Necessary Optimality Conditions for Nonsmooth Minimax Problems

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Under a suitable assumption necessary optimality conditions are derived for nonsmooth minimax problems involving infinitely many functions. The results obtained here generalize some necessary optimality conditions for mathematical programming and minimax problems.

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1. Introduction

Let C be a nonempty subset of a normed space X, and let Q be a compact topological space. For all $\alpha \in Q$, let f_α be an extended-real-valued function on X. We shall be concerned with the following minimax problem:

$$(P) \qquad \begin{array}{l} \text{minimize } \sup_{\alpha \in Q} f_\alpha(x), \\ \text{subject to } x \in C. \end{array}$$

Optimality conditions for minimax problems involving functions that are differentiable in the sense of Fréchet or Gâteaux are given by several authors, but in this paper we are interested in general necessary conditions of the type given in [4], [5], [10]. In recent years, in nonsmooth analysis a calculus for various directional derivatives and subgradients of locally Lipschitzian functions and even larger classes of functions has been developed (see e.g. [3], [8], [11]-[15]). The results obtained in [13], [15] yield necessary optimality conditions for (P) of the type mentioned above.

The purpose of this paper is to establish various necessary optimality conditions for (P) in a rather general setting.

The remainder of the paper is organized as follows. Section 2 is devoted to derive a general necessary optimality condition for (P) together with some examples. In Section 3, we give a necessary condition in terms of subgradients and polar cones. We also give here examples corresponding to the special cases introduced in Section 2. Finally, in Section 4 we establish necessary optimality conditions for a mathematical program with mixed constraints.

2. General Necessary Optimality Conditions

We assume that x_0 is a local minimizer for (P), and for x_0 fixed we define (see e.g. [3]):

(a) The contingent cone to C at x_0 is the set

$$K_C(x_0) := \{d \in X \mid \exists d_n \to d,\ \exists t_n \downarrow 0 \text{ such that } x_0 + t_n d_n \in C\};$$

(b) The Clarke tangent cone to C at x_0 is the set

$$T_C(x_0) := \{d \in X \mid \forall x_n \to x_0 \text{ with } x_n \in C,\ \forall t_n \downarrow 0,$$
$$\exists d_n \to d \text{ such that } x_n + t_n d_n \in C\}.$$

Define $Q_0 := \{\alpha \in Q \mid f_\alpha(x_0) = \sup_{\beta \in Q} f_\beta(x_0)\}$. Assume that for all $\alpha \in Q$ we have extended-real-valued functions $\varphi_\alpha(\cdot)$ on X such that

1) $\varphi_\alpha(\cdot)$ is convex along rays from the origin, $\varphi_\alpha(0) \le 0$ $(\forall \alpha \in Q)$;

2) The mapping $\alpha \mapsto \varphi_\alpha(d)$ is upper semicontinuous (u.s.c.) and real-valued for all $d \in K_C(x_0)$;

3) For all $\alpha \in Q \setminus Q_0$, $\varphi_\alpha(d) < +\infty$ $(\forall d \in K_C(x_0))$.

Suppose, in addition, that the mapping $\alpha \mapsto f_\alpha(x_0)$ is u.s.c..

Let us introduce the following

Assumption 2.1: *For all $d \in K_C(x_0)$ and sequences $d_n \to d$, $t_n \downarrow 0$ satisfying $x_0 + t_n d_n \in C$,*

$$\varphi_\alpha(d) \ge \limsup_{n\to\infty} \frac{f_\alpha(x_0 + t_n d_n) - f_\alpha(x_0)}{t_n}$$

uniformly in α.

Theorem 2.2: *Assume that Assumption 2.1 is fulfilled and for all $\alpha \in Q$, $f_\alpha(x_0)$ is finite. Then*

$$\sup_{\alpha \in Q_0} \varphi_\alpha(d) \ge 0 \quad (\forall d \in K_C(x_0)). \tag{2.1}$$

Proof: Suppose that (2.1) is false. So, there exists $\overline{d} \in K_C(x_0)$ and $\mu > 0$ such that

$$\varphi_\alpha(\overline{d}) \leq -\mu < 0 \ (\forall \alpha \in Q_0). \tag{2.2}$$

Define $\psi_\alpha(d) := f_\alpha(x_0)+\varphi_\alpha(d)$. It follows from (2.2) that for all $\alpha \in Q_0$, $\psi_\alpha(\overline{d}) \leq \widehat{m}-\mu$, where $\widehat{m} := \sup_{\alpha\in Q} f_\alpha(x_0)$. Note that $\widehat{m} < +\infty$, since the mapping $\alpha \mapsto f_\alpha(x_0)$ is real-valued and u.s.c., and Q is compact. We shall begin with showing that there is $\widehat{d} \in K_C(x_0)$ such that

$$\psi_\alpha(\widehat{d}) < \widehat{m} \ (\forall \alpha \in Q). \tag{2.3}$$

To do this, we set $U = \{\alpha \in Q \mid \varphi_\alpha(\overline{d}) < -\mu/2\}$. In view of (2.2) one has $Q_0 \subset U$. By virtue of the upper semicontinuity of the mapping $\alpha \mapsto \varphi_\alpha(\overline{d})$, $Q \setminus U$ is compact. Hence, by the upper semicontinuity of the mapping $\alpha \mapsto f_\alpha(x_0)$, we can find a constant $l > 0$ such that for all $\beta \in Q \setminus U$, $f_\beta(x_0) \leq \widehat{m} - l$, and therefore also

$$\psi_\beta(0) = f_\beta(x_0) + \varphi_\beta(0) \leq \widehat{m} - l. \tag{2.4}$$

Since $Q \setminus U$ is compact and the mapping $\alpha \mapsto \varphi_\alpha(\overline{d})$ is real-valued and u.s.c., we can find a constant $\gamma \in I\!R$ such that $\varphi_\beta(\overline{d}) \leq \gamma$ $(\forall \beta \in Q \setminus U)$, whence

$$\psi_\beta(\overline{d}) = f_\beta(x_0) + \varphi_\beta(\overline{d}) \leq \widehat{m} + \gamma. \tag{2.5}$$

For $\lambda \in (0,1]$, $d_\lambda := \lambda\overline{d} = \lambda\overline{d} + (1-\lambda)0 \in K_C(x_0)$. Then by virtue of the convexity along rays of $\psi_\alpha(\cdot)$ and the definition of U we get that for all $\alpha \in U$,

$$\begin{aligned} \psi_\alpha(d_\lambda) &\leq \lambda\psi_\alpha(\overline{d}) + (1-\lambda)\psi_\alpha(0) \\ &\leq \lambda(\widehat{m} - \frac{\mu}{2}) + (1-\lambda)\widehat{m} \\ &= \widehat{m} - \frac{1}{2}\lambda\mu < \widehat{m}. \end{aligned} \tag{2.6}$$

For $\beta \in Q \setminus U$, it follows from (2.4) and (2.5) that

$$\begin{aligned} \psi_\beta(d_\lambda) &\leq \lambda(\widehat{m} + \gamma) + (1-\lambda)(\widehat{m} - l) \\ &= \widehat{m} - l + \lambda(\gamma + l). \end{aligned}$$

For λ small enough $(0 < \lambda \leq \lambda_1)$, $-l + \lambda(\gamma + l) < 0$, which implies $\psi_\beta(d_\lambda) < \widehat{m}$. This together with (2.6) gives (2.3), whence

$$\sup_{\alpha \in Q} \left(f_\alpha(x_0) + \varphi_\alpha(\widehat{d}) \right) < \widehat{m}.$$

Then for some number $\widehat{\mu} > 0$ we obtain

$$f_\alpha(x_0) + \varphi_\alpha(\widehat{d}) \leq \widehat{m} - \widehat{\mu} \ (\forall \alpha \in Q). \tag{2.7}$$

Since $\widehat{d} \in K_C(x_0)$, there exist sequences $d_n \to \widehat{d}$, $t_n \downarrow 0$ such that $x_0 + t_n d_n \in C$. Taking account of Assumption 2.1, we get

$$\limsup_{n \to \infty} \frac{f_\alpha(x_0 + t_n d_n) - f_\alpha(x_0)}{t_n} \leq \varphi_\alpha(\widehat{d}) \tag{2.8}$$

uniformly in α. Combining (2.7) and (2.8) yields that

$$\limsup_{n \to \infty} \frac{f_\alpha(x_0 + t_n d_n) - f_\alpha(x_0)}{t_n} \leq \widehat{m} - \widehat{\mu} - f_\alpha(x_0)$$

uniformly in α. Consequently, for $\varepsilon > 0$ there is a natural number N (not depending on α) such that for all $n \geq N$,

$$\frac{f_\alpha(x_0 + t_n d_n) - f_\alpha(x_0)}{t_n} \leq \widehat{m} - \widehat{\mu} - f_\alpha(x_0) + \varepsilon. \tag{2.9}$$

We can assume that $t_n \leq 1$, as $t_n \downarrow 0$. So, observing that $\widehat{m} - f_\alpha(x_0) \geq 0$, from (2.9) it follows that

$$\begin{aligned} f_\alpha(x_0 + t_n d_n) &\leq f_\alpha(x_0) + t_n(\widehat{m} - \widehat{\mu} - f_\alpha(x_0) + \varepsilon) \\ &= f_\alpha(x_0) + t_n(\widehat{m} - f_\alpha(x_0)) - t_n(\widehat{\mu} - \varepsilon) \\ &\leq f_\alpha(x_0) + \widehat{m} - f_\alpha(x_0) - t_n(\widehat{\mu} - \varepsilon) \\ &= \widehat{m} - t_n(\widehat{\mu} - \varepsilon). \end{aligned}$$

For $\varepsilon < \widehat{\mu}$, one has $\widehat{\mu} - \varepsilon > 0$. Hence, for a number n fixed we get

$$\sup_{\alpha \in Q} f_\alpha(x_0 + t_n d_n) \leq \widehat{m} - t_n(\widehat{\mu} - \varepsilon),$$

whence

$$\sup_{\alpha \in Q} f_\alpha(x_0 + t_n d_n) < \sup_{\alpha \in Q} f_\alpha(x_0),$$

which conflicts with the hypothesis that x_0 is a local minimizer for (P). □

In the sequel we shall deal with a number of special cases of Theorem 2.2.

Examples 1. Let the functions f_α ($\alpha \in Q$) be Fréchet differentiable at x_0. Taking $\varphi_\alpha(d) := \langle f'_\alpha(x_0), d\rangle$ we have

$$\lim_{n\to\infty} \frac{f_\alpha(x_0 + t_n d_n) - f_\alpha(x_0)}{t_n} = \langle f'_\alpha(x_0), d\rangle, \tag{2.10}$$

for all $d \in K_C(x_0)$, $d_n \to d$, $t_n \downarrow 0$ satisfying $x_0 + t_n d_n \in C$. Hence, if we assume that (2.10) is uniform in α, then Assumption 2.1 is satisfied.

We can formulate an immediate consequence of Theorem 2.2 as follows.

Corollary 2.3: *Let x_0 be a local minimizer for (P). Assume that (2.10) is uniform in α; the mappings $\alpha \mapsto f_\alpha(x_0)$ and $\alpha \mapsto \langle f'_\alpha(x_0), d\rangle$ are u.s.c. for all $d \in K_C(x_0)$. Then*

$$\sup_{\alpha\in Q_0} \langle f'_\alpha(x_0), d\rangle \geq 0 \ \forall d \in K_C(x_0).$$

2. Let the functions f_α ($\alpha \in Q$) be Hadamard differentiable at x_0. This means that for each $\alpha \in Q$ there exists $\nabla f_\alpha(x_0) \in X^*$ such that for all $d \in X$,

$$\lim_{d'\to d,\ t\downarrow 0} \frac{f_\alpha(x_0 + td') - f_\alpha(x_0)}{t} = \langle \nabla f_\alpha(x_0), d\rangle. \tag{2.11}$$

It is obvious that if the limit (2.11) is uniform in α ($\forall d \in K_C(x_0)$, then Assumption 2.1 is satisfied. This together with Theorem 2.2 yields the following

Corollary 2.4: *Let x_0 be a local minimizer for (P). Assume that the limit (2.11) is uniform in α for all $d \in K_C(x_0)$; the mappings $\alpha \mapsto f_\alpha(x_0)$ and $\alpha \mapsto \langle \nabla f_\alpha(x_0), d\rangle$ are u.s.c. for all $d \in K_C(x_0)$. Then*

$$\sup_{\alpha\in Q_0} \langle \nabla f_\alpha(x_0), d\rangle \geq 0 \ \forall d \in K_C(x_0).$$

This is a generalization of a result in [7].

3. Let the functions f_α ($\alpha \in Q$) be Lipschitz in a neighborhood of x_0 with Lipschitzian constants L_α such that $\sup_{\alpha\in Q} L_\alpha < +\infty$. For each $\alpha \in Q$, Clarke's directional derivative of f_α at x_0 with respect to d, denoted by $f^0_\alpha(x_0; d)$, is defined by

$$f^0_\alpha(x_0; d) := \limsup_{x\to x_0, t\downarrow 0} \frac{f_\alpha(x + td) - f_\alpha(x)}{t}. \tag{2.12}$$

Proposition 2.5: *Assume that the limit (2.12) is uniform in α for all $d \in K_C(x_0)$. Then Assumption 2.1 is satisfied for $\varphi_\alpha(\cdot) := f^0_\alpha(x_0; \cdot)$.*

Proof: Let $L := \sup_{\alpha \in Q} L_\alpha$. Since (2.12) is uniform in α it follows

$$\begin{aligned}\limsup_{d_n \to d,\ t_n \downarrow 0} \frac{f_\alpha(x_0 + t_n d_n) - f_\alpha(x_0)}{t_n} &\leq \limsup_{d_n \to d,\ t_n \downarrow 0} \frac{f_\alpha(x_0 + t_n d) - f_\alpha(x_0) + t_n L \|d_n - d\|}{t_n} \\ &= \limsup_{t_n \downarrow 0} \frac{f_\alpha(x_0 + t_n d) - f_\alpha(x_0)}{t_n} \\ &\leq \limsup_{x \to x_0,\ t \downarrow 0} \frac{f_\alpha(x + td) - f_\alpha(x)}{t} = f^0_\alpha(x_0; d)\end{aligned}$$

uniformly in α. Hence Assumption 2.1 is satisfied. □

Now we can state a consequence of Theorem 2.2 as follows

Corollary 2.6: *Let x_0 be a local minimizer for (P). Assume that the limit (2.12) is uniform in α and the mappings $\alpha \mapsto f_\alpha(x_0)$ and $\alpha \mapsto f^0_\alpha(x_0; d)$ are u.s.c. for all $d \in K_C(x_0)$. Then*

$$\sup_{\alpha \in Q_0} f^0_\alpha(x_0; d) \geq 0 \ (\forall \alpha \in K_C(x_0)). \tag{2.13}$$

4. Let the functions f_α $(\alpha \in Q)$ be directionally Lipschitzian at x_0. Let f be an extended-real-valued funtion on a normed space X. Let $x_0 \in X$ be a point at which f is finite and $d \in X$. We recall [3] that the Rockafellar directional derivative of f at x_0 in the direction d is defined by

$$f^\uparrow(x_0; d) := \inf\{r \mid (d, r) \in T_{\text{epi} f}(x_0, f(x_0))\}.$$

It can be defined directly by the characterization of $f^\uparrow(x_0; \cdot)$ as follows

$$f^\uparrow(x_0; d) = \lim_{\varepsilon \downarrow 0} \limsup_{(x,\gamma) \downarrow_f x_0,\ \lambda \downarrow 0} \inf_{\omega \in d + \varepsilon B} \frac{f(x + \lambda\omega) - \gamma}{\lambda},$$

where $(x, \gamma) \downarrow_f x_0$ means that $(x, \gamma) \in$epi$f$, $x \to x_0$ and $\gamma \to f(x_0)$; B stands for the open unit ball.

If f is lower semicontinuous (l.s.c.) at x_0, this definition reduces to

$$f^\uparrow(x_0; d) = \lim_{\varepsilon \downarrow 0} \limsup_{x \downarrow_f x_0,\ \lambda \downarrow 0} \inf_{\omega \in d + \varepsilon B} \frac{f(x + \lambda\omega) - f(x)}{\lambda},$$

where $x \downarrow_f x_0$ means that $x \to x_0$ and $f(x) \to f(x_0)$.

The following properties of $f^{\uparrow}(x_0;\cdot)$ will be used later (see e.g. [13]):

(a) $f^{\uparrow}(x_0;\cdot)$ is l.s.c., convex and positively homogeneous;

(b) $f^{\uparrow}(x_0;0) = \begin{cases} 0, & \text{if } f^{\uparrow}(x_0;\cdot) \text{ is proper;} \\ -\infty, & \text{else} \end{cases}$

(c) $\operatorname{epi} f^{\uparrow}(x_0;\cdot) = T_{\operatorname{epi} f}(x_0, f(x_0))$.

For further discussions on the Rockafellar derivative, we refer the reader to Ward and Borwein [13], Rockafellar [11], Clarke [3], Hiriart-Urruty [8].

We also recall [3] that the function f is said to be directionally Lipschitzian at x_0 with respect to d, if $f(x_0)$ is finite and

$$f^{+}(x_0;d) := \limsup_{(x,\gamma)\downarrow_f x_0,\ \omega\to d,\ t\downarrow 0} \frac{f(x+t\omega)-\gamma}{t} < +\infty.$$

If f is l.s.c. at x_0, this reduces to

$$f^{+}(x_0,d) = \limsup_{x\downarrow_f x_0,\ \omega\to d,\ t\downarrow 0} \frac{f(x+t\omega)-f(x)}{t} < +\infty.$$

Denote by $\mathrm{D}_f(x_0)$ the set of all vectors d such that f is directionally Lipschitzian at x_0 with respect to d. Notice that (see [3]):

$$f^{\uparrow}(x_0;d) = f^{+}(x_0;d) \ (\forall d \in \mathrm{D}_f(x_0)).$$

With the class of directionally Lipschitzian functions we introduce the following

Assumption 2.7: *The limit*

$$f_\alpha^{+}(x_0;d) = \limsup_{x\downarrow_{f_\alpha} x_0,\ \omega\to d,\ t\downarrow 0} \frac{f_\alpha(x+t\omega)-f_\alpha(x)}{t}$$

is uniform in α $(\forall d \in K_C(x_0))$.

Proposition 2.8: *Assume that the functions* f_α $(\alpha \in Q)$ *are l.s.c. at* x_0. *Then Assumption 2.7 implies Assumption 2.1 with* $\varphi_\alpha(d) := f_\alpha^{+}(x_0;d)$.

Proof: Suppose that Assumption 2.7 is true. For $\overline{d} \in K_C(x_0)$, it holds that

$$\begin{aligned} f_\alpha^{+}(x_0;d) &= \limsup_{x\downarrow_{f_\alpha} x_0,\ \omega\to d, t\downarrow 0} \frac{f_\alpha(x+t\omega)-f_\alpha(x)}{t} \\ &\geq \limsup_{d_n\to d,\ t_n\downarrow 0} \frac{f_\alpha(x_0+t_n d_n)-f_\alpha(x_0)}{t_n} \end{aligned}$$

uniformly in α. Thus Assumption 2.1 is satisfied. □

We now can state a consequence of Theorem 2.2 for the directionally Lipschitzian case as follows.

Corollary 2.9: *Let x_0 be a local minimizer for (P). Assume that the functions f_α $(\alpha \in Q)$ are l.s.c. at x_0 and directionally Lipschitzian at x_0 with respect to all directions $d \in K_C(x_0)$, and Assumption 2.7 holds. Suppose, in addition, that the mappings $\alpha \mapsto f_\alpha(x_0)$ and $\alpha \mapsto f_\alpha^+(d)$ are u.s.c. for all $d \in K_C(x_0)$. Then*

$$\sup_{\alpha \in Q_0} f_\alpha^+(x_0; d) \geq 0 \ (\forall d \in K_C(x_0)). \tag{2.14}$$

5. In the case where the functions f_α $(\alpha \in Q)$ are not necessarily directionally Lipschitzian at x_0, we introduce the following

Assumption 2.10: *For all $d \in K_C(x_0)$, $d_n \to d$, $t_n \downarrow 0$ satisfying $x_0 + t_n d_n \in C$,*

$$f_\alpha^\uparrow(x_0; d) \geq \limsup_{n \to \infty} \frac{f_\alpha(x_0 + t_n d_n) - f_\alpha(x_0)}{t_n}$$

uniformly in α.

Under this assumption Assumption 2.1 is satisfied for $\varphi_\alpha := f_\alpha^\uparrow(x_0; \cdot)$.

Corollary 2.11: *Let x_0 be a local minimizer for (P) and f_α be finite at x_0 $(\forall \alpha \in Q)$. Assume that Assumption 2.10 holds; $f_\alpha^\uparrow(x_0; d) < +\infty$ $(\forall \alpha \in Q \setminus Q_0)$ for all $d \in K_C(x_0)$; the mappings $\alpha \mapsto f_\alpha(x_0)$ and $\alpha \mapsto f_\alpha^\uparrow(x_0; d)$ are u.s.c. for all $d \in K_C(x_0)$. Then*

$$\sup_{\alpha \in Q_0} f_\alpha^\uparrow(x_0; d) \geq 0 \ (\forall d \in K_C(x_0)).$$

6. Let us consider the case

$$\varphi_\infty(d) = \sup_{\alpha \in Q_0} f_\alpha^\uparrow(x_0; d).$$

Instead of Assumption 2.1 we introduce the following

Assumption 2.12:

(i) $$\liminf_{d' \to d,\ d'' \to d,\ t \downarrow 0} \left[\frac{f_\alpha(x_0 + td') - f_\alpha(x_0 + td'')}{t} \right] = 0$$

uniformly in α ($\forall d \in K_C(x_0)$); $f_\alpha(x_0 + td')$ *and* $f_\alpha(x_0 + td'')$ *are finite for* $\lambda > 0$ *in a neighborhood of* 0 *and* (d', d'') *in a neighborhood of* (d, d) ($\forall \alpha \in Q$; $\forall d \in K_C(x_0)$).

(ii) $$\sup_{\alpha \in Q_0} f_\alpha^\uparrow(x_0; d) \geq (\sup_{\alpha \in Q} f_\alpha)^\uparrow(x_0; d) \ (\forall d \in K_C(x_0)).$$

Proposition 2.13: *Let* x_0 *be a local minimizer for* (P) *and* f_α *be finite at* x_0 ($\forall \alpha \in Q$). *Assume that Assumption 2.12 is fulfilled. Then*

$$\sup_{\alpha \in Q_0} f_\alpha^\uparrow(x_0; d) \geq 0 \ (\forall d \in K_C(x_0)). \tag{2.15}$$

Proof: Suppose that (2.15) is false. This means that there exists $\overline{d} \in K_C(x_0)$ and $\mu > 0$ such that

$$\sup_{\alpha \in Q_0} f_\alpha^\uparrow(x_0; \overline{d}) \leq -\mu < 0. \tag{2.16}$$

Let $f_\infty := \sup_{\alpha \in Q} f_\alpha$. It follows from Assumption 2.12 (ii) and (2.16) that

$$f_\infty^\uparrow(x_0; \overline{d}) \leq -\mu,$$

that is

$$(\overline{d}, -\mu) \in \text{epi} f_\infty^\uparrow(x_0; \cdot).$$

Moreover,

$$\text{epi} f_\infty^\uparrow(x_0; \cdot) = T_{\text{epi} f_\infty}(x_0, f_\infty(x_0)). \tag{2.17}$$

Since $\overline{d} \in K_C(x_0)$, there exist sequences $d_n \to d$ and $t_n \downarrow 0$ such that $x_0 + t_n d_n \in C$. It follows from (2.17) that there exist sequences $d'_n \to \overline{d}$, $\mu_n \to -\mu$ such that

$$(x_0, f_\infty(x_0)) + t_n(d'_n, \mu_n) \in \text{epi} f_\infty,$$

which implies that

$$\frac{f_\infty(x_0 + t_n d'_n) - f_\infty(x_0)}{t_n} \leq \mu_n. \tag{2.18}$$

By Assumption 2.12 (i), for $\varepsilon > 0$ there exists a subsequence $\{n_k\}$ (not depending on α) of the set $\{1, 2, \ldots\}$ such that

$$\frac{f_\alpha(x_0 + t_{n_k} d_{n_k}) - f_\alpha(x_0 + t_{n_k} d'_{n_k})}{t_{n_k}} < \varepsilon \ (\forall \alpha \in Q). \tag{2.19}$$

Combining (2.18) and (2.19) yields that

$$\frac{f_\infty(x_0 + t_{n_k} d_{n_k}) - f_\infty(x_0)}{t_{n_k}} < \mu_{n_k} + \varepsilon.$$

Since $\mu_{n_k} \to -\mu$, there is a natural number N (not depending on α) such that for all $k \geq N$,

$$\frac{f_\infty(x_0 + t_{n_k} d_{n_k}) - f_\infty(x_0)}{t_{n_k}} < -\mu + 2\varepsilon,$$

which implies that

$$f_\infty(x_0 + t_{n_k} d_{n_k}) - f_\infty(x_0) < t_{n_k}(2\varepsilon - \mu).$$

For ε small enough, $2\varepsilon - \mu < 0$. Hence, for a number k fixed we get

$$f_\infty(x_0 + t_{n_k} d_{n_k}) - f_\infty(x_0) < 0,$$

which contradicts the hypothesis that x_0 is a local minimizer for (P). □

Remark 2.14: *Proposition 2.13 includes Theorem 6 [8] as a special case.*

3. Necessary Conditions in Terms of Subgradients

Denote by X^*_σ the topological dual of X, endowed with weak* topology.

Assume now that φ_α is sublinear ($\forall \alpha \in Q$). Let $\partial\varphi_\alpha(0) := \{x^* \in X^* \mid \langle x^*, d\rangle \leq \varphi_\alpha(d)\ \forall d \in X\}$.

Theorem 3.1: *Let x_0 be a local minimizer for (P) and let M be a closed convex subcone of $K_C(x_0)$ with vertex at the origin. Assume that all the hypotheses of Theorem 2.2 hold; $\partial\varphi_\alpha(0) \neq \emptyset$ and $\varphi_\alpha(d) = \sup_{x^* \in \partial\varphi_\alpha(0)} \langle x^*, d\rangle$ ($\forall \alpha \in Q_0$). Then*

$$0 \in \mathrm{cl}\big[\mathrm{co}\big(\bigcup_{\alpha \in Q_0} \partial\varphi_\alpha(0)\big) - M^*\big], \tag{3.1}$$

where co and cl denote convex hull and weak closure, respectively; M^* is the polar cone of M,*

$$M^* := \{x^* \in X^* \mid \langle x^*, d\rangle \geq 0\ \forall d \in M\}.$$

Proof: Taking account of Theorem 2.2 we get

$$\sup_{\alpha \in Q_0} \varphi_\alpha(d) \geq 0\ (\forall d \in M). \tag{3.2}$$

We now assume that (3.1) is false, i.e.,

$$0 \notin \mathrm{cl}\big[\mathrm{co}(\bigcup_{\alpha \in Q_0} \partial\varphi_\alpha(0)) - M^*\big]. \tag{3.3}$$

The right hand side of (3.3) is weak* closed convex. So from a standard separation theorem for convex sets (see e.g. Theorem 3.6 [5]) there exist $d_0 \in (X_\sigma^*)^* = X$ and $\gamma \in I\!R$ such that

$$0 > \gamma \geq \langle \xi, d_0 \rangle \ \text{ for all } \xi \in \mathrm{co}(\bigcup_{\alpha \in Q_0} \partial\varphi_\alpha(0)) - M^*.$$

Since M^* is a cone containing the origin this implies

$$0 \geq \langle \xi, d_0 \rangle \ \forall \xi \in -M^*, \tag{3.4}$$

$$0 > \gamma \geq \langle \xi, d_0 \rangle \ \forall \xi \in \bigcup_{\alpha \in Q_0} \partial\varphi_\alpha(0). \tag{3.5}$$

It follows from (3.4) that $d_0 \in -(-M^*)^* = M$. If follows from (3.5) that

$$0 > \gamma \geq \sup_{\xi \in \partial\varphi_\alpha(0)} \langle \xi, d_0 \rangle = \varphi_\alpha(d_0) \ (\forall \alpha \in Q_0),$$

whence

$$0 > \sup_{\alpha \in Q_0} \varphi_\alpha(d_0).$$

This contradicts (3.2). □

Remark 3.2: *Under all hypotheses of Theorem 3.1, (3.1) is equivalent to (3.2).*

Indeed, Theorem 3.1 shows that (3.2) implies (3.1). We shall prove that, conversely, (3.1) implies (3.2). Assume now that (3.1) is true. Observe that if A is any subset of X^*, then $\overline{a}$ being a element of the weak* closure of A implies that for all $d \in X$ and $\varepsilon > 0$ there exists $a \in A$ such that $|\langle a - \overline{a}, d\rangle| \leq \varepsilon$. Hence for every $d \in X$ and $\varepsilon > 0$ there exist finitely many $\alpha_i \in Q_0$, $\xi_i \in \partial\varphi_{\alpha_i}(0)$, $\lambda_i \geq 0$ satisfying $\sum_i \lambda_i = 1$, and $m^* \in M^*$ such that

$$-\varepsilon \leq \sum_i \lambda_i \langle \xi_i, d\rangle - \langle m^*, d\rangle \leq \sum_i \lambda_i \varphi_{\alpha_i}(d) - \langle m^*, d\rangle.$$

Choosing $d \in M$ we get

$$-\varepsilon \leq \sum_i \lambda_i \varphi_{\alpha_i}(d) \leq \sum_i \lambda_i \sup_{\alpha \in Q_0} \varphi_\alpha(d) = \sup_{\alpha \in Q_0} \varphi_\alpha(d).$$

Since ε is arbitrary this implies

$$\sup_{\alpha \in Q_0} \varphi_\alpha(d) \geq 0.$$

Hence (3.1) implies (3.2). □

Remark 3.3: *If* $cl\big(co\bigcup_{\alpha\in Q_0}\partial\varphi_\alpha(0)\big)$ *is weak* compact, then (3.1) becomes*

$$0 \in \mathrm{cl}\big(\mathrm{co}\bigcup_{\alpha\in Q_0}\partial\varphi_\alpha(0)\big) - M^*. \tag{3.6}$$

Indeed, from the general fact that $\mathrm{cl}(A+B) = \mathrm{cl}(\mathrm{cl}A+B)$, if we take $A = \mathrm{co}\bigcup_{\alpha\in Q_0}\partial\varphi_\alpha(0)$, $B = -M^*$, then $\mathrm{cl}A + B$ is weak* closed. So the assertion is proved. □

Corollary 3.4: *Let x_0 be a local minimizer for (P) and M be a closed convex subcone of $K_C(x_0)$ with vertex at the origin. Assume that all the hypotheses of Theorem 2.2 hold; φ_α is l.s.c., proper and upper-bounded in a neighborhood of 0 ($\forall\alpha\in Q_0$); the mapping $\alpha\mapsto\partial\varphi_\alpha(0)$ is u.s.c. from Q into X^*_σ. Then*

$$0 \in \mathrm{cl}\big(\mathrm{co}\bigcup_{\alpha\in Q_0}\partial\varphi_a(0)\big) - M^*.$$

Proof: From Theorem 3.1 and Remark 3.3 we need only show that $\mathrm{cl}\big(\mathrm{co}\bigcup_{\alpha\in Q_0}\partial\varphi_\alpha(0)\big)$ is weak* compact.

By virtue of Proposition 2.1.4 [3], $\partial\varphi_\alpha(0)$ ($\forall\alpha\in Q$) are nonempty weak* compact subset and

$$\varphi_\alpha(d) = \max_{x^*\in\partial\varphi_\alpha(0)}\langle x^*, d\rangle.$$

Making use of the compactness of Q and the upper semicontinuity of the mapping $\alpha\mapsto\partial\varphi_\alpha(0)$ we get $\bigcup_{\alpha\in Q}\partial\varphi_\alpha(0)$ is weak* compact (see e.g. [2]). By Alaoglu's Theorem [1] $\bigcup_{\alpha\in Q}\partial\varphi_\alpha(0)$ is norm-bounded. So $\mathrm{cl}\big(\mathrm{co}\bigcup_{\alpha\in Q_0}\partial\varphi_a(0)\big)$ is norm-bounded. Hence $\mathrm{cl}\big(\mathrm{co}\bigcup_{\alpha\in Q_0}\partial\varphi_a(0)\big)$ is weak* compact. □

Examples

We now turn back to the situations investigated in the examples of Section 2.

1. The functions f_α ($\alpha\in Q$) are Fréchet differentiable at x_0 with the derivatives $f'_\alpha(x_0)$. We suppose that (2.10) is uniform in α, the mappings $\alpha\mapsto f_\alpha(x_0)$ and $\alpha\mapsto\langle f'_\alpha(x_0), d\rangle$ are u.s.c. for all $d\in K_C(x_0)$. Then, by Theorem 3.1 we get

$$\mathrm{cl}\,\mathrm{co}\{f'_\alpha(x_0),\ \alpha\in Q_0\}\cap M^* \neq \emptyset. \tag{3.7}$$

This is a generalization of a geometrical necessary condition in [5].

2. The functions $f_\alpha(\alpha \in Q)$ are Lipschitz in a neighborhood of x_0.

The Clarke subgradient of f at x_0 is defined by

$$\partial f_\alpha(x_0) := \{x^* \in X^* \mid \langle x^*, d\rangle \leq f_\alpha^0(x_0; d)\ \forall \alpha \in X\}.$$

Then,

$$\partial f_\alpha(x_0) = \partial f_\alpha^0(x_0; 0).$$

Hence, by Corollary 2.6 and Corollary 3.4 we get the following

Corollary 3.5: *Let x_0 be a local minimizer for (P) and M be a closed convex subcone of $K_C(x_0)$ with vertex at the origin. Assume that all the hypotheses of Corollary 2.6 hold. Suppose, furthermore, that the mapping $\alpha \mapsto \partial f_\alpha(x_0)$ is u.s.c. from Q into X_σ^*. Then*

$$0 \in \text{cl co} \bigcup_{\alpha \in Q_0} \partial f_\alpha(x_0) - M^*. \tag{3.8}$$

3. The functions f_α $(\alpha \in Q)$ are directionally Lipschitzian at x_0.

The generalized subgradient of f_α at x_0 is defined by

$$\partial f_\alpha(x_0) := \{x^* \in X^* \mid \langle x^*, d\rangle \leq f_\alpha^\uparrow(x_0; d)\ \forall d \in X\}.$$

Then, if $f_\alpha^\uparrow(x_0; \cdot)$ is proper,

$$\partial f_\alpha(x_0) = \partial f_\alpha^\uparrow(x_0; 0)$$

(see e.g. [13]). By Corollary 2.9 and Corollary 3.4 we obtain the following

Corollary 3.6: *Let x_0 be a local minimizer for (P) and M be a closed convex subcone of $K_C(x_0)$ with vertex at the origin. Assume that all the hypotheses of Corollary 2.9 hold; the mapping $\alpha \mapsto \partial f_\alpha(x_0)$ is u.s.c. from Q into X_σ^*; $f_\alpha^+(x_0; \cdot)$ $(\forall \alpha \in Q_0)$ are finite on X. Then*

$$0 \in \text{cl co} \bigcup_{\alpha \in Q_0} \partial f_\alpha(x_0) - M^*. \tag{3.9}$$

4. The case: $\varphi_\infty(d) = \sup_{\alpha \in Q_0} f_\alpha^\uparrow(x_0; d)$.

Combining Proposition 2.13 and Corollary 3.4 we get

Corollary 3.7: *Let x_0 be a local minimizer for (P), f_α is finite at x_0 $(\forall \alpha \in Q)$, and M be a closed convex subcone of $K_C(x_0)$ with vertex at the origin. Assume that*

Assumption 2.12 holds; the mapping $\alpha \mapsto \partial f_\alpha(x_0)$ is u.s.c. from Q into X^*_σ; $f^\uparrow_\alpha(x_0;\cdot)$ is proper and upper-bounded in a neighborhood of 0 $(\forall \alpha \in Q_0)$. *Then*

$$0 \in \text{cl co} \bigcup_{\alpha \in Q_0} \partial f_\alpha(x_0) - M^*.$$

4. A Constrained Mathematical Program

Let us consider the following problem:

$$(MP) \qquad \begin{array}{l} \text{minimize } f(x) \\ \text{subject to} \\ \sup_{\alpha \in Q} f_\alpha(x) \le 0 \text{ and } x \in C. \end{array}$$

where f is an extended-real-valued function on X, f_α $(\alpha \in Q)$ and C are as in (P).

Assume that φ is a u.s.c., positively homogenuous function on X satisfying the following

Assumption 4.1: *(i) For all $d \in K_C(x_0)$, $d_n \to d$, $t_n \downarrow 0$ satisfying $x_0 + t_n d_n \in C$,*

$$\varphi(d) \ge \limsup_{n \to \infty} \frac{f(x_0 + t_n d_n) - f(x_0)}{t_n};$$

(ii) cl$\{d \in K_C(x_0) \mid \sup_{\alpha \in Q_0} \varphi_\alpha(d) < 0\} \supset \{d \in K_C(x_0) \mid \sup_{\alpha \in Q_0} \varphi_\alpha(d) \le 0\}$, where cl indicates the norm-closure.

Theorem 4.2: *Let x_0 be a local minimizer for (MP) and f, f_α $(\forall \alpha \in Q)$ be finite at x_0. Assume that Assumptions 2.1 and 4.1 are fulfilled. Then*

$$\varphi(d) \ge 0 \textit{ for all } d \in K_C(x_0) \textit{ satisfying } \sup_{\alpha \in Q_0} \varphi_\alpha(d) \le 0. \tag{4.1}$$

Proof: We first prove that $\varphi(d) \ge 0$ for all $d \in K_C(x_0)$ satisfying $\sup_{\alpha \in Q_0} \varphi_\alpha(d) < 0$.

Suppose that this is false. So there is $\overline{d} \in K_C(x_0)$ satisfying $\sup_{\alpha \in Q_0} \varphi_\alpha(\overline{d}) < 0$ such that $\varphi(\overline{d}) < 0$. Hence, for some $\mu > 0$, $\varphi(\overline{d}) \le -\mu < 0$. Define $\psi_\alpha(d) = f_\alpha(x_0) + \varphi_\alpha(d)$. In the same way as in the proof of Theorem 2.2 we can find $\widehat{d} = \widehat{\lambda}\overline{d} \in K_C(x_0)$ such that

$$\sup_{\alpha \in Q} \psi_\alpha(\widehat{d}) < \widehat{m}, \quad \text{where } \widehat{m} = \sup_{\alpha \in Q} f_\alpha(x_0).$$

Since $\widehat{d} \in K_C(x_0)$, there exist sequences $d_n \to \widehat{d}$, $t_n \downarrow 0$ such that $x_0 + t_n d_n \in C$. Making use of Assumption 2.1, by an argument analogous to that used for the proof of Theorem 2.2 we can find a natural number N_1 such that for all $n \geq N_1$,

$$\sup_{\alpha \in Q} f_\alpha(x_0 + t_n d_n) - \sup_{\alpha \in Q} f_\alpha(x_0) < 0.$$

Hence $x_0 + t_n d_n$ is a feasible point of (MP). On the other hand, since $\varphi(\overline{d}) \leq -\mu$ it follows from the positively homogeneity of φ, that

$$\varphi(\widehat{d}) = \widehat{\lambda}\varphi(\overline{d}) \leq -\widetilde{\mu} < 0,$$

where $\widetilde{\mu} = \widehat{\lambda}\mu$. By Assumption 4.1 (i), for $\varepsilon > 0$ there is a natural number $N_2(\geq N_1)$ such that for all $n \geq N_2$,

$$\frac{f(x_0 + t_n d_n) - f(x_0)}{t_n} \leq -\widetilde{\mu} + \varepsilon,$$

whence

$$f(x_0 + t_n d_n) - f(x_0) \leq t_n(-\widetilde{\mu} + \varepsilon).$$

Consequently, for $\varepsilon < \widetilde{\mu}$ and a number n fixed, we get

$$f(x_0 + t_n d_n) - f(x_0) < 0,$$

which contradicts the hypothesis that x_0 is a local minimizer for (MP). So, we have proved that

$$\varphi(d) \geq 0 \ \forall d \in \{d \in K_C(x_0) \mid \sup_{\alpha \in Q_0} \varphi_\alpha(d) < 0\}.$$

Since φ is u.s.c., it follows that

$$\varphi(d) \geq 0 \ \forall d \in \mathrm{cl}\{d \in K_C(x_0) \mid \sup_{\alpha \in Q_0} \varphi_\alpha(d) < 0\}.$$

By Assumption 4.1 (ii), we get

$$\varphi(d) \geq 0 \ \forall d \in \{d \in K_C(x_0) \mid \sup_{\alpha \in Q_0} \varphi_\alpha(d) \leq 0\}$$

□

Corollary 4.3: *Let x_0 be a local minimizer for (MP). Assume that all the hypotheses of Corollary 2.9 and Assumption 4.1 (i), (ii) are fulfilled. Then*

$$f^{\uparrow}(x_0; d) \geq 0 \text{ for all } d \in K_C(x_0) \text{ satisfying } \sup_{\alpha \in Q_0} f_\alpha^{\uparrow}(x_0; d) \leq 0.$$

By arguments analogous to that used for the proof of Proposition 2.13 we get the following result.

Proposition 4.4: *Let x_0 be a local minimizer for (MP) and f, f_α $(\alpha \in Q)$ be finite at x_0 $(\forall \alpha \in Q)$). Assume that Assumptions 2.12 and 4.1 (i), (ii) hold. Then*

$$f^{\uparrow}(x_0; d) \geq 0 \text{ for all } d \in K_C(x_0) \text{ satisfying } \sup_{\alpha \in Q_0} f_\alpha^{\uparrow}(x_0; d) \leq 0.$$

To derive a necessary optimality condition for (MP) in terms of subgradients, we now assume that φ and φ_α are sublinear $(\forall \alpha \in Q)$. Let

$$\partial\varphi(0) := \{x^* \in X^* \mid \langle x^*, d\rangle \leq \varphi(d) \; \forall d \in X\}.$$

Theorem 4.5: *Let x_0 be a local minimizer for (MP), f, f_α $(\alpha \in Q)$ be finite at x_0, and M be a closed convex subcone of $K_C(x_0)$ with vertex at the origin. Assume that Assumptions 2.1 and 4.1 are fulfilled. Suppose, furthermore, that $\partial\varphi(0)$ is nonempty, weak* compact and $\varphi(d) = \sup_{x^* \in \partial\varphi(0)} \langle x^*, d\rangle$; for each $\alpha \in Q_0$, $\partial\varphi_\alpha(0)$ is nonempty and $\varphi_\alpha(d) = \sup_{x^* \in \partial_\alpha(0)} \langle x^*, d\rangle$. Then*

$$0 \in \partial\varphi(0) + \mathrm{cl}\big[\mathrm{cc}\big(\bigcup_{\alpha \in Q_0} \partial\varphi_\alpha(0)\big) - M^*\big], \tag{4.2}$$

where cc and cl denote convex conical hull and weak closure, respectively.*

Proof: By Theorem 4.2 we get

$$\varphi(d) \geq 0 \text{ for all } d \in M \text{ satisfying } \sup_{\alpha \in Q_0} \varphi_\alpha(d) \leq 0. \tag{4.3}$$

Assume now that (4.2) is not true. So 0 doesn't belong to the set on the right hand side. The latter is weak* closed, since $\partial\varphi(0)$ is weak* compact. Moreover it is convex. So from a standard separation theorem (see e.g. Theorem 3.6 [5]) there exist $d_0 \in (X^*_\sigma)^* = X$ and $\gamma \in I\!R$ such that

$$0 > \gamma \geq \langle \xi, d_0 \rangle \ \forall \xi \in \partial\varphi(0) + \mathrm{cc}(\bigcup_{\alpha \in Q_0} \partial\varphi_\alpha(0)) - M^*.$$

Since $\mathrm{cc}(\bigcup_{\alpha \in Q_0} \partial\varphi_\alpha(0))$ and M^* are cones it follows from this that

$$0 > \gamma \geq \langle \xi, d_0 \rangle \ \forall \xi \in \partial\varphi(0),$$

$$0 \geq \langle \xi, d_0 \rangle \ \forall \xi \in \bigcup_{\alpha \in Q_0} \partial\varphi_\alpha(0),$$

$$0 \geq \langle \xi, d_0 \rangle \ \forall \xi \in -M^*.$$

The first of these inequalities implies $\varphi(d_0) < 0$. The second implies $\varphi_\alpha(d_0) \leq 0$ for all $\alpha \in Q_0$. The third implies $d_0 \in M^{**} = M$, a contradiction with (4.3). □

Remark 4.6: *We remark that, conversely, (4.2) implies (4.3).*

Indeed, assume that (4.2) is true. Observe that if A is any subset of X^*, then $\overline{a}$ being an element of the weak* closure of A implies that for all $d \in X$ and $\varepsilon > 0$ there exists $a \in A$ such $|\langle a - \overline{a}, d \rangle| \leq \varepsilon$. Hence, if (4.2) is true, then for every $d \in X$ and $\varepsilon > 0$ there exist $\xi \in \partial\varphi(0)$, finitely many $\alpha_i \in Q_0$, $\xi_i \in \partial\varphi_{\alpha_i}(0)$, $\lambda_i \geq 0$ and $m^* \in M^*$ such that

$$-\varepsilon \leq \langle \xi, d \rangle + \sum_i \lambda_i \langle \xi_i, d \rangle - \langle m^*, d \rangle \leq \varphi(d) + \sum_i \lambda_i \varphi_{\alpha_i}(d) - \langle m^*, d \rangle.$$

In particular if $d \in M$ and satisfies $\varphi_\alpha(d) \leq 0$ $(\forall \alpha \in Q_0)$ we obtain

$$-\varepsilon \leq \varphi(d).$$

Since ε is arbitrary, this implies $0 \leq \varphi(d)$. Hence (4.3) is satisfied. So (4.2) is equivalent to (4.3) □

Remark 4.7: *If we assume that φ is l.s.c., proper, sublinear, then it can be expressed by*

$$\varphi(d) = \sup_{x^* \in \partial\varphi(0)} \langle x^*, d \rangle,$$

where $\partial\varphi(0)$ is nonempty, weak closed (see e.g. [3]). If we suppose, in addition, that φ is upper-bounded in a neighborhood of 0, or that φ is a finite function on X, then $\partial\varphi(0)$ is weak* compact (see e.g. Theorem 5 [8] and Proposition 2.1.4 [3]).*

We derive a consequence of Theorem 4.5 for the directionally Lipschitzian case of f_α $(\forall \alpha \in Q)$.

Corollary 4.8: *Let x_0 be a local minimizer for (MP) and M be a closed convex subcone of $K_C(x_0)$ with vertex at the origin. Assume that all hypotheses of Corollary 4.3 hold; $f^{\uparrow}(x_0;\cdot)$ is proper and upper-bounded in a neighborhood of 0. Then*

$$0 \in \partial f(x_0) + \mathrm{cl}\big[\mathrm{cc}\big(\bigcup_{\alpha\in Q_0} \partial f_\alpha(x_0)\big) - M^*\big].$$

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