

Nr. 191/95

## Unification of Gravity and Yang-Mills-Higgs Gauge Theories

Thomas Ackermann<sup>1</sup> and Jürgen Tolksdorf<sup>2</sup> \*

<sup>1</sup> Fakultät für Mathematik & Informatik, Universität Mannheim  
D-68159 Mannheim, F.R.G.

<sup>2</sup> Centre de Physique Théorique, CNRS Luminy, Case 907  
F-13288 Marseille Cedex 9, France

**Abstract.** In this letter we show how the action functional of the standard model and of gravity can be derived from a specific Dirac operator. Far from being exotic this particular Dirac operator turns out to be structurally fixed by the Yukawa coupling term. The main feature of our approach to the standard model of elementary particle physics is that it naturally includes gravity.

Keywords: *gravity, Yang-Mills-Higgs, gauge theories, Dirac operator*

---

<sup>1</sup> e-mail: ackerm@euler.math.uni-mannheim.de

<sup>2</sup> e-mail: tolkdorf@cptsu4.univ-mrs.fr

\* Supported by the European Communities, contract no. ERB 401GT 930224

# 1 Introduction

Usually, the bosonic part of the standard model of elementary particles is based on the Yang-Mills-Higgs action which is interpreted as a specific functional  $\mathcal{I}_{YMH}$  defined on the set of triple  $\{g_{\mu\nu}, A_\mu, \varphi\}$ . Here,  $g_{\mu\nu}$  denotes the metric on spacetime  $\mathcal{M}$  (and is usually assumed to be fixed by the flat metric),  $A_\mu$  denotes the gauge potential and  $\varphi$  the Higgs field. In contrast to the bosonic part of the standard model, the fermionic action is defined with respect to a specific Dirac operator  $D_Y$  called the "Dirac-Yukawa" operator. This Dirac operator takes the specific form<sup>3</sup>:

$$D_Y = i\gamma^\mu(\partial_\mu + \omega_\mu + A_\mu) + i\Phi, \quad (1)$$

where  $\omega_\mu := -\frac{1}{4}\gamma^{ab}\omega_{ab\mu}$  denotes the spin connection defined by the Levi-Cevita connection on spacetime  $\mathcal{M}$  (which drops out in the case of  $g_{\mu\nu}$  being flat).  $\Phi$  denotes an odd (anti-hermitian) matrix containing the Higgs field  $\varphi$  in a certain representation and defines the "Yukawa coupling term":

$$i\bar{\psi}\Phi\psi \equiv \sum_{a,b=1}^N \bar{\psi}_a (i\Phi)_{ab} \psi_b. \quad (2)$$

Here, the spin degrees of freedom of the fermions  $\psi$  are suppressed and  $N$  denotes the dimension of the fermion representation of the gauge group  $G$ . Consequently, the fermionic action  $\mathcal{I}_{DY}$  - the Dirac-Yukawa action - can be considered as a specific functional defined on the set  $\{g_{\mu\nu}, A_\mu, \varphi\}$  as well. This is the "usual" point of view of how the action functional of the standard model is understood. Of course, this point of view corresponds to the fact that the specific form of the action serves as one of the main ingredients of a general Yang-Mills-Higgs model building kit.

But there is also another perspective of how the action of the standard model (without gravity!) can be viewed, namely the approach by Connes and Lott which uses the framework of Connes' non-commutative geometry (c.f. [C] and [CL]). In this approach the action functional is derived from a "K-cycle". Moreover, Connes has also mentioned that the specific Dirac operator:

$$D = \gamma^\mu(\partial_\mu + \omega_\mu + A_\mu) \quad (3)$$

is linked to the euclidean Einstein-Hilbert action  $\mathcal{I}_{EH}$  via the Wodzicki residue of the inverse of  $D^2$ . This was explicitly shown in [K] and [KW]. Obviously, this Dirac operator defines the "kinetic" part of the Dirac-Yukawa action. It might be worth mentioning that in the Connes-Lott approach to the standard model the spin connection  $\omega_\mu$  always drops out and therefore the metric  $g_{\mu\nu}$  on spacetime  $\mathcal{M}$  remains indefinite. If on the other hand, one uses the kinetic part  $D$  of the Dirac-Yukawa operator (1) in order to derive the Einstein-Hilbert action - as proposed by Connes - the information contained in the gauge field  $A_\mu$  is lost.

---

<sup>3</sup>Our conventions are specified in the next section.

Following Connes' point of view, where a Dirac operator (K-cycle) is on the input side and the action is on the output side, it is natural to ask whether there is a Dirac operator  $\tilde{D}$  from which both the Einstein-Hilbert and the Yang-Mills-Higgs action can be derived simultaneously. In this letter we answer this question affirmatively. Moreover, we show that this particular Dirac operator is completely determined by the Dirac-Yukawa operator (1).

## 2 Mathematical preliminaries

In order to be specific and to fix our notation we make the following assumptions: let spacetime  $\mathcal{M}$  be described by a four dimensional compact Riemannian spin-manifold without boundary. Then we use the following convention for the Clifford relation:  $\{\gamma^a, \gamma^b\} \equiv \gamma^a \gamma^b + \gamma^b \gamma^a = -2g^{ab}$ . The involution on the "spin-space"  $S (\simeq \mathbb{C}_L^2 \oplus \mathbb{C}_R^2, \text{ locally})$  is denoted by  $\gamma_5$  with the two conditions:  $\gamma_5^2 = \mathbb{1}_S$  and  $\gamma_5^* = \gamma_5$ . We also use the shorthand notation:  $2\gamma^{ab} := [\gamma^a, \gamma^b]$ . Furthermore it is assumed that the "gamma matrices"  $\gamma^a$  define a hermitian representation of the complexified Clifford algebra. Since we are only interested in local statements we also assume that the (hermitian) "gauge-bundle"  $E \mapsto \mathcal{M}$  is trivial, i.e. we assume that  $E = \mathcal{M} \times \mathbb{C}^N$ . In order to take parity violation into account we set:  $\mathbb{C}^N = \mathbb{C}^{n_1} \oplus \mathbb{C}^{n_2}$  ( $N := n_1 + n_2$ ). Consequently, the internal space  $E$  is also  $\mathbb{Z}_2$ -graded and we denote the corresponding involution on  $E$  by  $\chi$ . Since the spinor fields  $\psi = (\psi_L, \psi_R)$  are to be considered as elements of the tensor product  $\mathcal{E} := S \hat{\otimes} E$  of the spin space  $S$  and the internal space  $E$ <sup>4</sup> we can take into account the two  $\mathbb{Z}_2$ -gradings of the respective spaces  $S$  and  $E$  by using the "graded tensor product" - indicated by " $\hat{\otimes}$ " - instead of the usual tensor product<sup>5</sup>. Thus  $\psi(x) \in \mathcal{E}_x = S_x \hat{\otimes} E_x \simeq (\mathbb{C}_L^2 \oplus \mathbb{C}_R^2) \hat{\otimes} \mathbb{C}^N, x \in \mathcal{M}$ . The unitary representation of the gauge group  $G$  (which is assumed to be compact and semi-simple) on the internal space  $E$  is denoted by  $\rho$ , i.e.  $\rho : G \mapsto \mathbf{M}_N(\mathbb{C})$ <sup>6</sup>.

Since the covariant derivative<sup>7</sup>

$$\nabla_\mu := \partial_\mu + \omega_\mu \hat{\otimes} \mathbb{1}_E + \mathbb{1}_S \hat{\otimes} A_\mu \quad (4)$$

defining the Dirac operator  $D$  satisfies the crucial relation:

$$[\nabla_\mu, \gamma^\nu] = -\gamma^\sigma \Gamma_{\sigma\mu}^\nu, \quad (5)$$

it deserves a special name: it is called a Clifford connection. Here,  $\Gamma_{\sigma\mu}^\nu$  denotes the Christoffel symbol. The relation (5) fixes the spin part of a Clifford connection to be unambiguously

<sup>4</sup>I.e.  $\psi = \psi_{sa}$ , where "s" denotes the spin degree of freedom and therefore refers to the spin space  $S$  and "a" denotes the gauge degree of freedom and therefore refers to the internal space  $E$ .

<sup>5</sup>Let us recall that the graded tensor product is defined as follows:  $(A \hat{\otimes} B)(A' \hat{\otimes} B') := (-1)^{|B||A'|} AA' \hat{\otimes} BB'$ , for all  $A, A', B, B' \in \text{End}(\mathcal{E})$ , with  $|B|, |A'| \in \{0, 1\}$  depending on whether  $B, A'$  are even or odd elements with respect to  $\chi$  and  $\gamma_5$ . The total grading on  $\mathcal{E}$  is defined by  $\gamma_5 \hat{\otimes} \chi$ .

<sup>6</sup>From a mathematical point of view most of these assumptions are redundant. For the general setting we refer to [AT].

<sup>7</sup>Here,  $\mathbb{1}_S$  and  $\mathbb{1}_E$  denotes the identity elements in  $\text{End}(S) (\simeq \mathbf{M}_4(\mathbb{C}), \text{ locally})$  and  $\text{End}(E) \simeq \mathbf{M}_N(\mathbb{C})$ , respectively. Note that we have ignored the tensor product structure of  $\mathcal{E}$  in the introduction.

defined by the Levi-Cevita connection. In other words: two Clifford connections can differ only with respect to the gauge field  $A_\mu$ .

As a consequence of (5) we can introduce the following covariant section:

$$\xi_\mu := -\frac{1}{4}g_{\mu\nu}\gamma^\nu \hat{\otimes} \mathbb{I}_E \quad (6)$$

with the property of being covariantly constant with respect to any Clifford connection (4). Additionally it fulfills:

$$\gamma^\mu \xi_\mu = \mathbb{I}_E. \quad (7)$$

We now introduce the Wodzicki function  $\mathcal{W}_\mathcal{E}$ . Let us denote by  $\mathcal{D}(\mathcal{E})$  the affine space of all Dirac operators  $\tilde{D}$  defined on  $\mathcal{E}$  which satisfy<sup>8</sup>

$$[\tilde{D}, f] = \gamma^\mu \partial_\mu f \quad (8)$$

for all  $f \in C^\infty(\mathcal{M})$ . Then the Wodzicki function is the particular functional on  $\mathcal{D}(\mathcal{E})$

$$\begin{aligned} \mathcal{W}_\mathcal{E} : \mathcal{D}(\mathcal{E}) &\mapsto \mathbb{C} \\ \tilde{D} &\mapsto \mathcal{W}_\mathcal{E}(\tilde{D}) := -\frac{1}{6N} \text{Res}(\tilde{D}^{-2}). \end{aligned}$$

Here, *Res* denotes the Wodzicki residue which generalizes the residue of Adler and Manin (c.f. [W], [A] and [M]). It can be shown that  $\text{Res}(\tilde{D}^{-2})$  is strongly related to the subleading term of the asymptotic expansion of the heat kernel  $\exp(-\tau \tilde{D}^2)$  of  $\tilde{D}^2$  (c.f. [G] and [KW]). Moreover, using a generalized version of the Lichnerowicz formula for the decomposition of  $\tilde{D}^2$  we have shown that (c.f. [AT])

$$\mathcal{W}_\mathcal{E}(\tilde{D}) := \frac{1}{|\mathcal{E}|} \int_{\mathcal{M}} \text{tr}_\mathcal{E} \left( \mathcal{F}(\tilde{D}) - \frac{r_\mathcal{M}}{6} \mathbb{I}_\mathcal{E} \right) \sqrt{|g|} d^4x, \quad (9)$$

with  $|\mathcal{E}| := \dim \mathcal{E}$  and  $|g| := \det[g_{\mu\nu}]$ . Here,  $r_\mathcal{M}$  denotes the Ricci scalar of  $\mathcal{M}$  and the homomorphism  $\mathcal{F}$  takes the explicit form:

$$\begin{aligned} \mathcal{F}(\tilde{D}) &:= \frac{1}{4} r_\mathcal{M} \mathbb{I}_\mathcal{E} + \frac{1}{2} \gamma^{\mu\nu} \hat{\otimes} F_{\mu\nu} \\ &+ \frac{1}{2} [\gamma^\mu [\omega_\mu, \gamma^\nu], \omega_\nu] + \gamma^{\mu\nu} (\nabla_\mu \omega_\nu) - \frac{1}{2} \gamma^\mu [(\nabla_\nu \omega_\mu), \gamma^\nu] \\ &+ \frac{1}{2} \gamma^{\mu\nu} [\omega_\mu, \omega_\nu] + \frac{1}{4} g_{\mu\nu} \gamma^\sigma [\omega_\sigma, \gamma^\mu] \gamma^\kappa [\omega_\kappa, \gamma^\nu], \end{aligned} \quad (10)$$

with  $\omega_\mu := \xi_\mu(\tilde{D} - D)$ . Since  $\tilde{D} - D \in \text{End}(\mathcal{E})$ , here, the notation  $\xi_\mu(\tilde{D} - D)$  means the product in  $\text{End}(\mathcal{E})$ .  $D$  is defined with respect to any Clifford connection (4) and  $F_{\mu\nu}$  denotes the Yang-Mills field strength with respect to the gauge field  $A_\mu$ . Note that  $A_\mu \in \rho_*(\mathcal{G}) \subset \text{End}(E)$ , where

<sup>8</sup>This relation can actually be considered as a definition of a Dirac operator compatible with the given Clifford structure on  $\mathcal{E}$  since the space  $\mathcal{E}$  not only denotes a  $\mathbb{Z}_2$ -graded vectorbundle over  $\mathcal{M}$  but also a "Clifford modul".

$\mathcal{G}$  denotes the Lie-algebra of the gauge group  $G$  and  $\rho_*$  the corresponding homomorphism between  $\mathcal{G}$  and  $End(E)$  induced by the representation  $\rho$ . The covariant derivative  $'\nabla_\mu$  denotes the induced Clifford connection on  $T^*\mathcal{M} \otimes End(\mathcal{E})$ , i.e.

$$' \nabla_\mu \omega_\nu := [\nabla_\mu, \omega_\nu] - \omega_\sigma \Gamma_{\mu\nu}^\sigma. \quad (11)$$

Note that the second and the third term in (10) always drop out when calculating  $\mathcal{W}_\mathcal{E}$ .

We are now in the position to define our approach to the standard model. After we have fixed the general scheme we shall introduce a particular Dirac operator from which the full action functional of the standard model can be derived.

### 3 Model building kit

#### 3.1 The general scheme

In this subsection we introduce a new model building kit for gauge theories similar to the Connes-Lott scheme without, however, using the mathematical framework of non-commutative geometry (c.f. [CL], [SZ], [KS] and [IS]). The main feature of the kit proposed here is that it naturally includes gravity and therefore unifies the latter with gauge theories.

The "input" of our model building kit is given by the following triple:

$$(G, \rho, \tilde{D}). \quad (12)$$

Again  $G$  denotes a (compact, semi-simple) gauge group,  $\rho$  its representation concerning an internal space  $E$  and  $\tilde{D}$  any given Dirac operator on  $\mathcal{E} := S \hat{\otimes} E$  satisfying the relation (8), i.e.  $\tilde{D} \in \mathcal{D}(\mathcal{E})$ . The "rule" of our kit simply consists in calculating the Wodzicki function  $\mathcal{W}_\mathcal{E}(\tilde{D})$  concerning the Dirac operator  $\tilde{D}$ . This means that one has to calculate the traces over the gamma matrices regarding the last four terms in (10). The "output" is then a specific action functional for the gauge theory in question. Of course, the chosen Dirac operator also defines a particular fermionic action. This is our general scheme. We now use this general building kit in order to derive the combined Einstein-Hilbert-Yang-Mills-Higgs action. As we shall see this follows from (10) in case that we use a specific Dirac operator  $\tilde{D}$  which also defines the Dirac-Yukawa action.

#### 3.2 The Einstein-Yang-Mills-Higgs building kit

We define the following particular Dirac operator:

$$\tilde{D} := \begin{pmatrix} \lambda_1 D_{\hat{\Phi}} & \gamma^{\mu\nu} \hat{F}_{\mu\nu} \\ -\gamma^{\mu\nu} \hat{F}_{\mu\nu} & \lambda_2 D_{\hat{\Phi}} \end{pmatrix}. \quad (13)$$

Here, the diagonal part of our Dirac operator is defined by the Dirac-Yukawa operator  $D_{\hat{\Phi}} := -i D_\gamma$  where we have re-scaled the Yukawa coupling term  $\Phi \equiv \mathbb{I}_S \hat{\otimes} \phi$  (with  $\phi$  odd and anti-hermitian) by  $\hat{\Phi} := a\Phi$ .

The off-diagonal term is fixed by the element

$$\hat{F}_{\mu\nu} := [\nabla_\mu + \omega_\mu, \nabla_\nu + \omega_\nu] - \frac{1}{4} R_{ab\mu\nu} \gamma^{ab} \hat{\otimes} \mathbb{I}_E \quad (14)$$

which can be considered as a "relative curvature" on  $\mathcal{E}$ <sup>9</sup>. Here,  $\omega_\mu$  is given by

$$\omega_\mu = -\frac{ib}{4} g_{\mu\nu} \gamma^\nu \hat{\otimes} \phi, \quad (15)$$

$R_{ab\mu\nu}$  denotes the Riemannian curvature on  $\mathcal{M}$  and  $\lambda_1, \lambda_2, a, b$  are real constants. Because of (15) the Higgs field can be considered as defining a particular connection  $\tilde{\nabla}_\mu = \nabla_\mu + \omega_\mu$  on  $\mathcal{E}$ . Consequently, the Higgs field generalizes the Yang-Mills curvature on the gauge bundle  $E$ , whereby the whole Dirac operator  $\tilde{D}$  is fully defined with respect to the Yukawa coupling term  $\Phi$ . Let us emphasize that the Dirac-Yukawa operator is also mathematically distinguished since it is of "simple type" (c.f. [AT]). Therefore, the Dirac operator (13) is indeed natural.

Regarding this particular Dirac operator (13) the Wodzicki function (9) reads:

$$\begin{aligned} \mathcal{W}_\mathcal{E}(\tilde{D}) &= \alpha_o \lambda_1 \lambda_2 l_p^{-2} \int_{\mathcal{M}} r_{\mathcal{M}} \sqrt{|g|} d^4x \\ &+ \alpha_2 \int_{\mathcal{M}} \text{tr}(F^{\mu\nu} F_{\mu\nu}) \sqrt{|g|} d^4x \\ &+ \alpha_3 b^2 \int_{\mathcal{M}} \text{tr}((\nabla_\mu \phi)^* \nabla^\mu \phi) \sqrt{|g|} d^4x \\ &+ \int_{\mathcal{M}} \text{tr}(b^4 \alpha_4 (\phi^* \phi)^2 - \alpha_1 \lambda_1 \lambda_2 a^2 l_p^{-2} (\phi^* \phi)) \sqrt{|g|} d^4x, \end{aligned} \quad (16)$$

with  $\alpha_o = 1/12$ ,  $\alpha_1 = 1/\dim E$ ,  $\alpha_2 = (1/2)\alpha_1$ ,  $\alpha_3 = (9/8)\alpha_1$ ,  $\alpha_4 = (3/2)\alpha_3$ .

Here, the traces have to be performed within  $\text{End}(E)$  and therefore depend on the specific representation  $\rho$  which is chosen in the input. Note that we have already introduced a physical length scale which is fixed by the Planck length  $l_p$ .

Of course, the fermionic part of the standard model is given by the Dirac-Yukawa action:

$$\mathcal{I}_{DY} = \int_{\mathcal{M}} \bar{\psi} D\psi \sqrt{|g|} d^4x + b \int_{\mathcal{M}} \bar{\psi} (i\Phi) \psi \sqrt{|g|} d^4x. \quad (17)$$

We remark that the relative constant in the off-diagonal term of the Dirac operator  $\tilde{D}$  is fixed by the relative sign between the Einstein-Hilbert and the Yang-Mills action (which has to be positive here) and that the Dirac-Yukawa action must be real. Indeed, the Dirac operator (13) is anti-selfadjoint.

Let us denote by  $\mathcal{I}_{bosonic}$  and by  $\mathcal{I}_{fermionic}$  the bosonic and the fermionic action, respectively. If we then define  $\Psi := (\psi, \psi)$  we can summarize our result as follows:

$$\mathcal{I}_{bosonic} := \mathcal{W}_\mathcal{E}(\tilde{D}) \sim \mathcal{I}_{EHYM},$$

<sup>9</sup>Actually, it can be shown that this relative curvature is but the "super curvature"  $F_\phi := (\nabla^E + ib\phi)^2$  on the internal space  $E$ . Thus the Higgs field can also be interpreted geometrically as a part of the particular superconnection  $\nabla^E + ib\phi$  on  $E$  (c.f. [NS]). Note that  $\nabla^E + \phi$  uniquely defines the corresponding super connection of the Dirac-Yukawa operator  $D_Y$ .

$$\mathcal{I}_{fermionic} := \langle \Psi, i\tilde{D}\Psi \rangle \sim \mathcal{I}_{DY}. \quad (18)$$

These two action functionals will always be on the output side when the Einstein-Yang-Mills-Higgs building kit is concerned:

$$(G, \rho, \tilde{D}), \quad (19)$$

with  $\tilde{D}$  defined by (13). We emphasize that in this particular case the Higgs representation is always contained in the fermionic representation.

## 4 Conclusion

We have shown that the whole action functional of the standard model and of gravity can be derived from a certain Dirac operator. This operator, in turn, is defined by the Yukawa coupling term generating the masses of the fermions.

Moreover, our approach unifies gravity with Yang-Mills-Higgs gauge theories. The Einstein-Hilbert action arises as a natural “companion” of the Yang-Mills-Higgs functional. Therefore, the metric in Yang-Mills gauge theories is no longer fixed “by hand” but by the Einstein equation, which is very satisfying, conceptually.

From a geometrical point of view the full action functional of the standard model can be considered as encoded within the subleading term of the asymptotic expansion of the heat kernel corresponding to the particular Dirac operator which we have introduced in the last section. Consequently, it might be interesting also to investigate the next terms of the expansion which could serve as a kind of correction terms to the classical action.

## References

- [A] M.Adler, *Invent. Math.* 50, 219-248 (1979) On a trace functional for formal pseudo-diff. operators and the symplectic structure of Korteweg-de Vries type equations.
- [AT] T.Ackermann, J.Tolksdorf, The generalized Lichnerowicz formula and analysis of Dirac operators, Preprint CPT-94/P.3106 and Mannheimer Manuskripte 181, (1994)
- [C] A.Connes, Non-commutative geometry and physics, to be published in the proceedings of the 1992 Les Houches Summer School.
- [CL] A.Connes, J.Lott, Particle models and non-commutative geometry, *Nucl. Phys. B Proc. Supp.* 18B (1990) 29-47.
- [G] P.Gilkey, *Invariance Theory, The Heat Equation, And the Atiyah-Singer Index Theorem*, Publish or Perish (1984).
- [IS] B.Iochum, Th. Schücker, Yang-Mills-Higgs versus Connes-Lott, CPT-94/P.3090.
- [K] D.Kastler, The Dirac operator and gravitation, to appear in *Com. Math. Phys.*
- [KS] D.Kastler, Th.Schücker, A detailed account of Alain Connes' version of the standard model in non-commutative geometry, 4, CPT-94/P.3092 (1994).
- [KW] W.Kalau, M.Walze, Gravity, non-commutative geometry and the Wodzicki residue, to appear in *Journ. of Geometry and Physics*.
- [M] Yu.I.Manin, *J. Soviet Mathematics* 11, 1-122 (1979), Algebraic aspects of non-linear differential equations.
- [NS] Y.Ne'eman, S.Sternberg, *Internal Supersymmetry and Superconnections*, *Symplectic Geometry and Math. Physics*, Ed. by P.Donato, C.Durce, J.Elhadad, G.M.Tuymmann, Birkhäuser 1991.
- [SZ] Th.Schücker, J.-M.Zylinski, Connes' model building kit, *J. Geom. Phys.* 16 (1994) 1.
- [W] M.Wodzicki, Non-commutative residue I, *LNM* 1289 (1987), 320-399.