Symmetric Properties in Linear Programming Problems

Werner Oettli [1] and Maretsugu Yamasaki [2]

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[1] Lehrstuhl für Mathematik VII, Universität Mannheim
[2] Department of Mathematics, Shimane University

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Let X and $\tilde{X}$ be real linear spaces which are in duality with respect to a bilinear functional $\langle\cdot,\cdot\rangle$. Likewise let Y and $\tilde{Y}$ be real linear spaces which are in duality with respect to another bilinear functional, for simplicity also denoted by $\langle\cdot,\cdot\rangle$. We assume that the topologies on $X, \tilde{X}$ and $Y, \tilde{Y}$ are such that $X^* = \tilde{X}$, $\tilde{X}^* = X$, $Y^* = \tilde{Y}$, $\tilde{Y}^* = Y$. Let $A : X \to \tilde{Y}$ be a continuous linear mapping. The adjoint $A^* : Y \to \tilde{X}$ is determined by the relation $\langle A^*y, x\rangle := \langle Ax, y\rangle$ for all $x \in X$, $y \in Y$. We require that A^* is continuous and $(A^*)^* = A$. For any nonvoid closed convex cone $\alpha \subseteq X$ we denote by α^+ the polar cone of α, i.e.,

$$\alpha^+ := \{\xi \in \tilde{X} | \ \langle \xi, x\rangle \geq 0 \text{ for all } x \in \alpha\}.$$

According to the bipolar theorem, $(\alpha^+)^+ = \alpha$. Furthermore if $\alpha_1 \subseteq \alpha_2$, then $\alpha_2^+ \subseteq \alpha_1^+$, and if $x \in \alpha$, $x \neq 0$, and $\xi \in \text{ int } \alpha^+$, then $\langle \xi, x\rangle > 0$. Likewise for any nonvoid closed convex cone $\beta \subseteq Y$ we denote by β^+ the polar cone of β, i.e.,

$$\beta^+ := \{\eta \in \tilde{Y} | \ \langle \eta, y\rangle \geq 0 \text{ for all } y \in \beta\}.$$

The same comments as for α^+ apply.

Let $P \subseteq X$ be a fixed nonvoid closed convex cone with $\text{int } P^+ \neq \emptyset$. Let $Q \subseteq Y$ be a fixed nonvoid closed convex cone with $\text{int } Q^+ \neq \emptyset$. Let $\mathcal{P}$ be a family of nonvoid closed convex cones $\alpha \subseteq P$ with $\alpha \neq \{0\}$, and let $\mathcal{Q}$ be a family of nonvoid closed convex cones $\beta \subseteq Q$ with $\beta \neq \{0\}$. Finally let $f \in \text{ int } P^+$ and $g \in \text{ int } Q^+$ be given. For $\alpha \in \mathcal{P}$, $\beta \in \mathcal{Q}$ we consider the following mathematical programming problems:

(1) $$M(\alpha,\beta) := \sup \ \{\langle f, x\rangle | \ x \in \alpha, \ Ax + g \in \beta^+\},$$

(2) $$\check{M}(\alpha,\beta) := \sup \ \{\langle g, y\rangle | \ y \in \beta, \ A^*y + f \in \alpha^+\}.$$

We remark that problems (1) and (2) are not dual to each other in the usual linear programming sense. Rather, the standard dual of (1) is given by

$$(3) \qquad M^*(\alpha,\beta) := \inf \{\langle g,y\rangle |\ y \in \beta,\ A^*y + f \in -\alpha^+\},$$

and the standard dual of (2) is given by

$$(4) \qquad \check{M}^*(\alpha,\beta) := \inf \{\langle f,x\rangle |\ x \in \alpha,\ Ax + g \in -\beta^+\}.$$

Let us define

$$M(\mathcal{P},\mathcal{Q}) := \sup \{M(\alpha,\beta)|\ \alpha \in \mathcal{P},\ \beta \in \mathcal{Q}\},$$

$$\check{M}(\mathcal{P},\mathcal{Q}) := \sup \{\check{M}(\alpha,\beta)|\ \alpha \in \mathcal{P},\ \beta \in \mathcal{Q}\}.$$

We shall study the symmetric property $M(\mathcal{P},\mathcal{Q}) = \check{M}(\mathcal{P},\mathcal{Q})$.

Lemma 1. *$M(\alpha,\beta) > 0$ and $\check{M}(\alpha,\beta) > 0$ for all $\alpha \in \mathcal{P}, \beta \in \mathcal{Q}$.*

Proof: Let $\tilde{x} \in \alpha,\ \tilde{x} \neq 0$. Since $g \in \text{ int } Q^+ \subseteq \text{ int } \beta^+$ we can choose $\lambda > 0$ so small that $\lambda A\tilde{x} + g \in \beta^+$. Set $x_0 := \lambda\tilde{x}$. Then x_0 satisfies the constraints of (1), and from $x_0 \in \alpha,\ x_0 \neq 0,\ f \in \text{ int } P^+ \subseteq \text{ int } \alpha^+$ follows $\langle f, x_0\rangle > 0$. Thus $M(\alpha,\beta) > 0$. A symmetric argument shows $\check{M}(\alpha,\beta) > 0$. q.e.d.

We introduce several conditions:

(A.1) For all $\alpha \in \mathcal{P}$, if $x \in \alpha,\ x \neq 0,\ \xi \in -\alpha^+,\ \langle \xi, x\rangle = 0$, then there exists $\tilde{\alpha} \in \mathcal{P}$ such that $\xi \in \tilde{\alpha}^+$.

(A.2) For all $\beta \in \mathcal{Q}$, if $y \in \beta,\ y \neq 0,\ \eta \in -\beta^+,\ \langle \eta, y\rangle = 0$, then there exists $\tilde{\beta} \in \mathcal{Q}$ such that $\eta \in \tilde{\beta}^+$.

Condition (A.1) will be satisfied in particular, if $\mathcal{P}$ contains all cones of the type $\alpha(\overline{x}) := \{\lambda\overline{x}|\ \lambda \geq 0\}$ with $\overline{x} \in P,\ \overline{x} \neq 0$. Indeed, in this case, if x and ξ obey the hypothesis of (A.1), then with $\tilde{\alpha} := \alpha(x)$ we have $\tilde{\alpha} \in \mathcal{P}$ and $\xi \in \tilde{\alpha}^+$, as requested. Likewise condition (A.2) will be satisfied, if $\mathcal{Q}$ contains all cones of the type $\beta(\overline{y}) := \{\lambda\overline{y}|\ \lambda \geq 0\}$ with $\overline{y} \in Q,\ \overline{y} \neq 0$.

(B.1) For all $\alpha \in \mathcal{P},\ \beta \in \mathcal{Q}$ the duality theorem holds for (1) and (3), i.e., the linear programming problems (1) and (3) have optimal solutions, and the optimal values $M(\alpha,\beta)$ and $M^*(\alpha,\beta)$ are equal.

(B.2) For all $\alpha \in \mathcal{P}$, $\beta \in \mathcal{Q}$ the duality theorem holds for (2) and (4), i.e., the linear programming problems (2) and (4) have optimal solutions, and the optimal values $\check{M}(\alpha,\beta)$ and $\check{M}^*(\alpha,\beta)$ are equal.

Conditions (B.1) and (B.2) will be discussed below.

Theorem 1. *If conditions* (A.1), (A.2), (B.1), (B.2) *are fulfilled, then the equality* $M(\mathcal{P},\mathcal{Q}) = \check{M}(\mathcal{P},\mathcal{Q})$ *holds.*

Proof: Let $\alpha \in \mathcal{P}$, $\beta \in \mathcal{Q}$. Then from (B.1) problem (1) has an optimal solution $\overline{x}$, problem (3) has an optimal solution $\overline{y}$, and

$$\langle f,\overline{x}\rangle = M(\alpha,\beta) = M^*(\alpha,\beta) = \langle g,\overline{y}\rangle.$$

From Lemma 1 follows $\overline{x} \neq 0$. From the constraints of (1) and (3) follows

$$\langle f,\overline{x}\rangle \leq -\langle A^*\overline{y},\overline{x}\rangle = -\langle A\overline{x},\overline{y}\rangle \leq \langle g,\overline{y}\rangle.$$

Combined with $\langle f,\overline{x}\rangle = \langle g,\overline{y}\rangle$ this gives $\langle A^*\overline{y} + f,\overline{x}\rangle = 0$. Since $\overline{x} \in \alpha$, $\overline{x} \neq 0$ and $A^*\overline{y} + f \in -\alpha^+$, it follows from (A.1) that $A^*\overline{y} + f \in \tilde{\alpha}^+$ for some $\tilde{\alpha} \in \mathcal{P}$. From this and $\overline{y} \in \beta$ follows

$$\begin{aligned} M(\alpha,\beta) = \langle g,\overline{y}\rangle &\leq \sup \{\langle g,y\rangle \mid y \in \beta,\ A^*y + f \in \tilde{\alpha}^+\} \\ &= \check{M}(\tilde{\alpha},\beta) \leq \check{M}(\mathcal{P},\mathcal{Q}). \end{aligned}$$

Hence $M(\mathcal{P},\mathcal{Q}) \leq \check{M}(\mathcal{P},\mathcal{Q})$. A symmetric argument, using (A.2) and (B.2), gives $\check{M}(\mathcal{P},\mathcal{Q}) \leq M(\mathcal{P},\mathcal{Q})$. Therefore $M(\mathcal{P},\mathcal{Q}) = \check{M}(\mathcal{P},\mathcal{Q})$. q.e.d.

Now we look for a condition which ensures that (B.1) and (B.2) are satisfied simultaneously.

Lemma 2. *The following conditions are equivalent:*

(C.1) $\quad A^*y \in \operatorname{int}(-P^+) \quad$ for all $y \in Q$, $y \neq 0$;

(C.2) $\quad Ax \in \operatorname{int}(-Q^+) \quad$ for all $x \in P$, $x \neq 0$.

Proof: Because of symmetry it suffices to show that (C.2) implies (C.1). Let (C.2) hold. Assume, for contradiction, that there exists $y \in Q$, $y \neq 0$ with $A^*y \notin \text{ int } (-P^+)$. Then from the separation theorem for convex sets there exists $x \in (X^*)^* = X$, $x \neq 0$, such that

$$\langle x, A^*y \rangle \geq 0 \geq \langle x, \xi \rangle \quad \text{for all } \xi \in -P^+.$$

This implies $\langle Ax, y \rangle \geq 0$ and $x \in (P^+)^+ = P$. But from (C.2) follows then $Ax \in \text{ int } (-Q^+)$, and therefore $\langle Ax, y \rangle < 0$, a contradiction. q.e.d.

Theorem 2. *Let* (C.1) *or* (C.2) *hold. Then both conditions* (B.1) *and* (B.2) *are satisfied.*

Proof: From Lemma 2 we may assume that both (C.1) and (C.2) are satisfied. Let $\alpha \in \mathcal{P}$, $\beta \in \mathcal{Q}$. a) Choose $x_0 := 0$. Then $x_0 \in \alpha$ and $Ax_0 + g = g \in \text{ int } Q^+$. Now choose $\tilde{y} \in \beta$, $\tilde{y} \neq 0$. Then $\tilde{y} \in Q$, and by (C.1), $A^*\tilde{y} + U \subseteq \text{ int } (-P^+)$ for some neighborhood U of the origin. Choose $\lambda > 0$ so large that $f \in \lambda U$, and set $y_0 := \lambda\tilde{y}$. Then $y_0 \in \beta$ and $A^*y_0 + f \in \lambda A^*\tilde{y} + \lambda U \subseteq \text{ int } (-P^+)$. Since $P^+ \subseteq \alpha^+$ and $Q^+ \subseteq \beta^+$ we have altogether obtained x_0, y_0 such that

$$x_0 \in \alpha, \quad Ax_0 + g \in \text{ int } \beta^+,$$

$$y_0 \in \beta, \quad A^*y_0 + f \in \text{ int } (-\alpha^+).$$

These are the regularity conditions which ensure that the duality theorem holds for (1) and (3) – see [2, p. 164], [3]. Hence (B.1) is satisfied. b) Using (C.2) instead of (C.1) we obtain y_0 and x_0 such that

$$y_0 \in \beta, \quad A^*y_0 + f \in \text{ int } \alpha^+,$$

$$x_0 \in \alpha, \quad Ax_0 + g \in \text{ int } (-\beta^+).$$

These are the regularity conditions which ensure that the duality theorem holds for (2) and (4). Hence (B.2) is satisfied . q.e.d.

We turn now to the situation where $Y = X$, $\tilde{Y} = \tilde{X}$, so that $A : X \to \tilde{X}$ and $A^* : X \to \tilde{X}$. Instead of simply specializing the previous results we consider a somewhat different problem. From now on let $\mathcal{P}$ be a family of nonvoid closed convex cones $\alpha \subseteq X$. Let $f \in \tilde{X}$, $g \in \check{X}$ be given arbitrarily. For all $\alpha \in \mathcal{P}$ we consider the problems

$$L(\alpha) := \sup \{\langle f, x\rangle \mid x \in \alpha,\ Ax + g \in \alpha^+\}, \tag{5}$$

$$\check{L}(\alpha) := \sup \{\langle g, y\rangle \mid y \in \alpha,\ A^* y + f \in \alpha^+\}. \tag{6}$$

The linear programming dual of (5) is given by

$$L^*(\alpha) := \inf \{\langle g, y\rangle \mid y \in \alpha,\ A^* y + f \in -\alpha^+\}, \tag{7}$$

and the linear programming dual of (6) is given by

$$\check{L}^*(\alpha) := \inf \{\langle f, x\rangle \mid x \in \alpha,\ Ax + g \in -\alpha^+\}. \tag{8}$$

We define

$$L(\mathcal{P}) := \sup \{L(\alpha) \mid \alpha \in \mathcal{P}\}, \tag{9}$$

$$\check{L}(\mathcal{P}) := \sup \{\check{L}(\alpha) \mid \alpha \in \mathcal{P}\}, \tag{10}$$

and we want to establish the equality $L(\mathcal{P}) = \check{L}(\mathcal{P})$. We require the following conditions:

(D) For all $\alpha \in \mathcal{P}$ and all $x \in \alpha$ there exists $\tilde{\alpha} \in \mathcal{P}$ such that $x \in \tilde{\alpha} \subseteq \alpha$ and, whenever $\xi \in -\tilde{\alpha}^+$ and $\langle \xi, x\rangle = 0$, then $\xi \in \tilde{\alpha}^+$.

(E) For all $\alpha \in \mathcal{P}$ with $L(\alpha) > -\infty$ the duality theorem holds for (5) and (7), and for all $\alpha \in \mathcal{P}$ with $\check{L}(\alpha) > -\infty$ the duality theorem holds for (6) and (8).

(F) The suprema occuring in (9) and (10) are finite, and are assumed somewhere on $\mathcal{P}$.

Now we have:

Theorem 3. *Let conditions* (D), (E), (F) *be satisfied. Then* $L(\mathcal{P}) = \check{L}(\mathcal{P})$.

Proof: In accordance with condition (F) let $\alpha_1 \in \mathcal{P}$ be optimal for $L(\mathcal{P})$, so that $L(\mathcal{P}) = L(\alpha_1)$. In accordance with condition (E) let $\overline{x}$ be optimal for $L(\alpha_1)$, so that $L(\alpha_1) = \langle f, \overline{x} \rangle$. Given $x := \overline{x}$ and $\alpha := \alpha_1$ fix $\tilde{\alpha}$ in accordance with condition (D). Then $\overline{x} \in \tilde{\alpha} \subseteq \alpha_1$. From the constraints of $L(\alpha_1)$ one has $A\overline{x} + g \in \alpha_1^+ \subseteq \tilde{\alpha}^+$. Thus $\overline{x}$ satisfies also the constraints of $L(\tilde{\alpha})$, and therefore $\langle f, \overline{x} \rangle \leq L(\tilde{\alpha})$. But since $L(\mathcal{P}) = \langle f, \overline{x} \rangle$ it follows that $\langle f, \overline{x} \rangle = L(\tilde{\alpha})$, and $\overline{x}$ is also optimal for $L(\tilde{\alpha})$. In accordance with condition (E) let $\overline{y}$ be optimal for the dual $L^*(\tilde{\alpha})$, so that

$$\langle f, \overline{x} \rangle = L(\tilde{\alpha}) = L^*(\tilde{\alpha}) = \langle g, \overline{y} \rangle.$$

Then $A^*\overline{y} + f \in -\tilde{\alpha}^+$, and as in the proof of Theorem 1 follows $\langle A^*\overline{y} + f, \overline{x} \rangle = 0$. From condition (D) follows $A^*\overline{y} + f \in \tilde{\alpha}^+$. Consequently $\overline{y}$ satisfies also the constraints of $\check{L}(\tilde{\alpha})$, and therefore $\langle g, \overline{y} \rangle \leq \check{L}(\tilde{\alpha}) \leq \check{L}(\mathcal{P})$. Since $L(\mathcal{P}) = \langle g, \overline{y} \rangle$ it follows $L(\mathcal{P}) \leq \check{L}(\mathcal{P})$. A symmmetric argument gives $\check{L}(\mathcal{P}) \leq L(\mathcal{P})$. Hence the claimed equality is true. q.e.d.

Let us discuss condition (D). It is satisfied for instance, if $\alpha \subseteq P$ for all $\alpha \in \mathcal{P}$ and $\mathcal{P}$ contains all cones of the type $\alpha(\overline{x}) := \{\lambda \overline{x} | \ \lambda \geq 0\}$, $\overline{x} \in P$, where $P \subseteq X$ is a given nonvoid closed convex cone. Indeed, if $x \in \alpha$ for some $\alpha \in \mathcal{P}$, then choosing $\tilde{\alpha} := \alpha(x)$ one has $\tilde{\alpha} \in \mathcal{P}$, $x \in \tilde{\alpha} \subseteq \alpha$, and if $\langle \xi, x \rangle = 0$, then $\xi \in \tilde{\alpha}^+$ ($x = 0$ is permitted here since $\alpha = \{0\}$ is not excluded). Hence (D) is satisfied.

Another situation where (D) is satisfied is the following. Let K be a finite set and $X := I\!R^K$, $\mathcal{P}$ be the family of all cones of the type $\alpha(A) := \{x \in I\!R^K | \ x_i \geq 0$ for all $i \in A$, $x_i = 0$ for all $i \in K \setminus A\}$, where A runs over all subsets of K. Then $(\alpha(A))^+ = \{y \in I\!R^K | \ y_i \geq 0$ for all $i \in A\}$. For $x \in I\!R^K_+$ let supp $x := \{i \in K | \ x_i > 0\}$. Now if $x \in \alpha(A)$, then choosing $\tilde{\alpha} := \alpha(\text{supp } x)$ we have $\tilde{\alpha} \in \mathcal{P}$ and $x \in \tilde{\alpha} \subseteq \alpha(A)$. Moreover, if $\xi \in -\tilde{\alpha}^+$ and $\langle \xi, x \rangle = 0$, then $\xi_i = 0$ for all $i \in$ supp x, hence $\xi \in \tilde{\alpha}^+$: (D) is satisfied. In this situation the conclusion of Theorem 3 is equivalent with

$$\sup \{\langle f,x\rangle \mid x \in \mathbb{R}_+^K,\ (Ax+g)_i \geq 0 \text{ for all } i \in \text{ supp } x\}$$
$$= \sup \{\langle g,y\rangle \mid y \in \mathbb{R}_+^K,\ (A^*y+f)_j \geq 0 \text{ for all } j \in \text{ supp } y\},$$

provided that both suprema are finite and are assumed. An infinite-dimensional analog of this result with K a compact Hausdorff space and x, y Radon measures over K, has been given by Ohtsuka [4], and motivated the present investigation.

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Lehrstuhl für Mathematik VII
Universität Mannheim
Mannheim
Germany

and

Department of Mathematics
Shimane University
Matsue
Japan