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The Cantor set in probability theory

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Dedicated to Professor Horand Störmer on the occasion of his 65th birthday

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Abstract

The present paper discusses some aspects of the role of the Cantor set in probability theory. It contains a simple construction of the Cantor set which is used to construct a singular–continuous distribution and a singular martingale.

1 Introduction

The present paper discusses some aspects of the role of the Cantor set in probability theory. In Section 2 we present a simple construction of the Cantor set which is probably known but hard to find in the literature. This construction proceeds 'without dots' and yields some structures which are useful for further purposes. In Section 3 we construct a singular–continuous distribution which, in turn, shows that the Cantor set is uncountable, and in Section 4 we produce an increasing sequence of σ-algebras whose union is only an algebra and a singular positive martingale whose limit measure is countably additive.

The point that we wish to make is that one single concept yields important examples in different areas of probability theory: If singular–continuous distributions did not exist, then every distribution would be a mixture of a continuous and a discrete one; similarly, if singular martingales did not exist, then every positive martingale would be a sequence of successive conditional expectations.

General background on measure theory, probability theory, and martingale theory may be found in the monographs by Aliprantis/Burkinshaw [1], Bauer [2,3], Billingsley [4], Chung [6], Halmos [7], and Neveu [8].

2 The construction of the Cantor set

Let $\mathcal{C}$ denote the collection of all subsets of the interval $[0,1]$ which are the union of finitely many disjoint compact intervals of $[0,1]$ and define a map $\Psi : \mathcal{C} \longrightarrow \mathcal{C}$ by letting

$$\Psi\left(\sum_{i=1}^{m}[a_i,b_i]\right) \;:=\; \sum_{i=1}^{m}\left(\left[a_i,\frac{2a_i+b_i}{3}\right]+\left[\frac{a_i+2b_i}{3},b_i\right]\right).$$

Then, for each $A \in \mathcal{C}$, the sequence $\{\Psi^n(A)\}_{n\in\mathbf{N}}$ is decreasing and we have

$$\lambda(\Psi^n(A)) \;=\; \left(\frac{2}{3}\right)^n \cdot \lambda(A) \tag{1}$$

for all $n \in \mathbf{N}$.

For each $n \in \mathbf{N}$, define

$$C_n \;:=\; \Psi^n([0,1]).$$

Then C_n is compact and satisfies

$$\lambda(C_n) \;=\; \left(\frac{2}{3}\right)^n.$$

For later reference, let $\mathcal{C}_n$ denote the collection of the 2^n compact intervals partitioning C_n and define a function $f_n : \mathbf{R} \to \mathbf{R}$ by letting

$$f_n(x) \;:=\; \left(\frac{3}{2}\right)^n \cdot \chi_{C_n}(x)$$

and a set function $Q_n : \mathcal{B}(\mathbf{R}) \to [0,1]$ by letting

$$Q_n(A) \;:=\; \int_A f_n \, d\lambda.$$

Then Q_n is a continuous distribution with density f_n and, by identity (1), each $J \in \mathcal{C}_n$ satisfies

$$\begin{aligned} \int_J f_n \, d\lambda &= \left(\frac{3}{2}\right)^n \cdot \lambda(J) \\ &= \left(\frac{3}{2}\right)^{n+k} \cdot \lambda(\Psi^k(J)) \\ &= \int_{\Psi^k(J)} f_{n+k} \, d\lambda \\ &= \int_J f_{n+k} \, d\lambda \end{aligned} \tag{2}$$

and hence

$$\begin{aligned} Q_n(J) &= \int_J f_n \, d\lambda \\ &= \int_J f_{n+k} \, d\lambda \\ &= Q_{n+k}(J) \end{aligned} \tag{3}$$

for all $k \in \mathbf{N}$.

Define now

$$C \;:=\; \bigcap_{n \in \mathbf{N}} C_n.$$

Then C is compact and satisfies

$$\lambda(C) \;=\; 0. \tag{4}$$

The set C is called the *Cantor set*.

It is well–known that the limit of a pointwise convergent sequence of Riemann integrable functions may fail to be Riemann integrable unless convergence is uniform. Since a function is Riemann integrable if and only if it is almost surely continuous, each χ_{C_n} is Riemann integrable, and the same argument shows that the pointwise limit χ_C of the sequence $\{\chi_{C_n}\}_{n\in\mathbf{N}}$ is Riemann integrable as well although convergence is not uniform. This is an example of the insufficiency of Riemann integration theory.

3 A singular–continuous distribution

A distribution $Q : \mathcal{B}(\mathbf{R}) \to [0,1]$ is *singular–continuous* if it satisfies
- $Q(\mathbf{R}\backslash A) + \lambda(A) = 0$ for some $A \in \mathcal{B}(\mathbf{R})$, and
- $Q(\{x\}) = 0$ for all $x \in \mathbf{R}$.

Due to a famous decomposition theorem, every distribution has a unique representation as a mixture of a continuous distribution, a singular–continuous distribution, and a discrete distribution; see Chung [6]. This shows the importance of the following result:

3.1 Theorem. *There exists a singular–continuous distribution.*

Proof. For each $n \in \mathbf{N}$, let F_n denote the distribution function of Q_n. We claim that $\{F_n\}_{n\in\mathbf{N}}$ is a uniform Cauchy sequence. To see this, consider $n, k \in \mathbf{N}$.
For $x \in \mathbf{R}\backslash[0,1]$, we clearly have

$$|F_{n+k}(x) - F_n(x)| \;=\; 0.$$

For $x \in [0,1]\backslash C_n$, identity (3) yields

$$\begin{aligned}
|F_{n+k}(x) - F_n(x)| &= |Q_{n+k}((-\infty,x]) - Q_n((-\infty,x])| \\
&= \left| \sum_{J\in\mathcal{C}_n, J\subseteq(-\infty,x]} Q_{n+k}(J) - \sum_{J\in\mathcal{C}_n, J\subseteq(-\infty,x]} Q_n(J) \right| \\
&= \left| \sum_{J\in\mathcal{C}_n, J\subseteq(-\infty,x]} (Q_{n+k}(J) - Q_n(J)) \right| \\
&= 0.
\end{aligned}$$

For $x \in C_n$, there exists a unique $J(x) \in \mathcal{C}_n$ satisfying $x \in J(x)$, and identity (3) yields

$$\begin{aligned}
|F_{n+k}(x) - F_n(x)| &= |Q_{n+k}((-\infty,x]) - Q_n((-\infty,x])| \\
&= |Q_{n+k}(J(x)\cap(-\infty,x]) - Q_n(J(x)\cap(-\infty,x])| \\
&\leq Q_{n+k}(J(x)) + Q_n(J(x)) \\
&= 2\cdot Q_n(J(x)) \\
&= 2\cdot\left(\frac{3}{2}\right)^n \cdot \lambda(J(x)) \\
&= 2\cdot\frac{1}{2^n}.
\end{aligned}$$

Therefore, we have

$$|F_{n+k}(x) - F_n(x)| \;\leq\; 2\cdot\frac{1}{2^n}$$

for all $x \in \mathbf{R}$, which proves our claim. It now follows easily that $\{F_n\}_{n\in\mathbf{N}}$ converges uniformly to a continuous function $F : \mathbf{R} \to [0,1]$ and that F is a distribution function.

Let $Q : \mathcal{B}(\mathbf{R}) \to [0,1]$ denote the distribution of F. For each $n \in \mathbf{N}$, the continuity of F and F_n yields

$$\begin{aligned} Q((a,b)) &= F(b) - F(a) \\ &= \lim_{n\to\infty} (F_n(b) - F_n(a)) \\ &= \lim_{n\to\infty} Q_n((a,b)) \\ &= 0 \end{aligned}$$

for each open interval $(a,b) \subseteq [0,1] \setminus C_n$, hence $Q([0,1] \setminus C_n) = 0$, and thus

$$Q(C_n) = 1.$$

This implies

$$Q(C) = 1, \tag{5}$$

and identities (4) and (5) yield

$$Q(\mathbf{R} \setminus C) + \lambda(C) = 0.$$

The continuity of F also yields

$$Q(\{x\}) = 0$$

for all $x \in \mathbf{R}$. Therefore, Q is singular–continuous. □

The function F and the distribution Q constructed in the proof of Theorem 3.1 are called the *Cantor function* and the *Cantor distribution*, respectively. The construction of F is similar as in Rudin [10]; see also Aliprantis/Burkinshaw [1], Billingsley [4], Chung [6], and Halmos [7].

3.2 Corollary. *The Cantor set C is uncountable.*

This is obvious from the fact that the Cantor distribution Q is continuous–singular and has all its mass concentrated on the Cantor set C.

4 A singular martingale

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{\mathcal{F}_n\}_{n\in\mathbf{N}}$ be an increasing sequence of sub-σ-algebras of $\mathcal{F}$. Define $\mathcal{F}_\infty := \bigcup_{n\in\mathbf{N}} \mathcal{F}_n$ and let P_∞ denote the restriction of P to the algebra $\mathcal{F}_\infty$.

If $\{X_n\}_{n\in\mathbf{N}}$ is a martingale with respect to the filtration $\{\mathcal{F}_n\}_{n\in\mathbf{N}}$, then the map $\mu_\infty : \mathcal{F}_\infty \to \mathbf{R}$, given by

$$\mu_\infty(A) \;:=\; \lim_{n\to\infty} \int_A X_n \, dP,$$

is a finitely additive set function which is positive if and only if the martingale is positive. The set function μ_∞ is called the *limit measure* of $\{X_n\}_{n\in\mathbf{N}}$ (although it may happen to be neither countably additive nor positive).

A positive martingale $\{X_n\}_{n\in\mathbf{N}}$ is *continuous* if its limit measure μ_∞ is continuous (in the sense that for each $\varepsilon > 0$ there exists some $\delta > 0$ such that $\mu_\infty(A) < \varepsilon$ holds for each $A \in \mathcal{F}_\infty$ satisying $P_\infty(A) < \delta$), and this is the case if and only if there exists an integrable random variable X satisfying $X_n = E^{\mathcal{F}_n} X$ for all $n \in \mathbf{N}$. A positive martingale $\{X_n\}_{n\in\mathbf{N}}$ is *singular* if its limit measure μ_∞ is singular (in the sense that for each $\varepsilon > 0$ there exists some $A \in \mathcal{F}_\infty$ satisfying $\mu_\infty(\Omega \setminus A) + P_\infty(A) < \varepsilon$), and this is the case if and only if the martingale converges almost surely to 0. The previous remarks indicate that many properties of a positive martingale are reflected by its limit measure; for further details, see Neveu [8] and Schmidt [11,12] and the references given there.

Due to the Lebesgue decomposition, every positive martingale has a unique representation as the sum of a continuous martingale and a singular martingale; see e. g. Schmidt [11,12]. This shows the importance of the following result:

4.1 Theorem. *There exists a singular positive martingale.*

Proof. Define $(\Omega, \mathcal{F}, P) := ([0,1], \mathcal{B}([0,1]), \lambda|_{\mathcal{B}([0,1])})$. Also, for each $n \in \mathbf{N}$, define $\mathcal{F}_n := \sigma(\mathcal{C}_n)$ and let X_n denote the restriction of f_n to $[0,1]$. Then the sequence $\{\mathcal{F}_n\}_{n\in\mathbf{N}}$ is strictly increasing, and it follows from identity (2) that $\{X_n\}_{n\in\mathbf{N}}$ is a positive martingale with respect to $\{\mathcal{F}_n\}_{n\in\mathbf{N}}$.
Since the Cantor distribution Q satisfies

$$\begin{aligned}
Q(A) &= \lim_{n\to\infty} Q_n(A) \\
&= \lim_{n\to\infty} \int_A f_n \, d\lambda \\
&= \lim_{n\to\infty} \int_A X_n \, dP \\
&= \mu_\infty(A)
\end{aligned}$$

for all $A \in \bigcup_{n \in \mathbf{N}} \mathcal{C}_n$ and hence for all $A \in \mathcal{F}_\infty$, it is plain that the restriction of Q to $\mathcal{F}_\infty$ is precisely the limit measure μ_∞ of $\{X_n\}_{n \in \mathbf{N}}$. Furthermore, since each of Q and λ is countably additive and since the sequence $\{C_n\}_{n \in \mathbf{N}} \subseteq \mathcal{F}_\infty$ decreases to $C \in \sigma(\mathcal{F}_\infty)$, we have

$$\begin{aligned}\lim_{n \to \infty} (\mu_\infty(\Omega \backslash C_n) + P_\infty(C_n)) &= \lim_{n \to \infty} (Q([0,1] \backslash C_n) + \lambda(C_n)) \\ &= Q([0,1] \backslash C) + \lambda(C) \\ &= 0.\end{aligned}$$

This implies that the martingale $\{X_n\}_{n \in \mathbf{N}}$ is singular. □

The singular martingale constructed in the proof of Theorem 4.1 is called the *Cantor martingale.* While the limit measure of the Cantor martingale is countably additive, a singular martingale whose limit measure is purely finitely additive can be found in Neveu [8]. In view of the Yosida–Hewitt decomposition of finitely additive set functions and the examples provided by the Cantor set, one is tempted to ask whether every positive martingale has a (necessarily unique) representation as the sum of a continuous martingale, a singular martingale whose limit measure is countably additive, and a singular martingale whose limit measure is purely finitely additive. Such a result would be a complete formal counterpart of the decomposition of distributions mentioned in Section 2, but it is not known whether it is true. However, it is easily seen from the results in Schmidt [12] that such a decomposition obtains if all martingales are replaced by asymptotic martingales (amarts).

As a final remark, let us note that the algebra $\mathcal{F}_\infty$ determined by the filtration $\{\mathcal{F}_n\}_{n \in \mathbf{N}}$ related to the Cantor martingale cannot be a σ-algebra. This is due to the fact that $\{\mathcal{F}_n\}_{n \in \mathbf{N}}$ is strictly increasing; see Broughton/Huff [5], Overdijk/Simons/ Thiemann [9], and Stoyanov [13]. In the present situation, the argument is particularly simple: Since each $\mathcal{F}_n$ is finite, $\mathcal{F}_\infty$ is countable. On the other hand, there exists an uncountable number of strictly decreasing sequences $\{J_n\}_{n \in \mathbf{N}}$ satisfying $J_n \in \mathcal{C}_n$ for all $n \in \mathbf{N}$, each of these sequences decreases to a nonempty compact set in $\sigma(\mathcal{F}_\infty)$, and any two such sets are disjoint. Therefore, $\sigma(\mathcal{F}_\infty)$ is uncountable and hence strictly larger than $\mathcal{F}_\infty$. Incidentally, this is another argument showing that the Cantor set is uncountable.

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