From a discrete setting to a smooth idealized skin

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In memoriam G. Reeb and J.L. Callot

0 Introduction

The purpose of these notes is to show how the geometric description of a deformable continuum, here an idealized skin, can be based on the collection of finitely many interacting particles which constitute the medium.

Let us make this more precise: We assume a finite collection $P' \subset \mathbb{R}^n$ of points to be given. Each point is thought of as the mean location of a material particle. The interaction shall be such that there is a smooth, compact, connected and oriented manifold $S' \subset \mathbb{R}^n$ of $dim\ S' \geq 1$ passing through P'. S' is the macroscopic, differential geometric visualization of the skin.

The problem we are confronted with is hence to derive, out of the interaction scheme of the particles, a differential geometric ingredient on S' characterizing a deformable medium in the sense of continuum mechanics. In doing so, we will not pass to a limit such as enlarging the number of interacting particles, e.g. Neather we will make use of an approximation.

Here is what we do: Let S be an abstract manifold diffeomorphic to S' and $P \subset S$ be a collection of points with the same cardinality as P'. $\mathbb{R}^n$ shall be equipped with a fixed scalar product $<,>$. We base our characterization of the discrete medium on the principle of virtual work (cf. [Bi1],[Bi2] and [M,H]). The virtual work on the discrete level is here assumed to be a special kind of a one-form A_P on the configuration space $E^\infty(P, \mathbb{R}^n)$, consisting of embeddings of P into $\mathbb{R}^n$ (to be specified below). (This makes it already clear that the realm in which we work is rather simplified from a continuum mechanical point of view. We, however, do so to present the general principles we develop here in a simple fashion). Since the configuration space is finite dimensional, A_P, assumed to be smooth, admits a configuration dependent smooth force Φ_P, formed with respect to the naturally given scalar product $\mathcal{G}_P$ on $E^\infty(P, \mathbb{R}^n)$. More precisely

$$A_P(j_P)(h_P) = \mathcal{G}_P\left(\Phi_P(j_P), h_P\right) := \sum_{q \in P} < \Phi_P(j_P)(q), h_P(q) > \qquad (0.1)$$

for any $j_P \in E^\infty(P, \mathbb{R}^n)$ and any h_P in the finite dimensional vector space $\mathcal{F}(P, \mathbb{R}^n)$ of all $\mathbb{R}^n$-valued maps of P. We think of h_P as a distortion of $j_P(P)$. Since Φ_P shall be an inner force, we assume that it is invariant under the translation group $\mathbb{R}^n$ of $\mathbb{R}^n$. Moreover, no constant distortion $z \in \mathbb{R}^n$ shall cause any virtual work at any configuration j_P, i.e. $A_P(j_P)(z) = 0$.

To specify the type of interaction we let P be the collection of all zero-simplices of an oriented, simplicial one-complex $\boldsymbol{L} \subset S$. We say that a particle at $q \in P$

interacts with one at q' provided q and q' bound the same one-simplex. A point q' is called a nearest neighbour of q if q and q' are the (mean) location of interacting particles. Thus we let the interaction scheme to be the one of nearest neighbour interaction (again a rather simple set-up). This sort of interaction, however, requires us to restrict A_P to a (rather small) open set $O_P \subset E^\infty(P, \mathbb{R}^n)$, since in practice the interactions are determined by distance depending potentials. $\boldsymbol{L}$ determines a Laplacian Δ_T, acting on $\mathcal{F}(P, \mathbb{R}^n)$. The assumption we made on Φ_P yields the representation

$$\Phi_P(j_P) = \Delta_T \mathcal{H}_P(j_P) \quad \forall\, j_P \in O_P \tag{0.2}$$

for some map $\mathcal{H}_P \in C^\infty(O_P, \mathcal{F}(P, \mathbb{R}^n))$. We called $\mathcal{H}_P$ the constitutive map of the medium. It is the equivalent to the first Piola-Kirchhoff stress tensor in continuum mechanics (cf. [Bi2]).

Exactly in the same way we characterize a deformable continuum: Let $E(S, \mathbb{R}^n)$ be the collection of all smooth embeddings of S into $\mathbb{R}^n$, a Fréchet manifold if endowed with the C^∞-topology. Fixing $j_0 \in E(S, \mathbb{R}^n)$ there is a Riemannian metric $m(j_0) := j^* <,>$ and hence the associated L_2-scalar product $\mathcal{G}(j_0)$ on $C^\infty(S, \mathbb{R}^n)$. Given a smooth internal force density Φ on an open set $O \subset E(S, \mathbb{R}^n)$ (accordingly restricted as Φ_P) there is a smooth map $\hat{\mathcal{H}} \in C^\infty(O, C^\infty(S, \mathbb{R}^n))$ such that

$$\Phi(j) = \Delta(j_0)\hat{\mathcal{H}}(j) \quad \forall\, j \in O; \tag{0.3}$$

here $\Delta(j_0)$ is the Laplacian of $m(j_0)$. The virtual work caused by Φ is called A. (We refer at this point to a reformulation of the classical Dirichlet integral for $\Delta(j_0)$ in the appendix).

The link between the two descriptions is as follows: Given A_P on O_P, we lift it up to some open set $O \subset E(S, \mathbb{R}^n)$. The idea is that $O \subset \mathcal{W}^\infty(j_0) \times \mathcal{F}^\infty(S, \mathbb{R}^n)^\perp$ where $\mathcal{F}^\infty(S, \mathbb{R}^n)^\perp \subset C^\infty(S, \mathbb{R}^n)$, say, is an infinite dimensional subspace and where $\mathcal{W}^\infty(j_0)$ is diffeomorphic to O_P, i.e. $\mathcal{W}^\infty(j_0)$ is a slice of some projection π_∞ of $C^\infty(S, \mathbb{R}^n)$ to a finite dimensional subspace $\mathcal{F}^\infty(S, \mathbb{R}^n)$, say. This slice is of the form $j_0 + \mathcal{W}^\infty(O')$ where $\mathcal{W}^\infty(O')$ is an open neighbourhood of $0 \in \mathcal{F}^\infty(S, \mathbb{R}^n) \subset C^\infty(S, \mathbb{R}^n)$. This subspace $\mathcal{F}^\infty(S, \mathbb{R}^n)$ is such that it is invariant under $\Delta(j_0)$, a requirement in accordance with (0.3), and is moreover isomorphic to $\mathcal{F}(P, \mathbb{R}^n)$ via the restriction map r. It hence is generated by $\mathbb{R}^n$ and some eigenvectors of $\Delta(j_0)$ with non-vanishing eigenvalues. The eigenvectors are chosen such that the $tr\ \Delta(j_0)|\mathcal{F}^\infty(S, \mathbb{R}^n)$ is as small as possible. The above mentioned space $\mathcal{F}^\infty(S, \mathbb{R}^n)^\perp$ is generated by all eigenvectors of $\Delta(j_0)$ not in $\mathcal{F}^\infty(S, \mathbb{R}^n)$. We let thus $O = \mathcal{W}^\infty(j_0) + O''$ with $O'' \subset \mathcal{F}^\infty(S, \mathbb{R}^n)^\perp$. The projection π_∞ selects hence a certain finite sum of terms in the Fourier series. Setting $r_\infty := r \circ \pi_\infty$ we let $A := r_\infty^* A_P$ on $r_\infty^{-1} O_P = O$. We form the pull back $r_\infty^* \mathcal{G}_P$ of the metric $\mathcal{G}_P$ by $r_\infty | \mathcal{W}^\infty(j_0)$ to $\mathcal{W}^\infty(j_0)$. Moreover, we observe that $r_\infty^* \mathcal{G}_P$ is related to $\mathcal{G}(j_0)$ by some $\rho \in C^\infty(S, \mathbb{R})$, turning Φ_P into a force density. Then we determine the constitutive map $\hat{\mathcal{H}}$ according to (0.3). The map $\hat{\mathcal{H}}$ characterizes by definition the skin made up by finitely many particles and determines hence its first Piola-Kirchhoff stress tensor (cf. [Bi2]).

Then we determine the exact parts $I\!\!D\bar{F}$ of A on $\mathcal{W}^\infty(j_0)$ respectively $I\!\!D\bar{F}_P$ of A_P on O_P, and obtain, by construction, that $\bar{F} = r_\infty^* \bar{F}_P$ holds on the slice $\mathcal{W}^\infty(j_0)$. ($\bar{F}$ is linked to the notion of free energy associated with some observable within a Gibbs statistics (cf. [Bi3])). $\bar{F}$ has the form

$$\bar{F} = \frac{1}{2}(a \cdot \mathcal{A}) - \frac{1}{2} \cdot G + \ const,$$

where a is the structural capillarity, determining the amount of work caused by distorting the area of S and where $\mathcal{A}$ is the area functional. G reflects in particular the (possible) non-linear dependence of A on the configuration $j \in \mathcal{W}^\infty(j_0)$.

We then illustrate the mechanism just described in case that the internal force Φ_P is determined by a potential.

Finally we define the notion of a fitting surface $j_0(S)$ passing through $j_0(P)$, within our framework. $j_0 \in O$ has to be an equilibrium configuration for which hence $A(j_0) = 0$ holds true and for which $\rho = 1$. For a dynamics we refer to [Bi,Sch].

1 The spaces of configurations

Throughout these notes S denotes a smooth, compact, oriented and connected manifold (without boundary) of $dim\ S \geq 1$. The space of configurations of S is $E(S, I\!\!R^n)$, the collection of all smooth embeddings of S into $I\!\!R^n$ equipped with the C^∞-topology. This set is open in $C^\infty(S, I\!\!R^n)$, the Fréchet space (carrying the C^∞-topology) of all smooth $I\!\!R^n$-valued maps of S (cf. [Bi,Fi1], [Bi,Sn,Fi],[G,H,V] and [M,H]).

The analogon of $C^\infty(S, I\!\!R^n)$ for a finite collection $P \subset S$ of points is $\mathcal{F}(P, I\!\!R^n)$, the $I\!\!R$-vector space of all $I\!\!R^n$-valued maps of P. The restriction map $r : C^\infty(S, I\!\!R^n) \longrightarrow \mathcal{F}(P, I\!\!R^n)$ is surjective. Due to the link between P and S we have in mind, we choose the configuration space of P to be $r\,(E(S, I\!\!R^n))$, called $E^\infty(P, I\!\!R^n)$. It is open in $\mathcal{F}(P, I\!\!R^n)$. Let $<,>$ be a fixed scalar product on S. Each $j \in E^\infty(S, I\!\!R^n)$ defines a scalar product on $C^\infty(S, I\!\!R^n)$ given by

$$\mathcal{G}(j)(h,k) = \int_S < h, k > \mu(j) \quad \forall\, h, k \in C^\infty(S, I\!\!R^n), \tag{1.1}$$

where $\mu(j)$ is the Riemannian volume form on S given by the Riemannian metric $m(j) := j^* <,>$. The metric $\mathcal{G}(j)$ depends smoothly on j.

2 Characterization of the media

We assume that the particles located (in the mean) at the points of P interact within the nearest neighbour interaction scheme. To make this precise we assume an oriented simplicial one-complex $\boldsymbol{L} \subset S$ to be given. P shall be the collection of all zero-simplices.

We hence have the finite dimensional spaces $\mathcal{F}(P, I\!R^n)$ and $C^1(\boldsymbol{L}, I\!R^n)$ of all $I\!R^n$-valued zero- and one-cochains respectively. These two spaces are connected with the simplicial coboundary operator

$$\partial^1 : \mathcal{F}(P, I\!R^n) \longrightarrow C^1(\boldsymbol{L}, I\!R^n).$$

Both spaces carry a metric namely $\mathcal{G}_P$ and $\mathcal{G}_P^1$ given respectively by

$$\mathcal{G}_P(h_P, k_P) = \sum_{q \in P} < h_P(q), k_P(q) > \quad \forall\, h_P, k_P \in \mathcal{F}(P, I\!R^n) \tag{2.1}$$

and

$$\mathcal{G}_P^1(\alpha_P, \beta_P) = \sum_{\sigma \in \boldsymbol{L}_1} < \alpha(\sigma), \beta(\sigma) > \quad \forall\, \alpha_P, \beta_P \in C^1(\boldsymbol{L}, I\!R^n) \tag{2.2}$$

with $\boldsymbol{L}_1$ being the collection of all one-simplices of $\boldsymbol{L}$. Defining the divergence δ^1 by

$$\mathcal{G}_P(\delta^1 \alpha_P, h_P) = \mathcal{G}^1(\alpha_P, \partial h_P) \tag{2.3}$$

for all $\alpha_P \in C^1(\boldsymbol{L}, I\!R^n)$ and all $h_P \in \mathcal{F}(P, I\!R^n)$ yields the Laplacian

$$\Delta_T := \delta^1 \circ \partial^1. \tag{2.4}$$

This Laplacian is the basic geometric ingredient to formulate the constitutive law, i.e. to define the type of the medium under consideration.

We assume that the medium is determined by a smooth map

$$\Phi_P : O_P \subset E^\infty(P, I\!R^n) \longrightarrow \mathcal{F}(P, I\!R^n) \tag{2.5}$$

defined on a specified open set O_P. Its value $\Phi_P(j_P)$ at each $j_P \in O_P$ is thought of as the **internal force** resisting any deformations $h_P \in \mathcal{F}(P, I\!R^n)$. The virtual work A_P caused by h_P is defined by

$$A_P(j_P)(h_P) = \mathcal{G}_P\left(\Phi_P(j_P), h_P\right) \tag{2.6}$$

for all $j_P \in O_P$ and all $h_P \in \mathcal{F}(P, I\!R^n)$. Internality of Φ shall be characterized by the following two requirements:

$$a) \quad \Phi_P \quad \text{is invariant under the translation group } I\!R^n \text{ of } I\!R^n \tag{2.7}$$

and

$$b) \quad A_P(j_P)(z) = 0 \quad \forall\, j_P \in O_P \quad \text{and} \quad \forall\, z \in I\!R^n. \tag{2.8}$$

The latter property says that constant deformations cause no virtual work and it is obviously equivalent with

$$b') \quad \sum_{q \in P} \Phi_P(j_P)(q) = 0 \quad \forall\, j_P \in O_P. \tag{2.9}$$

This, however, is the integrability condition for solving the equation

$$\Delta_T \mathcal{H}_P(j_P) = \Phi_P(j_P) \quad \forall\, j_P \in O_P. \tag{2.10}$$

As it is easy to verify, there is a solution $\mathcal{H}_P$ to (2.10) smooth on O_P. $\mathcal{H}_P : O_P \longrightarrow \mathcal{F}(P, \mathbb{R}^n)$ is called the **constitutive map** of the **discrete** medium. In characterizing the discrete medium we may thus specify either one of A_P, $\mathcal{H}_P$ or Φ_P. We call $j_P^0 \in O_P$ an **equilibrium configuration** if $\Phi_P(j_P^0) = 0$. The following is now obvious (cf. [B]):

Theorem 2.1 *Let $\mathcal{H}_P$ be a constitutive map on $O_P \subset E^\infty(P, \mathbb{R}^n)$ and let the number of nearest neighbours of any $q \in P$ be $k(q)$. Since for each $j_P \in O_P$*

$$\Delta_T \mathcal{H}_P(j_P)(q) = k(q) \cdot \mathcal{H}_P(j_P)(q) - \sum_{i=1}^{k(q)} \mathcal{H}_P(j_P)(q_i), \tag{2.11}$$

the left hand side is the resulting force of all the interaction forces $\mathcal{H}_P(j_P)(q) - \mathcal{H}_P(j_P)(q_i)$, off equilibrium, at j_P for all $i = 1, ..., k(q)$. Vice versa if all these interaction forces for all $q \in P$ are given, then $\mathcal{H}_P$ exists provided (2.9) is satisfied. If j_P^0 is an equilibrium configuration, we may assume that $\mathcal{H}_P(j_P) = 0$.

In the same spirit we characterize a deformable medium on S, i.e. a continuum. Any configuration j in an open subset $O \subset E(S, \mathbb{R}^n)$ yields a Riemannian metric $m(j) := j^* <,>$ of which its Laplacian is denoted by $\Delta(j)$.

An **internal force density** Φ is a smooth map $\Phi : O \longrightarrow C^\infty(S, \mathbb{R}^n)$ satisfying the following two conditions

$a)$ Φ is invariant under the translation group $\mathbb{R}^n$ of $\mathbb{R}_n$

$b)$ $\int_S < \Phi(j), z > \mu(j) = 0 \quad \forall j \in O \quad \text{and} \quad \forall z \in \mathbb{R}^n.$

The last requirement yields a smooth **constitutive map** $\mathcal{H} : O \longrightarrow C^\infty(S, \mathbb{R}^n)$ solving the equation

$$\Delta(j)\mathcal{H}(j) = \Phi(j) \quad \forall j \in O$$

(cf. [Bi1],[Bi2],[Hö] and [Bi,Fi2]). The configuration $j_0 \in O$ is called an **equilibrium configuration** if $\Phi(j_0) = 0$. If we want to describe the virtual work A given by

$$A(j)(h) = \mathcal{G}(j)(\Delta(j)\mathcal{H}(j), h) = 0 \quad \forall j \in O \quad \text{and} \quad \forall h \in C^\infty(S, \mathbb{R}^n)$$

with respect to a fixed configuration, $j_0 \in O$, say, we solve

$$det\ f(j) \cdot \Delta(j)\mathcal{H}(j) = \Delta(j_0)\hat{\mathcal{H}}(j_0). \tag{2.12}$$

Here $f(j) \in End\ TM$ is such that

$$m(j_0)\left(f^2(j)v, w\right) = m(j)(v, w) \quad \forall v, w \in T_qM \quad \text{and} \quad \forall q \in S$$

(cf. A1.3). Again there is a smooth solution $\hat{\mathcal{H}} : O \longrightarrow C^\infty(S, \mathbb{R}^n)$ to (2.12) (cf. [Bi,Fi2]). Thus we have

$$A(j)(h) = \mathcal{G}(j_0)(\Delta(j_0)\hat{\mathcal{H}}(j), h)$$

for all $j \in O$ and any $h \in C^\infty(S, \mathbb{R}^n)$.

This is a rather rough classification of deformable media; it would not be sufficient to describe plasticity e.g. To achieve the latter, one would have to replace $d\mathcal{H}$ and dh by general one-forms: Moreover, one should extend the whole formalism to the phase space to include fluid phenomena. The setting we choose is for the sake of simplicity. It can be generalized appropriately (cf. [W] and [Bi1]).

3 The link between the two descriptions

The link between the two descriptions is made by the restriction map $r : E(S, I\!R^n) \longrightarrow E^\infty(P, I\!R^n)$. We base our construction on a fixed $j_0 \in E^\infty(S, I\!R^n)$ (it will be an equilibrium configuration, occasionally). The Laplacian $\Delta(j_0)$ of the metric $m(j_0)$ admits a complete $\mathcal{G}(j_0)$-orthogonal eigensystem $\bar{e}_1, \bar{e}_2, ... \in C^\infty(S, I\!R^n)$ with respective eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots$. We form $r(\bar{e}_1), r(\bar{e}_2)...$ and select a finite subset of the eigensystem of $\Delta(j_0)$ as follows :

Let $e_1 = \bar{e}_1$. Then we take in the sequence $\bar{e}_1, \bar{e}_2, ...$ the vector $\bar{e}_{i_2}$, say, with the smallest index such that $r(\bar{e}_1)$ and $r(\bar{e}_{i_2})$ are linearly independent. Call $\bar{e}_{i_2}$ by e_2. Next let $\bar{e}_{i_3}$ be the one with the smallest index for which e_1, e_2 and $\bar{e}_{i_3}$ are linearly independent, call it e_3. We continue in this way to obtain $e_1, ..., e_{(s_0-1)\cdot n} \in C^\infty(S, I\!R^n)$ with s_0 is the number of points in P. Let $\mathcal{F}_0^\infty(S, I\!R^n)$ be its span. Setting $\mathcal{F}^\infty(S, I\!R^n) := \mathcal{F}_0^\infty(S, I\!R^n) \oplus I\!R^n$ we observe that $r_\infty := r|\mathcal{F}^\infty(S, I\!R^n)$ is an isomorphism. We proceed accordingly for $n = 1$. The following is obvious:

Lemma 3.1

$$\Delta(j_0)\left(\mathcal{F}^\infty(S, I\!R^n)\right) \subset \mathcal{F}^\infty(S, I\!R^n).$$

Now let $\mathcal{W}^\infty(0) \subset \mathcal{F}^\infty(S, I\!R^n)$ be an open neighbourhood of zero, chosen such that r maps $\mathcal{W}^\infty(j_0) := j_0 + \mathcal{W}^\infty(0)$ bijectively onto O_P. The manifold $\mathcal{W}^\infty(j_0)$ has $\mathcal{W}^\infty(j_0) \times \mathcal{F}^\infty(S, I\!R^n)$ as its tangent bundle.

Next we relate the two metrics $\mathcal{G}(j_0)$ and $\mathcal{G}_P$ on $\mathcal{W}^\infty(j_0)$ and O_P, respectively. In doing so it is enough to work with $\mathcal{F}(P, I\!R)$, the collection of all $I\!R^n$-valued maps of P and the space $\mathcal{F}^\infty(S, I\!R)$ defined in the same way as $\mathcal{F}^\infty(S, I\!R^n)$.

Clearly $r^*\mathcal{G}_P(h, k) = \mathcal{G}(j_0)(Qh, k)$ for any two $h, k \in \mathcal{F}^\infty(S, I\!R)$ and some $Q \in End\ \mathcal{F}^\infty(S, I\!R)$. Since the characteristic maps $\mathbf{1}_q$ and $\mathbf{1}_{q'}$ of any two $q, q' \in P$ are $\mathcal{G}_P$-orthogonal we find for each $q \in P$

$$Q(r_\infty^{-1}(\mathbf{1}_q)) = \rho_P(q) \cdot (r_\infty^{-1}(\mathbf{1}_q))$$

for some pointwise positive map $\rho_P : P \longrightarrow I\!R$. Since $r : C^\infty(S, I\!R) \longrightarrow \mathcal{F}(P, I\!R)$ is a surjection the following is easily verified:

Lemma 3.2 *There is a smooth pointwise positive map $\rho \in C^\infty(S, I\!R)$ such that*

$$r_\infty^*\mathcal{G}_P(\rho \cdot h, k) = \mathcal{G}(j_0)(h, k) \quad \forall\, h, k \in \mathcal{F}^\infty(S, I\!R^n). \qquad (3.1)$$.

Now we have the geometrical tools to lift a given internal force Φ_P, prescribed on O_P, to an internal force density Φ defined on some suitably chosen open set $O \subset E(S, I\!R^n)$ and determine its constitutive map: At first we let

$$A := r_\infty^* A_P,$$

where A_P is a given constitutive law on O_P. To construct $O \subset E(S, I\!R^n)$ we first observe that $C^\infty(S, I\!R^n) = \mathcal{F}^\infty(S, I\!R^n) \oplus \mathcal{F}^\perp(S, I\!R^n)$. Here $\mathcal{F}^\perp(S, I\!R^n) \subset C^\infty(S, I\!R^n)$ is generated by all eigenvectors of $\Delta(j_0)$ which are not in $\mathcal{F}^\infty(S, I\!R^n)$. Let π_∞ be the projection with $\mathcal{F}^\perp(S, I\!R^n)$ as its kernel (clearly $\pi_\infty \neq r_\infty^{-1} \circ r$). Now let O be such that $\mathcal{W}^\infty(j_0) \subset O \subset E(S, I\!R^n)$ and $O = \mathcal{W}^\infty(j_0) + (\mathcal{F}^\perp(S, I\!R^n) \cap O)$. Thus $j \in O$ is of the form $j = j_0 + l + k'$ with $l \in \mathcal{W}^\infty(j_0) - j_0 \subset \mathcal{F}^\infty(S, I\!R^n)$ and $k' \in \mathcal{F}^\perp(S, I\!R^n)$. We extend r_∞ to O by $r_\infty := r \circ \pi_\infty$. Now, we set

$$A = r_\infty^* A_P. \tag{3.2}$$

Clearly, $A \neq r^* A_P$ since $\mathcal{F}^\infty(S, I\!R^n)$ is not $\mathcal{G}(j_0)$-orthogonal to $ker\ r$. We call A in (3.2) a **finitely determined** constitutive law on S. Let us remark, that instead of working on all of O we continue to work mostly on $\mathcal{W}^\infty(j_0)$, for the sake of simplicity.

Let $r_\infty(j) = j_P$ for all $j \in \mathcal{W}^\infty(j_0)$. Now, the equation

$$A_P(j_P)(h_P) = \mathcal{G}_P(j_P)\,(\Phi_P(j_P), h_P) \quad \forall\, j_P \in O_P \text{ and all } h_P \in \mathcal{F}(P, I\!R^n)$$

for a smooth $\Phi_P : O_P \longrightarrow \mathcal{F}(P, I\!R^n)$ implies

$$A(j)(h) = r_\infty^* \mathcal{G}_P(\Phi_\infty(j), h) \quad \forall\, j \in \mathcal{W}^\infty(j_0) \text{ and all } h \in \mathcal{F}^\infty(S, I\!R^n)$$

for some $\Phi_\infty \in C^\infty\left(\mathcal{W}^\infty(j_0), \mathcal{F}(P, I\!R^n)\right)$. By lemma 3.2 the following is obvious:

Proposition 3.3 *Given a constitutive law A_P on $O_P \subset E^\infty(P, I\!R^n)$ then the internal force densities Φ on O, formed with respect to $\mathcal{G}(j_0)$ is smooth and is determined by*

$$\Phi = r_\infty^{-1} \circ r(\rho \cdot \Phi_\infty) \quad \text{with} \quad r_\infty \circ \Phi_\rho = \Phi_P. \tag{3.3}$$

The constitutive map $\hat{\mathcal{H}}$ satisfies

$$\Delta(j_0)\hat{\mathcal{H}} = \Phi.$$

Let $\hat{\mathcal{H}}$ be the constitutive map of $A = r_\infty^* A_P$, represented as $\hat{\mathcal{H}}(j) = \sum_{i=1}^{(s_o-1)\cdot n} \hat{\kappa}_i(j) \cdot e_i$ for all $j \in \mathcal{W}^\infty(j_0)$ (where s_0 equals the number of points in P). We now assume that $\hat{\kappa}_i(j) = 1$ for all $i = 1, ..., (s_o - 1) \cdot n$ and all $j \in O$, yielding $\hat{\mathcal{H}}_{geom}$, say. Then $\sum_{i=1}^{(s_o-1)\cdot n} \mathcal{G}(j_0)(\mathcal{H}_{geom}(j), e_i) = tr\ \Delta|\mathcal{F}^\infty(S, I\!R^n)$. Hence $\mathcal{F}^\infty(S, I\!R^n)$ is chosen such that the trace of $\Delta_{\mathcal{F}^\infty} := \Delta|\mathcal{F}^\infty(S, I\!R^n)$ is as small as possible. Calling the virtual work of $\hat{\mathcal{H}}_{geom}$ by A_{geom}, the virtual work A_{red} defined by the reduced constitutive map $\hat{\mathcal{H}}_{red} := \hat{\mathcal{H}} - \hat{\mathcal{H}}_{geom}$ depends only on physical grounds. Hence $j_0 \in O$ is an equilibrium configuration for A (with $\hat{\mathcal{H}}(j_0) = 0$), iff $\hat{\mathcal{H}}_{red} = -\hat{\mathcal{H}}_{geom}$.

4 Neumann decomposition of A_P and A

Let A_P be a smooth constitutive law on the closure $\bar{O}_P$ of O_P. We assume that $\bar{O}_P$ is a compact, connected, smooth manifold with boundary. We split A_P on $\bar{O}_P$ in the sense of Neumann into

$$A_P = \mathbb{D}\,\bar{F}_P + \Psi_P,$$

by solving the following (elliptic) Neumann problem:

$$\mathrm{div}_P A_P = \Delta\!\!\!/\,\bar{F}_P$$

with boundary condition

$$A_P(j_P)(\mathcal{N}_P(j_P)) = \mathbb{D}\,\bar{F}_P(j_P)(\mathcal{N}_P(j_P)) \text{ on } \partial\bar{O}_P.$$

Here div_P, $\Delta\!\!\!/\,_P$ and $\mathbb{D}$ are the divergence operator, the Laplacian of $\mathcal{G}_P$ and the Fréchet derivative on O_P, respectively. $\mathcal{N}_P$ is the outward directed unit normal field along $\partial\bar{O}_P$, assumed to be smooth. The one-form Ψ is divergence free and vanishes on $\mathcal{N}_P$.

Accordingly, we split A on $\bar{\mathcal{W}}^\infty(j_0)$, being diffeomorphic to $\bar{O}_P$ via r_∞, into:

$$A = \mathbb{D}\,\bar{F} + \Psi, \tag{4.1}$$

with

$$\mathrm{div}_\infty A = \Delta\!\!\!/\,_\infty \bar{F} \tag{4.2}$$

and the boundary condition

$$A(j)(\mathcal{N}(j)) = \mathbb{D}\,\bar{F}(j)(\mathcal{N}(j)) \quad \text{on} \quad \partial\bar{\mathcal{W}}^\infty(j_0)$$

with div_∞ and $\Delta\!\!\!/\,_\infty$ the divergence operator and the Laplacian of $r_\infty^*\mathcal{G}_P$, respectively. Notice that (4.1) is orthogonal in the following sense: Let $\mathcal{Z}_\Psi$ be such that $\Psi = r_\infty^*\mathcal{G}_P(\mathcal{Z}_\Psi, ...)$ and $Grad_\infty$ be the gradient of $\bar{F}$ formed with respect to $r_\infty^*\mathcal{G}_P$. Then

$$\int_{\bar{\mathcal{W}}^\infty(j_0)} r_\infty^*\mathcal{G}_P\left(Grad_\infty \bar{F}, \mathcal{Z}_\Psi\right) \mu_{\bar{\mathcal{W}}^\infty(j_0)} \tag{4.3}$$

$$= \int_{\bar{\mathcal{W}}^\infty(j_0)} \bar{F}\cdot \mathrm{div}_\infty \Psi \mu_{\bar{\mathcal{W}}^\infty(j_0)} + \int_{\partial\bar{\mathcal{W}}^\infty(j_0)} \bar{F}\cdot\Psi(\mathcal{N})\mu_{\partial\bar{\mathcal{W}}^\infty(j_0)} = 0,$$

where $\mu_{\bar{\mathcal{W}}^\infty(j_0)}$ and $\mu_{\partial\bar{\mathcal{W}}^\infty(j_0)}$ are the Riemannian volume forms of $r_\infty^*\mathcal{G}_P$ on $\bar{\mathcal{W}}^\infty(j_0)$ and $\partial\bar{\mathcal{W}}^\infty(j_0)$ respectively. By (3.2) and by construction we obviously have

Lemma 4.1

$$\bar{F} = \bar{F}_P \circ r_\infty \quad \text{and} \quad r_\infty^*\Psi_P = \Psi \tag{4.4}$$

on all of $\bar{\mathcal{W}}^\infty(j_0)$.

To understand $\bar{F}$ a little better, we introduce the **structural capillarity** of A: By construction $\bar{\mathcal{W}}^\infty(j_0)$ consists of smooth embeddings only. There is a smooth map $a \in C^\infty(\bar{\mathcal{W}}^\infty(j_0), I\!R)$ called the structural capillarity of A, for which

$$A(j)(j) = dim\ S \cdot a(j) \cdot \mathcal{A}(j) \quad \forall\, j \in \bar{\mathcal{W}}^\infty(j_0) \tag{4.5}$$

holds true; here $\mathcal{A} : E(S, I\!R^n) \longrightarrow I\!R$ is the area function of S associating to each $j \in E(M, I\!R^n)$ the area

$$\mathcal{A}(j) := \int_S \mu(j)$$

of S (cf.[Bi1] and [Bi2]). In (4.5) we have used the fact that the linear map $A(j)$ is for each $j \in \bar{\mathcal{W}}^\infty(j_0)$ defined on all of $C^\infty(S, I\!R^n)$; hence $A(j)(j)$ is well defined. $a(j) \cdot I\!D\,\mathcal{A}(j)(h)$ is the amount of the virtual work $A(j)(h)$ caused by distorting the area at j in the direction of $h \in C^\infty(S, I\!R^n)$. Since $\pi_\infty : \bar{\mathcal{W}}^\infty(j_0) \longrightarrow \mathcal{F}^\infty(S, I\!R^n)$ is the gradient of the map assigning to any $j \in \bar{\mathcal{W}}^\infty(j_0)$ the value $r_\infty^* \mathcal{G}_P(\pi_\infty(j), \pi_\infty(j))$, we deduce via (4.3) and (4.5) the following system of equations

$$A(j)(j) = I\!D\,\bar{F}(j)(j) = dim\ S \cdot a(j) \cdot \mathcal{A}(j) \quad \forall\, j \in \bar{\mathcal{W}}^\infty(j_0). \tag{4.6}$$

Thus

$$A = a \cdot I\!D\,\mathcal{A} + A_1 \quad \text{and} \quad I\!D\,\bar{F} = a \cdot I\!D\,\mathcal{A} + A_2 \tag{4.7}$$

with A_1 and A_2 being one-forms on $\bar{\mathcal{W}}^\infty(j_0)$. Approximating all sides of (4.6) at $j \in \bar{\mathcal{W}}^\infty(j_0)$ up to order two yields for any $h \in \bar{\mathcal{W}}^\infty(j_0)$ the system:

$$A(j)(j) + A(j)(h) + I\!D\,A(j)(h)(j) + I\!D\,A(j)(h)(h) + \frac{1}{2} I\!D^2 A(j)(h,h)(j)$$

$$= I\!D\,\bar{F}(j)(j) + I\!D\,\bar{F}(j)(h) + I\!D^2\bar{F}(j)(h,j) + I\!D^2\bar{F}(j)(h,h) + \frac{1}{2} I\!D^3\bar{F}(j)(h,h,j)$$

$$= dim\ S \cdot \Big((a \cdot \mathcal{A})(j) + I\!D\,(a \cdot \mathcal{A})(j)(h) + \frac{1}{2} I\!D^2 (a \cdot \mathcal{A})(j)(h,h) \Big).$$

The following is immediately verified:

Proposition 4.2 *Let $a \in C^\infty\left(\bar{\mathcal{W}}^\infty(j_0), C^\infty(S, I\!R^n)\right)$ be the structural capillarity of a finitely determined constitutive law A with $I\!D\,\bar{F}$ as its exact part. Then the following equations hold for a fixed $j \in \bar{\mathcal{W}}^\infty(j_0)$ and all $h \in \mathcal{F}^\infty(S, I\!R^n)$:*

$$\begin{aligned} A(j)(h) + I\!D\,A(j)(h)(j) &= I\!D\,\bar{F}(j)(h) + I\!D^2\bar{F}(j)(h,j) \\ &= dim\ S \cdot I\!D\,(a \cdot \mathcal{A})(j)(h) \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} I\!D\,A(j)(h)(h) + \frac{1}{2} I\!D^2 A(j)(h,h)(j) &= I\!D^2\bar{F}(j)(h,h) + \frac{1}{2} I\!D^3\bar{F}(j)(h,h,j) \\ &= \frac{1}{2} \cdot dim\ S \cdot I\!D^2(a \cdot \mathcal{A})(j)(h,h). \end{aligned} \tag{4.9}$$

Taking traces with respect to a $r_\infty^* \mathcal{G}_P$-orthogonal basis in $\mathcal{F}^\infty(S, I\!R^n)$ the system (4.9) yields via (4.2) immediately :

Corollary 4.3 *Under the suppositions of proposition 4.2 there is real-valued function G on $\bar{\mathcal{W}}^\infty(j_0)$ uniquely determined up to a constant such that*

$$- tr\ I\!D^2 A(j)(...,...)(j) = \Delta_\infty G(j) = -tr\ I\!D^3 \bar{F}(j)(...,...,j) \tag{4.10}$$

and hence

$$\bar{F}(j) + \frac{1}{2} G(j) = \frac{1}{2} \cdot dim\ S \cdot (a \cdot \mathcal{A})(j) + \ const \tag{4.11}$$

(with Neumann boundary condition) hold for any $j \in \bar{\mathcal{W}}^\infty(j_0)$.

To identify the function G we use (4.7). Decomposing both terms on the right hand sides in the decomposition (4.7) of $I\!D\bar{F}$ in the sense of Neumann yields

$$\bar{F} = \bar{F}_a + \bar{F}_{A_2} + \ const$$

with $I\!D\bar{F}_{A_2}$ being the exact part of A_2 and

$$\bar{F} = a \cdot \mathcal{A} + \bar{F}_{A_2} - \bar{F}_{\mathcal{A}} + \ const$$

with $I\!D\bar{F}_{\mathcal{A}}$ being the exact part of $\mathcal{A} \cdot I\!D a$. Hence (4.11) yields

Proposition 4.4

$$G = (dim\ S - 2) \cdot \bar{F}_a - 2 \cdot \bar{F}_{\mathcal{A}} - 2 \cdot \bar{F}_{A_2} + \ const. \tag{4.12}$$

Of some interest in elasticity theory are the **linear constitutive laws**. In case of a finitely generated constitutive law A on $\bar{\mathcal{W}}^\infty(j_0)$, linearity means

$$A(j_0 + l)(h) = A(j_0)(h) + I\!D A(j_0)(l)(h)$$

for all $l \in \bar{\mathcal{W}}^\infty(j_0) - j_0$ and for all $h \in \mathcal{F}^\infty(S, I\!R^n)$. If A is linear then G in (4.12) vanishes and (4.11) together with (4.9) yield hence

$$I\!D^3 \bar{F}(j_0) = 0.$$

We therefore obtain by proposition 4.2, corollary 4.3 and (4.6):

Proposition 4.5 *The structure of a linear, finitely generated constitutive law A is of the form (4.1) on $\bar{\mathcal{W}}^\infty(j_0)$, supplemented by the following equations*

$$I\!D A(j_0)(l)(h) = I\!D^2 \bar{F}(j_0)(l,h) = \frac{1}{2} \cdot I\!D^2 (a \cdot \mathcal{A})(j_0)(l,h)$$

and hence

$$A(j_0)(h) = I\!D \bar{F}(j_0)(h)$$

valid for all $l \in \bar{\mathcal{W}}^\infty(j_0)$ and for all $h \in \mathcal{F}^\infty(S, I\!R^n)$. Moreover

$$\bar{F} = \frac{1}{2} \cdot dim\ S \cdot (a \cdot \mathcal{A}) + \ const,$$

where a is the structural capillarity of A. Hence

$$a(j_0) = 0$$

if $j_0 \in \mathcal{W}^\infty(j_0)$ is an equilibrium configuration and if in addition $A = I\!D\bar{F}$ then

$$\bar{F}(j_0 + l) = F(j_0) + \frac{dim\ S}{4} \cdot I\!D^2 (a \cdot \mathcal{A})(j_0)(l,l) + \ const \quad \forall\, l \in \bar{\mathcal{W}}^\infty(j_0) - j_0.$$

5 Examples: Potentials

In what follows, we will illustrate the apparatus developed in the previous sections in a special situation: We assume that the smooth internal force Φ_P of a given constitutive law A_P on $O_P \subset E^\infty(P, I\!R^n)$ is caused by a smooth potential

$$V_P : \partial^1 O_P \longrightarrow I\!R.$$

The domain is open in $\partial^1 \mathcal{F}(P, I\!R^n) \subset C^1(\boldsymbol{L}, I\!R^n)$. There is a gradient $grad_{\mathcal{G}^1} V_P$ of V_P formed with respect to $\mathcal{G}^1$. This gradient with values in $\partial^1 \mathcal{F}(P, I\!R^n)$ splits for each simplex $\sigma \in \boldsymbol{L}_1$ in a $<,>$-orthogonal fashion into

$$grad_{\mathcal{G}^1} V_P(\partial^1 j_P)(\sigma) = \psi(\partial^1 j_P)(\sigma) \cdot \partial^1 j_P(\sigma) + \beta(j_P)(\sigma)$$

for each $j_P \in O_P$ and some $\beta(j_P) \in C^1(\boldsymbol{L}, I\!R^n)$. For simplicity let us assume that ψ is independent of $\partial^1 j_P$. Thus V_P splits accordingly into

$$V_P(\partial^1 j_P) = \frac{1}{2} \cdot \mathcal{G}^1 \left(\psi \cdot \partial^1 j_P, \partial^1 j_P\right) + V_P^1(\partial^1 j_P) \quad \forall j_P \in O_P.$$

In analogy to the situation of a force of a spring, the map $\psi : \boldsymbol{L}_1 \longrightarrow I\!R$ is called the **spring constant**. $j_P^0 \in O_P$ is an equilibrium configuration iff

$$\psi \cdot \partial^1 j_P = -grad_{\mathcal{G}^1} V_P^1(\partial^1 j_P).$$

Thus if $V_P^1 = 0$, an equilibrium configuration exists if $\psi = 0$ i.e. if $V_P = 0$. To determine the constitutive map $\hat{\mathcal{H}}_P$ we start from

$$I\!D\, V(\partial^1 j_P) = \mathcal{G}^1(\psi \cdot \partial^1 j_P, ...) + \mathcal{G}^1(\beta(\partial^1 j_P), ...)$$

and obtain by (2.3)

$$\Delta_T \hat{\mathcal{H}}(j_P) = \delta^1(\psi \cdot \partial^1 j_P) + \delta^1 \beta(\partial^1 j_P) \quad \forall j_P \in O_P.$$

Using the terminology of section four we thus have

$$\bar{F}_P = V_P + \ const.$$

Let $j_0 \in O$ be such that $r(j_0) = j_P^0$ for a given $j_P^0 \in O_P$. Setting $A = r_\infty^* A_P$ as in (3.2) and using (4.4) together with (4.6), the structural capillarity a of A on $\mathcal{W}^\infty(j_0)$ is determined for each $j \in \mathcal{W}^\infty(j_0)$ by the formula:

$$a(j) = \frac{1}{dim\ S \cdot \mathcal{A}(j)} \cdot \mathcal{G}^1(\psi \cdot \partial^1 j_P, \partial^1 j_P).$$

Here G is by (4.10) entirely determined by V_P^1, namely as

$$\Delta_\rho G(j) = -tr\ I\!D^3(V_P^1 \circ r_\infty)(j)(..., ...) \quad \forall j \in \mathcal{W}^\infty(j_0).$$

6 Fitting surfaces

Let $A_P \in A^1(E^\infty(P, I\!R^n))$ be a specified constitutive law with equilibrium configuration j_P^0. We lift A_P to O as in (3.2). This lift is called A. Moreover let $j_0 \in O$ be fixed.

For the purpose of the description of A_P on S, our developments presented so far offer to call $j_0(S) \subset I\!R^n$ to be a fitting surface passing through $j_P^0(P) \subset I\!R^n$ or, equivalently, j_0 to be a fitting configuration if the following is satisfied:

a) j_0 is an equilibrium configuration for $A := r_\infty^* A_P$
b) ρ in (3.1) is a constant equal to one.

In general j_0 satisfying (b) does not exist (cf. [G,R]).

Appendix

Here we will present what is called the **Dirichlet-integral** in fashions different from the usual one. Let $<,>$ be a fixed scalar product on $I\!R^n$. At first we consider $h \in C^\infty(S, I\!R^n)$ and a fixed embedding $j \in E(S, I\!R^n)$. The differential $dh : TS \longrightarrow I\!R^n$ can be represented via dj as

$$dh = c_h \cdot dj + dj \circ (C_h + B_h)$$

which applied to any tangent vector $v_q \in T_qS$ for any $q \in S$ reads as

$$dh\ v_q = c_h(q)\left((dj\ v_q)\right) + dj\left((C_h + B_h)v_q\right).$$

Here $c_h : S \longrightarrow so(n)$ is a smooth map sending vectors in djT_qS into vectors in the orthogonal complement $(djT_qS)^\perp$ and vice versa for any $q \in S$; thus c_h is an infinitesimal Gauss map. The maps C_h and B_h are both smooth (strong) bundle endomorphisms of TS, skew - respectively selfadjoint with respect to the pull back metric $j^*<,>$ denoted by $m(j)$. For this representation we refer to [Bi1],[Bi2],[Bi,Fi2] or [Bi,Sn,Fi]. For any $q \in S$ the endomorphism $c_h^2(q)$ on $I\!R^n$ is a selfadjoint endomorphism of djT_qS respectively $(djT_qS)^\perp$. The part of c_h^2 mapping (djT_qS) into itself is called $(c_h^2(q))^\top$. For any two $h, k \in C^\infty(S, I\!R^n)$ we define

$$dh \bullet dk := -tr(c_h \circ c_k)^\top - tr\ C_h \circ C_k + tr\ B_h \circ B_k = -\frac{1}{2} tr\ c_h \circ c_k - tr\ C_h \circ C_k + tr\ B_h \circ B_k$$

and observe that

$$\mathcal{G}(j)(dh, dk) := \int_S dh \bullet dk\ \mu(j) = \int_S < \Delta(j)h, k > \mu(j) \qquad \text{(A1.1)}$$

where $\mu(j)$ is the Riemannian volume element of $m(j)$. The operator $\Delta(j)$ is the Laplacian associated with $m(j)$. For (A.1.2) and (A.1.3) we refer to [Bi1],[Bi2] or [Bi,Fi2]. Clearly the metric $\mathcal{G}$, given by

$$\mathcal{G}(j)(h, k) = \int_S < h, k > \mu(j) \quad \forall E(S, I\!R^n),$$

is a weak Riemannian metric on $E(S, I\!R^n)$. The left hand side of (A1.1) is called the Dirichlet integral usually formulated via the Hodge star operator. Clearly $\mathcal{G}$ is a weak Riemannian metric on $\{dj | j \in E(S, I\!R^n)\}$.

Next we will represent the integral (A1.1) in a complete different way, based on the second derivative of $m(j)$ formed with respect to j. To this end let $j_0 \in E(S, I\!R^n)$ be fixed and let $h \in C^\infty(S, I\!R^n)$ be such that $j_0 + h \in E(S, I\!R^n)$. Then for any $v, w \in T_qS$ and any $q \in S$

$$\begin{aligned} m(j_0+h)(v,w) &= m(j_0)(v,w) + < dj_0\ v, dh\ w > \\ &\quad + < dh\ v, dj_0\ w > + < dh\ v, dh\ w > \\ &= m(j_0) + I\!D\, m(j_0)(h) + \frac{1}{2} I\!D^2 m(j_0)(h,h). \end{aligned} \tag{A1.2}$$

Writing

$$m(j_0+h)(v,w) = m(j_0)(f^2(j_0+h)v, w) \tag{A1.3}$$

for a well defined smooth strong bundle endomorphism $f(j_0+h)$ of TS, positive definite with respect to $m(j_0)$, we observe by (A1.2) that

$$\begin{aligned} m(j_0+h)(v,w) &= m(j_0)(f^2(j_0+h)v, w) \\ &= m(j_0)(v,w) + m(j_0)(I\!D\, f^2(j_0)(h)v, w) \\ &\quad + \frac{1}{2} m(j_0)(I\!D^2 f^2(j_0)(h,h)v, w) \end{aligned} \tag{A1.4}$$

for all $v, w \in T_qS$ and for all $q \in S$. Using (A1.1) we conclude that

$$< dh\ v, dh\ w > = < (c_h + \bar{B}_h + \bar{C}_h) \circ (c_h + \bar{B}_h + \bar{C}_h)^* \cdot dj_0\ v, dj_0\ w >$$

where $\bar{C}_h \cdot dj_0$ and $\bar{B}_h \cdot dj_0$ are defined by

$$\bar{C}_h \cdot dj_0 = dj_0 \circ C_h \quad \text{and} \quad \bar{B}_h \cdot dj_0 = dj_0 \circ B_h$$

and the requirement that both $\bar{C}_h$ and $\bar{B}_h$ vanish on the normal bundle of $TjTS$. By * we mean the adjoint. Therefore the following equations hold

$$\begin{aligned} < dh\ v, dh\ w > &= < -c_h^2 \cdot dj_0\ v, dj_0\ w > + < dj_0 \circ (B_h + C_h) \circ (B_h + C_h)^* v, dj_0\ w > \\ &= \frac{1}{2} m(j_0)(I\!D^2 f^2(j_0)(h,h)v, w). \end{aligned}$$

Since $c_h^2 \cdot dj_0 = (c_h^2)^\top \cdot dj_0$ we find for all $h \in C^\infty(S, I\!R^n)$

$$\frac{1}{2} I\!D^2 f^2(j_0)(h,h) = -dj_0^{-1} \circ c_h^2 \cdot dj_0 - C_h^2 + B_h^2 + C_h \circ B_h - B_h \circ C_h$$

and

$$f^2(j_0+h) = id + 2B_h - dj_0^{-1} \circ c_h^2 \cdot dj_0 - C_h^2 + B_h^2 + C_h \circ B_h - B_h \circ C_h.$$

Hence

$$dh \bullet dh = \frac{1}{2} tr\ I\!D^2 f^2(j_0)(h,h) = \frac{1}{2} I\!D^2 (tr\ f^2(j_0))(h,h)$$

and by polarization

$$dh \bullet dk = \frac{1}{2} tr\ I\!D^2 f^2(j_0)(h,k) = \frac{1}{2} I\!D^2 (tr\ f^2(j_0))(h,k).$$

Therefore we may state

Lemma:

Given any $j_0 \in E(S, \mathbb{R}^n)$ *and any two* $h, k \in C^\infty(S, \mathbb{R}^n)$

$$dh \bullet dk = \frac{1}{2} \mathbb{D}^2 (tr\ f^2(j_0))(h,k) = \frac{1}{2} tr\ \mathbb{D}^2 f^2(j_0)(h,k)$$

hold true and imply

$$\mathcal{G}(j_0)(dh, dk) = \frac{1}{2} \cdot \int_S \mathbb{D}^2 tr\ f^2(j_0)(h,k)\mu(j_0) = \int_S < \Delta(j_0)h, k > \mu(j_0)$$

for all $h, k \in C^\infty(S, \mathbb{R}^n)$. *Hence (A1.4) yields*

$$\int_S tr\ f^2(j_0 + h)\mu(j_0) = dim\ S \cdot \mathcal{A}(j_0) + \int_S tr\ \mathbb{D} f^2(j_0)(h)\mu(j_0) + \int_S < \Delta(j_0)h, h > \mu(j_0).$$

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