Memoryless distributions revisited

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Abstract

The present paper discusses the memoryless property of distributions on the real line. The main result asserts that every distribution Q which is memoryless on a set containing 0 must be concentrated on $\{0\}$ or $(0, \infty)$. It is then shown that this condition is necessary and sufficient for Q to be either the Dirac distribution in 0 or an exponential distribution. Corresponding results for the geometric distribution are given as well.

1 Introduction

The present paper discusses the memoryless property of distributions on the real line. The main result asserts that every distribution Q which is memoryless on a set containing 0 must be concentrated on $\{0\}$ or $(0,\infty)$. It is then shown that this condition is necessary and sufficient for Q to be either the Dirac distribution in 0 or an exponential distribution. Corresponding results for the geometric distribution are given as well.

2 Memoryless distributions

Throughout this paper, let $Q : \mathcal{B}(\mathbf{R}) \to [0,1]$ be a distribution and consider a set $S \in \mathcal{B}(\mathbf{R})$. The distribution Q is *memoryless on* S if

- $Q(S) = 1$ and
- the identity $Q((x+y,\infty)) = Q((x,\infty)) \cdot Q((y,\infty))$ holds for all $x, y \in S$.

Thus, if X is a random variable satisfying $P_X = Q$, then Q is memoryless on S if and only if

- $P(X \in S) = 1$ and
- the identity $P(X > x+y) = P(X > x) \cdot P(X > y)$ holds for all $x, y \in S$.

The preceding identity has an obvious interpretation in terms of conditional probabilities.

2.1 Examples.
(a) The Dirac distribution δ_0 is memoryless on $\mathbf{R}_+$ and on $\mathbf{N}_0$.
(b) Every exponential distribution $\mathbf{Exp}(\alpha)$ is memoryless on $\mathbf{R}_+$.
(c) Every geometric distribution $\mathbf{Geo}(\vartheta)$ is memoryless on $\mathbf{N}_0$.

Note that all distributions of Example 2.1 are concentrated on (a subset of) $\mathbf{R}_+$, and that none of them is memoryless on $\mathbf{R}$.

2.2 Theorem. *Let Q be memoryless on S and assume that $0 \in S$. Then Q satisfies either $Q(\{0\}) = 1$ or $Q((0,\infty)) = 1$.*

Proof. Assume that $Q((0,\infty)) < 1$. Since $0 \in S$, we have

$$Q((0,\infty)) = Q((0,\infty)) \cdot Q((0,\infty)),$$

hence

$$Q((0,\infty)) = 0,$$

and thus

$$\begin{aligned} Q((x,\infty)) &= Q((x,\infty)) \cdot Q((0,\infty)) \\ &= 0 \end{aligned}$$

for all $x \in S$. Define now $z := \inf S$ and choose a sequence $\{x_n\}_{n\in\mathbf{N}} \subseteq S$ which decreases to z. Then we have

$$\begin{aligned} Q((z,\infty)) &= Q\left(\bigcup_{n\in\mathbf{N}} (x_n,\infty)\right) \\ &= \sup_{\mathbf{N}} Q((x_n,\infty)) \\ &= 0 \end{aligned}$$

and hence $-\infty < z$. Since $Q(S) = 1$, the definition of z together with the previous identity yields

$$Q(\{z\}) = 1,$$

and thus $z \in S$. Finally, since $0 \in S$, we have either $z < 0$ or $z = 0$. But $z < 0$ implies $z \in (2z, \infty)$ and hence

$$\begin{aligned} Q(\{z\}) &\leq Q((2z,\infty)) \\ &= Q((z,\infty)) \cdot Q((z,\infty)), \end{aligned}$$

which is impossible. Therefore, we have $z = 0$, as was to be shown. □

3 The exponential distribution

The exponential distribution can be characterized as follows:

3.1 Theorem. *The following are equivalent:*
(a) *Q is memoryless on $(0,\infty)$.*
(b) *$Q = \mathbf{Exp}(\alpha)$ for some $\alpha \in (0,\infty)$.*
In this case, $\alpha = -\log Q((1,\infty))$.

Proof. Assume that (a) holds. By induction, we have

$$Q((n,\infty)) = Q((1,\infty))^n$$

and

$$Q((1,\infty)) = Q((1/n,\infty))^n$$

for all $n \in \mathbf{N}$. Thus, $Q((1,\infty)) = 1$ is impossible because of

$$\begin{aligned} 0 &= Q(\emptyset) \\ &= \inf_{\mathbf{N}} Q((n,\infty)) \\ &= \inf_{\mathbf{N}} Q((1,\infty))^n, \end{aligned}$$

and $Q((1,\infty)) = 0$ is impossible because of

$$\begin{aligned} 1 &= Q((0,\infty)) \\ &= \sup_{\mathbf{N}} Q((1/n,\infty)) \\ &= \sup_{\mathbf{N}} Q((1,\infty))^{1/n}. \end{aligned}$$

Therefore, we have

$$Q((1,\infty)) \in (0,1).$$

Define now $\alpha := -\log Q((1,\infty))$. Then we have $\alpha \in (0,\infty)$ and

$$Q((1,\infty)) = \exp(-\alpha),$$

and thus

$$\begin{aligned} Q((m/n,\infty)) &= Q((1,\infty))^{m/n} \\ &= (\exp(-\alpha))^{m/n} \\ &= \exp(-\alpha m/n) \end{aligned}$$

for all $m, n \in \mathbf{N}$. This yields

$$Q((x,\infty)) = \exp(-\alpha x)$$

for all $x \in (0,\infty) \cap \mathbf{Q}$. Finally, for each $z \in (0,\infty)$ we may choose a sequence $\{x_n\}_{n\in\mathbf{N}} \subseteq (0,\infty) \cap \mathbf{Q}$ which decreases to z, and we obtain

$$\begin{aligned} Q((z,\infty)) &= Q\left(\bigcup_{n\in\mathbf{N}} (x_n,\infty)\right) \\ &= \sup_{\mathbf{N}} Q((x_n,\infty)) \\ &= \sup_{\mathbf{N}} \exp(-\alpha x_n) \\ &= \exp(-\alpha z). \end{aligned}$$

Since $Q((0,\infty)) = 1$, it follows that $Q = \mathbf{Exp}(\alpha)$. Therefore, (a) implies (b). The converse is obvious from Example 2.1(b). □

The previous proof follows Barlow/Proschan [1; Theorem 3.2.2].

3.2 Corollary. *The following are equivalent:*
(a) *Q is memoryless on $\mathbf{R}_+$.*
(b) *Either $Q = \boldsymbol{\delta}_0$ or $Q = \mathbf{Exp}(\alpha)$ for some $\alpha \in (0, \infty)$.*

Proof. The assertion is immediate from Theorems 2.2 and 3.1. □

With regard to the previous result, note that the Dirac distribution $\boldsymbol{\delta}_0$ is the limit of the exponential distributions $\mathbf{Exp}(\alpha)$ as $\alpha \to \infty$.

3.3 Corollary. *There is no distribution which is memoryless on* $\mathbf{R}$.

Proof. If Q is memoryless on $\mathbf{R}$, then either $Q = \boldsymbol{\delta}_0$ or $Q = \mathbf{Exp}(\alpha)$ for some $\alpha \in (0, \infty)$, by Theorem 2.2 and Corollary 3.2; see also Nelsen [2]. On the other hand, none of these distributions is memoryless on $\mathbf{R}$. □

4 The geometric distribution

The results on the memoryless property of the exponential distribution presented in the previous section have a complete counterpart for the geometric distribution:

4.1 Theorem. *The following are equivalent:*
(a) *Q is memoryless on $\mathbf{N}$.*
(b) *$Q = \mathbf{Geo}(\vartheta)$ for some $\vartheta \in (0, 1]$.*
In this case, $\vartheta = 1 - Q((1, \infty))$.

The verification of Theorem 4.1 is straightforward. With regard to Corollary 3.2 and the subsequent remark, note that $\mathbf{Geo}(1) = \boldsymbol{\delta}_1$.

4.2 Corollary. *The following are equivalent:*
(a) *Q is memoryless on $\mathbf{N}_0$.*
(b) *Either $Q = \boldsymbol{\delta}_0$ or $Q = \mathbf{Geo}(\vartheta)$ for some $\vartheta \in (0, 1]$.*

To complete the discussion, we recall the following well–known relation between exponential and geometric distributions: If $Q = \mathbf{Exp}(\alpha)$ for some $\alpha \in (0, \infty)$ and if Q' denotes the unique distribution satisfying $Q'(\{n\}) = Q((n-1, n])$ for all $n \in \mathbf{N}$, then $Q' = \mathbf{Geo}(1 - \exp(-\alpha))$.

References

[1] **Barlow, R. E., and Proschan, F.:** *Statistical Theory of Reliability and Life Testing.* Silver Spring: To Begin With 1981.

[2] **Nelsen, R. B.**: *Consequences of the memoryless property for random variables.* Amer. Math. Monthly **94**, 981–984 (1987).

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