Mannheimer Manuskripte 140/92

April 1992

Solving a General Boundary Value Problem for the Exterior Derivative of a Differential Form

140/92

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ABSTRACT

This paper is concerned with the question to find a differential form ν of degree r solving the exterior differential equation $d\nu = \eta$ under general inhomogenous boundary conditions $\nu|_{\partial M} = \chi|_{\partial M}$. We give necessary and sufficient conditions for the forms η and χ and prove an existence theorem on a bounded Riemannian manifold. The proof is based on the Hodge-Kodaira decomposition theorem. The dual problem for differential forms and applications from vector analysis are investiated.

1. Introduction

In a preceding paper [Sch] the author showed how to solve – under appropriate integrability conditions – the general inhomogeneous boundary value problem

$$\operatorname{div} V = \Xi \qquad \text{with} \qquad V|_{\partial M} = Z|_{\partial M} \tag{1.1}$$

for a vector field $V \in \Gamma(TM)$ on a Riemannian manifold M with boundary ∂M. The analytic basis for doing so was the Hodge-Kodaira theorem for differential forms on bounded manifolds, which yields a simple decomposition of the prescribed zero form $\Xi \in \Omega^0(M)$. This paper is concerned with the boundary value problem

$$d\omega = \chi \qquad \text{with} \qquad \omega|_{\partial M} = \psi|_{\partial M} \tag{1.2}$$

for a differential forms $\omega \in \Omega^r(M)$. We give necessary integrability and show be means of the Hodge-Kodaira decomposition technique that they also suffice to solve (1.2).
To fix the notion we rewrite in section 2 some basic structures on $\Omega(M)$, the algebra of differential forms on a bounded Riemannian manifold M. For the boundary value of $\omega \in \Omega^r(M)$ we introduce the splitting $\omega|_{\partial M} = \jmath^*\omega + \mathbf{n}\omega$ into a tangential and normal part, needed to give an appropriate formulation of Stoke's theorem and Morrey's generalization of the Hodge-Kodaira decomposition for differential forms on M.
In section 3 we use this decomposition to solve boundary value problems with the exterior derivative $d\omega$ and the tangential component $\jmath^*\omega$ prescribed on M and ∂M, respectively. Such problems of Dirichlet type are well established – at least on the Euklidean $\mathbb{R}^n$. To consider general boundary conditions we further need special extension results for a differential form – here on existence of a differential form with its boundary value $\omega|_{\partial M}$ and the normal part of of its derivative $\mathbf{n}d\omega$ prescribed.
In section 4 we apply these technical results to a general inhomogeneous problem for the exterior derivative, i.e. we solve the boundary value problem (1.2) under appropriate integrability conditions. The existence theorem generalizes the well established ones on Neumann- and Dirichlet-problems for differential forms [Kre], where only $\mathbf{n}\omega$ or $\jmath^*\omega$ can be prescribed. It may have several applications in partial differential equations, by guaranteeing the solvability of any problem, which can be transformed into the exterior systems (1.2). As an immediate consequence we solve the Hodge-dual problem with $\delta\omega$ and $\omega|_{\partial M}$ prescribed.
We finish the paper with an application of the decomposition technique to classical problems for vector fields. By means of the isomorphism $\sharp$ vector fields and the one forms on a Riemannian manifold are identified such that the divergence and the curl can be expressed in terms of the co-differential operator and the exterior derivative, respectively. We give explicit integrability conditions for the problem (1.1) as well as the coresponding dual problem with $\operatorname{curl} W$ prescribed.

2. Differential forms on Riemannian manifolds with boundaries

The analytic foundation for our investigation on boundary value problems for differential forms is the Hodge-Kodaira decomposition theorem on manifolds with boundaries, cf. [Mo62]. To given an appropriate formulation we fix our notion according to [AMR] :
By a n-dimensional manifold M with boundary we mean a paracompact topological Hausdorff space, which is locally homeomorphic to an open subset of $\mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n \,|x_n \geq 0\}$, such that boundary points $p \in \partial M$ are mapped to $\mathbb{R}^n_0 = \{\mathbf{x} \in \mathbb{R}^n \,|x_n = 0\}$. The differentiable structure on M is defined with respect to the differentiable structure on $\mathbb{R}^n_+$, naturally induced from $\mathbb{R}^n$ by restriction. We assume M to be compact, orientable and (at least) of differentiability class C^2. Then there exists a Riemannian metric g of class C^1 on M and a coresponding a Riemannian volume form μ_M.
Considering the boundary ∂M there is the natural embedding $\jmath : \partial M \to M$, and for any vector field $Y \in \Gamma(TM)$ the push forward $\jmath_* Y$ is a field tangential to ∂M. On the other hand one has on ∂M a (unite) normal field $\mathcal{N} : \partial M \to (TM)|_{\partial M}$ of class C^1 (obeying $\jmath_* \mathcal{N} = 0$) which also induces a Riemannian volume element $\mu_\partial := \mathbf{i}_\mathcal{N} \mu_M$ on ∂M. Here $\mathbf{i}$ denotes the interior product.
As on manifolds without boundaries differential forms $\omega \in \Omega^r(M)$ are defined as (C^1)-sections in the bundle $\Lambda^r(M)$ of all anti-symmetric, r-linear forms on TM and the $\wedge$-product $\wedge : \Omega^r(M) \wedge \Omega^s(M) \to \Omega^{r+s}(M)$ is defined as usual. Furthermore we have – naturally induced from the metric on M – a product

$$\begin{aligned} \langle , \rangle_{\Omega^r} : \Omega^r(M) \times \Omega^r(M) &\longrightarrow \Omega^0(M) \\ \langle \omega, \eta \rangle_{\Omega^r} := & \sum_{1 \leq j_1 \dots \leq j_r \leq n} \omega(E_{j_1}, \dots, E_{j_r}) \cdot \eta(E_{j_1}, \dots, E_{j_r}) \end{aligned} \tag{2.1}$$

where the fields E_{j_k} are taken from a local g-orthonormal frame $(E_1, \dots E_n)$ on TM. One easily sees that this product does not depend on the choice of the frame. It is used to define the Hodge operator $\star : \Omega^r(M) \to \Omega^{n-r}(M)$ by demanding for all $\eta, \omega \in \Omega^r(M)$

$$\eta \wedge \star\omega = \langle \eta, \omega \rangle_{\Omega^r} \, \mu_M \tag{2.2}$$

and furthermore allows to equip $\Omega^r(M)$ with a scalar product

$$<< \omega, \eta >>:= \int_M \langle \omega, \eta \rangle_{\Omega^r} \, \mu_M \qquad (2.3).$$

For the boundary value of some $\omega \in \Omega^r(M)$, i.e. the restriction $\omega|_{\partial M}$, the pull back under the embedding $\jmath$ yields a r-form $\jmath^*\omega \in \Omega^r(\partial M)$ and hence a natural splitting into a tangential and a normal component :

$$\omega|_{\partial M} = \jmath^*\omega + \mathbf{n}\,\omega \qquad \text{with} \qquad \begin{cases} \jmath^*\omega(\mathcal{N}, Y_1, \dots Y_{r-1}) & = 0 \\ \mathbf{n}\,\omega(\jmath_* Y_1, \dots \jmath_* Y_r) & = 0 \end{cases} \quad \forall Y_k \in \Gamma(TM) \quad (2.4).$$

A crucial property of these components of the boundary value $\omega|_{\partial M}$, which easily can be shown by using a coordinate representation [Mo56,Sch], is the fact that they are adjoint to each other by means of the Hodge operator in the sense that

$$\jmath^* \star \omega \;=\; \star \mathbf{n}\, \omega \tag{2.5}$$

To get the notion of an exterior derivative $d : \Omega^r(M) \to \Omega^{r+1}(M)$ we observe that the derivative D of functions in direction of a vector field X_k as well as the Lie bracket $[X_i, X_j]$ can be given a proper meaning also on a bounded manifold. Hence we define :

$$\begin{aligned} d\omega(X_0, X_1, \ldots, X_r) = & \sum_{0\le k\le r} (-1)^k D(\omega(X_0, \ldots, \widehat{X}_k, \ldots, X_r))(X_k) \\ & \sum_{0\le i<j\le r} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_r) \end{aligned} \tag{2.6}$$

where $\widehat{X}_k$ means that X_k is omitted. Then it is possible to equip $\Omega^r(M)$ with appropriate Sobolev structures, defined as the respective completions

$$\begin{aligned} H^0\Omega^r(M) &= \overline{\{\omega \in \Omega^r(M) \mid \ll \omega, \omega \gg < \infty\}} \\ H^1\Omega^r(M) &= \overline{\{\omega \in \Omega^r(M) \mid (\ll \omega, \omega \gg \; + \; \ll d\omega, d\omega \gg) \; < \infty\}} \end{aligned} \tag{2.7}$$

Furthermore we call $\omega|_{\partial M}$ a boundary value of Sobolev class H^1 along ∂M iff

$$\int_{\partial M} \left(\langle \omega, \omega \rangle_{\Omega^r} \; + \; \langle d\omega, d\omega \rangle_{\Omega^{r+1}}\right) \mu_\partial \; < \infty \tag{2.8}$$

On a boundaryless manifold the co-differential operator can be given as the (H^0-)adjoint of the exterior derivative. For $\partial M \neq \emptyset$ we define $\delta : \Omega^{r+1}(M) \to \Omega^r(M)$ via Hodge operator as $\delta := (-1)^{nr+1} \star d \star$ and obtain from Stoke's theorem

$$\ll d\omega, \eta \gg \; = \; \ll \omega, \delta\eta \gg + \int_{\partial M} \jmath^*(\omega \wedge \star\eta) \qquad \forall \omega \in \Omega^r(M)\; \forall \eta \in \Omega^{r+1}(M) \tag{2.9}$$

Hence d and δ are no longer adjoint to each other; the boundary term, however, vanishes whenever $\jmath^*\omega = 0$ or $\mathbf{n}\,\eta = 0$ on ∂M.
The central ingredient to face boundary value problems for differential forms is Morrey's generalization of the Hodge-Kodaira decomposition for bounded manifolds. We set

$$\begin{aligned} \mathcal{D}^r(M) &:= \{d\alpha \mid \alpha \in H^1\Omega^{r-1}(M) \;\text{ with }\; \jmath^*\alpha = 0\} \\ \mathcal{E}^r(M) &:= \{\delta\beta \mid \alpha \in H^1\Omega^{r+1}(M) \;\text{ with }\; \mathbf{n}\,\beta = 0\} \\ \mathcal{H}^r(M) &:= \{\kappa \in H^1\Omega^r(M) \mid \;\text{ with }\; d\kappa = \delta\kappa = 0\} \end{aligned}$$

for the subspaces of closed ($\mathcal{D}^r(M)$) and co-closed ($\mathcal{E}^r(M)$) forms with an appropriate boundary behavior respectively for the space of harmonic fields ($\mathcal{H}^r(M)$). The Hodge-Kodaira decomposition theorem, cf. chapter 7.7 of [Mo62], states that there is an orthonormal decomposition with respect to the scalar product (2.3)

$$H^0\Omega^r(M) = \mathcal{D}^r(M) \oplus \mathcal{E}^r(M) \oplus \mathcal{H}^r(M) \tag{2.10}$$

such that each $\omega \in H^0\Omega^r(M)$ uniquely writes as

$$\omega = d\alpha + \delta\beta + \kappa \tag{2.11}$$

with $d\alpha \in \mathcal{D}^r(M)$, $\delta\beta \in \mathcal{E}^r(M)$ and $\kappa \in \mathcal{H}^r(M)$. Assuming M to be smooth and ω to be C^k-differentiable, one gets α and β of class C^{k+1} and κ of class C^k. On this basis all the results presented below can also be formulated in the C^k-category, cf. [Sch].

3. Boundary value problems of Dirichlet type and an extension problem

First we apply the decomposition technique to solve boundary value problems of the type

$$\begin{aligned} d\omega &= \chi && \text{on } M \\ \jmath^*\omega &= \jmath^*\psi && \text{on } \partial M \end{aligned} \tag{3.1}$$

where $\chi \in H^0\Omega^{r+1}(M)$ and the r-form ψ of Sobolev class H^1 on ∂M are prescribed. Since on ∂M the tangential component $\jmath^*\omega$ is given, we call such problems of Dirichlet type. If χ where of Sobolev class H^1 a necessary condition for solving (3.1) would be $d\chi = 0$; more general $\chi \in H^1\Omega^{r+1}(M)$ has to obey the integrability condition

$$<< \chi, \delta\gamma >> = 0 \qquad \forall \gamma \in H^1\Omega^{r+2}(M) \qquad \text{with} \qquad \mathbf{n}\,\gamma = 0 \qquad (3.2).$$

On the other hand we get from Stoke's theorem (2.9) by means of (2.5) the equation

$$<< \chi, \kappa >> = \int_{\partial M} \jmath^*\psi \wedge \star\mathbf{n}\,\kappa \qquad \forall \kappa \in \mathcal{H}^{r+1}(M) \tag{3.3}$$

as a second necessary condition for solving (3.1). In fact these two conditions are also sufficient to solve the Dirichlet problem :

Lemma 1

Let M be a compact, orientable Riemannian C^2-manifold with boundary. Given a differential form $\chi \in H^0\Omega^{r+1}(M)$ where $0 \le r \le N-1$ and a r-form ψ, which is of class H^1 along ∂M. If χ and ψ obey the integrability conditions (3.2) and (3.3), there exists some $\omega \in H^1\Omega^r(M)$ which solves the boundary value problem

$$\begin{aligned} d\,\omega &= \chi && \text{on } M \\ \jmath^*\omega &= \jmath^*\psi && \text{on } \partial M \end{aligned} \qquad (3.4).$$

Proof :
Applying the Hodge-Kodaira decompostion (2.11) to $\chi \in H^0\Omega^{r+1}(M)$ we get

$$\chi = d\alpha_\chi + \delta\beta_\chi + \kappa_\chi \qquad \text{with} \qquad \jmath^*\alpha_\chi = 0 \qquad \text{and} \qquad \mathbf{n}\,\beta_\chi = 0 \qquad (3.5).$$

Since this decomposition is $H^0\Omega^r(M)$-orthogonal we can conclude from the integrability condition (3.2) that $<< \delta\beta_\chi, \delta\beta_\chi >> = 0$ and in consequence $\delta\beta_\chi = 0$. On the other hand ψ has an extension $\overline{\psi} \in H^1\Omega^r(M)$ and we can decompose

$$d\overline{\psi} = d\alpha_\psi + \delta\beta_\psi + \kappa_\psi \qquad \text{with} \qquad \jmath^*\alpha_\psi = 0 \qquad \text{and} \qquad \mathbf{n}\,\beta_\psi = 0 \qquad (3.6).$$

Since $d\overline{\psi} \in H^0\Omega^{r+1}(M)$ is an exact form, we argue as above to show that also $\delta\beta_\psi = 0$. Defining then

$$\omega := \alpha_\chi - \alpha_\psi + \overline{\psi} \quad \text{where} \quad \begin{cases} d\omega \;=\; \chi - \kappa_\chi + \kappa_\psi & \text{on } M \\ \jmath^*\omega \;=\; \jmath^*\psi & \text{on } \partial M \end{cases} \qquad (3.7)$$

we get from (2.9) for any $\eta \in H^1\Omega^{r+1}(M)$:

$$<< \kappa_\psi - \kappa_\chi, \eta >> = << \omega, \delta\,\eta >> \; - \; << \chi, \eta >> + \int_{\partial M} \jmath^*(\omega \wedge \star\eta) \qquad (3.8).$$

If $\eta \in \mathcal{H}^{r+1}(M)$, i.e. a harmonic field, the right hand side vanishes by means of the integrability condition (3.3). Especially we may chose $\eta := (\kappa_\psi - \kappa_\chi)$ and conclude that $\kappa_\psi - \kappa_\chi = 0$. Hence ω as constructed above solves the problem (3.4). □

A similar problem, restricted to the Euklidean space $\mathbb{R}^n$, has been considered by [Kre]. He obtained – under appropriate (stronger) integrability conditions – an explicit expression for solutions of (3.4), for which he also could prescribe the value of the co-differential $\delta\omega$ on ∂M. In turn our result – based on a variational method [Mo62] – yields a result on a general Riemannian manifold but just gives necessary and sufficient conditions for the existence of solutions.

Furthermore this approach allows also to investigate more general boundary value problems, what requires the study a special extension problem for a differential form. In our case this concerns the question of existence of a differential form $\omega \in H^1\Omega^r(M)$ such that its boundary value $\omega|_{\partial M}$ and the normal components of its derivative $\mathbf{n}(d\omega)$ are prescribed. The coresponding result is due to Morrey :

Lemma 2

Let M be a compact, orientable Riemannian C^2-manifold with boundary and let η be a r-form on M with boundary value $\eta|_{\partial M}$ of class H^1 along ∂M. Then there exists a $(r-1)$-form $\sigma \in H^1\Omega^{r-1}(M)$ with $d\sigma \in H^1\Omega^r(M)$ such that

$$\sigma|_{\partial M} = 0 \qquad \text{and} \qquad \mathbf{n}\,(d\sigma) = \mathbf{n}\,\eta \qquad (3.9).$$

If M is smooth and η is C^k-differentiable σ and $d\sigma$ can be chosen of class C^k.

For a proof, based on a coordinate representation and a Friederichs mollifier argument on $\mathbb{R}^n$, we refer to [Mo56,Sch]. Remarkably any solution σ of (3.9) also has a vanishing tangential derivative. This is clear by observing that the exterior derivative d commutes with the pull back such that we can conclude from $\sigma|_{\partial M} \equiv 0$ that

$$\jmath^* d\sigma = d(\jmath^*\sigma) = 0 \tag{3.10}$$

We note that also lemma 1 implicitly is based on a special extension problem, namely to find a $H^1\Omega^r(M)$ extension $\overline{\psi}$ of the prescribed boundary value ψ. A careful analysis, based in a trace theorem, cf. [Ada], shows that in fact it suffies for lemma 1 to assume ψ to be of Sobolev class $H^{1/2}$ along ∂M in order to find ω of class H^1. Hence all the proceeding results also hold under appropriate weaker regularity assumptions along ∂M.

4. General boundary value problems and dual problems

Having the results of section 3 at hand, i.e. the solution theorem for the Dirichlet problem (3.4) and the extension result of lemma 2, we can consider a general problem for the exterior derivative on $\Omega^r(M)$:

Theorem 3

Let M be a compact, orientable Riemannian C^2-manifold with boundary. Given a differential form $\chi \in H^0\Omega^{r+1}(M)$ with $0 \leq r \leq n-1$ and a r-form ψ of class H^1 on ∂M, which obey the integrability conditions

$$\begin{aligned} \ll \chi, \delta\gamma \gg &= 0 && \forall \gamma \in H^1\Omega^{r+2}(M) \quad \text{with} \quad \mathbf{n}\gamma = 0 \\ \ll \chi, \kappa \gg &= \int_{\partial M} \jmath^*\psi \wedge \star\mathbf{n}\,\kappa && \forall \kappa \in \mathcal{H}^{r+1}(M) \end{aligned} \tag{4.1}$$

there exists a solution $\nu \in H^1\Omega^r(M)$ of the boundary value problem

$$\begin{aligned} d\nu &= \chi && \text{on } M \\ \nu|_{\partial M} &= \psi|_{\partial M} && \text{on } \partial M \end{aligned} \tag{4.2}$$

Proof :
Splitting the boundary value $\psi|_{\partial M}$ into its tangential and its normal part (2.4), we solve by means of lemma 1 the problem

$$\begin{aligned} d\omega &= \chi && \text{on } M \\ \jmath^*\omega &= \jmath^*\psi && \text{on } \partial M \end{aligned} \tag{4.3}$$

To control the normal component we observe from (3.7) that

$$\mathbf{n}\,\omega = \mathbf{n}\,(\alpha_\chi - \alpha_\psi + \overline{\psi}) \tag{4.4}$$

where $\overline{\psi}, \alpha_\chi, \alpha_\psi \in H^1\Omega^r(M)$ are determined from χ and ψ. With lemma 2 we can construct an appropriate $\sigma_\omega \in H^1\Omega^{r-1}(M)$, obeying also (3.10), such that

$$\jmath^*(d\sigma_\omega) = 0 \quad , \quad \mathbf{n}(d\sigma_\omega) = \mathbf{n}(\alpha_\psi - \alpha_{\overline{\psi}} + \psi) \quad \text{and} \quad \sigma_\omega|\partial M = 0 \qquad (4.5).$$

Hence $\nu := \omega - d\sigma_\omega$ solves the problem (4.2). □

We remark that the elliptic technique to solve boundary value problems [Hör] could not be applied here, since the exterior derivative "d" is not an elliptic operator. Theorem 3 may have a wide range of possible applications in partial differential equations, since it guaranties the solvability of any linear first order system, which can be transformed into an exterior system of the form (4.2).

In direct analogy to the duality between the Dirichlet- and the Neumann-problem for differential forms [Kre], we consider the Hodge-dual version of the general boundary value problem, i.e. the problem with the co-differential $\delta\omega$ together with the boundary value $\omega|\partial M$ prescribed. Using the identity $\star(\star\omega) = (-1)^{s(n-s)}\omega$ we show :

Corollary 4

Let M be a compact, orientable Riemannian C^2-manifold with boundary. Given a differential form $\xi \in H^0\Omega^{s-1}(M)$ with $1 \leq s \leq n$ and a s-form ϕ of class H^1 on ∂M, which obey the integrability conditions

$$\begin{aligned} << \xi, d\epsilon >> &= 0 && \forall \epsilon \in H^1\Omega^{s-2}(M) \quad \text{with} \quad \jmath^*\epsilon = 0 \\ (-1)^{ns+n+1} << \xi, \lambda >> &= \int_{\partial M} \star\mathbf{n}\,\phi \wedge \jmath^*\lambda && \forall \lambda \in \mathcal{H}^{s-1}(M) \end{aligned} \qquad (4.6),$$

there exists a solution $\omega \in H^1\Omega^s(M)$ of the boundary value problem

$$\begin{aligned} \delta\,\omega &= \xi && \text{on } M \\ \omega|_{\partial M} &= \phi|_{\partial M} && \text{on } \partial M \end{aligned} \qquad (4.7).$$

Proof :
Choosing $\xi = (-1)^{n+s+1} \star \chi$ and $\phi = \star\psi$ (where $s = n - r$) the integrability conditions (4.6) turn into

$$\begin{aligned} << \chi, \star d\epsilon >> &= 0 && \forall \epsilon \in H^1\Omega^{n-r-2}(M) \quad \text{with} \quad \mathbf{n}(\star\epsilon) = 0 \\ << \chi, \star\lambda >> &= \int_{\partial M} \jmath^*\psi \wedge \star\mathbf{n}(\star\lambda) && \forall \lambda \in \mathcal{H}^{n-r-1}(M) \end{aligned} \qquad (4.8).$$

This is equivalent to (4.1), such that theorem 3 can be applied, i.e. (4.2) has a solution ν. Then $\omega := \star\nu$ becomes the desired solution of (4.7), since $\delta\omega = (-1)^{n(s-1)+1} \star d \star \omega$ and the Hodge operator commutes with the restriction to the boundary. □

5. Two applications motivated from physics

As an illustration of the technique presented above we consider two boundary value problems for vector fields, which play an important role for physical applications. We observe that the Riemannian structure g on M induces an isomorphism between the vector fields of Sobolev class H^1 and one differential forms the same differentiability class :

$$\begin{aligned} \sharp : \Gamma(TM) &\longrightarrow \Omega^1(M) \\ X^\sharp(Y) &:= g(X,Y) \qquad \forall Y \in \Gamma(TM) \end{aligned} \tag{5.1}$$

One immediately shows, cf. (2.1) and (2.2), two algebraic properties of this operator

$$g(X,Y) = < X^\sharp, Y^\sharp >_{\Omega^1} \quad \text{and} \quad \star X^\sharp = \mathbf{i}_X \mu \tag{5.2}$$

and the fact that the co-differential on $\Omega^1(M)$ transforms by $\sharp$ into the divergence, i.e.

$$\delta(X^\sharp) = \operatorname{div} X \qquad \forall X \in \Gamma(TM) \tag{5.3}$$

Furthermore – on a 3-dimensional manifold – the cross-product of vector fields computes as $(X \times Y)^\sharp = \star(X^\sharp \wedge Y^\sharp)$ and the exterior derivative of a one form can be expressed in terms of the curl of the corresponding field by

$$d(X^\sharp) = \star(\operatorname{curl} X)^\sharp \qquad \forall W \in \Gamma(TM) \tag{5.4}$$

Corollary 5

Let on a compact, orientable Riemannian C^2-manifold M a real valued function $\Xi \in \Omega^0(M)$, a vector field $Y \in \Gamma(TM)$, both of Sobolev class H^1, and a vector field $Z \in \Gamma(TM)$ with $Z|_{\partial M}$ of class H^1 along ∂M be given.

a) The boundary value problem

$$\begin{aligned} \operatorname{div} V &= \Xi \qquad \text{on } M \\ V|_{\partial M} &= Z|_{\partial M} \qquad \text{on } \partial M \end{aligned} \tag{5.5}$$

has a solution $V \in \Gamma(TM)$ of class H^1 iff

$$\int_M \Xi \mu_M + \int_{\partial M} g(Z|_{\partial M}, \mathcal{N}) \mu_\partial = 0 \tag{5.6}$$

b) For $dimM = 3$ the boundary value problem

$$\begin{aligned} \operatorname{curl} W &= Y \qquad \text{on } M \\ W|_{\partial M} &= Z|_{\partial M} \qquad \text{on } \partial M \end{aligned} \tag{5.7}$$

has a solution $W \in \Gamma(TM)$ of class H^1 iff

$$\operatorname{div} Y = 0 \qquad \text{and} \tag{5.8.}$$

$$\int_M g(Y,L)\, \mu = \int_{\partial M} g(\mathcal{N}, Z \times L)\, \mu_\partial \qquad \forall \text{ harmonic vector fields } L \in \Gamma(TM)$$

Proof :
a) The exterior system corresponding to (5.5) becomes by (5.3)

$$\delta V^\sharp = \Xi \quad \text{with} \quad V^\sharp|_{\partial M} = Z^\sharp|_{\partial M} \tag{5.9}$$

Since $s = 1$ the first condition of (4.6) is empty and $\mathcal{H}^{s-1} = I\!R$, i.e. the harmonic fields are the constant functions. Furthermore we have $\mathbf{i}_{\mathcal{N}}\mu =: \mu_\partial$ and $\star\,\mathbf{n}(Z^\sharp) = g(Z, \mathcal{N}) \star \mathcal{N}^\sharp$ for all $Z \in \Gamma(TM)$. Then the remaining integrability condition for solving (5.9) reads as

$$c \int_M \Xi \mu_M + c \int_{\partial M} g(Z|_{\partial M}, \mathcal{N})\, \mu_\partial = 0 \qquad \forall c \in I\!R \qquad (5.10).$$

b) The exterior system coresponding to (5.7) becomes by (5.4)

$$d W^\sharp = \star Y^\sharp \quad \text{and} \quad V^\sharp|_{\partial M} = Z^\sharp|_{\partial M} \qquad (5.11).$$

Since $\star Y^\sharp$ is of class H^1 by assumption the first integrability condition of (4.1) turns into $d \star Y^\sharp = 0$, what is equivalent to $\operatorname{div} Y = 0$. Expressing the right hand side of the second condition of (4.1) in terms of the cross-product and using the fact that the harmonic vector fields on M are in one-to-one corespondence with the one forms $\star\kappa \in \mathcal{H}^1(M)$ we get

$$\int_M g(Y, L)\mu = \int_{\partial M} \jmath^*(\star(Z \times L)^\sharp) \qquad \forall \text{ harmonic vector fields } L \in \Gamma(TM) \qquad (5.12).$$

Arguing as for a) we see that this is equivalent to the second condition of (5.8). □

For the special case $Z|_{\partial M} \equiv 0$ these two problems are studied in [vWa] where also the great interest on such existence questions for fluid dynamics is indicated. The general problem with the full boundary value $Z|_{\partial M}$ prescribed – modulo demanding consistency with the integrability condition (5.6) respectively (5.8) – has (to the author's knowledge) not been studied in the literature, but should be of importance for several areas in classical field theory.

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