More Results on B-Splines via Recurrence Relations

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Abstract: The present paper is to be understood as a revision and continuation of a recent series of publications on recursively defined B-splines, due to C. deBoor and K. Höllig [4] and G. Ciascola [6, 7].
We define B-splines as solutions of a certain difference equation of evolution type, which is equivalent to the well-known B-spline recurrence relation. It turns out that this approach leads to very elementary and direct proofs, in particular without using the Marsden identity, of many important properties of the B-splines.
Special emphasis is also laid on the probability theoretic aspects of these functions. Among other results, we prove explicit formulas for the k^{th} moments, the variance and the distribution function of a B-spline.

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1. Introduction and Preliminaries

In a recent series of papers, C. de Boor & K. Höllig [4] and G. Ciascola [6, 7] developed quite nicely parts of the B-spline-theory, starting with the famous recurrence relation for B-splines (3.1) as *definition* of these functions. The present paper is to be understood as a revision and continuation of the above ones in that sense that we develop more results on B-splines – some are well-known from different approaches, but others are new – also using only the recurrence relation (3.1) as definition. In addition, we shall prove, in a more elementary way than in [6], the fact that the recursively defined functions possess the B-spline-properties (cf. Theorems 2.1 and 3.2). In particular, we don't use the Marsden identity, and we simplify the proof of the differential recursion formula significantly, compared with [7].

Even more, we first study, for arbitrary real functions, a certain difference equation of evolution type, which is equivalent to the B-spline recurrence relation, and ask for those initial values, for which the solutions of this difference equation are spline functions. As the main result of *section 2* it turns out that the only possible choice of initial values is given by the first order B-splines.

In *section 3*, we prove, in a very elementary way, that our recursively defined functions not only belong to the spline space $S_m(K)$, but also possess the minimal-support property, i.e. they coincide with the classical B-spline functions $Q_{\nu,m}$. In the rest of the section, further properties of the $Q_{\nu,m}$ are derived; among others, we shall prove an optimal upper bound for their norm and the representation of $Q_{\nu,m}(x)$ as linear combination of truncated power functions.

Section 4 is devoted to some connections of B-spline functions with certain linear functionals and, in particular, with divided differences. We shall derive representations of $Q_{\nu,m}(x)$ as a quotient of two determinants and, of course, as the divided difference of a truncated power function.

Finally, in *section 5* we present some results concerning integrals over B-splines. Since a B-spline can be interpreted as a probability density function, it is natural to ask for the corresponding distribution function as well as for the k^{th} moments and for the variance. These questions will be answered in Theorems 5.2 and 5.3.

Some of the assertions are connected with the corresponding results on complex B-splines, for which we refer the interested reader to [14].

We simplify the matter by restricting to the case of simple knots, mainly to avoid formulas which look complicated only because of the many multiple sub- and superscripts. In most cases our results can be carried over to multiple knots in an obvious way.

Let us assume that we are given real numbers x_μ , $\mu \in \mathbb{Z}$, the *knots*, satisfying

$$x_\mu \; < \; x_{\mu+1} \quad \text{for all} \;\; \mu \in \mathbb{Z}$$

and having no point of accumulation in $\mathbb{R}$. The set of these knots is denoted by

$$K \; = \; \{x_\mu\}_{\mu\in\mathbb{Z}} \; .$$

Furthermore we introduce the subintervals

$$I_\mu \; := \; \{x \in \mathbb{R};\; x_\mu < x \leq x_{\mu+1}\} \;\; \text{for all} \; \mu \in \mathbb{Z}$$

and for $\nu \in \mathbb{Z}$ and $m \in \mathbb{N}$ the polynomials

$$\omega_{\nu,m}(z) \; := \; (z - x_\nu)(z - x_{\nu+1}) \cdots (z - x_{\nu+m}) \; ,$$

where z is a complex variable. Since, for $m \geq 2$,

$$\begin{aligned} \omega_{\nu,m}(z) \; &= \; (z - x_{\nu+m})\, \omega_{\nu,m-1} \\ &= \; (z - x_\nu)\, \omega_{\nu+1,m-1} \; , \end{aligned}$$

we obtain for the derivatives of these functions the equations

$$\omega'_{\nu,m-1}(x_\mu) \; = \; \frac{\omega'_{\nu,m}(x_\mu)}{x_\mu - x_{\nu+m}} \quad \text{for} \;\; \mu = \nu, \nu+1, \ldots, \nu+m-1 \tag{1.1}$$

and

$$\omega'_{\nu+1,m-1}(x_\mu) \; = \; \frac{\omega'_{\nu,m}(x_\mu)}{x_\mu - x_\nu} \quad \text{for} \;\; \mu = \nu+1, \nu+2, \ldots, \nu+m\, . \tag{1.2}$$

Now let $m \in \mathbb{N}$. A real function s is called a *spline function* resp. a *spline* of order m with respect to the set K if it possesses the properties

1. The restriction of s to each intervall I_μ belongs to the space Π_{m-1} of polynomials of degree at most $m-1$, and, if $m > 1$,
2. we have $s \in C^{m-2}(\mathbb{R})$.

The real vector space of all these splines is denoted by $S_m(K)$. Due to H. B. Curry and I. J. Schoenberg [9] there is a comfortable basis of this infinite dimensional vector space, consisting of the so-called B-splines $Q_{\nu,m}(x)$, $\nu \in \mathbb{Z}$, each of which has finite support. In this paper we will define these special splines as solutions of a certain initial value problem for a difference equation of evolution type, and derive many of their well-known and also some new properties. The difference equation is equivalent to the famous recurrence relation for B-splines, due to C. deBoor [1, 2] and M. G. Cox [8].

2. The Initial Value Problem

Let

$$q = \{q_{\nu,m}\}_{\nu\in\mathbb{Z},m\in\mathbb{N}}$$

be a two-dimensional array, consisting of real functions $q_{\nu,m}$ on $\mathbb{R}$. The set of all these arrays will be denoted by A. We are interested in such $q \in A$ which satisfy for $m \geq 2$ and all $x \in \mathbb{R}$ the difference equation

$$q_{\nu,m}(x) = \left(\frac{x - x_\nu}{x_{\nu+m} - x_\nu}\right) q_{\nu,m-1}(x) + \left(\frac{x_{\nu+m} - x}{x_{\nu+m} - x_\nu}\right) q_{\nu+1,m-1}(x) \tag{2.1}$$

and have initial values

$$q_{\nu,1} \in S_1(K) \tag{2.2}$$

with the normalization

$$\int_{-\infty}^{+\infty} q_{\nu,1}(x)\,dx = 1 \quad \text{for all } \nu \in \mathbb{Z}. \tag{2.3}$$

It is obvious that any solution of the initial value problem (2.1), (2.2), restricted to the interval I_μ, belongs to the space Π_{m-1}. We will prove more:

Theorem 2.1: *Let $q \in A$ be a solution of the initial value problem (2.1)-(2.3). Then*

$$q_{\nu,m} \in S_m(K) \quad \textit{for all } \nu \in \mathbb{Z}, m \in \mathbb{N} \tag{2.4}$$

if and only if

$$q_{\nu,1}(x) = \begin{cases} (x_{\nu+1} - x_\nu)^{-1} & \text{for } x \in I_\nu, \\ 0 & \text{for } x \in \mathbb{R}, x \notin I_\nu, \end{cases} \tag{2.5}$$

for all $\nu \in \mathbb{Z}$.

Proof. We first prove the necessity of the condition (2.5). Since $q_{\nu,1} \in S_1(K)$, there exist real numbers $\alpha_{\nu,\mu}$ such that

$$q_{\nu,1}(x) = \alpha_{\nu,\mu} \quad \text{for } x \in I_\mu,\ \nu,\mu \in \mathbb{Z} .$$

We assume $q_{\nu,2} \in S_2(K)$ for all $\nu \in \mathbb{Z}$, i.e. in particular we have $q_{\nu,2} \in C(\mathbb{R})$. Equation (2.1) yields

$$q_{\nu,2}(x) = \left(\frac{x - x_\nu}{x_{\nu+2} - x_\nu}\right) \alpha_{\nu,\mu} + \left(\frac{x_{\nu+2} - x}{x_{\nu+2} - x_\nu}\right) \alpha_{\nu+1,\mu} , \qquad x \in I_\mu , \tag{2.6}$$

and

$$q_{\nu,2}(x) = \left(\frac{x - x_\nu}{x_{\nu+2} - x_\nu}\right) \alpha_{\nu,\mu-1} + \left(\frac{x_{\nu+2} - x}{x_{\nu+2} - x_\nu}\right) \alpha_{\nu+1,\mu-1} , \quad x \in I_{\mu-1}. \tag{2.7}$$

Because of the continuity of $q_{\nu,2}(x)$ at x_μ, we may put $x = x_\mu$ in (2.7) and let $x \to x_\mu$ in (2.6) to get the same value of $q_{\nu,2}(x_\mu)$. This leads to the equation

$$(x_\nu - x_\mu)\lambda_{\nu,\mu} = (x_{\nu+2} - x_\mu)\lambda_{\nu+1,\mu} \tag{2.8}$$

where $\lambda_{\nu,\mu}$ stands for the jump of $q_{\nu,1}$ at the point x_μ, i.e.

$$\lambda_{\nu,\mu} := \alpha_{\nu,\mu} - \alpha_{\nu,\mu-1} \quad \text{for } \nu,\mu \in \mathbb{Z} .$$

From (2.8) we conclude for all $\nu, \mu, k \in \mathbb{Z}$ the relation

$$(x_\nu - x_\mu)(x_{\nu+1} - x_\mu)\,\lambda_{\nu,\mu} = (x_k - x_\mu)(x_{k+1} - x_\mu)\,\lambda_{k,\mu} . \tag{2.9}$$

Hence

$$\lambda_{\nu,\mu} = 0 \quad \text{for } \mu \neq \nu, \nu+1,\ \mu \in \mathbb{Z} ,$$

which implies

$$q_{\nu,1}(x) = \begin{cases} \alpha_{\nu,\nu} & \text{for } x \in I_\nu, \\ \gamma_\nu & \text{for } x \notin I_\nu, x \in \mathbb{R} . \end{cases}$$

The normalization condition

$$\int_{-\infty}^{+\infty} q_{\nu,1}(x)\,dx = 1$$

now yields $\gamma_\nu = 0$ and $\alpha_{\nu,\nu} = (x_{\nu+1} - x_\nu)^{-1}$. This proves the necessity of (2.5).

In order to prove the sufficiency we assume $q_{\nu,1}$ to be given by (2.5). Then it follows easily by induction that

$$q_{\nu,m} = 0 \quad \text{for all} \;\; x \le x_\nu \;\; \text{and} \;\; x > x_{\nu+m} \,.$$

Thus the restriction $p^m_{\nu,\mu}(x) \in \Pi_{m-1}$ of $q_{\nu,m}(x)$ to the interval I_μ satifies

$$p^m_{\nu,\nu-1}(x) \equiv 0 \quad \text{and} \quad p^m_{\nu,\nu+m}(x) \equiv 0 \,.$$

To show $q_{\nu,m} \in C^{m-2}(\mathbb{R})$ for $m > 1$ we only need to prove the corresponding property in a neighborhood of the knots x_μ, $\mu = \nu, \nu+1, \ldots, \nu+m$. Equivalently we claim that for $x \in \mathbb{R}$ the relation

$$(p^m_{\nu,\mu} - p^m_{\nu,\mu-1})(x) = b^m_{\nu,\mu}(x - x_\mu)^{m-1} \tag{2.10}$$

holds true with

$$b^m_{\nu,\mu} := \frac{(-1)^m}{\omega'_{\nu,m}(x_\mu)} = \frac{(-1)^m}{\prod\limits_{j=\nu,j\neq\mu}^{\nu+m} (x_\mu - x_j)} \tag{2.11}$$

for $\mu = \nu, \nu+1, \ldots, \nu+m$.
The relation (2.10) is valid for $m = 1$, since

$$p^1_{\nu,\nu-1}(x) \equiv 0\,, \quad p^1_{\nu,\nu}(x) \equiv \frac{1}{x_{\nu+1} - x_\nu}\,, \quad p^1_{\nu,\nu+1}(x) \equiv 0\,,$$

$$\omega'_{\nu,1}(x_\nu) = x_\nu - x_{\nu+1}\,, \quad \omega'_{\nu,1}(x_{\nu+1}) = x_{\nu+1} - x_\nu\,,$$

and therefore

$$(p^1_{\nu,\nu} - p^1_{\nu,\nu-1})(x) = \frac{1}{x_{\nu+1} - x_\nu} = \frac{-1}{\omega'_{\nu,1}(x_\nu)}\,,$$

$$(p^1_{\nu,\nu+1} - p^1_{\nu,\nu})(x) = \frac{-1}{x_{\nu+1} - x_\nu} = \frac{-1}{\omega'_{\nu,1}(x_{\nu+1})}\,.$$

To apply the induction principle we use (2.1) to obtain

$$\begin{aligned}
(p^m_{\nu,\mu} - p^m_{\nu,\mu-1})(x) &= \\
&= \left(\frac{x - x_\nu}{x_{\nu+m} - x_\nu}\right)(p^{m-1}_{\nu,\mu} - p^{m-1}_{\nu,\mu-1})(x) + \left(\frac{x_{\nu+m} - x}{x_{\nu+m} - x_\nu}\right)(p^{m-1}_{\nu+1,\mu} - p^{m-1}_{\nu+1,\mu-1})(x) \\
&= \frac{(-1)^{m-1}(x - x_\mu)^{m-2}}{x_{\nu+m} - x_\nu} \cdot h^m_{\nu,\mu}(x)\,,
\end{aligned}$$

where, by induction hypothesis,

$$h^m_{\nu,\mu}(x) = \begin{cases} \dfrac{x - x_\nu}{\omega'_{\nu,m-1}(x_\nu)} & \text{for } \mu = \nu, \\ \dfrac{x - x_\nu}{\omega'_{\nu,m-1}(x_\mu)} + \dfrac{x_{\nu+m} - x}{\omega'_{\nu+1,m-1}(x_\mu)} & \text{for } \mu = \nu+1, \ldots, \nu+m-1, \\ \dfrac{x_{\nu+m} - x}{\omega'_{\nu+1,m-1}(x_{\nu+m})} & \text{for } \mu = \nu+m \end{cases}$$

holds.

A little computation, using (1.1) and (1.2), yields

$$h^m_{\nu,\mu}(x) = -\frac{x - x_\mu}{\omega'_{\nu,m}(x_\mu)}(x_{\nu+m} - x_\nu) \quad \text{for } \mu = \nu, \nu+1, \ldots, \nu+m,$$

which proves the relation (2.10) and hence the suffiency of (2.5). □

3. Basic Properties of Recursively Defined B-Splines

In this section we will study the properties of splines, which are defined as solutions of the difference equation (2.1) with initial values (2.5); these special functions will be denoted as *B-splines*. The following definition conincides with the recursive definition of B-splines as given, for example, in [4] and [6].

Definition 3.1: *Let x be an arbitrary real number and $K = \{x_\nu\}$ a given knot sequence. Then we define for $m \in \mathbb{N}$, $\nu \in \mathbb{Z}$, functions $Q_{\nu,m}(x)$ by*

$$Q_{\nu,1}(x) := \begin{cases} (x_{\nu+1} - x_\nu)^{-1} & \text{for } x \in I_\nu, \\ 0 & \text{for } x \in \mathbb{R}, x \notin I_\nu, \end{cases}$$

and, for $m \geq 2$,

$$Q_{\nu,m}(x) = \left(\frac{x - x_\nu}{x_{\nu+m} - x_\nu}\right) Q_{\nu,m-1}(x) + \left(\frac{x_{\nu+m} - x}{x_{\nu+m} - x_\nu}\right) Q_{\nu+1,m-1}(x). \quad (3.1)$$

According to Theorem 2.1, the functions $Q_{\nu,m}$ belong to the spline space $S_m(K)$; they will be denoted as <u>*B-splines of order m*</u>.

Remark. One could go even further and define B-splines of order zero by setting

$$Q_{\nu,0}(x) \;:=\; (x-x_\nu)_+^{-1} \;=\; \begin{cases} (x-x_\nu)^{-1}\,, & \text{for } x > x_\nu\,, \\ 0 & \text{for } x \le x_\nu\,. \end{cases} \tag{3.2}$$

Then (3.1) would produce all B-splines of order m for $m \ge 1$, but we do not think that the functions defined in (3.2) will be of any interest in practice. Therefore, in the following we shall always assume m to be greater than zero.

In the next theorem we give some fundamental properties of the functions $Q_{\nu,m}(x)$, which follow immediately from Definition 3.1.

Theorem 3.2:

a) The functions $Q_{\nu,m}(x)$ possess the minimal-support property

$$Q_{\nu,m}(x) \begin{cases} > 0\,, & \text{if } x_\nu < x < x_{\nu+m}\,, \\ = 0\,, & \text{if } x \le x_\nu \text{ or } x > x_{\nu+m}\,. \end{cases} \tag{3.3}$$

If $m \ge 2$, then also $Q_{\nu,m}(x_{\nu+m}) = 0$.

b) The B-splines form a partition of unity, i.e. for all $m \in \mathbb{N}$ and $x \in \mathbb{R}$

$$\sum_{\nu=-\infty}^{+\infty} (x_{\nu+m} - x_\nu)\, Q_{\nu,m}(x) \;\equiv\; 1\,.$$

c) For all $\nu \in \mathbb{Z}$, $m \in \mathbb{N}$, the B-spline $Q_{\nu,m}(x)$ satisfies

$$\max_{x\in\mathbb{R}} Q_{\nu,m}(x) \;\le\; \frac{1}{x_{\nu+m} - x_\nu}\,. \tag{3.4}$$

The bound (3.4) is optimal.

Proof. The minimal-support property as well as the partition of unity are easily verified by induction with repect to m, using the recurrence relation (3.1).

Inequality (3.4) now follows immediately from assertion b), since all terms $(x_{\nu+m} - x_\nu)Q_{\nu,m}(x)$ are nonnegative.

We now prove the optimality of (3.4). For $m = 1$ this is obvious, so assume $m \ge 2$. If we let the inner knots $x_{\nu+1}, \ldots, x_{\nu+m-1}$ of the support of $Q_{\nu,m}$ all converge to $x_{\nu+m}$, we obtain after a short calculation for the limit function $Q^*_{\nu,m}$:

$$\begin{aligned} Q^*_{\nu,m}(x) \;&:=\; \lim_{\{x_{\nu+1},\ldots,x_{\nu+m-1}\}\to x_{\nu+m}} Q_{\nu,m}(x) \\ &=\; \begin{cases} \dfrac{(x-x_\nu)^{m-1}}{(x_{\nu+m}-x_\nu)^m} & \text{for } x_\nu < x \le x_{\nu+m} \\ 0 & \text{for } x \le x_\nu \text{ or } x > x_{\nu+m}. \end{cases} \end{aligned}$$

Hence

$$Q^*_{\nu,m}(x_{\nu+m}) = \frac{1}{x_{\nu+m} - x_\nu} ,$$

which shows that (3.4) is best possible. □

The next theorem provides an explicit representation of the B-splines as linear combination of the truncated power functions. Note that we have used so far nothing else but the recurrence relation (3.1)!

Theorem 3.3: *For each $m \in \mathbb{N}$ and $\nu \in \mathbb{Z}$, the function $Q_{\nu,m}(x)$ can be written as*

$$Q_{\nu,m}(x) = \sum_{\mu=\nu}^{\nu+m} b^m_{\nu,\mu} (x - x_\mu)_+^{m-1} \tag{3.5}$$

where the coefficients $b^m_{\nu,\mu}$ are defined in (2.11).

Sketch of proof. The proof can be done by induction with respect to m, very much like the second part of the proof of Theorem 2.1. □

Corollary 3.4 (Derivative recursion formula): *For all $m \geq 2$ we have the relation*

$$\frac{d}{dx} Q_{\nu,m}(x) = \frac{m-1}{x_{\nu+m} - x_\nu} \cdot (Q_{\nu,m-1}(x) - Q_{\nu+1,m-1}(x)) . \tag{3.6}$$

Proof. Differentiation of the representation (3.5) yields

$$\frac{d}{dx} Q_{\nu,m}(x) = (m-1) \sum_{\mu=\nu}^{\nu+m} b^m_{\nu,\mu} (x - x_\mu)_+^{m-2} , \tag{3.7}$$

while on the other hand it is clear that the right hand side of (3.6) is of the form

$$(m-1) \sum_{\mu=\nu}^{\nu+m} c^m_{\nu,\mu} (x - x_\mu)_+^{m-2}$$

with the coefficients

$$c^m_{\nu,\mu} = \frac{1}{x_{\nu+m} - x_\nu} \cdot \begin{cases} b^{m-1}_{\nu,\nu} & , \text{if } \mu = \nu , \\ (b^{m-1}_{\nu,\mu} - b^{m-1}_{\nu+1,\mu}) & , \text{if } \nu + 1 \leq \mu \leq \nu + m - 1 , \\ b^{m-1}_{\nu+1,\nu+m} & , \text{if } \mu = \nu + m . \end{cases}$$

Obviously, $c^m_{\nu,\mu} = b^m_{\nu,\mu}$ for $\mu = \nu$ and $\mu = \nu + m$. But also for $\mu = \nu + 1, \ldots, \nu + m - 1$ we obtain

$$c^m_{\nu,\mu} = \frac{1}{x_{\nu+m} - x_\nu} \cdot \left(\frac{(-1)^{m-1}}{\omega'_{\nu,m-1}(x_\mu)} - \frac{(-1)^{m-1}}{\omega'_{\nu+1,m-1}(x_\mu)} \right)$$
$$= \frac{(-1)^{m-1}}{x_{\nu+m} - x_\nu} \cdot \left(\frac{x_\mu - x_{\nu+m}}{\omega'_{\nu,m}(x_\mu)} - \frac{x_\mu - x_\nu}{\omega'_{\nu,m}(x_\mu)} \right)$$
$$= \frac{(-1)^{m-1}}{\omega'_{\nu,m}(x_\mu)} = b^m_{\nu,\mu} ,$$

due to (1.1) and (1.2). □

Moreover, the explicit representation of the B-spline $Q_{\nu,m}(x)$ given in Theorem 3.3 enables us to give a very short proof of the famous contour integral formula for B-splines, which originally appeared in [10]. One should also compare the complex case treated in [14].

Theorem 3.5: *Let $\nu \in \mathbb{Z}$ and $m \in \mathbb{N}$. For $x \in \mathbb{R}$ let $C(x)$ denote a simply closed and rectifiable curve in the complex plane such that all knots x_μ with $x < x_\mu \leq x_{\nu+m}$ and no others lie inside of $C(x)$. Then the following contour integral representation for the B-spline $Q_{\nu,m}(x)$ is valid:*

$$Q_{\nu,m}(x) = \frac{1}{2\pi i} \int_{C(x)} \frac{(z-x)^{m-1}\, dz}{\omega_{\nu,m}(z)} ,$$

where the integration is carried out in the positive sense.

Proof. We denote the integrand by $\varphi(z)$. Then the residue of φ in the point x_μ, $\nu \leq \mu \leq \nu + m$, equals

$$\mathrm{Res}\,(\varphi, x_\mu) = \frac{(x_\mu - x)^{m-1}}{\omega'_{\nu,m}(x_\mu)}$$
$$= \frac{(-1)^{m-1}}{\omega'_{\nu,m}(x_\mu)} \cdot (x - x_\mu)^{m-1}$$
$$= -b^m_{\nu,\mu} \cdot (x - x_\mu)^{m-1} .$$

Now let k be any fixed integer with $\nu \leq k \leq \nu + m - 1$ and $x \in I_k$. Then the theorem of residues implies

$$\frac{1}{2\pi i}\int_{C(x)} \varphi(z)\,dz = \sum_{\mu=k+1}^{\nu+m} -b^m_{\nu,\mu}(x-x_\mu)^{m-1}$$

$$= \sum_{\mu=\nu}^{\nu+m} -b^m_{\nu,\mu}(x-x_\mu)^{m-1} - \sum_{\mu=\nu}^{k} -b^m_{\nu,\mu}(x-x_\mu)^{m-1}$$

$$= \sum_{\mu=\nu}^{k} b^m_{\nu,\mu}(x-x_\mu)^{m-1},$$

since $\sum_{\mu=\nu}^{\nu+m} -b^m_{\nu,\mu}(x-x_\mu)^{m-1} = 0$, due to Theorem 3.3. □

4. Some Connections with Linear Functionals and Divided Differences

Let $K=\{x_\mu\}$ again denote some fixed knot sequence. In this section we shall study linear functionals $L=L_{\nu,m}$ on the space $C(\mathbb{R})$ of the form

$$L(f) = \sum_{\mu=\nu}^{\nu+m} \alpha^m_{\nu,\mu} f(x_\mu), \tag{4.1}$$

with certain given coefficients $\alpha^m_{\nu,\mu}$.

Our first result is the following theorem, which has - in different forms - already appeared in the literature, see e.g. [5], [11], [14]:

Theorem 4.1: *Let the linear functional (4.1) satisfy the conditions*

$$L(g_k) = \begin{cases} 0, & \text{if } k=0,1,\ldots,m-1, \\ (-1)^m, & \text{if } k=m, \end{cases} \tag{4.2}$$

with $g_k(t) := t^k$ for $k=0,1,\ldots,m$. Then the coefficients $\alpha^m_{\nu,\mu}$ are uniquely determined and satisfy

$$\alpha^m_{\nu,\mu} = b^m_{\nu,\mu} \qquad \text{for } \mu=\nu,\ldots,\nu+m$$

with the numbers $b^m_{\nu,\mu}$ from (2.11).
In particular, we have

$$L(f) = (-1)^m \cdot \Delta(x_\nu,\ldots,x_{\nu+m};f), \tag{4.3}$$

where Δ denotes the usual $m-$ th order divided difference operator.

Proof. The $(m+1)$ conditions (4.2) lead to an $(m+1)\times(m+1)-$ system of linear equations for the determination of the coefficients $\alpha^m_{\nu,\mu}$. The matrix A of this system is a Vandermondian, hence regular, which proves the fact that the system has a unique solution.

For the determinant of A we have

$$\det(A) \;=\; \prod_{\nu\le k<j\le\nu+m} (x_j - x_k)\;,$$

and therefore the application of Cramer's rule to (4.2) leads to the solutions

$$\begin{aligned}
\alpha^m_{\nu,\mu} \;&=\; \frac{(-1)^{m+\mu-\nu}(-1)^m \prod\limits_{\substack{\nu\le k<j\le\nu+m\\ k,j\neq\mu}} (x_j - x_k)}{\det(A)} \\
&=\; \frac{(-1)^{\mu-\nu}}{\prod\limits_{\mu<j\le\nu+m} (x_j - x_\mu)\cdot \prod\limits_{\nu\le k<\mu} (x_\mu - x_k)} \\
&=\; \frac{(-1)^{\mu-\nu}}{\prod\limits_{\mu<j\le\nu+m} (x_\mu - x_j)\cdot(-1)^{\nu+m-\mu}\cdot \prod\limits_{\nu\le j<\mu} (x_\mu - x_j)} \;=\; b^m_{\nu,\mu}\;.
\end{aligned}$$

Finally, equation (4.3) is an immediate consequence of the well-known fact (see e.g. [11]) that the divided difference Δ satisfies the relations

$$\Delta(x_\nu,\dots,x_{\nu+m};g_k) \;=\; \begin{cases} 0, & \text{if } k=0,1,\dots,m-1,\\ 1, & \text{if } k=m\,. \end{cases} \tag{4.4}$$

□

Originally, B-splines have been introduced as divided differences of a certain truncated power function, and one would expect that from our definition the same representation can be derived. That this is indeed true shows the following corollary to Theorem 4.1.

Corollary 4.2: *The B-spline $Q_{\nu,m}(x)$ has the representation*

$$Q_{\nu,m}(x) \;\equiv\; (-1)^m\cdot\Delta(x_\nu,\dots,x_{\nu+m};(x-t)^{m-1}_+)\;;$$

here, the divided difference has to be taken with respect to t.

Proof. Follows directly from Theorems 3.3 and 4.1. □

As is well-known (cf. [5], [13]), a linear functional of the above type can also be written as a quotient of two determinants. In the case of the B-spline $Q_{\nu,m}(x)$ this is asserted in the next lemma, for which we need the following notation: For functions $v_0, \ldots, v_m$ and numbers $y_0, \ldots, y_m$, we set

$$\det\begin{pmatrix} v_0 & v_1 & \cdots & v_m \\ y_0 & \cdots & \cdots & y_m \end{pmatrix} := \det\begin{pmatrix} v_0(y_0) & \cdots & \cdots & v_0(y_m) \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ v_m(y_0) & \cdots & \cdots & v_m(y_m) \end{pmatrix}.$$

As in Corollary 4.2, we treat x as a parameter and take the determinant with respect to the variable t.

Lemma 4.3: *The B-spline $Q_{\nu,m}(x)$ has the representation*

$$Q_{\nu,m}(x) \equiv (-1)^m \cdot \frac{\det\begin{pmatrix} g_0 & \cdots & g_{m-1} & (x-t)_+^{m-1} \\ x_\nu & \cdots & \cdots & x_{\nu+m} \end{pmatrix}}{\det\begin{pmatrix} g_0 & g_1 & \cdots & g_m \\ x_\nu & \cdots & \cdots & x_{\nu+m} \end{pmatrix}}.$$

Proof. Obvious; see also [5] or [12]. □

5. Results Concerning Integrals over B-Splines

An important result in the theory of B-splines, originally due to Curry and Schoenberg, is the fact that the integral of $Q_{\nu,m}(x)$ over $\mathbb{R}$ does neither depend on the location of the knots $\{x_\nu\}$ nor on the index ν, but only on the order m; this is established in the following Lemma 5.1. Of course this could be deduced as a special case from Theorems 5.2 and 5.3, but we want to give a very simple direct proof, based on the representation formula (3.5).

Lemma 5.1: *For all $m \in \mathbb{N}$ and $\nu \in \mathbb{Z}$ we have*

$$\int_{-\infty}^{\infty} Q_{\nu,m}(x)\,dx = \frac{1}{m}. \tag{5.1}$$

Proof. We start with the formula (3.5) and obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} Q_{\nu,m}(x)\,dx &= \int_{x_\nu}^{x_{\nu+m}} Q_{\nu,m}(x)\,dx \\
&= \frac{1}{m}\cdot\sum_{\mu=\nu}^{\nu+m} b^m_{\nu,\mu}\,(x-x_\mu)^m_+ \Bigg|_{x_\nu}^{x_{\nu+m}} \\
&= \frac{1}{m}\cdot\sum_{\mu=\nu}^{\nu+m} b^m_{\nu,\mu}\,(x_{\nu+m}-x_\mu)^m\,. \qquad (5.2)
\end{aligned}$$

From Theorem 4.1 we know that the linear functional

$$L(f) = \sum_{\mu=\nu}^{\nu+m} b^m_{\nu,\mu}\,f(x_\mu) \qquad (5.3)$$

has the properties

$$L(g_k) = \begin{cases} 0, & \text{if } k=0,1,\ldots,m-1,\\ (-1)^m, & \text{if } k=m,\end{cases}$$

with $g_k(t)=t^k$ for $k=0,1,\ldots,m$. Therefore (5.2) equals

$$\frac{1}{m}\cdot L((-1)^m\,g_m(t)) = \frac{1}{m}\,,$$

which completes the proof of Lemma 5.1. □

In combination with Theorem 3.2 a), Lemma 5.1 says that the B-spline function $m\cdot Q_{\nu,m}(x)$ is a probability density. Therefore it seems natural to ask for the corresponding *distribution function* as well as for the k^{th} *moments*; these questions will be answered in the subsequent two theorems, the first of which is essentially due to C. de Boor, T. Lyche and L. L. Schumaker.

Theorem 5.2 (cf. C. de Boor, T. Lyche and L. L. Schumaker [3]): *For all* $\nu\in\mathbb{Z}$, $m\in\mathbb{N}$ *and* $x\in\mathbb{R}$,

$$m\int_{-\infty}^{x} Q_{\nu,m}(t)\,dt = \sum_{\mu=\nu}^{\infty}(x_{\mu+m+1}-x_\mu)Q_{\mu,m+1}(x) \qquad (5.4)$$

$$= \begin{cases} 0\,, & \textit{if } x\le x_\nu,\\ \displaystyle\sum_{\mu=\nu}^{k}(x_{\mu+m+1}-x_\mu)Q_{\mu,m+1}(x), & \textit{if } x\in I_k, k=\nu,\ldots,\nu+m\,,\\ 1\,, & \textit{if } x> x_{\nu+m+1}.\end{cases} \qquad (5.5)$$

Proof. Relation (5.4) can be easily deduced from Lemma 2.1 in [3], while (5.5) is then an immediate consequence of the minimal-support property of the functions $Q_{\mu,m+1}$ (Theorem 3.2 a) and b)). □

Continuing the study of the B-splines' probability theoretic aspects, we now prove an explicit formula for the so-called k^{th} moments of these functions.

Theorem 5.3: *For all $k \in \mathbb{N}_0$ the following explicit formula for the k^{th} moment of $Q_{\nu,m}(x)$ holds:*

$$\int_{-\infty}^{\infty} x^k \cdot Q_{\nu,m}(x)\,dx = \frac{(m-1)!\,k!}{(m+k)!} \cdot \Delta(x_\nu, \ldots, x_{\nu+m}; t^{m+k}) . \tag{5.6}$$

Remark. For $k = 1$, Theorem 5.3 provides a formula for the *mean* of the probability density function $m\,Q_{\nu,m}(x)$, i.e.

$$\rho_{\nu,m} := \int_{-\infty}^{\infty} x \cdot m\,Q_{\nu,m}(x)\,dx = \frac{x_\nu + x_{\nu+1} + \cdots + x_{\nu+m}}{m+1} .$$

Proof of Theorem 5.3. We use induction on m, so let $m = 1$. In this case the assertion is true, since for all $k \in \mathbb{N}_0$ the application of Definition 3.1 directly yields

$$\begin{aligned}
\int_{-\infty}^{\infty} x^k \cdot Q_{\nu,1}(x)\,dx &= \frac{1}{x_{\nu+1} - x_\nu} \cdot \int_{x_\nu}^{x_{\nu+1}} x^k\,dx \\
&= \frac{1}{k+1} \cdot \frac{x_{\nu+1}^{k+1} - x_\nu^{k+1}}{x_{\nu+1} - x_\nu} \\
&= \frac{0!\,k!}{(1+k)!} \cdot \Delta(x_\nu, x_{\nu+1}; t^{1+k}) .
\end{aligned}$$

Now let $m \geq 2$. Our induction hypothesis is that for all exponents $r \in \mathbb{N}_0$ and for all indices $\mu \in \mathbb{Z}$ the relation

$$\int_{-\infty}^{\infty} x^r \cdot Q_{\mu,m-1}(x)\,dx = \frac{(m-2)!\,r!}{(m+r-1)!} \cdot \Delta(x_\mu, \ldots, x_{\mu+m-1}; t^{m+r-1}) \tag{5.7}$$

holds. To prove (5.6), we use integration by parts and apply the differential recursion formula (3.6). Then

$$\int_{-\infty}^{\infty} x^k \cdot Q_{\nu,m}(x)\,dx = \frac{x^{k+1}}{k+1} \cdot Q_{\nu,m}(x)\Big|_{-\infty}^{\infty} - \frac{1}{k+1}\int_{-\infty}^{\infty} x^{k+1} \cdot Q'_{\nu,m}(x)\,dx$$

$$= -\frac{m-1}{(k+1)(x_{\nu+m}-x_\nu)}\left\{\int_{-\infty}^{\infty} x^{k+1} \cdot Q_{\nu,m-1}(x)\,dx - \right.$$

$$\left. - \int_{-\infty}^{\infty} x^{k+1} \cdot Q_{\nu+1,m-1}(x)\,dx \right\}.$$

Equation (5.7) yields

$$\int_{-\infty}^{\infty} x^k \cdot Q_{\nu,m}(x)\,dx = -\frac{(m-1)!\,k!}{(m+k)!\,(x_{\nu+m}-x_\nu)}\,\{\Delta(x_\nu,\ldots,x_{\nu+m-1};t^{m+k}) -$$

$$- \Delta(x_{\nu+1},\ldots,x_{\nu+m};t^{m+k})\}$$

$$= \frac{(m-1)!\,k!}{(m+k)!} \cdot \Delta(x_\nu,\ldots,x_{\nu+m};t^{m+k}),$$

where we have used the well-known recursion formula for divided differences, see e.g. [11]. □

Remark. The divided difference, applied to a monomial, can be computed recursively too. Let us consider the functions

$$f_k(t) := t^{k-1}\,\omega_{\nu,m}(t)$$

$$= t^{k-1}(t-x_\nu)\cdots(t-x_{\nu+m})$$

$$= t^{k-1}\left(t^{m+1} - \sum_{\mu=1}^{m+1}(-1)^{\mu+1}\lambda_\mu\, t^{m+1-\mu}\right),$$

where $\lambda_\mu = \lambda_{\mu,\nu,m}$ denotes the elementary symmetric functions of the quantities $x_\nu, x_{\nu+1}, \ldots, x_{\nu+m}$, e.g.

$$\lambda_1 = \sum_{\rho=\nu}^{\nu+m} x_\rho,$$

$$\lambda_2 = \sum_{\substack{\rho,\tau=\nu \\ \rho<\tau}}^{\nu+m} x_\rho x_\tau \,,$$

$$\lambda_3 = \sum_{\substack{\rho,\tau,\kappa=\nu \\ \rho<\tau<\kappa}}^{\nu+m} x_\rho x_\tau x_\kappa \,, \quad \text{etc.}$$

Then obviously

$$\Delta(x_\nu, \ldots, x_{\nu+m}; f_k) = 0 \quad \text{for all } k \in \mathbb{N} \,.$$

Due to the linearity of the functional Δ we get the relation

$$\Delta(x_\nu, \ldots, x_{\nu+m}; t^{m+k}) = \sum_{\mu=1}^{m+1} (-1)^{\mu+1} \lambda_\mu \, \Delta(x_\nu, \ldots, x_{\nu+m}; t^{m+k-\mu})$$

for $k \in \mathbb{N}$. Equation (4.4) now yields

$$\Delta(x_\nu, \ldots, x_{\nu+m}; t^{m+1}) = \lambda_1 \qquad \text{and}$$

$$\Delta(x_\nu, \ldots, x_{\nu+m}; t^{m+2}) = \lambda_1 \Delta(x_\nu, \ldots, x_{\nu+m}; t^{m+1}) - \lambda_2 \Delta(x_\nu, \ldots, x_{\nu+m}; t^m)$$

$$= \lambda_1^2 - \lambda_2 \,.$$

Using Theorem 5.3 we get for the 2^{nd} moment of the probability density $m\, Q_{\nu,m}(x)$ the value

$$\kappa_{\nu,m} := \int_{-\infty}^{\infty} x^2 \cdot m\, Q_{\nu,m}(x)\, dx = \frac{2}{(m+1)(m+2)} \cdot (\lambda_1^2 - \lambda_2) \,.$$

Thus for the variance

$$\sigma_{\nu,m}^2 := \int_{-\infty}^{\infty} (x - \rho_{\nu,m})^2 \cdot m\, Q_{\nu,m}(x)\, dx = \kappa_{\nu,m} - \rho_{\nu,m}^2$$

it follows, after a short computation,

$$\sigma_{\nu,m}^2 = \frac{m\lambda_1^2 - 2(m+1)\lambda_2}{(m+1)^2(m+2)} = \frac{1}{(m+1)^2(m+2)} \sum_{\substack{\rho,\tau=\nu \\ \rho<\tau}}^{\nu+m} (x_\rho - x_\tau)^2 \,.$$

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