$ locally, c.f. [BGV], a twisting graduation on \mathcal{E} corresponds to a graduation of the twisting part E .

3. The generalized Lichnerowicz formula

If $\nabla^\mathcal{E}: \Gamma(\mathcal{E}) \rightarrow \Gamma(T^*M \otimes \mathcal{E})$ is a Clifford connection on the Clifford module \mathcal{E} , there is the well-known decomposition-formula for the square of the corresponding Dirac operator $D_{\nabla^\mathcal{E}} := c \circ \nabla^\mathcal{E}$ due to Lichnerowicz [L]:

$$D_{\nabla^\mathcal{E}}^2 = \Delta^{\nabla^\mathcal{E}} + \frac{r_M}{4} + c(R_{\nabla^\mathcal{E}}^{\mathcal{E}/S}). \quad (3.1)$$

Here $\Delta^{\nabla^\mathcal{E}}$ denotes the connection laplacian associated to $\nabla^\mathcal{E}$ ⁽¹⁾, r_M is the scalar curvature of M and $c(R_{\nabla^\mathcal{E}}^{\mathcal{E}/S})$ denotes the image of the twisting curvature $R_{\nabla^\mathcal{E}}^{\mathcal{E}/S} \in$

⁽¹⁾ With respect to a local coordinate frame of TM , the connection laplacian $\Delta^{\nabla^\mathcal{E}}$ is explicitly given by $\Delta^{\nabla^\mathcal{E}} = -g^{\mu\nu}(\nabla_\mu^\mathcal{E} \nabla_\nu^\mathcal{E} - \Gamma_{\mu\nu}^\sigma \nabla_\sigma^\mathcal{E})$.

$\Omega^2(M, \text{End}_{C(M)}(\mathcal{E}))$ of the Clifford connection $\nabla^\mathcal{E}$ with respect to the quantisation map $c: \Lambda^* T^* M \rightarrow C(M)$, cf. [BGV].

In this section we will generalize formula (3.1) for an arbitrary Dirac operator \tilde{D} on \mathcal{E} which is compatible with the Clifford action. Using $\tilde{D} = c(dx^\mu) \tilde{\nabla}_\mu^\mathcal{E}$ and mimicking the first step in the computation of the Lichnerowicz formula (3.1), we obtain

$$\tilde{D}^2 = -g^{\mu\nu} \tilde{\nabla}_\mu^\mathcal{E} \tilde{\nabla}_\nu^\mathcal{E} + c(dx^\mu) [\tilde{\nabla}_\mu^\mathcal{E}, c(dx^\nu)] \tilde{\nabla}_\nu^\mathcal{E} + \frac{1}{2} c(dx^\mu) c(dx^\nu) [\tilde{\nabla}_\mu^\mathcal{E}, \tilde{\nabla}_\nu^\mathcal{E}]. \quad (3.2)$$

It is remarkable that none of the first two terms in (3.2) is globally defined, but only their sum. However, we observe that given a Clifford connection $\nabla^\mathcal{E}$ on \mathcal{E} , then (3.2) can be rearranged in such a way that $\tilde{D}^2 = \Delta^{\nabla^\mathcal{E}} + P + F'$ where $P: \Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{E})$ is a first-order differential operator and $F' \in \text{End}(\mathcal{E})$ with both terms depending on $(\tilde{\nabla}^\mathcal{E} - \nabla^\mathcal{E}) =: \omega \in \Omega^1(M, \text{End}^+(\mathcal{E}))$. More precisely we have

LEMMA 3.1. *Let $\nabla^\mathcal{E}: \Gamma(\mathcal{E}^\pm) \rightarrow \Gamma(T^*M \otimes \mathcal{E}^\pm)$ be a Clifford connection on the Clifford module \mathcal{E} and $\tilde{D} = c \circ \tilde{\nabla}$ a Dirac operator. If ω denotes the one-form on M with values in $\text{End}^+(\mathcal{E})$ uniquely determined by $\omega := \tilde{\nabla} - \nabla^\mathcal{E}$, then*

$$\tilde{D}^2 = \Delta^{\nabla^\mathcal{E}} - (B_\omega \nabla^\mathcal{E}) + \frac{r_M}{4} + c(R_{\nabla^\mathcal{E}}^{\mathcal{E}/S}) + F'_\omega \quad (3.3)$$

where $B_\omega: \Gamma(T^*M \otimes \mathcal{E}) \rightarrow \Gamma(\mathcal{E})$ with $B_\omega(dx^\nu \otimes s) = (2g^{\mu\nu} \omega_\mu - c(dx^\mu) [\omega_\mu, c(dx^\nu)]) s$ and $F'_\omega = c(dx^\mu) c(dx^\nu) ({}'\nabla_\mu \omega_\nu) + c(dx^\mu) \omega_\mu c(dx^\nu) \omega_\nu$ together with the Lichnerowicz terms $\frac{r_M}{4} + c(R_{\nabla^\mathcal{E}}^{\mathcal{E}/S})$ determine the first order resp. the endomorphism part with respect to a local coordinate system.

PROOF: By inserting $\tilde{\nabla} = \nabla^\mathcal{E} + \omega$ in (3.2) and by using that $\nabla^\mathcal{E}$ is a Clifford connection, we get

$$\begin{aligned} \tilde{D}^2 &= \Delta^{\nabla^\mathcal{E}} + \frac{1}{4} [c(dx^\mu), c(dx^\nu)] [\nabla_\mu^\mathcal{E}, \nabla_\nu^\mathcal{E}] - g^{\mu\nu} ([\nabla_\mu^\mathcal{E}, \omega_\nu] - \Gamma_{\nu\mu}^\sigma \omega_\sigma) \\ &\quad - g^{\mu\nu} \omega_\mu \omega_\nu - g^{\mu\nu} (2\omega_\mu \nabla_\nu^\mathcal{E}) + c(dx^\mu) [\omega_\mu, c(dx^\nu)] \nabla_\nu^\mathcal{E} \\ &\quad + \frac{1}{4} [c(dx^\mu), c(dx^\nu)] [\omega_\mu, \omega_\nu] + \frac{1}{2} [c(dx^\mu), c(dx^\nu)] [\nabla_\mu^\mathcal{E}, \omega_\nu] \\ &\quad + c(dx^\mu) [\omega_\mu, c(dx^\nu)] \omega_\nu \end{aligned} \quad (3.4)$$

with respect to the local coordinate frame $\{\partial_\mu\}$ of TM . Here $\Gamma_{\mu\nu}^\sigma$ are the Christoffel symbols defined by the Levi-Civita connection on M . Thus, the fifth together with

the sixth term on the right-hand-side define $B_\omega: \Gamma(T^*M \otimes \mathcal{E}) \rightarrow \Gamma(\mathcal{E})$. Furthermore we have to explain our 'short-hand' notation $(\nabla_\mu \omega_\nu)$:

$$\begin{aligned} -g^{\mu\nu}([\nabla_\mu^\mathcal{E} \omega_\nu] - \Gamma_{\mu\nu}^\sigma \omega_\sigma) &= -g^{\mu\nu}(\nabla_\mu^{\text{End } \mathcal{E}} \omega_\nu) - \Gamma_{\mu\nu}^\sigma \omega_\sigma =: -g^{\mu\nu}(\nabla_\mu \omega_\nu) \\ \frac{1}{2}[c(dx^\mu), c(dx^\nu)][\nabla_\mu^\mathcal{E} \omega_\nu] &= \frac{1}{2}[c(dx^\mu), c(dx^\nu)](\nabla_\mu^{\text{End } \mathcal{E}} \omega_\nu) - \Gamma_{\mu\nu}^\sigma \omega_\sigma \\ &=: \frac{1}{2}[c(dx^\mu), c(dx^\nu)](\nabla_\mu \omega_\nu). \end{aligned} \quad (3.5)$$

Equivalently one could describe (3.5) in a global manner by using the composition of the connection $\nabla^{T^*M \otimes \text{End } \mathcal{E}}: \Gamma(T^*M \otimes \text{End } \mathcal{E}) \rightarrow \Gamma(T^*M \otimes T^*M \otimes \text{End } \mathcal{E})$ together with the evaluation $-ev_g$ and the quantisation map c , respectively. Finally we use $\frac{1}{2}[c(dx^\mu), c(dx^\nu)] - g^{\mu\nu} = c(dx^\mu)c(dx^\nu)$ to obtain the first term of $F' \in \Gamma(\text{End}(\mathcal{E}))$. The computation of the second term of F' is straightforward.

□

Of course, in spite of being global, the decomposition formula (3.3) seems to be unsatisfactory since the first-order operator $B_\omega \nabla^\mathcal{E}$ does not vanish, generally ⁽²⁾. Also (3.3) term by term depends on the chosen Clifford connection $\nabla^\mathcal{E}$ and is therefore by no means an intrinsic property of the Dirac operator \tilde{D} . This shows that (3.3) can not serve as a generalisation of the Lichnerowicz formula (3.1). To remedy this flaw we need the following observation:

LEMMA 3.2. *Let Δ^{∇^E} and $\Delta^{\hat{\nabla}^E}$ be the connection laplacians defined with respect to the connections ∇^E and $\hat{\nabla}^E$ acting on sections of a vector bundle E over M . If furthermore $(\hat{\nabla}^E - \nabla^E) =: \alpha \in \Omega^1(M, \text{End } E)$, then*

$$\Delta^{\hat{\nabla}^E} = \Delta^{\nabla^E} - A_\alpha \nabla^E - \Theta_\alpha, \quad (3.6)$$

where $A_\alpha: \Gamma(T^*M \otimes E) \rightarrow \Gamma(E)$ with $A_\alpha(dx^\nu \otimes s_\nu) := 2g^{\mu\nu} \alpha_\mu s_\nu$ and the endomorphism $\Theta_\alpha := g^{\mu\nu}(\nabla_\mu^E \alpha_\nu) + g^{\mu\nu} \alpha_\mu \alpha_\nu$ are defined with respect to a local coordinate system.

(2) We will study the case $B_\omega = 0$ in the next section.

PROOF: Obviously we have $\hat{\nabla}^E = \nabla^E + \alpha$ and compute

$$\begin{aligned}
\Delta^{\hat{\nabla}^E} &:= -g^{\mu\nu}((\nabla_\mu^E + \alpha_\mu)(\nabla_\nu^E + \alpha_\nu) - \Gamma_{\mu\nu}^\sigma(\nabla_\sigma^E + \alpha_\sigma)) \\
&= -g^{\mu\nu}(\underbrace{\nabla_\mu^E \nabla_\nu^E - \Gamma_{\mu\nu}^\sigma \nabla_\sigma^E}_{= \Delta^{\nabla^E}} \\
&\quad - \underbrace{2g^{\mu\nu} \alpha_\mu \nabla_\nu^E}_{= A_\alpha \nabla^E} - \underbrace{g^{\mu\nu}((\nabla_\mu^{\text{End } E} \alpha_\nu - \Gamma_{\mu\nu}^\sigma \alpha_\sigma) + \alpha_\mu \alpha_\nu)}_{= \Theta_\alpha}),
\end{aligned} \tag{3.7}$$

where we have also used the definition of ∇^E given in (3.5). □

Note, that $A_\alpha \in \Gamma(TM \otimes \text{End } E)$ can be described also by $A_\alpha = 2 \alpha^\sharp$, where α^\sharp denotes the corresponding ‘dual’ element of $\alpha \in \Gamma(T^*M \otimes \text{End } E)$ in $\Gamma(TM \otimes \text{End } E)$ under the ‘musical’ isomorphism $T^*M \otimes \text{End } E \xrightleftharpoons[b \otimes \mathbb{I}]{\sharp \otimes \mathbb{I}} TM \otimes \text{End } E$ defined by the Riemannian metric g .

Guided by the previous lemma in the case of $E = \mathcal{E}$ and $\nabla^E = \nabla^\mathcal{E}$ the given Clifford connection, we define $\hat{\nabla}^\mathcal{E} := \nabla^\mathcal{E} + \frac{1}{2} B_\omega^\flat$, with $B_\omega \in \Gamma(TM \otimes \text{End } \mathcal{E})$ as above. Thus we have $\hat{\nabla}^\mathcal{E} = \tilde{\nabla}^\mathcal{E} + \varpi$ with

$$\varpi := -\frac{1}{2} \left(\frac{\partial}{\partial x^\nu} \otimes c(dx^\mu) [\omega_\mu, c(dx^\nu)] \right)^\flat = -\frac{1}{2} g_{\sigma\nu} dx^\sigma \otimes c(dx^\mu) [\omega_\mu, c(dx^\nu)]. \tag{3.8}$$

In fact, this one form ϖ ‘measures’ how much $\tilde{\nabla}^\mathcal{E}$ differs from being a Clifford connection because $\varpi = -\frac{1}{2} g_{\sigma\nu} dx^\sigma \otimes c(dx^\mu) ([\tilde{\nabla}_\mu^\mathcal{E}, c(dx^\nu)] - [\nabla_\mu^\mathcal{E}, c(dx^\nu)])$. Since any two Clifford connections on the Clifford module \mathcal{E} differ by an element of $\Omega^1(M, \text{End}_{C(M)}(\mathcal{E}))$, ϖ only depends on the initial connection $\tilde{\nabla}^\mathcal{E}$ ⁽³⁾. Consequently also $\hat{\nabla}^\mathcal{E}$ only depends on $\tilde{\nabla}^\mathcal{E}$. So we get as a result of the previous lemmas:

LEMMA 3.3. *Let \mathcal{E} be a Clifford module over an even dimensional Riemannian manifold M and $\tilde{D} = c \circ \tilde{\nabla}^\mathcal{E}$ a Dirac operator. If $\Delta^{\tilde{\nabla}^\mathcal{E}}$ denotes the connection laplacian corresponding to the above defined connection $\tilde{\nabla}$, then $\tilde{D}^2 = \Delta^{\tilde{\nabla}^\mathcal{E}} + F_{\tilde{\nabla}^\mathcal{E}}$, where $F_{\tilde{\nabla}^\mathcal{E}} \in \text{End}(\mathcal{E})$ is given by $F_{\tilde{\nabla}^\mathcal{E}} = \frac{r_M}{4} + c(R_{\nabla^\mathcal{E}}^{\mathcal{E}/S}) + F_\omega + \Theta_{\frac{1}{2}B_\omega}$ with respect to an arbitrary Clifford connection $\nabla^\mathcal{E}$ and $\omega := (\tilde{\nabla}^\mathcal{E} - \nabla^\mathcal{E})$.*

(3) In the further we will therefore write $\varpi_{\tilde{\nabla}^\mathcal{E}}$ to indicate this dependence.

PROOF: By lemma 3.1 we can decompose the square of the Dirac operator \tilde{D} as

$$\tilde{D}^2 = \Delta^{\nabla^\varepsilon} - (B_\omega \nabla^\varepsilon) + \frac{r_M}{4} + c(R_{\nabla^\varepsilon}^{\varepsilon/S}) + F'_\omega. \quad (3.9)$$

When applying lemma 3.2 to the two connection laplacians $\Delta^{\nabla^\varepsilon}$ and $\Delta^{\tilde{\nabla}^\varepsilon}$, which correspond to the given Clifford connection ∇^ε and the above defined connection $\tilde{\nabla} := \nabla^\varepsilon + \frac{1}{2}B_\omega^b$ with $\omega := (\tilde{\nabla}^\varepsilon - \nabla^\varepsilon)$ as in lemma 3.1, we obtain

$$\Delta^{\nabla^\varepsilon} - B_\omega \nabla^\varepsilon = \Delta^{\tilde{\nabla}^\varepsilon} + \Theta_{\frac{1}{2}B_\omega^b}, \quad (3.10)$$

which implies the desired result. □

Since the Dirac operator \tilde{D} as well as the connection laplacian $\Delta^{\tilde{\nabla}^\varepsilon}$ only depend on $\tilde{\nabla}^\varepsilon$, obviously so does the endomorphism $F_{\tilde{\nabla}^\varepsilon}$. This suggests an intrinsic meaning of this term. Using the definitions

$$\begin{aligned} F'_\omega &:= c(dx^\mu)c(dx^\nu)(\nabla_\mu^\varepsilon \omega_\nu) + c(dx^\mu)\omega_\mu c(dx^\nu)\omega_\nu \\ \frac{1}{2}B_\omega^b &:= \omega - \frac{1}{2}g_{\sigma\nu}dx^\sigma \otimes c(dx^\mu)[\omega_\mu, c(dx^\nu)] \\ \Theta_{\frac{1}{2}B_\omega^b} &:= \frac{1}{2}g^{\mu\nu}(\nabla_\mu^\varepsilon (B_\omega^b)_\nu) + \frac{1}{4}g^{\mu\nu}(B_\omega^b)_\mu (B_\omega^b)_\nu \end{aligned} \quad (3.11)$$

together with the identity $\frac{r_M}{4} + c(R_{\nabla^\varepsilon}^{\varepsilon/S}) = \frac{1}{4}[c(dx^\mu), c(dx^\nu)][\nabla_\mu^\varepsilon, \nabla_\nu^\varepsilon]$ we compute

$$\begin{aligned} F_{\tilde{\nabla}^\varepsilon} &= \frac{1}{4}[c(dx^\mu), c(dx^\nu)][\nabla_\mu^\varepsilon, \nabla_\nu^\varepsilon] + \frac{1}{2}[c(dx^\mu), c(dx^\nu)](\nabla_\mu^\varepsilon \omega_\nu) \\ &\quad + \frac{1}{4}[c(dx^\mu), c(dx^\nu)][\omega_\mu, \omega_\nu] + g^{\sigma\nu}[\tilde{\nabla}_\nu^\varepsilon, -\frac{1}{2}g_{\sigma\nu}c(dx^\mu)[\omega_\mu, c(dx^\nu)]] \\ &\quad + \frac{1}{4}g_{\mu\nu}c(dx^\kappa)[\omega_\kappa, c(dx^\mu)]c(dx^\sigma)[\omega_\sigma, c(dx^\nu)]. \end{aligned} \quad (3.12)$$

The first three terms of (3.12) join together to define $\frac{1}{4}[c(dx^\mu), c(dx^\nu)][\tilde{\nabla}^\varepsilon, \tilde{\nabla}^\varepsilon] = c(R^{\tilde{\nabla}^\varepsilon})$, i.e. the image of the curvature $R^{\tilde{\nabla}^\varepsilon} \in \Omega^2(M, \text{End } \mathcal{E})$ of the connection $\tilde{\nabla}^\varepsilon$ under the quantization map c . The forth term can be written as $ev_g \tilde{\nabla}^{T^*M \otimes \text{End } \mathcal{E}} \varpi_{\tilde{\nabla}^\varepsilon}$ with the tensor product connection $\tilde{\nabla}^{T^*M \otimes \text{End } \mathcal{E}} := \nabla \otimes \mathbb{I}_\mathcal{E} + \mathbb{I}_{T^*M} \otimes \tilde{\nabla}^\varepsilon$, cf. equation (3.5) and the remark thereafter. Using the pointwise defined product ‘ \cdot ’ in the

algebra bundle $T(M) \otimes \text{End}\mathcal{E}$, where $T(M)$ denotes the tensor bundle of T^*M , we can write $ev_g(\varpi_{\tilde{\nabla}^\mathcal{E}} \cdot \varpi_{\tilde{\nabla}^\mathcal{E}})$ for the last term. So we get

$$F_{\tilde{\nabla}^\mathcal{E}} = c(R^{\tilde{\nabla}^\mathcal{E}}) + ev_g \tilde{\nabla}^{T^*M \otimes \text{End}\mathcal{E}} \varpi_{\tilde{\nabla}^\mathcal{E}} + ev_g(\varpi_{\tilde{\nabla}^\mathcal{E}} \cdot \varpi_{\tilde{\nabla}^\mathcal{E}}). \quad (3.13)$$

Therefore we have shown

THEOREM 3.4. *Let \mathcal{E} be a Clifford module over an even dimensional Riemannian manifold M and $\tilde{D} = c \circ \tilde{\nabla}^\mathcal{E}$ a Dirac operator. Then*

$$\tilde{D}^2 = \Delta^{\tilde{\nabla}^\mathcal{E}} + c(R^{\tilde{\nabla}^\mathcal{E}}) + ev_g \tilde{\nabla}^{T^*M \otimes \text{End}\mathcal{E}} \varpi_{\tilde{\nabla}^\mathcal{E}} + ev_g(\varpi_{\tilde{\nabla}^\mathcal{E}} \cdot \varpi_{\tilde{\nabla}^\mathcal{E}}), \quad (3.14)$$

with $\varpi_{\tilde{\nabla}^\mathcal{E}} := -\frac{1}{2}g_{\nu\kappa}dx^\nu \otimes c(dx^\mu)([\tilde{\nabla}_\mu^\mathcal{E}, c(dx^\kappa)] + c(dx^\sigma)\Gamma_{\sigma\mu}^\kappa) \in \Omega^1(M, \text{End}\mathcal{E})$ and the connection $\hat{\nabla}^\mathcal{E} := \tilde{\nabla}^\mathcal{E} + \varpi_{\tilde{\nabla}^\mathcal{E}}$.

In this decomposition of the square of \tilde{D} , the last two terms obviously indicate the deviation of the connection $\tilde{\nabla}^\mathcal{E}$ being a Clifford connection. Only the second term of (3.14) is endowed with geometric significance. Of course, if $\tilde{\nabla}^\mathcal{E}$ is a Clifford connection, obviously $\varpi_{\tilde{\nabla}^\mathcal{E}} = 0$ and therefore (3.14) reduces to Lichnerowicz's formula $\tilde{D}^2 = \Delta^{\tilde{\nabla}^\mathcal{E}} + \frac{r_M}{4} + c(R_{\tilde{\nabla}^\mathcal{E}}^{\mathcal{E}/S})$. So we call (3.14) 'the generalized Lichnerowicz formula'.

4. The 'Super - Lichnerowicz formula'

In this section we will turn our attention to the generalized Lichnerowicz formula (3.14) in the context of 'super-geometry'. This is motivated by the well-known fact, that any Clifford superconnection A on a Clifford module \mathcal{E} uniquely determines a Dirac operator D_A due to the following construction

$$D_A: \Gamma(\mathcal{E}) \xrightarrow{A} \Omega^*(M, \mathcal{E}) \xrightarrow{c \otimes \mathbb{I}_\mathcal{E}} \Gamma(C(M) \otimes \mathcal{E}) \xrightarrow{c} \Gamma(\mathcal{E}), \quad (4.1)$$

i.e. there is a one-to-one correspondence between Clifford superconnections and Dirac operators, see [BGV] ⁽⁴⁾. Since (3.14) is already the decomposition of

⁽⁴⁾ This reference can serve also as an excellent introduction into the theory of superconnections and Clifford superconnections.

the square of an arbitrary Dirac operator on \mathcal{E} , we only have to adapt it to super-geometry. Thus, the expression ‘Super-Lichnerowicz formula’ is an abuse of notation here. To proceed, let again $\nu: C(M) \rightarrow T^*M \otimes C(M)$ be the linear bundle map defined in the second section. The following simple observation is crucial:

LEMMA 4.1. *Let A be a Clifford superconnection on the Clifford module \mathcal{E} . Then the operator ∇^A defined by the following composition*

$$\Gamma(\mathcal{E}) \xrightarrow{A} \Omega^*(M, \mathcal{E}) \xrightarrow{c \otimes \mathbb{I}_{\mathcal{E}}} \Gamma(C(M) \otimes \mathcal{E}) \xrightarrow{\nu \otimes \mathbb{I}_{\mathcal{E}}} \Gamma(T^*M \otimes C(M) \otimes \mathcal{E}) \xrightarrow{\mathbb{I}_{T^*M} \otimes c} \Gamma(T^*M \otimes \mathcal{E})$$

is a connection on \mathcal{E} with the property that the associated Dirac operator $D_{\nabla^A} := c \circ \nabla^A$ coincides with D_A .

PROOF: Because A satisfies Leibniz’s rule $A(fs) = df \otimes s + fAs$ for all $f \in C^\infty(\mathcal{E})$ and $s \in \Gamma(\mathcal{E})$, so does ∇^A . Therefore ∇^A is a connection on \mathcal{E} . To prove the second statement we simply remark, that the diagramm

$$\begin{array}{ccc} \Gamma(C(M) \otimes \mathcal{E}) & \xrightarrow{c} & \Gamma(\mathcal{E}) \\ \downarrow \nu \otimes \mathbb{I}_{\mathcal{E}} & & \uparrow c \\ \Gamma(T^*M \otimes C(M) \otimes \mathcal{E}) & \xrightarrow{\mathbb{I}_{T^*M} \otimes c} & \Gamma(T^*M \otimes \mathcal{E}) \end{array} \quad (4.2)$$

commutes.

□

In general, however, the connection ∇^A defined by a Clifford superconnection is not a Clifford connection, i.e. $[\nabla_\mu^A, c(dx^\nu)] \neq -c(dx^\sigma)\Gamma_{\sigma\mu}^\nu$. Using lemma 4.1, we can now ‘reformulate’ theorem 3.4 in the following way

THEOREM 4.2. *Let \mathcal{E} be a Clifford module over an even dimensional Riemannian manifold M , $A: \Omega^\pm(M, \mathcal{E}) \rightarrow \Omega^\mp(M, \mathcal{E})$ a Clifford superconnection and D_A the corresponding Dirac operator. Then*

$$D_A^2 = \Delta^{\hat{\nabla}^\mathcal{E}} + c(R^{\nabla^A}) + ev_g(\nabla^A)^{T^*M \otimes \text{End}\mathcal{E}} \varpi_{\nabla^A} + ev_g(\varpi_{\nabla^A} \cdot \varpi_{\nabla^A}), \quad (4.3)$$

with $\varpi_{\nabla^A} := -\frac{1}{2}g_{\nu\kappa}dx^\nu \otimes c(dx^\mu)([\nabla_\mu^A, c(dx^\kappa)]) + c(dx^\sigma)\Gamma_{\sigma\mu}^\kappa \in \Omega^1(M, \text{End}\mathcal{E})$ and the connection $\hat{\nabla}^\mathcal{E} := \nabla^A + \varpi_{\nabla^A}$.

5. Dirac operators of simple type

Now we will begin with our analysis of Dirac operators with regard to our decomposition formula (3.14). We start with the following

DEFINITION 5.1. A Dirac operator \tilde{D} acting on sections of a Clifford module \mathcal{E} is called 'of simple type' if the connection $\hat{\nabla}^{\mathcal{E}}$ which defines the connection laplacian $\Delta^{\hat{\nabla}^{\mathcal{E}}}$ in the decomposition formula (3.14) of \tilde{D}^2 is a Clifford connection.

Because it is essentially the section $B_{\omega} \in \Gamma(TM \otimes \text{End}\mathcal{E})$ which defines the connection $\hat{\nabla}^{\mathcal{E}}$ (cf. the definition of $\hat{\nabla}^{\mathcal{E}}$ before (3.8)), such Dirac operators are characterized by the following

LEMMA 5.2. Let \tilde{D} be a Dirac operator on a Clifford module \mathcal{E} . Then \tilde{D} is of simple type iff there exists a Clifford connection $\nabla^{\mathcal{E}}: \Gamma(\mathcal{E}) \rightarrow \Gamma(T^*M \otimes \mathcal{E})$ such that $B_{\omega} \in \Gamma(TM \otimes \text{End}\mathcal{E})$ in the decomposition (3.3) of \tilde{D}^2 vanishes.

Obviously any Dirac operator $D_{\nabla^{\mathcal{E}}}$ associated with a Clifford connection $\nabla^{\mathcal{E}}$ is of simple type. Now we ask whether there are still other Dirac operators of simple type on \mathcal{E} . If the twisting graduation of the Clifford module \mathcal{E} is non-trivial, we will answer this question affirmatively. Moreover we will show, that Dirac operators of simple type are in one-to-one correspondence with the set of pairs $\{(\nabla^{\mathcal{E}}, \phi)\}$. Here $\nabla^{\mathcal{E}}$ denotes a Clifford connection on the respective Clifford module \mathcal{E} and ϕ is a section of the endomorphism bundle $\text{End}_{C(M)}^-(\mathcal{E})$. In the following we denote by $\text{Sym}^2(T^*M)$ the bundle of symmetric two-tensors of T^*M over M . We further remark that there is a natural inclusion $\text{Sym}^2(T^*M) \hookrightarrow T^*M \otimes C^-(M)$.

LEMMA 5.3. Let \mathcal{E} be a Clifford module over an even dimensional Riemannian manifold M and let the linear map $\Xi: \Omega^1(M, \text{End}^+(\mathcal{E})) \rightarrow \Gamma(TM \otimes \text{End}(\mathcal{E}))$ be defined by $\omega \mapsto B_{\omega}$. Then $\ker \Xi \neq 0$ iff the twisting graduation of \mathcal{E} is non-trivial. Moreover $\omega \in \ker \Xi$ iff $\omega = w \otimes F$ with $w \in \Gamma(\text{Sym}^2(T^*M))$ and $F \in \Gamma(\text{End}_{C(M)}^-(\mathcal{E}))$.

PROOF: On an open subset $U \subset M$ any $\omega \in \Omega^1(M, \text{End}^+(\mathcal{E}))$ can be described by

$$\omega = \sum_{|I|} \omega_{\mu}^i dx^{\mu} \otimes c(dx^I) \hat{\otimes} F_i, \quad (5.1)$$

where $\omega_{\mu I}^i \in C^\infty(U)$ and $F_i \in \text{End}_{C(M)}(\mathcal{E})$, $I = (i_1, \dots, i_{|I|})$ is a multi-index, $|I|$ denotes its length, $c(dx^I) := c(dx^{i_1}) \cdots c(dx^{i_{|I|}})$ and \sum' signifies that the sum is taken over strictly increasing indices ⁽⁴⁾. By the definition of B_ω (see lemma 3.1) we get

$$\begin{aligned} B_\omega^\nu s_\nu &= B(dx^\nu \otimes s_\nu) = \sum_{|I|}' (2g^{\mu\nu} \omega_{\mu I}^i c(dx^I) \hat{\otimes} F_i s_\nu \\ &\quad - \sum_{|I|}' \omega_{\mu I}^i (-1)^{|F_i| |c(dx^\nu)|} c(dx^\mu) c(dx^I) c(dx^\nu) - c(dx^\mu) c(dx^\nu) c(dx^I) \hat{\otimes} F_i s_\nu. \end{aligned} \quad (5.2)$$

Here the sign $(-1)^{|F_i| |c(dx^\nu)|}$ is a consequence of using the \mathbb{Z}_2 -graded tensor product in (5.1). We have to determine the solutions of the equation $B_\omega^\nu = 0$ for all $\nu \in \{1, \dots, 2n\}$.

1. case: $\text{End}^+(\mathcal{E}) \cong C^+(M) \otimes \text{End}_{C(M)}(\mathcal{E}) \cong C^+(M) \hat{\otimes} \text{End}_{C(M)}^+(\mathcal{E})$, i.e. the twisting graduation of \mathcal{E} is trivial.

Using $F_i \in \text{End}_{C(M)}^+(\mathcal{E})$, i.e. $|F_i| = 0$ for all i , together with the Clifford relation $c(dx^{i_k})c(dx^\nu) + c(dx^\nu)c(dx^{i_k}) = -2g^{i_k\nu}$ we obtain from (3.8)

$$\sum_{|I| \text{ even}}' \left(g^{\mu\nu} \omega_{\mu I}^i c(dx^I) + (-1)^k g^{i_k\nu} \omega_{\mu I}^i c(dx^\mu) c(dx^{i_1}) \dots c(\widehat{dx^{i_k}}) \dots c(dx^{i_{|I|}}) \right) = 0,$$

where the hat indicates that the corresponding factor has been omitted. In the following we suppress the endomorphism-index i for convinience. Since each coefficient $\omega_{\mu I} = \omega_{\mu i_1 \dots i_{|I|}}$ is totally antisymmetric in the multi-index I , we have $(-1)^k \omega_{\mu i_1 \dots i_k \dots i_{|I|}} = -\omega_{\mu i_k i_1 \dots \widehat{i_k} \dots i_{|I|}}$. With the definition of $\sum'_{|I|}$ and using the short-hand notation $c(dx^{i_1 \dots \widehat{i_k} \dots i_{|I|}}) := c(dx^{i_1}) \dots c(\widehat{dx^{i_k}}) \dots c(dx^{i_{|I|}})$, we therefore

(4) Without using the Einstein summation convention this means

$$\sum_{|I|}' \omega_{\mu I}^i dx^\mu \otimes c(dx^I) \hat{\otimes} F_i := \sum_{|I|} \sum_{\substack{\mu, i \\ i_1 < \dots < i_{|I|}}} \omega_{\mu I}^i dx^\mu \otimes c(dx^I) \hat{\otimes} F_i.$$

obtain for the second term above:

$$\begin{aligned}
\sum_{|I| \text{ even}}' (-1)^k g^{i_k \nu} \omega_{\mu} c(dx^\mu) c(dx^{i_1 \dots \widehat{i_k} \dots i_{|I|}}) &= \sum_{\substack{i_1 < \dots < i_{|I|} \\ |I| \text{ even}}} (-1)^k g^{i_k \nu} \omega_{\mu i_1 \dots i_k \dots i_{|I|}} c(dx^\mu) c(dx^{i_1 \dots \widehat{i_k} \dots i_{|I|}}) \\
&= \sum_{\substack{i_2 < \dots < i_{|I|} \\ |I| \text{ even}}} - \left(\frac{1}{|I|} \omega_{\mu}{}^\nu{}_{i_2 \dots i_{|I|}} c(dx^\mu) c(dx^{i_2 \dots i_{|I|}}) \right) \\
&= \sum_{\substack{\mu < i_2 < \dots < i_{|I|} \\ |I| \text{ even}}} - \omega_{\mu}{}^\nu{}_{i_2 \dots i_{|I|}} c(dx^{\mu i_2 \dots i_{|I|}}) \\
&\quad + \sum_{\substack{i_3 < \dots < i_{|I|} \\ |I| \text{ even}}} \omega_{\mu}{}^\nu{}_{\mu i_3 \dots i_{|I|}} c(dx^{i_3 \dots i_{|I|}}).
\end{aligned}$$

Consequently for $\mu = i_1$ we have to solve

$$\sum_{|I| \text{ even}}' (\omega_{i_1 \dots i_{|I|}}{}^\nu - \omega_{i_1}{}^\nu{}_{i_2 \dots i_{|I|}}) c(dx^I) + \sum_{\substack{i_3 < \dots < i_{|I|} \\ |I| \text{ even}}} \omega_{i_1}{}^\nu{}_{i_1 i_3 \dots i_{|I|}} c(dx^{i_3 \dots i_{|I|}}) = 0 \quad (5.3)$$

for all $1 \leq \nu \leq 2n$. Now let $|I| > 0$. Since the indices are ordered and therefore the vectors $c(dx^I)$ resp. $c(dx^{i_3 \dots i_{|I|}})$ are linear independent in the Clifford algebra, equation (5.3) shows that

$$\omega_{i_1 \nu i_2 \dots i_{|I|}} = \omega_{\nu i_1 i_2 \dots i_{|I|}} \quad (5.4)$$

must hold for all i_1, ν , i.e. each coefficient $\omega_{i_1 \nu i_2 \dots i_{|I|}}$ is symmetric in the first two indices i_1, ν . As a consequence $\omega_{i_1 \nu i_2 \dots i_{|I|}}$ resp. $\omega_{i_1 \nu i_1 i_3 \dots i_{|I|}}$ are also totally symmetric in the multi-index $(i_3 \dots i_{|I|})$ for all $|I|$ even and greater than zero. This, however, contradicts the total antisymmetry of the multi-index $I = (i_1, \dots, i_{|I|})$. Therefore $\omega_{\mu i_1 \dots i_{|I|}} = 0$ for all $2 \leq |I| \leq 2n$ and all i . For $|I| = 0$ we have $c(dx^I) := \mathbb{1}_{C(M)}$ by definition and therefore obviously $\omega_{\mu}^i = 0$. In sum we obtain only the trivial solution $\omega = 0$. Consequently $\ker \Xi = 0$.

2. case: $\text{End}^+(\mathcal{E}) \cong (C(M)^+ \otimes \text{End}_{C(M)}^+(\mathcal{E})) \oplus (C(M)^- \otimes \text{End}_{C(M)}^-(\mathcal{E}))$, i.e. the twisting graduation of \mathcal{E} is non-trivial.

We now only have to check the case of $\omega \in \Omega^1(M, C(M)^- \otimes \text{End}_{\bar{C}(M)}(\mathcal{E}))$. Using $F_i \in \text{End}_{\bar{C}(M)}(\mathcal{E})$, i.e. $|F_i| = 1$ for all i together with the Clifford relation we obtain from (3.8) as above

$$\sum_{|I| \text{ odd}} \left(g^{\mu\nu} \omega_{\mu I}^i c(dx^I) + (-1)^k g^{i k \nu} \omega_{\mu I}^i c(dx^\mu) c(dx^{i_1}) \dots c(\widehat{dx^{i_k}}) \dots c(dx^{i_{|I|}}) \right) = 0.$$

For $|I| > 1$ we can argue as in the even case above concluding $\omega_{\mu i_1 \dots i_{|I|}}^i = 0$ for all i . In the case $|I| = 1$, however, we obtain

$$g^{\mu\nu} \omega_{\mu i_1}^i c(dx^{i_1}) - g^{\nu i_1} \omega_{\mu i_1}^i c(dx^\mu) = 0.$$

Therefore each coefficient $\omega_{\mu\nu}^i = \omega_{\nu\mu}^i$ is symmetric and so $\omega = \omega_{\mu i_1}^i dx^\mu \otimes c(dx^{i_1}) \hat{\otimes} F_i \in \text{Sym}^2(T^*M) \hat{\otimes} \text{End}_{\bar{C}(M)}(\mathcal{E})$ lies in the kernel of \mathcal{E} .

□

COROLLARY 5.4. *There exist Dirac operators of simple type not corresponding to Clifford connections on a Clifford module \mathcal{E} iff the twisting graduation of \mathcal{E} is non-trivial. Moreover if \tilde{D} is such a Dirac operator, then $\tilde{D} = D_{\nabla^\mathcal{E}} + \mathbb{I}_{C(M)} \hat{\otimes} \Phi$, where $D_{\nabla^\mathcal{E}}$ is a Dirac operator defined by a Clifford connection $\nabla^\mathcal{E}$ and $\Phi \in \Gamma(\text{End}_{\bar{C}(M)}(\mathcal{E}))$.*

PROOF: By the previous lemmas 5.2 and 5.3 the first statement is obvious. So we only have to prove the second one.

Let $\omega := \omega_{\mu\nu}^i dx^\mu \otimes c(dx^\nu) \hat{\otimes} F_i$ with $w^i := \omega_{\mu\nu}^i dx^\mu \otimes c(dx^\nu) \in \Gamma(\text{Sym}^2(T^*M)) \subset \Gamma(T^*M \otimes C^-(M))$ and $F_i \in \Gamma(\text{End}_{\bar{C}(M)}(\mathcal{E}))$ be as in the previous lemma. With the help of the Clifford relations we compute

$$\begin{aligned} c(\omega) &= \omega_{\mu\nu}^i c(dx^\mu) c(dx^\nu) \hat{\otimes} F_i \\ &= \frac{1}{2} \omega_{\mu\nu}^i (c(dx^\mu) c(dx^\nu) + c(dx^\nu) c(dx^\mu)) \hat{\otimes} F_i \\ &= -g^{\mu\nu} \omega_{\mu\nu}^i \hat{\otimes} F_i \\ &= -\text{tr}(w^i) \hat{\otimes} F_i \\ &=: \mathbb{I}_{C(M)} \hat{\otimes} \Phi \end{aligned} \tag{5.5}$$

with $\Phi := \sum_i -\text{tr}(\omega^i) \cdot F_i \in \Gamma(\text{End}_{\bar{C}(M)}(\mathcal{E}))$. Now let $\nabla^\mathcal{E}$ be a Clifford connection and let us define $\tilde{\nabla}^\mathcal{E} := \nabla^\mathcal{E} + \omega$. We obtain $\tilde{D} = c(\tilde{\nabla}) = c(\nabla^\mathcal{E}) + c(\omega) = D_{\nabla^\mathcal{E}} + c(\omega)$ for the corresponding Dirac operator and the above computation (5.5) proves the second statement.

□

Note that $\tilde{\nabla}^\mathcal{E}$ is not a Clifford connection unless $\omega = 0$. In the case of a twisted spinor bundle $\mathcal{E} = S \otimes E$ the corresponding Clifford superconnection A is given by

$$A := \nabla^S \otimes \mathbb{I}_E + \mathbb{I}_S \otimes (\nabla^E + \Phi),$$

i.e. A is uniquely determined by the superconnection $A^E := \nabla^E + \Phi$ on the twisting bundle E .

We now study the decomposition formula (3.14) for the square of such Dirac operators which are distinguished by lemma 5.3 resp. corollary 5.4. Let therefore again $\omega \in \Omega^1(M, \text{End}^+(\mathcal{E}))$ be given by $\omega := w^i \hat{\otimes} F_i$ with $w^i := \omega_{\mu\nu}^i dx^\mu \otimes c(dx^\nu) \in \Gamma(\text{Sym}^2(T^*M)) \subset \Gamma(T^*M \otimes C^-(M))$ and $F_i \in \Gamma(\text{End}_{\bar{C}(M)}(\mathcal{E}))$. Then we find that $[c(dx^\mu), c(dx^\nu)][\omega_\mu, \omega_\nu] = 0$. Thus, this term does not contribute to the endomorphism part $F_{\tilde{\nabla}^\mathcal{E}}$ in (3.12). Using this, straightforward computation yields

$$\tilde{D}^2 = \Delta^{\nabla^\mathcal{E}} + \frac{r_M}{4} + c(R_{\nabla^\mathcal{E}}^{\mathcal{E}/S}) + c \nabla^{\text{End}\mathcal{E}}(\mathbb{I}_{C(M)} \hat{\otimes} \Phi) + \mathbb{I}_{C(M)} \hat{\otimes} \Phi^2 \quad (5.6)$$

which can be seen as a characteristic feature of Dirac operators of simple type. Now, if one takes the connection laplacian $\Delta^{\nabla^\mathcal{E}}$ together with the Lichnerowicz part $\frac{r_M}{4} + c(R_{\nabla^\mathcal{E}}^{\mathcal{E}/S})$, then - by using (3.1) - we obtain the

COROLLARY 5.5. *A Dirac operator \tilde{D} acting on sections of a Clifford module \mathcal{E} is of simple type iff there exists a Clifford connection $\nabla^\mathcal{E}$ on \mathcal{E} and a morphism $\Phi \in \Gamma(\text{End}_{\bar{C}(M)}(\mathcal{E}))$ such that $\tilde{D}^2 = D_{\nabla^\mathcal{E}}^2 + c \nabla^{\text{End}\mathcal{E}}(\mathbb{I}_{C(M)} \hat{\otimes} \Phi) + \mathbb{I}_{C(M)} \hat{\otimes} \Phi^2$.*

We note that the endomorphism part in the decomposition formula (5.6) of the square of a Dirac operator of simple type is of Clifford degree ≤ 2 . Getzler has recognized in [G], that this is the essential information needed to prove the local Atiyah-Singer index theorem for Dirac operators associated with Clifford connections, which is provided in this case by the 'usual' Lichnerowicz formula (3.1). As

far as we know it has not yet been proven whether this refined index theorem also holds for some Dirac operator not associated to a Clifford connection. However, because of our observation mentioned above and our formula (5.6) instead of the usual Lichnerowicz formula (3.1), the techniques of [BGV], chapter 4, can be adapted even to all Dirac operators of simple type. So we state:

THEOREM 5.6. *For Dirac operators of simple type acting on sections of a Clifford module \mathcal{E} the local Atiyah-Singer index theorem holds.*

6. The Wodzicki function

In this section, we define the Wodzicki function $W_{\mathcal{E}}$ on the space of all Dirac operators $\mathcal{D}(\mathcal{E})$ acting on sections of a Clifford module \mathcal{E} via the non-commutative residue, which has been studied extensively by Guillemin and Wodzicki (cf. [Gu], [W1], [W2]). This function is closely related to (gravity-) action functionals of physics as already mentioned in the introduction and might also be useful to investigate the space of all Dirac operators. This will be the subject of a forthcoming paper. As will be seen, the generalized Lichnerowicz formula that we have derived in section 3 also applies to calculate the Wodzicki function explicitly.

Let E and F be finite dimensional complex vector bundles over a compact, n -dimensional manifold N . The non-commutative residue of a pseudo-differential operator $P \in \Psi\text{DO}(E, F)$ can be defined by

$$\text{res}(P) := \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \int_{S^*M} \text{tr}(\sigma_{-n}^P(x, \xi)) \, dx d\xi, \quad (6.1)$$

where $S^*M \subset T^*M$ denotes the co-sphere bundle on M and σ_{-n}^P is the component of order $-n$ of the complete symbol $\sigma^P := \sum_i \sigma_i^P$ of P . Here the integral is normalized by $\text{vol}(S^{n-1})^{-1} = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}}$. In his thesis, Wodzicki has shown that the linear functional $\text{res}: \Psi\text{DO}(E, F) \rightarrow \mathbb{C}$ is in fact the unique trace (up to multiplication by constants) on the algebra of pseudo-differential operators $\Psi\text{DO}(E, F)$.

Now let M be a Riemannian manifold of even dimension $2n$ and let \mathcal{E} be a Clifford module over M . If D is a Dirac operator acting on sections of \mathcal{E} and $A \in$

$\Gamma(\text{End}^-(\mathcal{E}))$, then $\tilde{D} := D + A$ is another Dirac operator corresponding to the same Clifford action on \mathcal{E} . As we have already mentioned (cf. section 2) the converse is also true: Any two Dirac operators D_0 and D_1 differ by a section $A \in \Gamma(\text{End}^-(\mathcal{E}))$. Consequently the set of all Dirac operators $\mathcal{D}(\mathcal{E})$ corresponding to the same Clifford action on a Clifford module \mathcal{E} is an affine space modelled on $\Gamma(\text{End}^-(\mathcal{E}))$. Hence, for any Dirac operator $D \in \mathcal{D}(\mathcal{E})$ we have the natural identifications

$$\Gamma(\text{End}^-(\mathcal{E})) \cong T_D \mathcal{D}(\mathcal{E}) \cong \mathcal{D}(\mathcal{E}). \quad (6.2)$$

DEFINITION 6.1. *Let \mathcal{E} be a Clifford module of rank r over a Riemannian manifold M with $\dim M = 2n$ and let $\mathcal{D}(\mathcal{E})$ be the space of all Dirac operators on \mathcal{E} . The Wodzicki function $W_{\mathcal{E}}$ on $\mathcal{D}(\mathcal{E})$ is the complex-valued function $W_{\mathcal{E}}: \mathcal{D}(\mathcal{E}) \rightarrow \mathbb{C}$ defined by $W_{\mathcal{E}}(D) := -\frac{2}{r(2n-1)} \text{res}(D^{-2n+2})$.*

Notice that in the case of fixing a hermitian inner product $(\cdot, \cdot)_{\Gamma(\mathcal{E})}$ on $\Gamma(\mathcal{E})$ one can introduce the (formal) adjoint operator P^* of $P \in \Psi\text{DO}(\mathcal{E})$. Since $\text{res}(P^*) = \overline{\text{res}(P)}$ (cf. [W2]), where the bar denotes complex conjugation, the Wodzicki function $W_{\mathcal{E}}$ is real for self-adjoint Dirac operators.

As an intermediate step to compute the Wodzicki function $W_{\mathcal{E}}$ explicitly, we need the following expression for the diagonal part $\phi_1(x, x, \tilde{D}^2)$ of the subleading term ϕ_1 in the heat-kernel expansion of the square of an arbitrary Dirac operator \tilde{D} :

LEMMA 6.2. *Let \mathcal{E} be a Clifford module over an even-dimensional compact Riemannian manifold M and let $\tilde{D} = c \circ \tilde{\nabla}^{\mathcal{E}}$ be a Dirac operator. Then the diagonal part of the subleading term ϕ_1 in the asymptotic expansion of the heat-kernel of \tilde{D}^2 is given by*

$$\phi_1(x, x, \tilde{D}^2) = \frac{1}{6} r_M(x) - \left(c(R^{\tilde{\nabla}^{\mathcal{E}}}) + \text{ev}_g \tilde{\nabla}^{T^*M \otimes \text{End } \mathcal{E}} \varpi_{\tilde{\nabla}^{\mathcal{E}}} + \text{ev}_g(\varpi_{\tilde{\nabla}^{\mathcal{E}}} \cdot \varpi_{\tilde{\nabla}^{\mathcal{E}}}) \right)(x),$$

with $\varpi_{\tilde{\nabla}^{\mathcal{E}}} := -\frac{1}{2} g_{\nu\kappa} dx^\nu \otimes c(dx^\kappa) ([\tilde{\nabla}_\mu^{\mathcal{E}}, c(dx^\kappa)] + c(dx^\sigma) \Gamma_{\sigma\mu}^\kappa) \in \Omega^1(M, \text{End } \mathcal{E})$.

PROOF: Let $\hat{\Delta}$ be a generalized laplacian acting on sections of a hermitian vector bundle E over M and let $\phi_1(x, x, \hat{\Delta})$ denote the diagonal part of the subleading term ϕ_1 in the heat-kernel expansion of $\hat{\Delta}$. It is well-known that

$$\phi_1(x, x, \hat{\Delta}) = \frac{1}{6} r_M(x) \cdot \mathbb{I}_E - F(x). \quad (6.3)$$

Again, r_M denotes the scalar curvature of M and $F \in \Gamma(\text{End } E)$ is determined by the unique decomposition of the generalized laplacian $\hat{\Delta}$:

$$\hat{\Delta} = \Delta^{\hat{\nabla}^E} + F \quad (6.4)$$

Here $\Delta^{\hat{\nabla}^E}$ denotes the connection laplacian associated with the connection $\hat{\nabla}^E$ on the hermitian bundle E .

In our case we have $E = \mathcal{E}$ and $\hat{\Delta} = \tilde{D}^2$. Using the generalized Lichnerowicz formula (3.14) the further proof of this lemma is obvious.

□

REMARK 6.3. For an arbitrary generalized laplacian $\hat{\Delta}$ on a hermitian vector bundle E it is only the decomposition (6.4) which is proven to exist - neither the connection $\hat{\nabla}^E$ nor the endomorphism $F \in \Gamma(\text{End } E)$ are known explicitly (cf. [BGV], Proposition 2.5). Consequently in general the subleading term $\phi_1(x, x, \hat{\Delta})$ in the heat-kernel expansion of $\hat{\Delta}$ can not be computed with the help of (6.4). Therefore lemma 6.2 is also interesting on its own.

According to [KW] the diagonal part $\phi_1(x, x, \hat{\Delta})$ of the subleading term in the heat-kernel expansion of any generalized laplacian $\hat{\Delta}$ acting on sections of a hermitian vector bundle E over an even dimensional Riemannian manifold M with $\dim M = 2n \geq 4$ can be related to the non-commutative residue

$$\text{res}(\hat{\Delta}^{-n+1}) = \frac{2n-1}{2} \int_M *tr(\phi_1(x, x, \hat{\Delta})). \quad (6.6)$$

Here $*$ denotes the Hodge-star operator defined by the Riemannian metric on M . Together with the previous lemma we therefore obtain:

THEOREM 6.4. *Let \mathcal{E} be a Clifford module over a compact, Riemannian manifold M with $\dim M = 2n \geq 4$ and let $\tilde{D} = c \circ \tilde{\nabla}^{\mathcal{E}}$ be a Dirac operator. Then*

$$W_{\mathcal{E}}(\tilde{D}) = \int_M * \left(-\frac{1}{6} r_M + \frac{1}{r} tr(c(R^{\tilde{\nabla}^{\mathcal{E}}}) + ev_g \tilde{\nabla}^{T^*M \otimes \text{End } \mathcal{E}} \varpi_{\tilde{\nabla}^{\mathcal{E}}} + ev_g(\varpi_{\tilde{\nabla}^{\mathcal{E}}} \cdot \varpi_{\tilde{\nabla}^{\mathcal{E}}})) \right),$$

with $\varpi_{\tilde{\nabla}^{\mathcal{E}}} := -\frac{1}{2} g_{\nu\kappa} dx^\nu \otimes c(dx^\kappa) ([\tilde{\nabla}_\mu^{\mathcal{E}}, c(dx^\kappa)] + c(dx^\sigma) \Gamma_{\sigma\mu}^\kappa) \in \Omega^1(M, \text{End } \mathcal{E})$.

Now let (\mathcal{E}_1, c_1) and (\mathcal{E}_2, c_2) be Clifford modules of rank r_1 resp. r_2 over M . Here $c_i: C(M) \times \mathcal{E}_i \rightarrow \mathcal{E}_i$ for $i = 1, 2$ denotes the respective Clifford action of the Clifford bundle $C(M)$. Obviously the direct sum bundle $\mathcal{E} := \mathcal{E}_1 \oplus \mathcal{E}_2$ together with the Clifford action $c := c_1 \oplus c_2$ is also a Clifford module. Given Dirac operators $\tilde{D}_i := c_i \circ \tilde{\nabla}^{\mathcal{E}_i}$ with $i = 1, 2$ corresponding to the connections $\tilde{\nabla}^{\mathcal{E}_i}$ on \mathcal{E}_i and $a^{21} \in \Omega^1(M, \text{Hom}(\mathcal{E}_2, \mathcal{E}_1))$ respectively $a^{12} \in \Omega^1(M, \text{Hom}(\mathcal{E}_1, \mathcal{E}_2))$, the following operator

$$\tilde{D} := \begin{pmatrix} \tilde{D}_1 & c_1 \circ a^{21} \\ c_2 \circ a^{12} & \tilde{D}_2 \end{pmatrix}, \quad (6.7)$$

is a Dirac operator on $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$.

We now want to calculate the Wodzicki function $W_{\mathcal{E}} = W_{\mathcal{E}_1 \oplus \mathcal{E}_2}$ of this Dirac operator \tilde{D} . Given that $\tilde{\nabla}^{\mathcal{E}} := \tilde{\nabla}^{\mathcal{E}_1} \oplus \tilde{\nabla}^{\mathcal{E}_2}$ be the direct sum connection, an easy computation shows

$$\varpi_{\tilde{\nabla}^{\mathcal{E}}} = \begin{pmatrix} \varpi_{\tilde{\nabla}^{\mathcal{E}_1}} & b^{21} \\ b^{12} & \varpi_{\tilde{\nabla}^{\mathcal{E}_2}} \end{pmatrix} \in \Omega^1(M, \text{End } \mathcal{E}), \quad (6.8)$$

with $b^{ji} := -\frac{1}{2} dx^\nu \otimes c_i(dx^\mu)(a_\mu^{ji} c_j(dx^\nu) - c_i(dx^\nu) a_\mu^{ji}) \in \Omega^1(M, \text{Hom}(\mathcal{E}_j, \mathcal{E}_i))$ for $i, j \in \{1, 2\}$ and $i \neq j$. Denoting $a := \begin{pmatrix} 0 & a^{21} \\ a^{12} & 0 \end{pmatrix}$ and $b := \begin{pmatrix} 0 & b^{21} \\ b^{12} & 0 \end{pmatrix}$, so $a, b \in \Omega^1(M, \text{End } \mathcal{E})$, we further obtain:

$$\begin{aligned} \text{tr}(ev_g(\varpi_{\tilde{\nabla}^{\mathcal{E}}} \cdot \varpi_{\tilde{\nabla}^{\mathcal{E}}})) &= \text{tr}(ev_g(\varpi_{\tilde{\nabla}^{\mathcal{E}_1}} \cdot \varpi_{\tilde{\nabla}^{\mathcal{E}_1}})) + \text{tr}(ev_g(\varpi_{\tilde{\nabla}^{\mathcal{E}_2}} \cdot \varpi_{\tilde{\nabla}^{\mathcal{E}_2}})) \\ &\quad + \text{tr}(ev_g(b \cdot b)) \end{aligned} \quad (6.9)$$

$$\begin{aligned} \text{tr}(ev_g(\tilde{\nabla}^{T^*M \otimes \text{End } \mathcal{E}} \varpi_{\tilde{\nabla}^{\mathcal{E}}})) &= \text{tr}(ev_g(\tilde{\nabla}^{T^*M \otimes \text{End } \mathcal{E}_1} \varpi_{\tilde{\nabla}^{\mathcal{E}_1}})) \\ &\quad + \text{tr}(ev_g(\tilde{\nabla}^{T^*M \otimes \text{End } \mathcal{E}_2} \varpi_{\tilde{\nabla}^{\mathcal{E}_2}})) \end{aligned} \quad (6.10)$$

and

$$\begin{aligned} \text{tr}(c(R^{\tilde{\nabla}^{\mathcal{E}}})) &= \text{tr}(c_1(R^{\tilde{\nabla}^{\mathcal{E}_1}})) + \text{tr}(c_2(R^{\tilde{\nabla}^{\mathcal{E}_2}})) \\ &\quad + \text{tr}(c([a \wedge a])). \end{aligned} \quad (6.11)$$

Here $[\cdot \wedge \cdot]$ denotes the multiplication in the graded Lie algebra $\Omega^*(M, \text{End } \mathcal{E})$.

Thus we are able to prove the following

LEMMA 6.5. *Let (\mathcal{E}_1, c_1) and (\mathcal{E}_2, c_2) be Clifford modules of rank r_1 resp. r_2 over a compact Riemannian manifold M of even dimension with $2n \geq 4$ and let $\tilde{D}_1 := c_1 \circ \tilde{\nabla}^{\mathcal{E}_1}$ resp. $\tilde{D}_2 := c_2 \circ \tilde{\nabla}^{\mathcal{E}_2}$ be Dirac operators. Then $\tilde{D} := \begin{pmatrix} \tilde{D}_1 & c_1 \circ a^{21} \\ c_2 \circ a^{12} & \tilde{D}_2 \end{pmatrix}$ with $a := \begin{pmatrix} 0 & a^{21} \\ a^{12} & 0 \end{pmatrix} \in \Omega^1(M, \text{End } \mathcal{E})$ defines a Dirac operator on $\mathcal{E} := \mathcal{E}_1 \oplus \mathcal{E}_2$ and*

$$W_{\mathcal{E}}(\tilde{D}) = \frac{r_1}{r} W_{\mathcal{E}_1}(\tilde{D}_1) + \frac{r_2}{r} W_{\mathcal{E}_2}(\tilde{D}_2) + \frac{1}{r} \int_M \text{tr} \left(c([a \wedge a]) + \text{ev}_g \begin{pmatrix} b^{21}, b^{12} & 0 \\ 0 & b^{12}, b^{21} \end{pmatrix} \right)$$

with $b^{ji} := -\frac{1}{2} g_{\kappa\nu} dx^\kappa \otimes c_i(dx^\mu)(a_\mu^{ji} c_j(dx^\nu) - c_i(dx^\nu) a_\mu^{ji}) \in \Omega^1(M, \text{Hom}(\mathcal{E}_j, \mathcal{E}_i))$ for $i, j \in \{1, 2\}$, $i \neq j$ and $r := (r_1 + r_2)$.

7. The Wodzicki function for special Dirac operators

We now apply theorem 6.4 resp. lemma 6.5 to calculate the Wodzicki function of different types of Dirac operators. These examples are inspired by physics, where they might be used to derive gravity and combined gravity/Yang-Mills actions, respectively (cf. [K], [KW], [AT1], [AT2]). In the following, let M be a compact Riemannian manifold with $\dim M = 2n \geq 4$.

• **Dirac operators of simple type.** Let \mathcal{E} be a Clifford module of rank r over M and \tilde{D} a Dirac operator of simple type acting on sections of \mathcal{E} . By corollary 5.5 there exists a Clifford connection $\nabla^{\mathcal{E}}$ on \mathcal{E} and an endomorphism $\Phi \in \text{End}_{\bar{C}(M)}(\mathcal{E})$, such that we obtain the decomposition formula (5.6) for the square of \tilde{D} :

$$\tilde{D}^2 = \Delta^{\nabla^{\mathcal{E}}} + \frac{r_M}{4} + c(R_{\nabla^{\mathcal{E}}}^{\mathcal{E}/S}) + c \nabla^{\text{End } \mathcal{E}} (\mathbb{I}_{C(M)} \hat{\otimes} \Phi) + \mathbb{I}_{C(M)} \hat{\otimes} \Phi^2.$$

Using theorem 6.4 we then calculate

$$\begin{aligned} W_{\mathcal{E}}(\tilde{D}) &= \int_M * \left(-\frac{1}{6} r_M + \frac{1}{r} \text{tr} \left(\frac{r_M}{4} \mathbb{I}_{\mathcal{E}} + c(R_{\nabla^{\mathcal{E}}}^{\mathcal{E}/S}) + c \nabla^{\text{End } \mathcal{E}} (\mathbb{I}_{C(M)} \hat{\otimes} \Phi) + \mathbb{I}_{C(M)} \hat{\otimes} \Phi^2 \right) \right) \\ &= \int_M * \left(\frac{1}{12} r_M + \frac{2^n}{r} \text{tr}(\Phi^2) \right) \end{aligned}$$

for the Wodzicki function, since both $\text{tr}(c(R_{\nabla^\mathcal{E}}^{\mathcal{E}/S}))$ and $\text{tr}(c\nabla^{\text{End } \mathcal{E}}(\mathbb{I}_{C(M)} \hat{\otimes} \Phi))$ vanish.

Now let us take the zero-morphism for Φ . In that exceptional case \tilde{D} is associated with the Clifford connection $\nabla^\mathcal{E}$. Therefore we obtain $W_\mathcal{E}(\tilde{D}) = \frac{1}{12} \int_M *r_M$. Thus, the Wodzicki function evaluated on Dirac operators corresponding to Clifford connections reproduce the classical Einstein-Hilbert functional on M . This was already mentioned by Connes and explicitly shown in [K] and [KW]. In the case of M being a spin-manifold and $\mathcal{E} = S \otimes E$ with $E := M \times \mathbb{C}^2 = (M \times \mathbb{C}) \oplus (M \times \mathbb{C})$, then $\text{End } \mathcal{E} = C(M) \hat{\otimes} M_2(\mathbb{C})$. Here $M_2(\mathbb{C})$ denotes the algebra of two by two matrices over the complex numbers. Interestingly, we thus recover the situation of a ‘non-commutative two-point space’ as considered in [CFF] and [KW]. Further specializing the Dirac operator of simple type by $\Phi := \phi \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, where $\phi \in C^\infty(M)$ denotes a complex-valued function as in the mentioned references, we obtain

$$W_\mathcal{E}(\tilde{D}) = \int_M *(\frac{1}{12}r_M + \phi^2). \quad (7.1)$$

This was interpreted in [KW] as Einstein-Hilbert gravity action with cosmological constant.

• **Dirac operators with torsion.** Let now M be a spin manifold in addition. The Levi-Civita connection $\nabla: \Gamma(TM) \rightarrow \Gamma(T^*M \otimes TM)$ on M induces a connection $\nabla^S: \Gamma(S) \rightarrow \Gamma(T^*M \otimes S)$ on the spinor bundle S which is compatible with the hermitian metric $\langle \cdot, \cdot \rangle_S$ on S . Adding a torsion term $t \in \Omega^1(M, \text{End } TM)$ to the Levi-Civita connection, we obtain a new covariant derivative $\tilde{\nabla} := \nabla + t$ on the tangent bundle TM . Since t is really a one-form on M with values in the bundle of skew endomorphism $Sk(TM)$ (cf. [GHV]), $\tilde{\nabla}$ is in fact compatible with the Riemannian metric g and therefore it also induces a connection $\tilde{\nabla}^S = \nabla^S + T$ on the spinor bundle. Here $T \in \Omega^1(M, \text{End } S)$ denotes the ‘lifted’ torsion term $t \in \Omega^1(M, \text{End } TM)$. However, in general this induced connection $\tilde{\nabla}^S$ is neither compatible with the hermitian metric $\langle \cdot, \cdot \rangle_S$ nor is it a Clifford connection.

The most general Dirac operator on the spinor bundle S corresponding to a metric connection $\tilde{\nabla}$ on TM can be defined by $\tilde{D} := c \circ \tilde{\nabla}^S$. Thus the Wodzicki function

$W_{\mathcal{E}}$ of \tilde{D} yields

$$W_{\mathcal{E}}(\tilde{D}) = \int_M * \left(\frac{1}{12} r_M + \frac{1}{2^n} (-t_{abc} t^{abc} + 2t_{abc} t^{acb}) \right). \quad (7.2)$$

From a physical point of view, this can be interpreted as the action functional of a modified Einstein-Cartan theory (cf. [AT1]).

• **An Einstein-Yang-Mills Dirac operator.** Let (\mathcal{E}, c) be a Clifford module of rank $r = 2^n \cdot rk(\mathcal{E}/S)$, where \mathcal{E}/S denotes the twisting part of \mathcal{E} . As already mentioned, the direct sum $\tilde{\mathcal{E}} := \mathcal{E} \oplus \mathcal{E}$ is also a Clifford module. Given a Clifford connection $\nabla^{\mathcal{E}}$ on \mathcal{E} , obviously $\tilde{\nabla}^{\tilde{\mathcal{E}}} := \begin{pmatrix} \nabla^{\mathcal{E}} & 0 \\ 0 & \nabla^{\mathcal{E}} \end{pmatrix}$ defines a Clifford connection on $\tilde{\mathcal{E}}$. Define

$$\tilde{\nabla}^{\tilde{\mathcal{E}}} := \nabla^{\tilde{\mathcal{E}}} + \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \quad (7.3)$$

with $a := dx^{\mu} \otimes c(dx^{\nu}) \hat{\otimes} R_{\mu\nu} \in \Omega^1(M, \text{End} \mathcal{E})$. Here $R = R_{\nabla^{\mathcal{E}}}^{\mathcal{E}/S} \in \Omega^2(M, \text{End}_{C(M)}(\mathcal{E}))$ denotes the twisting curvature of $\nabla^{\mathcal{E}}$. We now consider the associated Dirac operator

$$\tilde{D} := \tilde{c} \circ \tilde{\nabla}^{\tilde{\mathcal{E}}} = \begin{pmatrix} D_{\nabla^{\mathcal{E}}} & c \circ a \\ c \circ (-a) & D_{\nabla^{\mathcal{E}}} \end{pmatrix}. \quad (7.4)$$

An easy computation yields $\varpi_{\nabla^{\mathcal{E}}} = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$. Thus, with regard to our lemma 6.5 we obtain

$$\begin{aligned} tr \tilde{c} \left(\left[\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \right] \right) &= -2^n tr \begin{pmatrix} R_{\mu\nu} R^{\mu\nu} & 0 \\ 0 & R_{\mu\nu} R^{\mu\nu} \end{pmatrix} \\ tr ev_g \begin{pmatrix} -a \cdot a & 0 \\ 0 & -a \cdot a \end{pmatrix} &= -2^n tr \begin{pmatrix} R_{\mu\nu} R^{\mu\nu} & 0 \\ 0 & R_{\mu\nu} R^{\mu\nu} \end{pmatrix}. \end{aligned} \quad (7.5)$$

When we use lemma 6.5 together with our first example, the Wodzicki function $W_{\tilde{\mathcal{E}}}$ for the above defined Dirac operator \tilde{D} yields

$$W_{\tilde{\mathcal{E}}}(\tilde{D}) = \int_M * \left(\frac{1}{12} r_M - \frac{2}{rk(\mathcal{E}/S)} tr(R_{\mu\nu} R^{\mu\nu}) \right). \quad (7.6)$$

Thus we recover the combined Einstein-Hilbert/Yang-Mills action. Note, that this example can also be understood in the sense of a non-commutative two-point space (compare the first example).

Although being a gauge theory, it is well-known that the classical theory of gravity as enunciated by Einstein stands apart from the non-abelian gauge field theory of

Yang and Mills, which encompasses the three other fundamental forces: the weak, strong and electromagnetic interactions. Interesting, as this example shows, they both have a common ‘root’: the special Dirac operator \tilde{D} considered above. We hope that this may shed new light on the gauge theories in question. Moreover, in a physical sense the above derivation of (7.6) via the Wodzicki function $W_{\mathcal{E}}$ can be understood as unification of Einstein’s gravity- and Yang-Mills theory as it is shown in [AT2].

8. Conclusion

In this paper we studied Dirac operators acting on sections of a Clifford module \mathcal{E} over a Riemannian manifold M . We obtained an intrinsic decomposition of their squares, which is the generalisation of the well-known Lichnerowicz formula [L]. This enabled us to distinguish Dirac operators of simple type. For each operator of this natural class the local Atiyah-Singer index theorem was shown to hold. Furthermore we defined a complex-valued function $W_{\mathcal{E}}: \mathcal{D}(\mathcal{E}) \rightarrow \mathbb{C}$ on the space of all Dirac operators on \mathcal{E} , the Wodzicki function, via the non-commutative residue. If M is compact and $\dim M = 2n \geq 4$, we derived an expression for $W_{\mathcal{E}}$ in theorem 6.4. For certain Dirac operators we calculated this function explicitly. From a physical point of view, this provides a method to reproduce gravity, resp. combined gravity/Yang-Mills actions out of the Dirac operators in question. Therefore we expect new insights in the interrelation of Einstein’s gravity and Yang-Mills gauge theories.

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