No.123/91

A symplectic setting for the formulation of a dynamics of smoothly deformable media

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Abstract

The quality of a smoothly deformable medium in a Riemannian manifold N is characterized by a smooth one form F on the space of configurations, a Fréchet manifold. F characterizes the work done at a configuration under an infinitesimal distortion. In fact F determines a smooth constitutive vector field $\mathcal{H}$ on that Fréchet manifold. A symplectic setting is deduced to determine the equation of motion of the medium subjected to F. The equation - not hamiltonian in general - is such that $\nabla_{\frac{d}{dt}} \dot{\sigma}(t)$ balances along the curve of configurations σ the (internal) force density $\Delta(\sigma(t))\ \mathcal{H}(\sigma(t))$ with $\Delta(\sigma(t))$ the Laplacian determined by $\sigma(t)$ and ∇ the covariant derivative on the space of configurations. We exhibit the effect to this equation of the work done by distorting the volume. No balance laws are presupposed in deriving the equation. Two types of such laws are derived from the symplectic setting on one hand and on the other from the Noether theorem, provided symmetry groups are present.

Introduction

In these notes we formulate in the realm of symplectic geometry a dynamics for smoothly deformable media. The equations of motion will not be Hamiltonian in general. Let us proceed to more details:

The description of the quality of any such medium expressed within the setting of global differential geometry, relies on the space of configuration described next: The deformable media at hand are supposed to maintain the shape of a compact, smooth connected and orientable manifold M' which moves and deforms in a smooth connected, orientable Riemannian manifold N. This is to say that a configuration is a smooth embedding j of a fixed manifold M into N. The manifold M shares the topological and geometric qualities of M'. The collection E(M,N) of all these embeddings forms a Fréchet manifold if endowed with the C^∞ - topology, (cf sec. 1). (To discuss the influence of internal structures or any sorts of defects e.g the space of configuration would have to be appropriately generalized.)

A constitutive law which characterizes the quality of the medium is supposed to be a smooth one - form F on $E(M,N)$ which admits an integral representation via a so called constitutive field $\mathcal{H}$ (a smooth vector field on $E(M,N)$) as explained in sec. 7. The real number $F(j)(l)$ is interpreted as the work done under an (infinitesimal) distortion l at the configuration j. In case the Riemannian manifold is an Euclidean space and F is invariant under the Euclidean group then a symmetric stress tensor field T can be used to compute the work mentioned (cf. [Sch] and [Tr]). However this method fails in case of a general Riemannian manifold N because of the lack of enough isometries. Using one - forms as the basic constitutive ingredient, as mentioned above, yields a formalism rich enough to handle the rather general situation. This way of describing the quality of a deformable medium is used in the literature e.g. in [He], [E,S], [Bi 1] to [Bi 4] and in [Bi,Sc,So].
To explain what's meant by an integral representation we first introduce the metric $\mathcal{G}$ on which this representation relies.

After the basic geometric preliminaries are collected, we introduce in sec. 3 the bundle $\mathcal{A}^1_E(M,TN)$ of smooth TN-valued one-forms on M mapping M into N which cover embeddings. This space is fibred over $E(M,N)$ by the Fréchet spaces $\mathcal{A}^1_j(M,TN) \cong \mathcal{A}^1(M,j^\star TN)$. On these fibres, a "dot metric" $\mathcal{G}$ is defined by

$$\mathcal{G}(J)(\alpha,\beta) = \int_M \alpha \cdot \beta \, \mu(j) \, ,$$

for $\alpha, \beta \in \mathcal{A}^1_j(M,TN)$. Here $\alpha \cdot \beta$ is a smooth real-valued function on M which is symmetric and bilinear in α, β and whose construction is based on the classical "trace inner product" for bundle endomorphisms of the Riemannian bundle TN (cf sec. 3). If $\mathcal{G}$ is restricted to the subspace $\mathcal{L}_E(M,TN)$ consisting of all ∇l with $l \in C^\infty(M,TN)$, one obtains a generalization of the classical Dirichlet integral (cf. [Bi 2]).

By $\mathcal{G}$-representable one-forms F on $E(M,N)$, that is a one - form F admitting an integral representation, we mean the following: There exists a smooth map $\alpha : E(M,N) \longrightarrow \mathcal{A}^1_E(M,TN)$ such that for $j \in E(M,N)$ and $l \in C^\infty_j(M,TN)$, the real number $F(j)(l)$ can be expressed as

$$F(j)(l) = \int_M \alpha(j) \cdot \nabla l \, \mu(j) = \mathcal{G}(j)(\alpha(j), \nabla l) \, , \tag{0.1}$$

where ∇l is the covariant derivative of l along j induced by the Levi-Cività connection of N.

A crucial step is the following result: For any $\mathcal{G}$-representable one-form F, there exists a smooth vector field $\mathcal{H}$ on $E(M,N)$ such that

$$F(j)(l) = \int_M \nabla \mathcal{H}(j) \cdot \nabla l \, \mu(j) = \mathcal{G}(j)(\nabla \mathcal{H}(j), \nabla l) \tag{0.2}$$

holds for $j \in E(M,N)$ and $l \in C_j^\infty(M,TN)$. The existence of such a field $\mathcal{H}$ follows from the fact that α in (0.1) defines an elliptic problem whose solvability is guaranteed by [Hö 2]. Then (0.2) may be rewritten in the form

$$F(j)(l) = \int_M < \Delta(j)\mathcal{H}(j), l > \mu(j) \ ,$$

where $< \ , \ >$ denotes the Riemannian metric in N. Here $\Delta(j)$ is the Laplacian determined by ∇ and $j^\star < \ , \ >$ on M. In physical terms, if F describes the deformable medium in N, then $\Delta(j)\mathcal{H}(j)$ is the (internal) force density acting up on M. The existence of force densities at each configuration $j \in E(M,N)$ associated with F is equivalent with the $\mathcal{G}$ - representability of F (cf. [Bi 4] and [Bi,Fi 2]). If $N = I\!R^n$ and $< \ , \ >$ is a scalar product, then representing F by (0.1) means that the center of mass of the medium is fixed (cf. [Bi 4]).

In sec. 8 we exhibit the volume sensitive part of F (which will make its appearance in the equation of motion subjected to F in a very specific fashion, cf. below): At each $j \in E(M,N)$ the constitutive field $\mathcal{H}(j)$ contains as a component a real multiple $\mathbf{a}(j) \cdot \overline{j}$ of $\overline{j}$, the solution of $\Delta(j)\overline{j} = tr \ S(j)$ where the right hand side is the mean curvature field. This splitting is of an L_2 type with respect to $\mathcal{G}$.

The dynamics we introduce in sec. 9 is based on a symplectic form on $TE(M,N)$ and on a metric $\mathcal{B}$ on $E(M,N)$ (introduced in sec. 4) which relies itself on a density map ρ defined for each configuration and which obeys a continuity equation. The integral $\int_M \rho(j)\mu(j)$ is the total mass $\mathbf{m}(j)$ on $j(M)$. Due to the continuity equation $\mathbf{m}(j)$ is independent of $j \in E(M,N)$.

The symplectic form ω on TN (cf. sec. 5) determined by pulling back the canonical symplectic form on $T^\star N$ via the metric on N yields, in conjunction with the density map ρ, a symplectic form $\omega_{\mathcal{B}}$ on $TE(M,N)$.

The work form $\mathcal{W}_F$ on $TE(M,N)$ is given by

$$\mathcal{W}_F = d\mathcal{E}_{kin} - \pi_E^\star F$$

where $d\mathcal{E}_{kin}$ denotes the differential (on $C^\infty(M,TN)$) of the kinetic energy $\mathcal{E}_{kin}$ given by $\mathcal{B}$ and where $\pi_E^\star F$ denotes the pullback of F onto $TE(M,N)$ by the canonical projection π_E.

The Eulerfield $\mathcal{X}_F \in \Gamma(T^2E(M,N))$ (cf. sec. 9) is then defined by

$$\omega_{\mathcal{B}}(\mathcal{X}_F, \mathcal{Y}) = d\mathcal{E}_{kin}(\mathcal{Y}) - \pi_E^\star F(\mathcal{Y}).$$

The Eulerfield $\mathcal{X}_F$ is called Hamiltonian if F is exact. In case $F = 0$ then $\mathcal{X}_F$ is the spray $\mathcal{X}_{\mathcal{B}}$ of $\mathcal{B}$, of which the solution curves are geodesics. A smooth curve σ on $E(M,N)$ is a geodesic of $\mathcal{B}$ iff σ_p given by $\sigma_p(t) := \sigma(t)(p)$ for any t, is a geodesic in N for any fixed $p \in M$. Thus $\mathcal{X}_{\mathcal{B}}$ describes the free motion of the collection of mass points constituting the body in N. The term $\pi_E^\star F$ with $F \neq 0$ can hence be understood as being responsible for keeping these mass points together to form a body at each configuration.

Along a solution curve of $\mathcal{X}_F$ a balance law between the differential of kinetic energy and $\pi_E^\star F$ holds.

The volume sensitive part contributes to the equation of a motion σ, subjected to a constitutive law F, by the force density $\Delta(\sigma(t))\overline{\sigma}(t)$ for all t in the domain of σ. If F constitutes of the volume sensitive part only, then the equation of a motion subjected to F is a generalized wave equation. Generalized is meant in the sense, that the Laplacian is configuration dependent.

In sec. 11 we consider a constitutive law on $E(M, I\!R^n)$ determined by a stress tensor $T(j)$ smoothly depending on $j \in E(M, I\!R^n)$ and determine the constitutive field. The equation of a motion of a deformable material characterized by T takes a simple form. We refer to [H,M] for the various notions of stress tensors in elasticity theory.

In sec. 12 we discuss briefly the influence of groups of orientation preserving diffeomorphisms $Diff^+M$ on M and of orientation preserving isometries $\mathcal{J}$ on N respectively. In particular we determine via the moment maps a criterion for first integrals given by the elements of the respective Lie algebras, provided that F is invariant under either of the groups mentioned. The symplectic technics used here, show in particular in the context of first integrals a superiority over setting up dynamics by hand.

Finally sec. 13 contains the studies of motions subjected to a constitutive law F under the constraint, that it takes place on a fixed manifold $i(M)$ with $i \in E(M, N)$. In case $\mathcal{W}_F$ is the differential of the kinetic energy only, the resulting equations are Euler's equations of a perfect fluid provided the motion is further restricted to $i \circ Diff_{\mu(i)}M$ with $Diff_{\mu(i)}M$ the group of all $\mu(i)$ - preserving diffeomorphism on M, (cf [E,M]).

The somewhat broad complex of the first six sections is of an introductory nature and is intended to prepare the geometric tools necessary for a rigorous treatment of our setup. The objects introduced and discussed are in particular the configuration space $E(M, N)$ as a Fréchet manifold, the metrics $\mathcal{G}$ and $\mathcal{B}$ on $E(M, N)$ and the metric $\mathcal{g}$ involved in the integral representation of F. Moreover the one - and two - forms $\Theta_{\mathcal{B}}$ and $\omega_{\mathcal{B}}$ associated with $\mathcal{B}$ are derived and are used to determine the spray $\mathcal{X}_{\mathcal{B}}$ of $\mathcal{B}$.

Throughout these notes smoothness on infinite dimensional manifold is meant in the sense of [Bi,Sn,Fi] or [Fr,Kr].

Finally let us remark that $\bullet$ indicates the end of lemmata, propositions, theorems and $\Box$ the end of proofs respectively.

1. Geometric preliminaries and the Fréchet manifold $E(M,N)$

Let M be a compact, oriented, connected smooth manifold and N be a connected, smooth and oriented manifold with a Riemannian metric $< \, , \, >$ assumed to be fixed. For any $j \in E(M,N)$ we define a Riemannian metric $m(j)$ on M by setting

$$m(j)(X,Y) := < TjX, TjY >, \qquad \forall X,Y \in \Gamma(TM). \tag{1.1}$$

(More customary is the notation $j^\star < \, , \, >$ instead of $m(j)$.) We use $\Gamma(\mathbf{E})$ to denote the collection of all smooth sections of any smooth vector bundle $\mathbf{E}$ over a manifold Q with $\pi_Q : \mathbf{E} \longrightarrow Q$ the canonical projection.

Let ∇ be the Levi-Cività connection of the Riemannian manifold $(N, < \, , \, >)$. In this situation, the Levi-Cività connection $\nabla(j)$ of $(M, m(j))$ is obtained as follows:

$TN|j(M)$ splits into $Tj(TM)$ and its orthogonal complement $(Tj(TM))^\perp$ (the Riemannian normal bundle of j) and hence any $Z \in \Gamma(TN|j(M))$ has an orthogonal decomposition $Z = Z^\top + Z^\perp$, where the tangential component $Z^\top$ is of the form $Z^\top = Tj \cdot V$ for a unique $V \in \Gamma(TM)$.

If now $Y \in \Gamma(TM)$, then $TjY : M \longrightarrow TN$ is smooth and therefore, the above covariant derivative $\nabla(TjY)$ is well-defined. We use this to define the vector field $\nabla(j)_X Y$ on M by the equation

$$Tj(\nabla(j)_X Y) = \nabla_X(TjY) - (\nabla_X(TjX))^\perp \tag{1.2}$$

for all $X, Y \in \Gamma(TM)$. In fact $\nabla(j)$ is the Levi - Cività connection of $m(j)$ which is called d in case N is Euclidean, i.e if $N = \mathbb{R}^n$ and $< \, , \, >$ is a fixed scalar product. Instead of $(\nabla_X(T_jX))^\perp$ we write $S(j)(X,Y)$ and call $S(j)$ the second fundamental tensor of j.

It is well-known that the set $C^\infty(M,N)$ of smooth maps from M into N endowed with Whitney's C^∞-topology is a Fréchet manifold (cf.e.g.[Bi,Sn,Fi] or [Fr,Kr]). For a given $f \in C^\infty(M,N)$, the tangent space $T_f C^\infty(M,N)$ is the Fréchet space $C^\infty_f(M,TN) := \{ l \in C^\infty(M,TN) | \tau_N \circ l = f \} \cong \Gamma(f^\star TN)$ and the tangent bundle $TC^\infty(M,N)$ is identified with $C^\infty(M,TN)$, the topology again being the C^∞-topology.

The set $E(M,N)$ of all C^∞-embeddings from M to N is open in $C^\infty(M,N)$ and thus is a Fréchet manifold whose tangent bundle we denote by $C^\infty_E(M,TN)$. It is an open submanifold of $C^\infty(M,TN)$, fibred over $E(M,N)$ by "composition with π_N". Moreover $E(M,N)$ is a principal $Diff\ M$-bundle under the obvious right $Diff\ M$-action and the quotient $U(M,N) := E(M,N)/Diff\ M$ is the manifold of "submanifolds of type M" of N (cf. [Bi,Sn,Fi], ch.5, and [Bi,Fi 1]).
Lastly, the set $\mathcal{M}(M)$ of all Riemannian structures on M is a Fréchet manifold since it is an open convex cone in the Fréchet space $S^2(M)$ of smooth, symmetric bilinear forms on M. Moreover, the map

$$m : E(M,N) \longrightarrow \mathcal{M}(M)$$

is smooth (cf.[Bi,Sn,Fi]).

By an $\mathbf{E}$- valued one-form a on M, where $\mathbf{E}$ is a vector bundle over N, we mean a smooth map

$$a : TM \longrightarrow \mathbf{E}$$

for which $a|T_pM$ is linear for all $p \in M$. We denote the set of such one-forms by $\mathcal{A}^1(M,\mathbf{E})$ and now obtain the following description of its structure: The requirement that $a \in \mathcal{A}^1(M,\mathbf{E})$ should be linear along the fibres of TM means that there is a (smooth) map $f : M \longrightarrow N$ such that $a|T_pM$ is a linear map into $\mathbf{E}_{f(p)}$ for $p \in M$, in other words, that a is a bundle map from TM to $\mathbf{E}$ over f: This map $f \in C^\infty(M,N)$ satisfies $\pi_{\mathbf{E}} \circ a = f \circ \pi_M$ (where π_E, π_M are the respective bundle projections). The set of such one-forms is naturally identified with the Fréchet space $\mathcal{A}^1(M, f^\star\mathbf{E})$. This shows that

$$\mathcal{A}^1(M,\mathbf{E}) = \bigcup_f \{\mathcal{A}^1(M, f^\star\mathbf{E}) | f \in C^\infty(M,N)\}.$$

It is clear from the construction that there is a natural surjection

$$\Pi : \mathcal{A}^1(M,\mathbf{E}) \longrightarrow C^\infty(M,N)$$

whose fibres are the Fréchet spaces $\mathcal{A}^1(M, f^\star\mathbf{E})$, in fact $\mathcal{A}^1(M, f^\star\mathbf{E})$ is a vector bundle over $C^\infty(M,N)$ with projection Π (cf.[Bi,Fi 2]).

In the following three sections we will introduce three metrics on some special types of infinite dimensional manifolds and will prepare in this way the geometric background of the description of a dynamics for smoothly deformable media.

2. The metric $\mathcal{G}$ on $E(M,N)$

The Riemannian structure $<\ ,\ >$ of N induces a "Riemannian structure" $\mathcal{G}$ on $E(M,N)$ as follows: For $j \in E(M,N)$, let $\mu(j)$ be the Riemannian volume defined on M by the given orientation and the metric $m(j)$. For any two tangent vectors $l_1, l_2 \in C^\infty_j(M,TN)$, we set

$$\mathcal{G}(j)(l_1,l_2) := \int_M < l_1, l_2 > \mu(j). \tag{2.1}$$

It is clear that $\mathcal{G}(j)$ is a continuous, symmetric, positive-definite bilinear form on $C^\infty_j(M,TN)$ for each $j \in E(M,N)$.

The metric $\mathcal{G}$ possesses some invariance properties (which will become important in sec. 12): Let $Diff^+M$ be the group of orientation-preserving diffeomorphisms of M. As a subgroup of $Diff\ M$, it operates (freely) on the right on $E(M,N)$ by

$$\begin{aligned} \phi : E(M,N) \times Diff^+M &\longrightarrow E(M,N) \\ (j,\varphi) &\longrightarrow j \circ \varphi. \end{aligned} \tag{2.2}$$

For a fixed φ, we also write $R_\varphi j$ for $j \circ \varphi$.

Similarly, if $\mathcal{J}$ is any group of orientation-preserving isometries of N, then it operates on the left on $E(M,N)$ by

$$\begin{aligned} \mathcal{J} \times E(M,N) &\longrightarrow E(M,N). \\ (g,j) &\longrightarrow g \circ j \end{aligned} \tag{2.3}$$

We need the following rather obvious result (cf.[Bi,Fi 2]) for some basic invariance properties of one-forms on $E(M,N)$:

2.1 Proposition:
$\mathcal{G}$ is invariant under both $Diff^+M$ and $\mathcal{J}$.

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3. The fibred space $\mathcal{L}_E(M,TN)$ and its dot metric

To begin with, denote by $\mathcal{A}^1_E(M,TN)$ the subset of $\mathcal{A}^1(M,TN)$ consisting of all TN-valued one-forms covering smooth embeddings from M to N. This is the inverse image of $E(M,N)$ under the projection $\Pi : \mathcal{A}^1(M,TN) \longrightarrow C^\infty(M,N)$, hence is an open submanifold and, in fact, is itself a (Fréchet) vector bundle whose fibre at j we denote by $\mathcal{A}^1_j(M,TN)$.
By construction of $m(j)$, the map Tj is fibrewise isometric. This allows us to write $a \in \mathcal{A}^1_j(M,TN)$ in the form

$$\alpha = c(\alpha, Tj) \cdot Tj + Tj \cdot A(\alpha, Tj) \tag{3.1}$$

for a suitable bundle endomorphism $c(\alpha,Tj)$ of $TN|j(M)$, skew adjoint with respect to $<\ ,\ >$ and mapping $TjTM$ into its normal bundle $(TjTM)^\perp$ and vice versa and where $A(\alpha,Tj)$ is a strong bundle endomorphism of TM. These endomorphisms are uniquely determined and are smooth continuous linear functions of α as shown in [Bi 4]. The usual "trace inner product" for endomorphisms of TN and of TM then yields for any $j \in E(M,N)$ the *dot product*

$$\alpha \cdot \beta := -\frac{1}{2} tr\ c(\alpha, Tj) \cdot c(\beta, Tj) + tr\ A(\alpha, Tj) \cdot A^\star(\beta, Tj),$$

for any two $\alpha, \beta \in \mathcal{A}^1_j(M, TN)$. Here $A^\star$, the adjoint of A, is formed fibrewise with respect to $m(j)$. We define

$$\mathcal{g}(Tj)(\alpha, \beta) := \int_M \alpha \cdot \beta\ \mu(j). \tag{3.2}$$

This yields a smooth continuous, symmetric and positive-definite bilinear form on the Fréchet space $\mathcal{A}^1_j(M, TN)$, the *dot metric* (it is a generalization of the classical Dirichlet integral which will be apparent in the theorems (3.1) and (6.2) below). For the sake of simplicity we will write $\mathcal{g}(j)$ instead of $\mathcal{g}(Tj)$.

We shall also need a subfibration of $\mathcal{A}^1_E(M, TN)$ defined by

$$\mathcal{L}_E(M, TN) := \{\nabla l | l \in C^\infty_E(M, TN)\} \tag{3.3}$$

whose fibres we denote by $\mathcal{L}_j(M, TN)(= \mathcal{L}_E(M, TN) \cap \mathcal{A}^1_j(M, TN))$; evidently these are subspaces of the Fréchet spaces $\mathcal{A}^1_j(M, TN)$; cf.[Bi,Fi 2].

Next, we introduce the Laplacian $\Delta(j)$ which will depend on j via $m(j)$; (cf.[Ma]):
For $k \in C^\infty_j(M, TN)$, we define the covariant divergence by

$$\nabla^\star(j)k := 0 \tag{3.4}$$

as usual, while following [Ma], $\nabla^\star(j)\alpha$ for $\alpha \in \mathcal{A}^1_j(M, TN)$ is given locally by

$$\nabla^\star(j)\alpha := -\sum_{r=1}^{n} \nabla_{E_r}(\alpha)(E_r), \tag{3.5}$$

(E_r) a local orthonormal frame with respect to $m(j)$. In (3.5) we have used $\nabla_X \alpha$ to denote the more informative symbol $\nabla(j)_X \alpha$, defined in the standard manner by

$$\nabla(j)_X(\alpha)(Y) = \nabla_X(\alpha Y) - \alpha(\nabla(j)_X Y) \qquad \forall X, Y \in \Gamma(TM).$$

The definition of $\Delta(j)$ does not depend on the moving frames chosen (cf. [Bi,Fi 2])

The following theorem (cf. [Bi.4], [Bi,Fi 2] and [L.M] for the last assertion) will be a basic tool in our studies of one - forms on $E(M,N)$ (cf. [Bi 4] and [Bi,Fi 2]). It relates the metric $\mathcal{G}$ with $\mathcal{g}$:

3.1 Theorem:
For any $j \in E(M,N)$, any $\alpha \in \mathcal{A}^1_E(M,TN)$ and any two $h, l \in C^\infty_j(M,TN)$ the following relations hold

$$\mathcal{g}(j)(\alpha, \nabla l) = \mathcal{G}(j)(\nabla^\star(j)\alpha, l), \tag{3.6}$$

and

$$\mathcal{g}(j)(\nabla h, \nabla l) = \mathcal{G}(j)(\Delta(j)h, l). \tag{3.7}$$

Here ∇ denotes the Levi-Cività connection of the metric $<\ ,\ >$ on N. Let $\mathbf{K}_j := \{l \in C^\infty_j(M,TN) | \nabla l = 0\}$ for any $j \in E(M,N)$, then

$$l \in \mathbf{K}_j \iff \Delta(j)l = 0. \tag{3.8}$$

In fact $\dim \mathbf{K}_j < \infty$.

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We close this section by stating invariance properties. For the rather straight forward proof we refer to [Bi,Fi 2].

3.2 Proposition:
The metric $\mathcal{g}$ on $\mathcal{L}_E(M,TN)$ is invariant under $Diff^+M$ and any group $\mathcal{J}$ of orientation-preserving isometries on N.

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4) The metric $\mathcal{B}$ and its covariant derivative on $E(M,N)$

The dynamics of deformable media to be introduced later relies on the metric $\mathcal{B}$ on $E(M,N)$. This metric will be based on a density map ρ. A smooth map

$$\rho : E(M,N) \longrightarrow C^\infty(M, I\!R)$$

is called a *density map* if the following is satisfied:

$$\rho(j)(p) > 0 \qquad \forall j \in E(M,N) \text{ and } \forall p \in M. \tag{4.1}$$

$$\boldsymbol{d}\rho(j)(k) = -\frac{\rho(j)}{2} tr_{m(j)} \boldsymbol{d}m(j)(k) \qquad \forall j \in E(M,N) \text{ and } \forall k \in C^\infty_j(M,TN). \tag{4.2}$$

$tr_{m(j)}$ denotes the trace formed with respect to $m(j)$. The symbol $\boldsymbol{d}$ denotes the differential of maps of which the domain is a Fréchet manifold and which assume values in a Fréchet space. If both domain and range are finite dimensional the usual d replaces $\boldsymbol{d}$. We will construct a density map next. For any $j' \in E(M,N)$ we express $m(j')$ via a smooth strong bundle endomorphism $f^2(j')$ of TM selfadjoint with respect to $m(j)$ as

$$m(j')(v_p, w_p) = m(j)\big(f^2(j')(p)v_p, w_p\big) \qquad \forall v_p, w_p \in T_pM \text{ and } \forall p \in M \tag{4.3}$$

and observe that the Riemannian volume forms $\mu(j)$ and $\mu(j')$ are linked by

$$\mu(j') = det\ f(j')\mu(j). \tag{4.4}$$

Fixing a map $\rho(j) \in C^\infty(M, I\!R)$ for some fixed $j \in E(M,N)$ satisfying (4.1) then

$$\rho : E(M,N) \longrightarrow C^\infty(M, I\!R)$$

given for any $j' \in E(M,N)$ by

$$\rho(j') := \frac{1}{det\ f(j')} \rho(j) \tag{4.5}$$

satisfies both (4.1) and (4.2).

To construct the metric $\mathcal{B}$ we fix a density map ρ on $E(M,N)$ once and for all (unless specified otherwise).

The metric $\mathcal{B}$ is then defined by

$$\mathcal{B}(j)(l_1, l_2) := \int_M \rho(j) < l_1, l_2 > \mu(j) \tag{4.6}$$

for each $j \in E(M,N)$ and for each pair $l_1, l_2 \in C^\infty(M,TN)$. This metric depends smoothly on all of its variables.

To find the Levi-Cività connection and the one- and two - forms associated with $\mathcal{B}$ we need to differentiate $\mathcal{B}$ which we regard as a map

$$\mathcal{B} : C^\infty_E(M,TN) \underset{\text{E(M,N)}}{\times} C^\infty_E(M,TN) \longrightarrow I\!R.$$

The domain is the fibred product of $C^\infty_E(M,TN)$ with itself over $E(M,N)$. Now let $l(t) \in C^\infty_E(M,TN)$, varying smoothly in $t \in I\!R$ and let $j(t) := \pi_N \circ l(t)$. Setting $j(0) = j, l(0) = l$ and $\dot{l}(0) = 0$, then we verify:

$$
\begin{aligned}
\frac{d}{dt}\mathcal{B}(j(t))(l(t),l(t))|_{t=0} &= \int\limits_M \rho(j)\frac{d}{dt} < l(t), l(t) > |_{t=0}\mu(j) \\
&= 2\int\limits_M \rho(j) < \nabla_{\frac{d}{dt}} k, l > \mu(j) \\
&= 2\mathcal{B}(j)(\nabla_{\frac{d}{dt}} k, l) = 2\mathcal{B}(j)(k^{vert}, l)
\end{aligned}
\tag{4.7}
$$

where *vert* denotes the pointwise formed vertical component of k in T^2N (with respect to the connection given by $<\ ,\ >$, cf [G,H,V]). It is regarded at each $p \in M$ as a tangent vector to $l(p) \in T_{\pi_N \circ l(p)}N$ and hence as an element of $T_{\pi_N \circ l(p)}N$. Thus given any two $\mathcal{K}_1, \mathcal{K}_2 \in \Gamma(C^\infty_E(M,TN))$ and any $h \in C^\infty_j(M,TN)$ the following formula

$$
d\mathcal{B}(h)(\mathcal{K}_1,\mathcal{K}_2) = \mathcal{B}(j)\big((T\mathcal{K}_1 \circ h)^{vert}, \mathcal{K}_2\big) + \mathcal{B}(j)\big((\mathcal{K}_2 \circ h)^{vert}, \mathcal{K}_1\big) \tag{4.8}
$$

holds true. This shows that the covariant derivative

$$
\nabla : \Gamma C^\infty(M,TN) \longrightarrow \Gamma C^\infty(M,TN)
$$

given by

$$
\nabla_k \mathcal{L}(p) = \Big(T_l\mathcal{L}\big(k(p)\big)\Big)^{vert} \qquad \forall l \in E(M,N) \text{ and } \forall p \in M \tag{4.9}
$$

for any choices of $\mathcal{L}$ and $k \in \Gamma C^\infty(M,TN)$ is metric and obviously torsion free. Here $T_l\mathcal{L}$ denotes the tangent map of $\mathcal{L}$ on $E(M,N)$ at l and *vert* means again the vertical component formed in T^2N. This type of connection is unique for $\mathcal{B}$, as easily seen by following the proof of the analogous statement for finite dimensional manifolds. Therefore we have (cf. [Bi 4]):

4.1 Lemma:
The covariant derivative ∇ given by (4.9) is the Levi-Cività connection of the metric $\mathcal{B}$.

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5) The one - and two - forms associated with $\mathcal{B}$ and the spray of $\mathcal{B}$

In the category of finite dimensional manifold the canonical one - and two - forms respectively are associated with the cotangent bundle. In the infinite dimensional setting however the notion of the cotangent bundle is involved with several sorts of complications. In case of a metric the situation is rather simple: We equip the set

$$\mathcal{B}^\flat(TE(M,N)) := \{\mathcal{B}(j)(l,\ldots) \mid j \in E(M,N) \text{ and } l \in T_jE(M,N)\} \tag{5.1}$$

with the C^∞-topology and obtain a Fréchet manifold, the geometric dual of $TE(M,N)$. It is a smooth vector bundle and we use it as a replacement of the cotangent bundle of $E(M,N)$. The one - form $\Theta_{\mathcal{B}}$ associated with $\mathcal{B}$ on this bundle is defined in analogy to the finite dimensional case: It is the pull back by

$$\begin{aligned} \mathcal{B}^\flat : TE(M,N) &\longrightarrow \mathcal{B}^\flat(TE(M,N)) \\ l &\mapsto \mathcal{B}(\pi_N \circ l)(l,\ldots) \end{aligned} \tag{5.2}$$

of the canonical one form on $\mathcal{B}^\flat(TE(M,N))$, i.e $\Theta_{\mathcal{B}}$ is given by

$$\begin{aligned} \Theta_{\mathcal{B}}(l)(k) &= -\mathcal{B}(j)(l, T\pi_E(k)) \\ &= -\mathcal{B}(j)(l, T\pi_N \circ k). \end{aligned} \tag{5.3}$$

Here $\pi_E : TE(M,N) \longrightarrow E(M,N)$ and $\pi_N : TN \longrightarrow N$ are the canonical projections. The two - form $\omega_{\mathcal{B}}$ associated with $\mathcal{B}$ is defined by

$$\omega_{\mathcal{B}} := \boldsymbol{d}\Theta_{\mathcal{B}} \tag{5.4}$$

where $\boldsymbol{d}$ also denotes the exterior differential for forms on $TE(M,N)$. We obtain a more explicit formula if we execute the differentiations on the right hand side of (5.4). To this end let $\mathcal{K}_1, \mathcal{K}_2 \in \Gamma(T^2E(M,N)) = \Gamma(TC^\infty_E(M,TN))$ and $l \in C^\infty_j(M,N)$, then

$$\begin{aligned} \boldsymbol{d}\Theta_{\mathcal{B}}(\mathcal{K}_1,\mathcal{K}_2)(l) = &- \int_M \rho(j)\boldsymbol{d}\langle \pi_{TN} \circ \mathcal{K}_2, T\pi_N \circ \mathcal{K}_2\rangle(\mathcal{K}_1(l))\mu(j) \\ &+ \int_M \rho(j)\boldsymbol{d}\langle \pi_{TN} \circ \mathcal{K}_1, T\pi_N \circ \mathcal{K}_1\rangle(\mathcal{K}_2(l))\mu(j) \\ &+ \int_M \rho(j)\langle l, T\pi_N[\mathcal{K}_1,\mathcal{K}_2](l)\rangle\mu(j). \end{aligned} \tag{5.5}$$

This shows that for any $j \in E(M,N)$, for any $l \in C^\infty_j(M,N)$ and for any two $k_1, k_2 \in C^\infty_l(M,T^2N)$

$$\begin{aligned} \omega_{\mathcal{B}}(l)(k_1,k_2) &= \mathcal{B}(j)(k_2^{vert}, T\pi_N \circ k_1) - \mathcal{B}(j)(k_1^{vert}, T\pi_N \circ k_2) \\ &= \int_M \rho(\pi_N \circ l)\omega^\flat(k_1,k_2)\mu(\pi_N \circ l) \end{aligned} \tag{5.6}$$

where $\omega^\flat$ is the pullback of the canonical two - form on the cotangent bundle T^*N of N by the diffeomorphism

$$\begin{aligned} <\,,\,>^\flat : TN &\longrightarrow T^*N \\ v &\mapsto <v,.>. \end{aligned}$$

Fundamental in our setup of a dynamics will be the notion of the spray $\mathcal{X}_{\mathcal{B}}$ of $\mathcal{B}$. It will govern the free motion. It is defined by

$$\omega(\mathcal{X}_{\mathcal{B}}, \mathcal{Y}) = d\mathcal{E}_{kin}(\mathcal{Y}) \qquad \forall \mathcal{Y} \in \Gamma T^2 E(M,N) \tag{5.7}$$

with

$$\mathcal{E}_{kin}(l) := \frac{1}{2}\mathcal{B}(l,l) \qquad \forall l \in C^\infty(M,TN). \tag{5.8}$$

Since

$$d\mathcal{E}_{kin}(l)(\mathcal{X}(l)) = \int_M \rho(\pi_N \circ l)\langle \mathcal{X}(l)^{vert}, T\pi_N \circ l\rangle \mu(\pi_N \circ l) \qquad \forall l \in C^\infty(M,TN)$$

holds for any $\mathcal{X} \in \Gamma(T^2E(M,N)) = \Gamma(TC^\infty_E(M,TN))$ we deduce from (5.6)

$$< \mathcal{X}(l)^{vert}, T\pi_N \circ l >= \omega^b(S_N \circ l, \mathcal{X}(l)) \tag{5.9}$$

and therefore

$$\mathcal{B}(\mathcal{X}(l)^{vert}, T\pi_N \circ l) = \int_M \rho(\pi_N \circ l)\omega^b(S_N \circ l, \mathcal{X}(l))\mu(\pi_N \circ l)$$

where S_N is the spray of $< \, , \, >$ on TN, a second order vector field on N. If $\mathcal{X}_{\mathcal{B}}$ exists, then it is unique. Hence we conclude from (5.9)

$$\mathcal{X}_{\mathcal{B}}(l) = S_N \circ l \qquad \forall l \in C^\infty(M,TN). \tag{5.10}$$

A smooth curve

$$\sigma : (-\lambda, \lambda) \longrightarrow E(M,N) \text{ with } \lambda \in \mathbb{R}^+$$

is called a geodesic iff

$$\mathcal{X}_{\mathcal{B}}(\dot\sigma(t)) = S_N \circ \dot\sigma(t) \qquad \forall t \in (-\lambda, \lambda). \tag{5.11}$$

Since $\nabla_{\frac{d}{dt}}\dot\sigma(t) = T_{\frac{d}{dt}}\dot\sigma(t) - \mathcal{X}_{\mathcal{B}}(\dot\sigma(t))$, where $T_{\frac{d}{dt}}\dot\sigma(t)$ denotes the tangent map $T\dot\sigma : \mathbb{R} \times \mathbb{R} \longrightarrow TE(M,N)$ evaluated at $(t,1)$, equation (5.11) turns into

$$\nabla_{\frac{d}{dt}}\dot\sigma(t) = 0. \tag{5.12}$$

Hence we have in more simple terms and in accordance with Lemma 4.1:

5.1 Proposition:
A smooth curve $\sigma : (-\lambda, \lambda) \longrightarrow E(M,N)$ is a geodesic of $\mathcal{B}$ with the initial conditions

$$\sigma(0) = j \text{ and } \dot{\sigma}(0) = l$$

iff

$$\sigma_p : (-\lambda, \lambda) \longrightarrow N$$
$$t \mapsto \sigma(t)(p)$$

is a geodesic in N for any $p \in M$, satisfying the initial conditions $\sigma_p(0) = j(p)$ and $\dot{\sigma}_p(0) = l(p)$.

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The above proposition implies in particular, that the spray $\mathcal{X}_\mathcal{B}$ admits locally a unique flow on $C^\infty_E(M, TN)$.

Now we have the tools necessary to introduce the notion of a constitutive law and to begin with the description of smooth deformable media.

6) One - forms on $E(M, N)$, the notion of a constitutive law

In this section we review the notion of a constitutive law for a medium, smoothly deformable in the Riemannian manifold N. The sorts of constitutive laws we have in mind will be special one - forms on $E(M, N)$ in accordance with the definition as given e.g. in [E,S]. Recall from section 1 that the tangent bundle of $E(M, N)$ is identified with $C^\infty_E(M, TN)$; accordingly, we define one-forms on $E(M, N)$ as follows:

A (scalar) *one-form on* $E(M, N)$ is a smooth function

$$F : C^\infty_E(M, TN) \longrightarrow \mathbb{R}$$

with the property that for each $j \in E(M, N)$, the restriction $F(j) := F|C^\infty_j(M, TN)$ is linear in $l \in C_j(M, TN)$. In particular, $F(j)$ is a continuous linear map on this fibre.

As done in [He], [E,S], [Bi 1] to [Bi 4] and in [Bi,Sc,So] the quality of a deformable media is given by specifying a one - form F on $E(M, N)$. The real $F(j)(l)$ is interpreted as the *work* done under the (infinitesimal) distortion $l \in C^\infty_j(E, TN)$ at the configuration $j \in E(M, N)$.

For our purposes, it will be sufficient to limit attention to a smaller class of such one-forms. They are precisely those which yield a well defined force density at each configuration (cf. theorem 6.2 below). More precisely:

6.1 Definition:
The one-form F on $E(M,N)$ is said to be g-*representable* if there exists a smooth section $\alpha : E(M,N) \longrightarrow \mathcal{A}^1_E(M,TN)$ of the bundle $(\mathcal{A}^1_E(M,TN), \Pi, E(M,N))$, such that

$$F(j)(l) = \int_M \alpha(j) \cdot \nabla l \mu(j) = g(j)(a(j), \nabla l) \tag{6.1}$$

for $j \in E(M,N)$ and $l \in C^\infty_j(M,TN)$. The section α is called the $g-$ *kernel* of F.

For instance, suppose that $\mathcal{H}$ is a smooth section of $C^\infty_E(M,TN)$ over $E(M,N)$, i.e a smooth vector field; for the existence of such fields cf. [Bi,Fi 2]. Then $\alpha(j) = \nabla\mathcal{H}(j)$ will provide a g-kernel and the right-hand side of (6.1) will define a g - representable one-form. In fact, this example can be shown to characterize the representability of one-forms, cf. below. Let us denote by $\mathcal{A}^1_g(E(M,N), I\!R)$ the collection of all smooth g-representable one-forms on $E(M,N)$. We now point out that any kernel α of a smooth one-form $F \in \mathcal{A}^1_g(E(M,N), I\!R)$ can be replaced by $\nabla\mathcal{H}$ where $\mathcal{H} : E(M,N) \longrightarrow C^\infty_E(M,TN)$ is a smooth vector field. This means that for any $j \in E(M,N)$ the following formulas

$$\int_M \alpha(j) \cdot \nabla l \ \mu(j) = \int_M \nabla\mathcal{H}(j) \cdot \nabla l \ \mu(j) \tag{6.2}$$

or equivalently,

$$g(j)(\alpha(j), \nabla l) = g(j)(\nabla\mathcal{H}(j), \nabla l) \tag{6.3}$$

have to hold for all $l \in C^\infty_j(M,TN)$. To prove this, we are required to solve

$$\Delta(j)\mathcal{H}(j) = \nabla^\star \alpha. \tag{6.4}$$

The existence of a solution $\mathcal{H}(j)$ is guaranteed for each $j \in E(M,N)$ by the theory of elliptic problems comprehensively described in [Hö 2] (cf. also [Bi,Fi 2]). We may therefore state:

6.2 Theorem:
Any $F \in \mathcal{A}^1_g(E(M,N), I\!R)$ admits a smooth vector field $\mathcal{H} : E(M,N) \longrightarrow C^\infty_E(M,TN)$ for which

$$F(j)(l) = \int_M \nabla\mathcal{H}(j) \cdot \nabla l \ \mu(j) \tag{6.5}$$

and hence

$$F(j)(l) = \int_M < \Delta(j)\mathcal{H}(j), l > \mu(j) \tag{6.6}$$

hold for all variables of F. The map $\Delta(j)\mathcal{H}(j) \in T_jE(M,N)$ is called the (internal) *force density* at the configuration $j \in E(M,N)$.

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Equation (6.6) shows that $F(j)$ is for each $j \in E(M,N)$ continuous on $L^2_j(M,TN)$, the L^2 - completion of $C^\infty_j(M,TN)$ formed with respect to $\mathcal{G}(j)$. Given vice versa smooth force density $\phi(j)$ depending smoothly on the configuration then there is a $\mathcal{H}(j)$ depending smoothly on j, such that $\Delta(j)\phi(j) = \mathcal{H}(j)$, provided $\phi(j)$ satisfies an integrability condition (cf [Hö 2]). If $\phi(j)$ does not satisfy such a condition, then an L^2 - orthogonal splitting of $\phi(j)$ allows to exhibit a component of $\phi(j)$ which does satisfy an integrability condition (cf. [Bi 4] and [Bi,Fi 1]). This component might be called the *internal* force density at j.

6.3 Definition:
A one - form $F \in A(E(M,N), I\!R)$ is called a constitutive law iff $F \in A^1_{\mathcal{G}}(E(M,N), I\!R)$. Any smooth vector field $\mathcal{H} : E(M,N) \longrightarrow TE(M,N)$, for which $\nabla\mathcal{H}$ is a $\mathcal{G}$ - kernel of F is called a *constitutive* field.

The following corollary is an easy consequence of proposition 2.1 :

6.4 Corollary:
Let G and K be groups acting on M and on N via the homomorphism

$$\Phi : G \longrightarrow Diff^+M \qquad \text{and} \qquad \Psi : K \longrightarrow \mathcal{J}$$

respectively, where $\mathcal{J}$ is a group of orientation preserving isometries of N. If $F \in \mathcal{A}^1_{\mathcal{G}}(E(M,N), I\!R)$ is invariant at each $j \in E(M,N)$ under Φ and Ψ respectively, then there is a smooth vector field $\mathcal{H} : E(M,N) \longrightarrow C^\infty_E(M,TN)$ such that

$$F(j)(l) = \int_M \nabla\mathcal{H}(j) \cdot \nabla l \ \mu(j)$$

satisfying

$$\mathcal{H}(j \circ \Phi(g)) = \mathcal{H}(j) \circ \Phi(g), \qquad \forall g \in G \tag{6.7}$$

as well as

$$\mathcal{H}(\Psi(\iota) \circ j) = T\Psi(\iota) \circ \mathcal{H}(j), \qquad \forall \iota \in \mathcal{J} \tag{6.8}$$

for each $j \in E(M,N)$.

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7) Constitutive fields of exact constitutive laws

Again F denotes a constitutive law of a deformable medium with smooth constitutive field $\mathcal{H}$. Let us assume that F is exact, i.e

$$F = d\mathcal{U} \tag{7.1}$$

with $\mathcal{U} \in C^\infty(E(M,N), I\!R)$, called a *potential.* $\mathcal{U}$ allows a representation as

$$\mathcal{U}(j) = \int_M \mathbf{e}(j)\mu(j) \qquad \forall j \in E(M, I\!R^n) \tag{7.2}$$

with the map $\mathbf{e}$, called the *mean density* of $\mathcal{U}$, defined by

$$\mathbf{e}(j) := \frac{\mathcal{U}(j)}{\mathcal{A}(j)} \qquad \forall j \in E(M,N) \tag{7.3}.$$

Here

$$\mathcal{A}(j) := \int_M \mu(j) \tag{7.4}$$

is called the volume of M. Clearly $\mathbf{e}(j) : M \longrightarrow I\!R$ is a constant map for each $j \in E(M,N)$ and varies smoothly in $j \in E(M,N)$. Hence F allows the following descriptions

$$\begin{aligned} F(j)(l) = d\mathcal{U}(j)(l) &= \int_M \left(d\mathbf{e}(j)(l) + \frac{1}{2}\mathbf{e}(j) \cdot tr_{m(j)} dm(j)(l) \right) \mu(j) \\ &= \mathcal{G}(j)(\Delta(j)\mathcal{H}(j), l) \end{aligned} \tag{7.5}$$

for any $j \in E(M,N)$ and for any $l \in C_j^\infty(M,TN)$. Equation (7.5) shows the existence of the $\mathcal{G}$ - gradient Grad $\mathcal{U}$ of $\mathcal{U}$.

Firstly let us investigate the term involving the differential of the metric in (7.5). To this end we decompose any $l \in C_j^\infty(M,TN)$ according to sec. 1 into

$$l = TjX(j,l) + l^\perp$$

for a unique $X(j,l) \in \Gamma(TM)$. The upper index $\perp$ denotes the pointwise formed normal component to $TjTM$. Moreover we let $W(j,l^\perp) : TM \longrightarrow TM$ be the uniquely determined strong bundle map of TM for which

$$\left(\nabla_X l^\perp\right)^\top = TjW(j,l)X \qquad \forall X \in \Gamma(TM). \tag{7.6}$$

The upper index $\top$ means here the pointwise formed component tangential to $j(M)$. Then

$$dm(j)(l)(X,Y) = L_{X(j,l)}m(j)(X,Y) + m(j)\left(W(j,l),X,Y\right) \qquad \forall X,Y \in \Gamma TM. \tag{7.7}$$

Due to the theorem of Gauss and the fact that $d\mathbf{e}(j) = 0$ the following equation holds for any $j \in E(M,N)$:

$$\frac{1}{2}\mathbf{e}(j) \cdot \int_M tr_{m(j)} \boldsymbol{d}m(j)(l)\mu(j) = \mathbf{e}(j) \cdot \int_M tr\ W(j,l)\mu(j) \qquad \forall l \in C_j^\infty(M,TN). \tag{7.8}$$

Hence

$$\int_M tr\ W(j,l)\mu(j) = \boldsymbol{d}\mathcal{A}(j)(l) \qquad \forall j \in E(M,N), \forall l \in C_j^\infty(M,TN) \tag{7.9}$$

holds true. Moreover $\boldsymbol{d}\mathcal{A}$ can be represented as

$$\boldsymbol{d}\mathcal{A}(j)(l) = \int_M Tj \cdot \nabla(j) l \mu(j) = \mathcal{G}(j)(\Delta(j)\bar{j}, l) \qquad \forall j \in E(M,N) \tag{7.10}$$

with

$$\Delta(j)\bar{j} = -\nabla^\star(Tj) \tag{7.11}$$

for some $\bar{j} \in C_j^\infty(M,TN)$ determined up to a harmonic field along j (the integrability conditions of (7.11) are obviously satisfied). Clearly $\Delta(j)\bar{j}$ is pointwise normal to $Tj(M) \subset TN$ as one immediately deduces from the theorem of Gauss. If not specified otherwise, we let $\bar{j}$ be $\mathcal{G}$ - othogonal to K_j (cf. (3.8)) for any $j \in E(M,N)$. Hence $\bar{j}$ depends smoothly on j. Since $S(j)(X,Y) := (\nabla_X(Tj)Y)^\perp$ is symmetric in X and Y, we find

$$\nabla^\star(Tj) = -tr\ S(j) = -\sum_{i=1}^{dim\ M} S(j)(E_i, E_i) \quad , \tag{7.12}$$

$E_i, \ldots, E_{dim\ M}$ being a moving orthogonal frame on M. The vector field $tr\ S(j)$ along j is called the mean curvature vector field (cf [L]). Clearly Tj and $\nabla\bar{j}$ are identical. To illustrate $\bar{j}$, let N be Euclidean and of $codim\ M = 1$. If $N(j)$ denotes the positively unit normal vector field along j then we may set $j = \bar{j}$ implying $\Delta(j)j = H(j) \cdot N(j)$ with $H(j)$ the trace of the Weingarten map $W(j)$ as seen from (7.9) and (7.10). Moreover, $\bar{j}$ being $\mathcal{G}$ - orthogonal to K_j requires $\bar{j} \in E(M,N)$ to have the center of mass fixed at $0 \in \mathbb{R}^n$, cf. [Bi 4]. Leaving this special case and turning back to our general investigation we deduce from (7.8) and (7.10)

$$\frac{1}{2}\mathbf{e}(j) \cdot \int_M tr_{m(j)} \boldsymbol{d}m(j)(l)\mu(j) = \int_M < \mathbf{e}(j)\Delta(j)\bar{j}, l > \mu(j). \tag{7.13}$$

To rewrite $\int_M \boldsymbol{d}\mathbf{e}(j)(l)\mu(j)$ in terms of $<\ ,\ >$ we introduce

$$\text{Grad}\ \mathbf{e} : E(M,N) \longrightarrow C_E^\infty(M,TN),$$

the $\mathcal{G}$ - gradient of $\mathbf{e}$, by setting

$$\boldsymbol{d}\mathbf{e}(j)(l) = \int\limits_M < \text{Grad } \mathbf{e}(j), l > \mu(j) \tag{7.14}$$

for any $j \in E(M,N)$ and for any $l \in C_j^\infty(M,TN)$. Equation (7.14) does not show the existence of Grad $\mathbf{e}$ at all. Nevertheless if it would exist we would be able to conclude

$$\Delta(j)\mathcal{H}(j) = \mathcal{A}(j) \cdot \text{Grad } \mathbf{e}(j) + \mathbf{e}(j) \cdot \Delta(j)\bar{j} \qquad \forall j \in E(M,N). \tag{7.15}$$

In particular we would find

$$\mathcal{A}(j) \cdot \text{Grad } \mathbf{e}(j) = \Delta(j)\left(\mathcal{H}(j) - \mathbf{e}(j)\bar{j}\right) \qquad \forall j \in E(M,N). \tag{7.16}$$

However, the right hand side of (7.16) does exist. Thus Grad $\mathbf{e}$ can be computed from (7.16) and (7.15) is the gradient of $\mathbf{e}(j) \cdot \mathcal{A}(j)$ with respect to $\mathcal{G}$. From (7.16) we immediately derive the existence of some smooth vector field

$$\hat{\mathbf{e}} : E(M,N) \longrightarrow C_E^\infty(M,TN)$$

given by $\hat{\mathbf{e}}(j) := \mathcal{A}(j)^{-1} \cdot \left(\mathcal{H}(j) - \mathbf{e}(j) \cdot \bar{j}\right)$ for all $j \in E(M,N)$ which satisfies

$$\text{Grad } \mathbf{e}(j) = \Delta(j)\hat{\mathbf{e}}(j). \tag{7.17}$$

These observations yield a description of a constitutive field of an exact constitutive law, in which the mean density $\mathbf{e}$ is of interest:

7.1 Proposition:
Let F be an exact constitutive law i.e $F = \boldsymbol{d}\mathcal{U}$ for some smooth map $\mathcal{U} : E(M,N) \longrightarrow \mathbb{R}$. Then F admits a constitutive field $\mathcal{H}$ of the form

$$\mathcal{H}(j) = \mathbf{e}(j) \cdot \bar{j} + \mathcal{A}(j) \cdot \hat{\mathbf{e}}(j) \qquad \forall j \in E(M,N) \tag{7.18}$$

with $\mathbf{e} : E(M,N) \longrightarrow \mathbb{R}$ the mean density of $\mathcal{U}$,(cf (7.3)), and $\hat{\mathbf{e}} : E(M,N) \longrightarrow C_E^\infty(M,TN)$ the smooth vector field linked to $\mathbf{e}$ by

$$\boldsymbol{d}\mathbf{e}(j)(l) = \mathcal{G}(j)(\Delta(j)\hat{\mathbf{e}}, l) = \int\limits_M \nabla\hat{\mathbf{e}}(j) \cdot \nabla l \mu(j) \quad , \tag{7.19}$$

valid for all $j \in E(M,N)$ and for all $l \in C_j^\infty(M,TN)$.

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Another type of splitting of $F = \boldsymbol{d}\mathcal{U}$ with constitutive field $\mathcal{H}$ is based on the proportionality to $\boldsymbol{d}\mathcal{A}$. Let $\mathbf{a} : E(M,N) \longrightarrow \mathbb{R}$ be given by

$$\mathbf{a}(j) := \mathcal{G}(j)(\nabla\mathcal{H}(j), \nabla\bar{j}) \tag{7.20}$$

for any $j \in E(M,N)$. In fact

$$F(j)(\bar{j}) = \mathbf{a}(j) \cdot \|\nabla\bar{j}\|^2_{\mathcal{G}} = dimM \cdot \mathcal{A}(j) \cdot \mathbf{a}(j) \qquad \forall j \in E(M,N),$$

showing that $\mathbf{a}(j) \in C^\infty(E(M,N), I\!R)$. By $\|l\|^2_{\mathcal{G}}$ we mean $\int \nabla l \cdot \nabla l \mu(j)$, the $\mathcal{G}$ - norm of ∇l. The splitting we have in mind is then described in the following obvious proposition:

7.2 Proposition:
Any constitutive law F of the form $F = d\mathcal{U}$ with $\mathcal{U} \in C^\infty(E(M,N), I\!R)$ splits uniquely into

$$F = d(\mathbf{a} \cdot \mathcal{A}) + d\psi \tag{7.21}$$

where

$$\mathbf{a}(j) := F(j)(\bar{j}) / \|\nabla\bar{j}\|^2_{\mathcal{G}} \qquad \forall j \in E(M,N) \tag{7.22}$$

and $\psi \in C^\infty(E(M,N), I\!R)$ is uniquely determined up to a constant map. Moreover F admits a smooth constitutive field $\mathcal{H}$ decomposed for any $j \in E(M,N)$ into

$$\mathcal{H}(j) = \mathbf{a}(j) \cdot \bar{j} + \mathcal{A}(j) \cdot \hat{\mathbf{a}}(j) + \hat{\psi}(j). \tag{7.23}$$

Here $\hat{\mathbf{a}}(j)$ and $\hat{\psi}(j)$ are for each $j \in E(M,N)$ defined by

$$\text{Grad } \mathbf{a}(j) = \Delta(j)\hat{\mathbf{a}}(j) \quad \text{and} \quad \text{Grad } \psi(j) = \Delta(j)\hat{\psi}(j). \tag{7.24}$$

Moreover

$$\mathcal{A}(j) \cdot \mathcal{G}(j)(\Delta(j)\hat{\mathbf{a}}(j), \bar{j}) = -\mathcal{G}(j)(\Delta(j)\hat{\psi}(j), \bar{j}) \tag{7.25}$$

holds for any $j \in E(M,N)$.

●

8) The volume sensitive part of a constitutive law

Let F be any constitutive law with constitutive field $\mathcal{H}$. First of all we split off $d\mathcal{A}$ from F, based on (7.11). We do it as follows: Recall that $L_j^2(M,TN)$ be the space of all vector fields l of M along j for which $\int\limits_M < l,l > \mu(j)$ is finite. Then taking the component of $\Delta(j)\mathcal{H}(j)$ along $\overline{j}$ in $L_j^2(M,TN)$ for each j yields

$$\Delta(j)\mathcal{H}(j) = \mathbf{a}(j)\cdot\Delta(j)\overline{j} + \Delta(j)\mathcal{H}_1(j) \tag{8.1}$$

for a well defined $\mathbf{a}(j) \in I\!R$ and some $\mathcal{H}_1(j) \in C_j^\infty(M,N)$ for which $\mathcal{H}_1(j)$ is orthogonal to $\Delta(j)\overline{j}$ in $L_j^2(M,TN)$. Hence $F(j)$ decomposes into

$$F(j)(l) = \mathbf{a}(j)\cdot d\mathcal{A}(j) + \int\limits_M \nabla\mathcal{H}_1(j)\cdot\nabla l\mu(j) \qquad \forall l \in C_j^\infty(M,TN). \tag{8.2}$$

$\mathbf{a}\cdot d\mathcal{A}$ is called the volume sensitive part of F. Due to (6.3) we have in particular

$$F(j)(\overline{j}) = \mathcal{G}(j)(\Delta(j)\mathcal{H}(j),\overline{j}) = \mathbf{a}(j)\cdot\|\nabla\overline{j}\|_{\sigma}^2. \tag{8.3}$$

Since both $\|\nabla(j)\overline{j}\|$ and $\overline{j}$ vary smoothly in j, the map $\mathbf{a} : E(M,N) \longrightarrow I\!R$ is smooth as well. The vector field $\mathcal{H}^a : E(M,N) \longrightarrow C_E^\infty(M,TN)$ assigning to each j the map $\mathbf{a}(j)\cdot\overline{j}$ is called the *volume sensitive part* of $\mathcal{H}$. In fact it only depends on F and not on the particular constitutive field $\mathcal{H}$. By looking at (8.1) and (8.3) we have the following:

8.1 Theorem:
For each constitutive law F, any constitutive field $\mathcal{H}$ determines uniquely a smooth map $\mathbf{a} : E(M,N) \longrightarrow I\!R$ given for each $j \in E(M,N)$ by

$$\mathbf{a}(j) := F(j)(\overline{j})/\|\nabla\overline{j}\|_{\sigma}^2 \tag{8.4}$$

and splits uniquely into

$$\mathcal{H}(j) = \mathbf{a}(j)\cdot\overline{j} + \mathcal{H}_1(j) \qquad \forall j \in E(M,N) \tag{8.5}$$

where $\mathcal{H}_1$ has vanishing volume sensitive part and is L_2 - orthogonal to $\Delta(j)\overline{j}$. The volume sensitive part $\mathcal{H}^a$ of $\mathcal{H}$ as well as $\mathcal{H}_1$ vary smoothly in $j \in E(M,N)$.

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8.2 Remark:
If $n = 1 + dim\ M$ then due to (7.11) and (7.9) $\Delta(j)\cdot\overline{j} = H(j)N(j)$ implying via (7.10)

$$F(j)(H(j)\cdot N(j)) = \mathbf{a}(j)\cdot\|\Delta(j)\overline{j}\|^2 \qquad \forall j \in E(M,I\!R^n)$$

where $N(j)$ is the positively oriented unit normal vector field and $H(j)$ denotes the trace of the Weingarten map and where $\|\Delta(j)\overline{j}\|^2 := \int < \Delta(j)\overline{j},\Delta(j)\overline{j} > \mu(j)$.

9) The dynamics determined by a constitutive law

As we have mentioned in the previous section, a constitutive law on $E(M,N)$ of a smoothly deformable medium is defined to be a smooth one - form $F : TE(M,N) \longrightarrow I\!R$ admitting a smooth constitutive vector field $\mathcal{H} \in \Gamma C^\infty_E(M,TN)$.

The *work form*

$$\mathcal{W}_F : C^\infty(M,T^2N) \longrightarrow I\!R$$

- the fundamental ingredient of our set up of a dynamics - is given by

$$\mathcal{W}_F(l)(k) := d\mathcal{E}_{kin}(l)(k) - \big(\pi_E^\star F\big)(l)(k) \tag{9.1}$$

for any $l \in C^\infty_E(M,TN)$ and for any $k \in C^\infty_E(M,T^2N)$.

The dynamics determined by F is given by the unique vector field $\mathcal{X}_F$ (if it exists at all) for which

$$\omega_B(\mathcal{X}_F,\mathcal{X}) = \mathcal{W}_F(\mathcal{X}) \qquad \forall \mathcal{X} \in \Gamma T^2E(M,N). \tag{9.2}$$

The following theorem shows the existence of $\mathcal{X}_F$ and moreover expresses its simple form:

9.1 Theorem:

Given a constitutive law F on $E(M,N)$ with constitutive field $\mathcal{H}$ then

$$\mathcal{X}_F(l) = \mathcal{X}_B(l) + \frac{1}{\rho(\pi_N \circ l)} \cdot \big(\Delta(\pi_N \circ l)\mathcal{H}(\pi_N \circ l)\big)^{vert} \qquad \forall l \in C^\infty(M,TN) \tag{9.3}$$

where *vert* denotes the pointwise formed vertical lift of $\Delta(\pi_N \circ l)\mathcal{H}(\pi_N \circ l)$ on N determined by ∇.

•

Proof:

If $\mathcal{X}_F$ exists, then again it is unique. Using (5.6) and (9.2) we verify (9.3) for any $l \in C^\infty(M,N)$ by the following calculation:

$$
\begin{aligned}
\omega_{\mathcal{B}}(\mathcal{X}_F(l),\mathcal{X}(l)) =& \int_M \rho(\pi_N \circ l)\omega(\mathcal{X}_{\mathcal{B}}(l),\mathcal{X}(l))\mu(\pi_N \circ l) \\
&+ \int_M \omega((\Delta(\pi_N \circ l)\mathcal{H}(\pi_N \circ l))^{vert},\mathcal{X}(l))\mu(\pi_N \circ l) \\
=& \boldsymbol{d}\mathcal{E}_{kin}(\pi_N \circ l)(\mathcal{X}(l)) \\
&- \int_M \langle \Delta(\pi_N \circ l)\mathcal{H}(\pi_N \circ l), T\pi_N(\mathcal{X}(l))\rangle \mu(\pi_N \circ l) \\
&+ \int_M \langle \mathcal{X}(l)^{vert}, T\pi_N \circ (\Delta(\pi_N \circ l)\mathcal{H}(\pi_N \circ l)^{vert})\rangle \mu(\pi_N \circ l).
\end{aligned}
\tag{9.4}
$$

Since the last summand is zero we find for each $l \in C^\infty(M,TN)$

$$
\begin{aligned}
\omega_{\mathcal{B}}(\mathcal{X}_F(l),\mathcal{X}(l)) =& \boldsymbol{d}\mathcal{E}_{kin}(\pi_N \circ l)(\mathcal{X}(l)) \\
&- \int_M \langle \Delta(\pi_N \circ l)\mathcal{H}(\pi_N \circ l), T\pi_N \circ \mathcal{X}(l)\rangle \mu(\pi_N \circ l) \\
=& \boldsymbol{d}\mathcal{E}_{kin}(\pi_N \circ l)(\mathcal{X}(l)) - F(\pi_N \circ l)(T\pi_N \circ \mathcal{X}(l)) \quad ,
\end{aligned}
$$

establishing the claim.

□

9.2 Definition:
The equation of a motion $\sigma : (-\lambda,\lambda) \longrightarrow E(M,N)$ subjected to F is given by

$$
\ddot{\sigma}(t) = \mathcal{X}_F(\dot{\sigma}(t)) \qquad \forall t \in (-\lambda,\lambda), \tag{9.5}
$$

combined with initial conditions.

We therefore have :

9.3 Theorem:
The equation of a motion $\sigma : (-\lambda,\lambda) \longrightarrow E(M,N)$ subjected to a given constitutive law F with constitutive field $\mathcal{H} \in \Gamma(C_E(M,TN))$ and with the initial data $\sigma(0) = j \in E(M,N)$ as well as $\dot{\sigma}(0) = l_0 \in C_j^\infty(M,TN)$ is given by

$$
\ddot{\sigma}(t) = \mathcal{X}_{\mathcal{B}}(\dot{\sigma}(t)) + \frac{1}{\rho(\sigma(t))} \cdot (\Delta(\sigma(t))\mathcal{H}(\sigma(t)))^{vert} \qquad \forall t \in (-\lambda,\lambda) \tag{9.6}
$$

or equivalently by

$$\nabla_{\frac{d}{dt}}\dot{\sigma}(t) = \frac{1}{\rho(\sigma(t))}\Delta(\sigma(t))\mathcal{H}(\sigma(t)). \tag{9.7}$$

The motion σ is free i.e a geodesic iff $F = 0$.

•

Equation (9.6) coincides with the equation of a motion subjected to a constitutive law of [Bi 4]. There the equation was derived by d'Alembert's principle and not on a geometric basis as done here. The present method has the advantage that first integrals can be derived from invariance under group actions, while the one in [Bi 4] does not admit a comparable mechanism.

10) Refinements of the equation of motion

The above decomposition theorems (8.1) and (8.2) yield immediately refinements of the equations of motions subjected to a constitutive law:

10.1 Theorem:
Let F be a constitutive law with constitutive field $\mathcal{H}$. Any motion $\sigma : (-\lambda, \lambda) \longrightarrow E(M, N)$ (with any initial condition) is subjected to F iff

$$\ddot{\sigma}(t) = S_N \circ \dot{\sigma}(t) + \frac{\mathbf{a}(\sigma(t))}{\rho(\sigma(t))} \cdot \big(\Delta(\sigma(t))\overline{\sigma}(t)\big)^{vert} + \frac{1}{\rho(\sigma(t))} \cdot \big(\Delta(\sigma(t))\mathcal{H}_1(\sigma(t))\big)^{vert} \qquad \forall t \in (-\lambda, \lambda). \tag{10.1}$$

or equivalently

$$\nabla_{\frac{d}{dt}}\dot{\sigma}(t) = \frac{\mathbf{a}(\sigma(t))}{\rho(\sigma(t))} \cdot \Delta(\sigma(t))\overline{\sigma}(t) + \frac{1}{\rho(\sigma(t))}\Delta(\sigma(t))\mathcal{H}_1(\sigma(t)) \qquad \forall t \in (-\lambda, \lambda). \tag{10.2}$$

Moreover the following balance law

$$\frac{1}{2}d\mathcal{E}_{kin}(\sigma(t))(\dot{\sigma}(t)) = \mathbf{a}(\sigma(t)) \cdot d\mathcal{A}_F(\sigma(t))\dot{\sigma}(t) + \pi_N^{\star} F_1(\sigma(t))\dot{\sigma}(t) \qquad \forall t \in (-\lambda, \lambda) \tag{10.3}$$

holds true. F_1 is the constitutive law associated with $\mathcal{H}_1$, the volume insensitive part of $\mathcal{H}$.

•

10.2 Corollary:
If the constitutive map of F is of the form

$$\mathcal{H}(j) = \mathbf{a}(j) \cdot \overline{j} \qquad \forall j \in E(M,N) \tag{10.4}$$

that is if $F(j) = \mathbf{a}(j) \cdot \boldsymbol{d}\mathcal{A}(j)$ then the motion σ, subjected to F, satisfies for any $t \in (-\lambda, \lambda)$

$$\nabla_{\frac{d}{dt}} \dot{\sigma}(t) = \frac{\mathbf{a}(t)}{\rho(\sigma(t))} \cdot \Delta(\sigma(t))\overline{\sigma}(t) \tag{10.5}$$

as well as the balance law

$$\frac{1}{2}\boldsymbol{d}\mathcal{E}_{kin}(\sigma(t))(\dot{\sigma}(t)) = \mathbf{a}(\sigma(t)) \cdot \boldsymbol{d}\mathcal{A}(\sigma(t))(\dot{\sigma}(t)). \tag{10.6}$$

If N is Euclidean we may let $\overline{j} = j$ for all $j \in E(M,N)$ and conclude

$$\ddot{\sigma}(t) = \frac{\mathbf{a}(\sigma(t))}{\rho(\sigma(t))} \cdot \Delta(\sigma(t))\sigma(t) \qquad \forall t \in (-\lambda, \lambda). \tag{10.7}$$

In case M is of codimension one then (10.5) rewrites as

$$\nabla_{\frac{d}{dt}} \dot{\sigma}(t) = \frac{\mathbf{a}(\sigma(t))}{\rho(\sigma(t))} \cdot H(\sigma(t)) \cdot N(\sigma(t)) \qquad \forall t \in (-\lambda, \lambda) \tag{10.8}$$

where $N(\sigma(t))$ is the positively oriented unit normal of $j(M)$ in N.

•

An immediate connection between the density map and the vanishing of F at some "equilibrium" configuration in the Euclidean case is described next:

10.3 Corollary:
Let F be given by (10.4). If the motion σ, subjected to F, given by (10.5) satisfies $\ddot{\sigma}(t_0) = 0$ for some $t_0 \in (-\lambda, \lambda)$ then

$$S_N \circ \dot{\sigma}(t_0) = 0 \text{ and } \mathbf{a}(\sigma(t_0)) \cdot \Delta(\sigma(t_0))\overline{\sigma}(t_0) = 0 \tag{10.9}$$

hold true. In particular if $(N, < \, , \, >)$ is Euclidean i.e if $N = \mathbb{R}^n$ with $< \, , \, >$ a scalar product and if the codimension of M in N is one then

$$\mathbf{a}(\sigma(t_0)) \cdot H(\sigma(t_0)) \cdot N(\sigma(t_0)) = 0 \tag{10.10}$$

and consequently

$$\frac{\mathbf{a}(\sigma(t_0))}{\rho(\sigma(t_0))} \cdot \boldsymbol{d}\rho(\sigma(t_0))(N(\sigma(t_0))) = 0 \tag{10.11}$$

implying $\boldsymbol{d}\rho(\sigma(t_0)) = 0$ if $\mathbf{a}(\sigma(t_0)) \neq 0$.

•

The proof of (10.9) and (10.11) is immediate from section one and (4.2).

11) F determined by a symmetric stress tensor T

In this section $N = \mathbb{R}^n$ and $< \, , \, >$ denotes a fixed scalar product and $\mathcal{J}$ the group of all orientations preserving linear isomorphism on $\mathbb{R}^n$. As shown in [Sch] and [Tr] a constitutive law invariant under $\mathcal{J}$ and satisfying an additional requirement can be described in this case by a symmetric stress tensor field T. Let us determine the equation of motion subjected to a $\mathcal{J}$ - invariant constitutive law: We are given a smooth map

$$T : E(M, \mathbb{R}^n) \longrightarrow S^2(M).$$

(Recall from section one that $S^2(M)$ is the Fréchet space of all smooth symmetric two - tensors on M endowed with the C^∞ - topology.) The symmetric tensor $T(j)$ is called the stress tensor at the configuration $j \in E(M, \mathbb{R}^n)$. As a general reference for the notion of a stress tensor we recommend [M,H].

The work $F : TE(M, \mathbb{R}^n) = E(M, \mathbb{R}^n) \times C^\infty(M, \mathbb{R}^n) \longrightarrow \mathbb{R}$ associated with T is according to [L,L] defined by

$$F(j)(l) = \frac{1}{2} \int_M T(j) \cdot \boldsymbol{d}m(j)(l)\mu(j) \tag{11.1}$$

for any $j \in E(M, \mathbb{R}^n)$ and any $l \in C^\infty(M, \mathbb{R}^n)$. The dot - product on the right hand side is defined as follows: Both $T(j)$ and $\frac{1}{2}\boldsymbol{d}m(j)(l)$ are represented via $m(j)$ as bundle endomorphisms $Q(j)$ and $B(dl, dj)$ of TM respectively. Both of these endomorphisms are selfadjoint with respect to $m(j)$. In fact $B(dl, dj)$ is the selfadjoint component of $A(dl, dj)$ introduced in (3.1). Then

$$\frac{1}{2}T(j) \cdot \boldsymbol{d}m(j)(l) = tr\ Q(j) \cdot B(dl, dj) \qquad \forall j \in E(M, \mathbb{R}^n). \tag{11.2}$$

A constitutive field of F is constructed by solving

$$\Delta(j)\mathcal{H}_T(j) = \nabla^\star Q(j) \tag{11.3}$$

for any $j \in E(M, \mathbb{R}^n)$. Choosing $\mathcal{H}_T(j) \in im\Delta(j)$ yields the smoothness in j. The constitutive law F is then rewritten as

$$
\begin{aligned}
F(j)(l) &= \frac{1}{2}\int\limits_M T(j)\cdot \boldsymbol{d}m(j)(l)\mu(j) = \int\limits_M < \Delta(j)\mathcal{H}_T(j), l > \mu(j) \\
&= \int\limits_M d\mathcal{H}_T(j)\cdot dl\mu(j)
\end{aligned}
\tag{11.4}
$$

for all variables of F.

We consider next the volume sensitive part only. By (8.4)

$$
F(j)(j) = \mathcal{G}(j)\big(\Delta(j)\mathcal{H}(j), j\big) = \mathbf{a}(j)\cdot \|dj\|^2_{\sigma} \tag{11.5}
$$

is valid for each $j \in E(M, I\!R^n)$. This yields immediately:

11.1 Proposition:
Given any smooth stress tensor $T : E(M, I\!R^n) \times C^\infty(M, I\!R^n) \longrightarrow S^2(M)$, there are uniquely defined smooth maps $\mathbf{a} : E(M, I\!R^n) \longrightarrow I\!R$ and $T_1 : E(M, I\!R^n) \longrightarrow S^2(M)$ such that for each $j \in E(M, I\!R^n)$ the following equations hold:

$$
\mathbf{a}(j)\cdot \|dj\|^2_{\sigma} = \frac{1}{2}\int\limits_M T(j)\cdot \boldsymbol{d}m(j)(dj)\mu(j) \tag{11.6}
$$

$$
T(j) = \mathbf{a}(j)\cdot m(j) + T_1(j) \tag{11.7}
$$

and

$$
\mathcal{H}_T(j) = \mathbf{a}(j)\cdot j + \mathcal{H}_1(j) \tag{11.8}
$$

where $\mathcal{H}_1$ and T_1 have vanishing volume sensitive parts.

•

The following is now immediately:

11.2 Theorem:
Let

$$
T : E(M, I\!R^n) \longrightarrow S^2(M)
$$

be a smooth stress tensor. Then the equation of a motion $\sigma : (-\lambda, \lambda) \longrightarrow E(M, I\!R^n)$ subjected to F, as defined by (9.5), is

$$
\begin{aligned}
\ddot{\sigma}(t) &= -\frac{1}{\rho(\sigma(t))} \cdot \operatorname{div}_{\sigma(t)} T(\sigma(t)) \\
&= -\frac{1}{\rho(\sigma(t))} \operatorname{div}_{\sigma(t)} \big(\mathbf{a}(\sigma(t)) \cdot m(\sigma(t))\big) - \frac{1}{\rho(\sigma(t))} \cdot \operatorname{div}_{\sigma(t)} T_1(\sigma(t)) \\
&= \frac{\mathbf{a}(\sigma(t))}{\rho(\sigma(t))} \cdot \Delta(\sigma(t))\sigma(t) + \frac{1}{\rho(\sigma(t))} \Delta(\sigma(t))\mathcal{H}_1(\sigma(t)).
\end{aligned}
\tag{11.9}
$$

•

12) Symmetry groups

Given any density map $\rho : E(M,N) \to C^\infty(M, \mathbf{R}^n)$, the metric $\mathcal{B}$ on $E(M,N)$ associated with ρ is invariant under $Diff^+M$, the group of all diffeomorphisms preserving the orientation of M. This is immediate from the solution of the continuity equation (4.2) and the transformation formula of the integral. Equation (5.6) moreover shows immediately the invariance of $\omega_{\mathcal{B}}$ under $Diff^+M$.

Let us suppose that we are given a constitutive law F being invariant under $\mathrm{Diff}^+ M$, meaning that $(R\psi)^\star F = F$ for all $\psi \in Diff^+M$, where $R\psi$ denotes the right translation by ψ on $E(M,N)$ (cf. sec. 2). More explicitly, $Diff^+M$ invariance of F means

$$
F(j \circ \psi)(l \circ \psi) = F(j)(l) \qquad \forall j \in E(M,N) \text{ and } \forall \psi \in Diff^+M \tag{12.1}
$$

The work form $\mathcal{W}_F$ is invariant under $Diff^+M$. Differentiating (12.1) with respect to ψ yields for any $j \in E(M,N)$ and any $l \in C^\infty_j(M,TN)$ the equation

$$
\nabla_{TjX}\big(F(j)\big)(l) + F(j)\big(\nabla_X l\big) = 0 \qquad \forall X \in \Gamma TM. \tag{12.2}
$$

Here $\Gamma(TM)$ is identified with $T_{id} Diff^+M$.

The symplectic formalism yields a smooth moment map

$$
\mathbf{J} : C^\infty_E(M,TN) \longrightarrow \mathcal{B}^\flat(\Gamma(TM)) \tag{12.3}
$$

given for any $l \in C^\infty_j(M,TN)$ and any $j \in E(M,N)$ by the equation

$$
\mathbf{J}(l)(X) = \theta_{\mathcal{B}}(l)(TlX) = -\mathcal{B}(\pi \circ l)(l, T\pi_N \circ TlX) = -\mathcal{B}(j)(l, TjX), \tag{12.4}
$$

The relation between $\mathbf{J}$ and integrals of a motion is as follows:

12.1 Lemma:
Let F be $Diff^+M$ invariant. For any $X \in \Gamma(TM)$ the map $\mathbf{J}_X : C^\infty_E(M,TN) \longrightarrow \mathbb{R}$ defined by

$$
\mathbf{J}_X(l) := \mathbf{J}(l)(X) \qquad \forall l \in C^\infty_E(M,TN) \tag{12.5}
$$

is constant on any motion subjected to F iff

$$F(j)(TjX) = 0 \qquad \forall j \in E(M,N) \text{ and } \forall X \in \Gamma TM.$$

If F is exact and the potential is $Diff^+M$ invariant, then $\mathbf{J}_X$ is a first integral of the motion for each $X \in \Gamma TM$.

•

Proof: Let us compute $\boldsymbol{d}\mathbf{J}_X$: For any $k \in C_l^\infty(M,TN)$ we have

$$\begin{aligned}\boldsymbol{d}\mathbf{J}_X(l)(k) &= \boldsymbol{d}(\theta_{\mathcal{B}}(l)(T(\pi_N \circ l)X))(k) = -\boldsymbol{d}\mathcal{B}(\pi_N \circ l)(l, T(\pi_N \circ l)X)(k)\\ &= -(\mathcal{B}(k^{vert}, T(\pi_N \circ l)X) + \mathcal{B}(l, T(\pi_N \circ k^{vert}X)) = -\mathcal{B}(k^{vert}, T(\pi_N \circ l)X).\end{aligned} \tag{12.6}$$

Because of $\mathcal{X}_{\mathcal{B}}(l)^{vert} = 0$ the choice $k := \mathcal{X}_F(l)$ yields

$$\begin{aligned}\boldsymbol{d}J_X(l)(\mathcal{X}_F(l)) &= -\,\mathcal{B}(\mathcal{X}_F(l)^{vert}, T(\pi_N \circ l)X) - \mathcal{B}(\mathcal{X}_{\mathcal{B}}(l)^{vert}, T(\pi_N \circ l)X)\\ &= F(\pi_N \circ l)(T(\pi_N \circ l)X)\end{aligned} \tag{12.7}$$

for all $l \in C_E^\infty(M,TN)$. The validity of the assertion is now immediate. □

Next we consider a more general situation coming up rather frequently. If we have differentiable groups $\mathcal{D}$ and I together with the respective smooth representation

$$a : \mathcal{D} \longrightarrow Diff^+M \text{ and } b : I \longrightarrow \mathcal{J}$$

then both a and b yield moment maps. Following the same routine in the proof of Lemma 12.1 we derive the following:

12.2 Theorem:
Let F be invariant under both $a(\mathcal{D})$ and $b(I)$. The respective moment maps of a and b yield first integrals of any motion subjected to F for each of the elements in the respective Lie - algebras if for any $j \in E(M,N)$

$$F(j)(Tj\dot{a}X) = 0 \qquad \forall X \in \Gamma(TM) \tag{12.8}$$

as well as

$$F(j)(\dot{b}(c)\cdot\bar{j}) = 0 \qquad \forall c \in T_{id}I \tag{12.9}$$

hold. Here $\dot{a}$ and $\dot{b}$ denote the representation of the respective Lie algebras determined by a and b.

•

13) The restriction of a motion subjected to F on a fibre in the principal bundle $E(M,N)$

Each fibre in the principal bundle $E(M,N)$ is of the form $i \circ Diff\ M$ with fixed $i \in E(M,N)$.

In this section we impose on the motion $\sigma : (-\lambda, \lambda) \longrightarrow E(M,N)$ subjected to F the constraint that

$$\sigma(t)(M) = i(M) \qquad \forall t \in (-\lambda, \lambda).$$

To find the equation of such a motion σ we proceed analogous as in the previous sections: We let $\mathcal{B}^i$ be the metric on $i \circ Diff\ M$ obtained by restricting $\mathcal{B}$ to this fibre. This yields immediately the symplectic structure ω^i on $T(j \circ Diff\ M)$, the pullback of $\omega_{\mathcal{B}}$ by the tangent map of the inclusion map $i \circ Diff\ M \hookrightarrow E(M,N)$. Moreover let F^i be the pullback of F by the inclusion map mentioned.

Observing that any tangent vector to $i \circ \psi \in i \circ Diff\ M$ with $\psi \in Diff\ M$ is of the form $T(j \circ \psi)X$ for some $X \in \Gamma(TM)$, the one - form F^i is given by

$$F^i(i \circ \psi)(T(i \circ \psi)X) = \int_M < \Delta(i \circ \psi)\mathcal{H}(i \circ \psi), T(i \circ \psi)X > \mu(j). \tag{13.1}$$

There is a connection on $E(M,N)$ induced by the orthogonal projection of TN to $TiTM$: Given any $l \in C^\infty_{i\circ\psi}(M,TN)$ with $\psi \in Diff\ M$ we let the component $l^\top$ of l in $T_{i\circ\psi} i \circ Diff\ M$ be given by

$$l^\top(p) = T(i \circ \psi)X(l^\top, j) \in T(i \circ \psi)(TM) \qquad \forall p \in M, \tag{13.2}$$

for a well defined vector field $X(l^\top, j) \in \Gamma(TM)$. Clearly the projection from $TE(M,N)$ to $T(i \circ Diff\ M)$ given by $\top$ is $Diff\ M$ invariant for each $i \in E(M,N)$.

Let $\mathcal{E}^i_{kin}$ denote the kinetic energy on $T(i \circ Diff^+ M)$ given by $\mathcal{B}^i$. Its Euler field on $Ti \circ Diff\ M$ is the spray $\mathcal{X}^i$ of $\mathcal{B}^i$. It is of the form

$$\mathcal{X}^i(T(i \circ g)X) = T^2(i \circ g)S^i(T(i \circ g))^{-1}T(i \circ g)X \tag{13.3}$$

with S^i the spray of $m(i)$ on TM and $X \in \Gamma(TM)$. We need one more geometric notion to formulate our equations: Let ∇^i denote the covariant derivative of Levi - Cività of $\mathcal{B}^i$ on $i \circ Diff\ M$. Due to general principles in Riemannian geometry and the fact that $\bar{j}$ is normal to $TjTM$ we immediately find for any $j \in E(M,N)$ the following :

13.1 Theorem:
The equation of motion $\sigma : (-\lambda, \lambda) \longrightarrow E(M,N)$, subjected on one hand to a given constitutive law F on $E(M;N)$ with constitutive field $\mathcal{H}$ and on the other to the constraint

$$\sigma(t)(M) = i(M) \qquad \forall t \in (-\lambda, \lambda) \tag{13.4}$$

for a fixed $i \in E(M, I\!R)$, read as

$$\nabla^i_{\frac{d}{dt}} \dot{\sigma}(t) = \left(\left(\Delta(\sigma(t)) \mathcal{H}(\sigma(t)) \right)^{vert} \right)^{\top} \tag{13.5}$$

for all $t \in (-\lambda, \lambda)$, with S_M the spray of $m(i)$ on TM.

•

If we subject moreover the motion to the further constraint that σ maps into $i \circ Diff_{\mu(i)} M$ with $Diff_{\mu(i)} M$, the group of all $\mu(i)$ preserving diffeomorphisms of M, then we obtain Euler's equation of a perfect fluid on $i(M)$ as in [E,M]), provided $\mathcal{H} = 0$

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