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YANG-MILLS AND DIRAC FIELDS IN THE MINKOWSKI SPACE-TIME

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Abstract

An existence and uniqueness theorem for the Cauchy problem for the evolution component of the coupled Yang-Mills and Dirac equations in the Minkowski space is proved in a Sobolev space for the temporal gauge condition. The constraint set $\mathbf{C}$ is shown to be a smooth submanifold of $\mathbf{P}$ preserved by the evolution.

The Lie algebra $gs(\mathbf{P})$ of infinitesimal gauge symmetries of $\mathbf{P}$ is identified. Its topology is of Beppo Levi type. The group $GS(\mathbf{P})$ of gauge symmetries is of Lie type; its topology is induced by the topology of its Lie algebra.

The constraint equations define a closed ideal $gs(\mathbf{P})_0$ of $gs(\mathbf{P})$. It generates a closed connected subgroup $GS(\mathbf{P})_0$ of $GS(\mathbf{P})$, which is shown to act properly in $\mathbf{P}$. The reduced phase space is the space of $GS(\mathbf{P})_0$ orbits in the constraint set $\mathbf{C}$. It is a smooth quotient manifold of $\mathbf{C}$ endowed with an exact symplectic form.

The quotient group $GS(\mathbf{P})/GS(\mathbf{P})_0$ is isomorphic to the structure group G of the theory. Its action in the reduced phase space is Hamiltonian. The associated conserved quantities are colour charges. Only the charges corresponding to the centre of the Lie algebra $gs(\mathbf{P})/gs(\mathbf{P})_0$ admit well defined local charge densities.

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1. Introduction.

An understanding of the quantum theory of a physical system is greatly facilitated by the knowledge of the classical phase space of the system. The present state of knowledge of classical Yang-Mills theory, comprehensively reviewed in [1], consists of an impressive array of results obtained under a great variety of assumptions. A systematic study of the theory under a consistent set of assumptions is lacking. This paper aims at filling this gap, at least partially, for minimally interacting Yang-Mills and Dirac fields in the Minkowski space-time.

The main stage of the Hamiltonian formulation of field theory is its extended phase space. Its choice determines the rest of the structure of the theory. Let us consider criteria we could use in order to make a "correct" choice. An obvious criterion is absence of the second class constraints, [2,3]. Since the extended phase space consists of the Cauchy data for the field equations, we require the existence and uniqueness theorems for the evolution equations. Moreover, gauge invariance implies uniqueness of solutions up to a gauge transformation, hence the uniqueness results require an imposition of a gauge condition.

Thus, an extended phase space depends on the choice of the Cauchy surface, the choice of a gauge condition, and the choice of the function space in which we can prove the existence and uniqueness theorems. Existence and uniqueness problems for the full set of Yang-Mills equations in the Minkowski space-time have been studied in several papers, [4-8]. We shall use the phase space of [6] and [9], which we enlarge to include the Dirac fields.

Let G be the structure group of the theory, presented as a matrix group, and $\mathfrak{g}$ its Lie algebra. We assume that G is compact, connected and that $\mathfrak{g}$ is endowed with an ad-invariant metric. The usual $(3+1)$ splitting of Minkowski space $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$ leads to the splitting of the Yang-Mills field

$$A_\mu = (A_0, \mathbf{A}) \tag{1.1}$$

into the scalar potential A_0 and the vector potential $\mathbf{A} = A_i dx^i$. We choose the temporal gauge condition

$$A_0 = 0\,. \tag{1.2}$$

It leads to a representation of the field strength $F_{\mu\nu}$ in terms of the "electric" field $\mathbf{E}$ and the "magnetic" field $\mathbf{B}$ with components

$$B_j = \tfrac{1}{2}\epsilon_j{}^{kl} F_{kl} = (\mathrm{curl}\,\mathbf{A})_j + [\mathbf{A}\times, \mathbf{A}]_j\ , \tag{1.3}$$

where we use the Euclidean metric in $\mathbb{R}^3$ to identify vector fields and forms, and $\times$ denotes the cross product.

The field equations split into the evolution equations

$$\partial_t \mathbf{A} = \mathbf{E} \ , \tag{1.4}$$

$$\partial_t \mathbf{E} = \operatorname{curl} \mathbf{B} - [\mathbf{A}\times, \mathbf{B}] + \mathbf{j} \ , \tag{1.5}$$

$$\partial_t \Psi = -\gamma^0(\gamma^j \partial_j + im + \gamma^j \mathbf{A}_j)\Psi \ , \tag{1.6}$$

and the constraint equation

$$\operatorname{div} \mathbf{E} + [\mathbf{A}; \mathbf{E}] = j^0 . \tag{1.7}$$

Here $\mathbf{A}$, $\mathbf{E}$, and $\mathbf{B}$ are treated as time dependent vector fields on $\mathbb{R}^3$ with values in the structure algebra $\mathfrak{g}$ of the theory, and Ψ is a time dependent spinor field with values in the space of the fundamental representation of G. Moreover $[\mathbf{A}; \mathbf{E}]$ means the Lie algebra bracket contracted over the vector indices. The source terms are given in terms of a basis $\{T^a\}$ of the Lie algebra $\mathfrak{g}$ by

$$j^0 = \Psi^\dagger (I \otimes T^a) \Psi T_a \qquad \text{and} \qquad j^k = \Psi^\dagger (\gamma^0 \gamma^k \otimes T^a) \Psi T_a \ . \tag{1.8}$$

In order to handle the non-linear terms in the evolution equations we choose $\mathbf{A}$ and Ψ in the Sobolev space $H^2(\mathbb{R}^3)$, and $\mathbf{E}$ in $H^1(\mathbb{R}^3)$, [10]. Hence, we are lead to the extended phase space

$$\mathbf{P} = \{(\mathbf{A}, \mathbf{E}, \Psi) | \mathbf{A} \in H^2(\mathbb{R}^3), \ \mathbf{E} \in H^1(\mathbb{R}^3), \ \Psi \in H^2(\mathbb{R}^3)\} \ , \tag{1.9}$$

in which we establish the existence and uniqueness of solutions of the Yang-Mills and Dirac system given by Eqs. (1.4) - (1.7). It is endowed with a one form θ defined by

$$\langle \theta(\mathbf{A}, \mathbf{E}, \Psi) | (\delta\mathbf{A}, \delta\mathbf{E}, \delta\Psi) \rangle = \int_{\mathbb{R}^3} (\mathbf{E} \cdot \delta\mathbf{A} + \Psi^\dagger \delta\Psi) d_3x \ , \tag{1.10}$$

where $\mathbf{E} \cdot \delta\mathbf{A}$ means the contraction in both, the vector and the Lie algebra indices. The exterior differential of θ

$$\omega = d\theta \tag{1.11}$$

is a weakly symplectic form on $\mathbf{P}$.

Following the approach of [11], we define the group $GS(\mathbf{P})$ of gauge symmetries of the phase space $\mathbf{P}$ as a connected group of gauge transformations

$$\mathbf{A} \mapsto \phi \mathbf{A} \phi^{-1} + \phi \operatorname{grad} \phi^{-1} \ , \quad \mathbf{E} \mapsto \phi \mathbf{E} \phi^{-1} \ , \quad \Psi \mapsto \phi \Psi \ , \tag{1.12}$$

where ϕ is a map from $\mathbb{R}^3$ to the structure group G, which preserve $\mathbf{P}$.

The infinitesimal action of the elements ξ of the Lie algebra $gs(\mathbf{P})$ of $GS(\mathbf{P})$ is given by

$$\mathbf{A} \mapsto \mathbf{A} - D_{\mathbf{A}}\xi \ , \quad \mathbf{E} \mapsto \mathbf{E} - [\mathbf{E}, \xi] \ , \quad \Psi \mapsto \Psi + \xi\Psi \ , \tag{1.13}$$

where

$$D_{\mathbf{A}}\xi = \mathrm{grad}\,\xi + [\mathbf{A}, \xi] \tag{1.14}$$

is the covariant differential of ξ with respect to the connection defined by $\mathbf{A}$. It gives rise to a vector field $\xi_{\mathbf{P}}$ on $\mathbf{P}$ such that

$$\xi_{\mathbf{P}}(\mathbf{A}, \mathbf{E}, \Psi) = -(D_{\mathbf{A}}\xi)\frac{\delta}{\delta \mathbf{A}} - [\mathbf{E}, \xi]\frac{\delta}{\delta \mathbf{E}} + \xi\Psi\frac{\delta}{\delta \Psi} \ . \tag{1.15}$$

The topology of $gs(\mathbf{P})$ is determined by the requirement that the action (1.14) should preserve $\mathbf{P}$ and be continuous. We show that

$$gs(\mathbf{P}) = \{\xi : \mathbb{R}^3 \longrightarrow \mathfrak{g} \mid \mathrm{grad}\,\xi \in H^2(\mathbb{R}^3, \mathfrak{g})\} \ , \tag{1.16}$$

so that $gs(\mathbf{P})$ is the intersection of three Beppo Levi spaces, [12]. Following [13] we topologize $gs(\mathbf{P})$ by the norm

$$\|\xi\|_{\mathcal{B}^3} = \int_{\mathbf{B}_1} |\xi| d_3x + \|\mathrm{grad}\,\xi\|_{H^2} \ , \tag{1.17}$$

where $\|\cdot\|_{H^2}$ is the norm of the Sobolev $H^2(\mathbb{R}^3)$, and B_1 is the unit ball in $\mathbb{R}^3$ centred at the origin.

The topology of the Lie algebra $gs(\mathbf{P})$ induces a topology of the group $GS(\mathbf{P})$, cf. [4,14]. With this topology $GS(\mathbf{P})$ is a group of Lie type, that is the exponential map maps a neighbourhood of zero in $gs(\mathbf{P})$ homeomorphically onto a neighbourhood of identity in $GS(\mathbf{P})$. Moreover, the action (1.12) is continuous and proper, which ensures the existence of slices, [15,16].

The action of $GS(\mathbf{P})$ preserves the 1-form θ, Eq. (1.10). Hence, it is Hamiltonian with the equivariant momentum map $J : \mathbf{P} \rightarrow gs(\mathbf{P})'$ such that

$$\langle J(\mathbf{A}, \mathbf{E}, \Psi)|\xi\rangle = \langle \theta | \xi_{\mathbf{P}}(\mathbf{A}, \mathbf{E}, \Psi)\rangle = \int_{\mathbb{R}^3} (-\mathbf{E} \cdot D_{\mathbf{A}}\xi + \Psi^{\dagger}\xi\Psi) d_3x \ , \tag{1.18}$$

where $gs(\mathbf{P})'$ denotes the (topological) dual of $gs(\mathbf{P})$. For each $\xi \in gs(\mathbf{P})$, the momentum corresponding to ξ is a function $J_\xi : \mathbf{P} \rightarrow \mathbb{R}$, given by

$$J_\xi(\mathbf{A}, \mathbf{E}, \Psi) = \langle J(\mathbf{A}, \mathbf{E}, \Psi)|\xi\rangle \ . \tag{1.19}$$

Its Hamiltonian vector field coincides with the vector field $\xi_{\mathbf{P}}$ giving the infinitesimal action of ξ in $\mathbf{P}$,

$$\xi_{\mathbf{P}} \,\lrcorner\, \omega = dJ_\xi \ . \tag{1.20}$$

Integrating by parts on the right hand side of Eq. (1.18) we obtain

$$\langle J(\mathbf{A},\mathbf{E},\Psi)|\xi\rangle = \int_{I\!R^3}\Big((\operatorname{div}\mathbf{E}+[\mathbf{A};\mathbf{E}])\xi+\Psi^\dagger\xi\Psi\Big)d_3x - \int_{S_\infty} n\mathbf{E}\,\xi\,dS\,, \tag{1.21}$$

where

$$\int_{S_\infty} n\mathbf{E}\,\xi\,dS = \lim_{r\to\infty}\int_{S_r} n\mathbf{E}\,\xi\,dS \tag{1.22}$$

is the flux of the normal component of $\mathbf{E}\,\xi$ through the sphere at spatial infinity. Since $\Psi^\dagger\xi\Psi = -j^0\,\xi$ the constraint equation (1.7) is equivalent to

$$\langle J(\mathbf{A},\mathbf{E},\Psi)|\xi\rangle = -\int_{S_\infty} n\mathbf{E}\,\xi\,dS \quad \forall\,\xi\in gs(\mathbf{P})\,. \tag{1.23}$$

The constraint set of the theory consists of all Cauchy data in $\mathbf{P}$ satisfying the constraint equation,

$$\mathbf{C} = \{(\mathbf{A},\mathbf{E},\Psi)\in\mathbf{P}\mid \operatorname{div}\mathbf{E}+[\mathbf{A},\mathbf{E}]=j^0\}\,. \tag{1.24}$$

It is a smooth co-isotropic submanifold of $\mathbf{P}$ preserved by the evolution. We define

$$gs(\mathbf{P})_0 = \{\xi\in gs(\mathbf{P})|\langle J(\mathbf{A},\mathbf{E},\Psi)|\xi\rangle = 0 \quad \forall\,(\mathbf{A},\mathbf{E},\Psi)\in\mathbf{C}\}\,. \tag{1.25}$$

As shown in [11], $gs(\mathbf{P})_0$ is a closed ideal in $gs(\mathbf{P})$. Using the results of [13], we will show that $gs(\mathbf{P})_0$ is the closure in the norm (1.17) of the space of smooth, compactly supported maps from $I\!R^3$ to $\mathfrak{g}$, and $gs(\mathbf{P})$ has a decomposition

$$gs(\mathbf{P}) = gs(\mathbf{P})_0\oplus\mathfrak{g}\,, \tag{1.26}$$

where $\mathfrak{g}$ is interpreted as the spaces of constant maps from $I\!R^3$ to the structure algebra.

$GS(\mathbf{P})_0$ is the closed connected subgroup of $GS(\mathbf{P})$ with the Lie algebra $gs(\mathbf{P})_0$. Its action in $\mathbf{P}$ is Hamiltonian with the momentum map $J_0:\mathbf{P}\to gs(\mathbf{P})'_0$ given by the composition of $J:\mathbf{P}\to gs(\mathbf{P})'$ with the dual of the inclusion map of $gs(\mathbf{P})_0$ into $gs(\mathbf{P})$. By construction, cf. (1.25), the constraint set $\mathbf{C}$ is the zero level of J_0,

$$\mathbf{C} = J_0^{-1}(0)\,. \tag{1.27}$$

The reduced phase space is the space $\check{\mathbf{P}}$ of $GS(\mathbf{P})_0$ orbits in $\mathbf{C}$,

$$\check{\mathbf{P}} = \mathbf{C}/GS(\mathbf{P})_0\,. \tag{1.28}$$

We show that $\check{\mathbf{P}}$ is a Hausdorff manifold and the projection map $\rho:\mathbf{C}\to\check{\mathbf{P}}$ is a submersion. The restrictions of forms θ and ω to $\mathbf{C}$ push forward to forms $\check{\theta}$ and $\check{\omega}$ on $\check{\mathbf{P}}$, respectively, that is

$$\theta|\mathbf{C} = \rho^*\check{\theta} \qquad \text{and} \qquad \omega|\mathbf{C} = \rho^*\check{\omega}\,. \tag{1.29}$$

Moreover, $\check{\omega}$ is a weakly symplectic form on $\check{\mathbf{P}}$, and

$$\check{\omega} = d\check{\theta} \ . \tag{1.30}$$

Thus, the reduced phase space of the theory is a symplectic manifold with an exact symplectic form.

On the constraint set $\mathbf{C}$ the evolution equations are Hamiltonian with H given by the usual expression

$$H = \int_{\mathbb{R}^3} \Big(\tfrac{1}{2}(\mathbf{E}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{B}) + \Psi^\dagger \gamma^0 \big(\gamma^k (i\partial_k + \mathbf{A}_k) + m\big)\Psi\Big) d_3x \ . \tag{1.31}$$

Since the Hamiltonian H is invariant under the action of $GS(\mathbf{P})_0$, it follows that its restriction to $\mathbf{C}$ pushes forward to a function $\check{H}$ on the reduced phase space $\check{\mathbf{P}}$, that is

$$H|\mathbf{C} = \rho^* \check{H} \ . \tag{1.32}$$

The field equations in the reduced phase space are Hamiltonian, with respect to the weak symplectic form $\check{\omega}$, with Hamiltonian $\check{H}$.

The subalgebra $gs(\mathbf{P})_0$ is an ideal in $gs(\mathbf{P})$. It follows that $GS(\mathbf{P})_0$ is a normal subgroup of $GS(\mathbf{P})$. The quotient group

$$Colour(\mathbf{P}) = GS(\mathbf{P})/GS(\mathbf{P})_0 \tag{1.33}$$

acts in the reduced phase space $\check{\mathbf{P}}$, and this action is Hamiltonian. For each $[\xi]$ in the Lie algebra

$$colour(\mathbf{P}) = gs(\mathbf{P})/gs(\mathbf{P})_0 \ , \tag{1.34}$$

the corresponding conserved quantity is the colour charge $\check{J}_{[\xi]}$. Given $\check{p} \in \check{\mathbf{P}}$ and $[\xi] \in colour(\mathbf{P})$,

$$\check{J}_{[\xi]}(\check{p}) = \langle J(\mathbf{A}, \mathbf{E}, \Psi)|\xi\rangle \ , \tag{1.35}$$

for any $(\mathbf{A}, \mathbf{E}, \Psi) \in \mathbf{C}$ projecting to $\check{p}$, and any $\xi \in gs(\mathbf{P})$ projecting to $[\xi]$. By (1.26) $colour(\mathbf{P})$ is isomorphic to the structure algebra $\mathfrak{g}$ so that the colour charges can be parametrized by elements of $\mathfrak{g}$. In particular, taking into account Eq. (1.23), we can write

$$\check{J}_{[\xi]}(\check{p}) = -\int_{S_\infty} n\mathbf{E}\zeta \, dS \ , \tag{1.36}$$

where ζ is the constant component of ξ in the decomposition (1.26). In view of this, the colour charge is a global quantity expressing the asymptotic behaviour of the fields. It is of interest to find out if they can be expressed as integrals of well defined colour charge densities. We have shown in [11] that such well defined colour charge densities, depending locally on the fields $(\mathbf{A}, \mathbf{E}, \Psi)$, exist if and only if $[\xi]$ is in the centre of $colour(\mathbf{P})$, that is if ζ is in the centre of $\mathfrak{g}$.

The above result shows that only the colour charges corresponding to the elements of the centre of the colour algebra admit gauge invariant colour charge densities, which satisfy local Poisson bracket relations. If ζ is not in the centre of $\mathfrak{g}$, we cannot construct a corresponding gauge invariant colour charge density which would satisfy local Poisson bracket relations. Upon quantization, such a colour charge density cannot be considered to be an observable, since it would violate the locality axiom of Quantum Field Theory, [17].

This paper is organized as follows. In Section 2 we prove the existence and uniqueness theorems for the evolution equations in our phase space **P**. Section 3 is devoted to the study of the gauge symmetry group. Constraints and reduction are discussed in Section 4. Dynamical variables are considered in Section 5.

2. Existence and uniqueness results.

To study of evolution equations we use the Helmholtz decomposition of vector fields in the Sobolev space $H^s(I\!R^3)$,

$$\mathbf{X} = \mathbf{X}^L + \mathbf{X}^T \ , \tag{2.1}$$

where $\mathbf{X}^L$ is the longitudinal, and $\mathbf{X}^T$ is the transverse part of $\mathbf{X}$, i.e.

$$\operatorname{curl} \mathbf{X}^L = 0 \qquad \text{and} \qquad \operatorname{div} \mathbf{X}^T = 0 \ . \tag{2.2}$$

$\mathbf{X}^L$ and $\mathbf{X}^T$ are uniquely determined by $\operatorname{div}\mathbf{X}$ and $\operatorname{curl}\mathbf{X}$, respectively. For details see the Appendix.

Following the approach of Eardley and Moncrief, [6], we solve the constraint equation and obtain a curl-free field $\mathbf{E}^C \in H^2(I\!R^3)$, depending on $\mathbf{A}$, $\mathbf{E}$ and Ψ, such that

$$\operatorname{div} \mathbf{E}^C(\mathbf{A}, \mathbf{E}, \Psi) = \Psi^\dagger (I \otimes T^a) \Psi T_a - [\mathbf{A}; \mathbf{E}] \ . \tag{2.3}$$

Since the longitudinal part $\mathbf{E}^L$ of $\mathbf{E}$ is uniquely determined by the divergence of $\mathbf{E}$, we can replace $\mathbf{E}^L$ in Eq. (1.4) by $\mathbf{E}^C(\mathbf{A}, \mathbf{E}, \Psi)$ and obtain

$$\partial_t \mathbf{A} = \mathbf{E}^C(\mathbf{A}, \mathbf{E}, \Psi) + \mathbf{E}^T \ . \tag{2.4}$$

The system (2.3), (2.4), (1.5), and (1.6) is equivalent to the original system (1.5) - (1.7).

Linearizing the evolution equations (2.4), (1.5) and (1.6), and splitting $\mathbf{A}$ and $\mathbf{E}$ into their longitudinal and transverse components, we obtain

$$\frac{d}{dt}\begin{bmatrix}\mathbf{A}^L\\ \mathbf{E}^L\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \ , \quad \frac{d}{dt}\begin{bmatrix}\mathbf{A}^T\\ \mathbf{E}^T\end{bmatrix} = \mathcal{T}(\mathbf{A}^T, \mathbf{E}^T) \ , \quad \frac{d}{dt}\Psi = \mathcal{D}\Psi \ , \tag{2.5}$$

where

$$\mathcal{T}(\mathbf{A}^T, \mathbf{E}^T) = \begin{bmatrix}0 & 1\\ \Delta & 0\end{bmatrix} \begin{bmatrix}\mathbf{A}^T\\ \mathbf{E}^T\end{bmatrix} \tag{2.6}$$

$$\text{and} \qquad \mathcal{D}\Psi = -\gamma^0(\gamma^j \partial_j + im)\Psi \ . \tag{2.7}$$

We shall study the linearized equations in the Hilbert spaces

$$\mathbf{H}_L = \{(\mathbf{A}^L, \mathbf{E}^L) \in H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)\}\,, \tag{2.8}$$

$$\mathbf{H}_T = \{(\mathbf{A}^T, \mathbf{E}^T) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\}\,, \tag{2.9}$$

$$\mathbf{H}_D = \{\Psi \in L^2(\mathbb{R}^3)\}\,, \tag{2.10}$$

endowed with the scalar products

$$\langle(\mathbf{A}^L, \mathbf{E}^L), (\widetilde{\mathbf{A}}^L, \widetilde{\mathbf{E}}^L)\rangle_L = \langle \mathbf{A}^L, \widetilde{\mathbf{A}}^L\rangle_{H^2} + \langle \mathbf{E}^L, \widetilde{\mathbf{E}}^L\rangle_{H^1}\,, \tag{2.11}$$

$$\langle(\mathbf{A}^T, \mathbf{E}^T), (\widetilde{\mathbf{A}}^T, \widetilde{\mathbf{E}}^T)\rangle_T = \langle \mathbf{A}^T, \widetilde{\mathbf{A}}^T\rangle_{H^1} + \langle \mathbf{E}^T, \widetilde{\mathbf{E}}^T\rangle_{L^2}\,, \tag{2.12}$$

$$\langle \Psi, \widetilde{\Psi}\rangle_D = \langle \Psi, \widetilde{\Psi}\rangle_{L^2}\,. \tag{2.13}$$

The linearized dynamics for the longitudinal components is trivial. The time evolutions of the transverse part and the Dirac field is determined by the operators $\mathcal{T}$ and $\mathcal{D}$, respectively.

Proposition 2.1

The operator $\mathcal{T}$ with domain

$$\mathbf{D}_T = \{(\mathbf{A}^T, \mathbf{E}^T) \in H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)\} \tag{2.14}$$

is the generator of a continuous group of transformations in $\mathbf{H}_T$.

Proof.

Consider first the operator

$$\widetilde{\mathcal{T}} = \begin{bmatrix} 0 & 1 \\ \Delta & 0 \end{bmatrix} \tag{2.15}$$

on the full space $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ with domain $H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$. It is the infinitesimal generator corresponding to the wave equation. By standard arguments, [18], we infer that $\widetilde{\mathcal{T}}$ is dissipative, and that

$$\mathrm{range}(\widetilde{\mathcal{T}} - \lambda I) = H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \qquad \text{and} \qquad \ker(\widetilde{\mathcal{T}} - \lambda I) = \{0\} \tag{2.16}$$

for some $\lambda > 0$. For each pair of transversal fields $(\mathbf{X}^T, \mathbf{Y}^T) \in \mathbf{H}_T$ there exists some $(\widetilde{\mathbf{A}}, \widetilde{\mathbf{E}}) \subset H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ such that

$$(\widetilde{\mathcal{T}} - \lambda I)(\widetilde{\mathbf{A}}, \widetilde{\mathbf{E}}) = (\mathbf{X}^T, \mathbf{Y}^T)\,. \tag{2.17}$$

Since Δ maintains the Helmholtz decomposition of $(\widetilde{\mathbf{A}}, \widetilde{\mathbf{E}}) = (\widetilde{\mathbf{A}}^T + \widetilde{\mathbf{A}}^L, \widetilde{\mathbf{E}}^T + \widetilde{\mathbf{E}}^L)$, this implies that

$$(\widetilde{\mathbf{A}}^L, \widetilde{\mathbf{E}}^L) \in \ker(\widetilde{\mathcal{T}} - \lambda I) = \{0\}\,. \tag{2.18}$$

Therefore, since $\mathcal{T} = \widetilde{\mathcal{T}}|_{\mathbf{D}_T}$,

$$\mathrm{range}(\mathcal{T} - \lambda I) = \mathrm{range}(\widetilde{\mathcal{T}} - \lambda I)|_{\mathbf{D}_T} = \mathbf{H}_T\,. \tag{2.19}$$

Moreover, $\mathcal{T}$ is dissipative on the Hilbert space $\mathbf{H}_{\mathcal{T}}$. Then the Lummer-Phillips theorem implies that $\mathcal{T}$ generates a one parameter semigroup of continuous transformations $\exp(t\mathcal{T})$.

Q.E.D.

Proposition 2.2.
(i) The operator $\mathcal{D}$, with domain

$$\mathbf{D}_D = \{\Psi \in H^1(\mathbb{R}^3)\} \ , \tag{2.20}$$

is the generator of a continuous group of (unitary) transformations $\exp(t\mathcal{D})$ in $\mathbf{H}_D$.
(ii) $\exp(t\mathcal{D})$ restricts to a group of continuous transformations in the Hilbert space $H^2(\mathbb{R}^3)$.

Proof.
(i) We know from [19] that the operator $\mathcal{D}$ with domain $\mathbf{D}_D$ is skew-adjoint in $\mathbf{H}_D$. Thus, $\mathcal{D}$ generates a group $\exp(t\mathcal{D})$ of unitary transformations in $\mathbf{H}_D$.
(ii) The operator $\mathcal{D} : H^1(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is continuous, and its square

$$\mathcal{D}^2 = \Delta - m^2 : H^2(\mathbb{R}^3) \longrightarrow L^2(\mathbb{R}^3) \tag{2.21}$$

is continuous and elliptic. With the elliptic a priori estimate this implies that

$$C_1 \|\mathcal{D}^2\Psi\|_{L^2} \leq \|\Psi\|_{H^2} \leq C_2\big(\|\mathcal{D}^2\Psi\|_{L^2} + \|\Psi\|_{H^1}\big) \ . \tag{2.22}$$

Moreover, from the identity $\gamma^i\gamma^k = -\delta^{ik} + \frac{1}{2}[\gamma^i, \gamma^k]$, we obtain

$$\|\mathcal{D}\Psi\|_{L^2}^2 = \sum_{j=1}^{3} \|\partial_j\Psi\|_{L^2}^2 - A(\Psi) + m^2\|\Psi\|_{L^2}^2 \tag{2.23}$$

for all $\Psi \in H^2(\mathbb{R}^3)$, where

$$2A(\Psi) = \sum_{j,k=1}^{3} \langle[\gamma^j, \gamma^k]\partial_k\Psi, \partial_j\Psi\rangle_{L^2} \ . \tag{2.24}$$

Integration by parts shows that $A(\Psi)$ vanishes for all Ψ in $C^\infty(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$. Thus, by a density argument, $A(\Psi) = 0$ for all $\Psi \in H^1(\mathbb{R}^3)$. Therefore

$$C_3\|\mathcal{D}\Psi\|_{L^2} \leq \|\Psi\|_{H^1} \leq C_4\|\mathcal{D}\Psi\|_{L^2} \tag{2.25}$$

and

$$\|\Psi\|_{H^2} \leq C_5\big(\|\mathcal{D}^2\Psi\|_{L^2} + \|\mathcal{D}\Psi\|_{L^2}\big) \ . \tag{2.26}$$

Since $\exp(t\mathcal{D})$ is a unitary operator, which commutes on the domain $\mathbf{D}_D$ with its generator $\mathcal{D}$, cf. [20], we can estimate for all $\Psi \in H^2(\mathbb{R}^3)$:

$$\begin{aligned} \|\exp(t\mathcal{D})\Psi\|_{H^2} &\leq C_5\big(\|\mathcal{D}^2\exp(t\mathcal{D})\Psi\|_{L^2} + \|\mathcal{D}\exp(t\mathcal{D})\Psi\|_{L^2}\big) \\ &= C_5\big(\|\mathcal{D}^2\Psi\|_{L^2} + \|\mathcal{D}\Psi\|_{L^2}\big) \leq C_6\|\Psi\|_{H^2} \ . \end{aligned} \tag{2.27}$$

Q.E.D.

The results obtained up to now are summarized below.

Corollary 2.3
The operators $\mathcal{T}$ and $\mathcal{D}$ define a linear operator

$$\mathcal{S} = 0 \oplus \mathcal{T} \oplus \mathcal{D} \tag{2.28}$$

in $\mathbf{H} = \mathbf{H}_L \times \mathbf{H}_T \times \mathbf{H}_D$ with domain $\mathbf{D} = \mathbf{H}_L \times \mathbf{D}_T \times \mathbf{D}_D$, which generates a one parameter group $\exp(t\mathcal{S})$ of continuous transformations in $\mathbf{H}$. The action of $\exp(t\mathcal{S})$ in $\mathbf{H}$ preserves the domain $\mathbf{D}$ of $\mathcal{S}$. The extended phase space $\mathbf{P}$, defined by (1.9), is a dense subspace of $\mathbf{D}$, and it is preserved by $\exp(t\mathcal{S})$. The restriction of $\exp(t\mathcal{S})$ to $\mathbf{P}$ is a continuous one parameter group $\mathcal{U}(t)$ of continuous transformations in $\mathbf{P}$,

$$\mathcal{U}(t) = \exp(t\mathcal{S})|_{\mathbf{P}} : \mathbf{P} \longrightarrow \mathbf{P} \text{ such that } (\mathbf{A}, \mathbf{E}, \Psi) \mapsto \mathcal{U}(t)(\mathbf{A}, \mathbf{E}, \Psi) . \tag{2.29}$$

$(\mathbf{A}(t), \mathbf{E}(t), \Psi(t)) = \mathcal{U}(t)(\mathbf{A}, \mathbf{E}, \Psi)$ is the unique solution of the linear evolution equations (2.5), (2.6) and (2.7) with initial condition $(\mathbf{A}, \mathbf{E}, \Psi)$.

Having solved the linearized problem, we return to the full set of evolution equations. With

$$(\mathbf{A}, \mathbf{E}, \Psi)_t := (\mathbf{A}(t), \mathbf{E}(t), \Psi(t)) , \tag{2.30}$$

we can rewrite Eqs. (2.4), (1.5), (1.6) in an abstract form as

$$\frac{d}{dt}(\mathbf{A}, \mathbf{E}, \Psi)_t = \mathcal{S}(\mathbf{A}, \mathbf{E}, \Psi)_t - \mathcal{F}((\mathbf{A}, \mathbf{E}, \Psi)_t) , \tag{2.31}$$

where $\mathcal{F}$ describes the nonlinearity of the theory. We split the nonlinear term $\mathcal{F} = \mathcal{F}_{YM} + \mathcal{F}_C$ into the pure Yang-Mills part $\mathcal{F}_{YM}$ and the coupling part $\mathcal{F}_C$ describing the interaction between the Yang-Mills field and the Dirac field, where

$$\mathcal{F}_{YM}(\mathbf{A}, \mathbf{E}, \Psi) = (\mathbf{E}^C(\mathbf{A}, \mathbf{E}, \Psi) ; -[\mathbf{A}\times, \mathbf{B}] - \text{curl}\,[\mathbf{A}\times, \mathbf{A}] ; 0) , \tag{2.32}$$

$$\mathcal{F}_C(\mathbf{A}, \mathbf{E}, \Psi) = (0 ; \mathbf{j} ; \; -\gamma^0\gamma^j \mathbf{A}_j \Psi) . \tag{2.33}$$

Here $\mathbf{B}$ and $\mathbf{j}$ are given by (1.3) and (1.8), respectively. In order to solve the system (2.31) we apply the method of nonlinear semigroups, cf. [21]. It requires the knowledge of some analytic properties of the nonlinearity.

Proposition 2.4
$\mathcal{F}$ maps $\mathbf{P}$ to $\mathbf{P}$, and is continuous, Lipschitz and smooth with respect to the norm

$$||(\mathbf{A}, \mathbf{E}, \Psi)||^2_{\mathbf{P}} = ||\mathbf{A}||^2_{H^2} + ||\mathbf{E}||^2_{H^1} + ||\Psi||^2_{H^2} . \tag{2.35}$$

This result was proved for the pure Yang-Mills part $\mathcal{F}_{YM}$ of $\mathcal{F}$ in [6], and for the coupling term $\mathcal{F}_C$ it was established in [22]. (The proof given there under bag boundary conditions literally generalizes to $\mathbb{R}^3$.) This enables us to infer the existence and uniqueness of solutions of minimally coupled Yang-Mills and Dirac equations from the corresponding results for nonlinear semigroups, which can be summarized as follows : If $\mathcal{S}$ generates a one parameter semigroup on $\mathbf{P}$, and $\mathcal{F} : \mathbf{P} \to \mathbf{P}$ is Lipschitz, then, for each initial data in $\mathbf{P}$, the dynamical system (2.31) has a unique solution, [23]. Thus, we have

Theorem 2.5.
For every initial condition $(\mathbf{A}, \mathbf{E}, \Psi) \in \mathbf{P}$ of gauge and matter fields in $\mathbb{R}^3$, there exists a unique continuous curve $(\mathbf{A}(t), \mathbf{E}(t), \Psi(t))$ in $\mathbf{P}$ satisfying the modified Yang-Mills and Dirac equations (2.4), (1.5) and (1.6). This time evolution is well defined for all $t \in [0, T)$, where the maximal time of existence $T \in (0, \infty]$ is determined by the initial condition.

It has been established in [6] that this evolution preserves the constraint equation (1.7), and that the modified evolution equations (2.4), (1.5) and (1.6), and the original equations (1.4) through (1.6) coincide on solutions of the constraint equation.

Observe that the modified time evolution of the Yang-Mills-Dirac system discussed here gives rise to local diffeomorphisms of the phase space $\mathbf{P}$. To see this, we consider the map

$$(\mathbf{A}, \mathbf{E}, \Psi) \longmapsto (\mathbf{A}, \mathbf{E}, \Psi)_t = \mathcal{U}(t)(\mathbf{A}, \mathbf{E}, \Psi) + \int_0^t \mathcal{U}(t-s)\mathcal{F}((\mathbf{A}, \mathbf{E}, \Psi)_s)ds \,. \tag{2.36}$$

Differentiation this map in the direction of an arbitrary vector $(\delta\mathbf{A}, \delta\mathbf{E}, \delta\Psi)$ in $\mathbf{P}$ we obtain

$$\begin{aligned}&\Big((\mathbf{A}, \mathbf{E}, \Psi),(\delta\mathbf{A}, \delta\mathbf{E}, \delta\Psi)\Big) \longmapsto \\ &\qquad \mathcal{U}(t)(\delta\mathbf{A}, \delta\mathbf{E}, \delta\Psi) + \int_0^t \mathcal{U}(t-s)D\big(\mathcal{F}((\mathbf{A}, \mathbf{E}, \Psi)_s)\big)(\delta\mathbf{A}, \delta\mathbf{E}, \delta\Psi)ds \,,\end{aligned} \tag{2.37}$$

which is continuous, since $\mathcal{F}$ is smooth. A corresponding argument for the higher derivatives implies that the time evolution (2.36) is smooth. Since the dynamics is reversible, this shows that it is a local diffeomorphism. It should be emphasized that this local diffeomorphism is not a symplectomorphism. The obtained evolution is Hamiltonian only on the constraint set.

3. Gauge symmetries.

The group $GS(\mathbf{P})$ of gauge symmetries of the phase space $\mathbf{P}$ consists of the gauge transformations (1.12) which preserve the extended phase space $\mathbf{P}$. The action of its Lie algebra $gs(\mathbf{P})$ is given by (1.13). Since the Yang-Mills potentials $\mathbf{A}$ in $\mathbf{P}$ are of Sobolev class $H^2(\mathbb{R}^3)$, it follows from (1.14) and (1.15) with $\mathbf{A} = 0$ that $\xi \in gs(\mathbf{P})$ only if

$\operatorname{grad}\xi \in H^2(\mathbb{R}^3, \mathfrak{g})$. Thus, $gs(\mathbf{P})$ is contained in the intersection of three Beppo Levi spaces,

$$gs(\mathbf{P}) \subseteq \bigcap_{m=1}^{3} BL_m(L^2(\mathbb{R}^3, \mathfrak{g})) \,, \tag{3.1}$$

where $BL_m(L^2(\mathbb{R}^3, \mathfrak{g}))$ is the space of $\mathfrak{g}$-valued distributions on $\mathbb{R}^3$ with square integrable partial derivatives of order m, cf. [12]. The space $BL_1(L^2(\mathbb{R}^3, \mathfrak{g}))$ has decomposition of Aikawa type [13],

$$BL_1(L^2(\mathbb{R}^3, \mathfrak{g})) = H^1(\mathbb{R}^3, \mathfrak{g}) \oplus \mathfrak{g} \,, \tag{3.2}$$

see also Lemma A.1 of the Appendix. Here $\mathfrak{g}$ is interpreted as the spaces of constant maps from $\mathbb{R}^3$ to the structure algebra. For the intersection of three Beppo Levi spaces we write

$$\mathcal{B}^3(\mathbb{R}^3, \mathfrak{g}) := \bigcap_{m=1}^{3} BL_m(L^2(\mathbb{R}^3, \mathfrak{g})) \,. \tag{3.3}$$

This space is topologized by the norm

$$\|\xi\|_{\mathcal{B}^3} = \int_{B_1} |\xi| d_3x + \|\operatorname{grad}\xi\|_{H^2} \,. \tag{3.4}$$

As shown in Proposition A.2 of the Appendix, the decomposition (3.2) implies that

$$\mathcal{B}^3(\mathbb{R}^3, \mathfrak{g}) = H^3(\mathbb{R}^3, \mathfrak{g}) \oplus \mathfrak{g} \,. \tag{3.5}$$

Therefore each $\xi \in gs(\mathbf{P})$ splits uniquely into

$$\xi = \xi_0 + \zeta \quad \text{where} \quad \xi_0 \in H^3(\mathbb{R}^3, \mathfrak{g}) \text{ and } \zeta \in \mathfrak{g} \,. \tag{3.6}$$

By the Sobolev embedding theorem ξ_0 is in $C^1(\mathbb{R}^3, \mathfrak{g})$, so that all $\xi \in \mathcal{B}^3(\mathbb{R}^3, \mathfrak{g})$ are continuous maps from $\mathbb{R}^3$ the structure Lie algebra $\mathfrak{g}$ with continuous first derivatives.

Proposition 3.1

$$gs(\mathbf{P}) = \mathcal{B}^3(\mathbb{R}^3, \mathfrak{g}) \,, \tag{3.7}$$

and its action in $\mathbf{P}$ is continuous.

Proof.

Since $\mathcal{B}^3(\mathbb{R}^3, \mathfrak{g}) \subset C^1(\mathbb{R}^3, \mathfrak{g})$ we can estimate

$$\begin{aligned} \|[\mathbf{A}, \xi]\|_{H^2} &\leq 2\|\xi\|_{L^\infty} \|\mathbf{A}\|_{H^2} \\ \|[\mathbf{E}, \xi]\|_{H^1} &\leq 2\|\xi\|_{L^\infty} \|\mathbf{E}\|^2_{H^1} \\ \|\xi\Psi\|_{H^2} &\leq \|\xi\|_{L^\infty} \|\Psi\|^2_{H^2} . \end{aligned} \tag{3.8}$$

Moreover, $\|\xi\|_{L^\infty} \leq C\|\xi\|_{\mathcal{B}^3}$, by Proposition A.2. This implies that the infinitesimal action of $\xi \in \mathcal{B}^3(\mathbb{R}^3, \mathfrak{g})$ given by (1.13) preserves $\mathbf{P}$, so that $\mathcal{B}^3(\mathbb{R}^3, \mathfrak{g}) \subseteq gs(\mathbf{P})$. This, and the inclusion (3.1) implies (3.7). The estimates (3.8), then also ensure the continuity of the action. Q.E.D.

Since $H^3(\mathbb{R}^3, \mathfrak{g})$ is the closure of the set of compactly supported maps in the H^3 topology, it follows from (1.23) that for all ξ_0 in the $H^3(\mathbb{R}^3, \mathfrak{g})$-component of $gs(\mathbf{P})$

$$\langle J(\mathbf{A}, \mathbf{E}, \Psi) | \xi_0 \rangle = 0 \tag{3.9}$$

if $(\mathbf{A}, \mathbf{E}, \Psi) \in \mathbf{C}$. Let $gs(\mathbf{P})_0$ be the ideal defined by the constraints, Eq. (1.27). Then Eq. (3.9) implies that $gs(\mathbf{P})_0 \subseteq H^3(\mathbb{R}^3, \mathfrak{g})$. In turn, if $\zeta \in \mathfrak{g}$ is a constant Lie algebra element, then $\langle J(\mathbf{A}, \mathbf{E}, \Psi) | \zeta \rangle \equiv 0$ on $\mathbf{C}$ if and only of $\zeta = 0$. Consequently

$$gs(\mathbf{P})_0 = H^3(\mathbb{R}^3, \mathfrak{g}) . \tag{3.10}$$

Summarizing this, we have

Proposition 3.2

The symmetry algebra $gs(\mathbf{P})$ decomposes into

$$gs(\mathbf{P}) = gs(\mathbf{P})_0 \oplus \mathfrak{g} , \tag{3.11}$$

and the topology of the Lie algebra $gs(\mathbf{P})_0$ is equivalent to the $H^3(\mathbb{R}^3, \mathfrak{g})$ topology.

Corollary 3.3

The infinitesimal action of $gs(\mathbf{P})_0$ in $\mathbf{P}$ is free.

Proof.

Suppose that the infinitesimal action of $\xi_0 \in gs(\mathbf{P})_0$ has a fixed point $(\mathbf{A}, \mathbf{E}, \Psi)$. It follows from (1.13) that ξ_0 is covariantly constant with respect to the connection given by $\mathbf{A}$,

$$D_{\mathbf{A}} \xi_0 = 0 . \tag{3.12}$$

Since the scalar product in $\mathfrak{g}$ is ad-invariant, this implies that $|\xi_0(x)|^2 = const.$, and

$$\|\xi_0\|_{L^2} = \infty . \tag{3.13}$$

This contradicts Eq. (3.10). Q.E.D.

The topology of the gauge group on non-compact manifolds with a Sobolev Lie algebra has been studied in [4] and [14]. Here we adapt the approach of [4] to our case. Let the structure group be $G \subset M_k^k$, where M_k^k denotes the space of $k \times k$ matrices, and let $C_c^\infty(\mathbb{R}^3, M_k^k)$ be the space of smooth maps $\phi : \mathbb{R}^3 \to M_k^k$ which are constant outside a compact set. The set $C_c^\infty(\mathbb{R}^3, G)$ forms a group under a pointwise multiplication. We denote by e the identity in $C_c^\infty(\mathbb{R}^3, G)$, that is the map associating to each $x \in \mathbb{R}^3$ the identity in G. The norm $\|\cdot\|_{\mathcal{B}^3}$, given by (3.4), naturally extends to the space of M_k^k valued maps on $\mathbb{R}^3$. Therefore it also defines a norm on the space $C_c^\infty(\mathbb{R}^3, G)$.

One parameter subgroups of $C_c^\infty(\mathbb{R}^3, G)$ are of the form $\exp(t\xi)$, where ξ is in the dense subalgebra $C_c^\infty(\mathbb{R}^3, \mathfrak{g})$ of $gs(\mathbf{P})$. The topology of $gs(\mathbf{P})$, given by the norm (3.4), introduces a uniform structure in $C_c^\infty(\mathbb{R}^3, G)$, with a neighbourhood basis at e consisting of the sets

$$N_\epsilon = \{\exp(\xi) \mid \xi \in C_c^\infty(\mathbb{R}^3, \mathfrak{g}),\ \|\xi\|_{\mathcal{B}^3} < \epsilon\} \qquad \text{with } \epsilon > 0 . \tag{3.14}$$

In order to show that the completion of $C_c^\infty(\mathbb{R}^3, G)$ in this uniform structure is a topological group, relatively to the canonically extended multiplication, we need to show :

Proposition 3.4

The mapping $\exp(\xi) \mapsto \exp(\xi)^{-1}$ is uniformly continuous relative to N_1. That is, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for every $\exp(\xi) \in N_1$,

$$\exp(\xi)^{-1} N_\delta \exp(\xi) \subseteq N_\epsilon . \tag{3.15}$$

Proof.

Let $\phi \in N_\epsilon \subset C^\infty(\mathbb{R}^3, G)$ then

$$\phi = \exp(\xi) = \sum_{n=0}^\infty \frac{1}{n!} \xi^n , \tag{3.16}$$

and

$$\operatorname{grad} \phi = \operatorname{grad} \exp(\xi) = \sum_{n=1}^\infty \sum_{k=0}^{n-1} \frac{1}{n!} \xi^k (\operatorname{grad} \xi)\, \xi^{n-k-1} . \tag{3.17}$$

Using the estimates of Lemma A.3 this implies that

$$\|\operatorname{grad} \phi\|_{H^2} \le \exp(C\|\xi\|_{\mathcal{B}^3})\, \|\operatorname{grad} \xi\|_{H^2} \tag{3.18}$$

and

$$\|\phi\|_{\mathcal{B}^3} \le \sum_{n=0}^\infty \frac{1}{n!} (C\|\xi\|_{\mathcal{B}^3})^n < e^{C\epsilon} . \tag{3.19}$$

For each $\zeta \in gs(\mathbf{P})$ we then obtain by using Lemma A.3 once more :

$$\|\exp(\xi)^{-1} \zeta \exp(\xi)\|_{\mathcal{B}^3} \le C^2 \|\exp(-\xi)\|_{\mathcal{B}^3} \|\zeta\|_{\mathcal{B}^3} \|\exp(\xi)\|_{\mathcal{B}^3} < C^2 e^{2C\epsilon} \|\zeta\|_{\mathcal{B}^3} \tag{3.20}$$

This proves (3.15) with $\delta = \epsilon (C e^{C\epsilon})^{-2}$. Q.E.D.

By a result of [24], Proposition 3.4 implies that the completion of $C_c^\infty(\mathbb{R}^3, G)$ in this uniform structure is a topological group, relatively to the canonically extended multiplication. It is a Banach-Lie group, whose Lie algebra is canonically isomorphic to the Banach-Lie algebra $gs(\mathbf{P})$. In view of this we set :

Definition 3.5

The group $GS(\mathbf{P})$ of gauge symmetries is the completion of the group $C_c^\infty(\mathbb{R}^3, G)$ in the uniform structure defined by the topology of the Lie algebra $gs(\mathbf{P})$.

The exponential map $\exp : gs(\mathbf{P}) \to GS(\mathbf{P})$ maps the unit ball in $gs(\mathbf{P})$ onto the neighbourhood of identity given by the completion $\overline{N}_1$ of N_1. Since G is connected, it follows that $C_c^\infty(\mathbb{R}^3, G)$ is connected, and $GS(\mathbf{P})$ is connected. Therefore, $GS(\mathbf{P})$ is the union of the sets

$$N_1^m = \{\phi_1 \cdot \phi_2 \cdot \ldots \cdot \phi_m | \phi_i \in N_1\} . \tag{3.21}$$

The inequality (3.18) together with (3.21) implies that, for each $\phi \in GS(\mathbf{P})$,

$$\operatorname{grad} \phi \in H^2(\mathbb{R}^3) . \tag{3.22}$$

Moreover, since G is compact, it is bounded in M_k^k, and the Sobolev embedding theorem implies that each $\phi \in GS(\mathbf{P})$ is a bounded continuous map. Hence, $\|\phi\|_{\mathcal{B}^3}$ is finite for every ϕ in $GS(\mathbf{P})$. We can give an alternative characterization of the topology of $GS(\mathbf{P})$.

Proposition 3.6

A sequence $\phi_k \in GS(\mathbf{P})$ converges to ϕ in $GS(\mathbf{P})$ if and only if the sequence of maps $\phi_k : \mathbb{R}^3 \to G$ converges to ϕ in the topology defined by the norm $\|\cdot\|_{\mathcal{B}^3}$.

Proof.

Suppose that ϕ_k converges to ϕ in the uniform topology of $GS(\mathbf{P})$. For sufficiently large k,

$$\phi_k = \phi \exp(\xi_k) , \tag{3.23}$$

where the sequence ξ_k converges to zero in the topology of $gs(\mathbf{P})$. The estimate (A.24) of Lemma A.3 implies that

$$\|\phi_k - \phi\|_{\mathcal{B}^3} \leq C\|\phi\|_{\mathcal{B}^3} \left\| \sum_{n=1}^{\infty} \frac{1}{n!} (\xi_k)^n \right\|_{\mathcal{B}^3} \leq C\|\phi\|_{\mathcal{B}^3} \left| 1 - e^{C\|\xi_k\|_{\mathcal{B}^3}} \right| . \tag{3.24}$$

For $\xi_k \to 0$ in the norm topology of $gs(\mathbf{P})$ this converges to zero, which implies that $\phi_k \to \phi$ in the topology defined by $\|\cdot\|_{\mathcal{B}^3}$.

Conversely, suppose that $\|\phi_k - \phi\|_{\mathcal{B}^3} \to 0$. Then

$$\|e - \phi^{-1}\phi_k\|_{\mathcal{B}^3} \leq \epsilon_k \tag{3.25}$$

with $\epsilon_k \to 0$ as k goes to infinity. Eq. (3.23) yields

$$\xi_k = \log(\phi^{-1}\phi_k) = -\sum_{n=1}^{\infty} \frac{(e - \phi^{-1}\phi_k)^n}{n} \tag{3.26}$$

for k sufficiently large. Therefore, by (A.24),

$$\|\xi_k\|_{\mathcal{B}^3} \leq \sum_{n=1}^{\infty} \frac{\left(C\|e - \phi^{-1}\phi_k\|_{\mathcal{B}^3}\right)^n}{C\,n} \leq -\frac{1}{C}\log(1 - C\epsilon_k) \tag{3.27}$$

This implies that $\xi_k \to 0$ in the topology of $gs(\mathbf{P})$, and hence $\phi_k \to \phi$ in the uniform topology of $GS(\mathbf{P})$. Q.E.D.

Theorem 3.7

The action of $GS(\mathbf{P})$ in $\mathbf{P}$, given by (1.13), is continuous and proper.

Proof.

Let ϕ_n be a sequence in $GS(\mathbf{P})$ converging to ϕ, and $p_n = (\mathbf{A}_n, \mathbf{E}_n, \Psi_n)$ a sequence converging to $p = (\mathbf{A}, \mathbf{E}, \Psi)$. From (1.13) we obtain by using the estimate (A.23) and the fact that the inversion $\phi \mapsto \phi^{-1}$ in $GS(\mathbf{P})$ is continuous :

$$\begin{aligned}
&\|(\phi_n\mathbf{A}_n\phi_n^{-1} + \phi_n\mathrm{grad}\,\phi_n^{-1}) - (\phi\mathbf{A}\phi^{-1} - \phi\mathrm{grad}\,\phi^{-1})\|_{H^2} \\
&\leq \|\phi_n\mathbf{A}_n\phi_n^{-1} - \phi_n\mathbf{A}\phi_n^{-1}\|_{H^2} + \|\phi_n\mathbf{A}\phi_n^{-1} - \phi_n\mathbf{A}\phi^{-1}\|_{H^2} + \|\phi_n\mathbf{A}\phi^{-1} - \phi\mathbf{A}\phi^{-1}\|_{H^2} \\
&\qquad + \|\phi_n\mathrm{grad}\,\phi_n^{-1} - \phi_n\mathrm{grad}\,\phi^{-1}\|_{H^2} + \|\phi_n\mathrm{grad}\,\phi^{-1} - \phi\mathrm{grad}\,\phi^{-1}\|_{H^2} \\
&\leq C\Big(\|\phi_n\|_{\mathcal{B}^3}^2\|\mathbf{A}_n - \mathbf{A}\|_{H^2} + \left(\|\phi_n\|_{\mathcal{B}^3} + \|\phi\|_{\mathcal{B}^3}\right)\|\mathbf{A}\|_{H^2}\|\phi_n - \phi\|_{\mathcal{B}^3} \\
&\qquad + \|\phi_n\|_{\mathcal{B}^3}\|\mathrm{grad}\,\phi_n - \mathrm{grad}\,\phi\|_{H^2} + \|\phi_n - \phi\|_{\mathcal{B}^3}\|\mathrm{grad}\,\phi\|_{H^2}\Big)
\end{aligned} \tag{3.28}$$

Writing symbolically ϕp for the action of ϕ on p, and $(\phi p)_{\mathbf{A}}$ for its $\mathbf{A}$ component, this implies that

$$\|(\phi_n p_n)_{\mathbf{A}} - (\phi p)_{\mathbf{A}}\|_{H^2} \leq C'\left(\|\mathbf{A}_n - \mathbf{A}\|_{H^2} + \|\phi_n - \phi^{-1}\|_{\mathcal{B}^3}\right), \tag{3.29}$$

since $|\phi_n\|_{\mathcal{B}^3}$ is bounded. Correspondingly we estimate with (A.22) and (A.23),

$$\begin{aligned}
\|(\phi_n p_n)_{\mathbf{E}} - (\phi p)_{\mathbf{E}}\|_{H^1} &\leq C'\left(\|\mathbf{E}_n - \mathbf{E}\|_{H^1} + |\phi_n - \phi^{-1}\|_{\mathcal{B}^3}\right) \\
\|(\phi_n p_n)_{\Psi} - (\phi p)_{\Psi}\|_{H^2} &\leq C'\left(\|\Psi_n - \Psi\|_{H^2} + \|\phi_n - \phi^{-1}\|_{\mathcal{B}^3}\right).
\end{aligned} \tag{3.30}$$

Therefore $\|\phi_n p_n - \phi p\| \to 0$ as $n \to \infty$, which proves the continuity of the action.

Let $p_n = (\mathbf{A}_n, \mathbf{E}_n, \Psi_n)$ converge in $\mathbf{P}$ to $p = (\mathbf{A}, \mathbf{E}, \Psi)$, and ϕ_n be a sequence in $GS(\mathbf{P})$ such that $\phi_n p_n$converges to $\widetilde{p}$. It is to show that ϕ_n converges to $\phi \in GS(\mathbf{P})$ and $\widetilde{p} = \phi p$. The argument used in [22] for compact domains implies that, for every compact domain $M \subset \mathbb{R}^3$, the restrictions $\phi_n|M$ converge in $H^2(M)$ to a map $\phi_M \in H^2(M)$. Since, $M \subseteq \widetilde{M}$ implies that $\phi|\widetilde{M}$ restricted to M coincides with ϕ_M, it follows that there exists a continuous map $\phi : \mathbb{R}^3 \to G$ such that ϕ_M is the restriction of ϕ to M. The proof that $\mathrm{grad}\,\phi_n$ converges to $\mathrm{grad}\,\phi$ in the $H^2(\mathbb{R}^3)$ topology is the same as in the compact case, [22]. Hence, Proposition 3.6 implies that ϕ_n converges to ϕ in the uniform topology. Q.E.D.

Let $C_0^\infty(I\!R^3, G)$ be the subgroup of $C_c^\infty(I\!R^3, G)$ consisting of maps $\phi : I\!R^3 \to G$ which are the identity in G outside a compact set.

Definition 3.8

$GS(\mathbf{P})_0$ is the closure of $C_0^\infty(I\!R^3, G)$ in the uniform topology.

Proposition 3.9

$GS(\mathbf{P})_0$ is a closed subgroup of $GS(\mathbf{P})$ with Lie algebra $gs(\mathbf{P})_0$.

Proof.

By construction, $GS(\mathbf{P})_0$ is a closed subgroup of $GS(\mathbf{P})$. The result of Proposition 3.6 implies that its Lie algebra is the closure in the $\mathcal{B}^3$ topology of the Lie algebra $C_0^\infty(I\!R^3, \mathfrak{g})$ of smooth maps $\xi : I\!R^3 \to \mathfrak{g}$ with compact support. By Aikawa's decomposition (Lemma A.1) and Eq. (3.10) this coincides with $gs(\mathbf{P})_0$. Q.E.D.

Theorem 3.10

The action of $GS(\mathbf{P})_0$ in $\mathbf{P}$ is proper and free.

Proof.

Since $GS(\mathbf{P})_0$ is a closed subgroup of $GS(\mathbf{P})$ which acts properly in $\mathbf{P}$, it follows that the action of $GS(\mathbf{P})_0$ in $\mathbf{P}$ is proper. In order to prove that the action of $GS(\mathbf{P})_0$ is free we observe from Corollary 3.3 that the infinitesimal action is free. Since $GS(\mathbf{P})_0$ is connected, every $\phi \in GS(P)_0$ is of the form

$$\phi = (\exp \xi_1) \cdot (\exp \xi_2) \ldots (\exp \xi_n)$$

for some $\xi_1, \ldots \xi_n$ in $gs(\mathbf{P})_0$. Hence the action of $GS(\mathbf{P})_0$ is free. Q.E.D.

4. The reduced phase space and dynamical variables.

The constraint set $\mathbf{C}$ is the zero level of the momentum mapping for the Hamiltonian action of the group $GS(\mathbf{P})_0$ in $\mathbf{P}$. According to Eq. (1.27) $\mathbf{C} = J_0^{-1}(0)$ where $J_0 : \mathbf{P} \to gs(\mathbf{P})_0'$ is given by the composition of the momentum map $J : \mathbf{P} \to gs(\mathbf{P})'$ with the dual of the inclusion map of $gs(\mathbf{P})_0$ into $gs(\mathbf{P})$.

Since the Lie algebra $gs(\mathbf{P})_0$ has the Sobolev $H^3(I\!R^3, \mathfrak{g})$ topology, and its action in $\mathbf{P}$ is free, one can use the general theory on momentum map constructions, [25]. A direct proof of this result for Yang-Mills fields, given in [9], extends without major changes to interacting Yang-Mills and Dirac fields. In particular, J_0 is $GS(\mathbf{P})_0$ equivariant, which implies

Proposition 4.1

The constraint set $\mathbf{C} = J_0^{-1}(0)$ is a smooth submanifold of the extended phase space $\mathbf{P}$, and the action of $GS(\mathbf{P})_0$ in $\mathbf{P}$ preserves $\mathbf{C}$.

Since $\mathbf{C}$ is the zero level of the momentum map J_0 for the Hamiltonian action of $GS(\mathbf{P})_0$ in $\mathbf{P}$, it is natural to define the reduced phase space $\check{\mathbf{P}}$ as the space of $GS(\mathbf{P})_0$ orbits in $\mathbf{C}$,

$$\check{\mathbf{P}} = \mathbf{C}/GS(\mathbf{P})_0 \, . \tag{4.1}$$

We denote by $\rho : \mathbf{C} \to \check{\mathbf{P}}$ the canonical projection, associating to each point $p \in \mathbf{C}$ the $GS(\mathbf{P})_0$ orbit through p. Since $\mathbf{C}$ is a submanifold of $\mathbf{P}$, and the action of $GS(\mathbf{P})_0$ in $\mathbf{P}$ is free and proper, the characterization of Theorem 4 of [16] literally applies the case under consideration with the normal subalgebra $h \subset \mathfrak{g}$ being the trivial algebra $\{0\}$. This implies the following theorem, which is analogous to the results of Mitter and Vialet obtained under different assumptions, [26].

Theorem 4.2

$\check{\mathbf{P}}$ is a smooth manifold, and the projection map $\rho : \mathbf{C} \mapsto \check{\mathbf{P}}$ is a submersion. The restrictions to $\mathbf{C}$ of the forms θ and ω on $\mathbf{P}$ push forward to $\check{\mathbf{P}}$, giving rise to forms $\check{\theta}$ and $\check{\omega}$ such that

$$\theta|\mathbf{C} = \pi^*\check{\theta} \qquad \text{and} \qquad \omega|\mathbf{C} = \pi^*\check{\omega} \, , \tag{4.2}$$

Moreover $\check{\omega} = d\check{\theta}$, and $\check{\omega}$ is weakly symplectic.

The full gauge symmetry group $GS(\mathbf{P})$ preserves the constraint set $\mathbf{C}$, so one could consider the space of $GS(\mathbf{P})$ orbits in $\mathbf{C}$ as a candidate for the reduced phase space. This choice would not be as natural as the one we made, since the conserved quantities J_ς corresponding to constant maps $\zeta : \mathbb{R}^3 \to \mathfrak{g}$ need not vanish. Conceptually, dividing $\mathbf{C}$ by $GS(\mathbf{P})$ would be analogous to dividing $\mathbf{C}$ by all the symmetries of the theory including translations and rotations in $\mathbb{R}^3$. Moreover, the action of $GS(\mathbf{P})$ need not be free and the space $\mathbf{C}/GS(\mathbf{P})$ need not be a manifold.

5. Dynamical variables.

The most commonly discussed dynamical variables in Yang-Mills theory are energy, linear momenta, and colour charges. They correspond to time translations, Euclidean motions in $\mathbb{R}^3$ and gauge transformations, respectively.

The time evolution in $\mathbf{P}$ is $GS(\mathbf{P})_0$ invariant, and it induces a 1-parameter group of diffeomorphisms of the reduced phase space space $\check{\mathbf{P}}$. The modified evolution equations in $\mathbf{P}$ are $GS(\mathbf{P})_0$ invariant and Hamiltonian on the constraint set $\mathbf{C}$. The Hamiltonian H, given by (1.31) is $GS(\mathbf{P})_0$ invariant, and it pushes forward to a function $\check{H}$ on $\check{\mathbf{P}}$ such that

$$\check{H}(\rho(\mathbf{A}, \mathbf{E}, \Psi)) = \int_{\mathbb{R}^3} \frac{1}{2}(\mathbf{E}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{B}) + \Psi^\dagger \gamma^0 [\gamma^k (i\partial_k + \mathbf{A}_k) + m] \Psi d_3 x \tag{5.1}$$

The time evolution induces a 1-parameter group of symplectomorphisms of $(\check{\mathbf{P}}, \check{\omega})$ generated by $\check{H}$.

Since $GS(\mathbf{P})_0$ is a normal subgroup of $GS(\mathbf{P})$, the quotient group

$$Colour(\mathbf{P}) = GS(\mathbf{P})/GS(\mathbf{P})_0 \quad . \tag{5.2}$$

acts in $\check{\mathbf{P}}$. Proposition 3.2 implies that its Lie algebra is isomorphic to $\mathfrak{g}$, that is

$$colour(\mathbf{P}) = gs(\mathbf{P})/gs(\mathbf{P})_0 \simeq \mathfrak{g} \; . \tag{5.3}$$

The action is Hamiltonian with the momentum map $\check{J} : \check{\mathbf{P}} \to colour(\mathbf{P})^*$ such that, $J_\zeta|\mathbf{C} = \check{J}_{[\xi]} \circ \rho$ where $\zeta \in [\xi]$. Therefore, for each $[\xi] \in colour(\mathbf{P})$ and each $\check{p} \in \check{\mathbf{P}}$,

$$\check{J}_{[\xi]}(\check{p}) = J_\zeta(p) \, , \tag{5.4}$$

where $\zeta \in [\xi]$ is the physical colour charge labeled by $[\xi] \in colour(\mathbf{P})$ in the classical state $p \in \rho^{-1}(\check{p})$. Eq. (1.25) implies that

$$\check{J}_{[\xi]}(\rho(\mathbf{A}, \mathbf{E}, \Psi)) = -\int_{S_\infty} n\mathbf{E}\zeta \, dS \quad \forall \, \zeta \in [\xi]. \tag{5.5}$$

Using Stokes' Theorem and the constraint equation we can rewrite the expression for the colour charge as a volume integral

$$\check{J}_{[\xi]}(\rho(\mathbf{A}, \mathbf{E}, \Psi)) = \int_{I\!R^3} (-\mathbf{E} \cdot D_{\mathbf{A}}\zeta + \Psi^\dagger \zeta \Psi) d_3x \quad \forall \, \zeta \in [\xi]. \tag{5.6}$$

It is the same expression as (1.18). The integrand on the right hand side depends on the choice of $\zeta \in [\xi]$. However, due to the constraint equation (1.7), the integral depends only on the class $[\xi] \in colour(\mathbf{P})$.

The expression (5.1) for the Hamiltonian is given by an integral over $I\!R^3$ in which the integrands are $GS(\mathbf{P})_0$ invariant local functions of the fields. The same holds for the expressions for the linear momenta. Hence, these integrands push forward to functions on the reduced phase space $\check{\mathbf{P}}$ describing physically observable quantities, namely energy density and the linear momentum density. Since they are given by local functions of the field variables, they satisfy local Poisson bracket relations.

It is tempting to consider the integrand on the right hand side of (5.6) as a function $\check{q}_{[\xi]} : \check{\mathbf{P}} \to L^1(I\!R^3)$ such that

$$\check{q}(\rho(\mathbf{A}, \mathbf{E}, \Psi)) = -\mathbf{E} \cdot D_{\mathbf{A}}\xi + \Psi^\dagger \xi \Psi \tag{5.7}$$

describes the physical density of the colour charge labeled by $[\xi]$ in the classical state $\rho(\mathbf{A}, \mathbf{E}, \Psi)$. For $\check{q}(\rho(\mathbf{A}, \mathbf{E}, \Psi))$ to be well defined it should be independent of the choice of $(\mathbf{A}, \mathbf{E}, \Psi)$ in the $GS(\mathbf{P})_0$ orbit $\rho(\mathbf{A}, \mathbf{E}, \Psi)$ and of the choice of $\zeta \in [\xi]$. The right hand side of (5.7) depends on these choices, but it is invariant under the joint action of $GS(\mathbf{P})_0$ in the orbit $\rho(\mathbf{A}, \mathbf{E}, \Psi)$ and in the coset $[\xi]$. Hence, $\check{q}(\rho(\mathbf{A}, \mathbf{E}, \Psi))$ is well defined only if ζ is a function of $(\mathbf{A}, \mathbf{E}, \Psi)$,

$$\zeta = f_{[\xi]}(\mathbf{A}, \mathbf{E}, \Psi) \, ,$$

which intertwines the action of $GS(\mathbf{P})_0$ in the orbit $\rho(\mathbf{A}, \mathbf{E}, \Psi)$ and the adjoint action in $[\xi]$. In [11] we have shown

Lemma 5.1

A function $f_{[\xi]} : \mathbf{C} \to [\xi]$ which intertwines the action of $GS(\mathbf{P})_0$ in the orbit $\rho(\mathbf{A}, \mathbf{E}, \Psi)$ and the adjoint action in $[\xi]$, that is

$$\phi^{-1} f_{[\xi]}(\mathbf{A}, \mathbf{E}, \Psi)\phi = f_{[\xi]}(\phi^{-1}\mathbf{A}\phi + \phi^{-1}\mathrm{grad}\,\phi, \phi^{-1}\mathbf{E}\phi, \phi\Psi) \tag{5.8}$$

for every $\phi \in GS(\mathbf{P})_0$, can locally depend on the fields $(\mathbf{A}, \mathbf{E}, \Psi)$ only if $[\xi]$ is in the centre of *colour*$(\mathbf{P})$.

If $[\xi]$ is not in the centre of *colour*$(\mathbf{P})$, then every choice of $f_{[\xi]} : \mathbf{C} \to [\xi]$ depends in a non-local way on the fields $(\mathbf{A}, \mathbf{E}, \Psi)$. In this case the corresponding colour charge density $\check{q}_{[\xi]}$ depends also non-locally on $(\mathbf{A}, \mathbf{E}, \Psi)$. Hence, upon quantization, which maps the Poisson brackets to the commutation relations up to local Schwinger terms, the corresponding quantum operator cannot be considered an local observable since it would violate the locality axiom of field theory.

The above argument does not prevent observable colour charge density operators for $[\xi]$ in the centre of *colour*$(\mathbf{P})$, if we choose $f_{[\xi]}$ depending locally on $(\mathbf{A}, \mathbf{E}, \Psi)$. Actually, the only such choice is a constant map, that is $\zeta = f_{[\xi]}(\mathbf{A}, \mathbf{E}, \Psi)$ independent of $(\mathbf{A}, \mathbf{E}, \Psi)$. Moreover, by (5.3), $\zeta \in [\xi]$ implies that ζ is a constant map from $I\!R^3$ to $\mathfrak{g}$, and it follows that the value of ζ is in the centre of $\mathfrak{g}$. In this case, for every Yang-Mills potential $\mathbf{A}$,

$$D_{\mathbf{A}}\zeta = 0\,, \tag{5.9}$$

and

$$\check{q}(\rho(\mathbf{A}, \mathbf{E}, \Psi)) = \Psi^\dagger\zeta\Psi \tag{5.10}$$

is the usual expression for the charge density of the matter fields. This suggests that the phenomenon analogous to the confinement of non-central colour charges appears already on the classical level, [11].

Appendix : Aikawa's decomposition and estimates for Beppo Levi spaces

Let $\mathcal{S}$ denote the Schwarz space of vector valued smooth fast falling distributions on $I\!R^3$. The Fourier transformation $\mathbf{X} \mapsto \mathcal{F}(\mathbf{X})$ is a homeomorphism from $\mathcal{S}$ to $\mathcal{S}$ which extends to a unitary map from $L^2(I\!R^3)$ to $L^2(I\!R^3)$. Given a vector field $\mathbf{X} \in \mathcal{S}$, we write $\widetilde{\mathbf{X}} = \mathcal{F}(\mathbf{X})$ and split this field into

$$\widetilde{\mathbf{X}} = \widetilde{\mathbf{X}}^L + \widetilde{\mathbf{X}}^T \tag{A.1}$$

where the respective components of $\widetilde{\mathbf{X}}_j$ are given as

$$(\widetilde{\mathbf{X}}^L(k))_j = \frac{k_j}{|k|^2} \sum_{i=1}^{3} k_i \widetilde{\mathbf{X}}_i(k) \tag{A.2}$$

$$(\widetilde{\mathbf{X}}^T(k))_j = \sum_{i=1}^{3} \Big(\delta_{ij} - \frac{k_i k_j}{|k|^2} \Big) \widetilde{\mathbf{X}}_i(k) \; . \tag{A.3}$$

The Helmholtz decomposition $\mathbf{X} = \mathbf{X}^L + \mathbf{X}^T$ is defined via inverse Fourier transformation

$$\mathbf{X}^L = \mathcal{F}^{-1}(\widetilde{\mathbf{X}}^L) \quad \text{and} \quad \mathbf{X}^T = \mathcal{F}^{-1}(\widetilde{\mathbf{X}}^T) \; . \tag{A.4}$$

Since $|\widetilde{\mathbf{X}}^L(k)| \leq |\widetilde{\mathbf{X}}(k)|$ we can estimate

$$\|\mathbf{X}^L\|^2_{H^s} = \int (1 + |k|^2)^s |\widetilde{\mathbf{X}}^L| d_3k \leq \int (1 + |k|^2)^s |\widetilde{\mathbf{X}}| d_3k = \|\mathbf{X}\|^2_{H^s} \; , \tag{A.5}$$

and similarly

$$\|\mathbf{X}^T\|^2_{H^s} \leq \|\mathbf{X}\|^2_{H^s} \; . \tag{A.6}$$

Furthermore

$$\mathcal{F}(\operatorname{div} \mathbf{X}) = \sum_{i=1}^{3} k_i \widetilde{\mathbf{X}}_i(k) \quad \text{and} \quad \mathcal{F}((\operatorname{curl} \mathbf{X})_l) = \sum_{i,j=1}^{3} \epsilon_{ijl} k_i \widetilde{\mathbf{X}}_j(k) \; . \tag{A.7}$$

This shows that $\mathbf{X}^L$ and $\mathbf{X}^T$ are uniquely determined by $\operatorname{div} \mathbf{X}$ and $\operatorname{curl} \mathbf{X}$, respectively.

Let $BL_1(L^2(I\!R^3, V))$ be the first Beppo Levi space of distributions with values in a normed vector space V which have a square integrable gradient, [12]. The space $BL_1(L^2(I\!R^3, V))$ is topologized by the norm

$$\|g\|_{\mathcal{B}^1} = \int_{B_1} |g| d_3x + \|\operatorname{grad} g\|_{L^2} \; . \tag{A.8}$$

Lemma A.1

The space $BL_1(L^2(I\!R^3, V))$ has a direct sum decomposition (Aikawa [13])

$$BL_1(L^2(I\!R^3, V)) = \overline{\mathcal{B}_0} \oplus V \tag{A.9}$$

where V is considered as the space of constant functions from $I\!R^3$ to V and $\overline{\mathcal{B}_0}$ is the closure of the space $C_0^\infty(I\!R^3, V)$ of smooth compactly supported functions in the topology of $BL_1(L^2(I\!R^3, V))$ given by the norm (A.8). Moreover

$$\overline{\mathcal{B}_0} = H^1(I\!R^3, V) \; .$$

Proof.

Given $f \in BL_1(L^2(I\!R^3, V))$, let f_n be a sequence in with $\operatorname{grad} f_n \in C_0^\infty(I\!R^3, V) \subset \mathcal{S}$ such that

$$\|\operatorname{grad} f_n - \operatorname{grad} f\|_{L^2} \longrightarrow 0\ . \tag{A.10}$$

Writing $\mathbf{Y}_n = \operatorname{grad} f_n$, the Helmholtz decomposition implies that $\mathbf{Y}_n = \mathbf{Y}_n^L$, and by (A.2)

$$(\widetilde{\mathbf{Y}}_j(k))_n = \frac{k_j}{|k|^2} \sum_{i=1}^{3} k_i (\widetilde{\mathbf{Y}}_j(k))_n\ . \tag{A.11}$$

Since $(\widetilde{\mathbf{Y}}(k))_n \in \mathcal{S}$ we can perform a Taylor expansion at $k = 0$. With Eq. (A.11) we obtain

$$(\widetilde{\mathbf{Y}}_j(k))_n = b_n k_j + o(|k|^2) \tag{A.12}$$

where the sequence b_n of Taylor coefficients is Cauchy, since the sequence $\widetilde{\mathbf{Y}}_n$ converges in the L^2 norm. Hence the sequence

$$\widetilde{h}_n(k) = \frac{1}{|k|^2} \sum_{j=1}^{3} k_j (\widetilde{\mathbf{Y}}_j(k))_n \tag{A.13}$$

is uniformly bounded by a Cauchy sequence d_n in a neighborhood of $k = 0$, that is $|\widetilde{h}_n(k)| \le d_n$ for $|k| < \epsilon$. By an inverse Fourier transformation of $\widetilde{h}_n(k)$ we define

$$h_n := \mathcal{F}^{-1}(\widetilde{h}_n(k))\ . \tag{A.14}$$

Denoting by B_ϵ the ball of radius $\epsilon < 1$ in k-space we get

$$\begin{aligned}
\|h_n\|^2_{H^1} &= \int_{I\!R^3} (1 + |k|^2) |\widetilde{h}_n(k)|^2 d_3k \\
&\le d_n^2 \int_{B_\epsilon} (1 + |k|^2) d_3k + \left(1 + \frac{1}{\epsilon^2}\right) \int_{I\!R^3 \setminus B_\epsilon} |k|^2 |\widetilde{h}_n(k)|^2 d_3k \\
&\le \frac{32\pi}{15} d_n^2 + \left(1 + \frac{1}{\epsilon^2}\right) \int_{I\!R^3} |\widetilde{\mathbf{Y}}_n(k)|^2 d_3k \\
&\le \frac{32\pi}{15} d_n^2 + \left(1 + \frac{1}{\epsilon^2}\right) \|\operatorname{grad} f_n\|^2_{L^2}\ .
\end{aligned} \tag{A.15}$$

This implies that h_n is a H^1 Cauchy sequence. Furthermore, $\operatorname{grad} f_n = \operatorname{grad} h_n$, so that $f_n = h_n + c_n$ with $c_n \in V$. By Eq. (A.10) implies that $h_n \to h$ in $H^1(I\!R^3, V)$ and $c_n \to c$ in V. Therefore $h_n \to h$ and $c_n \to c$ in the topology of $BL_1(I\!R^3, V)$, given by the norm (A.8), and hence $f = h + c$. This prove the Aikawa's decomposition (A.9).

Moreover, by the argument above, $h_n \to h$ also in the norm topology of $H^1(I\!R^3, V)$. Since $C_0^\infty(I\!R^3, V)$ is dense, this proves that $\overline{B_0}(I\!R^3, V) = H^1(I\!R^3, V)$. Q.E.D.

Let the intersection of three Beppo Levi spaces be denoted by

$$\mathcal{B}^3(\mathbb{R}^3, V) := \bigcap_{m=1}^{3} BL_m(L^2(\mathbb{R}^3, V)) \ . \tag{A.16}$$

and topologized by the norm

$$\|\xi\|_{\mathcal{B}^3} = \int_{B_1} |\xi| d_3x + \|\mathrm{grad}\, \xi\|_{H^2} \ . \tag{A.17}$$

Poposition A.2
The space $\mathcal{B}^3(\mathbb{R}^3, V)$ splits into

$$\mathcal{B}^3(\mathbb{R}^3, V) := H^3(\mathbb{R}^3, V) \oplus V \ . \tag{A.18}$$

Moreover, each $f \in \mathcal{B}^3(\mathbb{R}^3, V)$ is continuous and continuously differentiable, and

$$\|f\|_{L^\infty} \leq C\|f\|_{\mathcal{B}^3} \ . \tag{A.19}$$

Proof.
Intersecting the decomposition (A.9) with the two Beppo Levi spaces $BL_2(L^2(\mathbb{R}^3, V)) \cap BL_3(L^2(\mathbb{R}^3, V))$ implies that

$$f = f_0 + c_f \qquad \text{where} \qquad f_0 \in H^3(\mathbb{R}^3, V) \ \text{ and } \ c_f \in V \ . \tag{A.20}$$

This establishes the decomposition (A.18) algebraically. Since V is finite dimensional it is a split subspace, and its complement is closed. This establishes (A.18). The Sobolev imbedding theorem implies that f_0 is continuous and continuously differentiable. Therefore $\mathcal{B}^3(\mathbb{R}^3, V) \subset C^1(\mathbb{R}^3, V)$.

Moreover, the projections $\mathrm{pr}_1 : \mathcal{B}^3(\mathbb{R}^3, V) \to H^3(\mathbb{R}^3, V)$ and $\mathrm{pr}_0 : \mathcal{B}^3(\mathbb{R}^3, V) \to V$ are continuous. Together with the Sobolev embedding theorem this implies that

$$\|f\|_{\mathcal{B}^3} \geq C\big(\|\mathrm{pr}_1 f\|_{H^3} + |\mathrm{pr}_0 f|\big) \geq C'\big(\|f_0\|_{L^\infty} + |c_f|\big) \geq C'\|f_0 + c_f\|_{L^\infty} \tag{A.21}$$

This prove the estimate (A.19). Q.E.D.

Lemma A.3
Let f and g be maps from $\mathbb{R}^3$ to normed vector spaces, and $f \bullet g$ any pointwise multiplication with values in a normed vector space. If $f \in \mathcal{B}^3(\mathbb{R}^3, V)$ then

$$\|f \bullet g\|_{H^1} \leq C_1\|f\|_{\mathcal{B}^3}\|g\|_{H^1} \qquad \forall g \in H^1(\mathbb{R}^3, W) \ , \tag{A.22}$$

$$\|f \bullet g\|_{H^2} \leq C_2\|f\|_{\mathcal{B}^3}\|g\|_{H^2} \qquad \forall g \in H^2(\mathbb{R}^3, W) \ , \tag{A.23}$$

$$\|f \bullet g\|_{\mathcal{B}^3} \leq C_3\|f\|_{\mathcal{B}^3}\|g\|_{\mathcal{B}^3} \qquad \forall g \in \mathcal{B}^3(\mathbb{R}^3, W) \ . \tag{A.24}$$

Proof.

Since $f \in \mathcal{B}^3(\mathbb{R}^3, V)$ implies that $\|f\|_{L^\infty}$ is finite, it is obvious that

$$\|f \bullet g\|_{L^2} \leq \|f\|_{L^\infty}\|g\|_{L^2} \qquad \forall g \in H^1(\mathbb{R}^3, W) \; . \tag{A.25}$$

With an appropriately defined pointwise product $\bullet$ on the right hand side we have

$$\mathrm{grad}\,(f \bullet g) = \mathrm{grad}\,(f) \bullet g + f \bullet \mathrm{grad}\,(g) \; . \tag{A.26}$$

For $f \in \mathcal{B}^3(\mathbb{R}^3, V)$ the Cauchy-Schwarz inequality and (A.25) imply that

$$\|\mathrm{grad}\,(f \bullet g)\|_{L^2} \leq \|\mathrm{grad}\, f\|_{L^2}\|g\|_{L^2} + \|f\|_{L^\infty}\|\mathrm{grad}\, g\|_{L^2} \; , \tag{A.27}$$

and hence by (A.19)

$$\|f \bullet g\|_{H^1} \leq \|f\|_{L^\infty}\|g\|_{L^2} + \|\mathrm{grad}\, f\|_{L^2}\|g\|_{L^2} + \|f\|_{L^\infty}\|\mathrm{grad}\, g\|_{L^2} \leq C_1\|f\|_{\mathcal{B}^3}\|g\|_{H^1} \; .$$

This proves (A.22). Differentiating (A.26), we get

$$D\,\mathrm{grad}\,(f \bullet g) = D\,\mathrm{grad}\,(f) \bullet g + 2\,\mathrm{grad}\,(f) \bullet \mathrm{grad}\,(g) + f \bullet D\,\mathrm{grad}\,(g) \; . \tag{A.28}$$

As above, $f \in \mathcal{B}^3(\mathbb{R}^3, V)$ implies that

$$\begin{aligned}\|D\,\mathrm{grad}\,(f \bullet g)\|_{L^2} \leq \|D\,\mathrm{grad}\, f\|_{L^2}\|g\|_{L^2} &+ 2\,\|\mathrm{grad}\, f\|_{L^2}\|\mathrm{grad}\, g\|_{L^2} \\ &+ \|f\|_{L^\infty}\|D\,\mathrm{grad}\, g\|_{L^2} \; .\end{aligned} \tag{A.29}$$

Therefore, by (A.22) and (A.19),

$$\|f \bullet g\|_{H^2} \leq C_1\|f\|_{\mathcal{B}^3}\|g\|_{H^1} + 2\,\|\mathrm{grad}\, f\|_{H^1}\|g\|_{H^1} + \|f\|_{L^\infty}\|g\|_{H^2} \tag{A.30}$$

which implies (A.23). Finally the estimates given above yield

$$\begin{aligned}\|f \bullet g\|_{\mathcal{B}^3} \leq \|f\|_{L^\infty}\int_{B_1}|g|d_3x + \|f\|_{L^\infty}\|\mathrm{grad}\, g\|_{H^2} &+ \|g\|_{L^\infty}\|\mathrm{grad}\, f\|_{H^2} \\ &+ 2\|\mathrm{grad}\, f\|_{H^1}\|\mathrm{grad}\, g\|_{H^1} \; .\end{aligned} \tag{A.31}$$

Since $\|f\|_{L^\infty} \leq C\|f\|_{\mathcal{B}^3}$ this proves (A.24). Q.E.D.

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